Basic derivations for field theoretical Descriptions

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1 Maxwell

The Maxwell equations are probably the most general equations to describe electromagnetic fields. It's a system of 4 differential equations, which in the local form are:

$$\nabla \cdot D = \rho \tag{1}$$
$$\nabla \cdot B = 0 \tag{2}$$
$$\nabla \times E = -\delta_t B \tag{3}$$
$$\nabla \times H = J + \delta_t D, \tag{4}$$

(1) is called the Gauss's Law
(2) is called the Gauss's Law for magnetic fields
(3) is called MaxwellFaraday equation (Induction Law)
(4) is called Ampere's circuital law (with Maxwell's addition)

The charge retention describes that no macroscopic electrical charges can be created or destroyed. The charge retention $\nabla \cdot J = -\delta_t \rho$ is implicitly included in the last equation (4).

The electric field strength $\vec{E}$ is caused by the interaction of the electric flux with the electrical polarization of the material, which is caught in the permittivity equation for linear and isotropic polarizable material:

$$\vec{D} = \epsilon \vec{E} = \epsilon_r \epsilon_0 \vec{E}$$

Similarly, magnetic flux and field strength can mostly be described by a linear isotropic relationship:

$$\vec{B} = \mu \vec{H} = \mu_r \mu_0 \vec{H}$$

The more general forms involving polarization are

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

for the electric field and

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right)$$

for the magnetic field but will not be treated further here.

2 Electrostatics

In Electrostatics we only look at constant electrical fields free of magnetic effects. In quasi-electrostatics we additionally allow constant magnetic fields which indirectly implies that constant currents are allowed. The field is static (constant over time):

$$\delta_t \vec{D} = \delta_t \epsilon_r \epsilon_0 \vec{E} = 0$$

The effects of magnetic induction are insignificant:

$$\nabla \times E = -\delta_t B = 0$$

Not all Maxwell equations are necessary to describe the fields, but only:

$$\nabla \cdot D = \rho$$

and for quasi-electrostatics additionally:

$$\nabla \times H = J,$$

3 Point charge as a charge density

We start with equation (1):

$$\nabla \cdot \vec{D} = \rho$$

and ask ourselves, how can a point charge as a charge q with no volume be described by a charge density ρ: Let's suppose the charge has a spherical shape with radius r and thus the volume:

$$V = \frac{4}{3}\pi r^3.$$

The charge density ρ thus becomes:

$$\rho = \frac{q}{V} = \frac{3q}{4\pi r^3}$$

For the idealized representation as a point charge the radius r degrades to 0:

$$\lim_{r \to 0} \frac{3q}{4\pi r^3} = \infty$$

The charge density is infinity at the location of the charge (the origin) and 0 elsewhere, while the integral over any finite volume including the point of the charge has to be q. This can be represented by a dirac delta function that has exactly these properties:

$$\rho = q\delta(r)$$

4 Electric field of a point charge

Because the field is radial symmetric, the field of a point charge can be described by a sphere of constant charge density with the same radius as the point of interest for the field calculation:

$$\rho = q\delta(r)$$

The first Maxwell equation leads to:

$$\nabla \cdot \vec{D} = \rho = q\delta(r)$$

It's hard to read it out of this equation but this directly leads to the electrical flux $\vec{D}$ through vector analysis. The link is the divergence theorem that also connects the local form of the Maxwell equations we have used to the volumetric versions.

But in order to have a more intuitive way of getting there, we start differently. The total flux through a surface $A = \delta V$ of a volume V is defined by the charge it contains, which is the volume integral over the charge density ρ (this is actually Gauss's law):

$$\oint_A \vec{D}\vec{e}_n d^2A = \int_V \rho d^3V = \int_V \nabla \cdot \vec{D} d^3V$$

This already resembles the divergence theorem, which can also be proven and derived analytically. For any vector field $\vec{E}$:

$$\oint_A \vec{E}\vec{e}_n d^2A = \int_V \nabla \cdot \vec{E} d^3V$$

In our case, it's pretty simple: The total charge in any volume including the point charge is actually the point charge itself, as described before:

$$\int_V \rho d^3V = \int_V q\delta(r) d^3V = q$$

which in combination with Gauss's law gives:

$$\oint_A \vec{D}\vec{e}_n d^2A = \int_V \rho d^3V = q$$

If we set the Volume as a sphere with radius r and differentiate with respect to the sphere surface $A = \delta V = 4\pi|\vec{r}|^2$:

$$\frac{d^2}{dA} \oint_A \vec{D}\vec{e}_n d^2A = \vec{D} \cdot \vec{e}_n = \vec{D} \cdot \vec{e}_r = \frac{q}{4\pi|\vec{r}|^2}$$

because the field is radial symmetric and thus has only flux in a radial direction equally spread over the whole surface of the sphere. If we multiply it with the radial $\vec{e}_r$ direction, it leads to:

$$\vec{D} = \frac{q}{4\pi|\vec{r}|^2} \cdot \vec{e}_r$$

The corressponding electric field is:

$$\vec{E} = \frac{q}{4\pi\epsilon|\vec{r}|^2}\vec{e}_r = \frac{q}{4\pi\epsilon}\frac{\vec{r}}{|\vec{r}|^3}$$

because $\vec{e}_r = \frac{\vec{r}}{|\vec{r}|}$.

This resembles the electric field at point $\vec{r}$ of a single point charge located at the origin in an infinite space and is one of the most basic electrostatic equations. It decreases with a factor $|\vec{r}|^{-2}$ with the distance.

