Exclusive or

Definition: Let p and q be propositions. The proposition "p exclusive or q" denoted by $p \oplus q$, is true when exactly one of p and q is true and it is false otherwise.

p	**q**	**$p \oplus q$**
F	F	F
F	T	T
T	F	T
T	T	F

X-OR

- **For buying this Iphone, you have to pay by using bkash or by cash.**

- If you pay cash -- Iphone is yours
- If you pay by bkash – Iphone is yours

- If you don't pay using cash or bkash – Iphone is not yours
- Will you pay by using cash and bkash both ?

Implications or Conditional Statement

Definition: Let p and q be propositions. The proposition "p implies q" or "if p ...then q..." denoted by p $\rightarrow$ q is called implication. It is false when p is true and q is false and is true otherwise.

In p $\rightarrow$ q, p is called the **hypothesis** and q is called the **conclusion.**

p	q	p $\rightarrow$ q
F	F	T
F	T	T
T	F	F
T	T	T

Implication

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Implication

In p implies q or if p then q :

If p is true then we can argue about q and make decision about the compound proposition

If p or hypothesis is false then we can not argue about q or conclusion. That means we do not have any argument to prove the compound proposition false.

Implication

If you try for your exam, then you will succeed.

p = you try hard for your exam

q = you succeed

Case 1 : p true, q true

you tried hard for your exam and you succeed

So compound proposition $p \rightarrow q$ is true (you passed)

Case 1 : p true, q false

you tried hard for your exam and you did not succeed

So compound proposition $p \rightarrow q$ is false (you failed)

Implication

Case 3 : p false, q true

You did not try hard for your exam and you succeed

So compound proposition $p \rightarrow q$ is true (you passed)

As you did not try hard for your exam, so we could not argue about your result.

Here our hypothesis is false itself, so we could not argue about conclusion. So we will not argue and accept what the proposition says. That means we will accept proposition as true.

Case 4 : p false, q false

you did not try hard for your exam and you did not succeed

So compound proposition $p \rightarrow q$ is true (you failed)

Implications

$p \rightarrow q$ is read in a variety of equivalent ways:

- if p then q
- p only if q
- p is sufficient for q
- q whenever p

Examples:

– if Germany won 2010 world cup then 2 is a prime.

- If F then T ? **T**

− if today is Monday then 2 * 3 = 8.

- **F** $\rightarrow$ F? **T**

Implications

The **converse of $p \rightarrow q$ is $q \rightarrow p$.**

- The **contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$**
- The **inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$**

Examples:

- If it jams, the traffic moves slowly.

- p: it jams
- q: traffic moves slowly.

- $p \rightarrow q$

Implications

The converse:

If the traffic moves slowly then it jams.

- **$q \rightarrow p$**

The contrapositive:

- If the traffic does not move slowly then it does not jams.
- $\neg q \rightarrow \neg p$

The inverse:

- If it does not jams the traffic moves quickly.
- $\neg p \rightarrow \neg q$

Biconditional

Definition: Let p and q be propositions. The biconditional p $\leftrightarrow$ q (read p if and only if q), is true when p and q have the same truth values and is false otherwise.

p	**q**	**p $\leftrightarrow$ q**
F	F	T
F	T	F
T	F	F
T	T	T

Note: two truth values always agree**.**

Bi-Conditional

P if and only if q ----

- ❖ P only if q = if p then q
- ❖ p if q = If q then p

So p implies q and q implies p

For example :

p = You get promoted

q = You have connections

So $p \leftrightarrow q$ = You get promoted if and only if you have connections.

