Interpolation Error

$$\left| \underbrace{f(x) - P_n(x)}_{} \right| \hookrightarrow \text{error}$$

From Weierstrass Theorem, we know that if we increase the degree of polynomial, then the error reduces.

Ex: $|f(x) - P_2(x)| > |f(x) - P_{20}(x)|$

However, it is not true for all functions.

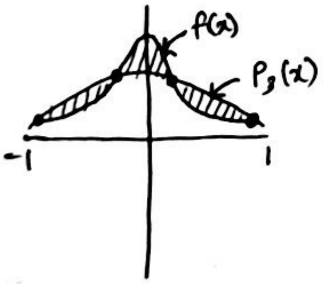
Convergence

↑ nodes ↓ error

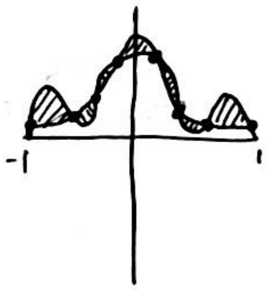
if, nodes $= \infty$, error $= 0$

let's take a function, $f(x) = \frac{1}{1+25x^2}$ on $[-1, 1]$

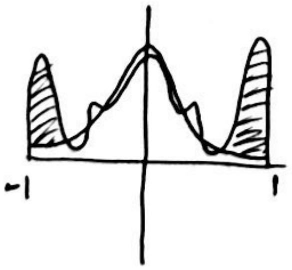
nodes = 4



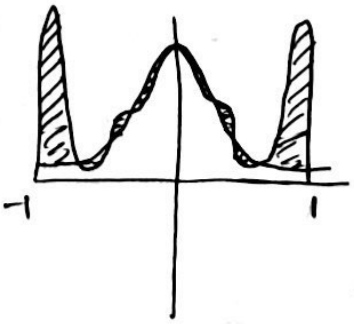
nodes = 7



nodes = 12



nodes = 16



equally spaced nodes.

From here, we can see that the error is decreasing/converging to f(x) in the middle but diverging/increasing more and more at the ends → the interval.

In short, there is a spike in the polynomial at the end points -1 and 1. This phenomena is known as Runge phenomena.

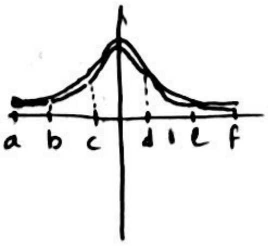
This occur due to:

- the function (f(x)) → symmetric function
- taking equally spaced nodes.

Since, the error is occurring at the corner points, we can take more nodes at those points to avoid/minimize this error. This means that we cannot take equally spaced nodes.

Solution:

a) Take piece wise interpolation → take small intervals and interpolate. lastly, add them up.



b) Take non-equal distant nodes → $\boxed{\text{Chebyshev Nodes}}$

→ we will take more nodes at end points

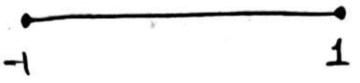
→ rather than taking equidistant nodes, we will take equal angled nodes.

Chebsher Nodes for Runge functions.

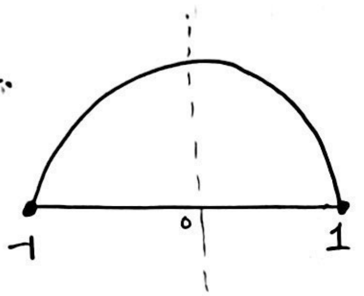
$$f(x) = \frac{1}{4+3x^2} \; ; \quad [-1, 1]$$

step: 1

Draw a line as per the interval.

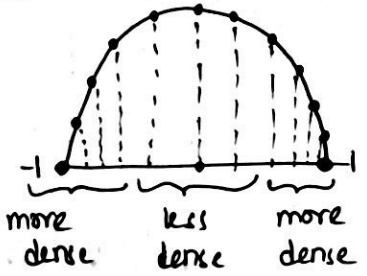


Step: 2



make a semi-circle with the end points

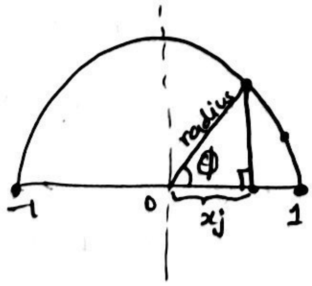
step: 3



now take equal angled nodes → "circumference e equal distance nodes nite hobe."

Then, draw lines (vertical lines).

So, from here, we can see that automatically, more nodes are present at the end points.



$$\cos\phi_j = \frac{x_j}{\text{radius}}$$

$$\boxed{\phi_j = \frac{(2j+1)\pi}{2(n+1)}}$$

$$\boxed{x_j = r\cos\phi_j + \text{center}}$$

Example

$$f(x) = \frac{1}{1+25x^2} \; ; \; [\overset{a}{-1}, \overset{b}{1}] \; ; \; \overset{\text{degree}}{n}=3$$

$n = 3$, nodes $= 4$ [nodes are not given for this ~~th~~ type of questions]

We need to find the Chebyshev nodes.

