How to calculate upper bound of truncation error when ξ is not mentioned?

Ex: $f(x) = \ln(x)$

calculate • upper bound error using central difference, forward difference and backward difference at $x = 1.0$, $h = 0.1$.

Formula for FD, BD $= \left| -\dfrac{f''(\xi)(h)}{2!} \right|$

formula for CD $= \left| -\dfrac{f'''(\xi)(h)^2}{3!} \right|$

ξ value $\rightarrow [x+h, x-h]$ = for CD

ξ value $\rightarrow [x+h, x]$ = for FD

ξ value $\rightarrow [x, x-h]$ = for BD

<u>Richardson Extrapolation</u> → only for Central Difference!

↳ more accurate value of $f'(x)$

from CD, we know:

$$D_h = f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

Previously, we have seen that the derivation of the formula was using Lagrange polynomial. In this case, we will used Taylor series instead.

Proof: Taylor Series $\Rightarrow f(x) = f(x_0) + f^{(1)}(x_0)(x-x_0) + \frac{f^2(x_0)(x-x_0)^2}{2!} + \ldots$

$$f(x+h) = f(x) + f'(x)\cdot h + \frac{f''(x)\cdot h^2}{2!} + \frac{f^3(x)\cdot h^3}{3!} + \frac{f^4(x)h^4}{4!} + \frac{f^5(x)h^5}{5!} + O(h^6) \quad \ldots\ldots\ldots \text{(i)}$$

$$f(x-h) = f(x) - f'(x)\cdot h + \frac{f''(x)h^2}{2!} - \frac{f^3(x)h^3}{3!} + \frac{f^4(x)h^4}{4!} - \frac{f^5(x)h^5}{5!} + O(h^6) \quad \ldots\ldots \text{(ii)}$$

$$f(x+h) - f(x-h) = 2f'(x)h + \frac{2f^3(x)h^3}{3!} + \frac{2f^5(x)h^5}{5!} + O(h)^7 \ldots\ldots \text{(iii)}$$

$$D_h = \frac{f(x+h) - f(x-h)}{2h}$$

[$O(h^6)$ term gets cancelled, however, to represent all the rest of the terms, $O(h)^7$ is used]

so, dividing equation (iii) by '2h' according to formula

$$D_h = \frac{1}{2h}\left[2f'(x)(h) + \frac{2f^3(x)h^3}{3!} + \frac{2f^5(x)h^5}{5!} + O(h)^7\right]$$

$$D_h = \underbrace{f'(x)}_{\text{actual derivative}} + \underbrace{\frac{f^3(x)h^2}{3!} + \frac{f^5(x)h^4}{5!} + O(h)^6}_{\text{truncation error.}}$$

From the error part, $\frac{f^3(x)h^2}{3!}$ is the dominating factor, so we need to remove this part.

$$D\frac{h}{2} = f'(x) + \frac{f^3(x)(h/2)^2}{3!} + \frac{f^5(x)(h/5)^4}{5!} + \underbrace{O(h)^6}_{} \;\ldots\ldots (iv)$$

keep this part as it is, you can also change it to $O(\frac{h}{2})^6$

Comparing equation (iii) & (iv), to remove the h^2 term, we need to multiply eq^n (iv) by 4.

$$4\,D(h/2) - D(h) = \underbrace{3f'(x)} + \left(\tfrac{1}{4}-1\right)\frac{f^5(x)(h)^4}{5!} + O(h)^6$$

now, with this actual value, no coefficient can be present, hence we divide it by 3.

$$\frac{4D(h/2)-D(h)}{3} = f'(x) + \frac{(\frac{1}{4}-1)}{3}\frac{f^5(x)(h)^4}{5!} + O(h)^6$$

$$\therefore D_n^{(1)} = \frac{4D(h/2)-D(h)}{3}$$

$$\boxed{\begin{aligned} D_h^{(1)} &= \frac{2^2 D(h/2) - D(h)}{2^2-1} \\ &= \frac{2^n D(h/2) - D(h)}{2^n-1}\end{aligned}}$$

$\Rightarrow$ same thing

If we take this combination, error gets reduced to an order of 4 (h^4), since $h^2 \rightarrow$ order 2 is eliminated.

To find, $D_h^{(2)}$: $D_h^{(1)} \rightarrow$ take this

Find $D_{h/2}^{(1)}$

Then take combinations in such way that h^4 term gets cancelled. Then error will be order 6.

