Chapter: 2

Polynomial Interpolation

$$P_n(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \dots + a_n x^n$$

↑ degree of polynomial

Ex: $P_3(x) = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3$

↑ degree $= 3$

$[a_0, a_1, a_2, a_3] \to$ coefficients / constants.

$\boxed{\text{degree} = n, \ \text{coefficient} = n+1}$

$P_{26}(x) \Rightarrow$ degree $= 26$, coefficient $= 27$

Vector Space : A region where we can:
- add vectors
- multiply with scalars.

Eg:
$$\begin{array}{r} 1 + x + x^2 \\ +x^3 \\ \hline 1 + x + x^2 + x^3 \end{array} \Rightarrow \text{new poly}^n$$

$$5(1 + x + x^2) = 5 + 5x + 5x^2$$

Basis : A set of vectors that spans the space.

$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

basis $= \{1, x, x^2, x^3\}$, degree $= 3$, coefficient $= 4$

dimensional space $= 4$

* Dimensional space is 1 more than degree.

Taylor Series

Functional space : $f(x)$ can go upto infinity.

for example:

$$f(x) = 1 + 2x + 4x^2 \underbrace{+ 14x^3 + 25x^4 + \cdots}_{\text{eliminated} \to \text{leads to error.}}$$

$$P_2(x) = 1 + 2x + 4x^2$$

We replicate $f(x)$ to $P_n(x)$ because we can only consider finite no. of terms for $P_n(x)$ and we can manipulate it.

Therefore, we can say:

$f(x) \in V^{\infty} \to$ ~~infinite~~ vector space of infinite dimension.

$P_n(x) \in V^{n+1} \to$ vector space of $(n+1)$ dimension.

If we increase the degree, the error reduces. This is stated by "Weierstrass Approximation Theorem".

Ex: $f(x) = 2 + 5x + 12x^2 + 25x^3 + 110x^4 + 117x^5 + \cdots$

$$P_2(x) = 2 + 5x + 12x^2$$

Vs

$$P_5(x) = 2 + 5x + 12x^2 + 25x^3 + 110x^4 + 117x^5$$

less error since more terms were considered.

decrease $\downarrow |f(x) - P_n^{\uparrow}(x)|$ increase

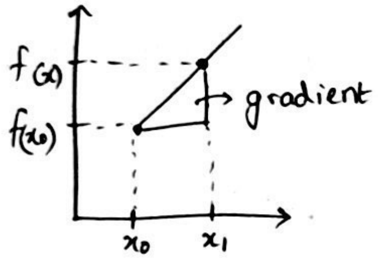
increase in degree of polynomial decreases the error.

Taylor Series

→ it's a way to approximate a function as an infinite sum of terms calculated from the function's derivatives at a single point.

actual Taylor series $\left\{ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \right.$

$f^n(a) \rightarrow$ nth derivative of $f(x)$ at $x=a$



$$\text{gradient} = f'(x_0) = \frac{y_2 - y_1}{x_2 - x_1}$$

$$f'(x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\boxed{f(x) = f'(x_0)(x - x_0) + f(x_0)}$$

The one that we use,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \frac{f'''(x_0)(x-x_0)^3}{3!} + \cdots$$

Proof

let,
$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots\cdots$$

$$f'(x) = 0 + a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2$$

$$f''(x) = 0 + 0 + 2a_2 + 3\cdot 2a_3(x - x_0)$$
$$f'''(x) = 3\cdot 2\cdot a_3$$

Now, let, $x = x_0$

then,

$$f'''(x_0) = 3\cdot 2\cdot a_3$$

$$f(x_0) = a_0$$

$$f'(x_0) = a_1$$

$$f''(x_0) = 2a_2$$

subject a_1, a_2, a_3.

then,

- $a_1 = f'(x_0)$
- $a_0 = f(x_0)$
- $2a_2 = f''(x_0)$

$$a_2 = \frac{f''(x_0)}{2!} \quad 2! \rightarrow 2\times 1 = 2$$

- $3\cdot 2\cdot a_3 = f'''(x_0)$

$$a_3 = \frac{f'''(x_0)}{3!} \quad 3! \rightarrow 3\times 2\times 1 = 6$$

Now, replace the a_0, a_1, a_2, a_3 with these.

$$\therefore f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \frac{f'''(x_0)(x-x_0)^3}{3!} + \cdots$$

Example : $f(x) = \sin(x)$

We will expand it using Taylor series.

