

# Basic Discrete Structure : Set

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Lecture 4

# Basic Discrete Structure

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**Discrete math =**

- study of the **discrete structures** used to **represent** discrete objects

Many **discrete structures** are built using **sets**

- **Sets = collection of objects**

Examples of discrete structures built with the help of sets:

- **Combinations**
- **Relations**
- **Graphs**

# Set

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## Definition:

**A set is a (unordered) collection of objects.**

**These** objects are sometimes called **elements or members of the set.**

- **Examples:**

- **Vowels in the English alphabet**

- $V = \{ a, e, i, o, u \}$

- **First seven prime numbers.**

- $X = \{ 2, 3, 5, 7, 11, 13, 17 \}$

# Representing Set

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Representing a set by:

- 1) Listing (enumerating) the members of the set.
- 2) Definition by property, using the set builder notation  
 $\{x \mid x \text{ has property } P\}$ .

Example:

- Even integers between 50 and 63.
  - 1)  $E = \{50, 52, 54, 56, 58, 60, 62\}$
  - 2)  $E = \{x \mid 50 \leq x < 63, x \text{ is an even integer}\}$

If enumeration of the members is hard we often use ellipses.

Example: a set of integers between 1 and 100

- $A = \{1, 2, 3, \dots, 100\}$

# Important set in discrete math

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- **Natural numbers:**

- $\mathbb{N} = \{0, 1, 2, 3, \dots\}$

- **Integers**

- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

- **Positive integers**

- $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$

- **Rational numbers**

- $\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$

- **Real numbers**

- $\mathbb{R}$

# Equality of Set

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Definition: Two sets are equal if and only if they have the same elements.

**Example:**

- $\{1,2,3\} = \{3,1,2\} = \{1,2,1,3,2\}$

Note: **Duplicates** don't contribute anything new to a set, so remove them. The **order** of the elements in a set doesn't contribute anything new.

**Example:** Are  $\{1,2,3,4\}$  and  $\{1,2,2,4\}$  equal?  
No!

# Universal set

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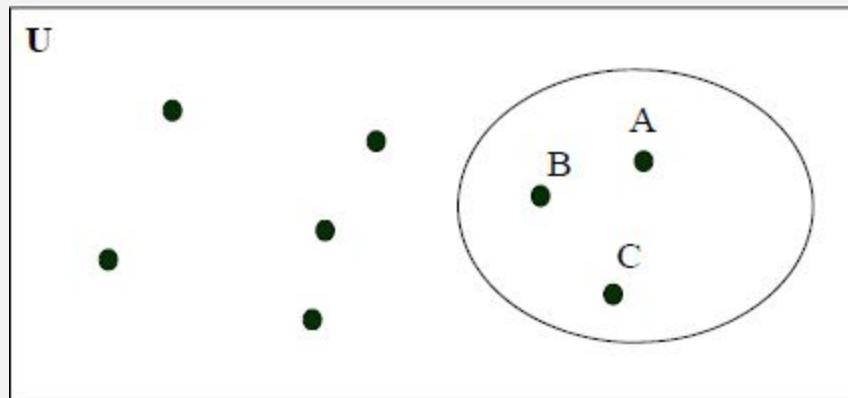
Special sets:

- The **universal set** is denoted by **U**: the set of all objects under the consideration.
- The **empty set** is denoted as  **$\emptyset$**  or **{ }**.

# Venn Diagram

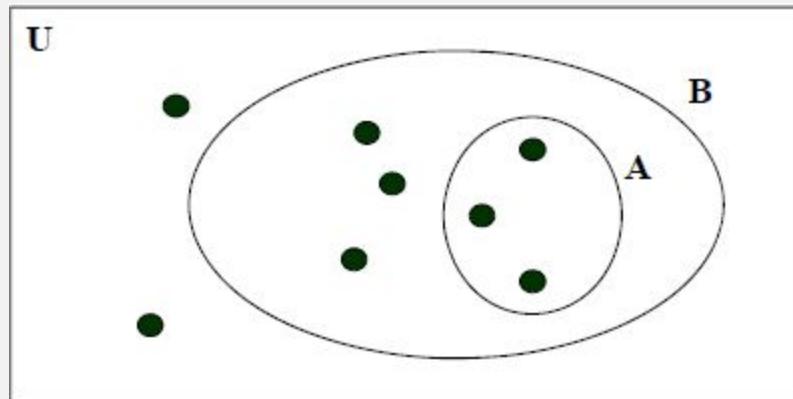
A set can be visualized using **Venn Diagrams**:

- $V=\{ A, B, C \}$



# A subset

**Definition:** A set A is said to be a subset of B if and only if every element of A is also an element of B. We use  $A \subseteq B$  to indicate A is a subset of B.



