(9) Solve for x and y. $\left(\frac{3}{2}+\frac{\sqrt{3}}{2}i\right)^{2024} = 3^{1012}(x+iy)$

Soln:

let, $z = \frac{3}{2} + \frac{\sqrt{3}}{2} i$

$$\text{Mod}(z) = \sqrt{\frac{9}{4}+\frac{3}{4}} = \sqrt{\frac{12}{4}} = \sqrt{3}$$

$$\text{Arg}(z) = \tan^{-1}\left|\frac{\sqrt{3}/2}{3/2}\right| = \tan^{-1}\left|\frac{1}{\sqrt{3}}\right| = \frac{\pi}{6}$$

So,

$$\left(\sqrt{3}\, e^{i\frac{\pi}{6}}\right)^{2024} = 3^{1012} e^{i\frac{506\pi}{3}}$$

$$= 3^{1012} e^{i\frac{504\pi+2\pi}{3}}$$

$$= 3^{1012} e^{i\left(\frac{2\pi}{3}+168\pi\right)}$$

$$= 3^{1012} e^{i\frac{2\pi}{3}}$$

$$= 3^{1012}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$= 3^{1012}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

So, $x = -\frac{1}{2}$

$y = \frac{\sqrt{3}}{2}$.

Variables and Functions

$z \rightarrow$ any one of a set of complex numbers

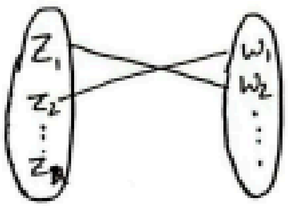
$z \rightarrow$ function, $w \longrightarrow w_1, w_2, w_3$

we can say w is a function of z.

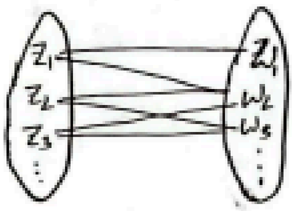
$w \downarrow$ dependent variable

$z \downarrow$ ~~Indep~~ Independent variable

Single and Multiple Valued Function



Single valued



multiple valued

Example:

$f(z) = z^2 \rightarrow$ Single valued function

$f(z) = z^{1/n} \rightarrow$ multiple valued function

$f(z) = \arg(z) \rightarrow$ 〃 〃 〃

$f(z) = \ln(z) \rightarrow$ 〃 〃 〃

$\hookrightarrow \ln z = \ln(re^{i\theta})$

$\Rightarrow \ln z = \ln r + \ln(e^{i\theta + 2ik\pi})$

$= \ln r + i(\theta + 2k\pi) \quad [k = 0, \pm 1, \pm 2, \ldots]$

Branch (of multiple valued function)

$$\ln(z) = \ln r + i(\theta + 2k\pi)$$
$$= \ln r + i\cdot\mu \qquad 0 \le \mu \le 2\pi$$

Example

$$f(z) = \arg(z)$$
$$\Rightarrow f(re^{i\theta}) = \theta + 2k\pi$$

(1) $0 \le \arg(z) < 2\pi$, $-\pi < \arg(z) \le \pi$

(*) Suppose $w = u+iv$ is the value of a function f at $z = x+iy$, so that,

$$\left.u+iv = f(x+iy)\right\} \begin{matrix} u = u(x,y) \\ v = v(x,y) \end{matrix}$$

Ex $f(z) = z^2 + 2z$

let, $z = x+iy$; $f(z) = (x+iy)$

$$f(z) = (x+iy)^2 + 2(x+iy)$$
$$= \underbrace{(x^2 + 2x - y^2)}_{u(x,y)} + i\underbrace{(2xy + 2y)}_{v(x,y)}$$

CamScanner

Q. If polar coordinates r and θ, instead of x and y are used,

$$u+iv = f(re^{i\theta}) \quad \Bigg\} \begin{array}{l} u = u(r,\theta) \\ v = v(r,\theta) \end{array}$$

$$f(re^{i\theta}) = r^2 e^{i2\theta} + 2r e^{i\theta}$$

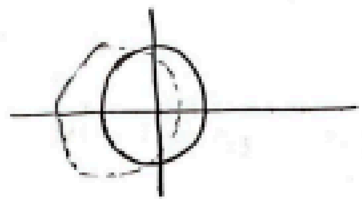
$$= r^2(\cos 2\theta + i\sin 2\theta) + 2r(\cos\theta + i\sin\theta)$$

$$= \underbrace{\{r^2\cos 2\theta + 2r\cos\theta\}}_{u(r,\theta)} + i\underbrace{\{r^2\sin 2\theta + 2r\sin\theta\}}_{v(r,\theta)}$$

Some example of complex function

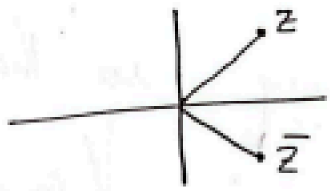
(i) $f(z) = z+1 = w$ Translation

$= (x+1) + iy$



(ii) $f(z) = \bar{z}$ Reflection

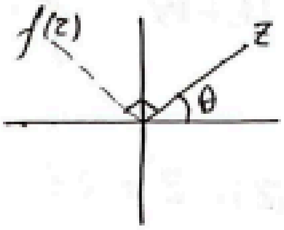
$= x - iy$



(iii) $f(z) = iz = ire^{i\theta}$

$= e^{i\frac{\pi}{2}} re^{i\theta}$

$= re^{i(\theta + \frac{\pi}{2})}$



Ex.

