

An Introduction to Reinforcement Learning

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Agenda

- What is Reinforcement Learning
- Markov Decision Processes
- Dynamic Programming
- Monte Carlo Methods
- Temporal Difference Learning
 - Sarsa
 - Q-learning
- More advanced approximation methods

What is reinforcement learning

Reinforcement Learning

- Reinforcement learning is learning what to do—how to map situations to actions to maximize a numerical reward signal.
- The learner is not told which actions to take, but instead must discover which actions yield the most reward by trying them.

Core Concepts

- **Agent** is the learner and decision maker.
 - It interacts with the environment continually to select actions
 - The environment responds to these actions and presents new situations to the agent.
- A **policy** defines the learning agent's way of behaving at a given time.
 - A mapping from states of the environment to actions to be taken when in those states.
- The **reward** signal indicates what is good immediately after each decision time.
- A **value function** specifies what is good in the long run.
 - The value of a state is the total amount of reward an agent can expect to accumulate over the future, starting from that state.

Markov Decision Processes (MDPs)

Core definitions

- MDPs are a classical formalization of sequential decision making or reinforcement learning problem

- Actions taken now influence not only immediate contributions (e.g. reward, cost, profit), but also future situations (states), and consequently, future contributions
- MDPs involve delayed contribution and the need to trade off immediate and delayed contributions

The agent–environment interaction in a decision process

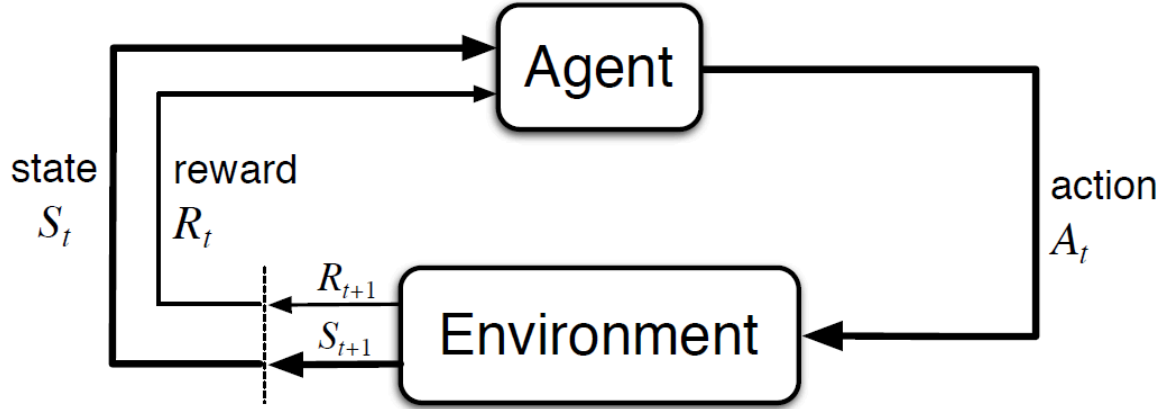


Figure 1: The agent–environment interaction in a decision process

Agent gives rise to a sequence that begins like this

$$S_0, A_0, R_1, S_1, A_1, R_2, S_2, A_2, R_3, \dots, S_t, A_t, R_{t+1}$$

The agent–environment interaction in a decision process (continued)

- An MDP can be described by the tuple $(S, A, p, r(a, s), \pi)$, where:
 - S is a finite set of states,
 - A is a finite set of actions,
 - $p(s', r \mid s, a)$ is the probability of transitioning to state $s' \in S$ and receiving reward $r(a, s)$ when taking action $a \in A$ in state $s \in S$,
 - $r(s, a)$ is the expected immediate reward received after taking action a in state s ,
 - π is the policy, a mapping from states to probabilities of selecting each possible action.
- The $p(s' \mid s, a)$ defines the *dynamics* of the MDP and p specifies a probability distribution for each choice of s and a :

$$\sum_{s' \in S} p(s' \mid s, a) = 1, \text{ for all } s \in S, a \in A(s)$$

The Goal of Agent

- At each time step, the reward is a simple number, $R_t \in \mathbb{R}$ passing from the environment to the agent.
- The agent's goal is to maximize the total (cumulative) amount of reward it receives in the long run.

- The aim is to maximize the *expected return*, G_t , which is defined as:

$$G_t \doteq R_{t+1} + R_{t+2} + R_{t+3} + \dots + R_T$$

Discounting technique is used to prioritize immediate rewards over future rewards.