Alternate way to define A is a subset of B:  
 $\forall x(x \in A) \rightarrow (x \in B)$

# Empty set/subset property

**Theorem :**  $\phi \in S$

- Empty set is a subset of any set.

**Proof:**

- Recall the definition of a subset: all elements of a set A must be also elements of B:  
$$\forall x(x \in A \rightarrow x \in B)$$

- We must show the following implication holds for any

$$\forall x(x \in \phi \rightarrow x \in S)$$

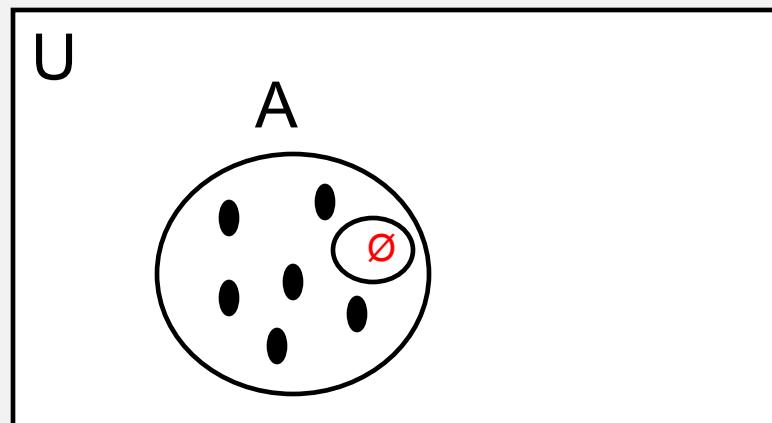
- Since the empty set does not contain any element,  $x \in \phi$  is always False

- Then the implication is always True.  $(F \rightarrow T/F = T)$   
**End of proof**

# Venn diagram of Empty set

**Theorem :**  $\emptyset \in S$

- Empty set is a subset of any set.



# Subset property

**Theorem:**  $S \subseteq S$

- Any set  $S$  is a subset of itself

**Proof:**

- the definition of a subset says: all elements of a set  $A$  must be also elements of  $B$ :  $\forall x(x \in A \rightarrow x \in B)$
- Applying this to  $S$  we get:
- $\forall x(x \in S \rightarrow x \in S)$  which is trivially **True**
- End of proof

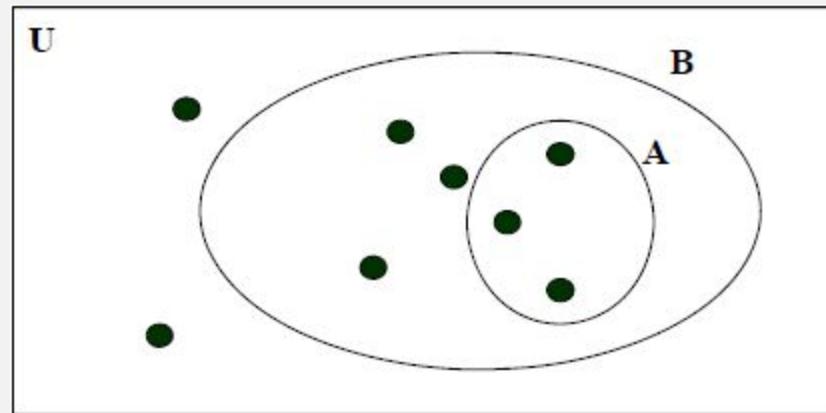
**Note on equivalence:**

- Two sets are equal if each is a subset of the other set.

# A proper Subset

## Definition:

A set A is said to be a **proper subset** of B if and only if  $A \subseteq B$  and  $A \neq B$ . We denote that A is a proper subset of B with the notation  $A \subset B$ .

