HOMEWORK #3

Mustafa Tokgöz 171044077
Introduction To Algorithm
Homework#3

Q1)

a) $T(n) = 27\,T(n/3) + n^2$

Answer: I use Master Teorem to salve this recuirence

$T(n) = a\,T(\frac{n}{b}) + n^d \quad a \geq 1,\ b > 1$

$a = 27 \quad b = 3 \quad d = 2$

if $a > b^d \Rightarrow T(n) = \Theta(n^{\log_b a})$ from Master Teoren

$27 > 3^2 \Rightarrow T(n) = \Theta(n^{\log_3 27}) = \boxed{\Theta(n^3)}$

b) $T(n) = 9\,T(n/4) + n$

Answer:

I use Master Teorem to salve this recursrence.

$T(n) = a\,T(\frac{n}{b}) + n^d \quad a \geq 1, b > 1$

$a = 9 \quad b = 4 \quad d = 1$

if $a > b^d \Rightarrow T(n) = \Theta(n^{\log_b a})$ from Master Teoren

$9 > 4^1 \Rightarrow \boxed{T(n) = \Theta(n^{\log_4 9}) = \Theta(n^{\log_2 3})}$

c) $T(n) = 2T\left(\frac{n}{4}\right) + n^{1/2}$

Answer:

I use Master Teorem to solve this recurrence.

$T(n) = aT\left(\frac{n}{b}\right) + n^d \quad a \geq 1 \quad b > 1$

$a = 2 \quad d = \frac{1}{2}$

$b = 4$

if $a = b^d$ then $T(n) = \Theta(n^d \log n)$

$\downarrow$

$2 = 2 \Rightarrow \boxed{T(n) = \Theta(\sqrt{n} \log n)}$

d) $T(n) = 2T(\sqrt{n}) + 1$

Answer $\Rightarrow$

$T(n) = 2T(n^{1/2}) + 1$

$T(2^m) = 2T(2^{m/2}) + 1$

Assume that $m = \log n$

$B(m) = T(2^m)$

$B(m) = 2B(m/2) + 1 \quad (2^{m/2} = m/2$ when $m = \log n)$

Then I use Master Teorem to solve

$T(n) = a\,T(\frac{n}{b}) + n^d \qquad a \geq 1,\ b > 1$

$B(m) = 2\,B(m/2) + 1$

$a = 2 \qquad b = 2 \quad d = 0$

$a > b^d \Rightarrow \Theta(n^{\log_b a})$

$2 > 2^0 \Rightarrow \Theta(m^{\log_2 2}) \Rightarrow$

$T(n) = \Theta(m) \Rightarrow T(n) = \boxed{\Theta(\log n)}$

e) $T(n) = 2T(n-2),\ T(0) = 1,\ T(1) = 1$

$T(n) = 2T(n-2)$

$T(n) = 2[2T(n-4)] = 2^2\,T(n-4)$

$T(n) = 2^3\,T(n-6)$

$\vdots \qquad \vdots$

$T(n) = 2^k\,T(n-2k)$

Assume that $n - 2k = 0 \quad n = 2k \Rightarrow k = n/2$

$T(n) = 2^{n/2}\,\underbrace{T(0)}_{1} \Rightarrow \boxed{T(n) = \Theta(2^{n/2})}$

f) $T(n) = 4T(n/2) + n$

Answer:

I use Master Teoren to salve this recurrence,

$$T(n) = aT(\tfrac{n}{b}) + n^d \quad a \geq 1, a > 1$$

$$a = 4 \quad b = 2 \quad d = 1$$

$$\text{if } a > b^d \Rightarrow T(n) = \Theta(n^{\log_b a})$$

$$\Rightarrow 4 > 2^1 \Rightarrow T(n) = \Theta(n^{\log_2 4}) = \boxed{\Theta(n^2)}$$

g) $T(n) = 2T(\sqrt[3]{n}) + 1, \; T(3) = 1$

Answer:

Assume that $m = \log_3 n$

$$T(n) = 2T(n^{1/3}) + 1$$

$$T(2^m) = 2T(2^{m/3}) + 1$$

$$B(m) = T(3^m)$$

$$B(m) = 2B(m/3) + 1 \quad (3^{m/3} = m/3 \text{ when } m = \log_3 n)$$

I use Master Teoren to salve this recurrence.

$$a = 2 \quad b = 3 \quad d = 0$$

$$2 > 3^0 \quad a > b^d \Rightarrow \Theta(m^{\log_b a}) \Rightarrow \Theta(m^{\log_3 2}) \Rightarrow$$

$$\boxed{T(n) = \Theta(\log_3 n^{0.63})}$$

Q2)

Answer:

How many lines function print?

