Filter Design Assignment Report

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Filter No.: 120

m = 120 - 75 = 45, $\implies$ q(m) = 4 & r(m) = 5

1 Bandpass filter

$\mathrm{B_L(m)} = 16.8$ & $\mathrm{B_H(m)} = 26.8$

1. *Un-normalized discrete time filter specifications*
   - Sampling frequency = 100 kHz
   - Equiripple passband
   - Passband: 16.8 kHz to 26.8 kHz
   - Stopband: 0 kHz to 14.8 kHz & 28.8 kHz to 50 kHz
   - Transition band: 14.8 kHz to 16.8 kHz & 26.8 kHz to 28.8 kHz
   - Passband magnitude response: 0.85 to 1.15
   - Stopband magnitude response: 0 to 0.15
2. *Normalized digital filter specifications*
   - Equiripple passband
   - Passband: 0.336π to 0.536π
   - Stopband: 0 to 0.296π & 0.576π to π
   - Transition band: 0.296π to 0.336π & 0.536π to 0.576π
   - Passband magnitude response: 0.85 to 1.15
   - Stopband magnitude response: 0 to 0.15
3. *Corresponding analog filter specifications*
   Using the bilinear transformation to map the normalized frequencies from the discrete domain to the analog domain,
   $$\Omega = tan(\frac{\omega}{2}) \tag{1}$$
   - Equiripple passband
   - Passband: 0.5829 to 1.12
   - Stopband: 0 to 0.5016 & 1.2726 to ∞
   - Transition band: 0.5016 to 0.5829 & 1.12 to 1.2726
   - Passband magnitude response: 0.85 to 1.15
   - Stopband magnitude response: 0 to 0.15

4. *The frequency transformation to be employed*
   To convert these bandpass analog filter specifications to lowpass analog filter specifications, we need to employ the frequency transform,

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \tag{2}$$

   where Ω is the bandpass filter frequency and Ω_L is the frequency of the equivalent lowpass filter. Ω_0 and B are parameters with the following values,

$$\Omega_0 = \sqrt{\Omega_{p1}\Omega_{p2}} \tag{3}$$

$$B = \Omega_{p2} - \Omega_{p1} \tag{4}$$

   where Ω_{p1} and Ω_{p2} are the frequencies of the edge of the passband, in increasing order, so that B is positive. In our case $\Omega_{p1} = 0.5829$ and $\Omega_{p2} = 1.12$, which gives $\Omega_0 = 0.808$ and B $= 0.5371$.

5. *The frequency transformed lowpass analog filter specifications*
   Under the above frequency transformation, the edge frequencies of passband and stopband get mapped as follows:

Ω	Ω_L
0^+	$-\infty$
0.5016 $(= \Omega_{s1})$	-1.4894 $(= -\Omega_{Ls1})$
0.5829 $(= \Omega_{p1})$	-1 $(= -\Omega_{Lp})$
1.12 $(= \Omega_{p2})$	1 $(= \Omega_{Lp})$
1.2726 $(= \Omega_{s2})$	1.4142 $(= \Omega_{Ls2})$
$+\infty$	$+\infty$

   Therefore, the specifications of the lowpass filter are as follows:

   - Equiripple passband
   - Passband: 0 to 1
   - Stopband: 1.4142 $(= min(\Omega_{Ls1}, \Omega_{Ls2}))$
   - Passband magnitude response: 0.85 to 1.15
   - Stopband magnitude response: 0 to 0.15

