Discrete Mathematics
Chapter 4 Homework

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Exercise Set 4.1

5. There are distinct integers m and n such that $\frac{1}{m} + \frac{1}{n}$ is an integer.

> **Ans:** Suppose $m = -2$ and $n = 2$, then:
> $$\frac{1}{m} + \frac{1}{n} = -\frac{1}{2} + \frac{1}{2} = 0$$
> The statement is true since $0 \in \mathbb{Z}$

13. For all integers m and n, if $2m + n$ is odd then m and n are both odd.

> **Ans:** The counterexample would have to satisfy the statement $\exists m, n \in Z$ such that $2m + n$ is odd and n is even or m is even
> Suppose $m = 4$ and $n = 1$, then
> $$2m + n = 2 \cdot 4 + 1 = 9$$
> 9 is odd by definition because $\exists l \in \mathbb{Z}$ such that $9 = 2l + 1$ and $m = 4$ is even by definition because $\exists k \in \mathbb{Z}$ such that $4 = 2k$ so $m = 4$ and $n = 1$ is a counterexample that disproves the statement.

41. **Theorem:** The product of an even integer and an odd integer is even. "**Proof:** Suppose m is an even integer and n is an odd integer. If $m \cdot n$ is even, then by definition of even there exists an integer r such that $m \cdot n = 2r$. Also since m is even, there exists an integer p such

that $m = 2p$, and since n is odd there exists an integer q such that $n = 2q + 1$. Thus

$$mn = (2p)(2q + 1) = 2r,$$

where r is an integer. By definition of even, then, $m \cdot n$ is an even, as was to be shown."

Ans: The proof confuses what is known and what is still to be shown. It is known that the definition of even for an integer n is $\exists k \in \mathbb{Z}$ such that $n = 2k$ but the proof assumes that is the case for $m \cdot n$ without showing or proving it.

57. If m and n are positive integers and mn is a perfect square, then m and n are perfect squares.

Ans: $\forall m, n \in \mathbb{Z}$, if $m \cdot n$ is a perfect square, then m and n are perfect squares.
The statement is false, we can prove this by finding a counterexample that satisfies: $\exists m, n \in \mathbb{Z}$ such that $m \cdot n$ is a perfect square and m is not a perfect square or n is not a perfect square

Proof. Suppose $m = 2$ and $n = 18$ then by definition of perfect square $\exists k \in \mathbb{Z}$ such that $mn = k^2$

$$mn = 2 \cdot 18$$

$$36 = 6^2$$

However that is not the case for m or n as we can see that $\nexists l \in \mathbb{Z}$ such that $m = l^2$ or $\nexists q \in \mathbb{Z}$ such that $n = q^2$

$$l = \sqrt{2}$$

$$q = \sqrt{18}$$

$\square$

58. The difference of the squares of any two consecutive integers is odd.

> **Ans:** $\forall m, n \in \mathbb{Z}$ if m and n are consecutive then $m^2 - n^2$ is odd
>
> *Proof.* Suppose m and n are consecutive integers
> Because m and n are consecutive, then by definition
>
> $$m - n = 1$$
> $$m = n + 1$$
>
> The difference between the squares of m and n can then be found by □

Exercise Set 4.2

16. The quotient of any two rational numbers is a rational number.

20. Given any two rational numbers r and s with $r < s$, there is another rational number between r and s.

38. Find the mistake in the following **"proof"** that the sum of any two rational numbers is a rational number
"**Proof:** Suppose r and s are rational numbers. Then $r = a/b$ and $s = c/d$ for some integers a, b, c and d with $b \neq 0$ and $d \neq 0$ (by definition of rational). Then

$$r + s = \frac{a}{b} + \frac{c}{d},$$

But this is a sum of two fractions, which is a fraction. So $r + s$ is a rational number since a rational number is a fraction"

Exercise Set 4.3

For 20 and 23 determine whether the statement is true or false. Prove the statement directly from the definitions if it is true, and give a counterexample if it is false.

20. The sum of any three consecutive integers is divisible by 3. (Two integers are **consecutive** if, and only if, one is one more than the other.)

23. A sufficient condition for an integer to be divisible by 8 is that it be divisible by 16.

48. Prove that for any nonnegative integer n, if the sum of the digits of n is divisible by 3, then n is divisible by 3. Exercise Set 4.4

    8. (a) 50 *div* 7
       (b) 50 *mod* 7

    17. Prove that the product of any two consecutive integers is even.

    21. Suppose b is an integer. If $b \ mod \ 12 = 5$, what is $8b \ mod \ 12$? In other words, if division of b by 12 gives a remainder of 5, what is the remainder when 8b is divided by 12?

    Ans: REPLACE WITH ANSWER

    26. Prove that a necessary and sufficient condition for a nonnegative integer n to be divisible by a positive integer d is that $n \ mod \ d = 0$

    43. Prove that If n is an odd integer, then $n^4 \ mod \ 16 = 1$

Exercise Set 4.5

7. Formulate the negation and prove by contradiction
   There is no least positive rational number

22. Consider the statement "For all real numbers r, if r^2 is irrational then r is irrational."
    (a) Write what you would suppose and what you would need to show to prove this statement by contradiction.
    (b) Write what you would suppose and what you would need to show to prove this statement by contraposition.

    Prove 24 and 29 in two ways: **a)** by contraposition and **b)** by contradiciton.

24. The reciprocal of any irrational number is irrational. (The **reciprocal** of a nonzero real number x is $1/x$
29. For all integers a, b, and c if $a \mid b$ and $a \nmid c$, then $a \nmid (b+c)$

Exercise Set 4.6

11. Determine whether the statement "The sum of any two positive irrational numbers is irrational." is true or false. Prove it if it is true and disprove it if it is false.
14. Consider the following sentence: If x is rational then $\sqrt{x}$ is irrational. Is this sentence always true, sometimes true and sometimes false, or always false? Justify your answer.
24. Prove that $\log_5(2)$ is irrational.
35. Prove that there is at most one real number b with the property that $br = r$ for all real numbers r.