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Categories

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1. (a) First we show the category is closed under composition.

Let $R : A \rightarrow B, S : B \rightarrow C$.

$$S \circ R = \{\langle a, c \rangle \in A \times C : \exists b(\langle a, b \rangle \in R \ \& \ \langle b, c \rangle \in S)\} \subseteq A \times C$$

Thus $S \circ R$ is an arrow in the category **Rel**.

Next we verify that the left and right identities are valid.

$$\begin{aligned} 1_B \circ R &= \{\langle a, b_2 \rangle \in A \times B : \exists b_1(\langle a, b_1 \rangle \in R \ \& \ \langle b_1, b_2 \rangle \in 1_B)\} \\ &= \{\langle a, b_2 \rangle \in A \times B : \exists b_1(\langle a, b_1 \rangle \in R \ \& \ b_1 = b_2)\} \\ &= \{\langle a, b \rangle \in A \times B : \exists b(\langle a, b \rangle \in R)\} \\ &= R \\ S \circ 1_B &= \{\langle b_2, c \rangle \in B \times C : \exists b_1(\langle b_2, b_1 \rangle \in 1_B \ \& \ \langle b_1, c \rangle \in S)\} \\ &= \{\langle b_2, c \rangle \in B \times C : \exists b_1(b_2 = b_1 \ \& \ \langle b_1, c \rangle \in S)\} \\ &= \{\langle b, c \rangle \in B \times C : \exists b(\langle b, c \rangle \in S)\} \\ &= S \end{aligned}$$

- (b) Let $f : A \rightarrow B$ be an arrow of **Sets**. $G(f)$ is by definition a subset of the product of $A \times B$, and hence is an arrow $A \rightarrow B$ in **Rel**.