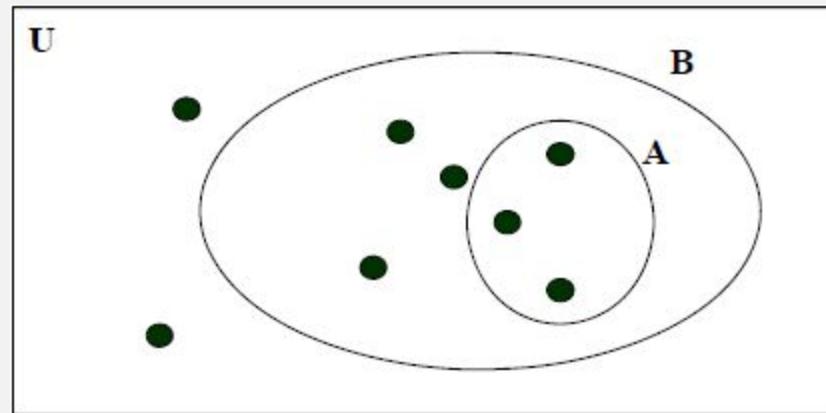


**Example:**  $A=\{1,2,3\}$   $B =\{1,2,3,4,5\}$   
Is:  $A \subset B$  ?

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**Example:**  $A=\{1,2,3\}$   $B =\{1,2,3,4,5\}$

Is:  $A \subset B$  ? Yes.

# Cardinality

Definition: Let  $S$  be a set. If there are exactly  $n$  distinct elements in  $S$ , where  $n$  is a nonnegative integer, we say  $S$  is a finite set and that  $n$  is the cardinality of  $S$ . The cardinality of  $S$  is denoted by  $|S|$ .

Examples:

- $V=\{1, 2, 3, 4, 5\}$   
 $|V| = 5$
- $A=\{1, 2, 3, 4, \dots, 20\}$   
 $|A| = 20$
- $|\emptyset| = 0$

# Infinite set

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Definition: A set is infinite if it is not finite.

Examples:

- The set of natural numbers is an infinite set.
- $N = \{1, 2, 3, \dots\}$
- The set of real numbers is an infinite set.

# Power set

Definition: Given a set  $S$ , the power set of  $S$  is the **set of all subsets** of  $S$ . The **power set is denoted by  $P(S)$** .

## Example

- What is the power set of  $\emptyset$  ?  $P(\emptyset) = \{\emptyset\}$ 
  - What is the cardinality of  $P(\emptyset)$  ?  $|P(\emptyset)| = 1$ .

Assume  $\{1,2,3\}$

- $P(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$
- $|P(\{1,2,3\})| = 8$

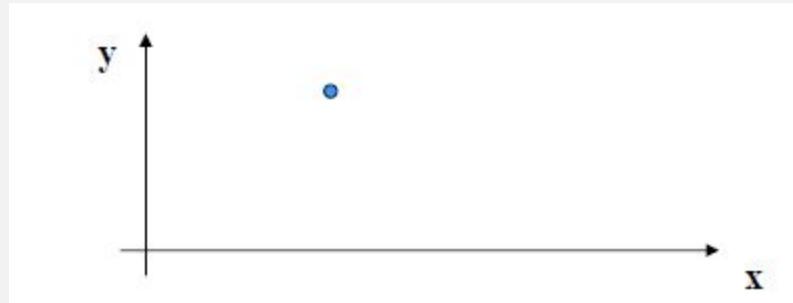
If  $S$  is a set with  $|S| = n$  then  $|P(S)| = ?$   $2^n$

# N-tuple

Sets are used to represent unordered collections.

- **Ordered-n tuples** are used to **represent an ordered collection**.

Definition: An ordered n-tuple  $(x_1, x_2, \dots, x_N)$  is the ordered collection that has  $x_1$  as its first element,  $x_2$  as its second element, ..., and  $x_N$  as its N-th element,  $N \geq 2$ .



**Example:** Coordinates of a point in the 2-D plane (12, 16)

# Cartesian Product

**Definition:** Let  $S$  and  $T$  be sets. The Cartesian product of  $S$  and  $T$ , denoted by  $S \times T$ , is the **set of all ordered pairs**  $(s,t)$ , where  $s \in S$  and  $t \in T$ . Hence,

- $S \times T = \{ (s,t) \mid s \in S \wedge t \in T \}$ .