6. *The analog lowpass filter transfer function $H_{analog,LPF}(s_L)$*
   Since the magnitude response is equiripple in the passband and monotonic in the stopband, we need to design a Chebyshev filter for this purpose. The absolute value of the transfer function for a lowpass Chebyshev filter is given by,

$$|H(j\Omega_L)| = \frac{1}{\sqrt{1 + \epsilon^2 C_N^2(\frac{\Omega}{\Omega_p})}} \tag{5}$$

   where $C_N(x)$ is the N^{th} order Chebyshev polynomial in x. From the above expression, we observe that $|H(j\Omega_L)| \leq 1$, so we only need to ensure that $|H(j\Omega_L)| \geq 0.85$ in the passband and $|H(j\Omega_L)| \leq 0.15$ in the stopband, which gives $\delta_1 = \delta_2 = 0.15$, when compared with the usual lowpass filter specifications. To meet the tolerance criterion, ϵ and N should satisfy the following inequalities,

$$\epsilon = \sqrt{D_1} \tag{6}$$

$$N \geq \frac{cosh^{-1}(\sqrt{\frac{D_2}{D_1}})}{cosh^{-1}(\frac{\Omega_s}{\Omega_p})} \tag{7}$$

where D_1 and D_2 depend upon δ_1 and δ_2 respectively as,

$$D_1 = \frac{1}{(1-\delta_1)^2} - 1 \tag{8}$$

$$D_2 = \frac{1}{\delta^2} - 1 \tag{9}$$

Using the values of δ_1, δ_2, Ω_s and Ω_p, we get $D_1 = 0.3841$ and $D_2 = 43.\bar{4}$. Therefore,

$$\epsilon = 0.6197 \tag{10}$$

$$N \geq \frac{cosh^{-1}(\sqrt{\frac{43.4444}{0.3841}})}{cosh^{-1}(\frac{1.4142}{1})}$$

$$\implies N \geq 3.4663$$

$$\implies N = 4 \tag{11}$$

Using these values of N and ϵ, we can find the poles of the analog domain, lowpass filter transfer function $H_{analog,LPF}(s_L)$. The location of the poles is given by,

$$s_i = -\Omega_p\Omega_i + j\Omega_p\Sigma_i, i = 0, 1, \cdots, 3(= N-1) \tag{12}$$

and Ω_i and Σ_i are obtained as,

$$\Sigma_i + j\Omega_i = \cos(A_i)\cosh(B) - j\sin(A_i)\sinh(B) \tag{13}$$

where A_i and B are given as,

$$A_i = (2i+1)\frac{\pi}{2N} \tag{14}$$

$$B = -\frac{1}{N}\sinh^{-1}(\frac{1}{\epsilon}) \tag{15}$$

Therefore, we have $B = -0.3141$, and

$$\Sigma_0 + j\Omega_0 = 0.9698 + j0.1222 \tag{16}$$

$$\Sigma_1 + j\Omega_1 = 0.4017 + j0.2949 \tag{17}$$

$$\Sigma_2 + j\Omega_2 = -0.4017 + j0.2949 \tag{18}$$

$$\Sigma_3 + j\Omega_3 = -0.9698 + j0.1222 \tag{19}$$

which gives,

$$s_0 = -0.1222 + j0.9698 \tag{20}$$

$$s_1 = -0.2949 + j0.4017 \tag{21}$$

$$s_2 = -0.2949 - j0.4017 \tag{22}$$

$$s_3 = -0.1222 - j0.9698 \tag{23}$$

Since this is an even order Chebyshev filter, its magnitude response has a minima at $\Omega = 0$. The point of maxima can be determined by minimizing the denominator of $|H(j\Omega_L)|$.

$$\implies \frac{d}{d\Omega_L}\cos^2(N\cos^{-1}(\frac{\Omega_L}{\Omega_p})) = 0$$

$$\implies \sin(2N\cos^{-1}(\frac{\Omega_L}{\Omega_p})) = 0$$

$$\cos^{-1}(\frac{\Omega_L}{\Omega_p}) = \frac{n\pi}{2N} \quad \text{for} \quad n = 0, 1, \cdots, N \tag{24}$$

since $0 \le \Omega_L \le \Omega_p$ in the passband. But as stated earlier, the magnitude response attains a minima at $\Omega_L = 0$, which corresponds to $n = N$. Generalizing this to all even n (which can be verified by calculating the double derivative), the magnitude response achieves a minima for even n and a maxima for odd n. Taking $n = 1$ to calculate the minima of the denominator (maxima of $|H(j\Omega_L)|$),

