Math 215 — In-class Problems February 3 Spring 2025

Proofs are an important part of mathematics because they officially establish the truth or falsity of a mathematical statement. Consider the following mathematical statement "If P, then Q". Here, P represents the *hypothesis*, and Q represents the *conclusion*.

To establish the truth of a mathematical statement, we need to produce an argument that shows that the hypothesis *logically* guarantees the conclusion. Here are four standard techniques used to prove mathematical statements:

1. **Direct proof** — Logically show that P implies Q (Monday's class)
2. **Proof by contraposition** — Logically show that not Q implies not P (Wednesday's class)
3. **Proof by contradiction** — Logically show that P and not Q imply S and not S. Here, S is some statement that is logically obtained from the assumptions (Wednesday's class)
4. **Proof by mathematical induction** (Friday's class)

We will also discuss proofs involving *sets* in this worksheet, and discuss how to show a mathematical statement is *false*. First, let's discuss the structure of a *direct proof*.

(0) Get into the habit of writing the word "Proof:" at the beginning of your proofs.

(i) The proof should ALWAYS start out by writing down the hypothesis and any other given conditions of the mathematical statement.

- E.g. "Suppose P...", "Let...", etc.

(ii) After writing down the hypothesis in (i), you need to think about what it *directly* implies. A good rule of thumb is to think about which definitions and/or theorems are applicable.

- E.g. "Then, by definition, we have...", "Then, theorem (cite its name or number) implies...", etc.

(iii) Use the implication(s) you obtained in (ii) to logically deduce additional implications. Keep doing this until you reach the conclusion Q.

(iv) Once you have logically deduced the conclusion, make sure to specify that the proof is finished. For example,

- "Therefore, Q... $\square$",
- "This completes the proof. $\square$",
- "By direct proof, we have shown that P implies Q. $\square$"

Remark: You will notice that at the end of a proof, there is a box-like symbol. This is a standard proof-writing convention that mathematicians have adopted to indicate to the reader that a proof is complete. This box is known as a "QED", which stands for *quot erat demonstrandum*, meaning "that which was to be demonstrated".

Warnings: When doing a (direct) proof, **do NOT assume the conclusion**. This is a common mistake for first-time proof writers, so avoid being tempted to make it. Remember, you are trying to provide a *general* argument that the hypothesis logically implies the conclusion. Also, establishing the validity of a statement with an example is NOT a proof.

Let's now see a few basic examples of how to do a direct proof. The following definition and theorem will be helpful in our examples.

Definition: Let n be an integer.

- n is **even** if $n = 2k$ for some integer k.
- n is **odd** if $n = 2k + 1$ for some integer k.

The Mean Value Theorem: Let f be a real-valued function that is continuous on $[a, b]$ and differentiable on (a, b). Then there is a real number c such that $a < c < b$ and $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Video Example 1: Let m be an even integer and n be an odd integer. The product mn is an even integer.

Video Example 2: Let f be a real-valued function that is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there is a real number c such that $a < c < b$ and $f'(c) = 0$.

Class Example 1: Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Prove that $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

Hint: You may assume that u_i and v_i are *real* numbers for any $1 \leq i \leq 3$.

Class Example 2: Let $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix}$ be a 2×2 matrix, $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, and c be a real-scalar. Prove that $A(c\vec{u}) = c(A\vec{u})$.

In several branches of mathematics, *sets* are the building blocks of many key concepts and results. Therefore, it is important to understand what sets are, specifically their fundamentals, and how we prove results or properties related to them.

Definition: A **set** is a collection of objects. The objects in a set are called the **elements** (or **members**) of the set.

Here are some examples of sets (some of which you may have seen previously):

1. $\mathbb{Z}$ — this is the **set of integers**
2. $\mathbb{R}$ — this is the **set of real numbers**
3. $\mathbb{R}^3$ — this is the **set of** 3×1 **column vectors with real number entries**
4. $S = \{a, b, c\}$
5. $\mathbb{N} = \{1, 2, 3, 4, ...\}$ — this is the **set of natural numbers**
6. $A = \{p \mid p \text{ is a positive integer}, p \text{ is prime, and } p \leq 10,000\}$ — note that the vertical bar — means "such that". This is an example of **set-builder notation**
7. $\emptyset = \{\quad\}$ — this is called the **empty set** because it has no elements

Definition: We say that two set A and B are **equal** if A and B contain exactly the same elements.