## Examples:

- $S = \{1,2\}$  and  $T = \{a,b,c\}$
- $S \times T = \{ (1,a), (1,b), (1,c), (2,a), (2,b), (2,c) \}$
- $T \times S = \{ (a,1), (a, 2), (b,1), (b,2), (c,1), (c,2) \}$
- Note:  $S \times T \neq T \times S$  !!!!

# Cardinality of a Cartesian Product

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- $|S \times T| = |S| * |T|$ .

## Example:

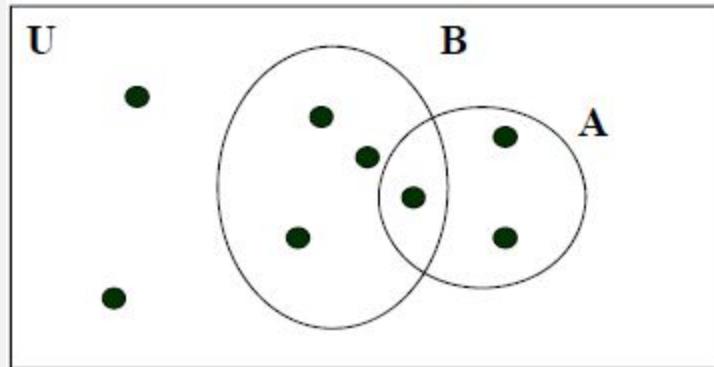
- $A = \{\text{John, Peter, Mike}\}$
- $B = \{\text{Jane, Ann, Laura}\}$
- $A \times B = \{(\text{John, Jane}), (\text{John, Ann}), (\text{John, Laura}), (\text{Peter, Jane}), (\text{Peter, Ann}), (\text{Peter, Laura}), (\text{Mike, Jane}), (\text{Mike, Ann}), (\text{Mike, Laura})\}$
- $|A \times B| = 9$
- $|A|=3, |B|=3 \rightarrow |A| |B|= 9$

Definition: A **subset of the Cartesian product  $A \times B$**  is called a **relation** from the set A to the set B.

# Set Operation

Definition: Let A and B be sets. The **union of A and B**, denoted by **A  $\cup$  B**, is the set that contains those elements that are in both A and B.

- Alternate:  $A \cup B = \{ x \mid x \in A \vee x \in B \}$ .



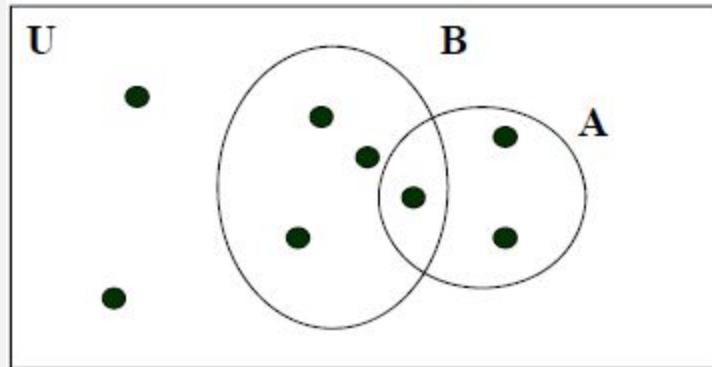
**Example:**

- $A = \{1,2,3,6\}$  and  $B = \{2,4,6,9\}$
- $A \cup B = \{1,2,3,4,6,9\}$

# Set Operation

Definition: Let A and B be sets. The **intersection of A and B**, denoted by  $A \cap B$ , is the set that contains those elements that are in both A and B.

- Alternate:  $A \cap B = \{ x \mid x \in A \wedge x \in B \}$ .



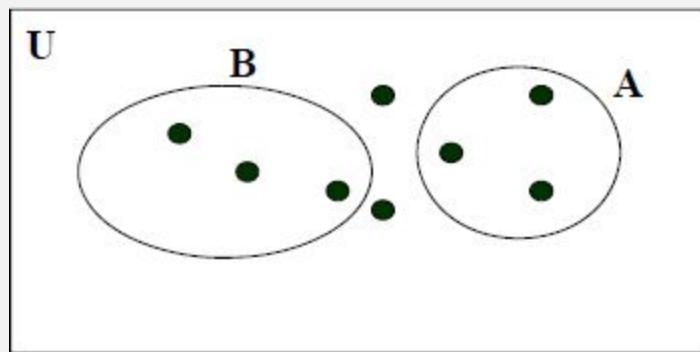
**Example:**

- $A = \{1, 2, 3\}$  and  $B = \{2, 4, 6, 9\}$
- $A \cap B = \{2\}$

# Disjoin Set

**Definition:** Two sets are called disjoint if their **intersection** is empty.

- Alternate: A and B are disjoint **if and only if**  $A \cap B = \emptyset$ .