$\boxed{D'(h) = D_h^{(1)} = \text{same}}$

Formula : $D^{(2)}_h = \dfrac{16\,D^{(1)}(h/2) - D^{(1)}(h)}{15}$

$f(x) = e^{x} \sin(x)$.

Question: Find $D'_{(h)}$ using Richardson extrapolation at $x = 1$, for

i) $h = 0.5$

ii) $h = 0.25$

we know Richardson extrapolation is only for central difference.

i). $f'(1)$; $h = 0.5$

$$f'(1) = \frac{f(1+0.5) - f(1-0.5)}{2 \times 0.5}$$

$$= \frac{e^{1.5}\sin(1.5) - e^{0.5}\sin(0.5)}{1}$$

$$= 3.68$$

ii) $f'(1)$; $h = 0.25$

$$f'(1) = \frac{f(1.25) - f(0.75)}{2 \times 0.25}$$

$$= 3.7385$$

Using Richardson to find more accurate value:

$h = 0.5$ $\quad D_h = 3.68$

$h = 0.25$ $\quad D_h = 3.7385$

$$D^{(1)}_h = \frac{4D(h/2) - D(h)}{3}$$

$$= \frac{4(3.7385) - 3.68}{3}$$

$$= 3.757$$

Example $h = 0.1 \quad f'(1) = 0.7$
$h = 0.2 \quad f'(1) = 0.5$

Using Richardson Extrapolation, find $f'(1)$.

$$D^{(1)}_{h} = \frac{4D(h/2) - D(h)}{3}$$

0.2

$$= \frac{4(0.7) - 0.5}{3}$$

$$= 0.77$$

Example: Given

x	$f(x)$
0.6	0.707178
0.8	0.8559892
0.9	0.925863
1.0	0.984007
1.1	1.033743
1.2	1.074575
1.4	1.127986

find $f'(1)$ for $h = 0.4$, $h = 0.2$, $h = 0.1$ using RE.

$h = 0.4$

$$D_h = f'(1) = \frac{f(1+0.4) - f(1-0.4)}{2 \times 0.4}$$

$$= (1.127986 - 0.707178)/2 \times 0.4$$

$$= 0.52601$$

$h = 0.2$

$$f'(1) = \frac{f(1+0.2) - f(1-0.2)}{2 \times 0.2}$$

$$= 0.5464$$

$h = 0.1$

$$f'(1) = \frac{f(1+0.1) - f(1-0.1)}{2 \times 0.1}$$

$$= 0.5394$$

h	D_h		
0.4	0.52601	$D^{(1)}_h$; $h=0.4$ = 0.553	$D^{(2)}_h$; $h=0.4$ = 0.535933
0.2	0.5464	$D^{(1)}_h$; $h=0.2$ = 0.537	
0.1	0.5394		

So,
$$D^{(1)}_h = \frac{4D(h/2) - D(h)}{3}$$
$$= \frac{4(0.5464) - 0.52601}{3} = 0.553$$

$$D^{(1)}_h = \frac{4D(h/2) - D(h)}{3}$$
$$= \frac{4(0.5394) - 0.5464}{3} = 0.537$$

Now, calculate $D^{(2)}_h$ using Richardson Extrapolation.

What will be the value of h?

$h = 0.4 \rightarrow$ the first value always!!

$\boxed{D'_h = D^{(1)}_h = \text{same}}$

Formula : $$D^{(2)}_h = \frac{16D'(h/2) - D'(h)}{15}$$
$$= \frac{16(0.537) - 0.553}{15}$$
$$= 0.535933$$

Example $f(x) = x^2 + e^x$

Compute $D^{(1)}_{0.2}$ and $D^{(2)}_{0.2}$ at $x = 1$ using Richardson Extrapolation.

$h = 0.2$

another $h = h/2 = 0.2/2 = 0.1$

$f'(1)$; $h = 0.2$

$$D_h = f'(1) = \frac{f(1.2) - f(0.8)}{2 \times 0.2}$$

$$= 4.7364$$

$f'(1)$; $h = 0.1$

$$D_h = f'(1) = \frac{f(1.1) - f(0.9)}{2 \times 0.1}$$

$$= 4.7228$$

$$D^{(1)}_{0.2} = \frac{4D(h/2) - D(h)}{3}$$

$$= \frac{4(4.7228) - 4.7364}{3}$$

$$= 4.7183$$

$$D^{(2)}_{0.2} = \frac{16D'(h/2) - D'(h)}{15}$$

$D'(h) = D^{(1)}_{0.2} = 4.7183$

Calc. $D'(h/2) = D^{(1)}_{0.1}$ using the same method and just plug in the value.