Discrete Mathematics
Chapter 3 Homework

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1. A menagerie consists of seven brown dogs, two black dogs, six gray cats, ten black cats, five blue birds, six yellow birds, and one black bird. Determine which of the following statements are true and which are false.
   (a) Every animal in the menagerie is brown or gray or black.
   (b) There is an animal in the menagerie that is neither a cat nor a dog.
   (c) No animal in the menagerie is blue.
   (d) There are in the menagerie a dog, a cat, and a bird that all have the same color.

Ans:
(a) False. There is a blue bird.
(b) True.
(c) False. There is a blue bird.
(d) True.

2. Find a counterexample to show that the statement below is false.

$$\forall a \in \mathbb{Z}, (a-1)/a \text{ is not an integer.}$$

Ans: The negation of the statement would be $\exists a \in \mathbb{Z}, (a-1)/a$ is an integer. If $a = 0$, $a - 1/1 = 0$ which is $\in \mathbb{Z}$. Since the negation is true for $a = 0$ the original statement is false.

3. Rewrite each of the following statements in the form "$\forall$ _____ x, _____.
   (a) Every real number is positive, negative, or zero.

(b) No logicians are lazy.

(c) The number -1 is not equal to the square of any real number.

Ans: A3. (double check a)

(a) $\forall$ real numbers $x, x > 0 \vee x < 0 \vee x = 0$

(b) $\forall$ logicians l, l is not lazy

(c) $\forall$ real numbers $x, x^2 \neq -1$

4. In any mathematics or computer science text other than this book, find an example of a statement that is universal but is implicitly quantified. Copy the statement as it appears and rewrite it making the quantification explicit. Give a complete citation for your example, including title, author, publisher, year, and page number.

Ans: Theorem 9.1: Convergence of a Monotone, Bounded Sequence
If a sequence s_n is bounded and monotone, it converges.

For every sequence s_n, if s_n is bounded and monotone, s_n converges.

Citation: Calculus 6th edition by Hughes-Hallet, Gleaason, McCallum et al., Wiley & Sons, Inc., 2013, page 495

5. Let $\mathbb{R}$ be the domain of the predicate variable x. Which of the following are true and which are false? Give counterexamples for the statements that are false.

(a) $ab = 0 \implies a = 0$ or $b = 0$

(b) $a < b$ and $c < d \implies ac < bd$

Ans:

(a) True.

(b) False.
Example (1)
$a = -2, b = -1, c = -5, d = -3$
$a < b$ and $c < d$
$ac \geq bd$

Example (2)
$a = 1, b = 2, c = -5, d = -4$
$a < b$ and $c < d$
$ac \geq bd$

6. Which of the following is a negation for "All dogs are loyal"? More than one answer may be correct.
   (a) All dogs are loyal.
   (b) No dogs are loyal.
   (c) Some dogs are disloyal.
   (d) Some dogs are loyal.
   (e) There is a disloyal animal that is not a dog.
   (f) There is a dog that is disloyal.
   (g) No animals that are not dogs are loyal.
   (h) Some animals that are not dogs are loyal.

Ans: (c) and (f)

7. Determine whether the proposed negations is correct. If it is not, write a correct negation.

   (a) Statement: The product of any irrational number and any rational number is irrational.
   Proposed negation: The product of any irrational number and any rational number is rational.

   (b) Statement: For all real numbers x_1 and x_2, if $x_1^2 = x_2^2$ then $x_1 = x_2$.
   Proposed negation: For all real numbers x_1 and x_2, if $x_1^2 = x_2^2$ then $x_1 \neq x_2$.

Ans:

(a) Incorrect.
There is a rational number that is the product of an irrational number and a rational number.

(b) Incorrect.
There exists real numbers x_1 and x_2 such that $x_1^2 = x_2^2$ and $x_1 \neq x_2$

8. Write a negation for each statement.
   (a) $\forall n \in \mathbb{Z}$, if n is prime then n is odd or $n = 2$
   (b) $\forall$ integers n, if n is divisible by 6, then n is divisible 2 and n is divisible by 3

Ans:

(a) $\exists n \in \mathbb{Z}$, such that n is prime and n is even and $n \neq 2$

(b) $\exists$ an integer n, such that n is divisible by 6 and n is not divisible by 2 or n is not divisible by 3

9. Give an example to show that a universal conditional statement is not logically equivalent to its inverse.

Ans:

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x) \not\equiv \forall x \in D, \text{ if } \neg P(x) \text{ then } \neg Q(x)$$

For all animals x, if x is a dog then it can be a pet
The inverse of the above statement would be:
For all animals x, if x is not a dog then it cannot be a pet
This is false as there are pets that are not dogs!

