27/8/18

Queuing Model
- Single server
  - M:M:1 | ∞:∞
  - M:M:1 | N:∞
- Multi server
  - M:M:C | ∞:∞
  - M:M:C | N:∞

Kendall's Notation

M:M:C | ∞:∞

$$p_n = R^n * p_0$$

Now we have three possibilities :-

Customer = Server $(n=c)$
Customer > Server. $(n>c)$
Customer < Server. $(n<c)$

When $n<c$.

$$\lambda_n = \lambda.$$
$$\mu_n = n\mu.$$

$$p_n = R^n p_0$$
$$= \frac{\lambda^n}{(n\mu)^n} p_0$$
$$= \frac{\lambda^n}{\mu^n \cdot n!} * p_0$$
$$\boxed{= \frac{\lambda^n}{\mu^n \cdot n!} \cdot p_0}$$

When $n \geq c$.

$$\lambda_n = \lambda$$
$$\mu_n = c\mu.$$

$$p_n = R^n p_0$$
$$= \frac{\lambda^n}{(c\mu)^n} \cdot p_0$$

When $n < c$

Suppose we have 2 customers & 4 server.

$$P^n = \frac{\lambda^n}{\mu \cdot 2\mu} P_0$$

$$= \frac{\lambda^n}{\mu^n (1 \cdot 2)} P_0$$

$$= \frac{\lambda^n}{\mu^n 2!} P_0$$

$$\boxed{P^n = \frac{\lambda^n}{\mu^n \cdot n!} \cdot P_0}$$

When $n > c$

Suppose we have 4 customers & 4 server -

$$P^n = \frac{\lambda^n \cdot P_0}{(\underset{\text{1st cust}}{\mu} \cdot \underset{\text{2nd cust}}{2\mu} \cdot \underset{\text{3rd cust}}{3\mu} \cdot \underset{\text{4th cust}}{4\mu}) \underbrace{(4\mu \cdot 4\mu \cdot 4\mu \cdots)}_{>4 \text{ cust.}}}$$

$$P^n \boxed{= \frac{\lambda^n}{\mu^n \cdot c! \cdot c^{n-c}} \cdot P_0}$$

$\left\{ \begin{array}{l} n = \text{no. of cu} \\ c = \text{no. of ser} \end{array} \right.$

Probability of 0 customer.

$$\Rightarrow p_0 + p_1 + p_2 + \cdots\cdots + p_\infty = 1.$$

<u>when $n < c$</u> .   <u>when $n \geq c$</u>

$$\Downarrow \qquad\qquad \Downarrow$$

$$\Rightarrow \sum_{n=0}^{n=c-1} p_0 \frac{R^n}{n!} + \sum_{n=c}^{\infty} \frac{R^n}{c! \cdot c^{n-c}} p_0 = 1$$

$$\Rightarrow p_0 \left[\sum_{n=0}^{c-1} \frac{R^n}{n!} + \sum_{n=c}^{\infty} \frac{R^{n+c-c}}{c!\, c^{n-c}} \right] = 1$$

$$\Rightarrow p_0 \left[\sum_{n=0}^{n=c-1} \frac{R^n}{n!} + \sum_{n=c}^{\infty} \frac{R^c}{c!} \left(\frac{R}{c}\right)^{n-c} \right] = 1$$

$$\Rightarrow p_0 \left[\sum_{n=0}^{n=c-1} \frac{R^n}{n!} + \frac{R^c}{c!} \sum_{n=c}^{\infty} \left(\frac{R}{c}\right)^{n-c} \right] = 1.$$

$$\Rightarrow p_0 \left[\sum_{n=0}^{n=c-1} \frac{R^n}{n!} + \frac{R^c}{c!} \left[\left(\frac{R}{c}\right)^0 + \left(\frac{R}{c}\right)^1 + \cdots\cdots + \infty \right] \right] = 1$$

$$p_0 = \boxed{\dfrac{1}{\displaystyle\sum_{n=0}^{c-1} \frac{R^n}{n!} + \frac{R^c}{c!}\left(\frac{1}{1-\frac{R}{c}}\right)}}$$

Expected length of the queue:

$$L_q = \sum n \cdot p_n$$

For case $n<c$ (There will be no queue formation so this formula doesn't hold).

