Derivatives

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad \Bigg\} \text{Provided that the limit exists.}$$

Example $f(z) = z^2$. Is f is differentiable at $z=0$?

Sol^n:

$$\cancel{\lim f(z)} \cdot \lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(\Delta z)^2 - 0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \Delta z$$

$$= 0.$$

(Q) Using definition show that $f(z) = \bar{z}$ is not differentiable at $z=0$.

Sol^n:

$$\lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z} - \bar{0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x + i\Delta y}{\Delta x - i\Delta y}$$

Approaching along real line, $\Delta y = 0$ and $\Delta x \to 0$,

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x + i.0}{\Delta x - i.0}$$

$$= 1$$

Approaching along imaginary line, $\Delta x = 0$ and $\Delta y \to 0$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{0 + i.\Delta y}{0 - i.\Delta y}$$

$$= -1$$

Ex Using the definitions, find the derivative of the function $f(z) = \dfrac{2z - 3i}{3z - 2i}$ at $z = -i$

Soln:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[f(z_0 + \Delta z) - f(z_0) \right]$$

$$\Rightarrow f'(-i) = \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[f(-i + \Delta z) - f(-i) \right]$$

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[\frac{2(-i + \Delta z) - 3i}{3(-i + \Delta z) - 2i} - \frac{-2i - 3i}{-3i - 2i} \right]$$

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[\frac{-5i + 2\Delta z}{-5i + 3\Delta z} - 1 \right]$$

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[\frac{-5i + 2\Delta z + 5i - 3\Delta z}{-5i + 3\Delta z} \right]$$

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[\frac{-\Delta z}{-5i + 3\Delta z} \right]$$

$$= -\lim_{\Delta z \to 0} \frac{1}{-5i + 3\Delta z} = \frac{1}{5i} = -\frac{i}{5}$$

⊛ Using the limit definition of derivative, find the derivative of $f(z) = z^2$.

Soln:

$$f(z) = z^2$$

let, $z = z_0$ be any arbitrary point.

Now,

$$f'(z_0) = \lim_{\substack{z \to z_0 \\ (\text{or } \Delta z \to 0)}} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z_0^2 + 2z_0\Delta z + \Delta z^2 - z_0^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (2z_0 + \Delta z)$$

$$= 2z_0 .$$

As. z_0 is any arbitrary complex number, so $f'(z) = 2z$ where $z \in \mathbb{C}$.

Ex.

Let, $f(z) = |z|^2$ or $f(z) = z \cdot \bar{z}$. Show that the derivative of $f(z)$ exists only at $z = 0$.

Sol^n: We know,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)\overline{(z_0+\Delta z)} - z_0\bar{z}_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)(\bar{z}_0+\overline{\Delta z}) - z_0\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0\bar{z}_0 + z_0\overline{\Delta z} + \Delta z\,\bar{z}_0 + \Delta z\,\overline{\Delta z} - z_0\bar{z}_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0\overline{\Delta z} + \Delta z\,\bar{z}_0 + \Delta z\,\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(\bar{z}_0 + z_0\frac{\overline{\Delta z}}{\Delta z} + \overline{\Delta z}\right)$$

Now, Approaching along real line, $\Delta y = 0$, $\Delta x \to 0$,

$$\lim_{\Delta z \to 0} \left(\bar{z}_0 + z_0 \cdot \frac{\overline{\Delta z}}{\Delta z} + \overline{\Delta z}\right) = \lim_{\Delta x \to 0} (x - iy + x + iy + \Delta x)$$

$$= x - iy + x + iy = \bar{z}_0 + z_0$$

we used,

$$z_0 = x + iy \qquad \Delta z = \Delta x + i\Delta y \qquad \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$\bar{z}_0 = x - iy \qquad \overline{\Delta z} = \Delta x - i\Delta y$$

Again, approaching from imaginary line, $\Delta x = 0$, $\Delta y \to 0$

$$\lim_{\Delta z \to 0} \left(\bar{z}_0 + z_0 \frac{\overline{\Delta z}}{\Delta z} + \overline{\Delta z} \right)$$

$$= \cancel{\lim_{\Delta x \to}} \lim_{\Delta y \to 0} \left(x - iy + (x+iy)(-1) - i\Delta y \right)$$

$$= \bar{z}_0 - z_0$$

If the limit exists, then both quantities ~~b~~ must be equal.

So, $\bar{z}_0 + z_0 = \bar{z}_0 - z_0$

$$\Rightarrow 2z_0 = 0$$

$$\Rightarrow z_0 = 0$$

Thus, derivative of $f(z) = |z|^2$ exists at $z_0 = 0$ only.