1
Categories

December 5, 2023

Definitions

1. A *category* consists of objects and arrows.
   For each arrow f, there exists a domain $\mathrm{dom}(f)$ and codomain $\mathrm{cod}(f)$.
   If $A = \mathrm{dom}(f)$ and $B = \mathrm{cod}(f)$, we may write $f : A \to B$.
   If $f : A \to B$ and $g : B \to C$, then there is a composite arrow $g \circ f : A \to C$.
   For each object A, there is an identity arrow $1_A : A \to A$.
   Objects and arrows must satisfy the following two equations, assuming each is well-defined:
   (a) $h \circ (g \circ f) = (h \circ g) \circ f$ (Associativity)
   (b) $f \circ 1_A = f = 1_B \circ f$ (Unit)
2. Given categories $\mathbf{C}$ and $\mathbf{D}$, a *functor* $F : \mathbf{C} \to \mathbf{D}$ is a mapping of objects to objects and arrows to arrows such that:
   (a) $F(f : A \to B) = F(f) : F(A) \to F(B)$
   (b) $F(1_A) = 1_{F(A)}$
   (c) $F(g \circ f) = F(g) \circ F(f)$

Exercises

1. (a) **Rel** has sets as objects and subsets of $A \times B$ as arrows $A \to B$. Such subsets are also called relations.
   The identity arrow 1_A is the relation $\{\langle a, a\rangle \in A \times A : a \in A\}$.
   The composite arrow of $R : A \to B$ and $S : B \to C$ is $\{\langle a, c\rangle \in A \times C : \exists b(\langle a, b\rangle \in R \,\&\, \langle b, c\rangle \in S)\}$.

   First we note the category is closed under composition since the composite arrow is a relation from A to C.

Second we verify that Unit holds.

$$\begin{aligned}1_B \circ R &= \{\langle a, b_2\rangle \in A \times B : \exists b_1(\langle a, b_1\rangle \in R \,\&\, \langle b_1, b_2\rangle \in 1_B)\}\\ &= \{\langle a, b_2\rangle \in A \times B : \exists b_1(\langle a, b_1\rangle \in R \,\&\, b_1 = b_2)\}\\ &= \{\langle a, b\rangle \in A \times B : \exists b(\langle a, b\rangle \in R)\}\\ &= R\\ S \circ 1_B &= \{\langle b_2, c\rangle \in B \times C : \exists b_1(\langle b_2, b_1\rangle \in 1_B \,\&\, \langle b_1, c\rangle \in S)\}\\ &= \{\langle b_2, c\rangle \in B \times C : \exists b_1(b_2 = b_1 \,\&\, \langle b_1, c\rangle \in S)\}\\ &= \{\langle b, c\rangle \in B \times C : \exists b(\langle b, c\rangle \in S)\}\\ &= S\end{aligned}$$

Finally we verify that Associativity holds. Let $T : C \to D$.

$$\begin{aligned}\langle a, d\rangle \in T \circ (S \circ R) &\iff \exists c(\langle a.c\rangle \in S \circ R \,\&\, \langle c, d\rangle \in T)\\ &\iff \exists c(\exists b(\langle a.b\rangle \in R \,\&\, \langle b, c\rangle \in S) \,\&\, \langle c, d\rangle \in T)\\ &\iff \exists b(\langle a.b\rangle \in R \,\&\, \exists c(\langle b, c\rangle \in S \,\&\, \langle c, d\rangle \in T))\\ &\iff \exists b(\langle a.b\rangle \in R \,\&\, \langle b, d\rangle \in T \circ S)\\ &\iff \langle a, d\rangle \in (T \circ S) \circ R\end{aligned}$$

(b) G : **Sets** $\to$ **Rel** is a mapping defined by taking objects to themselves and arrows in **Sets** as functions to its graph in **Rel**, that is for a function $f : A \times B$ in **Sets**
$G(f) = \{\langle a, f(a)\rangle \in A \times B : a \in A\}$

It immediate that G takes objects in **Sets** to objects in **Rel**.
Let $f : A \to B$ be an arrow of **Sets**. $G(f)$ is by definition a subset of the product of $A \times B$, and hence is an arrow $A \to B$ in **Rel**.

First, we show G preserves domains and codomains.

$$\begin{aligned}G(f : A \to B) &= \{\langle a, f(a)\rangle \in A \times B : a \in A\}\\ &\subseteq A \times B\end{aligned}$$

Second, we show G preserves identities.

$$\begin{aligned}G(1_A) &= \{\langle a, 1_A(a)\rangle \in A \times A : a \in A\}\\ &= \{\langle a, a\rangle \in A \times A : a \in A\}\end{aligned}$$

Third, we show G preserves compostions. Let $f : A \to B, g : B \to C$.

$$\begin{aligned}G(g \circ f) &= \{\langle a, g \circ f(a)\rangle \in A \times C : a \in A\}\\ &= \{\langle a, c\rangle \in A \times C : \exists b(f(a) = b \,\&\, g(b) = c)\}\\ &= \{\langle a, c\rangle \in A \times C : \exists b(\langle a, b\rangle \in G(f) \,\&\, \langle b, c\rangle \in G(g))\}\\ &= G(f) \circ G(g)\end{aligned}$$