Domain

Find the domain for each of the function,

① $f(z) = \frac{1}{z^2+1}$ ② $f(z) = \frac{z}{z+\bar{z}}$

(Singularities) $\mathbb{C} - \{-i, i\}$

Limit

Let, $f(z)$ be defined and single-valued in the neighborhood of $z = z_0$ with the possible exception of $z = z_0$. itself. We say that the number l is the limit of $f(z)$ as z approaches z_0 and write,

$$\lim_{z \to z_0} f(z) = l$$

Ex:
Prove that, $\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exist.

Soln: Let $z \to 0$ along x-axis. So, $y = 0$

$$\lim_{z \to 0} \frac{x - iy}{x + iy} = \lim_{z \to 0} \frac{x}{x} = \lim_{z \to 0} 1 = 1.$$

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let, $z \to 0$ along $y = x$.

$$\lim_{z\to 0} \frac{x - iy}{x + iy} = \lim_{z\to 0} \frac{x(1-i)}{x(1+i)}$$

$$= \lim_{z\to 0} \frac{(1-i)^2}{1+1}$$

$$= \frac{1-2i-1}{2} = -i$$

The two sided approach do not provide same result, the limit does not exists.

Ex.

(i) $\lim_{z \to 1+i} (z^2 - 5z + 16)$

(ii) $\lim_{z \to -2i} \frac{(2z+3)(z-1)}{z^2 - 2z + 4}$

(iii) $\lim_{z \to 2e^{i\pi/3}} \frac{z^3+8}{z^4+4z^2+16}$

Hint : $z^6 - 64 = (z^2-4)(z^4+4z^2+16)$

Soln:

$$z^4+4z^2+16 = \left(2e^{i\frac{\pi}{3}}\right)^4 + 4\left(2e^{i\pi/3}\right)^2 + 16$$

$$= 16\left\{e^{\frac{i4\pi}{3}} + e^{\frac{i2\pi}{3}} + 1\right\} = 16\left\{-\frac{1}{2} - \frac{\sqrt{3}}{2}i - \frac{1}{2} + \frac{\sqrt{3}}{2}i + 1\right\}$$

$$= 0$$

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Now,

$$\lim_{z \to 2e^{i\pi/3}} \frac{z^3+8}{z^4+4z^2+16}$$

$$= \lim_{z \to 2e^{i\pi/3}} \frac{(z^3+8)(z^2-4)}{z^6-64}$$

$$= \lim_{z \to 2e^{i\pi/3}} \frac{z^2-4}{z^3-8}$$

$$= \frac{4e^{i\frac{2\pi}{3}}-4}{8e^{i\pi}-8} = \frac{4(e^{i\frac{2\pi}{3}}-1)}{-16}$$

$$= \frac{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i-1\right)}{-4}$$

$$= \frac{3}{8} - \frac{\sqrt{3}}{8} i \quad . \text{ Ans.}$$

L'Hospital Rule

Let $f(z)$ and $g(z)$ be analytic in a region containing point z_0, then,

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}$$

Ex. $\displaystyle \lim_{z \to 1+i} \frac{z^2-z+1-i}{z^2-2z+2} = \lim_{z \to 1+i} \frac{2z-1}{2z-2} = \frac{1+2i}{2i} = 1+\frac{1}{2}i$

Note: The expression must be in the form of $\frac{0}{0}, \frac{\infty}{\infty}$ to use L'Hospital rule.

