Multi-variable Calculus
Problem Set 5

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1. Consider the function $g(x, y, z) = x^2 - y^2 - z^2$, and its level surface $g(x, y, z) = -1$. We are interested in the tangent plane to the level surface at the point $P = (1, 1, -\sqrt{3})$. You will find the tangent plane in two ways, using two-variable calculus as follows in this problem, and then using three-variable calculus in the next problem.
   (a) Can the level surface be written as a graph $z = f(x, y)$ for some function f?
   (b) Find a function $f(x, y)$ so that the part of the surface that contains P is the graph $z = f(x, y)$.
   (c) Find an equation for the tangent plane to $z = f(x, y)$ at the point P using the ideas of Section 14.3 (point-slope formula for tangent planes).

Ans:

(a) No it cannot. Rearranging $g(x, y, z) = -1$ we get $z^2 = x^2 + y^2 + 1$ and so $z = \pm\sqrt{x^2 + y^2 + 1}$ which is not the graph of a function. The surface level can be written as two graphs where $z = \sqrt{x^2 + y^2 + 1}$ and $z = -\sqrt{x^2 + y^2 + 1}$

(b) $f(x, y) = -\sqrt{x^2 + y^2 + 1}$ because $f(1, 1) = -\sqrt{(1)^2 + (1)^2 + 1} = -\sqrt{3}$ and so P is in this part of the level surface.

(c) The equation of the tangent plane at P where $(a, b) = (1, 1)$ is given by

$$f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(a - b)$$

$$f_x = -\frac{1}{2}(x^2 + y^2 + 1)^{-1/2} \cdot 2x = -\frac{x}{\sqrt{x^2 + y^2 + 1}}$$

$$f_y = -\frac{1}{2}(x^2 + y^2 + 1)^{-1/2} \cdot 2y = -\frac{y}{\sqrt{x^2 + y^2 + 1}}$$

$$f(1, 1) = -\sqrt{(1)^2 + (1)^2 + 1} = -\sqrt{3}$$

$$f_x(1, 1) = -\frac{1}{\sqrt{(1)^2 + (1)^2 + 1}} = -\frac{1}{\sqrt{3}}$$

$$f_y(1,1) = -\frac{1}{\sqrt{(1)^2 + (1)^2 + 1}} = -\frac{1}{\sqrt{3}}$$

So the equation of the tangent plane will then be

$$z = -\sqrt{3} - \frac{1}{\sqrt{3}}(x-1) - \frac{1}{\sqrt{3}}(y-1)$$

2. Now, let $g(x, y, z) = x^2 + y^2 - z^2$ and $P = (1, 1, -\sqrt{3})$ be as in Problem 1, and find the tangent plane to the graph of g at the point P using three-variable calculus, as follows.
   (a) Find the gradient vector $\nabla g(P)$.
   (b) Use your answer in (a) to find an equation for the tangent plane using the ideas of Section 14.5.
   (c) (Convince yourself that the equation is equivalent to the equation you got in Problem 1, i.e., they can be obtained from each other by simple algebraic manipulations, but don't write or submit any proofs.)

Ans:

(a)

$$\begin{aligned}
\nabla g &= < g_x, g_y, g_z > \\
g_x &= 2x, g_y = 2y, g_z = -2z \\
\nabla g &= < 2x, 2y, -2z > \\
\nabla g(P) &= < 2(1), 2(1), -2(-\sqrt{3}) > \\
\nabla g(P) &= < 2, 2, 2\sqrt{3} >
\end{aligned}$$

(b) The equation for the tangent plane to the graph of g is at the point (a, b, c) where $a = 1, b = 1$, and $c = -\sqrt{3}$ is given by

$$f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) = 0$$

By substuting for $f_x(a, b, c)$, $f_y(a, b, c)$, and $f_z(a, b, c)$ from $\nabla g(P)$ we get

$$2(x-1) + 2(y-1) + 2\sqrt{3}(z+\sqrt{3}) = 0$$

(c) *convincing myself*

3. A steel bar with square cross sections 5 cm by 5 cm and length 3 m is being heated. For each dimension, the bar expands 13×10^{-6} m for each 1 celsius rise in temperature. What is the rate of change in the volume of the steel bar with respect to temperature? You may want to use $l = l(T)$, $w = w(T)$, and $h = h(T)$ to denote the length, width and height of the bar, respectively, each of which viewed as a function of temperature T . (Hint: this is a problem about the chain rule.)

Ans: We have $V = l \cdot w \cdot h$ where $l = 3$ m, $w = 0.05$ m, and $h = 0.05$ m because $V(l, w, h)$ where $l = l(T)$, $w = w(T)$, and $h = h(T)$ then

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}T} &= \frac{\mathrm{d}V}{\mathrm{d}l} \cdot \frac{\mathrm{d}l}{\mathrm{d}T} + \frac{\mathrm{d}V}{\mathrm{d}w} \cdot \frac{\mathrm{d}w}{\mathrm{d}T} + \frac{\mathrm{d}V}{\mathrm{d}h} \cdot \frac{\mathrm{d}h}{\mathrm{d}T} \\ &= wh(13 \times 10^{-6}) + lh(13 \times 10^{-6}) + lw(13 \times 10^{-6}) \\ &= 13 \times 10^{-6}(0.05^2 + 3 \times 0.05 + 3 \times 0.05) \\ &= 3.9 \times 10^{-6} m^3/°C\end{aligned}$$

4. Consider $f(x, y) = \frac{k}{2}x^2 + \frac{1}{2}y^2 - xy$
   (a) Verify that for any value of $k, (0, 0)$ is a critical point.
   (b) Use the second derivative test to determine the values of k (if any) for which $0, 0)$ is
      - a local minimum,
      - a local maximum,
      - a saddle point.
   (c) In Question (4)b, the second derivative test gives no information when $k = 1$. For $k = 1$, find all critical points of f and then classify them by using software to plot the graph of f.