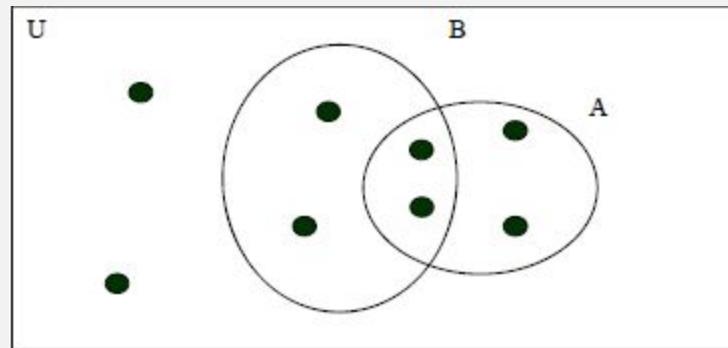
**Example:**

- $A=\{1,2,3,6\}$   $B=\{4,7,8\}$  Are these disjoint?
- Yes.
- $A \cap B = \emptyset$

# Cardinality of set union

**Cardinality of the set union.**

- $|A \cup B| = |A| + |B| - |A \cap B|$

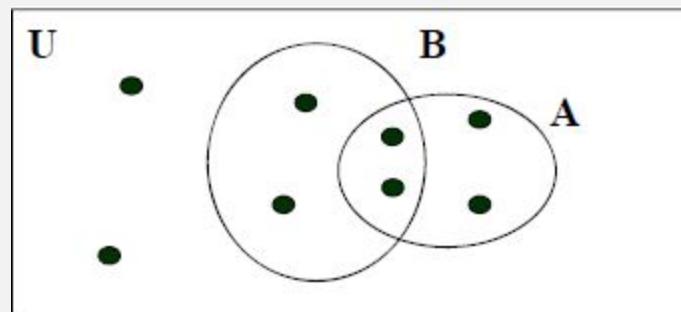


Why this formula? Correct for an over-count.

# Set Difference

Definition: Let A and B be sets. The difference of A and B, denoted by  $A - B$ , is the set containing those **elements that are in A but not in B**. The difference of A and B is also called the complement of B with respect to A.

- Alternate:  $A - B = \{x \mid x \in A \wedge x \notin B\}$



**Example:**  $A = \{1, 2, 3, 5, 7\}$   $B = \{1, 5, 6, 8\}$

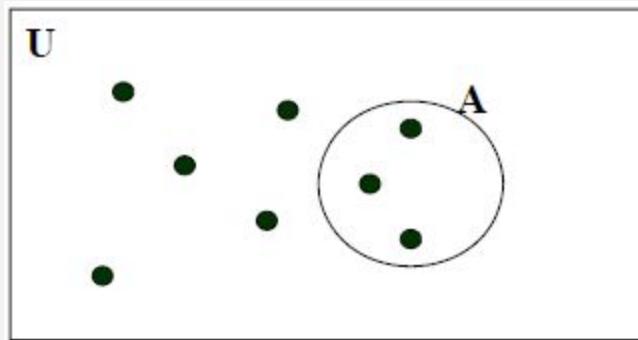
- $A - B = \{2, 3, 7\}$

# Complement of a Set

**Definition:** Let  $U$  be the universal set: the set of all objects under the consideration.

**Definition:** The complement of the set  $A$ , denoted by  $\tilde{A}$ , is the complement of  $A$  with respect to  $U$ .

