

# Discrete Mathematics

## Chapters 5.9, 6.1 & 6.2 Homework

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### Exercise Set 5.9

6. Define a set  $S$  recursively as follows:

- I. BASE:  $a \in S$
- II. RECURSION: If  $s \in S$ , then,
  - a.  $sa \in S$
  - b.  $sb \in S$
- III. RESTRICTION: Nothing is in  $S$  other than objects defined in I and II above.

Use structural induction to prove that every string in  $S$  begins with an  $a$ .

**Ans:**

*Proof.* Given any string  $s \in S$ , let the property be the claim that it begins with an  $a$ .

**Show that each object in the BASE case for  $S$  satisfies the property:** The only object in the base case is  $a$  which satisfies the property as it begins with  $a$ .

**Show that for each rule in the RECURSION for  $S$ , if the rule is applied to an object in  $S$  that satisfies the property, then the objects defined by the rule also satisfy the property:** The recursion for  $S$  consists of two rules denoted II(a) and II(b). Suppose  $s$  is a string in  $S$  that begins with  $a$ . In the case where the rule II(a) is applied to  $s$ , the result is the string  $sa$ , which also begins with  $a$ . In the case where the rule II(b) is applied to  $s$ , the result is the string  $sb$ , which also begins with  $a$ .

Thus when each rule in the RECURSION is applied to a string in  $S$  that begins with  $a$ , the result is also a string that begins with  $a$ . □

11. Define a set  $S$  recursively as follows:

- I. BASE:  $0 \in S$
- II. RECURSION: If  $s \in S$ , then,
  - a.  $s + 3 \in S$
  - b.  $s - 3 \in S$
- III. RESTRICTION: Nothing is in  $S$  other than objects defined in I and II above.

Use structural induction to prove that integer in  $S$  is divisible by 3.

**Ans:**

*Proof.* Given any integer  $s \in S$ , let the property be the claim that it is divisibly by 3.

**Show that each object in the BASE case for  $S$  satisfies the property:** The only object in the base case is 0 which satisfies the property because by definition  $3 \mid 0, \exists k \in \mathbb{Z}, 0 = 3k$ .

**Show that for each rule in the RECURSION for  $S$ , if the rule is applied to an object in  $S$  that satisfies the property, then the objects defined by the rule also satisfy the property:** The recursion for  $S$  consists of two rules denoted II(a) and II(b). Suppose  $s$  is an integer in  $S$  that is divisible by 3. Then by definition  $\exists l \in \mathbb{Z}, s = 3l$ . In the case where the rule II(a) is applied to  $s$ , the result is the integer  $s + 3 = 3l + 3 = 3(l + 1)$ , which is also divisible by 3 by definition because  $\exists m \in \mathbb{Z}, s + 3 = 3(l + 1)$  where  $m = l + 1$ . In the case where the rule II(b) is applied to  $s$ , the result is the integer  $s - 3 = 3l - 3 = 3(l - 1)$ , which is also divisible by 3 by definition because  $\exists n \in \mathbb{Z}, s - 3 = 3(l - 1)$  where  $n = l - 1$ . Thus when each rule in the RECURSION is applied to

an integer  $s$  that is divisible by 3, the result is also an integer that is divisible by 3.  $\square$

16. Give a recursive definition for the set of all strings of 0's and 1's for which all the 0's precede all the 1's.

**Ans:** Let  $S$  be the set of all strings of 0's and 1's for which all the 0's precede all the 1's. The following is the recursive definition of the set  $S$ :

I. BASE:  $0 \in S$

II. RECURSION: If  $s \in S$ , then,

a.  $s1 \in S$

b.  $0s \in S$

III. RESTRICTION: Nothing is in  $S$  other than objects defined in I and II above.

18. Give a recursive definition for the set of all strings of  $a$ 's and  $b$ 's that contain exactly one  $a$ .

**Ans:** Let  $S$  be the set of all strings of  $a$ 's and  $b$ 's that contain exactly one  $a$ . The following is the recursive definition of the set  $S$ :

I. BASE:  $a \in S$

II. RECURSION: If  $s \in S$ , then,

a.  $sb \in S$

b.  $bs \in S$

III. RESTRICTION: Nothing is in  $S$  other than objects defined in I and II above.

25. Student C tries to define a function  $G : \mathbb{Z}^+ \rightarrow \mathbb{Z}$  by the rule

$$G(n) = \begin{cases} 1 & \text{if } n \text{ is } 1 \\ G\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ 2 + G(3n - 5) & \text{if } n \text{ is odd and } n > 1 \end{cases}$$

for all integers  $n \geq 1$ . Student D claims that  $G$  is not well defined. Justify D's claim.

**Ans:** Suppose  $G$  is a function, then by definition of  $G$ ,

$$G(1) = 1,$$

$$G(2) = G(1) = 1,$$

$$G(3) = 2 + G(4) = 2 + G(2) = 2 + 1 = 3,$$

$$G(4) = G(2) = 1,$$

However,

$$G(5) = 2 + G(10) = 2 + G(5).$$

Subtracting  $G(5)$  from both sides gives  $0 = 2$  which is false. Since supposing  $G$  is a function leads to a logically false statement, then  $G$  cannot be a function and Student D's claim is justified.