- The idea is to multiply future rewards by a discount factor $\gamma \in (0, 1]$
- This makes future rewards worth less than immediate rewards.
- The return G_t with discounting is defined as:

$$G_t \doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

Policies and Value Functions

- A **policy** is a mapping from states to probabilities of selecting each possible action. If the agent is following policy π at time t , then $\pi(a \mid s)$ is the probability that $A_t = a$ if $S_t = s$
- **Value functions—functions of states** estimate *how good* it is for the agent to be in a given state (or how good it is to perform a given action in a given state).
- The value of a state s under a policy π , denoted v_π is expressed as

$$v_\pi(s) \doteq \mathbb{E}_\pi[G_t \mid S_t = s] = \mathbb{E}_\pi \left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \mid S_t = s \right], \text{ for all } s \in S,$$

where $\mathbb{E}[\cdot]$ denotes the expected value of a random variable given that the agent follows policy π , and t is any time step.

Bellman Equation

To solve stochastic sequential decision problems

1. We have to model the fact that new information becomes available after we make the decision a_t .
2. The result can be uncertainty in both the contribution earned, and in the determination of the next state we visit, S_{t+1} .

We estimate the expected future contributions (cumulative rewards) and add the immediate contribution received from the action taken in the current state following a given policy. This relationship is expressed by the Bellman equation, defined as:

$$V_\pi(S_t) = \max_{a_t \in A_t} \left(\underbrace{r_{t+1}(S_t, a_t)}_{\text{immediate contribution}} + \underbrace{\gamma E[V_{t+1}(S_{t+1}) \mid S_t, a_t]}_{\text{future contributions}} \right)$$

This equation can be solved backward or forward recursively. You get the value functions and at the same time an optimal policy at time t

$$a_t^*(S_t) = \arg \max_{a_t \in A_t} (r_{t+1}(S_t, a_t) + \gamma[V_{t+1}(S_{t+1}) | S_t, a_t]),$$

Dynamic Programming

Model-based methods (Dynamic Programming)

- They are used to derive optimal policies and value functions in Markov Decision Processes (MDPs) and reinforcement learning.

Bellman Equation

- The Bellman equations express the relationship between the value of a state and the values of its successor states. For the state-value function v_π , the Bellman equation is:

$$\begin{aligned} v_\pi(s) &\doteq \mathbb{E}_\pi [G_t | S_t = s] \\ &= \mathbb{E}_\pi [R_{t+1} + \gamma G_{t+1} | S_t = s] \\ &= \mathbb{E}_\pi [R_{t+1} + \gamma v_\pi(S_{t+1}) | S_t = s] \\ &= \sum_a \pi(a | s) \sum_{s', r} p(s', r | s, a) [r + \gamma v_\pi(s')]. \end{aligned}$$

Model-based methods (Dynamic Programming)

- An optimal policy is a policy that achieves the maximum expected return from any initial state.
- The optimal state-value function v_* is the maximum value function over all policies:

$$v_*(s) = \max_{\pi} v_\pi(s), \text{ for all } s \in S$$

- The optimal action-value function q_* is the maximum action-value function over all policies:

$$q_*(s, a) = \max_{\pi} q_\pi(s, a), \text{ for all } s \in S, a \in A(s)$$

We use Dynamic Programming (DP) to leverage value functions in the search for good policies.

- The Bellman optimality equation for v_* is:

$$\begin{aligned} v_*(s) &= \max_a \mathbb{E} [R_{t+1} + \gamma v_*(S_{t+1}) | S_t = s, A_t = a] \\ &= \max_a \sum_{s', r} p(s', r | s, a) [r + \gamma v_*(s')], \end{aligned}$$

- The Bellman optimality equation for q_* is:

$$q_*(s, a) = \sum_{s', r} p(s', r | s, a) [r + \gamma \max_{a'} q_*(s', a')]$$

Policy Evaluation (Prediction)

Policy evaluation is the computation of the state-value function v_π for an arbitrary policy π . We also refer to it as the *prediction problem.

$$\begin{aligned}
v_\pi(s) &\doteq \mathbb{E}_\pi [G_t \mid S_t = s] \\
&= \mathbb{E}_\pi [R_{t+1} + \gamma G_{t+1} \mid S_t = s] \\
&= \mathbb{E}_\pi [R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s] \\
&= \sum_a \pi(a \mid s) \sum_{s', r} p(s', r \mid s, a) [r + \gamma v_\pi(s')] .
\end{aligned}$$

where $\pi(a \mid s)$ is the probability of taking action a in state s under policy π , and the expectations are subscripted by π to indicate that they are conditional on π being followed.