Warning: A common misinterpretation of the previous definition is the belief that two sets are equal if they contain the "same number" of elements. This is actually NOT true. Consider $A = \{1, 2, 3\}$ and $B = \{1, 2, 2, 3, 3, 3\}$. Here, $A = B$ because *all* the elements of A can be found in B, and vice-vesa.

Definition: The symbol $\in$ means "**is an element of**", and the symbol $\notin$ means "**is not an element of**".

Video Example: Consider the set $S = \{1, 2, 3\}$. Then $1 \in S$ and $4 \notin S$.

Definition: Let A and B be two sets.

1. We say that A is a **subset** of B if all the elements of A are also elements of B. Mathematically-speaking, we write $A \subseteq B$.
2. We say that A is **not a subset** of B if there is an element of A that is not an element of B. Mathematically-speaking, we write $A \nsubseteq B$.

Video Example: Consider $A = \{a, b, c, d\}$ and $B = \{a, b, c, d, e, f, g\}$. Then $A \subseteq B$, and $B \nsubseteq A$.

Definition: Let A and B be sets. We say $A = B$ if $A \subseteq B$ and $B \subseteq A$.

Video Example: Let $A = \{x \in \mathbb{R} \mid x^2 = 4\}$ and $B = \{x \in \mathbb{R} \mid x = 2 \text{ or } x = -2\}$. Prove that $A = B$.

We can also work perform operations on two or more sets.

Definition: Let A and B be sets.

1. The **union** of A and B, written as $A \cup B$, is the set of all elements that are in A or in B or in both.
2. The **intersection** of A and B, written as $A \cap B$, is the set of all elements that are in both A and B.
3. We say that A and B are **disjoint** if A and B have no common elements. Here, $A \cap B = \emptyset$.
4. The **set difference** of A and B, written as $A \setminus B$ or $A - B$, is the set of elements of A which are not in B.
5. The **complement** of A, written as A^c or $\bar{A}$ or A', is the set that contains all the elements that are not in A.

Class Example 3: Let A and B be two non-empty sets. Prove that $(A \cup B)^c = A^c \cap B^c$.

Up to this point, we have looked at establishing the truth of a mathematical statement. We can also agree that proving that a mathematical statement is true requires work. So, what do we do when we want to show that a mathematical statement like "If P, then Q" is false?

To establish the falsity of a mathematical statement, we need to produce an argument/example that satisfies the hypothesis P, but fails the conclusion Q. That is, you need to logically show that P implies "not Q". Note that "not Q" is the **negation** of Q. This technique is known as **proof by counter-example**. Let's discuss the strategy for coming up with a counterexample.

(0) If a mathematical statement is false, write "Counterexample:" at the beginning.

(i) Make sure you fully understand all the conditions listed in the hypothesis P, and the meaning of the conclusion Q.

- It may help to write down the negation of Q as a reference.

(ii) Find some sort of an example that satisfies the conditions listed in the hypothesis. This may take some trial-and-error.

(iii) If the example you found in (ii) works, check that it does not satisfy the conclusion. In other words, see if it does the exact opposite of what the conclusion states.

- If the conclusion is satisfied, then go back to (ii).

(iv) If the example you found does not satisfy the conclusion, you are done. Here is some wording you should use:

- "This example satisfies the hypothesis, but fails the conclusion. Hence, the mathematical statement is false."
- "By this counterexample, the mathematical statement is false."

Example: Show that the following mathematical statements are false:

- Let $f(x) = x^2$. If $a < b$, then $f(a) < f(b)$.
- If f is a continuous function on its domain, then f is also differentiable on the domain.
- Let A be a 2×2 matrix, with at least one nonzero entry. If $\vec{x}$ is a *nontrivial* solution to $A\vec{x} = \vec{0}$, then every entry in $\vec{x}$ is nonzero.