### Exercise Set 6.1

4. Let  $A = \{n \in \mathbb{Z} : n = 5r \text{ for some integer } r\}$  and  $B = \{m \in \mathbb{Z} : m = 20s \text{ for some integer } s\}$ .
- (a) Is  $A \subseteq B$ ? Explain.
  - (b) Is  $B \subseteq A$ ? Explain.

**Ans:**

- (a) No,  $\exists n \in A, n \notin B$ . For example  $5 \in A, 5 \notin B$
- (b) Yes.  $\forall m \in B, m \in A$ . Because  $m \in B$ , then by definition of set  $B$ ,  $m = 20s, s \in \mathbb{Z}$  or  $m = 5 \cdot 4s = 5l, l \in \mathbb{Z}$  and therefore  $m \in A$  by the definition of set  $A$ .

7. Let

$$A = \{x \in \mathbb{Z} : x = 6a + 4 \text{ for some integer } a\},$$

$$B = \{y \in \mathbb{Z} : y = 18b - 2 \text{ for some integer } b\},$$

and

$$C = \{z \in \mathbb{Z} : z = 18c + 16 \text{ for some integer } c\}.$$

Prove or disprove each of the following statements.

- (a)  $A \subseteq B$
- (b)  $B \subseteq A$
- (c)  $B = C$

**Ans:**

- (a) *Proof.* Suppose  $x$  is a particular but arbitrarily chosen element of  $A$ , then by definition  $x = 6a + 4$  where  $a \in \mathbb{Z}$ . Suppose  $a = 0$ , and so  $x = 4$  meaning that  $4 \in A$ . However,  $4 \notin B$  because there is no integer solution for the equation  $18b - 2 = 4$ . Therefore  $A \not\subseteq B$  because  $\exists x \in A, x \notin B$ .  $\square$
- (b) *Proof.* Suppose  $y$  is a particular but arbitrarily chosen element of  $B$ , then by definition  $y = 18b - 2$  where  $b \in \mathbb{Z}$ . Let  $a = 3b - 1 \in \mathbb{Z}$  because  $b \in \mathbb{Z}$  and the products of integers are integers and so is the difference of integers. Then,  $6a + 4 = 6(3b - 1) + 4 = 18b - 2 = y$ , and so by definition of  $A$ ,  $y$  is an element of  $A$ .  $\square$

(c) *Proof.* Yes. If  $B \subseteq C \wedge C \subseteq B \rightarrow B = C$

**1) Proving that  $B \subseteq C$  :**

Suppose  $y$  is a particular but arbitrarily chosen element of  $B$ , then by definition  $y = 18b - 2$  where  $b \in \mathbb{Z}$ . Let  $c = b - 1 \in \mathbb{Z}$  because the difference of integers is also an integer. Then  $18c + 16 = 18(b - 1) + 16 = 18b - 2 = y$ , and so by definition of  $C$ ,  $y$  is an element of  $C$ .

**2) Proving that  $C \subseteq B$  :**

Suppose  $z$  is a particular but arbitrarily chosen element of  $C$ , then by definition  $z = 18c + 16$  where  $c \in \mathbb{Z}$ . Let  $b = c + 1 \in \mathbb{Z}$  because the sum of integers is also an integer. Then  $18(c + 1) - 2 = 18c + 16 = z$ , and so by definition of  $B$ ,  $z$  is an element of  $B$ .

Because  $B \subseteq C \wedge C \subseteq B$  then  $B = C$ .

□

12. Let the universal set be the set  $\mathbb{R}$  of all real numbers and let

$$A = \{x \in \mathbb{R} : -3 \leq x \leq 0\},$$

$$B = \{x \in \mathbb{R} : -1 < x < 2\},$$

and

$$C = \{x \in \mathbb{R} : 6 < x \leq 8\}.$$

Find each of the following:

