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Midterm 2 for CS323 Fall 2020

Answer to question 1 part (a)

i. Permuting the rows of A and b.
iii. Multiplying both sides of the equation from the left by a non-singular n x n matrix M.

Answer to question 1 part (b)

i. Symmetric.
iv. Positive semi-definite.

Answer to question 1 part (c)

i. cA, where c is any non-zero scalar.
iii. A^{-1}, the inverse of A.

Answer to question 1 part (d)

ii. 3

Answer to question 2 part (a)

- With the same matrix A, but a different right-hand side vector $b' = O(n^2)$
- With a different matrix A', but the same right-hand side vector $b = O(n^3)$

Answer to question 2 part (b)

$$M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

Answer to question 2 part (c)

First find $x = Bc$ using matrix-vector multiplication

Since finding $AX = B$ is quicker than finding A^{-1}, we find LU decomposition of A and then use it to solve $AX = B$

Find $Ly = x$ using forward substitution and $Uz = y$ using backward substitution to find $A^{-1}Bc$

Answer to question 3 part (a)

$$square\ matrix\ singular\ if\ columns\ linearly\ dependent$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix}-\begin{bmatrix}1\\2\\3\end{bmatrix}+\begin{bmatrix}0\\1\\2\end{bmatrix}=0$$

$$A\ is\ singular$$

Answer to question 3 part (b)

$$\begin{bmatrix}1\\1\\1\end{bmatrix}+\begin{bmatrix}1\\2\\3\end{bmatrix}+\begin{bmatrix}0\\1\\2\end{bmatrix}=\begin{bmatrix}2\\4\\6\end{bmatrix}$$

$$Infinite\ solutions\ as\ b\in range(A)$$

Answer to question 4

$$||Ax - b||_2^2 = ||A(x - x_0) + (Ax_0 - b)||_2^2$$

$$= \left(A(x - x_o) + (Ax_0 - b)\right)^T (A(x - x_0) + (Ax_0 - b))$$

$$= ||A(x - x_0)||_2^2 + ||Ax_0 - b||_2^2 + 2(x - x_0)^T A^T (Ax_o - b)$$

$$= ||A(x - x_0)||_2^2 + ||Ax_0 - b||_2^2 + 2(x - x_0)^T (A^T Ax_0 - A^T b)$$

$$= ||A(x - x_0)||_2^2 + ||Ax_0 - b||_2^2 + 2(x - x_0)^T (A^T b - A^T b)$$

$$||Ax - b||_2^2 = ||A(x - x_o)||_2^2 + ||Ax_0 - b||_2^2$$

Answer to question 5

$$\begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$l_{11}u_{11} = 0$$

$$l_{11}u_{12} = 1$$

$$l_{21}u_{11} = 1$$

$$l_{11}\ cannot\ be\ zero\ and\ u_{11} = 0$$

$$if\ u_{11} = 0\ then\ last\ equation\ invalid\ and\ this\ is\ a\ contradiction$$

Answer to question 6

```
import numpy as np

def jacobi(A, b, x0):
    D = np.diag(A)
    x = x0
    for i in range(2500):
        x_new = x + ((b - np.dot(A, x)) / D)
        if np.linalg.norm(x_new - x) < 10**-6:
            return x_new
        x = x_new
    return x

A = np.array([
            [5, 2, 1, 1],
            [2, 6, 2, 1],
            [1, 2, 7, 1],
            [1, 1, 2, 8]
            ])
b = np.array([29, 31, 25, 19])
x0 = np.array([1, 2, 3, 4])

ans = jacobi(A, b, x0)
print("\nFinal iteration = ", ans)
print("\n", np.dot(A, ans), " =? ", b)
print("\nYes, x^(k) is a solution to Ax = b")
```

From our test $x^{(k)}$ *is a solution to* $Ax = b$

On other tests, we find that it may not be a solution due to the A not meeting convergence requirements of Jacobi method.