Precedence of Logical Operator

Operators	Names	Precedence
$\neg$	**Negation**	**1**
$\land$	**Conjunction**	**2**
$\lor$	**Disjunction**	**3**
$\rightarrow$	**Implication**	**4**
$\leftrightarrow$	**Bi-condition**	**5**

Constructing the truth table

Example: Construct a truth table for $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

- Simpler if we decompose the sentence to elementary and intermediate propositions

p	q	$\neg p$	$p \rightarrow q$	$\neg p \leftrightarrow q$	$(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$
F	F				
F	T				
T	F				
T	T				

Constructing the truth table

Example: Construct a truth table for $(p \rightarrow q) \land (\neg p \leftrightarrow q)$

- Simpler if we decompose the sentence to elementary and intermediate propositions

p	**q**	**¬p**	**p → q**	**¬p ↔ q**	**(p → q) ∧ (¬p ↔ q)**
F	F	T			
F	T	T			
T	F	F			
T	T	F			

Constructing the truth table

Example: Construct a truth table for $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

- Simpler if we decompose the sentence to elementary and intermediate propositions

p	q	$\neg p$	$p \rightarrow q$	$\neg p \leftrightarrow q$	$(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$
F	F	T	T		
F	T	T	T		
T	F	F	F		
T	T	F	T		

Constructing the truth table

Example: Construct a truth table for $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

- Simpler if we decompose the sentence to elementary and intermediate propositions

p	q	$\neg p$	$p \rightarrow q$	$\neg p \leftrightarrow q$	$(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$
F	F	T	T	F	
F	T	T	T	T	
T	F	F	F	T	
T	T	F	T	F	

Constructing the truth table

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p	**q**	**¬p**	**p → q**	**¬p ↔ q**	**(p → q) ∧ (¬p ↔ q)**
F	F	T	T	F	F
F	T	T	T	T	T
T	F	F	F	T	F
T	T	F	T	F	F

Computer Representation of True and False

We need to encode two values True and False:

- Computers represents data and programs using 0s and 1s
- Logical truth values – True and False

- A bit is sufficient to represent two possible values:

– 0 (False) or 1(True)

- A variable that takes on values 0 or 1 is called a Boolean variable.

- Definition:A bit string is a sequence of zero or more bits. The length of this string is the number of bits in the string.

Bitwise operation

- T and F replaced with 1 and 0

p	q	p ∧ q	p ∨ q
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

p	¬p
0	1
1	0

Bitwise operation

- T and F replaced with 1 and 0

	1	0	1	1	0	0	1	1				1	0	1	1	0	0	1	1				1	0	1	1	0	0	1	1	
$\vee$	0	1	0	0	1	0	0	1			$\wedge$	0	1	0	0	1	0	0	1			$\oplus$	0	1	0	0	1	0	0	1	
	1	1	1	1	1	0	1	1				0	0	0	0	0	0	0	1				1	1	1	1	1	0	1	0	

Applications of propositional logic

- **Translation of English sentences**

- **Inference and reasoning:**

– new true propositions are inferred from existing ones

– Used in Artificial Intelligence:

- Builds programs that act intelligently
- Programs often rely on symbolic manipulations

- **Design of logic circuit**

Translation

❖ **refers to the process of expressing statements or sentences in natural language into a formal system of propositional symbols and logical connectives**

Assume a sentence:

If you are older than 13 or you are with your parents then you can attend a PG-13 movie.

Parse:

- If (you are older than 13 or you are with your parents) then (you can attend a PG-13 movie)

Atomic (elementary) propositions:

– A= you are older than 13

– B= you are with your parents

– C=you can attend a PG-13 movie

- Translation: $A \vee B \rightarrow C$

Application of Inference

involves drawing conclusions or making logical deductions based on given premises using valid logical rules

- Example 1:

Premise: It is not the case that I am not at home (¬¬P is true).
Using Double Negation, we can infer: Therefore, I am at home (P is true).

- Example 2:

Premise 1: If the store is open , then the lights are on .
Premise 2: The lights are not on.
So we can infer: Therefore, the store is not open (¬P is true).

- Example 3:

Premise 1: If it is Monday, then I have a meeting.
Premise 2: If I have a meeting, then I wear a suit.
So we can infer: Therefore, if it is Monday, then I wear a suit.