Ans:

(a) At a critical point $\nabla f(P) =< 0, 0 >$ we have $\nabla f =< kx - y, y - x >$. Substituting $x = 0$ and $y = 0$ we get $\nabla f =< k \cdot 0, 0 >$. Because for any $k \in \mathbb{R}, k \cdot 0 = 0$ then $\nabla f(P) =< 0, 0 >$ so P is a critical point for any value of k.

(b) The discriminant D is equal to

$$= f_{xx}(a, b)f_{yy}a, b - (f_{xy}(a, b))^2$$

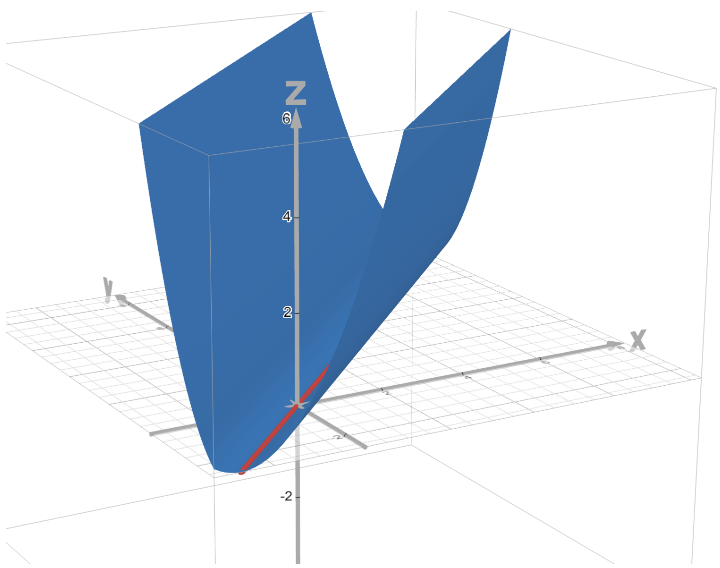
We have $f_x = kx - y$ and $f_y = y - x$, so

$$\begin{aligned} f_{xx} &= k \\ f_{yy} &= 1 \\ f_{xy} &= -1 \end{aligned}$$

So $D = k - (-1)^2 = k - 1$

- At a local minimum $D > 0$ and $f_{xx} > 0$ so $k - 1 > 0$ and $k > 0$ so the range of values will be $k > 1$
- At a local maximum $D > 0$ and $f_{xx} < 0$ so $k - 1 > 0$ or $k > 1$ and $k < 0$ which is a contradiction. So it is not possible to have a local maximum.
- At a saddle point $D < 0$ so $k - 1 < 0$ and so the range of values will be $k < 1$.

(c) Because $k = 1$, $\nabla f =< x - y, y - x >$ and because $\nabla f =< 0, 0 >$ at a critical point so $< x - y, y - x >=< 0, 0 >$. The critical points will then be the infinite set of all ordered pairs $a, b \in \mathbb{R}$ such that $a = b$. From the graph below we can see that these are all minimum points.



5. The following table gives selected values of the quadratic polynomial

$$P(x, y) = a + bx + cy + dx^2 + exy + fy^2$$

y/x	10	12	14
10	26	36	54
15	31	41	59
20	36	46	64

(a) Express P_x, P_{xx}, P_{xy} and P_{yy} in terms of a, b, c, d, e and f (and x, y). Show your work.

(b) Determine the signs of d and e (i.e., determine if they are positive, negative, or zero). Explain your reasoning. Hint: You'll want to use the table to estimate some second-order partial derivatives.

Ans:

(a)

$$
\begin{aligned}
P_x &= \frac{\partial(a + bx + cy + dx^2 + exy + fy^2)}{\partial x} = b + 2dx + ey \\
P_{xx} &= \frac{\partial(b + 2dx + ey)}{\partial x} = 2d \\
P_{xy} &= \frac{\partial(b + 2dx + ey)}{\partial y} = e \\
P_y &= \frac{\partial(a + bx + cy + dx^2 + exy + fy^2)}{\partial y} = c + ex + 2fy \\
P_{yy} &= \frac{\partial(c + ex + 2fy)}{\partial y} = 2f
\end{aligned}
$$

(b) Using the first row of the table we can estimate the value for P_{xx}. The value of $P(x, 10)$ changes from 26 to 36 as x increases from 10 to 12. So the rate of change with respect to $x \approxeq 5$. The value of $P(x, y)$ then changes from 36 to 54 as x increases from 12 to 14. So the rate of change with respect to $x \approxeq 9$. Because the rate of change of the rate of the change of x with respect to changing values of x is increasing so d must be positive.
Computing P_x at each row we get the same estimates for P_x namely 5 and 9, so changing the value of y has no effect on the rate of change of x and therefore $e = 0$.