$$\implies \cos^{-1}(\frac{\Omega_L}{\Omega_p}) = \frac{\pi}{8}$$

$$\implies \Omega_L = \cos(\frac{\pi}{8}) = 0.9239$$

$$\implies \prod_{i=0}^{i=3} |j0.9239 - s_{L,i}| = 0.2017$$

Thus, to normalize the passband gain to 1, the expression for the analog lowpass filter transfer function should be,

$$H_{analog,LPF}(s_L) = \frac{0.2017}{\prod_{i=0}^{i=3}(s_L - s_{L,i})} = \frac{0.2017}{s^4 + 0.8342s^3 + 1.3479s^2 + 0.6242s + 0.2373} \tag{25}$$

7. *The analog transfer function for the bandpass filter*
   To convert the analog lowpass filter transfer function to analog bandpass filter transfer function, the following transformation needs to be applied:

$$s_L \to \frac{s^2 + \Omega_0^2}{Bs} \tag{26}$$

where Ω_0 and B are given by equations (3) and (4) respectively. Under this transformation,

$$\frac{1}{s_L - s_L, i} \to \frac{Bs}{s^2 - s_{L,i}Bs + \Omega_0^2} \tag{27}$$

$$\begin{aligned} \implies H_{analog,BPF}(s) &= 0.2017 \prod_{i=0}^{i=3} \frac{Bs}{(s^2 - s_{L,i}Bs + \Omega_0^2)} \\ &= \frac{0.0168s^4}{s^8 + 0.45s^7 + 3s^6 + 0.97s^5 + 3.09s^4 + 0.64s^3 + 1.28s^2 + 0.13s + 0.18} \end{aligned} \tag{28}$$

8. *The discrete time filter transfer function*
   Applying the bilinear transformation converts the analog domain transfer function to discrete domain transfer function, which is:

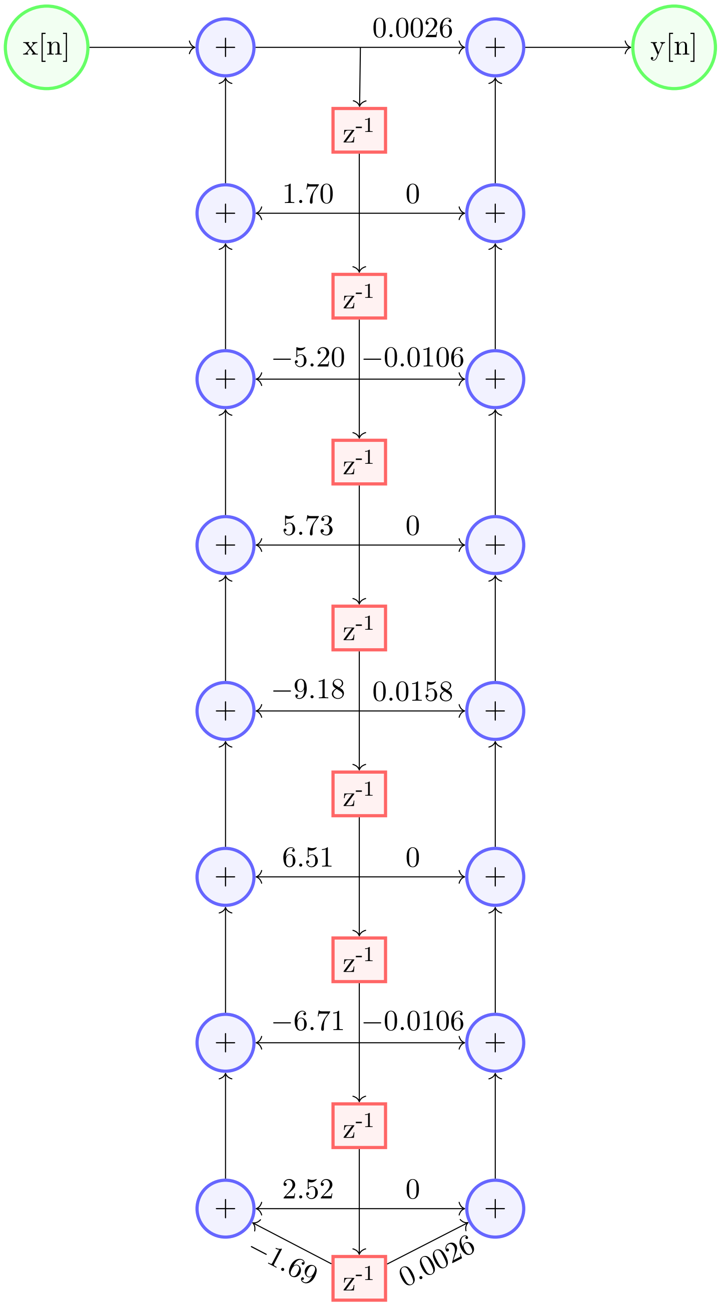
$$s \to \frac{1 - z^{-1}}{1 + z^{-1}} = \frac{z - 1}{z + 1} \tag{29}$$

