Discrete Mathematics
Chapters 5.9,6.1 & 6.2 Homework

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Exercise Set 5.9

6. Define a set S recursively as follows:

   I. BASE: $a \in S$

   II. RECURSION: If $s \in S$, then,
   a. $sa \in S$ b. $sb \in S$

   III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

   Use structual induction to prove that every string in S begins with an a.

> **Ans:**
>
> *Proof.* Given any string $s \in S$, let the property be the claim that it begins with an a.
> ***Show that each object in the BASE case for S satisfies the property:*** The only object in the base case is a which satisfies the property as it begins with a.
> ***Show that for each rule in the RECURSION for S, if the rule is applied to an object in S that satisfies the property, then the objects defined by the rule also satisfy the property:*** The recursion for S consists of two rules denoted II(a) and II(b). Suppose s is a string in S that begins with a. In the case where the rule II(a) is applied to s, the result is the string sa, which also begins with a. In the case where the rule II(b) is applied to s, the result is the string sb, which also begins with a.

Thus when each rule in the RECURSION is applied to a string in S that begins with a, the result is also a string that begins with a. $\square$

11. Define a set S recursively as follows:

    I. BASE: $0 \in S$

    II. RECURSION: If $s \in S$, then,
    a. $s + 3 \in S$ b. $s - 3 \in S$

    III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

    Use structual induction to prove that integer in S is divisible by 3.

Ans:

Proof. Given any integer $s \in S$, let the property be the claim that it is divisibly by 3.
Show that each object in the BASE case for S satisfies the property: The only object in the base case is 0 which satisfies the property because by definition $3 \mid 0, \exists k \in \mathbb{Z}, 0 = 3k$.
Show that for each rule in the RECURSION for S, if the rule is applied to an object in S that satisfies the property, then the objects defined by the rule also satisfy the property: The recursion for S consists of two rules denoted II(a) and II(b). Suppose s is an integer in S that is divisible by 3. Then by definition $\exists l \in \mathbb{Z}, s = 3l$. In the case where the rule II(a) is applied to s, the result is the integer $s+3 = 3l+3 = 3(l+1)$, which is also divisible by 3 by definition because $\exists m \in \mathbb{Z}, s+3 = 3(l+1)$ where $m = l+1$. In the case where the rule II(b) is applied to s, the result is the integer $s - 3 = 3l - 3 = 3(l - 1)$, which is also divisible by 3 by defnition because $\exists n \in \mathbb{Z}, s - 3 = 3(l - 1)$ where $n = l - 1$. Thus when each rule in the RECURSION is applied to

an integer s that is divisible by 3, the result is also an integer that is divisible by 3. $\square$

16. Give a recursive definition for the set of all strings of 0's and 1's for which all the 0's precede all the 1's.

Ans: Let S be the set of all strings of 0's and 1's for which all the 0's precede all the 1's. The following is the recursive definition of the set S:

I. BASE: $0 \in S$

II. RECURSION: If $s \in S$, then,

  a. $s1 \in S$
  b. $0s \in S$

III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

18. Give a recursive definition for the set of all strings of a's and b's that contain exactly one a.

Ans: Let S be the set of all strings of a's and b's that contain exactly one a. The following is the recursive definition of the set S:

I. BASE: $a \in S$

II. RECURSION: If $s \in S$, then,

  a. $sb \in S$
  b. $bs \in S$

III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

25. Student C tries to define a function $G : \mathbb{Z}^+ \to \mathbb{Z}$ by the rule

$$G(n) = \begin{cases} 1 & \text{if } n \text{ is } 1 \\ G\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ 2 + G(3n-5) & \text{if } n \text{ is odd and } n > 1 \end{cases}$$

for all integers $n \geq 1$. Student D claims that G is not well defined. Justify student D's claim.

Ans: Suppose G is a function, then by definition of G,

$$\begin{aligned} G(1) &= 1, \\ G(2) &= G(1) = 1, \\ G(3) &= 2 + G(4) = 2 + G(2) = 2 + 1 = 3, \\ G(4) &= G(2) = 1, \end{aligned}$$

However,

$$G(5) = 2 + G(10) = 2 + G(5).$$

Subtracting $G(5)$ from both sides gives $0 = 2$ which is false. Since supposing G is a function leads to a logically false statement, then G cannot be a function and student D's claim is justified.

