Multi-variable Calculus
Problem Set 2

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1. Sketch (by hand) a contour diagram for the function $f(x, y) = -x^2 - y^2 + 1$ with labeled contours $c = 0, -1, -2, -3, -4$. Draw at least four contour curves. Then write a sentence that describes the shapes of the contours and how they are spaced relative to each other.

Ans:

c	$f(x, y)$
0	$x^2 + y^2 = 1$
-1	$x^2 + y^2 = 0$
-2	$x^2 + y^2 = 3$
-3	$x^2 + y^2 = 4$
-4	$x^2 + y^2 = 5$

Concentric circles at (0,0) that get closer together as c decreases from 0 to -4.

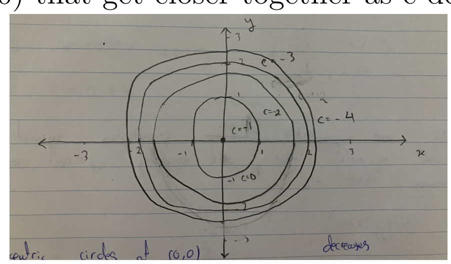

Figure 1

2. Repeat the instructions of Question (1) for the function $f(x, y) = y - x^2$ and the contours $c = -2, -1, 0, 1, 2$.

Ans:

c	$f(x, y)$
-2	$y = x^2 - 2$
-1	$y = x^2 - 1$
0	$y = x^2$
1	$y = x^2 + 1$
2	$y = x^2 + 2$

Contours are parabolas symmetric around y-axis that are equally spaced as c changes from -2 to 2

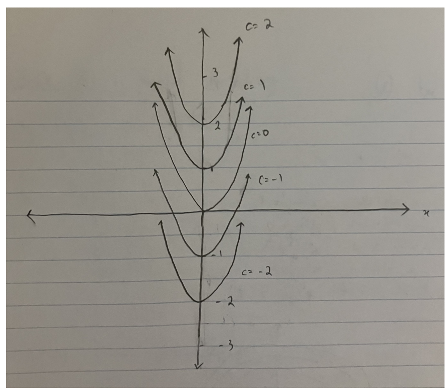

Figure 2

3. Let $f(x, y) = x^2 - y^2 = (x - y)(x + y)$. Use the factored form to sketch the contour given by $f(x, y) = 0$ and to find the regions in the xy-plane where $f(x, y) > 0$ and the regions where $f(x, y) < 0$. Explain how this sketch shows that the graph of $f(x, y)$ is saddle-shaped at the origin.

Ans: From the factored form we can see that for $f(x, y)$ to be more than 0 both $(x - y)$ and $(x + y)$ need to be more than 0 or both need to be less than 0
$f(x, y)$ is positive when $(x - y) > 0$ and $(x + y) > 0$ or $(x - y) < 0$ and $(x + y) < 0$
On the other side for $f(x, y)$ to be less than 0 we need $(x - y) < 0$ and $(x + y) > 0$ or $(x - y) > 0$ and $(x + y) < 0$
Since the contours are the lines $y = x$ and $y = -x$ we see that $f(x, y)$ is negative when $y > x$ and $y > -x$ or when $y < x$ and $y < -x$
We also see that $f(x, y)$ is positive when $x > y$ and $x > -y$ or when $x < y$ and

$x < -y$
The fact that $f(x, y)$ is saddle shaped at the origin can be seen when looking at how the contour values change if we move in any of the 4 sections on the origin. Above and below the point $(0, 0)$ the contours are negative meaning we are at shallower points but to the right and left of the point $(0, 0)$ we have positive contours meaning that we are at higher points.

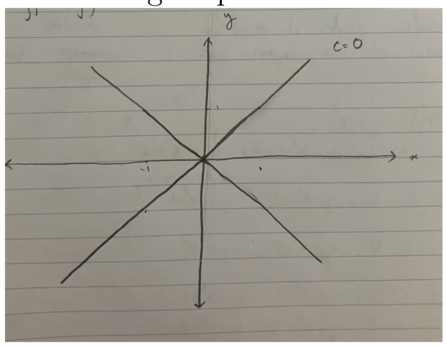

Figure 3

4. The figure on the next page shows the contour plot of a linear function $f(x, y)$. Find the formula for the linear function and express it in *Slope-Intercept Form.*

Ans: From the figure we can calculate $m = \frac{\Delta z}{\Delta x}$ and $n = \frac{\Delta z}{\Delta y}$

$$m = \frac{\Delta z}{\Delta x} = \frac{4-6}{0-1} = 2$$

$$n = \frac{\Delta z}{\Delta y} = \frac{4-2}{0-2} = -1$$

From the graph we can see that at $f(0, 0) = 4$ so $4 = 2(0) - (0) + c$ and $c = 4$
The slope-intercept form will then be

$$z = 2x - y + 4$$

5. Suppose a linear function has formula $f(x, y) = a + 10x - 5y$, but you don't know the value of a. Is it possible to find each of the following values, and if so, what is the value? Explain your reasoning.
   (a) $f(50, 62)$
   (b) $f(51, 60) - f(50, 62)$

Ans:

(a) Since we do not know where the plane $z = a + 10x - 5y$ cuts the z-axis at the origin our best estimate of the value of $f(50, 62)$ will be the following where a is some real number

$$f(50, 62) = 190 + a$$

(b) It is possible to find the value here because we are finding the difference of two equations where a is subtracted from a so we do not need to know what value it holds

$$f(51, 60) - f(50, 62) = (210 + a) - (190 + a)$$

$$= 210 - 190 + a - a$$

$$= 20$$