Discrete Mathematics
Chapters 5.4 & Supplement A

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Exercise Set 5.4

2. Suppose $b_1, b_2, b_3, ...$ is a sequence defined as follows:

$$\begin{aligned} b_1 =& 4, b_2 = 12 \\ b_k =& b_{k-2} + b_{k-1} \text{ for all integers } k \geq 3 \end{aligned}$$

Prove that b_n is divisible by 4 for all integers $n \geq 1$.

Ans:

Proof. Let the property $P(n)$ be the sentence

b_n is divisible by 4

Show that $P(1)$ and $P(2)$ are true:
We know that $b_1 = 4$ and $b_2 = 12$. Because 4 and 12 are divisible by 4 we know that $P(1)$ and $P(2)$ are true.
Show that for all integers $k \geq 2$, if $P(i)$ is true for all integers from 1 to through k, then $P(k+1)$ is also true.
Let k be any integer with $k \geq 2$, and suppose that b_i is divisible for all integers i with $1 \leq i \leq k$. We must show that $k+1$ is divisible by 4. We know that $b_{k+1} = b_{k-1} + b_k$ by definition of $b_1, b_2, b_3, ...$ Since $k-1 \leq k$ and $k \leq k$, by our inductive hypothesis b_{k-1} and b_k are both divisible by 4. By definition $b_{k-1} = 4q$ for some integer q and $b_k = 4s$ for some integer s so $b_{k+1} = 4q + 4s = 4(q+s)$ and so b_{k+1} is divisible by 4. $\square$

7. Suppose $g_1, g_2, g_3, \ldots$ is a sequence defined as follows:

$$\begin{aligned} g_1 =& 3, g_2 = 5 \\ g_k =& 3g_{k-1} - 2g_{k-2} \text{ for all integers } k \geq 3 \end{aligned}$$

Prove that $g_n = 2^n + 1$ all integers $n \geq 1$.

Ans:

Proof. Let $g_1, g_2, g_3, \ldots$ be the sequence defined by specifiying that $g_1 = 3$, $g_2 = 5$, and $g_k = 3g_{k-1} - 2g_{k-2}$ for all integers $k \geq 3$, and let the property $P(n)$ be the formula

$$g_n = 2^n + 1$$

We will use strong mathematical induction to prove that for all integers $n \geq 1$, $P(n)$ is true.
Show that $P(1)$ and $P(2)$ are true:
To establish $P(1)$ and $P(2)$, we must show that

$$g_1 = 2^1 + 1 \text{ and } g_2 = 2^2 + 1$$

But, by definition of $g_1, g_2, g_3, \ldots.$, we must have that $g_1 = 3$ and $g_2 = 5$. Since $2^1 + 1 = 2 + 1 = 3$ and $2^2 + 1 = 4 + 1 = 5$, the values of g_1 and g_2 agree with the values given by the formula. Show that for all integers $k \geq 2$, if $P(i)$ is true for all integers i from 1 through k, then $P(k+1)$ is also true:
Let k be any integer with $k \geq 2$ and suppose that

$$g_i = 2^i + 1 \text{ for all integers } i \text{ with } 1 \leq i \leq k$$

We must show that

$$g_{k+1} = 2^{k+1} + 1$$

But since $k \geq 2$, we have that $k+1 \geq 3$, and so

$$\begin{aligned} g_{k+1} &= 3g_k - 2g_{k-1} \\ &= 3(2^k + 1) - 2(2^{k-1} + 1) \\ &= 3 \cdot 2^k + 3 - 2^k - 2 \\ &= 3 \cdot 2^k - 2^k + 1 \\ &= (1+2)2^k - 2^k + 1 \\ &= 2^k + 2^{k+1} - 2^k + 1 \\ &= 2^{k+1} + 1 \end{aligned}$$

□

9. Define a sequence $a_1, a_2, a_3, \ldots$ as follows: $a_1 = 1, a_2 = 3$, and $a_k = a_{k-1} + a_{k-2}$ for all integers $k \geq 3$. Use strong mathematical induction to prove that $a_n \leq \left(\frac{7}{4}\right)^n$ for all integers $n \geq 1$.