Application of Inference

Assume we know the following sentences are true:
If you are older than 13 or you are with your parents then you can attend a PG-13 movie. You are older than 13.

Translation:
• If (you are older than 13 or you are with your parents) then (you can attend a PG-13 movie) . (You are older than 13).
– A= you are older than 13
– B= you are with your parents
– C=you can attend a PG-13 movie
• $(A \vee B \rightarrow C)$, A
• $(A \vee B \rightarrow C) \wedge A$ is true
• With the help of the logic we can infer the following statement (proposition):
– You can attend a PG-13 movie or C is True

Tautology and Contradiction

Definitions:
Tautology: A compound proposition that is always true for all possible truth values of the propositions is called a tautology.
Contradiction: A compound proposition that is always false is called a contradiction.
Contingency: A proposition that is neither a tautology nor contradiction is called a contingency (Sometimes true, sometimes false)
Satisfiable: at least one true result in truth table

Example: $p \lor \neg p$ is a tautology.
$p \land \neg p$ is a contradiction.

p	¬p	p ∨ ¬p	p ∧ ¬p
1	0	1	0
0	1	1	0

Equivalence

- ❑ two propositions are equivalent:
- ➢ their truth values in the truth table are the same.
- ➢ two propositions are considered equivalent if they are both true or both false under the same conditions

• Example: $p \rightarrow q$ is equivalent to $\neg q \rightarrow \neg p$ (contrapositive)

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	T	T

Equivalent statements are important for logical reasoning since they can be substituted and can help us to make a logical argument.

Logical Equivalence

Definition:

- The propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology (alternately, if they have the same truth table).
- The notation p <=> q denotes p and q are logically equivalent.

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$	$(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	T
T	T	T	T	T

Important Logical Equivalence

Double negation
– $\neg(\neg p) \iff p$

Commutative
– $p \lor q \iff q \lor p$
– $p \land q \iff q \land p$

Associative
– $(p \lor q) \lor r \iff p \lor (q \lor r)$
– $(p \land q) \land r \iff p \land (q \land r)$

Important Logical Equivalence

Distributive

– $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

– $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

• De Morgan

– $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

– $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

• Other useful equivalences

– $p \vee \neg p \Leftrightarrow T$

– $p \wedge \neg p \Leftrightarrow F$

– $p \rightarrow q \Leftrightarrow (\neg p \vee q)$

Showing Logical Equivalence

Show $(p \wedge q) \rightarrow p$ is a tautology

In other words $((p \wedge q) \rightarrow p \Leftrightarrow T)$

$(p \wedge q) \rightarrow p \Leftrightarrow \neg(p \wedge q) \vee p$ Useful

$\Leftrightarrow [\neg p \vee \neg q] \vee p$ DeMorgan

$\Leftrightarrow [\neg q \vee \neg p] \vee p$ Commutative

$\Leftrightarrow \neg q \vee [\neg p \vee p]$ Associative

$\Leftrightarrow \neg q \vee [T]$ Useful

$\Leftrightarrow T$ Domination

Logical Equivalence

$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws

Logical Equivalence

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Logical Equivalence

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Logical Equivalence

Show that $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$ are logically equivalent by developing a series of logical equivalences

$$
\begin{aligned}
\neg(p \lor (\neg p \land q)) &\equiv \neg p \land \neg(\neg p \land q) && \text{by the second De Morgan law} \\
&\equiv \neg p \land [\neg(\neg p) \lor \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \land (p \lor \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \land p) \lor (\neg p \land \neg q) && \text{by the second distributive law} \\
&\equiv \mathbf{F} \lor (\neg p \land \neg q) && \text{because } \neg p \land p \equiv \mathbf{F} \\
&\equiv (\neg p \land \neg q) \lor \mathbf{F} && \text{by the commutative law for disjunction} \\
&\equiv \neg p \land \neg q && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$