Interestingly, the derivations in this chapter also resemble a basic principle for electrostatics. The electric flux of a point charge q in an otherwise charge free space can be equivalently described by a constant charge density $\rho = \frac{q}{V} = \frac{3q}{4\pi|\vec{r}|^3}$ of a sphere in the subspace outside of this sphere.

5 Electric field of a dipole

An electrical dipole is the field of two contrarily signed but equally large point charges. Mostly, if people talk about dipoles, the far field is meant, which is the field far away from the two charges. But this is only an approximation that works if the distance to the charges r is much bigger than the distance between the two charges d.

Let's start with the general Electrical Field by two contrarily signed charges being set $\vec{d}$ apart from each other with one at position $\vec{r}$, by simply adding up the point charge fields:

$$\vec{E} = \frac{q}{4\pi\epsilon}\frac{\vec{r}-\vec{d}}{|\vec{r}-\vec{d}|^3} - \frac{q}{4\pi\epsilon}\frac{\vec{r}}{|\vec{r}|^3} = \frac{q}{4\pi\epsilon}\left(\frac{\vec{r}-\vec{d}}{|\vec{r}-\vec{d}|^3} - \frac{\vec{r}}{|\vec{r}|^3}\right)$$

The clue to simplify this eqation for the far field is a Taylor expansion on the denominator transformed to the shape $(1+s)^{-\frac{3}{2}}$ to get rid of the fraction:

$$(1+s)^{-\frac{3}{2}} = 1 - \frac{3}{2}s + \frac{15}{8}s^2 + ...$$

We apply this on $\frac{1}{|r-d|^3}$:

$$\begin{aligned}
\frac{1}{|r-d|^3} &= \left[\left(\vec{r}-\vec{d}\right)\cdot\left(\vec{r}-\vec{d}\right)\right]^{-\frac{3}{2}} \\
&= \left(\vec{r}\cdot\vec{r} - 2\vec{r}\cdot\vec{d} + \vec{d}\cdot\vec{d}\right)^{-\frac{3}{2}} \\
&= \left(|\vec{r}|^2 - 2\vec{r}\cdot\vec{d} + |\vec{d}|^2\right)^{-\frac{3}{2}} \\
&= |\vec{r}|^{-3}\left(1 - \frac{2\vec{r}\cdot\vec{d} + |\vec{d}|^2}{|\vec{r}|^2}\right)^{-\frac{3}{2}} \\
&= |\vec{r}|^{-3}\left[1 + \frac{3}{2}\frac{2\vec{r}\cdot\vec{d} + |\vec{d}|^2}{|\vec{r}|^2} + \frac{15}{8}\left(\frac{3}{2}\frac{2\vec{r}\cdot\vec{d} + |\vec{d}|^2}{|\vec{r}|^2}\right)^2 + ...\right]
\end{aligned}$$

and then insert this back into our basic equation:

$$\begin{aligned}
\vec{E} &= \frac{q}{4\pi\epsilon}\left(\frac{\vec{r}-\vec{d}}{|\vec{r}-\vec{d}|^3} - \frac{\vec{r}}{|\vec{r}|^3}\right) \\
&= \frac{q}{4\pi\epsilon}\left\{\left(\vec{r}-\vec{d}\right)|\vec{r}|^{-3}\left[1 - \frac{3}{2}\frac{2\vec{r}\cdot\vec{d}+|\vec{d}|^2}{|\vec{r}|^2} + \frac{15}{8}\left(\frac{3}{2}\frac{2\vec{r}\cdot\vec{d}+|\vec{d}|^2}{|\vec{r}|^2}\right)^2 + ...\right] - \frac{\vec{r}}{|\vec{r}|^3}\right\} \\
&= \frac{q}{4\pi\epsilon}\left\{\frac{\vec{r}}{|\vec{r}|^3} - \frac{\vec{r}}{|\vec{r}|^3} + 3\frac{\vec{r}\cdot\vec{d}}{|\vec{r}|^5}\cdot\vec{r} - \frac{\vec{d}}{|\vec{r}|^3} - 3\frac{\vec{r}\cdot\vec{d}}{|\vec{r}|^5}\cdot\vec{d} + ...\right\} \\
&= \frac{q}{4\pi\epsilon}\left\{3\frac{\vec{r}\cdot\vec{d}}{|\vec{r}|^5}\cdot\vec{r} - \frac{\vec{d}}{|\vec{r}|^3} + O\left[\left(\frac{|\vec{d}|}{|\vec{r}|}\right)^2\right]\right\} \\
&\approx \frac{q}{4\pi\epsilon}\left(3\frac{\vec{r}\cdot\vec{d}}{|\vec{r}|^5}\cdot\vec{r} - \frac{\vec{d}}{|\vec{r}|^3}\right)
\end{aligned}$$

if we omit terms of $O\left[\left(\frac{|\vec{d}|}{|\vec{r}|}\right)^2\right]$ because $|\vec{r}| \gg |\vec{d}|$. This equation is a simple linear approximation for the far field! If we define the dipole moment $\vec{p} = q\vec{d}$, we get the well known dipole equation for the electric field:

$$\vec{E} = \frac{1}{4\pi\epsilon}\left(3\frac{\vec{r}\cdot\vec{p}}{|\vec{r}|^5}\cdot\vec{r} - \frac{\vec{p}}{|\vec{r}|^3}\right)$$

It decreases with a factor $|\vec{r}|^{-3}$ with the distance, so faster than that of a unipole because the contrarily signed charges cancel each other out macroscopically.