- Alternate: Alternate:  $\overline{A} = \{x \mid x \notin A\}$



**Example:**  $U=\{1,2,3,4,5,6,7,8\}$   $A = \{1,3,5\}$

- $\tilde{A}=\{2,4,6,7,8\}$

# Generalized union

**Definition:** The union of a collection of sets is the set that contains those elements that are **members of at least one set** in the collection.

$$\bigcup_{i=1}^n A_i = \{A_1 \cup A_2 \cup \dots \cup A_n\}$$

**Example:**

- Let  $A_i = \{1, 2, \dots, i\} \quad i = 1, 2, \dots, n$

$$\bigcup_{i=1}^n A_i = \{1, 2, \dots, n\}$$

$$\begin{aligned} A_1 &= \{1\} \\ A_2 &= \{1, 2\} \\ A_3 &= \{1, 2, 3\} \\ &\dots \\ &\dots \\ A_n &= \{1, 2, 3, 4, \dots, n\} \end{aligned}$$

# Generalized intersection

**Definition:** The intersection of a collection of sets is the set that contains those elements **that are members of all sets in the collection.**

$$\bigcap_{i=1}^n A_i = \{A_1 \cap A_2 \cap \dots \cap A_n\}$$

**Example:**

- Let  $A_i = \{1, 2, \dots, i\} \quad i = 1, 2, \dots, n$

$$\bigcap_{i=1}^n A_i = \{1\}$$

$$\begin{aligned} A_1 &= \{1\} \\ A_2 &= \{1, 2\} \\ A_3 &= \{1, 2, 3\} \\ \cdots & \\ \cdots & \\ A_n &= \{1, 2, 3, 4, \dots, n\} \end{aligned}$$

# Computer representation of set

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How to represent sets in the computer?

- One solution: Data structures like a list
- A better solution: Assign a bit in a bit string to each element in the universal set and set the bit to 1 if the element is present otherwise use 0

**Example:**

All possible elements:  $U=\{1\ 2\ 3\ 4\ 5\}$

- Assume  $A=\{2,5\}$ 
  - Computer representation:  $A = 01001$
- Assume  $B=\{1,5\}$ 
  - Computer representation:  $B = 10001$

# Computer representation of set

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## Example:

- $A = 01001$
- $B = 10001$
- The **union** is modeled with a bitwise or  
 $A \cup B = 11001$
- The **intersection** is modeled with a bitwise and  
 $A \cap B = 00001$
- The **complement** is modeled with a bitwise negation  
 $\tilde{A} = 10110$

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# **Basic Discrete Structure :**

## **Set and Function**

# Quiz Problem 1

1. Suppose the following two statements are true.

I love Dad or I love Mum

If I love Dad then I love Mum

Does it necessarily follow that I love Dad? Does it necessarily follow that I love Mum? Use propositional logic to answer the questions.

P <b>(I love dad)</b>	Q <b>(I love mum)</b>	$P \vee Q$	$P \rightarrow Q$
F	F	F	T
F	T	T	T
T	F	T	F
T	T	T	T

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# Quiz Problem 2

2. Consider the following specification with

two predicate symbols  $p_1 \equiv (<)$ ,  $p_2 \equiv (=)$

two function symbols  $f_1 \equiv (+)$ ,  $f_2 \equiv (\times)$

two constant symbols  $c_1 \equiv (0)$ ,  $c_2 \equiv (1)$

Let domain of discourse be  $\langle \mathbb{Z}_+ \cup \{0\} \rangle$  where  $\mathbb{Z}_+ = \{1, 2, \dots\}$

What are the truth values of the following statements?

- i.  $\forall x p_1(c_1, x)$
- ii.  $\forall x \forall y \exists z (p_1(x, z) \wedge p_1(z, y))$
- iii.  $\exists x \forall y p_1(x, y)$
- iv.  $\forall x \forall y p_2(f_2(f_1(x, c_2), y), f_1(f_2(x, y), y))$

i.	False	$0 < x$

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i.	False	$0 < x$
ii.	False	$x < z \wedge z < y$

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- iv.  $\forall x \forall y p_2(f_2(f_1(x, c_2), y), f_1(f_2(x, y), y))$

i.	False	$0 < x$
ii.	False	$x < z \wedge z < y$
iii.	False	$x < y$
iv	True	$xy + y = xy + y$

# Quiz Problem 3

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3. Let A be the set of English words that contains x, and B be the set of English words that contain the letter q. Express each of these sets as a combination of A and B.
- i. The set of English words that do not contain the letter x.
  - ii. The set of English words that contain an x but not a q.
  - iii. The set of English words that do not contain either an x or a q.