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \frac{f'''(x_0)(x-x_0)^3}{3!} + \cdots$$

$\boxed{x_0 = 0} \rightarrow$ always.

$f(x) = \sin(x)$	$f(0) = \sin(0) = 0$
$f'(x) = \cos(x)$	$f'(0) = \cos(0) = 1$
$f''(x) = -\sin(x)$	$f''(0) = -\sin(0) = 0$
$f'''(x) = -\cos(x)$	$f'''(0) = -\cos(0) = -1$
$f^4(x) = \sin(x)$	$f^4(0) = \sin(0) = 0$
$f^5(x) = \cos(x)$	$f^5(0) = \cos(0) = 1$
$\vdots$	$\vdots$

Now, $f(x) = f(0) + f^{(1)}(0)(x-0) + \frac{f^{(2)}(0)(x-0)^2}{2!} + \frac{f^3(0)(x-0)^3}{3!}$

$$+ \frac{f^4(0)(x-0)^4}{4!} + \frac{f^5(0)(x-0)^5}{5!} + \cdots$$

$$f(x) = 0 + 1(x) + \frac{0x^2}{2!} + \frac{-1}{3!}(x)^3 + \frac{0}{4!}(x)^4 + \frac{1}{5!}(x)^5$$

$\therefore\ f(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \implies$ we have seen this in Chapter: 1

> Fact: when value of $x_0 = 0$, that specific Taylor series is known as MacLaurin Series.

Now, if we take $x = 0.1$:

1^{st} term $\Rightarrow f(0.1) = x = 0.1$

2 terms $\Rightarrow f(0.1) = x - \frac{1}{3!}x^3 = 0.1 - \frac{(0.1)^3}{3!} = 0.099833$

3 terms $\Rightarrow f(0.1) = 0.1 - \frac{(0.1)^3}{3!} + \frac{(0.1)^5}{5!} = 0.09983341$

Exact answer $= \sin(0.1) = 0.09983341$

Taylor's Theorem

~~tef~~ let f be $(n+1)$ times differentiable on (a,b) and let $f^{(n)}$ be continuous on $[a,b]$. If $x, x_0 \in [a,b]$, then there exists $\xi \in (a,b)$ such that:

xi

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{\text{Taylor's polynomial of degree 'n'}} + \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}}_{\text{Lagrange form of remainder}}$$

Ex: $f(x) = \sin(x)$

Let's take a polynomial of degree = 6.

So,

$$\underbrace{\left| f(x) - P_6(x) \right|}_{\text{error}} = \left| \frac{f^7(\xi)}{7!}(x-x_0)^7 \right|$$

Taylor series for $f(x) = \sin(x)$ upto degree 6 is:

$$= \underbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!} + 0 \cdot x^6}_{P_6(x)}$$

$f(x) = \sin(x)$
$f^1(x) = \cos(x)$
$f^2(x) = -\sin(x)$
$f^3(x) = -\cos(x)$
$f^4(x) = \sin(x)$
$f^5(x) = \cos(x)$
$f^6(x) = -\sin(x)$
$f^7(x) = -\cos(x)$

$$f(x) = P_6(x) + \frac{f^7(\xi)}{7!}(x-x_0)^7$$

$$\text{So, } \left| f(x) - P_6(x) \right| = \left| \frac{f^7(\xi)(x-x_0)^7}{7!} \right|$$

$$= \left| \frac{-\cos(x/\xi)(x-x_0)^7}{7!} \right|$$

same

we know,
max value
of $\cos(x) = 1$
min value $= -1$

$$= \left| \frac{-1}{5040}(0.1-0)^7 \right|$$

$$= 1.984 \times 10^{-11}$$

x_0 was 0
$x_1 = 0.1$
$(a,b) = (0, 0.1)$

This error is due to the truncation of taylor series.

There are 3 ways/techniques to find the coefficients on constants of a polynomial.

- Vandermonde Matrix
- Lagrange Polynomial
- Newton's divided difference.

Vandermonde Matrix

Ex: A dataset is given.

