Analytic/Holomorphic function

If the derivative

at all points of z of a region R the

to be analytic in R.

Example $f(z) = z^3 \Rightarrow f'(z) = 3z^2$

for any point z_0, $f(z_0) = 3z_0^2$, (deriv

Q. If $f(z) = u(x,y) + iv(x,y)$, find $f'(z)$.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$z = x + iy$
$\Delta z = \Delta x + i\Delta y$

$$= \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)]}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) - u(x,y)] + i[v(x+\Delta x, y+\Delta y) - v(x,y)]}{\Delta x + i\Delta y}$$

Approaching $\Delta z \to 0$ through $(\Delta x, 0)$,

$$f'(z) = \lim_{\Delta z \to 0}\left(\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\,\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\right)$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Approaching $\Delta z \to 0$ through $(0, \Delta y)$,

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x, y+\Delta y) - u(x,y)] + i[v(x, y+\Delta y) - v(x,y)]}{i\Delta y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Lecture - 06

* If $w = f(z) = u_{(x,y)} + iv(x,y)$ be analytic in Region R, u and v satisfy Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

* If u_x, u_y, v_x, v_y are continuous in R, then the Cauchy-Riemann equations are sufficient condition that $f(z)$ be analytic in R.

* Show that $f(z) = z^2$ is analytic.

$$f(x+iy) = (x+iy)^2 = x^2 + 2xiy + i^2y^2$$

$$= (x^2 - y^2) + i\,2xy$$

$\therefore u = x^2 - y^2, \; v = 2xy$

$\frac{\partial u}{\partial x} = 2x$

$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = 2x$

$\frac{\partial v}{\partial x} = 2y$ All of them are continuous.

So, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. Therefore, analytic.

Ⓧ Show that, $f(z) = |z|^2 - \bar{z}$ is not analytic.

Soln:

$$f(z) = |x+iy|^2 - \overline{(x+iy)}$$

$$= x^2+y^2 - x + iy$$

$$= (x^2+y^2-x) + iy$$

$\therefore \; u = x^2+y^2-x \;, \; v = y$

$u_x = 2x-1 \;, \; v_x = 0$

$u_y = 2y \;, \; v_y = 1$

Here,

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial x}$$

$\therefore f(z)$ is not analytic.

H.W.

Show that $f(z) = \frac{1}{z-3}$ is analytic except $z = 3$.