Under this transformation:

$$\frac{Bs}{s^2 - s_{L,i}Bs + \Omega_0^2} \to \frac{B(z^2 - 1)}{(\Omega_0^2 + 1 - s_{L,i}B)z^2 + 2(\Omega_0^2 - 1)z + (\Omega_0^2 + 1 + s_{L,i}B)} \tag{30}$$

$$\implies H(z) = 0.2017 \prod_{i=0}^{i=3} \frac{B(z^2-1)}{(\Omega_0^2+1-s_{L,i}B)z^2 + 2(\Omega_0^2-1)z + (\Omega_0^2+1+s_{L,i}B)}$$
$$= \frac{0.0016(z^8-4z^6+6z^4-4z^2+1)}{1.69z^8-2.52z^7+6.71z^6-6.51z^5+9.18z^4-5.73z^3+5.20z^2-1.70z+1} \tag{31}$$

9. *Direct Form II realization*



10. *FIR filter* The FIR filter transfer function needs to be obtained by applying the Kaiser window on the ideal bandpass impulse response and then taking its z-transform. The Kaiser window parameters can be calculated using the values of transition banswidth, passband and stopband tolerances as follows:

$$(2N+1) \geq 1 + \frac{A-8}{2.285\Delta\omega_t} \tag{32}$$

where,

$$\Delta\omega_t = \omega_s - \omega_p = 0.576\pi - 0.536\pi = 0.336\pi - 0.296\pi = 0.04\pi \tag{33}$$

$$A = -10\log_{10}\delta = -10\log_{10} 0.15 = 16.4782 \tag{34}$$

$$\implies (2N+1) \geq 1 + \frac{16.4782 - 8}{2.285x0.04\pi} = 30.5261$$

$$\implies N = 15 \tag{35}$$

A < 21, $\implies$ $\alpha = \beta = 0$. So, the Kaiser window reduces to a rectangular window of length 31. The impulse response of a bandpass filter with cutoff frequencies ω_{c1} and ω_{c2} ($\omega_{c2} > \omega_{c1}$) is given by:

$$h[n] = \frac{\sin(n\omega_{c2}) - \sin(n\omega_{c1})}{\pi n} \tag{36}$$

h[n] needs to be restricted from n = -N to n = N to achieve the desired specifications. The cutoff frequencies need to be defined as the average of the corresponding passband edge and stopband edge frequencies i.e. in the midst of the transition band. Therefore, $\omega_{c1} = 0.316\pi$ and $\omega_{c2} = 0.556\pi$, giving

$$\begin{aligned} h_{FIR}[n] &= \frac{\sin(0.556\pi n) - \sin(0.316\pi n)}{\pi n}, \quad && \forall|n| \leq N \\ &= 0, && \text{otherwise} \end{aligned} \tag{37}$$