	Age (y)	Salary ($)
these are called nodes/nodal points.	$20 \to x_0$	10000
	$25 \to x_1$	20000
	$30 \to x_2$	50000

degree of polynomial = nodes - 1

$\therefore$ nodes = 3, degree = 2

polynomial of degree $(n) \Rightarrow (n+1)$ nodes

For, this example,

$$P_{\underset{\text{degree}}{2}}(x) = a_0 + a_1 x^1 + a_2 x^2$$

$\boxed{\text{V.V.I}}$ * Always remember that the polynomial you have calculated should give the same value at the given points in the $f(x)$.

For ex: $\left.\begin{aligned} f(20) &= 10000 \\ \text{so, } P_2(20) &= 10000 \end{aligned}\right\}$ there 2 must match !!!

General form of Vandermonde matrix:

$$\underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}}_{\substack{\text{Vandermonde} \\ \text{Matrix, } V}} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}}_{F}$$

$$V \cdot A = F$$

$$A = V^{-1} \cdot F \quad [\text{matrix } V \text{ must be invertible}]$$

Ex: 1

	x	$f(x)$
$x_0 \leftarrow$	2	$5 \rightarrow f(x_0)$
$x_1 \leftarrow$	3	$6 \rightarrow f(x_1)$

nodes = 2

degree = 1

$\boxed{P_n(x) = f(x)}$ → must for given nodes.

$$P_1(x) = a_0 + a_1 x^1 \rightarrow \text{general form}$$

$$P_1(x_0) = a_0 + a_1 x_0 \Rightarrow 1a_0 + 2a_1 = 5 \quad \text{—— (i)}$$

$$P_1(x_1) = a_0 + a_1 x_1 \Rightarrow 1a_0 + 3a_1 = 6 \quad \text{—— (ii)}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Inverse of a 2x2 matrix

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \frac{1}{\text{determinant}} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

So, $$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}^{-1} = \frac{1}{3-2} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 3\times5-(2\times6) \\ -1\times5+1\times6 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$a_0 = 3,\ a_1 = 1$

$$P_1(x) = a_0 + a_1 x$$
$$= 3 + 1 \cdot x$$

Check
$x_0 = 2 \quad f(x_0) = 5$
$P_1(2) = 3 + 2 = 5$
$\therefore$ correct

- We can find $P_1(7/16/24)$ etc.

<u>Disadvantage</u>

→ In case of higher degree polynomial, matrix becomes very large

→ greater time & space complexity

→ if determinent is 0, matrix cannot be inversed.

Lagrange Basis

Previously,

$$P_n(x) = a_0 + a_1x^1 + a_2x^2 \ldots\ldots a_nx^n$$

Now, using lagrange basis:

lagrange basis : need to calculate.

$$P_n(x) = f(x_0)l_0(x) + f(x_1)l_1(x) + f(x_2)(l_2(x)) \ldots\ldots$$

given in question

Calculation

$$l_0(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x-x_2)\ldots(x_0-x_n)}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}$$

Ex:

	x	$f(x)$	
x_0	2	30	$f(x_0)$
x_1	5	45	$f(x_1)$
x_2	7	25	$f(x_2)$

nodes = 3
degree = 2

$$P_2(x) = l_0(x)\overset{\to 30}{f(x_0)} + l_1(x)\overset{\to 45}{f(x_1)} + l_2(x)\overset{\to 25}{f(x_2)}$$

[we are in x_0, so we avoid x_0 and take other nodes]

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-5)(x-7)}{(2-5)(2-7)}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-2)(x-7)}{(5-2)(5-7)}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-2)(x-5)}{(7-2)(7-5)}$$

$$\text{So, } P_2(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2)$$

$$= \left(\frac{(x-5)(x-7)}{15} \times 30\right) + \left(\frac{(x-2)(x-7)}{-6} \times 45\right) + \left(\frac{(x-2)(x-5)}{10} \times 25\right)$$

Example: 2

given $f(x) = \cos(x)$ $\quad x_0 = -\frac{\pi}{4},\ x_1 = 0,\ x_2 = \frac{\pi}{4}$

calculate $f(x_0), f(x_1), f(x_2) = \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}$ (radianmode!!!)

nodes = 3, degree = 2 [similar calculation]

Advantage: No need to inverse a matrix
↳ computationally expensive

Disadvantage: If we want to add new nodes, we need to redo our calculation from beginning.

