Stochastic Resonance in a Noise-Driven Excitable System

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1. Introduction and Methods

My aim is the study of the dynamics of the excitable Fitz Hugh–Nagumo system under both an external noise and a sinusoidal "force", whose period will be approximately the period of the noise-induced pulses that appear even in the absence of this driving force. This choice of driving frequency will lead to the *stochastic resonance* phenomenon: the noise-induced oscillations will synchronize with the external forcing.

The Fitz Hugh-Nagumo system is described by:

$$\varepsilon \frac{dx}{dt} = x - \frac{x^3}{3} - y \tag{1}$$

$$\frac{dy}{dt} = x + a + D\xi(t) \tag{2}$$

The main system I will work with is a simple modification of the equations above: I will add to the second dy/dt equation (the stochastic differential equation) an external periodic term of the form $Acos(\Omega t)$:

$$\varepsilon \frac{dx}{dt} = x - \frac{x^3}{3} - y \tag{3}$$

$$\frac{dy}{dt} = x + a + Acos(\Omega t) + D\xi(t) \tag{4}$$

being Ω a frequency near the "natural frequency" of the non-driven system. Thus, in order to choose the applied frequency for each noise amplitude D I will need to get the period of the pulses of the self-excited system previously.

Regarding the numerical details, equation (3) is an ordinary differential equation that can be easily integrated with the explicit Euler Method. I chose this method because the authors of reference [1] do the same. For that, a small enough time step $h = 0.0001$ was considered in order to assure accuracy and convergence of the solution. On the other hand, equation (4) was integrated using the Milstein algorithm for an Ito stochastic differential equation. The algorithms are hence:

$$x(t_{i+1}) = x(t_i) + \frac{h}{\varepsilon} q_1(x(t_i), y(t_i)) \qquad q_1(x,y) = x - \frac{x^3}{3} - y$$

$$y(t_{i+1}) = y(t_i) + hq_2(x(t_i), y(t_i), t) + Dh^{1/2} u_i \qquad q_2(x,y,t) = x + a + Acos(\Omega t)$$

u_i is a different uniform random number $U(0,1)$ for each time step and it is indeed the numerical implementation of a Gaussian noise $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$. In all the upcoming figures it was considered a time step $h = 0.0001$ and final time $t_f = 100$. For the coherence/time correlations analysis a much longer trajectory must be considered and that is why I took $t_f = 11000$ and a smaller time step $h = 0.001$ so as to avoid unnecessary long computational times. The remaining parameters used are $a = 1.05$ and $\varepsilon = 0.01$ as done in [1] and the amplitude of the external term is $A = 0.1$.

2. Results and Discussion

As shown in Figure 1, the first thing we can do is make a quick check of the trajectories. The initial conditions are set at the origin and for both small and large noise the system ends up in a periodic limit cycle. The fixed points of the deterministic system are $x^* = -a$ and $y^* = a^3/3 - a$ and, as expected, the larger the noise, the bigger the influence of the stochastic term $D\xi(t)$ and the farther the system is from the deterministic trajectory. The x and y coordinates have been measured every 20 time steps ($\Delta t = 0.002$).

