Multi-variable Calculus
Midterm I

October 21, 2024

Fall 2024

12.1 Functions of Two Variables

- Distance between point s (x, y, z) and a, b, c in 3-space

$$= \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$$

- Function notation: $f(x, y)$ is the value of the function f with inputs x and y
- Points in 3-space are specified by their coordinates relative to the x, y, z-axes
- Some planes in 3-space can be specified by simple equations given by one variable equal to a constant. For example, the xy-plane (where $z = 0$), the xz-plane where $(y = 0)$, and the yz-plane (where $x = 0$)

12.2 Graphs & Surfaces

- The graph of a function of two variables, f, is the set of all points (x, y, z) such that $z = f(x, y)$. In general the graph of a function of two variables is a surface in 3-space.
- For a function $f(x, y)$, the function we get by holding x fixed and letting y vary is called a **cross-section** of f with x fixed. The graph of the cross-section of $f(x, y)$ with $x = c$ is the curve or cross-section, we get by intersecting the graph of f with the plane $x = c$. We define a cross-section of f with y fixed similarly.
- Simple changes to a function, such as adding or multiplying by a constant, affect the graph by shifting, stretching or flipping, just as for functions of one variable.
- A cross-section of a function $f(x, y)$ is the one-variable function obtained by setting x or y equal to a constant
- A cylinder is the result of having one of the variables unspecified, such as $f(x, y) = x^2$

12.3 Contour Diagrams

- Contour diagrams are used to represent functions of two variables as they are difficult to see function behavior from a surface

- Contour lines, or level curves, are obtained from a surface by slicing it with horizontal planes. A contour diagram is a collection of level curves labeled with function values.
- A contour of the function $f(x, y)$ is the set of poiunts in the xy-plane satisfying $f(x, y) =$ constant. Contours can be thought of as horizontal slices of the graph of a function at a particular height
- A contour diagram for a function $f(x, y)$ is a graph of several contours for a selction of constants
- In a contour diagram with equally-spaced function values, contours that are closer together represent more rapid change of the function
- To find a contour algebraically, set the formula for $f(x, y)$ equal to a constant
- Sometimes contours can be seen numerically in a table of values by seeing where the same values occur in the table
- A Cobb-Douglas production function has the form

$$f(N, V) = cN^{\alpha}V^{\beta}$$

12.4 Linear Functions

- If a plane has slope m in the x-direction, has slope n in the y-direction, and passes through the point (x_0, y_0, z_0), then its equation is

$$z = z_0 + m(x - x_0) + n(y - y_0)$$

  This plane is the graph of the linear function

$$f(x, y) = z_0 + m(x - x_0) + n(y - y_0)$$

  If we write $c = z_0 - mx_0 - ny_0$, then we can write $f(x, y)$ in the equivalent form

$$f(x, y) = c + mx + ny$$

- A linear function can be recognized from its table by the following features:
  - Each row and each column is linear
  - All the rows have the same slope.
  - All the columns have the same slope. (although the slope of the rows and the slope of the columns are generally different)
- Contours of linear functions are parallel lines, evenly spaced.

12.5 Functions of Three Variables

- A level surface, or level set of a function of three variables, $f(x, y, z)$, is a surface of the form $f(x, y, z) = c$, where c is a constant. The function f can be represented by the family of level surfaces obtained by allowing c to vary.

- Catalog of surfaces

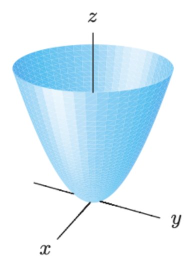

Figure 12.72: Elliptical paraboloid $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

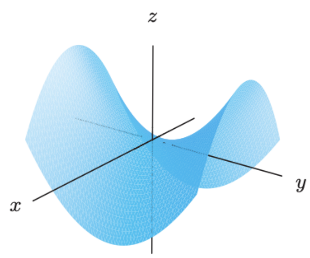

Figure 12.73: Hyperbolic paraboloid $z = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$