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Find

$$\lim_{z \to e^{i\pi/3}} \left(z - e^{i\pi/3}\right)\left(\frac{z}{z^3+1}\right)$$

$$= \lim_{z \to e^{i\pi/3}} \frac{z^2 - e^{i\pi/3} z}{z^3+1}$$

$$= \lim_{z \to e^{i\pi/3}} \frac{2z - e^{i\pi/3}}{3z^2}$$

$$= \frac{2e^{i\pi/3} - e^{i\pi/3}}{3\left(e^{i\pi/3}\right)^2}$$

$$= \frac{1}{3e^{i\pi/3}} = \frac{1}{3} e^{-i\pi/3}$$

$$= \frac{1}{3}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$

$$= \frac{1}{3}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

(R) $$\lim_{z \to 0} \frac{z - \sin z}{z^3}$$

$$= \lim_{z \to 0} \frac{1 - \cos z}{3z^2}$$

$$= \lim_{z \to 0} \frac{\sin z}{6z} = \lim_{z \to 0} \frac{\cos z}{6} = \frac{1}{6}$$

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H.W.

(*) $\lim_{z \to 0} \frac{\tan z - \sin z}{z^3}$

(*) $\lim_{z \to 0} \left(\frac{\sin z}{z}\right)^{\frac{1}{z^2}}$

Continuity

$f(z) \to$ be defined and single valued in a neighbourhood of $z = z_0$ as well as at $z = z_0$.

The function $f(z)$ is said to be continuous at $z = z_0$ if

(3 Conditions)

$$\lim_{z \to z_0} f(z) = f(z_0)$$

(*) Let, $f(z) = \frac{z^2+4}{z-2i}$ if $z \neq 2i$, while $f(2i) = 3+4i$.

(a) Prove that $\lim_{z \to 2i} f(z)$ exists and determine its value.

(b) Is $f(z)$ continuous at $z = 2i$?

(c) Is $f(z)$ Continuous at points $z \neq 2i$? Explain.

(d) Redefine the function to make it continuous.

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Ⓠ If $f(z) = \begin{cases} \frac{z^2-4}{z^2-3z+2}, & z \neq 2 \\ kz^2+6, & z=2 \end{cases}$, find k such that $f(z)$ becomes continuous at $z=2$.

Soln:

$$f(2) = k \cdot 2^2 + 6 = 4k+6$$

$$\lim_{z \to 2} f(z) = \lim_{z \to 2} \frac{z^2-4}{z^2-3z+2}$$

$$= \lim_{z \to 2} \frac{2z}{2z-3}$$

$$= \frac{4}{4-3} = 4$$

$\therefore 4k+6 = 4$

$\Rightarrow 4k = -2$

$\Rightarrow k = -\frac{1}{2}$.

Ex.

Ⓠ Is the function $f(z) = \frac{3z^4 - 3z^3 - 8z^2 - 2z + 5}{z-i}$ continuous at $z = i$

Ⓠ Show that, $f(z) = \frac{xy}{x^2+y^2}$ is not continuous at $z=0$.

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$$\lim_{z \to 0} \left(\frac{\sin z}{z} \right)^{1/z^2}$$

Sol^n:

Let, $$w = \lim_{z \to 0} \left(\frac{\sin z}{z} \right)^{1/z^2}$$

$$\Rightarrow \ln w = \lim_{z \to 0} \frac{1}{z^2} \ln\left(\frac{\sin z}{z} \right)$$

$$= \lim_{z \to 0} \frac{\ln(\sin z) - \ln(z)}{z^2}$$

$$= \lim_{z \to 0} \frac{\frac{\cos z}{\sin z} - \frac{1}{z}}{2z}$$

$$= \lim_{z \to 0} \frac{z\cos z - \sin z}{2z^2 \cdot \sin z}$$

$$= \lim_{z \to 0} \frac{-z\sin z + \cos z - \cos z}{4z\sin z + 2z^2\cos z}$$

$$= \lim_{z \to 0} \frac{-z\sin z}{4z\sin z + 2z^2\cos z}$$

$$= \lim_{z \to 0} \frac{-z\cos z - \sin z}{4\sin z + 4z\cos z + 4z\cos z - 2z^2\sin z}$$

$$= \lim_{z \to 0} \frac{-\cos z - \cos z + z\sin z}{4\cos z + 8\cos z + 8z(-\sin z) - 4z(\sin z) - 2z^2\cos z}$$

$$= \frac{-1-1}{4+8} = -\frac{1}{6}$$

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