Harmonic Function

If $\frac{\partial^2 u}{\partial x^2}$ ~~and~~, $\frac{\partial^2 v}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial^2 v}{\partial y^2}$ exists and continuous in a region R then u and v is harmonic function if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Ex. If $f(z) = u + iv$ ~~in~~ is analytic in a region R. Prove that u and v are harmonic in R if they have continuous second partial derivatives in R.

Soln: As $f(z)$ is analytic,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Now,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \cancel{\frac{\partial^2 v}{\partial y \partial x}}\ \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial x \partial y}$$

So, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Similarly, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Ex.

(a) Prove that, $u = e^{-x}(x\,\text{Sin}y - y\,\text{Cos}\,y)$ is harmonic.

(b) Find v such that $f(z) = u + iv$ is analytic.

(c) Find $f(z)$.

Soln:

(a) Given,

$$u = e^{-x}(x\,\text{Sin}y - y\,\text{Cos}y)$$

$$u_x = e^{-x}(\text{Sin}y - 0) + (x\,\text{Sin}y - y\,\text{Cos}\,y)\cdot(-e^{-x})$$

$$= e^{-x}(\text{Sin}y \cancel{+} - x\,\text{Sin}y + y\,\text{Cos}\,y)$$

$$u_{xx} = e^{-x}(-\text{Sin}y) + (\text{Sin}y - x\,\text{Sin}y + y\,\text{Cos}\,y)(-e^{-x})$$

$$= e^{-x}(-2\text{Sin}y + x\,\text{Sin}y - y\,\text{Cos}y)$$

Again,

$$u_y = e^{-x}(x\,\text{Cos}\,y + y\,\text{Sin}\,y - \text{Cos}\,y)$$

$$u_{yy} = e^{-x}(-x\,\text{Sin}y + y\,\text{Cos}\,y + \text{Sin}y + \text{Sin}y)$$

$$= e^{-x}(2\text{Sin}y - x\,\text{Sin}y + y\,\text{Cos}y)$$

Thus, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

So, $u(x,y)$ is harmonic function.

(b) As $f(z)$ is analytic,

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

Now,

$$v_y = e^{-x}\sin y - e^{-x}x\sin y + e^{-x}y\cos y$$

$$\Rightarrow v = \int \left(e^{-x}\sin y - e^{-x}x\sin y + e^{-x}y\cos y\right)dy + g(x)$$

$$= -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}\int y\cos y\,dy + g(x)$$

$$= -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}\left[y\sin y + \cos y\right] + g(x)$$

$$= xe^{-x}\cos y + e^{-x}y\sin y + g(x)$$

$$\int y\cos y\,dy$$
$$= y\sin y - \int 1\cdot\sin y\,dy$$
$$= y\sin y + \cos y$$

Again,

$$u_y = -v_x$$

$$\Rightarrow -v_x = e^{-x}\left(x\cos y + y\sin y - \cos y\right)$$

$$\cancel{\Rightarrow -\left(e^{-x}\cos y\right.}$$

$$\Rightarrow -\left(-xe^{-x}\cos y + e^{-x}\cos y - e^{-x}y\sin y + g'(x)\right) = e^{-x}\left(x\cos y + y\sin y - \cos y\right)$$

$$\Rightarrow g'(x) = 0 \Rightarrow g(x) = C$$

So, $v(x,y) = xe^{-x}\cos y + e^{-x} y \sin y + c$

(c)

$$u = e^{-x} x \sin y - e^{-x} y \cos y$$

$$v = xe^{-x}\cos y + e^{-x} y \sin y \cancel{+c}$$

We know,

$$f(z) = u + iv$$

$$= e^{-x}(x\sin y - y\cos y) + i\{xe^{-x}\cos y + e^{-x} y \sin y \cancel{+c}\}$$

$$= e^{-x}\left[\left\{x \cdot \frac{e^{iy}-e^{-iy}}{2i} - y\,\frac{e^{iy}+e^{-iy}}{2}\right\} + i\left\{x\frac{e^{iy}+e^{-iy}}{2} + y\frac{e^{iy}-e^{-iy}}{2i}\right\}\right]$$

$$= \frac{1}{2}e^{-x}\left[-ixe^{iy} + ixe^{-iy} - ye^{iy} - ye^{-iy} + ixe^{iy} + ixe^{-iy} + ye^{iy} - ye^{-iy}\right]$$

$$= \frac{1}{2}e^{-x}\left[ixe^{-iy} + ixe^{-iy} - ye^{-iy} - ye^{-iy}\right]$$

$$= \frac{1}{2}e^{-x}.e^{-iy}\left[2ix - 2y\right]$$

$$= ie^{-(x+iy)}(x+iy) = ie^{-z}.z \quad \text{Ans.}$$

H.W.

Determine whether u is harmonic. For each harmonic function find v such that $u+iv$ is analytic.

(i) $u = 3x^2y + 2x^2 - y^3 - 2y^2$

(ii) $u = xe^x \cos y - ye^x \sin y$

(iii) $u = e^{-2xy} \sin(x^2 - y^2)$

Sol^n:

$u_x = 6xy + 4x$ | $u_y = 3x^2 - 3y^2 - 4y$

$u_{xx} = 6y + 4$ | $u_{yy} = -6y - 4$

As, $u_{xx} + u_{yy} = 6y + 4 - 6y - 4 = 0$. So, u is harmonic.

We know,

$$u_x = v_y$$

$$\Rightarrow v_y = 6xy + 4x$$

$$\Rightarrow v = \int (6xy + 4x)\, dy + g(x)$$

$$= 6x\frac{y^2}{2} + 4xy + g(x) = 3xy^2 + 4xy + g(x)$$

Again,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow 3x^2 - 3y^2 - 4y = -\left(3y^2 + 4y + g'(x)\right)$$

$$\Rightarrow g'(x) = -3x^2$$

$$\Rightarrow g(x) = -3\int x^2\,dx$$

$$= -x^3 + C$$

$$\therefore v = 3xy^2 + 4xy - x^3 + C.$$