Consider a sequence of approximate value functions v_0, v_1, v_2, \dots . Each successive approximation is obtained by using the Bellman equation for v_π as an update rule:

$$\begin{aligned}
v_{k+1}(s) &\doteq \mathbb{E}_\pi [R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s] \\
&= \sum_a \pi(a \mid s) \sum_{s', r} p(s', r \mid s, a) [r + \gamma v_k(s')] .
\end{aligned}$$

Policy Evaluation (Prediction)

The sequence v_k can be shown in general to converge to v_π as $k \rightarrow \infty$ under the same conditions that guarantee the existence of v_π . This algorithm is called *iterative policy evaluation*.

i Iterative Policy Evaluation, for estimating $V \approx v_\pi$

Input:

π , the policy to be evaluated

Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation

Initialize $V(s)$ arbitrarily, for $s \in S$, and $V(\text{terminal}) = 0$

Loop:

$\Delta \leftarrow 0$

Loop for each $s \in S$:

$v \leftarrow V(s)$

$V(s) \leftarrow \sum_a \pi(a \mid s) \sum_{s', r} p(s', r \mid s, a) [r + \gamma V(s')]$

$\Delta \leftarrow \max(\Delta, |v - V(s)|)$

until $\Delta < \theta$

Policy Improvement

- We know how good it is to follow the current policy from s —that is $v_\pi(s)$ —but would it be better or worse to change to the new policy?

One way to answer this question is to consider selecting a in s and thereafter following the existing policy π .

This leads to the definition of the *q-value* of a state-action pair:

$$\begin{aligned}
q_\pi(s, a) &\doteq \mathbb{E} [R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \sum_{s', r} p(s', r \mid s, a) [r + \gamma v_\pi(s')] .
\end{aligned}$$

The policy improvement theorem states that if we improve the policy by acting greedily with respect to q_π , the new policy π' will be at least as good as π .

Formally, if

$$\begin{aligned}\pi'(s) &= \arg \max_a q_\pi(s, a) \\ &= \arg \max_a \mathbb{E} [R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \arg \max_a \sum_{s', r} p(s', r \mid s, a) [r + \gamma v_\pi(s')]\end{aligned}$$

then

$$v_{\pi'}(s) \geq v_\pi(s).$$

for all $s \in S$.

Policy Iteration

Once a policy, π , has been improved using v_π to yield a better policy, π' , we can then compute $v_{\pi'}$ and improve it again to yield an even better π'' .

$$\pi_0 \xrightarrow{E} v_{\pi_0} \xrightarrow{I} \pi_1 \xrightarrow{E} v_{\pi_1} \xrightarrow{I} \pi_2, \dots \xrightarrow{I} \pi_* \xrightarrow{E} v_*$$

where \xrightarrow{E} denotes a policy *evaluation* and \xrightarrow{I} denotes a policy *improvement*. This way of finding an optimal policy is called policy iteration.

Policy Iteration

A complete *policy iteration* algorithm

i Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization

$V(s) \in \mathbb{R}$ and $\pi(s) \in A(s)$ arbitrarily for all $s \in S$

$V(\text{terminal}) \doteq 0$

2. Policy Evaluation

Loop:

$\Delta \leftarrow 0$

Loop for each $s \in S$:

$v \leftarrow V(s)$

$V(s) \leftarrow \sum_{s',r} p(s', r | s, \pi(s)) [r + \gamma V(s')]$

$\Delta \leftarrow \max(\Delta, |v - V(s)|)$

Until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

3. Policy Improvement

policy-stable \leftarrow true

For each $s \in S$:

old-action $\leftarrow \pi(s)$

$\pi(s) \leftarrow \arg \max_a \sum_{s',r} p(s', r | s, a) [r + \gamma V(s')]$

If *old-action* $\neq \pi(s)$, then *policy-stable* \leftarrow false

If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$;

Else go to step 2

Value Iteration

- One drawback to policy iteration is that each of its iterations involves policy evaluation, which may itself be a protracted iterative computation requiring multiple sweeps through the state set.
- Value iteration is a special case of policy iteration where the policy evaluation step is truncated to just one sweep.
- This algorithm combines the policy improvement and truncated policy evaluation steps into a single update operation:

$$\begin{aligned} v_{k+1}(s) &= \mathbb{E} [R_{t+1} + \gamma v_k(S_{t+1}) | S_t = s, A_t = a] \\ &= \max_a \sum_{s',r} p(s', r | s, a) [r + \gamma v_k(s')] \end{aligned}$$

for all $s \in S$.

