Discrete Mathematics
Chapters 8.1,8.2,8.3 & 8.4 Homework

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Exercise Set 8.1

20. Let $A = \{-1, 1, 2, 4\}$ and $B = \{1, 2\}$ and define relations R and S from A to B as follows: For all $(x, y) \in A \times B$,

$$x \; R \; y \Leftrightarrow |x| = |y| \text{ and}$$
$$x \; S \; y \Leftrightarrow x - y \text{ is even}$$

State explicitly which ordered pairs are in $A \times B, R, S, R \cup S$, and $R \cap S$.

Ans:

$$\begin{aligned} A \times B &= \{(-1,1),(-1,2),(1,1),(1,2),(2,1),(2,2),(4,1),(4,2)\} \\ R &= \{(-1,1),(1,1),(2,2)\} \\ S &= \{(-1,1),(1,1),(2,2),(4,2)\} \\ R \cup S &= \{(-1,1),(1,1),(2,2),(4,2)\} \\ R \cap S &= \{(-1,1),(1,1),(2,2)\} \end{aligned}$$

Exercise Set 8.2

14. O is the relation defined on $\mathbb{Z}$ as follows: For all $m, n \in \mathbb{Z}$, $m \; O \; n \Leftrightarrow m - n$ is odd. Determine whether this relation is reflexive, symmetric, transitive, or none of these. Justify your answer.

Ans:
O is not reflexive: Suppose m is a particular but arbitrarily chosen integer. Now $m - m = 0$. But this is not odd. Hence $m - m$ is not odd and O is not reflexive.
O is symmetric: Suppose m and n are particular but arbitrarily chosen integers

that satisfy the condition $m\ O\ n$. By definition of O, since $m\ O\ n$ then $m-n$ is odd. By definition of odd, this means that $m-n=2k+1$ for some integer k. Multiplying both sides by -1 gives $n-m=2(-k-1)+1$ since $-k-1$ is an integer, this equation shows that $n-m$ is odd. Hence, by defintion of O, $n\ O\ m$.
O is not transitive: Suppose m, n and p are particular but arbitrarily chosen integers that satisfy condtion $m\ O\ n$ and $n\ O\ p$. By definition of O, since $m\ O\ n$ and $n\ O\ p$, then $m-n$ is odd and $n-p$ is also odd. By definition of odd, this means that $m-n=2k+1$ and $n-p=2q+1$ for some integers k and q. Adding the two equations gives $(m-n)+(n-p)=2k+1+2q+1$ and simplifying this gives $m-p=2(k+q+1)$. Since $k+q+1$ this equation shows that $m-p$ is even and not odd. Hence, by defintion of O, $m\ \not{O}\ p$.

36. If R is transitive, then R^{-1} is transitive. (Prove or disprove this statement.)

Ans: Suppose R is any relation on a set A that is transitive. By defintion of transitive, this means that for all x, y, and z in A, if $(x,y) \in R$ and $(y,z) \in R$ then $(x,z) \in R$. Suppose that $x\ R\ y$ and $y\ R\ z$. Because R is transitive, then $x\ R\ z$. The inverse relation by definition would contain the relations $y\ R^{-1}\ x$, $z\ R^{-1}\ y$, and $z\ R^{-1}\ x$. This means that for any x, y, and z in A if $z\ R^{-1}\ y$ and $y\ R^{-1}\ x$ then $z\ R^{-1}\ x$. Hence, R^{-1} is also transitive.

40. If R and S are reflexive, is $R \cup S$ reflexive? Why? (Assume R and S are relations on a set A. Prove or disprove the statement.)

Ans: Suppose not, suppose that R and S are reflexive and that $R \cup S$ is not reflexive. This means that there is an element x in $R \cup S$ such that $(x,x) \notin R$ or $(x,x) \notin S$. But we know that R and S are reflexive so it is not true that that there is an element x in R such that $(x,x) \notin R$ and it is not true that here is an element x in S such that $(x,x) \notin R$. This means that $R \cup S$ also has to be reflexive because all ordered pairs (x,x) in $R \cup S$ come are in R or S and so they are also reflexive.

Exercise Set 8.3

In 9 and 10 the relation R is an equivalence relation on the set A. Find the distinct equivalence classes of R.

9. $X = \{-1, 0, 1\}$ and $A = \mathscr{P}(X)$. R is defined on $\mathscr{P}(X)$ as follows: For all sets s and τ in $\mathscr{P}(X)$,

$$s \; R \; \tau \Leftrightarrow \text{ the sum of elements in } s \text{ equals the sum of the elements in } \tau$$

Ans:

$$\{\phi\}$$
$$\{\{-1\}, \{-1, 0\}\}$$
$$\{\{0\}, \{-1, 1\}, \{-1, 0, 1\}\}$$
$$\{\{1\}, \{0, 1\}\}$$

10. $A = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$. R is defined on A as follows: For all $m, n \in \mathbb{Z}$,

$$m \; R \; n \Leftrightarrow 3 \mid (m^2 - n^2).$$

39. The following argument claims to prove that the requirement that an equivalence relation be reflexive is redundant. In other words, it claims to show that if a relation is symmetric and transitive, then it is reflexive. Find the mistake in the argument. "**Proof:** Let R be a relation on a set A and suppose R is symmetric and transitive. For any two elements x and y in A, if $x \; R \; y$ then $y \; R \; x$ since R is symmetric. But then it follows by transitivity that $x \; R \; x$. Hence R is reflexive."

Exercise Set 8.4

Exercise Set 8.5

Extra-credit