This will be only applicable for $n \geq c$.

$$= \sum_{n=c}^{\infty} (n-c) R^n * p_0$$

Put $n-c=j$

$n=j+c$

$$= \sum_{j=0}^{\infty} j R^{c+j} * p_0$$

$$= p_0 \sum_{j=0}^{\infty} j R^{c+j}$$

$$\cancel{= p_0 \left[0 + R^c + \right.}$$

$$L_q = \cancel{p_0} \sum_{j=0}^{\infty} j \frac{\lambda^{c+j}}{\mu^{c+j} c! \, c^j} \cdot p_0 .$$

$$= \sum_{j=0}^{\infty} j \cdot \frac{\lambda^{c+j-1+1}}{\mu^{c+j-1+1} \cdot c! \cdot c^{j+1-1}} p_0 .$$

$$= \sum_{j=0}^{\infty} j \cdot \frac{R^{c+j-1+1}}{c! \cdot c^{j+1-1}} p_0 .$$

$$= \sum_{j=0}^{\infty} j \cdot \frac{p_0}{c!} \cdot \frac{R^{c+1}}{c} \cdot \frac{R^{j-1}}{c^{j-1}}$$

$$\frac{p_0}{c!} \frac{R^{c+1}}{c} \sum_{j=0}^{\infty} j \cdot \frac{R^{j-1}}{c^{j-1}}$$

$$= \frac{R^{c+1}}{c!\,c} \cdot p_0 \sum_{j=0}^{\infty} j * \left(\frac{R}{c}\right)^{j-1}$$

$$= \frac{R^{c+1}}{c!\cdot c} * p_0 \frac{d}{d(R/c)} \left[\sum_{j=0}^{\infty}\left(\frac{R}{c}\right)^{j}\right]$$

$$= \frac{R^{c+1}}{c!\cdot c} * p_0 \left[\frac{d}{d(R/c)}\left[1 + \frac{R}{c} + \cdots \infty\right]\right]$$

$$= \frac{R^{c+1}}{c!\cdot c} * p_0 \left[\frac{d}{d(R/c)}\left[\frac{1}{1-(R/c)}\right]\right]$$

$$= \frac{R^{c+1}}{c!\cdot c}\, p_0 \left[\frac{0 - 1\cdot(-1)}{\left(1-\left(\frac{R}{c}\right)\right)^2}\right]$$

$$= \frac{R^{c+1}}{c!\cdot c} \cdot p_0 \left[\frac{1}{\left(1-\frac{R}{c}\right)^2}\right]$$

$$= \frac{R^{C+1}}{(C-1)! \cdot \cancel{c \cdot c}} \cdot p_0 \left[\frac{\cancel{c^2}}{(C-R)^2} \right]$$

$$\boxed{L_q = \frac{R^{C+1}}{(C-1)!} * p_0 \left[\frac{1}{(C-R)^2} \right]}$$

$$\boxed{L_s = L_q + R}$$ When $n \geq c$

$$\boxed{W_q = \frac{L_q}{\lambda}}$$

$$\boxed{W_s = \frac{L_s}{\lambda}}$$

When $n \geq c$ When $n \geq c$

All the formula of L_q, L_s, W_q and W_s are applicable for $n \geq c$.

