B206 — Transitions to Higher Maths
Chapter 10

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2. Prove that $1^2 + 2^2 + 3^2 + 4^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$ for every positive integer n.

Ans:

Proposition. *If* $n \in \mathbb{N}$ *then* $1^2 + 2^2 + 3^2 + 4^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof. (1) - If $n = 1$, this statement is $1^2 = \frac{1(2)(3)}{6}$ or $1 = 1$ which is true.
(2) - We must prove $S_k \Rightarrow S_{k+1}$ for any $k \geq 1$. That is we must show that if $1^2 + 2^2 + 3^2 + 4^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$ then $1^2 + 2^2 + 3^2 + 4^2 + ... + (k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$. We use direct proof. Suppose $1^2 + 2^2 + 3^2 + 4^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$. Then

$$\begin{aligned}
1^2 + 2^2 + 3^2 + 4^2 + ... + (k+1)^2 &= \\
1^2 + 2^2 + 3^2 + 4^2 + ... + k^2 + (k+1)^2 &= \\
\frac{k(k+1)(2k+1)}{6} + (k+1)^2 &= \\
\frac{k(k+1)(2k+1) + 6(k+1)(k+1)}{6} &= \\
&= \frac{(k+1)\left[k(2k+1) + 6(k+1)\right]}{6} \\
&= \frac{(k+1)(2k+3)(k+2)}{6} \\
&= \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}
\end{aligned}$$

Thus $1^2 + 2^2 + 3^2 + 4^2 + ... + (k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$. This proves that $S_k \Rightarrow S_{k+1}$. It follows by induction that $1^2 + 2^2 + 3^2 + 4^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$ for every positive integer n. $\square$

8. If $n \in \mathbb{N}$, then $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ... + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$.

Ans:

Proposition. *If* $n \in \mathbb{N}$, *then* $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ... + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$.

Proof. (1) - If $n = 1$, this statement is $\frac{1}{2!} = 1 - \frac{1}{2!}$ or $\frac{1}{2} = \frac{1}{2}$ which is true.
(2) - We must prove $S_k \Rightarrow S_{k+1}$ for any $k \geq 1$. That is we must show that if $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ... + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$ then $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ... + \frac{k+1}{(k+1+1)!} = 1 - \frac{1}{(k+1+1)!}$.
We use direct proof. Suppose $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ... + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$. Then

$$\begin{aligned}
\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ... + \frac{k+1}{(k+1+1)!} &= \\
\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ... + \frac{k}{(k+1)!} + \frac{k+1}{(k+1+1)!} &= \\
1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+1+1)!} &= 1 - \frac{(k+2) + k + 1}{(k+1+1)!} \\
&= 1 - \frac{1}{(k+1+1)!}
\end{aligned}$$

Thus $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ... + \frac{k+1}{(k+1+1)!} = 1 - \frac{1}{(k+1+1)!}$. This proves that $S_k \Rightarrow S_{k+1}$. It follows by induction that $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ... + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$ for every positive integer n. □