Discrete Mathematics

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1 General Recursive Definitions and Structural Induction

Recursively defined sets

1. BASE: A statement that certain objects belong to the set.
2. RECURSION: A collection of rules indicating how to form new set objects from those already known to be in the set.
3. RESTRICTION: A statement that no objects belong to the set other than those coming from 1 and 2.

A string over S: Let S be a finite set with at least one element. A string over S is a finite sequnce of elemnts from S. The elements of S are **characters** of the string, and the **length** of a string is the number of characters it contains. The **null string over S** is defined to be the "string" with no characters. It is usually denoted ϵ and is said to have length 0.

Structural Induction for Recursively Defined Sets
Let S be a set that has been defined recursively, and consider a property that objects in S may or may not satisfy. To prove that every object in S satisfies the property:

1. Show that each object in the BASE for S satisfies the property;
2. Show that for each rule in the RECURSION, if the rule is applied to objects in S that satisfy the property, then the objects defined by the rule also satisfy the property.
   NOTE: Because no objects other than those obtained through the BASE and RECURSION conditions are contained in S, it must be the case that every object in S satisfies the property.

Recursive Function: A function is said to be defined recursively or to be a recursive function if its rule of definition refers to itself. Because of this self-reference, it is sometimes difficult to tell whether a given recursive function is well defined.

2 6.3: Disproofs and Algebraic Proofs

Finding counterexamples
Ask:$(A - B) \cup (B - C) =^{?} A - C$

There are obvious differences and so the equivalence is not true. For example if $A = \{2, 3, 5, 6\}, B = \{1, 2, 3, 4\}, C = \{3, 4, 6\}$
What does the venn diagram for four sets look like? Is $A \cap C \subseteq B \cup D$? Consider $A = C = \{1\}$ and $B = D = \phi$ so $A \cap C \not\subseteq B \cup D$ and this cannot be the Venn digaram for 4 sets as it doesn't have all the regions we need.

The Venn diagram for four sets with all regions present

$A \cap C = \{8, \mathbf{10}, 12, 13\}$
$B \cup D = \{1, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15\}$
$A \cap C \not\subseteq B \cup D$
2^n regions (2^{n-1}) for n sets
Algebraic Proofs
Is $(A \cup B) - C = (A - C) \cup (B - C)$?

$$\begin{aligned}
(A \cup B) &= (A \cup B) \cap C^c && \text{By definition of minus} \\
&= C^c \cap (A \cup B) && \text{By commutativity} \\
&= (A \cap C^c) \cup (C^c \cap B) && \text{By distribution} \\
&= (A \cap C^c) \cup (B \cap C^c) && \text{By commutativity} \\
&= (A - C) \cup (B - C) && \text{By definition of minus}
\end{aligned}$$

3 6.4: Boolean & Russel's Paradox

Boolean Algebra:
Is a set B with operations '+' and '$\cdot$' with the following properties:
0. Closure: $\forall x, y \in B, x + y \in B \wedge x \cdot y \in B$
1. Commutavity: $\forall x, y \in, x + y = y + x \wedge x \cdot y = y \cdot x$
2. Associativity: $\forall x, y, z \in B, (x + y) + z = x + (y + z) \wedge (x \cdot y) \cdot z = x \cdot (y \cdot z)$
3. Distributivity: $\forall x, y, z \in B, x + (y \cdot z) = (x + y) \cdot (x + z) \wedge x(y + z) = (x \cdot y) + (x \cdot z)$
4. Identity: $\exists 0, 1 \in B$ such that $\forall x \in B, x + 0 = x \wedge x \cdot 1 = x$ (0 is false like thing, 1 is true, + corresponds to or, $\cdot$ corresponds to and)
5. Complement: $\forall x \in B, \exists \overline{x}$ such that $x + \overline{x} = 1$ and $x \cdot \overline{x} = 0$
Canonical examples of boolean algebra

- $S = \{\text{propositional formulae}\}$, 0=false, 1=true, $+$ is $\vee$, $\cdot$ is $\wedge$, $\overline{x} = \neg x$
- $S = \{\text{sets}\}, 0 = \phi, 1 = S, +$ is $\cup$, $\cdot$ is $\cap$, $\overline{x} = x^c$.

Russel's Paradox:
$S = \{A : A \text{ is a set and } A \notin A\}$
$S \in S$: suppose $S \in S$. Then S is a set $S \notin S$ which is a contradiction. Alternatively, Suppose $S \notin s$. Then either S is not a set, or $S \in S$. If S is a set:$\Rightarrow\!\Leftarrow$

4 7.1-7.2: Functions

A function f from domain X to codomain $Y(f : x \rightarrow y)$ is a total, single-valued relation $(x, y) \in f, f(x) = y$ the latter can be read (f of x is y, the value of f at x is y, the output of f for input x is y, the image of x under f is y)
Range: $f(X) = \{f(x) : x \in X\}$ - the image of the domain Image of set under f

$$f(A) = \{f(x) : x \in A\}$$

Pre-image (also known as Inverse Image)

$$f^{-1}(C) = \{x \in X : f(x) \in C\}$$

$$f^{-1}(y) = f^{-1}(\{y\})$$

Remember: A function is not well defined if it is not total or signle-valued. Example $f(x) = x^2$ then f^-1 is not a well-defined function.
New terminology:

1. Injective (one-to-one:inverse is single valued): A function is injective if $f(x) = f(y) \Leftrightarrow x = y$
2. Surjective (onto: inverse is total): $\forall y \in Y, \exists x \in X$ such that $f(x) = y$.
3. Bijective (one-to one correspondence): Injective and surjective meaning that the inverse is a well-defined function
4. Identity function:

$$I_x : X \rightarrow X$$

$$I_x(a) = a \quad , \forall a \in X$$

Logarithm:

$$\log_b : \mathbb{R}^+ \rightarrow \mathbb{R}$$

$$\log_b x = y \Leftrightarrow b^y = x$$

$$\log_b(xy) = log_b x + log_b y$$

$$\log_b x^y = y \cdot log_b x$$

$$\log_b\left(\frac{x}{y}\right) = \log_b(xy^-1) = \log_b x + \log_b y^-1 = \log_b x - \log_b y$$