Value Iteration

- Value iteration is obtained simply by turning the Bellman optimality equation into an update rule.
- Also note how the value iteration update is identical to the policy evaluation update except that it requires the maximum to be taken over all actions.

i Value Iteration, for estimating $\pi \approx \pi_*$

Algorithm parameter:

A small threshold $\theta > 0$ determining the accuracy of estimation

Initialization:

Initialize $V(s)$ arbitrarily for all $s \in S^+$, except that $V(\text{terminal}) = 0$

Loop:

$\Delta \leftarrow 0$

Loop for each $s \in S$:

$v \leftarrow V(s)$

$V(s) \leftarrow \max_a \sum_{s',r} p(s', r | s, a)[r + \gamma V(s')]$

$\Delta \leftarrow \max(\Delta, |v - V(s)|)$

until $\Delta < \theta$

Output:

A deterministic policy, $\pi \approx \pi_*$, such that

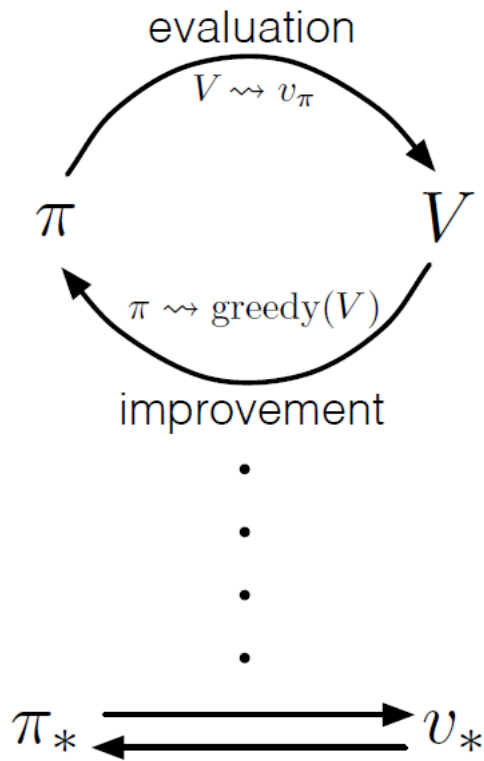
$$\pi(s) = \arg \max_a \sum_{s',r} p(s', r | s, a)[r + \gamma V(s')]$$

Let's code

Generalized Policy Iteration

- GPI is a general idea that describes how two processes — *evaluating a policy and improving it* — work together and influence each other.

- Most reinforcement learning algorithms use a value function to judge how good a policy is, and then update the policy based on that judgment.
- If both value estimation and policy improvement stabilize (i.e., stop changing), then the policy must be the best possible one for that value function—meaning the policy is optimal.



Model-free methods

- Monte Carlo Methods
- Sarsa
- Q-learning

Monte Carlo Methods

Monte Carlo (MC) Methods

- The term “Monte Carlo” is often used more broadly for any estimation method whose operation involves a significant random component.
- MC methods solve reinforcement learning problems by averaging results (returns) from sampled experiences sequences of states, actions, and rewards.
- They do not require knowledge of the environment’s dynamics, making them powerful for learning from real or simulated experiences.

Monte Carlo Prediction

- Suppose we wish to estimate $v_\pi(s)$, the values of a state s under policy π , given a set of episodes obtained by following π and passing through s .

i First-Visit Monte Carlo Prediction (for estimating $V \approx v_\pi$)

Input:

A policy π to be evaluated

Initialize:

$V(s) \in \mathbb{R}$ arbitrarily, for all $s \in S$

Returns (s) \leftarrow an empty list, for all $s \in S$

Loop forever (for each episode):

Generate an episode following π : $S_0, A_0, R_1, S_1, A_1, R_2, \dots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of the episode, $t = T - 1, T - 2, \dots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

Unless S_t appears in S_0, S_1, \dots, S_{t-1} :

Append G to Returns (S_t)

