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HW1 for CS323 Fall 2020

Answer to question 1 part (a)

Solving $ax^2 + bx + c = 0$ using quadratic equation

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Assume $\lim_{k\to\infty} = A$. Then taking limits on given equation gives

$$\lim_{k\to\infty} x_k^2 = \lim_{k\to\infty} \frac{ax_k^2 - c}{2ax_k + b}$$

$$A = \frac{aA^2 - c}{2aA + b}$$

$$2aA^2 + bA = aA^2 - c$$

$$aA^2 + bA = -c$$
$$aA^2 + bA + c = 0$$

Using the quadratic equation

$$\boldsymbol{A = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Answer to question 1 part (b)

Solving $x^2 - a = 0$ gives

$$\boldsymbol{x = \pm\sqrt{a}}$$

Assume $\lim_{k\to\infty} = A$. Then taking limits on given equation gives

$$\lim_{n\to k} x_k^2 = \lim_{n\to k} \frac{3x_k^2 + a}{4x_k}$$
$$A = \frac{3A^2 + a}{4A}$$
$$4A^2 = 3A^2 + a$$
$$A^2 = a$$
$$\boldsymbol{A = \pm\sqrt{a}}$$

<u>Answer to question 2 part (a)</u>

Non-linear equation

$$e^x = x + 2$$

Iterative method

$$x_{k+1} = e^{x_k} - 2$$

Guaranteed to converge?

$$|g'(x)| = |e^{x_k}| < 1$$

It is guaranteed to converge in the interval $(-\infty, \mathbf{0})$

Converge to a solution of the given equation?

Assume $\lim\limits_{k\to\infty} = A$. Then taking limits on given equation gives

$$\lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} e^{x_k} - 2$$

Since $x_k = x_{k+1}$ when $k \to \infty$

$$A = e^A - 2$$

$$e^A = A + 2$$

Putting back in original equation, we can see that equation still holds so its does converge to a solution of the given equation.

<u>Answer to question 2 part (b)</u>

Non-linear equation

$$x^3 = x^2 + 1$$

Iterative method

$$x_{k+1} = \frac{x_k}{1 + x_k^2}$$

Guaranteed to converge?

$$|g'(x)| = \left|\frac{1 - x^2}{(1 + x^2)^2}\right| < 1$$

It is guaranteed to converge on the interval $(-\infty, \mathbf{0}) \cup (\mathbf{0}, \infty)$

Converge to a solution of the given equation?

Assume $\lim\limits_{k \to \infty} = A$. Then taking limits on given equation gives

$$\lim_{k \to \infty} x_{k+1} = \lim_{k \to \infty} \frac{x_k}{1 + x_k^2}$$

Since $x_k = x_{k+1}$ when $k \to \infty$

$$A = \frac{A}{1 + A^2}$$

$$A = 0$$

Putting back in original equation, we can see that equation does not hold so it does not converge to a solution of the given equation.

<u>Answer to question 3 part (a)</u>

$$\sqrt[3]{a} = x$$

Non-linear equation

$$f(x) = x^3 - a = 0$$

According to the Newton's method, the iterative solution is

$$\begin{aligned} x_{k+1} &= x_k - \frac{f(x_k)}{f^{-1}(x_k)} \\ &= x_k - \frac{x_k^3 - a}{3x_k^2} \\ &= \boldsymbol{\frac{2x_k^3 + a}{3x_k^2}} \end{aligned}$$

<u>Answer to question 3 part (b)</u>

$$\ln a = x$$

Non-linear equation

$$f(x) = e^x - a = 0$$

According to the Newton's method, the iterative solution is

$$\begin{aligned} x_{k+1} &= x_k - \frac{f(x_k)}{f^{-1}(x_k)} \\ &= x_k - \frac{e^{x_k} - a}{e^{x_k}} \\ &= \boldsymbol{\frac{e^{x_k} x_k - e^{x_k} + a}{e^{x_k}}} \end{aligned}$$

Answer to question 3 part (c)

$$\arctan x = a$$
$$\tan a = x$$

Non-linear equation

$$f(a) = \tan a - x = 0$$

According to the Newton's method, the iterative solution is

$$a_{k+1} = a_k - \frac{f(a_k)}{f^{-1}(a_k)}$$

$$= a_k - \frac{\tan a_k - x}{\sec^2 a_k}$$

Since $\frac{1}{\sec^2 x} = \cos^2 x$

$$\boldsymbol{= a_k - (\tan a_k - x)(\cos^2 a_k)}$$

Answer to question 4 part (a)

$$\hat{x}_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$
$$x_{k+1} = \frac{x_k + \hat{x}_{k+1}}{2}$$

So,

$$x_{k+1} = \frac{x_k + x_k - \frac{f(x_k)}{f'(x_k)}}{2}$$

Assume $\lim_{k\to\infty} = A$. Then taking limits on given equation gives

$$\lim_{k\to\infty} x_k^2 = \lim_{k\to\infty} \frac{x_k + x_k - \frac{f(x_k)}{f'(x_k)}}{2}$$
$$A = \frac{A + A - \frac{f(A)}{f'(A)}}{2}$$

$$2A = 2A - \frac{f(A)}{f'(A)}$$

$$\frac{f(A)}{f'(A)} = 0$$

$$f(A) = 0$$

It will converge to a solution of $\boldsymbol{f(x) = 0}$

Answer to question 4 part (b)

We know convergence is guaranteed and Newton's method is a contraction if

- $f(a) = 0, i.e., a\ is\ a\ solution\ of\ f(x) = 0,$
- $f'(a) \neq 0, and$
- $f''is\ bounded\ near\ a\ (f''is\ continous)$

then,

$$\lim_{x \to a} g'(x) = \frac{f(a)f''(a)}{[f'(a)]^2} = 0$$

We know that convergence is guaranteed from part (a).

This method converges under the same conditions as Newton's method:

- **$\boldsymbol{f(x)}$ is continuous and differentiable**
- **$\boldsymbol{f'(x_k) \neq 0}$**
- **It has a second derivative**

Answer to question 4 part (c)

If the function is continuously differentiable and its derivative is not 0 at α and it has a second derivative at α then the convergence is quadratic.

So,

Order of convergence, $\boldsymbol{d = 2}$.