Diving into SPH

1. SPH: What I've understood
   - Recap
   - Equations
2. Postprocessing
3. What I tried

Navier-Stokes Equations

(Still N-S!)

The key idea is unchanged. You want to approximate the N-S equations as before, but not discretize your solution on a mesh. This is like any Lagrangian method. To recap the equations for continuum flow:

- Continuity Equation:

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u}$$

- Momentum Equation:

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{f}$$

Smoothed Particle Hydrodynamics (SPH)

What makes it different

- SPH is mesh-free!
- Free surface flows that are challenging in Eulerian methods are more or less natural in SPH

Kernels

Mathematical Form

In 3D (also what I've implemented)

$$W(q) = \frac{21}{64\pi h^3} \cdot \begin{cases} (2-q)^4(1+2q) & \text{if } 0 \leq q \leq 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where:

- Kernels just convolve over the function, and aim to be an approximation to the Dirac Delta function
- It has compact support, which means the tail is not infitine, it takes a certain range of particles around it

SPH Equations

Continuity

The approach in SPH is sto sum up contributions from neighbours.

$$\rho_a = \sum_b m_b \frac{W(\mathbf{r}_{ab}, h)}{\rho_b}$$

where:

- W is the smoothing kernel function.
- $\mathbf{r}_{ab}$ is the vector from particle a to b.
- h is the smoothing length.

SPH Equations

Momentum

The kernel functions can be differentiated - and this is used in the gradient for pressure in the momentum equation. Pressure is calculated from the equation of state.

$$\frac{D\mathbf{v}_a}{Dt} = -\sum_b m_b \left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2} \right) \nabla_a W_{ab}(h) + \nu \nabla^2 \mathbf{v}_a + \mathbf{f}_a$$

where:

- P_a and P_b, ρ_a and ρ_b are the pressures and densities of the current particle and its neighbours
- $\nabla_a W_{ab}(h)$ is the gradient of the smoothing kernel between particles a and b with smoothing length h.