Discrete Mathematics
Chapter 5.1 Homework

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Exercise Set 5.1

9. Find explicit formulas for the sequence of the form $a_1, a_2, a_3, \ldots$ with the initial terms below

$$1 - \frac{1}{2}, \frac{1}{2} - \frac{1}{3}, \frac{1}{3} - \frac{1}{4}, \frac{1}{4} - \frac{1}{5}, \frac{1}{5} - \frac{1}{6}, \frac{1}{6} - \frac{1}{7}$$

Ans:

$$a_k = \frac{1}{k} - \frac{1}{k+1} \text{ for all integers } k \geq 0$$

29. Evaluate the summation for the indicated values of the variable

$$1(1!) + 2(2!) + 3(3!) + \ldots + m(m!); m = 2$$

Ans:

$$\begin{aligned} &= 1(1!) + 2(2!) \\ &= 1 + 4 \\ &= 5 \end{aligned}$$

43. Write using summation or product notation

$$(1 - t) \cdot (1 - t^2) \cdot (1 - t^3) \cdot (1 - t^4)$$

Ans:

$$\prod_{k=1}^{4}(1-t^k)$$

52. Transform by making the change of variable $j = i - 1$

$$\sum_{i=1}^{n-1} \frac{i}{(n-1)^2}$$

Ans:

$$i = j + 1$$

$$j = i - 1$$

When $i = 1$, then $j = 1 - 1 = 0$ so the lower limit is 0. When $i = n - 1$, $j + 1 = n - 1$, $j = n - 2$ so the upper limit is $n - 2$. Substituting $i = j + 1$ we then have

$$\sum_{i=1}^{n-1} \frac{i}{(n-i)^2} = \sum_{j=0}^{n-2} \frac{j+1}{(n-j-1)^2}$$

55. Write as a single summation or product

$$2 \cdot \sum_{k=1}^{n}(3k^2+4) + 5 \cdot \sum_{k=1}^{n}(2k^2-1)$$

Ans: By Theorem 5.1.1 (2)

$$2 \cdot \sum_{k=1}^{n}(3k^2+4) + 5 \cdot \sum_{k=1}^{n}(2k^2-1) = \sum_{k}^{n} = (6k^2+8) + \sum_{k-1}^{n}(10k^2-5)$$

By Theorem 5.1.1 (1)

$$= \sum_{k=1}^{n} (6k^2 + 8 + 10k^2 - 5)$$

$$= \sum_{k=1}^{n} 16k^2 + 3$$

Compute 59 & 62. Assume the values of the variables are restricted so the functions are defined.

59.

$$\frac{4!}{0!}$$

Ans: $0! = 1$ by defintion and $4! = 4 \cdot 3 \cdot 2 \cdot 1$

$$= \frac{24}{1}$$

$$= 24$$

62.

$$\frac{n!}{(n-2)!}$$

Ans:

$$= \frac{n(n-1)(n-2)!}{(n-2)!}$$

$$= n(n-1)$$

$$= n^2 - n$$

74. Prove that if p is a prime number and r is an integer with $0 < r < p$, then $\binom{p}{r}$ is divisiple by p.

Ans:

Proof. Suppose p is a prime number and r is an integer with $0 < r < p$

$$\binom{p}{r} = \frac{p!}{r!(p-r)!}$$

Because p is prime its only factors are p and 1.

$$= p \cdot \frac{(p-1)!}{r!(p-r)!}$$

Because $r < p$, $(p-r)!$ is contained in $(p-1)!$ so we will get to a point where the fraction becomes $\frac{(p-1)(p-2)(p-r)!}{r!(p-r)!} = \frac{(p-1)(p-2)(...)}{r!}$
From definition of nCr we know that $\binom{p}{r} \in \mathbb{Z}$ and since $p \in \mathbb{Z}$ and since the product of two integers must be an integer then $\frac{(p-1)(p-2)(...)}{r!} \in \mathbb{Z}$
Therefore $\binom{p}{r} = pk$ for some integer k and so by definition $p \mid \binom{p}{r}$ □