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Midterm1 for CS323 Fall 2020

Answer to question 1 part (a)

i. It is always guaranteed to converge, while Newton's method is not.

ii. The Secant method can be used without having a formula for $f_0(x)$.

Answer to question 1 part (b)

iii. We can use the Bisection method without any knowledge of the derivative f0(x).

Answer to question 1 part (c)

ii. We may still be able to use Newton's method, if we have the explicit formula for f(x). However, the order of convergence would degrade to linear.

Answer to question 1 part (d)

ii. Such an interpolant generally exists, but it is not unique.

Answer to question 2 part (a)

Q. One advantage of Newton interpolation over the Vandermonde matrix method

- Enables both efficient evaluation and allows for incremental interpolation.

Q. One advantage of Lagrange interpolation over Newton interpolation.

- It is very cheap to determine the polynomial as we essentially write down a formula for the polynomial.

Answer to question 2 part (b)

Q. Is Bisection search a fixed-point method? Justify your answer.

Bisection search is not a fixed-point method.

The bisection algorithm locates a value by checking two sub-intervals $[a, x_m], [x_m, b]$ where x_m is the midpoint. It starts with two end points $a \; and \; b$ with different signs and linearly converges to a value $c \in (a, b)$ such that $f(c) = 0$

Answer to question 2 part (c)

Q. Give two reasons why the triangular matrix method for Newton interpolation might be preferable over the Vandermonde matrix method.

1. The triangular matrix method for Newton interpolation is a lower triangular matrix that can be solved in time $O(n^2)$ while the Vandermonde matrix solved in time $O(n^3)$.
2. As the degree of the polynomial increases, Vandermonde matrix is notorious for being challenging to solve and prone to large errors in the computed coefficients while triangular matrix for Newton interpolations with a large order starts to become singular.

Answer to question 3

$$x_{k+1} = g(x_k)$$

$$x_{k+1} = h(x_k)$$

$$x_{k+1} = g(x_k)h(x_k) = z(x_k)$$

Using the chain rule

$$z'(x_k) = g'(x_k)h(x_k) + g(x_k)h'(x_k)$$

Since $g(x_k)$ and $h(x_k)$ have quadratic convergence, $g'(a) = 0$ and $h'(a) = 0$ at solution $x = a$.

$$z'(a) = 0 * h(x_k) + g(x_k) * 0 = 0$$

$z'(a) = 0$ as well so it would exhibit a quadratic convergence.

Answer to question 4

$$(-2,-5),(-1,-6),(2,-8),(4,-4)$$

	1	2	3	4
x	**-2**	**-1**	**2**	**4**
y	**-5**	**-6**	**-8**	**-4**

Lagrange Interpolation:

$$P_n(x) = y_1 l_1(x) + y_2 l_2(x) + \cdots + y_{n+1} l_{n+1}(x)$$

Using given data:

$$y = -5l_1(x) - 6l_2(x) - 8l_3(x) - 4l_4(x)$$

The first Lagrange polynomial can be computed as follows:

$$l_1 = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} = \frac{(x+1)(x-2)(x-4)}{(-2+1)(-2-2)(-2-4)} = -\frac{(x+1)(x-2)(x-4)}{24}$$

The second Lagrange polynomial can be computed as follows:

$$l_2 = \frac{(t-t_1)(t-t_3)(t-t_4)}{(t_2-t_1)(t_2-t_3)(t_2-t_4)} = \frac{(x+2)(x-2)(x-4)}{(-1+2)(-1-2)(-1-4)} = \frac{(x+2)(x-2)(x-4)}{15}$$

The third Lagrange polynomial can be computed as follows:

$$l_3 = \frac{(t-t_1)(t-t_2)(t-t_4)}{(t_3-t_1)(t_3-t_2)(t_3-t_4)} = \frac{(x+2)(x+1)(x-4)}{(2+2)(2+1)(2-4)} = -\frac{(x+2)(x+1)(x-4)}{24}$$

The fourth Lagrange polynomial can be computed as follows:

$$l_4 = \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_4-t_1)(t_4-t_2)(t_4-t_3)} = \frac{(x+2)(x+1)(x-2)}{(4+2)(4+1)(4-2)} = \frac{(x+2)(x+1)(x-2)}{60}$$

Entire polynomial can be computed as:

$$\begin{aligned} y = -5\left(-\frac{(x+1)(x-2)(x-4)}{24}\right) - 6\left(\frac{(x+2)(x-2)(x-4)}{15}\right) \\ - 8\left(-\frac{(x+2)(x+1)(x-4)}{24}\right) - 4\left(\frac{(x+2)(x+1)(x-2)}{60}\right) \\ = \frac{3}{40}x^3 + \frac{19}{120}x^2 + \frac{-21}{20}x + \frac{-107}{15} \end{aligned}$$

Answer to question 5

$$f(x) = x^2 - 5 = 0$$

According to the Newton's method, the iterative solution is

$$x_{k+1} = x_k - \frac{f(x_k)}{f^{-1}(x_k)}$$

$$= x_k - \frac{x_k^2 - 5}{2x_k}$$

First part of the question

$$|g'(x)| = \left|1 - \frac{x^2 + 5}{2x^2}\right| < 1$$

It is a contraction on the interval:

$$\left(-\infty, -\sqrt{\frac{5}{3}}\right) \cup \left(\sqrt{\frac{5}{3}}, \infty\right)$$

$$\sqrt{\frac{5}{3}} = 1.29099$$

$g(x)$ is a contraction for $x >$ 1.29 so the Newton's method will always converge if started from an initial guess $x_0 \geq 2$.

Second part of the question

$$g(x_k) = x_k - \frac{x_k^2 - 5}{2x_k}$$

$$x^2 = 5$$

$$x = \pm\sqrt{5}$$

$$g'(a) = 0?$$

Using one of the solutions

$$g'(\sqrt{5}) = 1 - \frac{\left(\sqrt{5}\right)^2 + 5}{2\left(\sqrt{5}\right)^2} = 0$$

Hence, convergence will be quadratic.