The above value of N, in fact, did not satisfy the specifications and using trial and error, N = 20 was the least value of N satisfying the specifications. Hence, the discrete domain transfer function is,

$$H_{FIR}(z) = \sum_{n=-20}^{n=20} h_{FIR}[n]z^{-n} \tag{38}$$

2 Bandstop filter

$B_L(m) = 17.6$ & $B_H(m) = 27.6$

1. *Un-normalized discrete time filter specifications*
   - Sampling frequency = 100 kHz
   - Monotonic passband
   - Stopband: 17.6 kHz to 27.6 kHz
   - Passband: 0 kHz to 15.6 kHz & 29.6 kHz to 50 kHz
   - Transition band: 15.6 kHz to 17.6 kHz & 27.6 kHz to 29.6 kHz
   - Passband magnitude response: 0.85 to 1.15
   - Stopband magnitude response: 0 to 0.15
2. *Normalized digital filter specifications*
   - Monotonic passband
   - Stopband: 0.352π to 0.552π
   - Passband: 0 to 0.312π & 0.592π to π
   - Transition band: 0.312π to 0.352π & 0.552π to 0.592π
   - Passband magnitude response: 0.85 to 1.15
   - Stopband magnitude response: 0 to 0.15
3. *Corresponding analog filter specifications*
   - Monotonic passband
   - Stopband: 0.6171 to 1.1783
   - Passband: 0 to 0.5335 & 1.3406 to ∞
   - Transition band: 0.5335 to 0.6171 & 1.1783 to 1.3406
   - Passband magnitude response: 0.85 to 1.15
   - Stopband magnitude response: 0 to 0.15
4. *The frequency transformation to be employed*
   To convert these bandstop analog filter specifications to lowpass analog filter specifications, we need to employ the frequency transform,

   $$\Omega_L = \frac{B\Omega}{\Omega^2 - \Omega_0^2} \tag{39}$$

   where Ω is the bandstop filter frequency and Ω_L is the frequency of the equivalent lowpass filter. Ω_0 and B are parameters with the following values,

   $$\Omega_0 = \sqrt{\Omega_{p1}\Omega_{p2}} \tag{40}$$

   $$B = \Omega_{p2} - \Omega_{p1} \tag{41}$$

   where Ω_{p1} and Ω_{p2} are the frequencies of the edge of the passband, in increasing order, so that B is positive. In our case $\Omega_{p1} = 0.5335$ and $\Omega_{p2} = 1.3406$, which gives $\Omega_0 = 0.8457$ and B = 0.8071.

5. *The frequency transformed lowpass analog filter specifications*
Under the above frequency transformation, the edge frequencies of passband and stopband get mapped as follows:

Ω	Ω_L
0^+	0^-
$0.5335 \quad (= \Omega_{p1})$	$-1 \quad (= -\Omega_{Lp})$
$0.6171 \quad (= \Omega_{s1})$	$-1.4894 \quad (= -\Omega_{Ls1})$
$1.1783 \quad (= \Omega_{s2})$	$1.4127 \quad (= \Omega_{Ls2})$
$1.3406 \quad (= \Omega_{p2})$	$1 \quad (= \Omega_{Lp})$
$+\infty$	0^+

Therefore, the specifications of the lowpass filter are as follows:

- Monotonic passband
- Passband: 0 to 1
- Stopband: 1.4127 ($= min(\Omega_{Ls1}, \Omega_{Ls2})$)
- Passband magnitude response: 0.85 to 1.15
- Stopband magnitude response: 0 to 0.15

