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Course: 01:198:206 (04:18156) Section 4 Spring 2020

Homework 5

Answer to question 1

Bernoulli E(X) = p

Bernoulli Var(X) = $p - p^2$

Binomial E(Y) = n * p

Binomial Var(Y) = $np - np^2$

When X and Y are independent,

$\mathrm{E[XY]} = \mathrm{E[X]E[Y]}$

$E[X^2Y^2] = E[X^2]E[Y^2]$

$$\mathrm{Var}(XY) = E(X^2Y^2) - \left(E(XY)\right)^2$$

$$= \mathrm{Var}(X)\mathrm{Var}(Y) + \mathrm{Var}(X)\left(E(Y)\right)^2 + \mathrm{Var}(Y)\left(E(X)\right)^2$$

$$= \boldsymbol{(p - p^2)(np - np^2) + (p - p^2)(n^2p^2) + (np - np^2)(p^2)}$$

Since, $Var(aXY) = a^2Var(XY)$

Var(Z) = $\frac{\boldsymbol{(p-p^2)(np-np^2)+(p-p^2)(n^2p^2)+(np-np^2)(p^2)}}{n^2}$

Answer to question 2

Geometric distribution

P(biased heads) = $\frac{1}{4}$

P(heads) = $\frac{1}{2}$

P(biased) = $\frac{1}{2}$

P(fair) = $\frac{1}{2}$

Var(M) = E(M^2) – E^2(M)

E(fair) = $\frac{1}{1-\frac{1}{2}} = 2$

E(biased) = $\frac{1}{1-3/4} = 4$

E(X) = ½ * 2 + ½ * 4 = 3

Var(fair) = (1-p)/p^2 = 2

Var(biased) = (1-p)/p^2 = 12

E(fair^2) = 2+2 = 4

E(biased^2) = 4 + 12 = 16

E(X^2) = ½ * 4 + ½ * 16 = 10

Var(X) = E(X^2) – E(X)^2 = 10 – 3^2 = 10 – 9 = **1**

Answer to question 3

Upper bound to open door = n tries

P(T=1) = 1/n

P(T=2) = (n-1)/n * 1/(n-1) = 1/n

P(T=3) = (n-1)/n * (n-2)/(n-1) * 1/(n-2) = 1/n

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P(T=n) = ½

E(X) = 1*1/n + 2*1/n + 3*1/n + ….. + n*1/n = n(n+1)/2n = **(n+1)/2**

Var(T) = E(T^2) – E^2(T)

= $\frac{1}{n}\sum_{k=1}^{n} k^2 - \left(\frac{n+1}{2}\right)^2 = \frac{1}{n}\left(\frac{n(n+1)(2n+1)}{6}\right) - \left(\frac{n+1}{2}\right)^2 = \boldsymbol{\frac{n^2-1}{12}}$

Answer to question 4

Part (a)

3 spades = 13 choose 3 * 39 choose 2 = 211,926

4 spades = 13 choose 4 * 39 choose 1 = 27,885

5 spades = 13 choose 5 * 39 choose 0 = 1287

Total possibilities = 241,098

P(5 card poker hand with at least 3 spades) = 241098/(52choose5) = 3091/33320 = **0.0928**

Part (b)

E(spades in hand) = 5 * 13/52 = 1.25

$$\Pr(spades \geq 3) \leq \frac{1.25}{3} = \frac{5}{12} = \mathbf{0.416667}$$

Part (c)

Consider (spades – 1.25)^2 as a random variable

P(spades)	spades	Spades – 1.25	(spades-1.25)^2	P(spades)*(spades-1.25)^2
2109/9520	0	-1.25	1.5625	0.346146
1081/1020	1	-0.25	0.0625	0.066238
9139/33320	2	0.75	0.5625	0.154282
2717/33320	3	1.75	3.0625	0.249724
143/13328	4	2.75	7.5625	0.08114
33/66640	5	3.75	14.0625	0.006964

So E((spades-1.25)^2) = sum P(spades) * (spades-1.25)^2 = 0.904495

$\Pr(|spades - E(spades)| \geq 3) \leq \frac{0.904495}{3^2} =$ **0.100499**

Answer to question 5

Since Markov bounds only applied to non-negative random variable, we will move up the number line.

Y = X + 100

So, Y is strictly larger than 0

Then, E(Y) = E(X+100) = E(X) +100 = -60 + 100 = 40

$$P(X \geq \ -20) = P(Y \geq 80) \leq \frac{E(Y)}{80} = \frac{40}{80} = \mathbf{\frac{1}{2}}$$

Answer to question 6

E(one dice roll) = 1/6 * 1 + 1/6 * 2 + 1/6 * 3 + 1/6 * 4 + 1/6 * 5 + 1/6 * 6 = 7/2

E(X) = 100 * 7/2 = 350

Var(one dice roll) = E((R – 7/2)^2)

Consider (R-7/2)^2 as a random variable

R	R-7/2	(R-7/2)^2
1	-2.5	6.25
2	-1.5	2.25
3	-0.5	0.25
4	0.5	0.25
5	1.5	0.25
6	2.5	6.25

So E((R – 7/2)^2) = 1/6 * 6.25 + 1/6 * 2.25 + … + 1/6 * 6.25 = 35/12 = 2.92

Var(X) = 100 * 35/12

$$\mathrm{P}(|X-350| \geq 50) \leq \frac{100 * \frac{35}{12}}{50^2} = \boldsymbol{\frac{7}{60}}$$

Answer to question 7

$\mathrm{Var}[X] = E[(X)^2] - E[X]^2$

$\mathrm{Var}[Y] = E[(Y)^2] - E[Y]^2$

When X and Y are independent,

$E[XY] = E[X]E[Y]$

$$
\begin{aligned}
\mathrm{Var}[X - Y] &= E[(X - Y)^2] - E[X - Y]^2 \\
&= E[X^2 - 2XY + Y^2] - (E[X] - E[Y])^2 \\
&= E[X^2] - 2E[XY] + E[Y^2] - (E[X]^2 - 2E[X]E[Y] + E[Y]^2) \\
&= E[X^2] - E[X]^2 + E[Y^2] - E[Y]^2 \\
&= \mathbf{Var}[\mathbf{X}] + \mathbf{Var}[\mathbf{Y}]
\end{aligned}
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