1
Categories

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1. (a) First we show the category is closed under composition.
   Let $R : A \to B, S : B \to C$.
   $S \circ R = \{\langle a, c\rangle \in A \times C : \exists b(\langle a, b\rangle \in R \,\&\, \langle b, c\rangle \in S)\} \subseteq A \times C$
   Thus $S \circ R$ is an arrow in the category **Rel**.

   Next we verify that the left and right identities are valid.

$$\begin{aligned}
1_B \circ R &= \{\langle a, b_2\rangle \in A \times B : \exists b_1(\langle a, b_1\rangle \in R \,\&\, \langle b_1, b_2\rangle \in 1_B)\} \\
&= \{\langle a, b_2\rangle \in A \times B : \exists b_1(\langle a, b_1\rangle \in R \,\&\, b_1 = b_2)\} \\
&= \{\langle a, b\rangle \in A \times B : \exists b(\langle a, b\rangle \in R)\} \\
&= R \\
S \circ 1_B &= \{\langle b_2, c\rangle \in B \times C : \exists b_1(\langle b_2, b_1\rangle \in 1_B \,\&\, \langle b_1, c\rangle \in S)\} \\
&= \{\langle b_2, c\rangle \in B \times C : \exists b_1(b_2 = b_1 \,\&\, \langle b_1, c\rangle \in S)\} \\
&= \{\langle b, c\rangle \in B \times C : \exists b(\langle b, c\rangle \in S)\} \\
&= S
\end{aligned}$$

   (b) Let $f : A \to B$ be an arrow of **Sets**. $G(f)$ is by definition a subset of the product of $A \times B$, and hence is an arrow $A \to B$ in **Rel**.