28/8/18

Q $C = 2$; $\lambda = 10$; $\mu = 6$. (∞)

(i) $$p_0 = \frac{1}{\sum_{n=0}^{1} \frac{R^n}{n!} + \frac{R^c}{c!}\left(\frac{1}{1-\frac{R}{c}}\right)}$$

$R = \frac{10}{6}$

$$= \frac{1}{1+\left(\frac{10}{6}\right)^1 + \frac{\left(\frac{10}{6}\right)^2}{2!}\left(\frac{1}{1-\frac{10}{6\times 2}}\right)}$$

$$= \frac{1}{1+\cancel{\frac{10}{6}}^{5}_{3} + \frac{\cancel{100}^{25}}{\cancel{36}_{3}\times\cancel{2}} \times \frac{\cancel{12}}{\cancel{2}}}$$

$$= \frac{1}{\frac{3+5+25}{3}} = \frac{3}{33} = \frac{1}{11}$$

$= 0.090$

(ii) $$L_q = \frac{R^{c+1}}{(c-1)!} * p_0 \left[\frac{1}{(c-R)^2}\right]$$

33 | 125
99
260
11)60(
55

$$= \frac{\left(\frac{10}{6}\right)^3}{11} * \frac{1}{11} \times \left[\frac{1}{\left(2-\frac{10}{6}\right)^2}\right]$$

$$= \frac{\cancel{1000}^{\cancel{250}\ 125}}{\cancel{36\times 6}_{\,3}} \times \frac{1}{11} \times \frac{\cancel{36}}{\cancel{4}}$$

$$= \frac{125}{33} = 3.78 \rightarrow \underline{\text{Ans}}$$

(iii) $L_s = L_q + R$

$$= \frac{125}{33} + \frac{\cancel{10}\,5}{\cancel{6}\,3}$$

$$= \frac{125+55}{33} = \frac{\cancel{180}^{\,60}}{\cancel{33}_{\,11}}$$

$$= \frac{60}{11} = 5.43 \rightarrow \underline{\text{Ans}}$$

(iv) $W_q = \frac{L_q}{\lambda} = \frac{\cancel{125}^{\,25}}{33\times\cancel{10}_{\,2}} = \frac{25}{66} \rightarrow \underline{\text{Ans}}$

$= 0.378$

(v) $W_s = \frac{L_s}{\lambda} = \frac{6\cancel{0}}{11\times 1\cancel{0}} = \frac{6}{11} \rightarrow \underline{\text{Ans}}$

$= 0.54$.

$M : M : C \mid N : \infty \rightarrow$ Infinite population length

$\rightarrow$ finite queue length.

$\rightarrow$ No: of server.

Arrival rate λ/hr.

Service rate μ/hr

General formula.

$n < c$	$n \geq c$
$P_n = \frac{R^n}{n!} p_0$	$P_n = \frac{R^n}{c!\, c^{n-c}} p_0$
$\lambda_n = \lambda$	$\lambda_N = \lambda$.
$\mu_n = n\mu$.	$\mu_N = c\mu$.

Probability of no customer.

$$p_0 + p_1 + p_2 \cdots + p_N = 1$$

$$p_0 \left[\sum_{n=0}^{c-1} \frac{R^n}{n!} + \sum_{n=c}^{N} \frac{R^{n+c-c}}{c!\, c^{n-c}} \right]$$

$$= p_0 \left[\sum_{n=0}^{c-1} \frac{R^n}{n!} + \frac{R^c}{c!} \sum_{n=c}^{N} \left(\frac{R^{n-c}}{c^{n-c}} \right) \right] = 1$$

$$p_0 \left[\sum_{n=0}^{c-1} \frac{R^n}{n!} + \frac{R^c}{c!} \left[\left(\left(\frac{R}{c}\right)^0 + \left(\frac{R}{c}\right)^1 + \cdots + \left(\frac{R}{c}\right)^{N-c} \right] \right] = 1$$

$$p_0 \left[\sum_{n=0}^{c-1} \frac{R^n}{n!} + \frac{R^c}{c!} \left[\frac{1 - \left(\frac{R}{c}\right)^{N-c+1}}{1 - \left(\frac{R}{c}\right)} \right] \right] = 1$$

$$\boxed{\Rightarrow p_0 = \frac{1}{\sum_{n=0}^{c-1} \frac{R^n}{n!} + \frac{R^c}{c!} \left(\frac{1 - \left(\frac{R}{c}\right)^{N-c+1}}{1 - \left(\frac{R}{c}\right)} \right)}}$$

30/8/18.