Let, $w = e^z = f(z)$. What is

$f(\text{vertical line}) = ?$

$f(\text{horizontal line}) = ?$

Soln:

$$f(z) = e^{z}$$

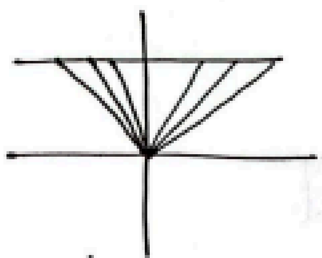
$$\Rightarrow f(z) = e^{x+iy}$$

$$\Rightarrow \rho e^{i\phi} = e^{x} \cdot e^{iy}$$

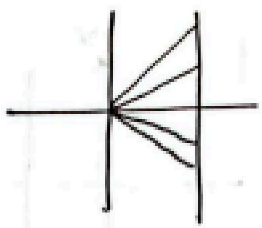
$$\rho = e^{x} \quad \text{and} \quad \phi = y + 2k\pi \quad ; \; k = 0, \pm 1, \pm 2, \ldots$$

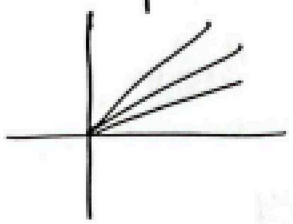
vertical line in xy plane $x = c_1$, e^{c_1}

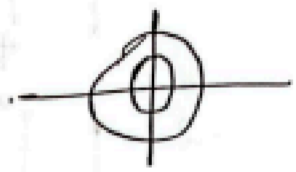
Horizontal line in xy-plane $y = c_2$



$$\phi = c_2 + 2\pi K$$







Problem

If c_1 and c_2 are any real constant determine the set of all points in the z-plane that maps into the lines (a) $u = c_1$ and (b) $v = c_2$ in the w-plane by $w = z^2$. Consider $(c_1 = 1, -2)$, $(c_2 = 4, -4)$.

Soln:

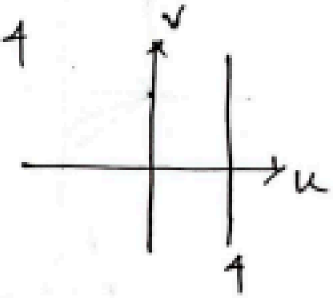
We have,

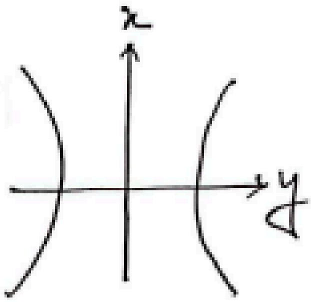
$$w = z^2 = (x+iy)^2$$
$$= x^2 - y^2 + 2ixy$$

So, $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$

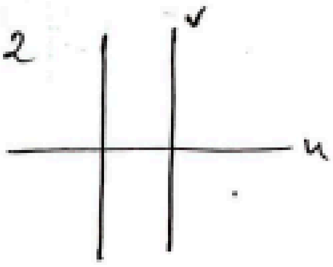
when,
$$u = c_1 \Rightarrow x^2 - y^2 = c_1$$
$$v = c_2 \Rightarrow xy = \frac{c_2}{2}$$

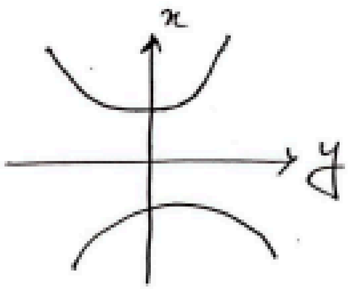
(i) $c_1 = 4$



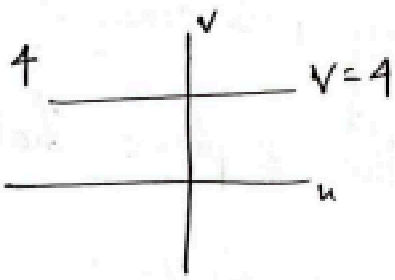


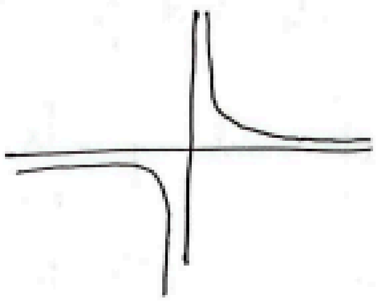
(ii) $c_1 = -2$



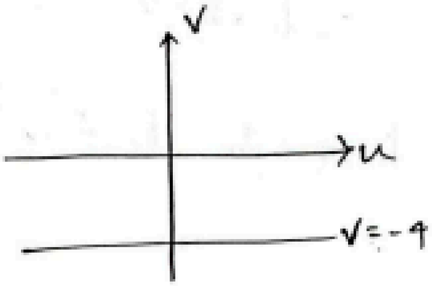


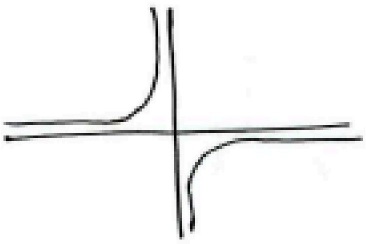
(iii) $c_2 = 4$





(iv) $c_2 = 4$





Elementary Functions

(i) Polynomial function

(ii) Rational Algebraic function

$$w = \frac{P(z)}{Q(z)}$$

(iii) Exponential Function

$$w = e^z$$

(iv) Trigonometric function

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

(v) Hyperbolic function

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}$$

$$\operatorname{sech} z = \frac{1}{\cosh z}, \quad \operatorname{cosech} z = \frac{1}{\sinh z}$$

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}$$

$$\cosh^2 z - \sinh^2 z = 1$$

Logarithmic Function

$$w = \ln(z) = \ln(re^{i\theta})$$