$j = 0, 1, 2, 3 \rightarrow 4$ nodes. $\rightarrow [x_0, x_1, x_2, x_3]$

$$\theta_j = \frac{(2j+1)\pi}{2(n+1)}$$

$$\theta_0 = \frac{(2\times 0+1)\pi}{2(3+1)} = \frac{\pi}{8}$$

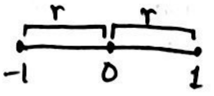
$$\theta_1 = \frac{3\pi}{8}, \; \theta_2 = \frac{5\pi}{8}, \; \theta_3 = \frac{7\pi}{8}$$

$$\text{radius} = \frac{b-a}{2} = \frac{1-(1)}{2} = 1$$

$$\text{centre} = \frac{a+b}{2} = \frac{-1+1}{2} = 0$$

Now,

radius $= 1 \rightarrow [-1, 1]$



$$x_0 = r \cos\theta_0 + \text{center}$$

$$x_0 = 1 \cos(\pi/8) + 0$$

$$x_1 = 1 \cos(3\pi/8) + 0$$

$$x_2 = 1 \cos(5\pi/8) + 0$$

$$x_3 = 1 \cos(7\pi/8) + 0$$

Chebyshev nodes

$\Rightarrow$ must calculate in radian mode!!!

Now, you know the values of nodes

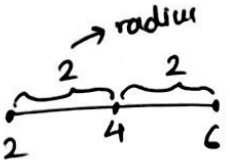
$\rightarrow x_0, x_1, x_2, x_3$

You can find the polynomial using any method.

$\rightarrow$ Vandermonde, Lagrange etc.

More examples

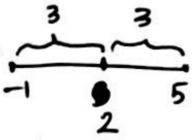
suppose interval = $[\overset{a}{2},\overset{b}{6}]$



radius = ?

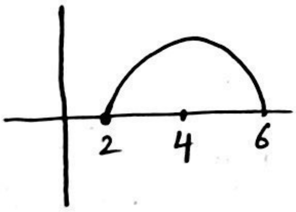
$$r = \frac{b-a}{2} = \frac{6-2}{2} = 2$$

interval = $[-1, 5]$



$$r = \frac{5-(-1)}{2} = 3$$

Ex:2 $f(x) = \frac{1}{2+3x^2}$; $n=3$; $[\overset{a}{2},\overset{b}{6}]$ $a \neq b$



$$r = \frac{6-2}{2} = 2$$

$$\text{centre} = \frac{a+b}{2} = \frac{2+6}{2} = 4$$

then,

$$x_j = r\cos\theta_j + \text{center}$$
$$= 2\cos\theta_j + \underline{\underline{4}} \longrightarrow \text{important !!!}$$

Previously, $[-1,1] \rightarrow$ values were same

$a = b$ [we donot consider the sign]

Hermite Interpolation

Previously, only one condition used to be satisfied, that is

$$P(x_i) = f(x_i)$$

now, along with the previous condition, one more condition needs to be fulfilled:

$$P'(x_i) = f'(x_i)$$

Before

given, (n+1) nodes, degree = $P_n(x)$

Now

given (n+1) nodes, degree = $P_{2n+1}(x)$

How?? $\Rightarrow$ n+1 for $f(x)$

$+$ n+1 for $f'(x)$

2n+2 $\Rightarrow$ nodes, so degree = $(2n+2)-1$

$= (2n+1)$

Using Hermite Basis

$$P_{2n+1}(x) = f(x_k)\,h_k(x) + f'(x_k)\,\hat{h}_k(x)$$

- $h_k(x) = \left[1 - 2(x-x_k)\, l_k'(x_k)\right] l_k^2(x)$
- $\hat{h}_k(x) = (x - x_k)\, l_k^2(x)$

Example

$f(x) = \sin(x).$ $\quad x = \underset{\searrow x_0}{0}, \underset{\searrow x_1}{\pi/2}$

$f(x_0) = \sin(0) = 0$

$f(x_1) = \sin(\pi/2) = 1$

nodes = 2, degree = 2(1)+1 = 3

n=1

$f'(x) = \cos(x)$

$f'(x_0) = \cos(0) = 1$

$f'(x_1) = \cos(\pi/2) = 0$

$P_3(x) = ?$

x	$f(x)$	$f'(x)$
0	0	1
$\pi/2$	1	0

$$P_3(x) = \underbrace{\overset{\to 0}{f(x_0)}\, h_0(x)}_{=0} + \overset{\to 1}{f'(x_0)}\, \hat{h}_0(x) + \overset{\to 1}{f(x_1)}\, h_1(x) + \underbrace{\overset{\to 0}{f'(x_1)}\, \hat{h}_1(x)}_{=0}$$