Note #.

Time	Velocity
0	0
10	227.4
15	362.8
20	517.35
22.5	602.97
30	901.67

Q:1 Find lagrange polyⁿ.
nodes = 6, degree = 5
Find $P_5(x)$

Q:2 Find $P_2(x)$?
Now, nodes = 6, but we need to use only 3. Which three? → Ans: any 3 nodes.

Q: 3 Find value of $P_2(x)$ when $x = 16$.

Then, we need to consider the differences.

meaning :

x	"
0	"
10	"
15	"
20	"
22.5	"
30	"

$x=16 \xrightarrow{6} 10$

$x=16 \xrightarrow{1} 15$

$x=16 \xrightarrow{4} 20$

$x=16 \xrightarrow{6.5} 22.5$

So, we use the nodes, $x = 10, 15, 20$.

if there are 2 nodes with same diffence, use any one.

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x_0	$f(x_0)$
x_1	$f(x_1)$

degree $= 1$

$$P_1(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$$

$$l_0(x) = \frac{(x - x_1)}{(x_0 - x_1)}$$

$l_0(x_0) = 1$ [x is x_0]

$l_0(x_1) = 0$ [x is x_1]

$$l_1(x) = \frac{(x - x_0)}{(x_1 - x_0)}$$

$l_1(x_0) = 0$ [x is x_0]

$l_1(x_1) = 1$ [x is x_1]

$$l_0(x_1) = \frac{(x_1 - x_1)}{(x_0 - x_1)} = 0$$

$$l_1(x_0) = \frac{(x_0 - x_0)}{(x_1 - x_0)} = 0$$

We can write it as: $l_0(x_i) \begin{cases} 0 & i=1 \\ 1 & i=0 \end{cases}$

$l_1(x_i) \begin{cases} 0 & i=0 \\ 1 & i=1 \end{cases}$

$\therefore l_n(x_j) = \delta_{nj}$ → $\boxed{\text{kronecker delta}}$

kronecker delta

$l_n(x_j) = \delta_{nj}$

$n = 0, 1 \quad j = i$

$\delta_{nj} = 0 \quad n \neq j$

$\delta_{nj} = 1 \quad n = j$

Newton's Divided Difference

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \ldots\ldots a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}).$$

Here,

$a_0 = f[x_0]$

$a_1 = f[x_0, x_1]$

$a_2 = f[x_0, x_1, x_2]$

$a_3 = f[x_0, x_1, x_2, x_3]$

So, $P_n(x) = f[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1) + \ldots\ldots$

Example

	x	$f(x)$	
x_0	-1	5	$f[x_0]$
x_1	0	1	$f[x_1]$
x_2	1	3	$f[x_2]$
x_3	2	11	$f[x_3]$

nodes $= 4$, degree $= 3$

So, $P_3(x) = f[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1) + f[x_0, x_1, x_2, x_3](x-x_0)(x-x_1)(x-x_2)$

From next page:

$$P_3(x) = 5 + (-4)(x+1) + 3(x+1)(x) + 0(x+1)(x)(x-1)$$

$\therefore$ $P_3(x) = 5 - 4(x+1) + 3x(x+1)$ Ans

$x_0 = -1$, $f[x_0] = 5$

$f[x_0, x_1] = \frac{1-5}{0-(-1)} = -4$ ⇑ $\frac{f[x_1]-f[x_0]}{x_1 - x_0}$

$x_1 = 0$, $f[x_1] = 1$

$f[x_1, x_2] = \frac{3-1}{1-0} = 2$

$f[x_0, x_1, x_2] = \frac{2-(-4)}{1-(-1)} = 3$ ⇑ $\frac{f[x_1,x_2]-f[x_0,x_1]}{x_2 - x_0}$

$x_2 = 1$, $f[x_2] = 3$

$f[x_2, x_3] = \frac{11-3}{2-1} = 8$

$f[x_1, x_2, x_3] = \frac{8-2}{2-0} = 3$

$f[x_0, x_1, x_2, x_3] = \frac{3-3}{11-(-1)} = 0$ ⇑ $\frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3 - x_0}$