i.	<b>U – A</b>

# Quiz Problem 3

---

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  - iii. The set of English words that do not contain either an x or a q.

i.	$U - A$
ii.	$A - B$

# Quiz Problem 3

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- i. The set of English words that do not contain the letter x.
  - ii. The set of English words that contain an x but not a q.
  - iii. The set of English words that do not contain either an x or a q.

i.	$U - A$
ii.	$A - B$
iii.	$U - (A \cap B)$

# Quiz Problem 4

4. Let  $A = \{a, b, c\}$ ,  $B = \{x, y\}$  and  $C = \{0, 1\}$ . Find  $C \times B \times A$ .

$C \times B$

$\{\{0,x\}\{0,y\}, \{1,x\}\{1,y\}\}$

# Quiz Problem 4

4. Let  $A = \{a, b, c\}$ ,  $B = \{x, y\}$  and  $C = \{0, 1\}$ . Find  $C \times B \times A$ .

$C \times B$	$\{\{0,x\}\{0,y\}, \{1,x\}\{1,y\}\}$
$C \times B \times A$	$\{\{0,x,a\}, \{0,x,b\}, \{0,x,c\}, \{0,y,a\}, \{0,y,b\}, \{0,y,c\}, \{1,x,a\}, \{1,x,b\}, \{1,x,c\}, \{1,y,a\}, \{1,y,b\}, \{1,y,c\}\}$

# Quiz Problem 5

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5. Let  $f$  be the function from  $\mathbf{R}$  to  $\mathbf{R}$  defined by  $y=f(m, n) = 2m - n$ . Write a method signature in C with appropriate return type and parameter list that could be used to realize the function.

```
double f(double m, double n);
```

# Definitions and notation

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Check these:

Is  $\{x\} \subseteq \{x\}$ ?  Yes

Is  $\{x\} \in \{x, \{x\}\}$ ?  Yes

Is  $\{x\} \subseteq \{x, \{x\}\}$ ?  Yes

Is  $\{x\} \in \{x\}$ ?  No

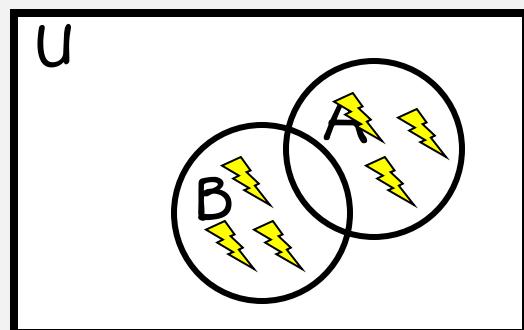
# Operators

like  
"exclusive or"

The **symmetric difference**,  $A \oplus B$ , is:

$$A \oplus B = \{ x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A) \}$$

$$= (A - B) \cup (B - A)$$



# Operators

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$$\begin{aligned}\text{Proof: } & \{ x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A) \} \\ &= (A - B) \cup (B - A)\end{aligned}$$

$$\begin{aligned}A \oplus B &= \{ x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A) \} \\ &= \{ x : (x \in A - B) \vee (x \in B - A) \} \\ &= \{ x : x \in ((A - B) \cup (B - A)) \} \\ &= (A - B) \cup (B - A)\end{aligned}$$

# Famous Identities

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- Identity

$$A \cap U = A$$

$$A \cup \emptyset = A$$

- Domination

$$A \cup U = U$$

$$A \cap \emptyset = \emptyset$$

- Idempotent

$$A \cup A = A$$

$$A \cap A = A$$

# Famous Identities

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- Excluded Middle       $A \cup \bar{A} = U$

- Uniqueness             $A \cap \bar{A} = \emptyset$

- Double complement     $\overline{\bar{A}} = A$

# Famous Identities

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- Commutativity

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

- Associativity

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

- Distributivity

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

# Famous Identities

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- DeMorgan's I

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$

- DeMorgan's II

$$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

# Inclusion/Exclusion

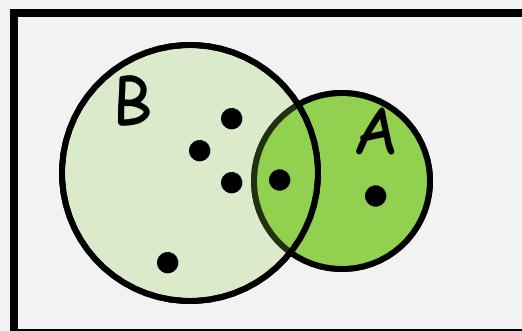
Example:

How many people are wearing a watch?