$$P_3(x) = \hat{h}_0(x) + h_1(x)$$

$$\boxed{h_1(x) = \left[1 - 2(x - x_1)\, l_1'(x_1)\right] l_1^2(x)}$$

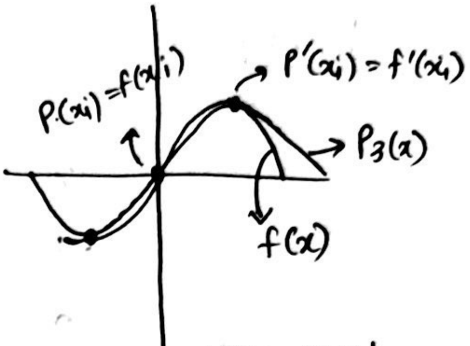
$$l_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 0}{\pi/2 - 0} = \frac{2}{\pi} x$$

$$l_1'(x) = \frac{2}{\pi}$$

$$h_1(x) = \left[1 - 2(x - \pi/2)\left(\frac{2}{\pi}\right)\right]\left[\frac{2}{\pi} x\right]^2$$

$$= \frac{4}{\pi^2} x^2 \left(3 - \frac{4}{\pi} x\right)$$

$$\hat{h}_0(x) = (x-x_0)\, l_0^2(x)$$
$$= (x-0)\left(1-\frac{2}{\pi}x\right)^2$$
$$= x\left(1-\frac{2x}{\pi}\right)^2$$



$$\therefore\ P_3(x) = \hat{h}_0(x) + h_1(x)$$
$$= x\left(1-\frac{2}{\pi}x\right)^2 + \frac{4}{\pi^2}x^2\left(3-\frac{4}{\pi}x\right)$$

very good interpolation with just 2 nodes.

Why Hermite Interpolation?

According to Weierstrass Theorem, $|f(x) - P_n(x)|$, a certain error is generated. If we increase the nodes, the error decreases. In our example, nodes = 2, so degree = 1. Using Hermite, with the same nodes, we get degree = 3. So, we can now decrease the error using the same data points.

Example: 2

	x	$f(x)$	$f'(x)$
x_0	-1	1	2
x_1	0	0	2
x_2	1	1	0

Use Hermite Interpolation to find the polynomial.

nodes = 3, degree = 2 = n

$$P_{2n+1}(x) = P_5(x) = h_0(x)f(x_0) + h_1(x)\overset{\nearrow 0}{f(x_1)} + h_2(x)f(x_2) + \hat{h}_0(x)f'(x_0) + \hat{h}_1(x)f'(x_1) + \hat{h}_2(x)\underset{\searrow 0}{f'(x_2)}$$

$$h_0(x) = \left[1 - 2(x-x_0)\,l_0'(x_0)\right] l_0^2(x)$$

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{1}{2}x(x-1)$$

$$l_0'(x) = \frac{1}{2}x(x-1) = \frac{1}{2}x^2 - \frac{1}{2}x = \frac{1}{2}\times 2x - \frac{1}{2} = x - \frac{1}{2}$$

$$l_0'(x_0) = x_0 - \frac{1}{2} = -1 - \frac{1}{2} = -\frac{3}{2}$$

$$\text{So, } h_0(x) = \left[1 - 2(x+1)\left(-\tfrac{3}{2}\right)\right]\left[\tfrac{1}{2}x(x-1)\right]^2 = \left[1+3(x+1)\right]\left[\tfrac{1}{2}x(x-1)\right]^2$$

$$h_2(x) = \left[1 - 2(x-x_2)\, l_2'(x_2)\right] l_2^2(x)$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+1)(x)}{2(1)} = \frac{1}{2}x(x+1)$$

$$l_2'(x) = \frac{1}{2}x^2 + \frac{1}{2}x$$
$$= x + \frac{1}{2}$$

$$l_2'(x_2) = 1 + 1/2 = 3/2$$

$$\therefore\ h_2(x) = \left[1 - 2(x-1)(3/2)\right]\left[\tfrac{1}{2}x(x+1)\right]^2$$
$$= \left[1 - 3(x-1)\right]\left[\tfrac{1}{2}x(x+1)\right]^2$$

$$\hat{h}_0(x) = (x-x_0)(l_0(x))^2$$
$$= (x+1)\left(\tfrac{1}{2}x(x-1)\right)^2$$

$$\hat{h_1}(x) = (x-x_1)(l_1(x))^2$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+1)(x-1)}{1(-1)} = 1 - x^2$$

$$\therefore\ \hat{h}_1(x) = (x)(1-x^2)^2$$

$$P_5(x) = \left[1 + 3(x+1)\right]\left[\tfrac{1}{2}x(x-1)^2\right](1) + \left[1 - 3(x-1)\right]\left[\tfrac{1}{2}x(x+1)\right]^2(1)$$
$$+ (x+1)\left(\tfrac{1}{2}x(x-1)\right)^2(2) + (x)(1-x^2)^2(2)$$