Discrete Mathematics
Chapters 6.3,6.4,7.1 & 7.2 Homework

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Exercise Set 6.3

For 20& 21 prove the statement that is true and find a counterexample for the statement that is false. Assume all sets are subsets of a universal set U.

20. For all sets A and B, $\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B)$

21. For all sets A and B, $\mathscr{P}(A \times B) = \mathscr{P}(A) \times \mathscr{P}(B)$

32. For all sets A and B, $(A - B) \cup (A \cap B) = A$. (Construct an algebraic proof and cite a property from Theorem 6.2.2 for every step)

43. $((A \cap (B \cup C)) \cap (A - B)) \cap (B \cup C^c)$ (Simplify the given expression. Cite a property from Theorem 6.2.2 for every step)

Exercise Set 6.4

11. Let $S = \{0, 1\}$ and define operations $+$ and $\cdot$ on S by the following tables:

+	**0**	**1**
0	0	1
1	1	1

·	**0**	**1**
0	0	0
1	0	1

(a) Show that the elements of S satisfy the following properties:

(i) the commutative law for $+$

(ii) the commutative law for $\cdot$

(iii) the associative law for $+$

(iv) the commutative law for $\cdot$

(v) the distributive law for $+$ over $\cdot$

(vi) the distributive law for $\cdot$ over $+$

(b) Show that 0 is an identity element for $+$ and that 1 is an identity element for $\cdot$.

(c) Define $\overline{0} = 1$ and $\overline{1} = 0$. Show that for all a in S, $a + \overline{a} = 1$ and $a \cdot \overline{a} = 0$. It follows from parts (a)-(c) that S is a Boolean algebra with operations $+$ and $\cdot$.

12. Prove that the associative laws for a Boolean algebra can be omitted from the definition. That is, prove that the associative laws can be derived from the other laws in the definition.

19. (a) Assuming that the following sentence is a statement, prove that $1 + 1 = 3$:

    If this sentence is true, then $1 + 1 = 3$.

    (b) What can you deduce from part (a) about the status of "This sentence is true"? Why? (This example is known as **Löb's paradox.**)

Exercise Set 7.1

14. Let $J_5 = \{0, 1, 2, 3, 4\}$, and define functions $h : J_5 \to J_5$ and $k : J_5 \to J_5$ as follows: For each $x \in J_5, h(x) = (x + 3)^3 \mod 5$ and $k(x) = (x^3 + 4x^2 + 2x + 2) \mod 5$. Is $h = k$? Explain.

24. If b and y are positive real numbers such that $\log_b y = 2$, what is $\log_{b^2}(y)$? Why?

28. Student C tries to define a function $h : Q \to Q$ by the rule

$$h\left(\frac{m}{n}\right) = \frac{m^2}{n}, \text{ for all integers } m \text{ and } n \text{ with } n \neq 0.$$

    Student D claims that h is not well defined. Justify student

43. Given a set S and a subset A, the **characteristic function of A,** denoted χ_A, is the function defined from S to $\mathbb{Z}$ with the property that for all $u \in S$,

$$\chi_A(u) = \begin{cases} 1 & \text{if } u \in A \\ 0 & \text{if } u \notin A \end{cases}$$

    Show that each of the following holds for all subsets A and B of S and all $u \in S$.

    (a) $\chi_{A \cap B}(u) = \chi_A(u) \cdot \chi_B(u)$

    (b) $\chi_{A \cup B} = \chi_A(u) + \chi_B(u) - \chi_A(u) \cdot \chi_B(u)$

Exercise Set 7.2

12. (a) Define $F : \mathbb{Z} \to \mathbb{Z}$ by the rule $F(n) = 2 - 3n$, for all integers n.

    (i) Is F one-to-one? Prove or give a counterexample.

    (ii) Is F onto? Prove or give a counterexample.

    (b) Define $G : \mathbb{R} \to \mathbb{R}$ by the rule $G(x) = 2 - 3x$ for all real numbers x. Is G onto? Prove or give a counterexample.

24. Define $J : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ by the rule $J(r, s) = r + \sqrt{2}s$ for all $(r, s) \in \mathbb{Q} \times \mathbb{Q}$.

(a) Is J one-to-one? Prove or give a counter example

(b) Is J onto? Prove or give a counterexample.

31. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both onto, is $f + g$ also onto? Justify your answer.

33. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and c a nonzero real number. If f is onto, is $c \cdot f$ also onto? Justify your answer.