Ans:

Proof. Let $a_1, a_2, a_3, \ldots$ be the sequence defined by specifying that $a_1 = 1, a_2 = 3$ and $a_k = a_{k-1} + a_{k+2}$ for all integers $k \geq 3$, and let the property $P(n)$ be the inequality

$$a_n \leq \left(\frac{7}{4}\right)^n$$

We will use strong mathematical induction to prove that for all integers $n \geq 1$, $P(n)$ is true.
Show that $P(1)$ and $P(2)$ are true:
To establish $P(1)$ and $P(2)$ we must show that

$$a_1 \leq \left(\frac{7}{4}\right)^1 \text{ and } a_2 \leq \left(\frac{7}{4}\right)^2$$

But by definition of $a_1, a_2, a_3, ...,$ we have that $a_1 = 1$ and $a_2 = 3$. Since $1 < \frac{7}{4}$ and $3 < (\frac{7}{4})^2$, the inequality holds for a_1 and a_2.
Show that for all integers $k \geq 2$, if $P(i)$ is true for all integers from 1 through k, then $P(k+1)$ is also true:
Let k be any integer with $k \geq 1$ and suppose that

$$a_i \leq \left(\frac{7}{4}\right)^i \text{ for all integers } i \text{ with } 1 \leq i \leq k$$

We must show that

$$a_{k+1} \leq \left(\frac{7}{4}\right)^{k+1}$$

But since $k \geq 2$, we have $k+1 \geq 3$, and so

$$a_{k+1} = a_k + a_{k-1} \leq \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1}$$

$$\left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1} \leq \left(\frac{7}{4}\right)^{k+1}$$

$$a_{k+1} \leq \left(\frac{7}{4}\right)^{k+1}$$

$\square$

18. Compute $9^0, 9^1, 9^2, 9^3, 9^4$, and 9^5. Make a conjecture about the units digit of 9^n where n is a positive integer. Use strong mathematical induction to prove your conjecture.

Ans:

9^0	1
9^1	9
9^2	81
9^3	729
9^4	6561
9^5	59049

Conjecture: The units digit of 9^n equals 1 if n is even and equals 9 if n is odd.

Proof by strong mathematical induction: Let the property $P(n)$ be the sentence

> The units digit of 9^n equals 1 if n is even and equals 9 if n is odd.

Show that $P(1)$ and $P(2)$ are true:
When $n = 1, 9^n = 9^1 = 9$, and the units digit is 9. When $n = 2$, then $9^n = 9^2 = 81$, and the units digits is 1. Thus $P(1)$ and $P(2)$ are true.
Show that for any integer $k \geq 2$, if the property is true for all integers i with $0 \leq i \leq k$ then it is true for $k + 1$:
Let k be any integer with $k \geq 2$, and suppose that for all integers i with $0 \leq i \leq k$, the units digit of 9^i equals 1 if i is even and equals 9 if i is odd. We must show that the units digit of 9^{k+1} equals 1 if $k + 1$ is even and equals 9 if $k + 1$ is odd.
Case 1 ($k + 1$ is even):
In this case, k is odd, and so, by inductive hypothesis, the units digit of 9^k is 9. Thus $9^k = 10q+9$ for some nonnegative integer q. It

follows that $9^{k+1} = 9^k \cdot 9 = (10q+9) \cdot 9 = 90q+81 = 10(9q+8)+1$. Thus the units digit of 9^{k+1} is 1.
Case 2 ($k+1$ is odd):
In this case, k is even, and so, by inductive hypothesis, the units digit of 9^k is 1. Thus $9^k = 10q+1$ for some nonnegative integer q. It follows that $9^{k+1} = 9^k \cdot 9 = (10q+1) \cdot 9 = 90q+9 = 10(9q)+9$. Thus the units digit of 9^{k+1} is 9.

$\square$

19. Find the mistake in the following "proof" that purports to show that every nonnegative integer power of every nonzero real number is 1.
"Proof: Let r be any nonzero real number and let the property $P(n)$ be the equation $r^n = 1$.
Show that $P(0)$ is true: P(0) is true because $r^0 = 1$ by defintion of zeroth power.
Show that for all integers $k \geq 0$, if $P(i)$ is true for all integers i from 0 through k, then $P(k+1)$ is also true: Let k be any integer with $k \geq 0$ and suppose that $r^i = 1$ for all integers i from 0 through k. This is the inductive hypothesis. We must show that $r^{k+1} = 1$. Now

$$\begin{aligned} r^{k+1} &= r^{k+k-(k-1)} && (k+k-(k-1) = k+k-k+1 = k+1) \\ &= \frac{r^k \cdot r^k}{r^{k-1}} && \text{(by the laws of exponents)} \\ &= \frac{1 \cdot 1}{1} && \text{(by inductive hypothesis)} \\ &= 1. \end{aligned}$$

Thus $r^{k+1} = 1$ *[as was to be shown].*
[Since we have proved the basis step and the inductive step of the strong mathematical induction, we conclude that the given statement is true.]"

21. Use the well-ordering principle for the integers to prove the existence

part of the unique factorization of integers theorem: Every integer greater than 1 is either prime or a product of prime numbers.

Ans: It is possible to think of examples where the basis step is not true. For instance, $P(1)$ is false because $r^1 = r$. This is only true when $r = 1$ and it is false for every other real number.