<u>Expected length of the Queue :-</u>

$$L_q = \sum (n-c) p_n$$

(This will only be valid $n > c$)

$$= \sum_{n=c}^{N} (n-c) \frac{R^n}{c!\, c^{n-c}} p_0$$

$n = c$ to $n = N$

$n - c = 0$ to $n - c = N - c$

$\Downarrow$

j

$$= \sum_{j=0}^{j=N-c} j \frac{R^{c+j+1-1}}{c!\, c^{j+1-1}} p_0$$

$$= \frac{R^{c+1}}{c! \cdot c} p_0 \sum_{j=0}^{j=N-c} j \cdot \left(\frac{R}{c}\right)^{j-1}$$

$$\because \frac{dx^n}{dx} = n x^{n-1}$$

$$= \frac{R^{c+1}}{c! \cdot c} p_0 \times \sum_{j=0}^{j=N-c} \frac{d \left(\frac{R}{c}\right)^j}{d(R/c)}$$

$$= \frac{R^{c+1}}{c! \cdot c} p_0 \times \frac{d}{d(R/c)} \sum_{j=0}^{j=N-c} \left(\frac{R}{c}\right)^j$$

$$= \frac{R^{c+1}}{c! \cdot c} p_0 \times \frac{d}{d(R/c)} \left[1 + \left(\frac{R}{c}\right)^1 + \left(\frac{R}{c}\right)^2 + \cdots \left(\frac{R}{c}\right)^{N-c} \right]$$

$$= \frac{R^{c+1}}{c! \cdot c} p_0 \left[\frac{d}{d(R/c)} \left(\frac{1 - \left(\frac{R}{c}\right)^{N-c+1}}{1 - \left(\frac{R}{c}\right)} \right) \right]$$

$$= \frac{R^{c+1}}{c! \cdot c} p_0 \left[\frac{\left(1 - \left(\frac{R}{c}\right)\right)\left(-(N-c+1)\right)\left(\frac{R}{c}\right)^{N-c} + \left(1 - \left(\frac{R}{c}\right)^{N-c+1}\right)}{\left(1 - \left(\frac{R}{c}\right)\right)^2} \right]$$

$$= \frac{R^{c+1}}{c! \cdot c} p_0 \left[\frac{(c-N-1)\left(\frac{R}{c}\right)^{N-c}\left(1 - \frac{R}{c}\right) + 1 - \left(\frac{R}{c}\right)^{N-c+1}}{\left(1 - \frac{R}{c}\right)^2} \right]$$

$$= \frac{R^{c+1}}{c! \cdot \cancel{c}} p_0 \cdot c^{\cancel{2}} \left[\frac{(c-N-1)\left(\frac{R}{c}\right)^{N-c}\left(1 - \frac{R}{c}\right) + 1 - \left(\frac{R}{c}\right)^{N-c+1}}{(c-R)^2} \right]$$

$$L_q = \frac{R^{c+1}}{(c-1)!} P_0 \left[\frac{\left(1-\left(\frac{R}{c}\right)^{n-c+1}\right) - (n-c+1)\left(1-\frac{R}{c}\right)}{(c-R)^2} \right]$$

$$L_s = L_q + R.$$

$$\lambda_{eff} = \lambda(1 - P_n)$$

$$W_s = \frac{L_s}{\lambda_{eff}}$$

$$W_q = \frac{L_q}{\lambda_{eff}}$$

Assignment-1.

Write all the formulas of all 4 models done till now.

Q There is a restaurant and people are coming by car. When one car will go then another car will come in and get served.

IAT	$P(x)$
1	0·4
2	0·3
3	0·15
4	0·15

ST	$P(x)$
1	0·2
2	0·4
3	0·4

RN

AT	37	60	79	21	85	71	48	39	31	35
ST	66	74	90	25	29	72	17	55	15	36

Calculate for next 10 cars.