6. *The analog lowpass filter transfer function $H_{analog,LPF}(s_L)$*
Since the magnitude response is monotonic in the passband and monotonic in the stopband, we need to design a Butterworth filter for this purpose. The absolute value of the transfer function for a lowpass Butterworth filter is given by,

$$|H(j\Omega_L)| = \frac{1}{\sqrt{1 + (\frac{\Omega_L}{\Omega_c})^{2N}}} \tag{42}$$

From the above expression, we observe that $|H(\Omega_L)| \leq 1$, so we only need to ensure that $|H(\Omega_L)| \geq 0.85$ in the passband and $|H(\Omega_L)| \leq 0.15$ in the stopband, which gives $\delta_1 = \delta_2 = 0.15$, when compared with the usual lowpass filter specifications. To meet the tolerance criterion, Ω_c and N should satisfy the following inequalities,

$$N \geq \frac{ln(\sqrt{\frac{D_2}{D_1}})}{ln(\frac{\Omega_s}{\Omega_p})} \tag{43}$$

$$\frac{\Omega_p}{D_1^{1/2N}} \leq \Omega_c \leq \frac{\Omega_s}{D_2^{1/2N}} \tag{44}$$

where D_1 and D_2 depend upon δ_1 and δ_2 respectively as,

$$D_1 = \frac{1}{(1-\delta_1)^2} - 1 \tag{45}$$

$$D_2 = \frac{1}{\delta^2} - 1 \tag{46}$$

Using the values of δ_1, δ_2, Ω_s and Ω_p, we get $D_1 = 0.3841$ and $D_2 = 43.\bar{4}$. Therefore,

$$N \geq \frac{ln(\sqrt{\frac{43.4444}{0.3841}})}{ln(\frac{1.4127}{1})}$$

$$\implies N \geq 6.8427$$

$$\implies N = 7 \tag{47}$$

$$\implies \frac{1}{0.3841^{1/14}} \leq \Omega_c \leq \frac{1.4127}{43.4444^{1/14}}$$

$$\implies 1.0707 \leq \Omega_c \leq 1.0791$$

$$\text{Taking } \Omega_c = \frac{(1.0707 + 1.0791)}{2} = 1.0749 \tag{48}$$

Using these values of N and Ω_c, we can find the poles of the analog domain, lowpass filter transfer function $H_{analog,LPF}(s_L)$. The location of the poles is given by,

$$s_i = j\Omega_c e^{(2i+1)\frac{\pi}{2N}}, i = 0, 1, \cdots, 6(= N - 1) \tag{49}$$

which gives,

$$s_i = \Omega_c(-\sin(\frac{(2i+1)\pi}{2N}) + j\cos(\frac{(2i+1)\pi}{2N})), i = 0, 1, \cdots, 6(= N - 1) \tag{50}$$

To normalize the dc gain to 1, the expression for the analog lowpass filter transfer function should be,

$$\begin{aligned} H_{analog,LPF}(s_L) &= \Omega_c^N \prod_{i=0}^{i=6} \frac{1}{(s_L - s_{L,i})} \\ &= \frac{1.658}{s^7 + 4.83s^6 + 11.667s^5 + 18.12s^4 + 19.48s^3 + 14.49s^2 + 6.93s + 1.66} \end{aligned} \tag{51}$$

7. *The analog transfer function for the bandpass filter* To convert the analog lowpass filter transfer function to analog bandpass filter transfer function, the following transformation needs to be applied:

$$s_L \to \frac{Bs}{s^2 + \Omega_0^2} \tag{52}$$

where Ω_0 and B are given by equations (3) and (4) respectively. Under this transformation,

$$\frac{1}{s_L - s_L, i} \to \frac{s^2 + \Omega_0^2}{-s_{L,i}s^2 + Bs - s_{L,i}\Omega_0^2} \tag{53}$$

$$\implies H_{analog,BPF}(s) = 1.658 \prod_{i=0}^{i=6} \frac{s^2 + \Omega_0^2}{-s_{L,i}s^2 + Bs - s_{L,i}\Omega_0^2} \tag{54}$$

$$= \frac{1s^{14} + 5s^12 + 10.7s^10 + 12.8s^8 + 9.2s^6 + 3.9s^4 + 0.9s^2 + 0.0957}{s^{14} + 3.4s^{13} + 10.7s^{12} + 20.7s^{11} + 35.7s^{10} + 46s^9 + 52.7s^8 + 47.2s^7 + 37.7s^6 + 23.5s^5 + 13.1s^4 + 5.4s^3 + 2s^2 + 0.4516s + 0.1}$$

