Transitions to Higher Mathematics
Portfolio Draft

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1 Chapter 4

Problem 8

Proposition. *Suppose a is an integer. If $5 \mid 2a$ then $5 \mid a$*

Proof. Suppose $a \in \mathbb{Z}$ and $5 \mid 2a$.
Because $5 \mid 2a$ then by definition of divisibility $2a = 5k$ for some $k \in \mathbb{Z}$. Because $5k = 2a$ then $5k$ is even by definition. This means k is even and so by definition k can be written as $k = 2c$ for some $c \in \mathbb{Z}$. We now have $2a = 5k = 5 \cdot (2c)$ which then becomes $2a = 10c$. Dividing both sides by 2 gives $a = 5c$ and thus we have written a as c multiples of 5 where $c \in \mathbb{Z}$. Therefore, $5 \mid a$ by definition.

$\square$

Problem 18

Proposition. *Suppose x and y are positive real numbers. If $x < y$, then $x^2 < y^2$.*

Proof. Suppose $x, y \in \mathbb{R}^+$ and $x < y$.
Consider the inequality $x < y$. Multiplying both sides by x gives $x^2 < xy$. The sign of the inequality does not change as $x > 0$. Taking $x < y$ again and multiplying both sides by y gives $xy < y^2$. The sign of the inequality does not change as $y > 0$. We can combine the two inequalities to get $x^2 < xy < y^2$. Therefore, $x^2 < y^2$. $\square$

2 Chapter 5

Problem 10

Proposition. *Suppose $x, y, z \in \mathbb{Z}$ and $x \neq 0$. If $x \nmid yz$, then $x \nmid y$ and $x \nmid z$.*

Proof. Suppose it is not true that $x \nmid y$ and $x \nmid z$. That is, by DeMorgan's law through negating the "and", suppose $x \mid y$ or $x \mid z$.
Case 1: Suppose $x \mid y$. Then $y = ax$ for some $a \in \mathbb{Z}$. Multiplying both sides by z gives $yz = axz$ or $yz = (az)x$. Therefore $x \mid yz$.
Case 2: Suppose $x \mid z$. Then $z = bx$ for some $b \in \mathbb{Z}$. Multiplying both sides by y gives $yz = bxz$ or $yz = (bz)x$. Therefore $x \mid yz$.
Case 1 and Case 2 show that $x \mid yz$. Therefore it is not true that $x \nmid yz$. $\square$

Problem 22

Proposition. *Let $a \in \mathbb{Z}, n \in \mathbb{N}$. If a has a remainder r when divided by n, then $a \equiv r \pmod{n}$.*

Proof. Suppose $a \in \mathbb{Z}, n \in \mathbb{N}$ and that a has a remainder r when divided by n. By the division algorithm we have $a = qn + r$ where $q \in \mathbb{Z}$ and $0 \leq r < n$. We can rearrange this to get $a - r = qn$. Therefore $n \mid (a - r)$. Hence $a \equiv r \pmod{n}$ by definition. ☐

3 Chapter 6

Problem 4

Proposition. $\sqrt{6}$ *is irrational*

Proof. Suppose, for the sake of contradiction, that $\sqrt{6}$ is rational. Therefore by definition $\sqrt{6} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and that a and b have no common divisiors. Squaring both sides gives $6 = \frac{a^2}{b^2}$. This then gives $6b^2 = a^2$. The left-handside of this equation is even because 6 is even the product of an even integer with any integer is even.

Lemma. *The product of an even integer a and any other integer b is even*
Case 1: *Suppose $a, b \in \mathbb{Z}$ and that a is even and b is odd. Therefore by definition $a = 2k, k \in \mathbb{Z}$ and $b = 2q + 1, q \in \mathbb{Z}$. The product ab is equal to $2k \cdot (2q + 1) = 4kq + 2k = 2(2kq + k)$ and is thus even by definition.*
Case 2: *Suppose $a, b \in \mathbb{Z}$ and that both a and b are even. Therefore by definition $a = 2r, r \in \mathbb{Z}$ and $b = 2s, s \in \mathbb{Z}$. The product ab is equal to $2r \cdot 2s = 4rs = 2(2rs)$ and is thus even by definition.*

This means that a^2 is even and so by the previous lemma a is also even so $a = 2l, l \in \mathbb{Z}$. We can substitute this in $6b^2 = a^2$ to get $6b^2 = 4l^2$. Dividing both sides by 2 gives $3b^2 = 2l^2$. Here the right-hand side is even and so the left-hand side must also be even. Because $3b^2$ is even then either 3 or b^2 is even. We know that 3 is odd and so b^2 must be even. Therefore by our previous lemma, b must also be even and so $b = 2w, w \in \mathbb{Z}$. So we have shown that both a and b are even. This means that they have a common factor of 2 but this contradicts our initial suppositioin that a and b have no common divisors. Therefore, $\sqrt{6}$ must be irrational. ☐

Problem 8

Proposition. *Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$ then a or b is even.*

Proof. Suppose, for the sake of contradiction, that $a^2 + b^2 = c^2$ and that a and b are odd. Therefore by definition $a = 2d + 1, d \in \mathbb{Z}$ and $b = 2e + 1, e \in \mathbb{Z}$. Subsituting this into $a^2 + b^2 = c^2$ gives $c^2 = (2d+1)^2 + (2e+1)^2 = 4d^2 + 4d + 1 + 4e^2 + 4e + 1 = 4(d^2 + d + e^2 + e) + 2$ The integer c is either odd or even.
Case 1: c is odd, and so by definition $c = 2q + 1, q \in \mathbb{Z}$. Thus $c^2 = 4q^2 + 4q + 1 = 4(d^2 + d + e^2 + e) + 2$. This means that $4q^2 + 4q = 4(d^2 + d + e^2 + e) + 1 = 2(2(d^2 + d + e^2 + e)) + 1$. Which is a contradiction because the left-hand side is even and the right-hand side is odd.
☐

4 Chapter 7

Problem 6

Proposition. *Suppose* $x, y \in \mathbb{R}$. *Then* $x^3 + x^2y = y^2 + xy$ *if and only if* $y = x^2$ *or* $y = -x$

Proof. Suppose $x, y \in \mathbb{R}$ and that $x^3 + x^2y = y^2 + xy$. We then have $x^2(x + y) = y(x + y)$. If $x + y \neq 0$, dividing both sides gives $x^2 = y$. If $x + y = 0$ we then get $y = -x$. Therefore if $x^3 + x^2 + y = y^2 + xy$ then $y = x^2$ or $y = -x$.
Suppose $x, y \in \mathbb{R}$ and either $x = y^2$ or $y = -x$. If $y = x^2$ then $x^3+x^2y = x^3+x^2(x^2) = x^3+x^4$. and $y^2 + xy = x^4 + x^3$. If $y = -x$ then $x^3 + x^2(-x) = x^3 - x^3 = 0$ and $y^2 + xy = x^2 - x^2 = 0$. Therefore if $y = x^2$ or $y = -x$ then $x^3 + x^2 + y = y^2 + xy$. □

5 Chapter 8

Problem 2

Proposition. $\{6n : n \in \mathbb{Z}\}$=$\{2n : n \in \mathbb{Z}\}\cap\{3n : n \in \mathbb{Z}\}$

Proof. Every element in $\{6n : n \in \mathbb{Z}\}$ can be written as $3n \cdot 2$ or $2n \cdot 3$ therefore it must also exist in $\{2n : n \in \mathbb{Z}\}\cap\{3n : n \in \mathbb{Z}\}$ □