# Diving into SPH

## Navier-Stokes Equations
(Still N-S!)

The key idea is unchanged. You want to approximate the N-S equations as before, but not discretize your solution on a mesh. This is like any Lagrangian method. To recap the equations for continuum flow:

- Continuity Equation:

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u}$$

- Momentum Equation:

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{f}$$

# Smoothed Particle Hydrodynamics (SPH)
## What makes it different

- SPH is mesh-free!
- Free surface flows that are challenging in Eulerian methods are more or less natural in SPH

## Kernels
Mathematical Form

In 3D and 4th order (also what I've used)

$$W(q) = \frac{21}{64\pi h^3} \cdot \begin{cases} (2-q)^4(1+2q) & \text{if } 0 \le q \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where:

- Kernels just convolve over the function, and aim to be an approximation to the Dirac Delta function
- It has compact support, which means the tail is not infitine, it takes a certain range of particles around it
- A kernel function must be consistent, i.e.:

$$\int_{-\infty}^{\infty} W(q) \, dq = 1 \tag{2}$$

# SPH Equations
## Continuity

The approach in SPH is sto sum up contributions from neighbours.

$$\rho_a = \sum_b m_b \frac{W(\mathbf{r}_{ab}, h)}{\rho_b} \tag{3}$$

where:

- $W$ is the smoothing kernel function.
- $\mathbf{r}_{ab}$ is the vector from particle $a$ to $b$.
- $h$ is the smoothing length.

## SPH Equations
Momentum

The kernel functions can be differentiated - and this is used in the gradient for pressure in the momentum equation. Pressure is calculated from the equation of state.

$$\frac{D\mathbf{v}_a}{Dt} = -\sum_b m_b \left( \frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2} \right) \nabla_a W_{ab}(h) + \nu \nabla^2 \mathbf{v}_a + \mathbf{f}_a \qquad (4)$$

where:

- $P_a$ and $P_b$, $\rho_a$ and $\rho_b$ are the pressures and densities of the current particle and its neighbours
- $\nabla_a W_{ab}(h)$ is the gradient of the smoothing kernel between particles $a$ and $b$ with smoothing length $h$.

# Numerical quantities
## Kernel Weighted Interpolation

- Postprocessing of SPH would involve interpolation of the same nature to obtain a numerical quantity from particle quantities
- We renormalize the interpolation using a Shepherd summation in the denominator
- For instance, the numerical pressure $P_a$ at particle $a$ can be computed as a weighted sum of neighbour particles

$$P_a = \frac{\sum_b P_b \frac{m_b}{\rho_b} W_{ab}}{\sum_b \frac{m_b}{\rho_b} W_{ab}}$$