26. Suppose $P(n)$ is a property such that

    1. $P(0), P(1), P(2)$ are all true,
    2. for all integers $k \geq 0$, if $P(k)$ is true, then $P(3k)$ is true. Must it follow that $P(n)$ is true for all integers $n \geq 0$? If yes, explain why; if no, give a counterexample.

Ans: No, $P(n)$ is not necessarily true for all integers. Consider the property $P(n)$ which represents the statement "n is less than 3". $P(3n)$ is not true for $n = 1$ or $n = 2$.

32. Prove that if a statement can be proved by ordinary mathematical induction, then it can be proved by the well-ordering principle.

Ans:

Proof. Suppose not. That is suppose there is a statement the form "For all integers $n \geq a$, a property $P(n)$ is true." that can be proved by ordinary mathematical induction and that can not be proved by the well-ordering principle. Because the statement can be proved by ordinary mathematical indudction we know the following:

1. $P(a)$ is true.
2. For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

Then, suppose that the set S is the set of all integers greater than or equal to a for which $P(n)$ is false. Because of our supposition the set S does not have a least element but because our statement is proved by ordinary mathematical induction then the set S must be empty, and so the well-ordering principle is not violated and this is a contradiction of our supposition. □

Supplement A

2. Show that if the predicate is true before entry to the loop, then it is also true after exit from the loop

loop: **while**$(m \geq 0)$ and $m \leq 100)$

$m := m + 4$

$n := n - 2$

end while

predicate: $m + n$ is odd

Ans:

Proof. Suppose the predicate $m + n$ is odd is true before entry to the loop. Then

$$m_{old} + n_{old} = 2q + 1 \text{ for some integer } q$$

After execution of the loop,

$$m_{new} = m_{old} + 4 \text{ and } n_{new} = n_{old} - 2$$

so

$$\begin{aligned} m_{new} + n_{new} &= m_{old} + 4 + n_{old} - 2 \\ &= m_{old} + n_{old} + 2 = 2q + 1 + 2 \\ &= 2(q + 1) + 1 \end{aligned}$$

Therefore $m_{new} + n_{new} = 2(q+1)+1$ and is thus odd by definition. $\square$

Exercises 7 contains a while loop annotated with a pre- and a post-condition and also a loop invariant. Use the loop invariant theorem to prove the correctness of the loop with respect to the pre- and post-conditions.

7. *[Pre-condition: largest = A[1] and i=1]*
   while $(i \neq m)$
   1. $i := i + 1$
   2. **if** $A[i] >$ largest **then** largest: $= A[i]$

   end while

   [Post-condition: largest = maximum value of A[1], A[2],...,A[m]]
   loop invariant: $I(n)$ is "largest = maximum value of $A[1], A[2], A[n+1]$ and $i = n + 1$"

Ans:
I. Basis Property
$I(0)$ is largest = maximum value of $A[1]$ and $i = 1$. According to the pre-condition the largest = A[1] and $i = 1$. Therefore $I(0)$ is true.
II. Inductive Property
Suppose k is any nonnegative integer such that $G \wedge I(k)$ is true before iteration of the loop. Then as execution reaches the top of the loop, $i \neq m$, largest $=A[k]$, and $i = k + 1$. Since $i \neq m$, the guard is passed and statement 1 is executed. Now before execution of statement 1,

$$i_{old} = k + 1$$

so execution of statement 1 has the following effect:

$$i_{new} = i_{old} + 1$$

Similarly, before execution of statement 2

$$largest_{old} = \text{ maximum value of } A[1], A[2], ..., A[k+1]$$

so after exeuction of statement 2,

$$largest_{new} = \text{ maximum value of } A[1], A[2], ..., A[k+2]$$

Hence after loop iteration, the two statements $i = k+2$ and largest = maximum value of $A[1], A[2], ..., A[k+2]$ are true, and so $I(k+1)$ is true.

III. Eventual Falsity of Guard

The guard G is the condition $i \neq m$, and m is a nonnegative integer. By I and II, it is known that

> for every integer $n \geq 0$, if the loop is iterated n times then largest = maximum value of $A[1], A[2], ..., A[n+1]$ and $i = n+1$

So after m iterations of the loop, $i = m$. Thus G becomes false after m iterations of the loop

IV. Correctness of the Post-Condition

According to the post-condition, the value of largest after execution of the loop should be the maximum of $A[1], A[2], ..., A[m]$. But when G is false $i = m$. And when $I(N)$ is true, $i = N+1$ and largest = maximum value of $A[1], A[2], ..., A[N+1]$. Since both conditions are satisfied, $m = i = N+1$ and largest = maximum value of $A[1], A[2], ..., A[m]$ as required.