Lecture - 02

De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Let, $z = re^{i\theta}$

$$\Rightarrow z^n = r^n e^{in\theta}$$
$$= r^n(\cos\theta + i\sin\theta)^n$$
$$= r^n(\cos n\theta + i\sin n\theta)$$

Roots of Complex number

A number w is called n-th root of the complex number z if $w^n = z$ and we write $w = z^{1/n}$.

Ex. Find the n-th root of the complex number of the form $z = re^{i\theta}$.

Sol^n: let $z_0 = r_0 e^{i\theta_0}$ be the n-th root of z.

So, $z_0^n = z \Rightarrow z_0^n = re^{i\theta}$

$\Rightarrow (r_0 e^{i\theta_0})^n = re^{i\theta}$

$\Rightarrow r_0^n e^{in\theta_0} = re^{i\theta}$

which implies that, $r_0^n = r \Rightarrow r_0 = r^{1/n}$

and $n\theta_0 = \theta + 2K\pi$ $\{$where $K = 0, \pm 1, \pm 2, \dots\}$

Now,

for $K = 0$, $n\theta_0 = \theta \Rightarrow \theta_0 = \frac{\theta}{n}$

for $K = 1$, $n\theta_0 = \theta + 2\pi \Rightarrow \cancel{\theta}\ \theta_0 = \frac{\theta}{n} + \frac{2\pi}{n}$

for $K = 2$, $n\theta_0 = \theta + 4\pi \Rightarrow \theta_0 = \frac{\theta}{n} + \frac{4\pi}{n}$

$\vdots$

for $K = n-1$, $n\theta_0 = \theta + 2\pi(n-1) \Rightarrow \theta_0 = \frac{\theta}{2n} + \frac{2\pi(n-1)}{n}$

for $K = n$, $n\theta_0 = \theta + 2n\pi \Rightarrow \theta_0 = \frac{\theta}{n} + 2\pi$

for $K = n+1$, $n\theta_0 = \theta + 2\pi(n+1) \Rightarrow \theta_0 = \frac{\theta}{n} + \frac{2\pi}{n} + 2\pi$

for $K = -1$, $n\theta_0 = \theta - 2\pi$

$$\Rightarrow \theta_0 = \frac{\theta}{n} - \frac{2\pi}{n}$$

$$= \frac{\theta}{n} + \frac{2\pi(n-1)}{n}$$

n-th root of complex number lie on a circle of radius $r^{1/n}$ and form a n-sided regular polygon.

Qs. $\rightarrow$ Why regular polygon.

Prob.

(i) Find all values of z such that $z^5 = -32$.

(ii) Locate these values in the complex plane.

Soln: (i) Let, $z_0 = -32$

Now, $\text{Mod}(z_0) = 32$

$$\text{Arg}(z_0) = \pi - \tan^{-1}\left(\left|\frac{0}{-32}\right|\right)$$

$$= \pi$$

So, $z^5 = -32 = 32e^{i\pi}$

$$\Rightarrow (re^{i\theta})^5 = 32e^{i\pi}$$

$$\Rightarrow r^5 e^{i5\theta} = 32e^{i\pi}$$

So, $r^5 = 32 \Rightarrow r = 2$ and $5\theta = \pi + 2k\pi$; $k = 0, 1, 2, 3, 4$

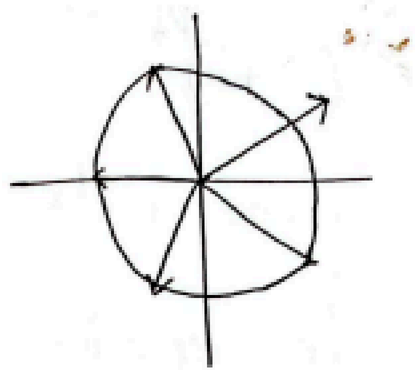
For, $k = 0 \rightarrow \theta = \frac{\pi}{5}$

For, $k = 1 \rightarrow \theta = \frac{3\pi}{5}$

For, $k = 2 \rightarrow \theta = \pi$

For, $k = 3 \rightarrow \theta = \frac{7\pi}{5}$

For, $k = 4 \rightarrow \theta = \frac{9\pi}{5}$



So roots are,

$z_1 = 2e^{i(\pi/5)}$, $z_2 = 2e^{i(3\pi/5)}$, $z_3 = 2e^{i\pi}$, $z_4 = 2e^{i7\pi/5}$, $z_5 = 2e^{i9\pi/5}$

Solution of $az^2+bz+c=0$ is,

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

For this we need the square root of complex number (i.e., $z^{1/2}$).