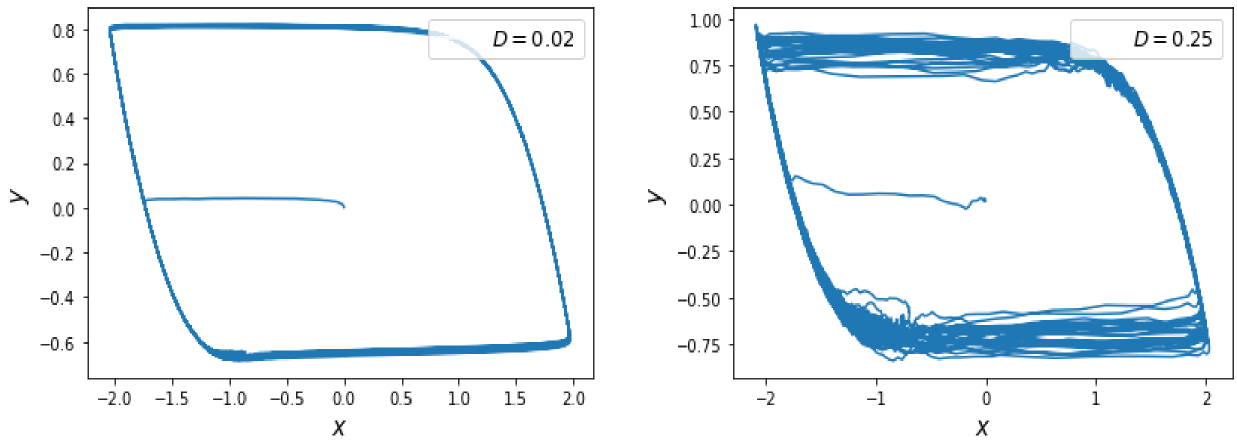

Figure 1: Trajectory of the system for small and large noise.

The next thing we can do is the study of the periodicity and the coherence with the external force of the y coordinate pulses and to this effect we first need to get the period of the original non-forced system described by (1) and (2). Using the same noise amplitudes as in the paper, namely $D = [0.02,\ 0.07,\ 0.25]$ I get, respectively, $T = [5,\ 4,\ 3.5]$.

Figures 2 a) and b) reveal an important difference between the coherence resonance (no external force) and the stochastic resonance phenomena: for the first case the period of the peaks is not well defined for all noise amplitudes whereas in the driven case the period of the peaks is always coherent with the external driving. Graphically the difference is manifest since in Figure 2 a) the peaks (the excitations) are irregular. As long as we choose a convenient frequency for the $Acos(\Omega t)$ term, the oscillations are always coherent for any value of D. For example, for $D = 0.02$ I chose a period $T = 3.3$ and the results pulses are of the same period. And how can we numerically measure this coherence? A straightforward method is the correlation time function $C(\tau)$ shown in Figures 2 c) and d):

$$C(\tau) = \frac{\langle \tilde{y}(t)\tilde{y}(t+\tau)\rangle_t}{\langle \tilde{y}^2\rangle} \quad ; \quad \tilde{y} = y - \langle y\rangle \tag{5}$$

The reason why I plot the square of $C(\tau)$ is simply because the characteristic correlation time τ_c is merely the following integral:

$$\tau_c = \int_0^\infty C(\tau)^2 d\tau \tag{6}$$

The computation of this correlation function $C(\tau)$ requires either an average over a long trajectory or an average over a set of many shorter trajectories. I opted for the first option by generating one trajectory with final time $t_f = 11,000$ and $550,000$ time values. In order to properly calculate the correlation times it is crucial to get accurate correlation plots. And from equation (6) it is clear why we need a long trajectory for that purpose (at least 1000 times the characteristic correlation time): the average over t stops when $t + \tau = t_f$ (the final points of the trajectory have no existing forward points to be correlated with).

The magnitude of the correlation times τ_c shown in the bottom figures of Figure 2 are consistent with the pulses depicted above them: for perfectly regular and periodic peaks we expect a large correlation time, while for irregular peaks and a not well defined period the correlation time must be smaller. The correlations measure somehow how similar the values are and thus for periodic series we have more similarity and higher correlation.