IAT	$P(x)$	C.P.	RI
1	0·4	0·40	0-39
2	0·3	0·70	40-69
3	0·15	0·85	70-84
4	0·15	1·0	85-99

ST	$P(x)$	C.P	RI
1	0·2	0·2	0-19
2	0·4	0·6	20-59
3	0·4	1·0	60-99

	①		②	
Cars	RN	IAT	RN	ST
1	–	–	66	3
2	37	1	74	3
3	60	2	90	3
4	79	3	25	2
5	21	1	29	2
6	85	4	72	3
7	71	3	17	1
8	48	2	55	2
9	39	1	15	1
10	31	1	36	2

Cars	IAT	AT	ST	Time SB	WT Time	Time serviced TSE	Time spent in sys	Idle time
1	–	0	3	0	0	3	3	0
2	1	1	3	3	2	6	5	0
3	2	3	3	6	3	9	6	0
4	3	6	2	9	3	11	5	0
5	1	7	2	11	4	13	6	0
6	4	11	3	13	2	16	5	0
7	3	14	1	16	2	17	3	0
8	2	16	2	17	1	19	3	0
9	1	17	1	19	2	20	3	0
10	1	18	2	20	2	22	4	0
	18		22		21		40	0

$$\text{Avg. wt time} = \frac{21}{10} = 2.1$$

31/8/18

Tutorial - 3

(1)

Time b/w Arrivals	Probability	CP	RI
5	0.25	0.25	0-24
8	0.40	0.65	25-64
10	0.20	0.85	65-84
15	0.15	1.0	85-99

(2)

ST	P(x)	CP	RI
10	0.30	0.30	0-29
15	0.28	0.58	30-57
20	0.25	0.83	58-82
45	0.17	1.0	83-99

(3)

ST	P(y)	CP	RI
12	0.35	0.35	0-34
17	0.25	0.60	35-59
23	0.20	0.80	60-79
25	0.20	1.0	80-99

RNAT	35	81	20	23	31
ST(x)	15	16	10	13	46
ST(y)	17	17	12	20	48

Simulate the environment for 5 customers.

(4) (5)

Customer	RN	IAT	RN	ST(x)
1	-	-	15	10
2	35	8	16	10
3	81	10	10	10
4	20	5	13	10
5	23	5	46	15

(6)

RN	ST(y)
17	12
17	12
12	12
20	12
48	17

Q. Calculate (1) AWT for customer in first and second ~~customer~~ server

(2) The fraction of idle time of both ~~customers~~ server.