$V(S_t) \leftarrow \text{average}(\text{Returns}(S_t))$

Monte Carlo Control

Alternating complete steps of policy evaluation and policy improvement are performed, beginning with an arbitrary policy π_0 and ending with the optimal policy and optimal action-value function:

$$\pi_0 \xrightarrow{E} q_{\pi_0} \xrightarrow{I} \pi_1 \xrightarrow{E} q_{\pi_1} \xrightarrow{I} \pi_2, \dots, \xrightarrow{I} \pi_* \xrightarrow{E} q_*$$

i Monte Carlo ES (Exploring Starts), for estimating $\pi \approx \pi_*$

Initialize:

$\pi(s) \in A(s)$ arbitrarily, for all $s \in S$

$Q(s, a) \in \mathbb{R}$ arbitrarily, for all $s \in S, a \in A(s)$

Returns $(s, a) \leftarrow$ empty list, for all $s \in S, a \in A(s)$

Loop forever (for each episode):

Choose $S_0 \in S, A_0 \in A(S_0)$ randomly, such that all pairs have probability > 0

Generate an episode from S_0, A_0 , following π :

$S_0, A_0, R_1, \dots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T - 1, T - 2, \dots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

Unless the pair (S_t, A_t) appears in

$S_0, A_0, S_1, A_1, \dots, S_{t-1}, A_{t-1}$:

Append G to Returns (S_t, A_t)

$Q(S_t, A_t) \leftarrow$ average (Returns (S_t, A_t))

$\pi(S_t) \leftarrow \arg \max_a Q(S_t, a)$

Let's code

Monte Carlo Control without Exploring Starts

- In on-policy control methods the policy is generally **soft**, meaning that $\pi(a | s) > 0$ for all $s \in S$ and all $a \in A(s)$, but gradually shifted closer and closer to a deterministic policy.
- The on-policy method we present in this section uses ϵ -greedy policies, meaning that most of the time they choose an action that has maximal estimated action value, but with probability ϵ they instead select an action at random. That is, all nongreedy actions are given the minimal probability of selection, $\frac{\epsilon}{|A(s)|}$, and the remaining bulk of the probability $1 - \epsilon + \frac{\epsilon}{|A(s)|}$ is given to the greedy action.
- ϵ -greedy policies are examples of ϵ -soft policies, defined as policies for which $\pi(a | s) \geq \frac{\epsilon}{|A(s)|}$ for all states and actions, for some $\epsilon > 0$. Among ϵ -soft policies, ϵ -greedy policies are in some sense those that are closest to greedy.

Monte Carlo Control without Exploring Starts

i On-policy First-Visit MC Control (for ϵ -soft policies), estimates $\pi \approx \pi_*$

Algorithm parameter:

Small $\epsilon > 0$

Initialize:

$\pi \leftarrow$ an arbitrary ϵ -soft policy

$Q(s, a) \in \mathbb{R}$ arbitrarily, for all $s \in S, a \in A(s)$

Returns $(s, a) \leftarrow$ empty list, for all $s \in S, a \in A(s)$

Repeat forever (for each episode):

Generate an episode following π : $S_0, A_0, R_1, \dots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T - 1, T - 2, \dots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

Unless the pair (S_t, A_t) appears in $S_0, A_0, \dots, S_{t-1}, A_{t-1}$:

Append G to Returns (S_t, A_t)

$Q(S_t, A_t) \leftarrow$ average (Returns (S_t, A_t))

$A^* \leftarrow \arg \max_a Q(S_t, a)$ (ties broken arbitrarily)

For all $a \in A(S_t)$:

$$\pi(a | S_t) \leftarrow \begin{cases} 1 - \epsilon + \epsilon / |A(S_t)| & \text{if } a = A^* \\ \epsilon / |A(S_t)| & \text{if } a \neq A^* \end{cases}$$

where $|A(s)|$ is the number of actions available in state s .

The ϵ -greedy policy ensures that all actions are tried, but actions with higher value estimates are tried more frequently. This balances exploration (trying new actions) and exploitation (choosing the best-known action).

Temporal-Difference Learning

Temporal-Difference Learning

- TD learning is a combination of Monte Carlo ideas and dynamic programming (DP) ideas.
- Like Monte Carlo methods, TD methods can learn directly from raw experience without a model of the environment's dynamics.
- Like DP, TD methods update estimates based in part on other learned estimates, without waiting for a final outcome (they bootstrap).