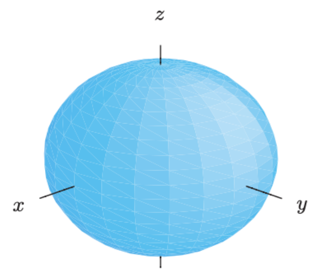

Figure 12.74: Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

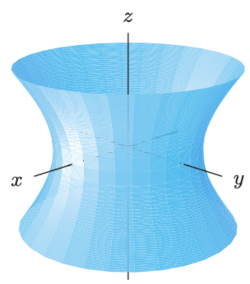

Figure 12.75: Hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

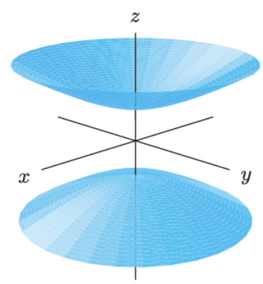

Figure 12.76: Hyperboloid of two sheets $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

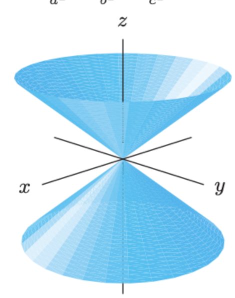

Figure 12.77: Cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

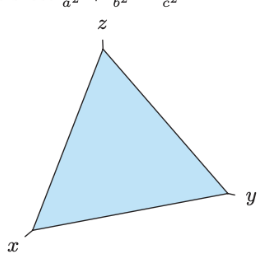

Figure 12.78: Plane $ax + by + cz = d$

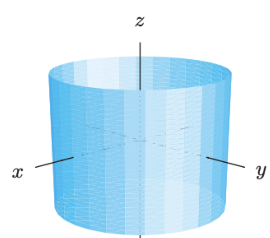

Figure 12.79: Cylindrical surface $x^2 + y^2 = a^2$

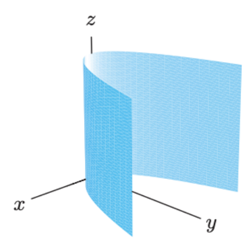

Figure 12.80: Parabolic cylinder $y = ax^2$

- A single surface that is the graph of a two-variable function $f(x, y)$ can be thought of as one member of the family of level surfaces representing the three-variable function

$$g(x, y, z) = f(x, y) - z$$

  The graph of f is the level surface $g = 0$

- Every surface that is the graph of a function $z = g(x, y)$ can be rewritten as a level surface by writing $f(x, y, z) = g(x, y) - z = 0$.

- Not every level surface $f(x, y, z) = c$ can be rewritten as the graph of a function $z = g(x, y)$. That is, level surfaces of 3-variable functions can describe more surfaces that can be described as graphs of 2-variable functions $z = g(x, y)$.

13.1 Displacement Vectors

- The displacement vector from one point to another is an arrow with its tail at the first point and its tip at the second. The magnitude (or length) of the displacement vector is the distance between the points and is represented by the length of the arrow. The direction of the displacement vector is the direction of the arrow.

- Displacement vectors which point in the same direction and have the same magnitude are considered to be the same, even if they have the same magnitude are considered to be the same, even if they do not coincide.

- The sum, $\vec{v} + \vec{w}$, of two vectors $\vec{v}$ and $\vec{w}$ is the combined displacement resulting from first applying $\vec{v}$ and then $\vec{w}$. The sum $\vec{w} + \vec{v}$ gives the same displacement.

- The difference, $\vec{w} - \vec{v}$, is the displacement vector that, when added to $\vec{w} = \vec{v} + (\vec{w} - \vec{v})$, gives $\vec{w}$. That is, $\vec{w} = \vec{v} + (\vec{w} - \vec{v})$

- The zero vector, $\vec{0}$, is a displacement vector with zero length

Second Order Partials

- Recall in Calc 1, given $y = f(x)$
  $\frac{dy}{dx} = f'(x)$
  $\frac{d^2y}{dx^2} = f''(x)$

- Can we compute 2nd derivatives for $z = f(x, y)$?

- $$\frac{\partial \left[\frac{\partial f}{\partial x}\right]}{\partial x} = \frac{\partial^2 f}{\partial x^2} = (f_x)_x = f_{xx}$$

- $$\frac{\partial \left[\frac{\partial f}{\partial y}\right]}{\partial y} = \frac{\partial^2 f}{\partial y^2} = (f_y)_y = f_{yy}$$

- $$\frac{\partial \left[\frac{\partial f}{\partial x}\right]}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy}$$

- $$\frac{\partial \left[\frac{\partial f}{\partial y}\right]}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx}$$