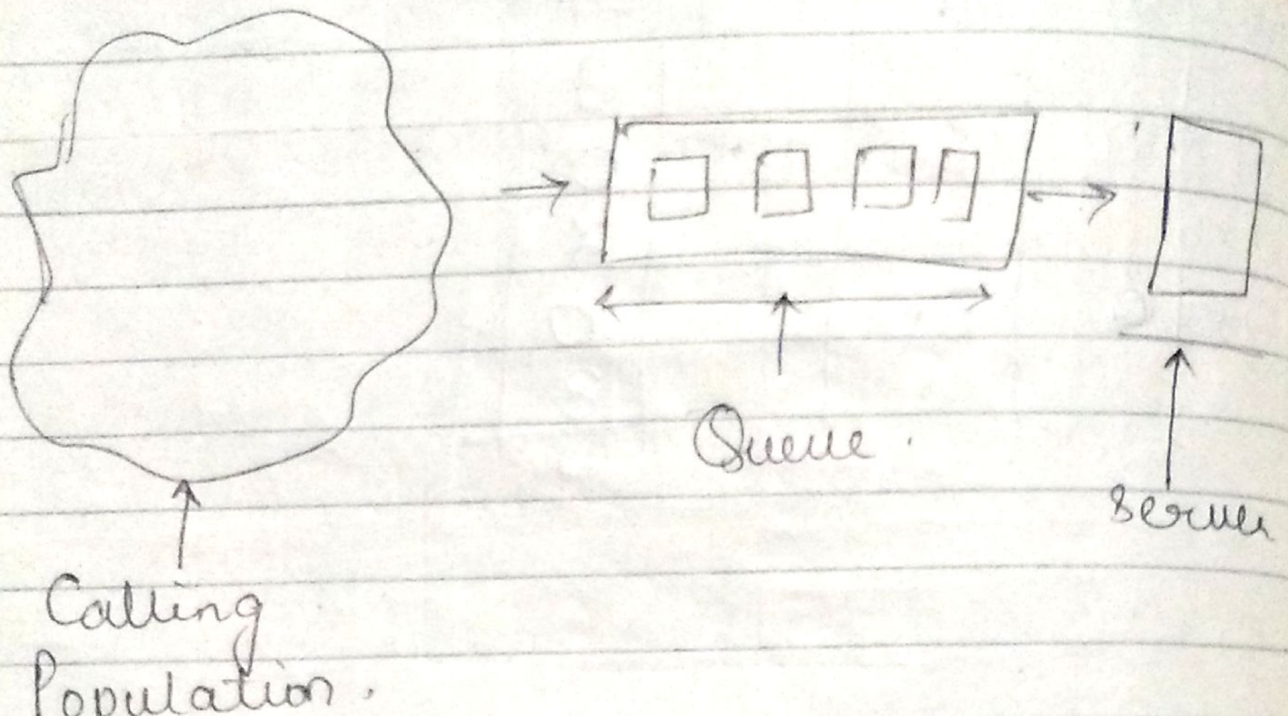
(X) Y

Custom	IAT	AT	ST (x)	STB (x)	WT (x)	STE (x)	TTIS (x)	IT (x)	STB (y)	WT (y)	STE (y)	TTIS (y)	IT (y)
1	–	0	10	0	0	10	10	0					8
2	8	8						8	8	0	20	12	0
3	10	18	10	18	0	28	10	0					3
4	5	23						0	23	0	35	12	0
5	5	28	15	28	0	43	15	0					8
								8					19

① AWT(x) = 0 AWT(y) = 0

② $FIT(x) = \frac{8}{43} \times 100$ $FIT(y) = \frac{19}{43} \times 100$

$= \frac{800}{43} =$ $\frac{1900}{43} =$



Event :-

① Arrival event
② Departure event.

State of system

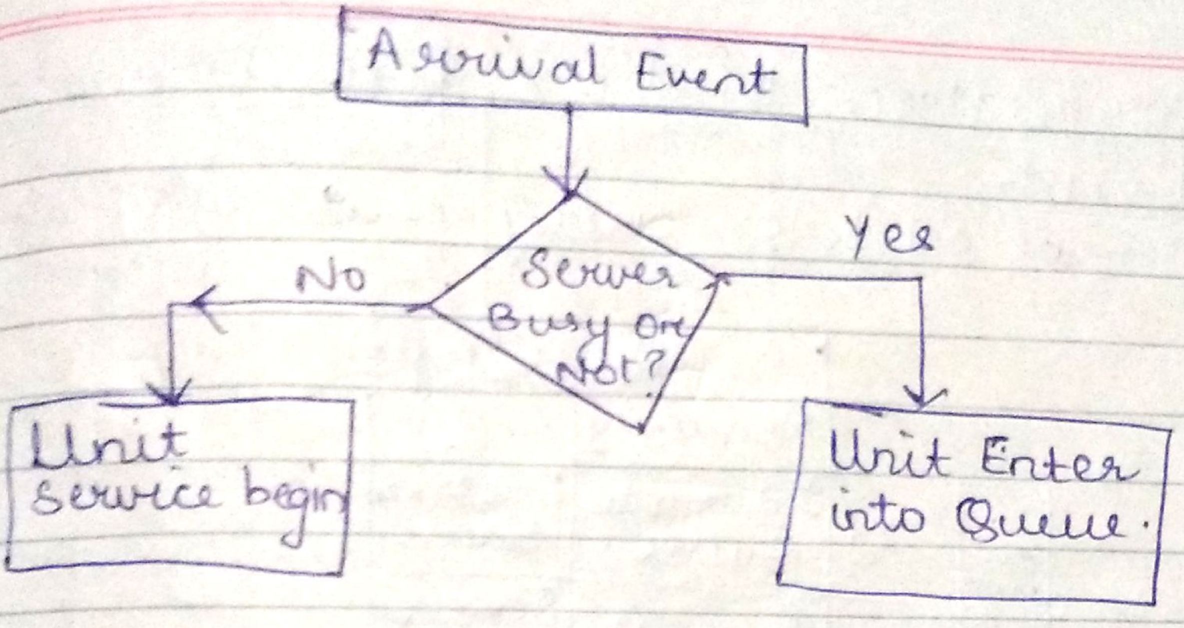
① No:- of People in queue
② Is server busy or idle

