YZV202E - OPTIMIZATION FOR DATA SCIENCE

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Problem 1

According to Sylvester's criterion, a symmetric matrix M is positive definite if and only if all its leading principal minors are strictly positive, so having all the leading principle minors of matrix M strictly positive a sufficient condition for positive definiteness.If matrix M is positive definite, then below inequality holds for all non-zero vectors in $\mathbb{R}^n$.

$$x^T M x > 0$$

Having all the leading principle minors of matrix M non-negative is not a sufficient condition for positive semi-definiteness.
Counter Example:

$$M = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

The leading principle minors of M are 1 and 0 (Determinants of its sub-matrices by selecting the first k rows and columns). Let say x = $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$x^T M x = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -2 < 0$$

Problem 2

a) i)

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix}$$

ii)

$$\begin{pmatrix} -5 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

iii)

$$\begin{pmatrix} 1 & 5 & 7 & 3 \\ 0 & -7 & 0 & 6 \\ 4 & 0 & -1 & 9 \\ 2 & 8 & 0 & -3 \end{pmatrix}$$

Problem 4

$$A = U \cdot D \cdot U^{-1}$$
$$A^2 = U \cdot D \cdot U^{-1} \cdot U \cdot D \cdot U^{-1} = U \cdot D \cdot \mathbb{I} \cdot D \cdot U^{-1} = U \cdot D^2 \cdot U^{-1}$$
$$A^n = U \cdot D^n \cdot U^{-1}$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_{n=0}^{\infty} \frac{U \cdot D^n \cdot U^{-1}}{n!} = U \cdot (\sum_{n=0}^{\infty} \frac{D^n}{n!}) \cdot U^{-1}$$
$$\sum_{n=0}^{\infty} \frac{D^n}{n!} = e^D$$
$$e^A = U \cdot e^D \cdot U^{-1}$$

Problem 5

a) To find the stationary points, we should find the points where partial derivatives of f with respect to x_1 and x_2 are both zero.

$$\frac{\partial f}{\partial x_1} = \frac{x_1^3}{3} - x_1, \ \text{roots:} \sqrt{3}, -\sqrt{3}, 0$$
$$\frac{\partial f}{\partial x_2} = \frac{x_2^2}{2} + 2x_2 + \frac{3}{2}, \ \text{roots:} -1, -3$$

As I stated above roots of first equation are $\sqrt{3}$,- $\sqrt{3}$,0. Roots of second equation are -1 and -3. So stationary points are: (0,-1),(0,-3),($\sqrt{3}$,-1),($\sqrt{3}$,-3),(- $\sqrt{3}$,-1),(- $\sqrt{3}$,-3),

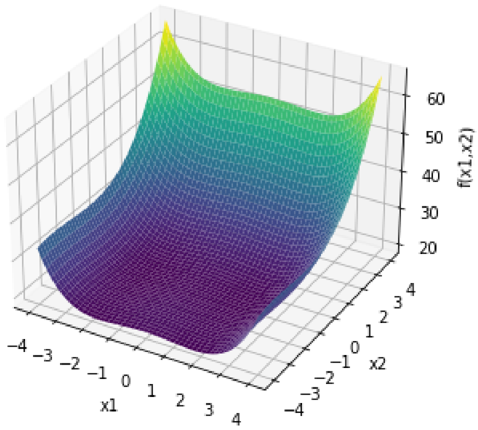
$$f(0,-1) = \frac{58}{3}$$
$$f(0,-3) = 20$$
$$f(\sqrt{3},-1) = \frac{223}{12}$$
$$f(\sqrt{3},-3) = \frac{231}{12}$$
$$f(-\sqrt{3},-1) = \frac{223}{12}$$
$$f(-\sqrt{3},-3) = \frac{231}{12}$$

b) To determine whether these stationary points are local minimum, local maximum or saddle point, we should compute hessian matrix at stationary points. Stationary point is a local minimum if the hessian matrix is positive definite, a local maximum if the hessian matrix is negative definite , and a saddle point if the hessian matrix is indefinite.

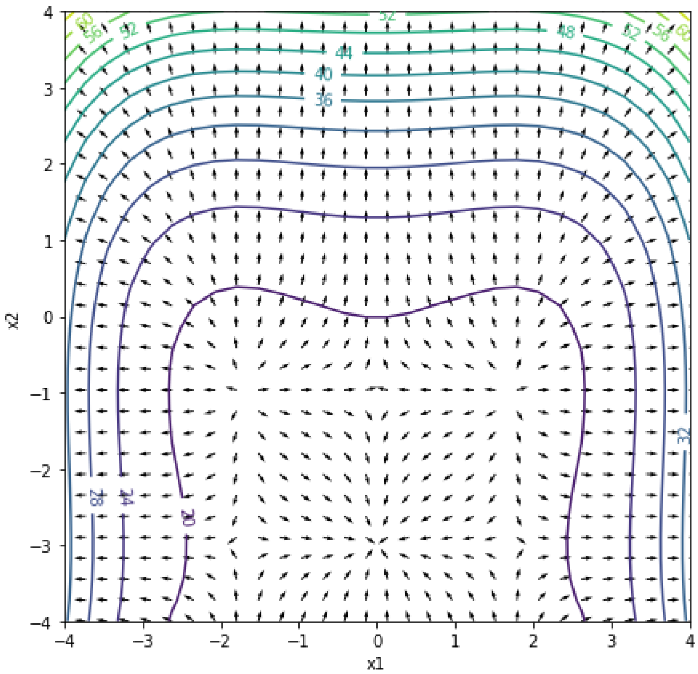
$$\begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} x_1^2 - 1 & 0 \\ 0 & x_2 + 2 \end{pmatrix}$$

For the stationary point (0,-1), eigenvalues: -1,1;indefinite . So it is a saddle point.
For the stationary point (0,-3), eigenvalues: -1,-1; negative definite. So it is a local maximum.
For the stationary point ($\sqrt{3}$,-1),eigenvalues: 1,2; positive definite. So it is a local minimum.
For the stationary point ($\sqrt{3}$,-3), eigenvalues: -1,2 ; indefinite. So it is a saddle point.
For the stationary point (- $\sqrt{3}$,-1), eigenvalues: 1,2; positive definite. So it is a local minimum.
For the stationary point (- $\sqrt{3}$,-3), eigenvalues: -1,2; indefinite. So it is a saddle point.

c) 3D surface containing all stationary points:



d) Contour plot and the gradient directions:



Problem 8

1) I live in Başakşehir, which is far from the city center. It takes a long time to travel anywhere from my home.

2) Parameters: Starting location (My home), destination location, distance to destination from my home
   Variables: Cost, Traffic conditions on my route, choice of transportation, departure time
   Objective: Optimizing my transportation to reduce travel time, minimize cost, and increase convenience.