- Fact Let $P = (a, b)$, then,

$$f_{xy}(P) = f_{yx}(P) \text{ if } f \text{ is smooth at P}$$

- Assume f is smooth unless the question says otherwise
- Quadratic Approximation (also called the second-order Taylor polynomial) of a function $z = f(x, y)$ at a point $P = (a, b)$ is given by:

$$Q_{(a,b)} = L_{(a,b)}(x, y) + \frac{1}{2} f_{xx}(a, b)(x - a)^2 + f_{xy}(a, b)(x - a)(y - b) + \frac{1}{2} f_{yy}(a, b)(y - b)^2$$

$$L_{a,b}(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Taylor polynomial of a polynomial is a polynomial itself

Critical Points & 2nd Derivative Test

- In Calc I, you found & classified critical points
- How do we find critical points for $f(x, y)$?
- $P = (a, b)$ is critical point for $f(x, y)$ IF $f_x(P) = 0$ AND $f_y(P) = 0$
- NOTE: P is critical point of $f(x, y)$ if $\nabla f(P) =< 0, 0 >$
- Example

$$f(x, y) = x^2 - 2x + y^2 - 4y + 5$$
$$f_x = 2x - 2$$
$$f_y = 2y - 4$$

Set $f_x = 0$ and $f_y = 0$

$$2x - 2 = 0 \Longrightarrow x = 1$$
$$2y - 4 = 0 \Longrightarrow y = 2$$

We found a critical point $P(1, 2)$ but what is happening at P? We need to employ the 2nd derivative test

- Let $P = (a, b)$ be a critical point for $f(x, y)$
  Put D = $f_{xx}(P) \cdot f_{yy}(P) - [f_{xy}(P)]^2$ - Discriminant
- 1) If $D > 0$ and $f_{xx}(P) < 0$, then f has a local maximum at P
- 2) If $D > 0$ and $f_{xx}(P) > 0$, then f has a local minimum at P
- 3) If $D < 0$ and then f has a saddle point at P

- 4) If $D = 0$, then test is inconclusive and we need to try other methods.
  Going back to our example we see that

$$D = f_{xx}(P) \cdot f_{yy}(P) - [f_{xy}(P)]^2$$

$$D = 2 \cdot 2 - 0^2$$

$$D > 0 \ \ \& \ \ f_{xx}(P) > 0$$

  Since $D > 0$ and $f_{xx} > 0$, then by the 2nd derivative test $(1, 2)$ is a local minimum.

Optimization
Steps: Let $f(x, y)$ on a given region R.

1. Find critical points.... Set $\nabla f(x, y) =< 0, 0 >$
2. Check if critical points are in R
3. Classify critical points using the 2nd derivative test
4. *Check the values along the boundary ∂R, i.e. find boundary points
   NOTE: A boundary may or may not be given. If a boundary is not given, then you can assume R is the entire xy-plane
5. Compare function output values in region R, and if possible, on the boundary (smallest output $\rightarrow$ global minimum, largest output $\rightarrow$ global maximum)

Example: Find global maximum & global minimum on the region given $-1 \leq x \leq 1$ & $-1 \leq y \leq 1$ for $f(x, y) = x^2 - y^2$

1. $\nabla f(x, y) =< 2x, -2y >$
   Set $< 2x, -2y >=< 0, 0 >$
   $2x = 0$ & $2y = 0$
   $x = 0$ & $y = 0$
   $P = (0, 0)$ is a critical point
2. P is in R (because of range for x and y)
3. $f_{xx} = 2$
   $f_{yy} = -2$
   $f_{xy} = 0$
   $D < 0$ so P=(0,0) is a saddly by second derivative test
4. There is a boundary
   How do we find boundary points?
   Case 1: Let $x = \pm 1$

$f(x,y) = 1 - y^2, -1 \leq y \leq 1$
Optimize $1 - y^2$ for $-1 \leq y \leq 1$
Local maximum at $y = 0$
Boundary points (1,0), (-1,0)
Case 2: Let $y = \pm 1$
$f(x,y) = x^2 - 1, -1 \leq x \leq 1$
Optimize $x^2 - 1$ for $-1 \leq x \leq 1$
Local minimum at $x = 0$
Boundary points $(0,1)$, $(0,-1)$

5. Compute $f(0,0), f(1,0), f(-1,0), f(0,1), f(0,-1)$
   Global maximum is at (1,0) and (-1,0)
   Global minimum is at (0,1) and (0,-1)