Function $f(n)$ prints lines (n is a power of 2)

when $n = 1 \Rightarrow 1 = 2^0$

$n = 2 \Rightarrow 2 = 2^1$

$n = 4 \Rightarrow 8 = 2^3$

$n = 8 \Rightarrow 64 = 2^6$

$n = 16 \Rightarrow 1024 = 2^{10}$

$\vdots$

I fomulize this sequence $\rightarrow (\log n = \log_2 n)$

$$\boxed{\text{How many lines} = 2^{\frac{\log n (\log n + 1)}{2}}}$$

And the recurrence =

$$\boxed{T(n) = n \cdot T(n/2)}$$

backword Substitution

$T(n) = n \cdot (\frac{1}{2} \cdot T(n/4)) = n^2/2 \cdot T(n/4)$

$T(n) = n^3/2^4 \cdot T(n/8)$

$T(n) = n^4/2^8 \cdot T(n/2^4)$

$\vdots$

$T(n) = n^k / 2^{\log 2^{k-1} \cdot \frac{(\log 2^{k-1} + 1)}{2}} \cdot T(n/2^k)$

Assume $n = 2^k$ $\quad k = \log n \quad T(n) = n^{\log n} / 2^{(\log n - 1) \cdot \frac{((\log n - 1) + 1)}{2}} \cdot T(1)$

$$\Rightarrow \boxed{T(n) = \Theta(n^{\log n} / 2^{\frac{\log^2 n - 1 + \log n - 1}{2}})}$$

Q3)

the recurrence of the function f(array) =>

0 .... ceil(2n/3) = 2n/3 number
floor(n/3) ..... n = 2n/3 number
0 ..... ceil(2n/3) = 2n/3 number

So recurrence $T(n) = 3\,T(2n/3) + 1$

I use Master teorem to solve this recurrence

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + n^d \quad a \geq 1,\ b > 1$$

$$a = 3 \qquad b = \frac{3}{2} \qquad d = 0$$

$$3 > \left(\frac{3}{2}\right)^0$$

$$a > b^d \Rightarrow T(n) = \Theta(n^{\log_b a}) = \Theta(n^{\log_{3/2} 3})$$

$$= \boxed{T(n) = \Theta(n^{2.70})}$$

Q4)

$i+3$ $(i+1).i$
~~i+1~~ $2(i+1)$

Answer:

When number of element is 10 with random numbers then swap operation number of Insertion Sort is 20. But Quicksort's is 16. when number of element is 100 then swap operation of Insertion sort is 2545 but in quicksort, the number of swap operation is 291. So number of swap operation of quicksort is less than insertion sort's when number of element is bigger than 10.

Insertion sort average case complexity analysis

1 to i There are i+1 cases

$$P(\tau_i = J) = \begin{cases} \frac{1}{i+1} & \text{if } 1 < J < i-1 \\ \frac{2}{i+1} & \text{if } J = i \end{cases}$$

$$\frac{i(i+3)}{2(i+1)} = \frac{i}{2} + 1 - \frac{1}{i+1} \qquad E[\tau] = \sum_{i=1}^{n-1} E[\tau_i] = \sum_{i=1}^{n-1} \left(\frac{i}{2} + 1 - \frac{1}{i+1}\right)$$

$$\frac{n \cdot n-1}{4} + n - 1 - \underbrace{\sum_{i=1}^{n-1} \frac{1}{i+1}}_{H_{n-1}}$$

$$\frac{n \cdot (n-1)}{4} + n - 1 - (H_{n-1}) = \frac{n \cdot n-1}{4} + n - H_n$$

$$\Downarrow$$

$\boxed{\Theta(n^2) \text{ Average case}}$

Quicksort Average Case:

$$T(n) = \sum_{|I|=n} P(I) \times T(I)$$

Smallest pivot $\quad T(0) + T(n-1) + n+1$

$T(1) + T(n-2) + n+1$

$\vdots$

$+ \quad T(n-1) + T(0) + n+1$

$$\sum_{k=0}^{n-1} 2T(k) + (n+1)$$

$$T(n) = \frac{1}{n}\left(\sum_{k=0}^{n-1} 2T(k) + n(n+1)\right)$$

$$n \cdot T(n) = \sum_{k=0}^{n-1} 2T(k) + n(n+1)$$

$$(n-1) \cdot T(n-1) = \sum_{k=0}^{n-2} 2T(k) + n(n-1)$$

$=$

$$n \cdot T(n) = (n+1)T(n-1) + 2n$$

$$\frac{T(n)}{n+1} = \frac{T(n-1)}{n} + \frac{2}{n+1}$$

$$T(n) = \Theta(n \log n)$$

Quicksort is better algoritkm becouse of the avarage case of it and number of swap operation.

5)

Answer)

a) This is divide and conquer algorithm So, recurrence is $T(n) = 5T(\frac{n}{3}) + n^2$

$a=5 \quad b=3 \quad d=2$

$a < b^d \Rightarrow O(n^d)$

$5 < 3^2 \Rightarrow \boxed{T(n) = O(n^2)}$

b) This is divide and conquer algorithm. So, recurrence is $T(n) = 2T(n/2) + n^2$

$a=2 \quad b=2 \quad d=2$

$a < b^d \Rightarrow O(n^d)$

$2 < 2^2 \Rightarrow \boxed{T(n) = O(n^2)}$.

c) $T(n) = T(n-1) + n$

$T(n) = T(n-2) + n + n - 1 = T(n-2) + 2n - 1$

$T(n) = T(n-3) + n + n + n - 1 - 1 = T(n-3) - 2$

$\vdots$

$T(n) = T(n-k) + n \cdot n - k - 1$

Assume $n=k$ then $T(n) = T(0) + n^2 - n - 1$

$\boxed{\text{Then } T(n) = O(n^2)}$

$\rightarrow$ All of them are $O(n^2)$ so I can choose any of them. Therefore I choose b).