8. *The discrete time filter transfer function* Applying the bilinear transformation converts the analog domain transfer function to discrete domain transfer function, which is:

$$s \to \frac{1 - z^{-1}}{1 + z^{-1}} = \frac{z - 1}{z + 1} \tag{55}$$

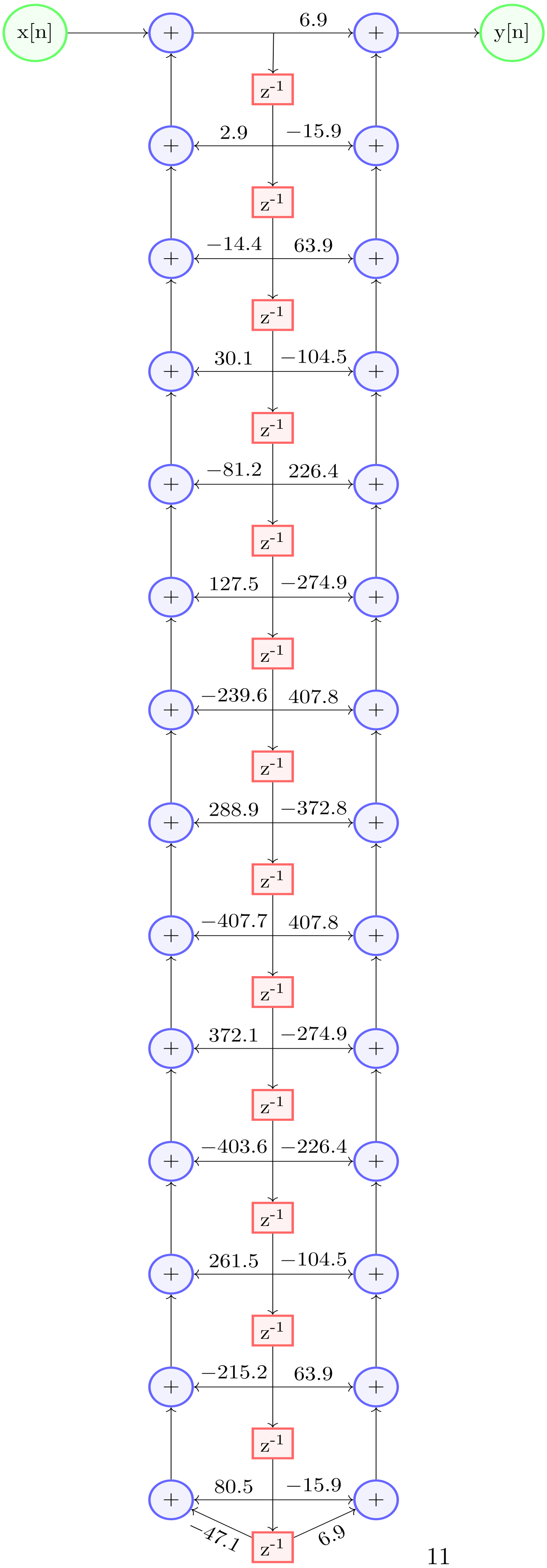
Under this transformation:

$$\frac{s^2 + \Omega_0^2}{-s_{L,i}s^2 + Bs - s_{L,i}\Omega_0^2} \to \frac{(\Omega_0^2 + 1)z^2 + 2(\Omega_0^2 - 1)z + (\Omega_0^2 + 1)}{(B - s_{L,i}(\Omega_0^2 + 1))z^2 - 2s_{L,i}(\Omega_0^2 - 1)z - (B + s_{L,i}(\Omega_0^2 + 1))} \tag{56}$$