The two possible events that can affect the state of the system :-

① Arrival event:- The entry of unit into the system.

② Departure :- The completion of service

Flow chart of arrival event

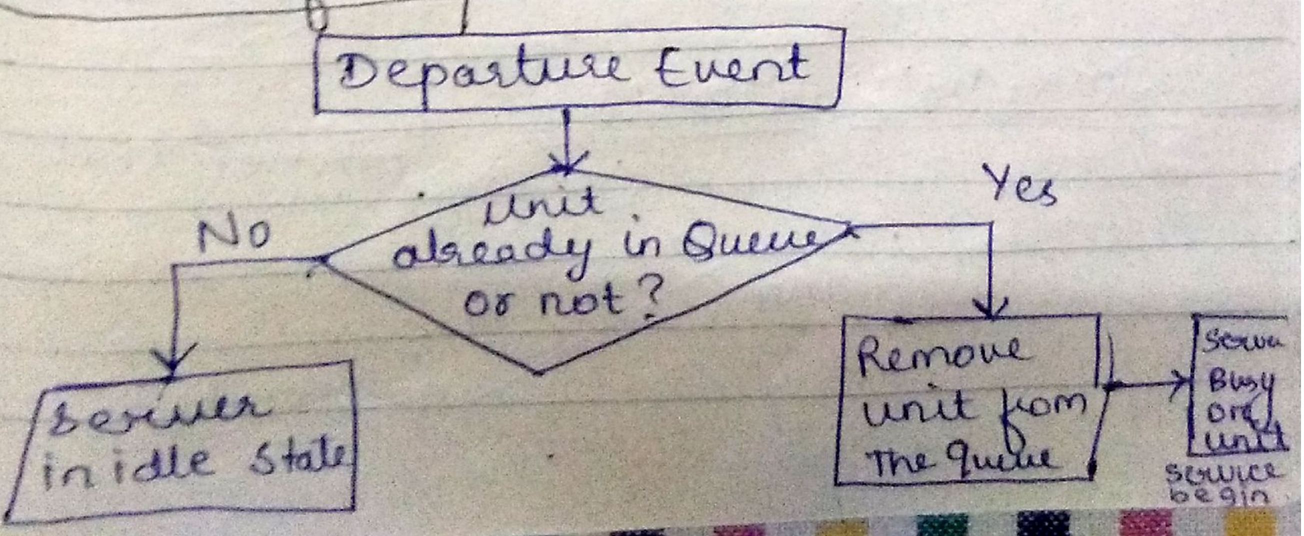


In case of arrival the unit may find the server to be busy or ideal.

Ideal → It means that unit's service begin immediately.

Busy :- It means that the unit joins the server queue and waits for its Turn for service.

Flowchart of Departure event.



Cor-relation Matrix for Arrival Event.

(i.e, we check for every new entry into the system)

		Queue length	
		Non-empty	Empty
Server	Busy	Enter into Queue.	Enter into Queue.
	Idle	Impossible case.	Service Begin.

Cor-relation Matrix for Departure Event. (i.e, we will check for every departure)

		Queue.	
		NEQ	EQ
Server Outcome	Busy	Possible	Possible
	Idle	Impossible	Impossible

Inventory Model.

Demand
- Independent
- Dependent

Eg:- If a car is completely build
i.e, "dependent demand.

Now that car needs petrol to run
then it is dependent demand

Types of inventory.