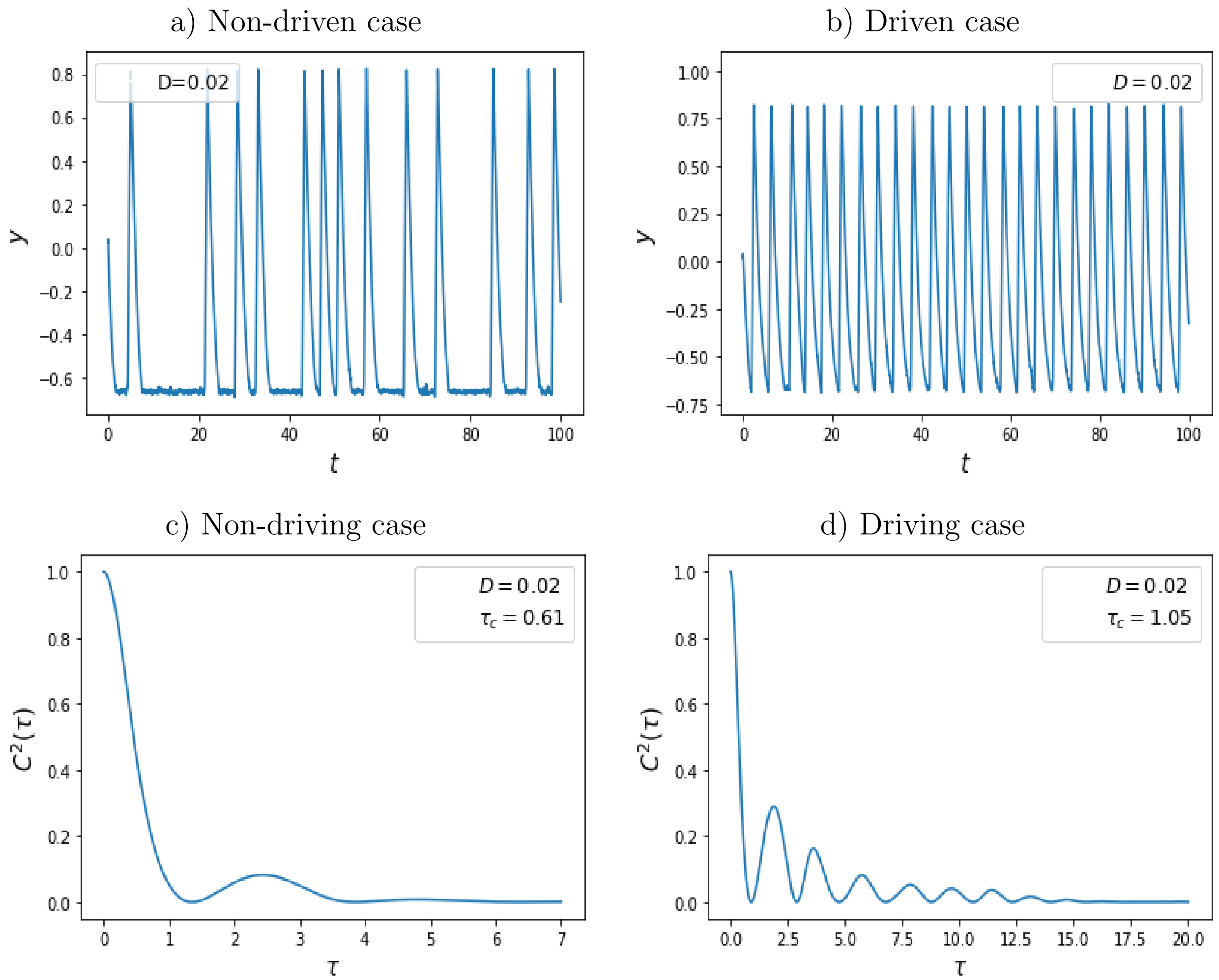

Figure 2: y coordinate pulses and the square of the correlation function for forced and non-forced systems.

Finally let's go back to the original non-forced Fitz Hugh-Nagumo system described by (1) and (2) and let's compute the optimal noise for which the pulses are almost perfectly periodic. This is done too in reference [1] and my aim is to compare my results with theirs. For that I consider 7 values of the noise amplitude D in the interval $[0.02, 0.14]$ and for each noise I compute the characteristic correlation time τ_c (6). We must thus evaluate 7 integrals not for a well defined function $f(x)$ as usual but for a vector of values $C(\tau)$. The separation between correlation values is $\Delta\tau = 0.02$, the same as the y coordinate vector. The integration method used is the trapezoids method, whose maximal error is:

$$f''(\xi)\frac{b^3}{12N^2} \qquad N = \frac{b}{\Delta\tau} \tag{7}$$

being b the final value of the integration interval (different for each noise because the decay rate is different), N the number of sub-intervals of integration and ξ the value of τ for which the second derivative is maximum. Thus, in order to estimate and plot the maximum numerical errors we need to get for each noise the maximum second derivative of $C^2(\tau)$:

$$f''(x_i) = \frac{f^2(x_{i-1}) - 2f^2(x_i) + f^2(x_{i+1})}{h^2}$$

For all noises the value $f''(\xi)$ was about 10 in absolute value and it corresponds to the first fast decay of $C^2(\tau)$ observed in Figure 2. Moreover, all b's are of order 10^1 and finally the maximum errors of this integration method is of order 10^{-3} as can be deduced from equation (7).

Figure 3 clearly depicts a maximum coherence/correlation time for a noise amplitude $D = 0.07$. The fitting is done with a fifth order polynomial and, unlike the non-driven case seen in Figure 2, the pulses are almost perfectly periodic as we had in the stochastic resonance case. Finally we can understand why this phenomena is called *coherence resonance*: there is an optimal noise value for which the coherence is maximum and the excitations are equally spaced. In the reference [1] the optimal value of D is 0.06, very similar to the one I found.

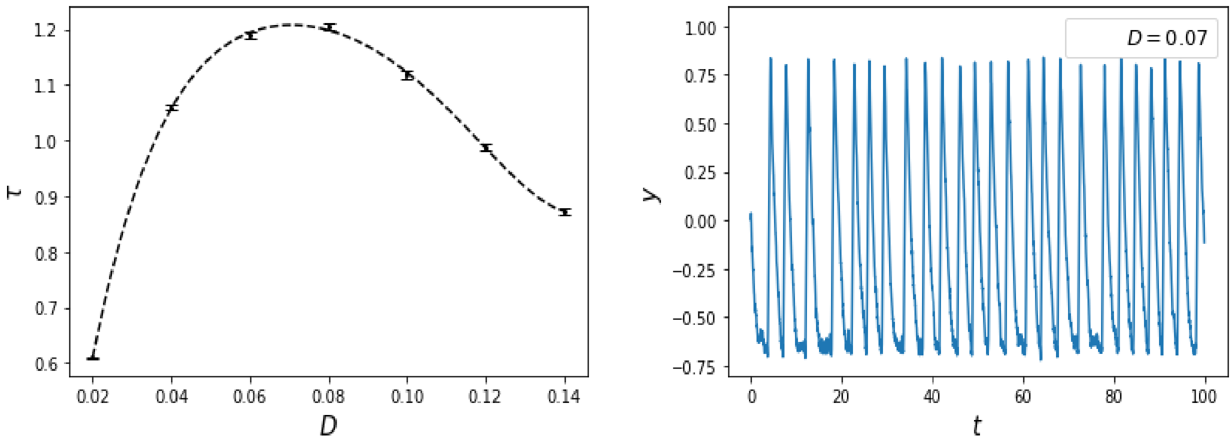

Figure 3: Correlation times for different noises and pulses for the optimal noise $D = 0.07$

3. Conclusions

I analyzed two systems: the non-driven Fitz Hugh-Nagumo excitable system and the same system with an external term $Acos(\Omega t)$ in the y component. In the first case the pulses were not totally periodic for any noise amplitude D but rather there was a clear coherence maximum for $D = 0.07$. In this case we come across with some sort of noise-induced order for this optimal noise value. In the second case, however, we always achieve perfectly periodic pulses in the $y - t$ plane as long as the applied frequency Ω is always similar to the noise-induced oscillations and the resulting pulses are of the same period as the external "force". The correlation time analysis does not reveal any clear maximum for different values of D.

References

[1] A. Pikovsky, J. Kurths, *Coherence resonance in a noise-driven excitable system.* Phys. Rev. Lett. 78 (5) (1997) 775–778.