$$\implies H(z) = 1.658 \prod_{i=0}^{i=6} \frac{(\Omega_0^2 + 1)z^2 + 2(\Omega^2 - 1)z + (\Omega_0^2 + 1)}{(B - s_{L,i}(\Omega_0^2 + 1))z^2 - 2s_{L,i}(\Omega_0^2 - 1)z - (B + s_{L,i}(\Omega_0^2 + 1))} \tag{57}$$

$$= \frac{6.9z^{14} - 15.9z^{13} + 63.9z^{12} - 104.5z^{11} + 226.4z^{10} - 274.9z^9 + 407.8z^8 - 372.8z^7 + 407.8z^6 - 274.9z^5 + 226.4z^4 - 104.5z^3 + 63.9z^2 - 15.9z + 6.9}{47.1z^{14} - 80.5z^{13} + 215.2z^{12} - 261.5z^{11} + 403.6z^{10} - 372.1z^9 + 407.7z^8 - 288.9z^7 + 239.6z^6 - 127.5z^5 + 81.2z^4 - 30.1z^3 + 14.4z^2 - 2.9z + 1}$$

9. *Direct Form II realization*


x[n]
y[n]
6.9
z-1
2.9
−15.9
−14.4
63.9
30.1
−104.5
−81.2
226.4
127.5
−274.9
−239.6
407.8
288.9
−372.8
−407.7
407.8
372.1
−274.9
−403.6
−226.4
261.5
−104.5
−215.2
63.9
80.5
−15.9
−47.1
6.9

10. *FIR filter* The FIR filter transfer function needs to be obtained by applying the Kaiser window on the ideal bandstop impulse response and then taking its z-transform. The Kaiser window parameters can be calculated using the values of transition banswidth, passband and stopband tolerances as follows:

$$(2N+1) \geq 1 + \frac{A-8}{2.285\Delta\omega_t} \tag{58}$$

where,

$$\Delta\omega_t = \omega_p - \omega_s = 0.592\pi - 0.552\pi = 0.352\pi - 0.312\pi = 0.04\pi \tag{59}$$

$$A = -10\log_{10}\delta = -10\log_{10}0.15 = 16.4782 \tag{60}$$

$$\implies (2N+1) \geq 1 + \frac{16.4782-8}{2.285x0.04\pi} = 30.5261$$

$$\implies N = 15 \tag{61}$$

A < 21, $\implies$ $\alpha = \beta = 0$. So, the Kaiser window reduces to a rectangular window of length 31. The impulse response of a bandstop filter with cutoff frequencies ω_{c1} and ω_{c2} ($\omega_{c2} > \omega_{c1}$) is given by:

$$h[n] = \delta[n] - \frac{\sin(n\omega_{c2}) - \sin(n\omega_{c1})}{\pi n} \tag{62}$$

h[n] needs to be restricted from n = -N to n = N to achieve the desired specifications. The cutoff frequencies need to be defined as the average of the corresponding passband edge and stopband edge frequencies i.e. in the midst of the transition band. Therefore, $\omega_{c1} = 0.332\pi$ and $\omega_{c2} = 0.572\pi$, giving

$$\begin{aligned} h_{FIR}[n] &= \delta[n] - \frac{\sin(0.572\pi n) - \sin(0.332\pi n)}{\pi n}, \quad && \forall |n| \leq 15 \\ &= 0, && \text{otherwise} \end{aligned} \tag{63}$$

The above value of N, in fact, did not satisfy the specifications and using trial and error, N = 20 was the least value of N satisfying the specifications.Hence, the discrete domain transfer function is,

$$H_{FIR}(z) = \sum_{n=-20}^{n=20} h_{FIR}[n]z^{-n} \tag{64}$$