HW 4.4 ROB NAVARRO

12. PROVE $f_1^2 + f_2^2 + \dots + f_n^2 = f_n f_{n+1}$ WHEN n IS A POSITIVE INT

LET $P(n)$: $f_1^2 + f_2^2 + \dots + f_n^2 = f_n f_{n+1}$

BASE CASE: $f_1^2 = f_1 \cdot f_2$; $f_1 = 1$ $f_2 = 1$ $f_1^2 = 1$

INDUCTIVE CASE: ASSUMING $P(n)$ IS TRUE

$$f_1^2 + f_2^2 + \dots f_n^2 + f_{n+1}^2 = f_{n+1} f_{n+2}$$

$$\begin{aligned} f_1^2 + f_2^2 + \dots + f_n^2 + f_{n+1}^2 &= f_1^2 + f_2^2 + \dots + f_n^2) + f_{n+1}^2 \\ &= f_n f_{n+1} + f_{n+1}^2 \\ &= f_{n+1}(f_n + f_{n+1}) \\ &= f_{n+1} f_{n+2} \checkmark \end{aligned}$$

26. a) LET S BE THE SUBSET OF THE SET OF ORDERED PAIRS DEFINED BY

BASE CASE: $(0,0) \in S$

RECURSIVE CASE: IF $(a,b) \in S$, then $(a+2, b+3) \in S$ and $(a+3, b+2) \in S$

SHOW THAT $5 \mid (a+b)$ when $(a,b) \in S$

BASE CASE: $5 \mid (0+0) = 0$

INDUCTIVE CASE: ASSUME THAT $5 \mid a+b$ SO $a+b = 5k$

case 1) $5 \mid (a+2+b+3)$

$= a+b+5 = 5k+5 = 5(k+1) \checkmark$

case 2) $5 \mid (a+3+b+2)$ where $k+1$ is some int

$= a+b+5 = 5k+5 = 5(k+1) \checkmark$

43. USE STRUCTURAL INDUCTION TO SHOW THAT $n(T) \geq 2h(T)+1$, WHERE T IS A FULL BINARY TREE, $n(T)$ EQUALS THE # OF VERTICES OF T, AND $h(T)$ IS the HEIGHT OF T.

BASE CASE: THE ROOT r IS A LEAF OF THE FULL BINARY TREE WITH EXACTLY ONE VERTEX r. THIS TREE HAS NO INTERNAL VERTICES.

RECURSIVE CASE: THE SET OF LEAVES OF THE TREE $T = T_1 \cdot T_2$ IS THE UNION OF THE SETS OF LEAVES OF T_1 AND OF T_2. THE INTERNAL VERTICES OF T ARE THE ROOT r of T AND THE UNION OF THE SET OF INTERNAL OF T_1 AND THE SET VERTICIES OF T_2

PROOF

BASE CASE: $n(T) = 1$ AND $h(T) = 0$; $1 \geq 2 \cdot 0 + 1$

RECURSIVE CASE: ASSUME THE RESULTS HOLD FOR ALL BINARY TREES SMALLER THAN T.

$$n(T) \geq 2h(T) + 1$$

$$n(T_1) \geq 2h(T_1) + 1 \qquad n(T_2) \geq 2h(T_2) + 1$$

$$n(T) = 1 + n(T_1) + n(T_2) \quad \text{and} \quad 1 + \max(h(T_1), h(T_2))$$

$$\begin{aligned} n(T) &= 1 + n(T_1) + n(T_2) \\ &\geq 1 + 2h(T_1) + 1 + 2h(T_2) + 1 \\ &\geq 1 + 2\max(h(T_1), h(T_2)) + 2 \\ &= 1 + 2(\max(h(T_1), h(T_2)) + 1) \\ &= 1 + 2h(T) \checkmark \end{aligned}$$

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44. USE STRUCTURAL INDUCTION TO SHOW THAT $l(T)$, the number OF LEAVES OF A FULL BINARY TREE T, IS ONE MORE THAN $i(T)$, the NUMBER OF INTERNAL VERTICES OF T.

BASE CASE: A SINGLE root r HAS NO INTERNAL VERTICIES ✓

INDUCTIVE CASE: ASSUME THAT RESULTS HOLD FOR ALL FULL BINARY TREES SMALLER THAN T.

$$l(T_1) = i(T_1) + 1 \qquad l(T_2) = i(T_2) + 1 \qquad \begin{aligned} l(T) &= l(T_1) + l(T_2) \\ i(T) &= i(T_1) + i(T_2) + 1 \end{aligned}$$

$$\begin{aligned} l(T) &= l(T_1) + l(T_2) \\ &= i(T_1) + 1 + i(T_2) + 1 \\ &= i(T) + 1 \checkmark \end{aligned}$$