$x_3 = 2$, $f[x_3] = 11$

* $x_4 = 4$, $f[x_4] = 20$

$f[x_3, x_4] = \frac{9}{2}$

$f[x_2, x_3, x_4] = -\frac{7}{6}$

$f[x_1, x_2, x_3, x_4] = -\frac{25}{24}$

<u>Q:1</u> Find polyⁿ using Newton's method → Done

<u>Q:2</u> Find value at $P_n(10)$? $x = 10$, find value.

you have to know

<u>Q:3</u> a new node $x = 4$, $f(x) = 20$ has been added, find the new polyⁿ.

degree = prev. + 1

$\Rightarrow P_4(x) \rightarrow$ 5 nodes now.

$x_4 = 4$, $f[x_4] = 20$

$f[x_0, x_1, x_2, x_3, x_4] = \frac{-\frac{25}{24} - 0}{4-(-1)} = -\frac{5}{24}$

So, $P_4(x) = 5 + (-4)(x+1) + 3(x+1)(x) + 0 + \left(-\frac{5}{24}\right)(x+1)(x)(x-1)(x-2)$

Ans

Advantage: New data points can be incorporated easily.

$\Downarrow$

No need to calculate from beginning.

Cauchy's Theorem

$$\underbrace{\left| f(x) - P_n(x) \right|}_{\text{error}} = \left| \frac{f^{n+1}(\xi)}{(n+1)!} (x-x_0)(x-x_1)\cdots(x-x_n) \right|$$

Example: Find the upper bound of error using Cauchy's Theorem.

Note: We are always concerned with the max. error. Hence we always find upper bound of error.

let, $f(x) = \cos(x)$ $\quad x = [\overset{x_0}{-\frac{\pi}{4}}, \overset{x_1}{0}, \overset{x_2}{\frac{\pi}{4}}]$ at

interval $= (-1, 1) \Rightarrow \xi / x \in [-1, 1]$

So,

$$\left| f(x) - P_2(x) \right| = \left| \frac{f^3(\xi)}{3!} (x-x_0)(x-x_1)(x-x_2) \right|$$

nodes = 3

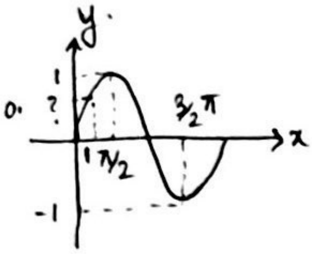
$f(x) = \cos(x)$

$f^3(x) = \sin(x)$

$$= \left| \frac{\sin(\xi)}{3!} \left(x+\frac{\pi}{4}\right)(x)\left(x-\frac{\pi}{4}\right) \right|$$

Now,

$$\underbrace{\left|\frac{\sin(\xi)}{3!}\right|}_{\substack{\sin(x) \\ max = 1 \\ min = -1}} \underbrace{\left| x\left(x+\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)\right|}_{\text{let this be } w(x).}$$



now, value of x/ξ is limited to 1 since interval is given.

but $\frac{\pi}{2} = 1.57 > 1$

sin(x) has max value when $x = \frac{\pi}{2}$ but we can not use that.

$\therefore$ we will use sin(1) ← max value possible.

$\sin(1) = 0.8415$

So, $\left|\frac{\sin(1)}{3!}\right|$

$= \left|\frac{0.8415}{6}\right|$

for any function, we get the max. value by:

$$\frac{dy}{dx} = 0.$$

So, $w(x) = x\left(x^2 - \frac{\pi^2}{16}\right)$

$$\frac{d(w)}{d(x)} = x^3 - \frac{\pi^2}{16}x$$
$$= 3x^2 - \frac{\pi^2}{16}$$

$$3x^2 - \frac{\pi^2}{16} = 0$$

$$x = \pm \frac{\pi}{4\sqrt{3}}$$

Possible values of (w(x))

x	$w(x)$
$\frac{\pi}{4\sqrt{3}}$	0.186
$-\frac{\pi}{4\sqrt{3}}$	0.186
1	0.383 ✓
−1	0.383

we took modulus

from given value of x.

So, max/upper bound error

$$= \left|\frac{0.8415}{6} \times 0.383\right| \quad \text{(Ans)}$$