Exercise Set 6.1

4. Let $A = \{n \in \mathbb{Z} : n = 5r \text{ for some integer } r\}$ and $B = \{m \in \mathbb{Z} : m = 20s \text{ for some integer } s\}$.
   (a) Is $A \subseteq B$? Explain.
   (b) Is $B \subseteq A$? Explain.

Ans:

(a) No, $\exists n \in A, n \notin B$. For example $5 \in A, 5 \notin B$

(b) Yes. $\forall m \in B, m \in A$. Because $m \in B$, then by definition of set B, $m = 20s, s \in \mathbb{Z}$ or $m = 5 \cdot 4s = 5l, l \in \mathbb{Z}$ and therefore $m \in A$ by the definition of set A.

7. Let

$$A = \{x \in \mathbb{Z} : x = 6a + 4 \text{ for some integer } a\},$$
$$B = \{y \in \mathbb{Z} : y = 18b - 2 \text{ for some integer } b\},$$

and

$$C = \{z \in \mathbb{Z} : z = 18c + 16 \text{ for some integer } c\}.$$

Prove or disprove each of the following statements.

(a) $A \subseteq B$

(b) $B \subseteq A$

(c) $B = C$

Ans:

(a) *Proof.* Suppose x is a particular but arbitrarily chosen element of A, then by definition $x = 6a + 4$ where $a \in \mathbb{Z}$. Suppose $a = 0$, and so $x = 4$ meaning that $4 \in A$. However, $4 \notin B$ because there is no integer solution for the equation $18b-2 = 4$. Therefore $A \nsubseteq B$ because $\exists x \in A, x \notin B$. □

(b) *Proof.* Suppose y is a particular but arbitrarily chosen element of B, then by definition $y = 18b - 2$ where $b \in \mathbb{Z}$. Let $a = 3b - 1 \in \mathbb{Z}$ because $b \in \mathbb{Z}$ and the products of integers are integers and so is the difference of integers. Then, $6a + 4 = 6(3b - 1) + 4 = 18b - 2 = x$, and so by definition of A, x is an element of A. □

(c) *Proof.* Yes. If $B \subseteq C \wedge C \subseteq B \rightarrow B = C$

1) Proving that $B \subseteq C$:

Suppose y is a particular but arbitrarily chosen element of B, then by definition $y = 18b - 2$ where $b \in \mathbb{Z}$. Let $c = b - 1 \in \mathbb{Z}$ because the difference of integers is also an integer. Then $18c + 16 = 18(b-1) + 16 = 18b - 2 = y$, and so by definition of C, y is an element of C.

2) Proving that $C \subseteq B$:

Suppose z is a particular but arbitrarily chosen element of C, then by definition $z = 18c + 16$ where $c \in \mathbb{Z}$. Let $b = c + 1 \in \mathbb{Z}$ because the sum of integers is also an integer. Then $18(c+1) - 2 = 18c + 16 = z$, and so by definition of B, z is an element of B.

Because $B \subseteq C \wedge C \subseteq B$ then $B = C$. $\square$

12. Let the universal set be the set $\mathbb{R}$ of all real numbers and let

$$A = \{x \in \mathbb{R} : -3 \leq x \leq 0\},$$
$$B = \{x \in \mathbb{R} : -1 < x < 2\},$$

and

$$C = \{x \in \mathbb{R} : 6 < x \leq 8\}.$$

Find each of the following:

(a) $A \cup B$

(b) $A \cap B$

(c) A^c

(d) $A \cup C$

(e) $A \cap C$

(f) B^c

(g) $A^c \cap B^c$

(h) $A^c \cup B^c$

(i) $(A \cap B)^c$

(j) $(A \cup B)^c$

Ans:

(a) $\{x \in \mathbb{R} : -3 \leq x < 2\}$

(b) $\{x \in \mathbb{R} : -1 < x \leq 0\}$

(c) $\{x \in \mathbb{R} : x < -3 \vee x > 0\}$

(d) $\{x \in \mathbb{R} : -3 \leq x \leq 0 \vee 6 < x \leq 8\}$

(e) $\{\}$

(f) $\{x \in \mathbb{R} : x \leq -1 \vee x \geq 2\}$

(g) $\{x \in \mathbb{R} : x < -3 \vee x \geq 2\}$

(h) $\{x \in \mathbb{R} : x \leq -1 \vee x > 0\}$

(i) $\{x \in \mathbb{R} : x \leq -1 \vee x > 0\}$

(j) $\{x \in \mathbb{R} : x < -3 \vee x \geq 2\}$

17. Consider the Venn diagram shown below. For each of (b)-(f), copy the diagram and shade the region corresponding to the indicated set.

(b) $B \cup C$

(c) A^c

(d) $A - (B \cup C)$

(e) $(A \cup B)^c$

(f) $A^c \cap B^c$

Ans:
b) c)
d) e)
f)

20. Let $B_i = \{x \in \mathbb{R} : 0 \leq x \leq i\}$ for all integers $i = 1, 2, 3, 4$.
    (a) $B_1 \cup B_2 \cup B_3 \cup B_4 =?$
    (b) $B_1 \cap B_2 \cap B_3 \cap B_4 =?$
    (c) Are B_1, B_2, B_3, and B_4 mutually disjoint? Explain?

Ans:
(a) $\{x \in \mathbb{R} : 0 \leq x \leq 4\}$
(b) $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$
(c) No, they are not mutually disjoint. There is at least one set $B_i \cap B_j$ where $i \neq j$ that is not empty. For example, $B_1 \cap B_4 = \{0 \leq x \leq 1\}$.

30. Let $\mathbb{Z}$ be the set of all integers and let

$$A_0 = \{n \in \mathbb{Z} : n = 4k, \text{ for some integer } k\},$$
$$A_1 = \{n \in \mathbb{Z} : n = 4k + 1, \text{ for some integer } k\},$$
$$A_2 = \{n \in \mathbb{Z} : n = 4k + 2, \text{ for some integer } k\},$$
$$A_3 = \{n \in \mathbb{Z} : n = 4k + 3, \text{ for some integer } k\},$$

Is $\{A_0, A_1, A_2, A_3\}$ a partition of $\mathbb{Z}$? Explain your answer.

Ans: For $\{A_0, A_1, A_2, A_3\}$ to be a partition of $\mathbb{Z}$ then, by definition, the following has to be true:

1. $\mathbb{Z} = A_0 \cup A_1 \cup A_2 \cup A_3$

2. The sets A_0, A_1, A_2 and A_3 are mutually disjoint.

Yes. By the quotient remainder theorem, when an integer n is divided by 4 there exists unique integers q and r such that $n = 4q + r$ where $0 \leq r \leq 4$. Clearly, the only integer values for r are $0, 1, 2, 3$, and 4. Therefore n can be written in exactly one of the following forms $n = 4q$ or $n = 4q + 1$ or $n = 4q + 2$ or $n = 4q+3$. Because k and q are both some integers we can say that $k = q$ and therefore each unique representation of n corresponds to A_0, A_1, A_2, A_3 respectively. This means that there is no integer n that is in both sets A_i and A_j whenever $i \neq j$ and so the sets sets A_0, A_1, A_2 and A_3 are mutually disjoint. Moreover, because any integer n can be written in exactly one of the forms $n = 4k$ or $n = 4k + 1$ or $n = 4k + 2$ or $n = 4k + 3$ then the union of the sets must be equal to $\mathbb{Z}$.

Exercise Set 6.2

14. For all sets A, B, and C, if $A \subseteq B$ then $A \cup C \subseteq B \cup C$

Ans:

Proof. Suppose A, B, and C are sets and $A \subseteq B$. Let $x \in A \cup C$ and so by definition of union $x \in A$ or $x \in C$. If $x \in A$ then, by $A \subseteq B, x \in B$ and consequently $x \in B \cup C$. On the other hand, if $x \in C$ then $x \in B \cup C$. Therefore in the two cases all the elements $x \in A \cup C$ are also in $B \cup C$. □

15. For all sets A and B, if $A \subseteq B$ then $B^c \subseteq A^c$.

Ans:

Proof. Suppose A and B are sets and that $A \subseteq B$ and also suppose, for the sake of contradiction, that $B^c \not\subseteq A^c$. Because $B^c \not\subseteq A^c$ then there must be an element $x \in B^c$ such that $x \notin A^c$. But by definition of complement $x \in B^c \Leftrightarrow x \notin B$ and $x \notin A^c$ is the negation of $x \in A^c$ which is equivalent to $x \in A$. But from the definition of $A \subseteq B$ we have that $\forall x \in A, x \in B$. So that means there is an element x such that $x \notin B$ and $x \in B$ which is a contradiction meaning that our supposition is false and the original statement is true. □

26. For all sets, A, B, and C,

$$(A - C) \cap (B - C) \cap (A - B) = \phi$$

Ans:

Proof. Suppose, for the sake of contradiction, that there is an element $x \in (A - C) \cap (B - C) \cap (A - B)$

$$\begin{aligned}
&x \in (A \cap C^c) \cap (B \cap C^c) \cap (A \cap B^c) && \text{(By definition of difference)}\\
&x \in (A \cap C^c) \cap (C^c \cap B) \cap (B^c \cap A) && \text{(By commutativity)}\\
&x \in (A \cap C^c) \cap C^c \cap (B \cap B^c) \cap A && \text{(By associativity)}\\
&x \in (A \cap C^c) \cap C^c \cap \phi \cap A && \text{(By complement)}\\
&x \in (A \cap C^c) \cap C^c \cap \phi && \text{(By universal bound)}\\
&x \in (A \cap C^c) \cap \phi && \text{(By universal bound)}\\
&x \in A \cap (C^c \cap \phi) && \text{(By associativity)}\\
&x \in A \cap \phi && \text{(By universal bound)}\\
&x \in \phi && \text{(By universal bound)}
\end{aligned}$$

This means that we have found an element x that is in the empty set ϕ. This contradicts the definition of ϕ and thus our supposition is false and the original statement must be true. □

41. For all integers $n \geq 1$, if A and $B_1, B_2, B_3, ...$ are any sets, then

$$\bigcap_{i=1}^{n}(A \times B_i) = A \times \left(\bigcap_{i=1}^{n} B_i\right)$$

Ans:

Proof. Suppose A and $B_1, B_2, B_3, ...$ are any sets.

$$\textbf{Proving that: } \bigcap_{i=1}^{n}(A \times B_i) \subseteq A \times \left(\bigcap_{i=1}^{n} B_i\right) :$$

Suppose (x, y) is any element in $\bigcap_{i=1}^{n}(A \times B_i)$. By definition of general intersection, $(x, y) \in A \times B_i$ for all $i = 1, 2, ..., n$. By definition of Cartesian product, this implies that (1) $x \in A$ and (2) $y \in B_i$ for all $i = 1, 2, ..., n$. By definition of general intersection (2) implies that $y \in \bigcap_{i=1}^{n} B_i$. Thus $x \in A$ and $y \in \bigcap_{i=1}^{n} B_i$, and so by definition of Cartesian product, $(x, y) \in A \times (\bigcap_{i=1}^{n} B_i)$

$$\textbf{Proving that: } A \times \left(\bigcap_{i=1}^{n} B_i\right) \subseteq \bigcap_{i=1}^{n}(A \times B_i) :$$

Suppose (x, y) is any element in $A \times (\bigcap_{i=1}^{n} B_i)$. By definition of Cartesian product, (1) $x \in A$ and (2) $y \in \bigcap_{i=1}^{n} B_i$. By definition of general intersection (2) implies that $y \in B_i$ for all $i = 1, 2, 3, ..., n$. Thus $x \in A$ and $y \in B_i$ for all $i = 1, 2, 3, ..., n$, and so, by definition of Cartesian product, $(x, y) \in A \times B_i$ for all $i = 1, 2, 3, ..., n$. It follows from the definition of general intersection that $(x, y) \in \bigcap_{i=1}^{n} (A \times B_i)$

Conclusion:

Since both containments have been proved, it follows by definition of set equality that $\bigcap_{i=1}^{n}(A \times B_i) = A \times (\bigcap_{i=1}^{n} B_i)$. $\square$