Multi-variable Calculus
Problem Set 4

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1. The energy, E, of a body of mass m moving with speed v is given by the formula

$$E = f(m, v) = mc^2 \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right)$$

The speed v is nonnegative and less than the speed of light, c, which is a constant.

(a) Find $\frac{\partial E}{\partial m}$. What would you expect the sign of $\frac{\partial E}{\partial m}$ to be? Explain.

(b) Find $\frac{\partial E}{\partial v}$. What would you expect the sign of $\frac{\partial E}{\partial v}$ to be? Explain.

Ans:

(a)

$$\frac{\partial E}{\partial m} = c^2 \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right)$$

The sign of $\frac{\partial E}{\partial m}$ is positive because as the mass m of the object increases, the energy E increases.

(b)

$$\frac{\partial E}{\partial v} = \frac{\partial \left(mc^2 \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right) \right)}{\partial v} = \frac{\partial \left(mc^2 (1 - \frac{v^2}{c^2})^{-\frac{1}{2}} - mc^2 \right)}{\partial v}$$

$$= \frac{\partial \left(mc^2 (1 - \frac{v^2}{c^2})^{-\frac{1}{2}} \right)}{\partial v} - \frac{\partial (mc^2)}{\partial v}$$

$$= -\frac{1}{2} mc^2 \left(1 - \frac{v^2}{c^2} \right)^{-\frac{3}{2}} \left(-\frac{1}{c^2} \right)$$

$$= \frac{1}{2} m \left(1 - \frac{v^2}{c^2} \right)^{-\frac{3}{2}}$$

The sign of $\frac{\partial E}{\partial v}$ is positive because as the velocity v of the object increases, the energy E increases.

2. Let $f(x,y) = x^2e^{xy}$ and $P = (1,0)$
   (a) Find the equation of the plane tangent to the graph of f at P.
   (b) Use Part (a) to approximate $f(1.1, 0.8)$.

Ans:

(a) The equation of the tangent plane z is equal to $f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$, where $(a,b) = (1,0)$.

- $f_x = \frac{\partial(x^2e^{xy})}{\partial x} = 2xe^{xy} + x^2ye^{xy} = (2x+x^2)e^{xy}$
- $f_y = \frac{\partial(x^2e^{xy})}{\partial y} = x^3e^{xy}$

$$f(a,b) = f(1,0) = 1^2e^{1\times 0} = 1$$
$$f_x(a,b) = 2(1)e^{1\times 0} + 1^2 \times 0 \times e^{1\times 0} = 2$$
$$f_y(a,b) = 1^3e^{1\times 0} = 1$$
$$z = 1 + 2(x-1) + 1(y-0) = 1 + 2(x-1) + y$$
$$z = 1 + 2(x-1) + y$$

(b) $f(1.1, 0.8) \approx 1 + 2(1.1-1) + 0.8$
$f(1.1, 0.8) \approx 2$

3. A one-meter long bar is heated unevenly, with its temperature in $°C$ at a distance x metres from one end at time t given by

$$H(x,t) = 100e^{-0.1t}\sin(\pi x) \qquad 0 \le x \le 1$$

   (a) Sketch a graph of H against x for $t = 0$ and $t = 1$.
   (b) Calculate $H_x(0.2, t)$ and $H_x(0.8, t)$. What is the practical interpretation (in terms of temperature) of these two partial derivatives? Explain why each one has the sign it does.
   (c) Calculate $H_t(x,t)$. What is its sign? What is its interpretation in terms of temperature?

Ans:

(a)

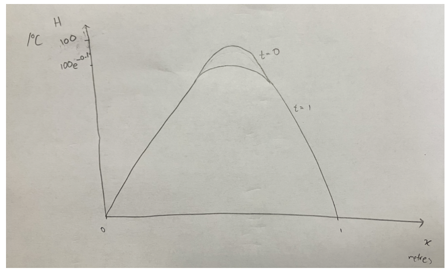

Figure 1

(b)

$$H_x = \frac{\partial\left(100e^{-0.1t}\sin(\pi x)\right)}{\partial x}$$

$H_x = 100\pi \cdot \cos(\pi x) \cdot e^{-0.1t}$
$H_x(0.2, t) = 100\pi \cdot \cos(0.2\pi) \cdot e^{-0.1t} \approx 254.2 \cdot e^{-0.1t}$ °C/meter
$H_x(0.8, t) = 100\pi \cdot \cos(0.8\pi) \cdot e^{-0.1t} \approx -254.2 \cdot e^{-0.1t}$ °C/meter
These two derivatives show how the temperature in °C changes with respect to the distance x in meters from one end at a fixed moment of time t. $H_x(0.2, t)$ is positive because the temperature increases where you are 0.2m away from one end but $H_x(0.8, t)$ is negative because the temperature decreases when you are 0.8m from one end.

(c)

$$H_t(x, t) = -0.1 \times 100 \times e^{-0.1t} \times \sin(\pi x)$$

$H_t(x, t) = -10e^{-0.1t} \times \sin(\pi x)°C/$ unit of time
The sign is negative because $\sin(\pi x)$ is positive for $0 \leq x \leq 1$ and t is positive. This means that with respect to time, or as the time increases, the temperature of the bar decreases.

4. A student was asked to find the equation of the tangent plane to the surface $z = x^3 - y^2$ at the point $(x, y) = (2, 3)$. The student's answer was

$$z = 3x^2(x-2) - 2y(y-3) - 1.$$

(a) At a glance, how do you know this is wrong?
(b) What mistake did the student make?
(c) Answer the question correctly.

Ans:

(a) The slopes of a linear function must be constants and not variables.

(b) They forgot to evaluate $f_x(a, b)$ and $f_y(a, b)$ at the point $(x, y) = (2, 3)$.

(c)
$$z = 3(2^2)(x - 2) - 2(3)(y - 3) - 1$$
$$z = 12(x - 2) - 6(y - 3) - 1$$