10. There is a triangle x such that for all circles y, y is above x. (Tarski World in Figure 3.3.1)

Ans: $\exists x$, x is a triangle such that $\forall y$, y is a circle, y is above x

choose x	, then given $y =$	check that y is above x
f or i	a	yes
	b	yes
	c	yes

For triangles f or i, no matter what circle y we choose, we find that the circle is above the triangle.

For the following questions, (a) rewrite the statement in English without using the symbol $\forall$ or $\exists$ or variables and expressing your answer as simply as possible, and (b) write a negation for the statement.

11. $\exists$ a real number u such that $\forall$ real numbers $v, uv = v$.

Ans:

(a) There is at least one real number such that when it is multiplied by any other real number, the result is the other real number.

(b) $\forall$ real numbers u, $\exists$ a real number v such that $uv \neq v$
For all real numbers there is at least one other real number such that their product is not equal to the other real number.

12. $\exists x \in \mathbb{R}^+$ such that $\forall y \in \mathbb{R}^+, x \leq y$

Ans:

(a) There is at least one positive real number that is less than or equal to all the other positive real numbers

(b) $\forall x \in \mathbb{R}^+, \exists y \in \mathbb{R}^+$ such that $x > y$
For any real positive number we can find another real positive number that is lower than it

13. Use the laws for negating universal and existential statements to derive the following rule:

$$\neg(\exists x \in D(\exists y \in E(P(x,y)))) \equiv \forall x \in D(\forall y \in E(\neg(P(x,y))))$$

Ans: $\neg(\exists x \in D(\exists y \in E(P(x,y))))$
$\forall x \in D(\neg(\exists y \in E(P(x,y))))$
$\forall x \in D(\forall y \in E(\neg(P(x,y))))$

14. The following is the definition for $\lim_{x \to a} f(x) = L$:

For all real numbers $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for all real numbers x, if $a - \delta < x < a + \delta$ and $x \neq a$ then $L - \varepsilon < f(x) < L + \varepsilon$.

Write what it means for $\lim_{x\to a} f(x) \neq L$. In other words, write the negation of the definition.

Ans: There exists a real number $\varepsilon > 0$ such that for all real numbers $\delta > 0$ there exists a real number x such that $x \leq a-\delta$ or $x \geq a+\delta$ or $x = a$ and $f(x) \leq L-\varepsilon$ or $f(x) \geq L+\varepsilon$

Some of the follwoing arguments are valid by universal modus ponens or universal modus tollens; others are invalid and exhibit the converse or the inverse error. State which are valid and which are invalid. Justify your answers

15. All honest people pay their taxes.
    Darth is not honest.
    $\therefore$ Darth does not pay his taxes.

Ans: Let $H(p)$ be p is honest and $T(p)$ be p pays their taxes.
$\forall$ people p, if $H(p)$ then $T(p)$
$\neg H(d)$ for a particular person, Darth
$\therefore \neg T(d)$
The argument is invalid as it exhibits the inverse error.

16. If a compilation of a computer program produces error messages, then the program is not correct.
    Compilation of this program does not produce error messages.
    $\therefore$ This program is correct.

Ans: Let $P(x)$ be x is a compilation of that produces error messages and $Q(x)$ be x is not correct.
$\forall x$, if $P(x)$ then $Q(x)$
$\neg P(x)$ for a particular computer program
$\therefore \neg Q(x)$
The argument is invalid as it exhibits the inverse error.

17. If an infinite series converges, then its terms go to 0.
    The terms of the infinite series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ do not go to 0.
    $\therefore$ The infinite series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ does not converge.

Ans: Let $P(x)$ be x is an infinite series that converges and $Q(x)$ be x is a series whose terms go to 0.
$\forall x$, if $P(x)$ then $Q(x)$
$\neg Q(x)$ for a particular infinite series
$\therefore \neg P(x)$
The argument is valid by modus tollens.

18. Derive the validity of universal modus tollens from the validity of universal instantiation and modus tollens.

Ans: Given:
1. $p \longrightarrow q$
$\neg q$
$\therefore \neg p$
2. $\forall x \in D, P(x)$
$x \in D$
$\therefore P(x)$
To be found:
$\forall x, P(x) \longrightarrow Q(x)$
$\neg Q(a)$
$\neg P(a)$

Let $\forall a \in D, \neg Q(a)$
By (2) we see that $\neg Q(a)$ and by (1) we can see than $\neg Q(a)$ implies $\neg Q(p)$