1. Raw materials inventory
2. Finished product "
3. Intermediate " "
4. Transportation "

Q Perform simulation of inventory system given daily demand represent by random numbers:-

4 3 18 22 55

and demand prob. is:-

Demand	P(n)
1	0.17
2	0.33
3	0.50

given that initial inventory = 6 units.
Find out the shortage occuring which day.

Demand	P(x)	CP(x)	RN(I)
1	0.17	0.17	0 - 16
2	0.33	0.50	17 - 49
3	0.50	1.00	50 - 99

Days	RN	Demand	Start stock	Ending	Shortage
1	4	1	6	5	0
2	3	1	5	4	0
3	18	2	4	2	0
4	22	2	2	0	0
5	55	3	0	0	3

The shortage occurred in 5th day and of 3 units.

Q: The time for completing the process is exponentially distributed and the mean used is 15 minutes. Every wash load goes through every step before another wash load begins the process.

According to Poisson process, 1 load arrives on every 40 minutes on an average for a wash.

① What is the average time a load waits to begin the wash.

② What is the avg no of loads in the system.

$\frac{1}{\cancel{60}} = \frac{40}{\lambda}$ $\lambda = 1$

Q What is the avg time required to wash a lod?

$M : M : 1 / \infty$

$\mu = \frac{1}{15}$ /min

$\lambda = \frac{1}{40}$ /min

(i) $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$

$= \frac{\frac{1}{40}}{\frac{1}{15}\left(\frac{1}{15}-\frac{1}{40}\right)} = \frac{\frac{1}{\cancel{40}}}{\frac{1}{15}\left(\frac{40-15}{\cancel{600}}\right)}$

$= \frac{\cancel{15} \times \cancel{15}\,3}{\cancel{25}\,5}$

$= 9$ minutes

(ii) $L_s = \frac{\lambda}{\mu-\lambda}$

$= \frac{\frac{1}{40}}{\frac{25}{600}} = \frac{\cancel{600}}{\cancel{40} \times \cancel{25}}$

$= \frac{3}{5} = 0.6$

(iii) $W_s = \frac{L_s}{\lambda} = \frac{3}{\cancel{5}} \times 4\cancel{0}$

$= 24$ minutes

7/9/18

Q ~~service distribution of Able.~~

Random digits for arrival.

26 98 90 26 42 | 74 80 68 2

Random digits for service.

95 21 51 92 89 38 | 13 61 50 4

Inter arrival distribution of cars

TBA	Prob.	CP	RD
1	0.25	0.25	1-25
2	0.40	0.65	26-65
3	0.20	0.85	66-85
4	0.15	1.00	86-00

Service distribution of Able.

ST	Prob.	CP	RD
2	0.30	0.30	1-30
3	0.28	0.58	31-58
4	0.25	0.83	59-83
5	0.17	1.00	84-00

given in ques → Able is the first server and Baker the second.

Service distribution of Baker

ST	Prob.	CP	RD
3	0.35	0.35	01-35
4	0.25	0.60	36-60
5	0.20	0.80	61-80
6	0.20	1.00	81-00

Baker | Able

Cust	IAT	AT	ST	STB	STE	WT	TTIS	IT	ST	STB	WT	STE	TTIS	IT
1	–	0	6	0	6	0								
2	2	2	–	–	–	–			2	2	0	4		
3	4	6	4	6	10	0								
4	4	10	6	10	16	0								
5	2	12							5	12	0	17		
6	2	14												

Able | Baker

Cust	IAT	AT	ST	STB	STE	WT	TTIS	IT	ST	STB	WT	STE	TTIS	IT
1	–	0	5	0	5	0	5	0						2
2	2	2						1	3	2	0	5.	3	7
3	4	6	3	6	9	0	3	0						–
4	4	10	5	10	15	0	5	0						–
5	2	12						0	6	12	0	18	6	–
6	2	14	3	15	18	1	4	0						5
7	3	17	2	18	20	1	3	0						–
8	3	20	4.	20	24	0	4	0						–
9	3	23							4	23	0	27	4	–
10	1	24	3	24	27	0	3	0						1

3