(a)  $A \cup B$

(b)  $A \cap B$

(c)  $A^c$

(d)  $A \cup C$

(e)  $A \cap C$

(f)  $B^c$

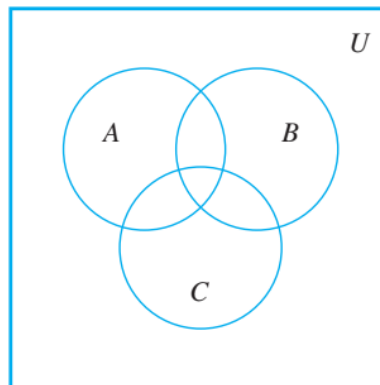
(g)  $A^c \cap B^c$

- (h)  $A^c \cup B^c$
- (i)  $(A \cap B)^c$
- (j)  $(A \cup B)^c$

**Ans:**

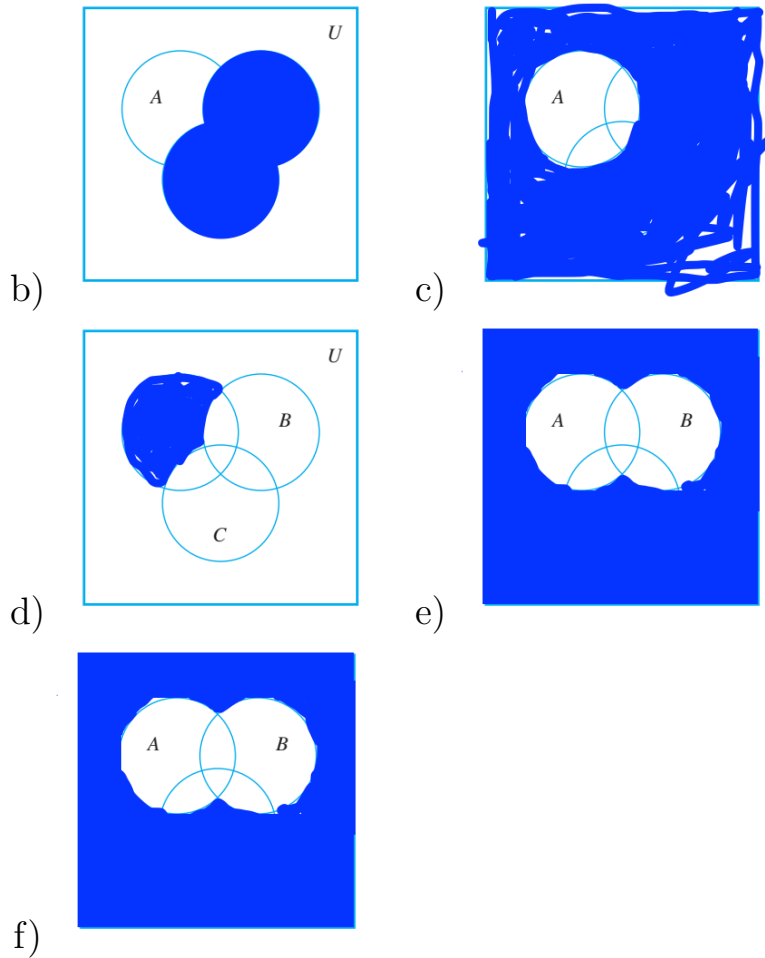
- (a)  $\{x \in \mathbb{R} : -3 \leq x < 2\}$
- (b)  $\{x \in \mathbb{R} : -1 < x \leq 0\}$
- (c)  $\{x \in \mathbb{R} : x < -3 \vee x > 0\}$
- (d)  $\{x \in \mathbb{R} : -3 \leq x \leq 0 \vee 6 < x \leq 8\}$
- (e)  $\{\}$
- (f)  $\{x \in \mathbb{R} : x \leq -1 \vee x \geq 2\}$
- (g)  $\{x \in \mathbb{R} : x < -3 \vee x \geq 2\}$
- (h)  $\{x \in \mathbb{R} : x \leq -1 \vee x > 0\}$
- (i)  $\{x \in \mathbb{R} : x \leq -1 \vee x > 0\}$
- (j)  $\{x \in \mathbb{R} : x < -3 \vee x \geq 2\}$

17. Consider the Venn diagram shown below. For each of (b)-(f), copy the diagram and shade the region corresponding to the indicated set.



- (b)  $B \cup C$
- (c)  $A^c$
- (d)  $A - (B \cup C)$
- (e)  $(A \cup B)^c$
- (f)  $A^c \cap B^c$

**Ans:**



20. Let  $B_i = \{x \in \mathbb{R} : 0 \leq x \leq i\}$  for all integers  $i = 1, 2, 3, 4$ .

- (a)  $B_1 \cup B_2 \cup B_3 \cup B_4 = ?$
- (b)  $B_1 \cap B_2 \cap B_3 \cap B_4 = ?$



(c) Are  $B_1, B_2, B_3$ , and  $B_4$  mutually disjoint? Explain?

**Ans:**

(a)  $\{x \in \mathbb{R} : 0 \leq x \leq 4\}$

(b)  $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$

(c) No, they are not mutually disjoint. There is at least one set  $B_i \cap B_j$  where  $i \neq j$  that are not empty. For example,  $B_1 \cap B_4 = \{0 \leq x \leq 1\}$ .

30. Let  $\mathbb{Z}$  be the set of all integers and let

$$A_0 = \{n \in \mathbb{Z} : n = 4k, \text{ for some integer } k\},$$

$$A_1 = \{n \in \mathbb{Z} : n = 4k + 1, \text{ for some integer } k\},$$

$$A_2 = \{n \in \mathbb{Z} : n = 4k + 2, \text{ for some integer } k\},$$

$$A_3 = \{n \in \mathbb{Z} : n = 4k + 3, \text{ for some integer } k\},$$

Is  $\{A_0, A_1, A_2, A_3\}$  a partition of  $\mathbb{Z}$ ? Explain your answer.