Exercise

1. Express Each of the following complex numbers in polar form,

   (a) $2+2\sqrt{3}\,i$, (b) $-5+5i$, (c) $-\sqrt{6}-\sqrt{2}\,i$

2. Find each of the indicated roots and locate them graphically,

   (a) $(-1+i)^{1/3}$ (b) $(-2\sqrt{3}-2i)^{1/4}$

3. Find square roots of $-15-8i$.

4. Solve the equation, $z^2+(2i-3)z+5-i=0$

5. Find all the 10 th roots of unity.

6. Represent graphically the set of values of z for which,

   (a) $\left|\frac{z-3}{z+3}\right|=2$, (b) $\left|\frac{z-3}{z+3}\right|<2$

7. Describe and graph the locus represented by each of the following,

ⓐ $|z+2i|+|z-2i|=6$, ⓑ $|z-3|-|z+3|=4$

8. Describe graphically the region represented by each of the following,

ⓐ $1<|z+i|\leq 2$, ⓑ $Re\{z^2\}>1$, ⓒ $Im\{z^2\}=4$

9. Find the Sixth roots of $-27i$

10. Find cube roots of $-11-2i$.

11. Find the indicated roots and locate them graphically.

ⓐ $(64)^{1/6}$ ⓑ $(i)^{2/3}$ ⓒ $\{-1+\sqrt{3}i\}^{1/6}$

2.b

$$(-2\sqrt{3}-2i)^{1/4}$$

Now, $Z = -2\sqrt{3} - 2i$

So, $\text{Mod}(Z) = \sqrt{12+4} = 4$

$\text{Arg}(Z) = \tan^{-1}\left(\left|\frac{-2}{-2\sqrt{3}}\right|\right) - \pi$

$= \frac{\pi}{3} - \pi = -\frac{2\pi}{3}$

Let, $Z_0 = r_0 e^{i\theta_0}$ be the 4th root of z, so,

$$Z_0 = (-2\sqrt{3}-2i)^{1/4}$$

$$\Rightarrow Z_0^4 = 4e^{-i\frac{2\pi}{3}}$$

$$\Rightarrow r_0^4 e^{i4\theta_0} = 4e^{-i\frac{2\pi}{3}}$$

So, $r_0^4 = 4 \Rightarrow r_0 = \sqrt{2}$ and, $4\theta_0 = -\frac{2\pi}{3} + 2k\pi$

for $k=0$, $\theta_0 = -\frac{\pi}{6}$

for $k=1$, $\theta_0 = \frac{\pi}{3}$

for $k=2$, $\theta_0 = \frac{5\pi}{6}$

for $k=3$, $\theta_0 = \frac{4\pi}{3}$

So, 4 roots are,

$Z_1 = \sqrt{2}\, e^{-i\frac{\pi}{6}}$, $Z_2 = \sqrt{2}\, e^{i\pi/3}$

$Z_3 = \sqrt{2}\, e^{i5\pi/6}$, $Z_4 = \sqrt{2}\, e^{i\frac{4\pi}{3}}$

4.

$$z^2 + (2i-3)z + 5 - i = 0$$

$$\Rightarrow z = \frac{-(2i-3) \pm \sqrt{(2i-3)^2 - 4.(5-i)}}{2.1}$$

$$= \frac{-(2i-3) \pm \sqrt{-15-8i}}{2}$$

Root of $\sqrt{-15-8i}$ (H.W.)

$$\left[(-15-8i)^{1/2} = \pm(1-4i)\right)$$

6.

ⓐ $\left|\dfrac{z-3}{z+3}\right| = 2$

$$\Rightarrow |z-3| = 2|z+3|$$

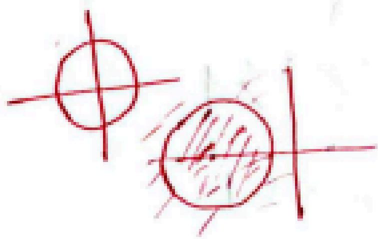
$$\Rightarrow |(x-3)+iy| = 2\,|(x+3)+iy|$$

$$\Rightarrow \sqrt{(x-3)^2+y^2} = 2\sqrt{(x+3)^2+y^2}$$

$$\Rightarrow (x-3)^2+y^2 = 4\{(x+3)^2+y^2\}$$

$$\Rightarrow x^2+y^2+10x+9 = 0$$

$$\Rightarrow (x+5)^2+y^2 = \cancel{0}\,16$$



$\Rightarrow |z+5| = 4$. a circle of radius 4 with center at $(-5, 0)$