TD Prediction

Whereas Monte Carlo methods must wait until the end of the episode to determine the increment to $V(S_t)$ (only then is G_t known), TD methods need to wait only until the next time step. At time $t + 1$ they immediately form a target and make a useful update using the observed reward R_{t+1} and the estimate $V_{S_{t+1}}$. The simplest TD method makes the update:

$$V(S_t) \leftarrow V(S_t) + \alpha [R_{t+1} + \gamma V(S_{t+1}) - V(S_t)]$$

immediately on transition to S_{t+1} and receiving R_{t+1} . In effect, the target for the Monte Carlo update is G_t , whereas the target for the TD update is $R_{t+1} + \gamma V(S_{t+1})$. This TD method is called TD(0), or *one-step* TD.

TD Prediction

i Tabular TD(0) for Estimating v_π

Input:

The policy π to be evaluated

Algorithm parameter:

Step size $\alpha \in (0, 1]$

Initialize:

$V(s)$ arbitrarily for all $s \in S^+$, except that $V(\text{terminal}) = 0$

Loop for each episode:

Initialize S

Loop for each step of episode:

$A \leftarrow$ action given by π for S

Take action A , observe R, S'

$V(S) \leftarrow V(S) + \alpha[R + \gamma V(S') - V(S)]$

$S \leftarrow S'$

Until S is terminal

Sarsa: On-policy TD Control

we consider transitions from state–action pair to state–action pair, and learn the values of state–action pairs.

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha [R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)]$$

i SARSA (On-Policy TD Control), for estimating $Q \approx q^*$

Algorithm parameters:

Step size $\alpha \in (0, 1]$, small $\varepsilon > 0$

Initialize:

$Q(s, a)$ arbitrarily, for all $s \in S^+, a \in A(s)$

$Q(\text{terminal}, \cdot) = 0$

Loop for each episode:

Initialize S

Choose A from S using a policy derived from Q (e.g., ε -greedy)

Loop for each step of episode:

Take action A , observe R, S'

Choose A' from S' using a policy derived from Q (e.g., ε -greedy)

$Q(S, A) \leftarrow Q(S, A) + \alpha[R + \gamma Q(S', A') - Q(S, A)]$

$S \leftarrow S'; \quad A \leftarrow A'$

Until S is terminal

Let's code

Q-learning: Off-policy TD Control

Q-learning is defined by

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha [R_{t+1} + \gamma \max_a Q(S_{t+1}, a) - Q(S_t, A_t)]$$

The Q-learning algorithm is shown below in procedural form.

i Q-learning (Off-Policy TD Control), for estimating $\pi \approx \pi_*$

Algorithm parameters:

Step size $\alpha \in (0, 1]$, small $\varepsilon > 0$

Initialize:

$Q(s, a)$ arbitrarily for all $s \in S^+$, $a \in S(s)$,
except that $Q(\text{terminal}, \cdot) = 0$

Loop for each episode:

Initialize S

Loop for each step of episode:

Choose A from S using a policy derived from Q (e.g., ε -greedy)

Take action A , observe R, S'

$Q(S, A) \leftarrow Q(S, A) + \alpha[R + \gamma \max_a Q(S', a) - Q(S, A)]$

$S \leftarrow S'$

Until S is terminal

Let's code

Recommended materials

Readings:

- Reinforcement Learning: An Introduction by Richard S. Sutton and Andrew G. Barto
- Reinforcement Learning and Stochastic Optimization by Warren B. Powell

Tutorials:

<https://www.youtube.com/watch?v=0MNVhXEX9to&list=PLMrJAKhIeNNQe1JXNvaFvURxGY4gE9k74&index=1>

<https://www.youtube.com/watch?v=i7q8bISGwMQ&t=910s>

<https://www.youtube.com/watch?v=sJIFUTITfBc>

<https://www.youtube.com/watch?v=0iqz4tcKN58>

<https://www.youtube.com/watch?v=wDVteayWWvU>

Any questions or thoughts?

Categories of RL methods

Model-based vs Model-free Methods

Model-based

- The agent knows/learns the model of the environment

- They then compute the policy using the ADP methods or the model-free methods on simulated data

Pros:

- Sample efficient
- Safer exploration

Cons:

- Prone to the model errors
- Learning a model is challenging

Model-free

- The agent does not know the model of the environment
- They learn the values or policies from trial-and-error interactions with the environment

Pros:

- Do not need a model
- Flexible

Cons:

- Sample inefficient: requires a lot of interactions with the environment
- Slow convergence