How many people are wearing sneakers?

How many people are wearing  
a watch OR sneakers?

What's wrong?



$$|A \cup B| = |A| + |B| \quad 7$$

$$|A \cup B| = |A| + |B| - |A \cap B| \quad 6$$

# Inclusion/Exclusion

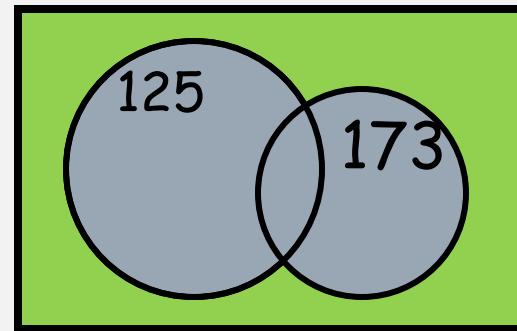
Example:

There are 217 cs majors.

157 are taking cs125.

145 are taking cs173.

98 are taking both.

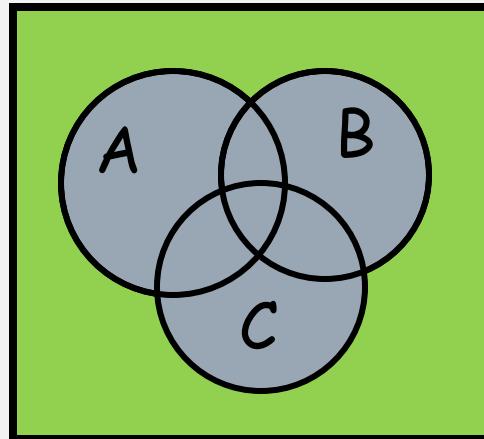


How many are taking neither?

$$217 - (157 + 145 - 98) = 13$$

# Generalized Inclusion/Exclusion

Suppose we have:



$$\begin{aligned}A &= \{0, 2, 4, 6, 8\}, \\B &= \{0, 1, 2, 3, 4\}, \\C &= \{0, 3, 6, 9\}.\end{aligned}$$

And I want to know  $|A \cup B \cup C|$

$$\begin{aligned}|A \cup B \cup C| &= |A| + |B| + |C| \\&\quad - |A \cap B| - |A \cap C| - |B \cap C| \\&\quad + |A \cap B \cap C|\end{aligned}$$

$$|A \cup B \cup C| = 5+5+4-3-2-2+1 \equiv 8 \equiv \{0, 1, 2, 3, 4, 6, 8, 9\}.$$

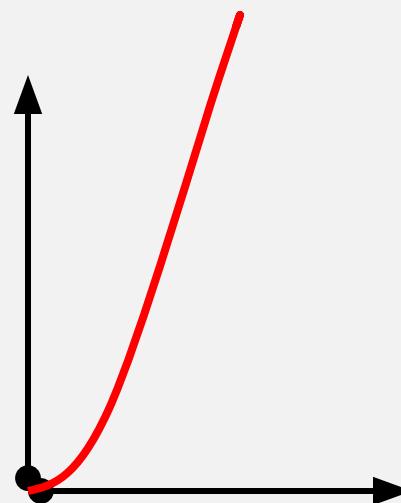
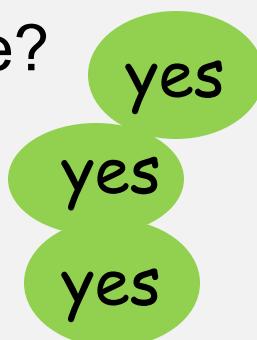
# Functions - examples

Suppose  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ ,  $f(x) = x^2$ .

Is  $f$  one-to-one?

Is  $f$  onto?

Is  $f$  bijective?



# Functions - examples

Suppose  $f: \mathbb{R} \rightarrow \mathbb{R}^+$ ,  $f(x) = x^2$ .

Is  $f$  one-to-one?

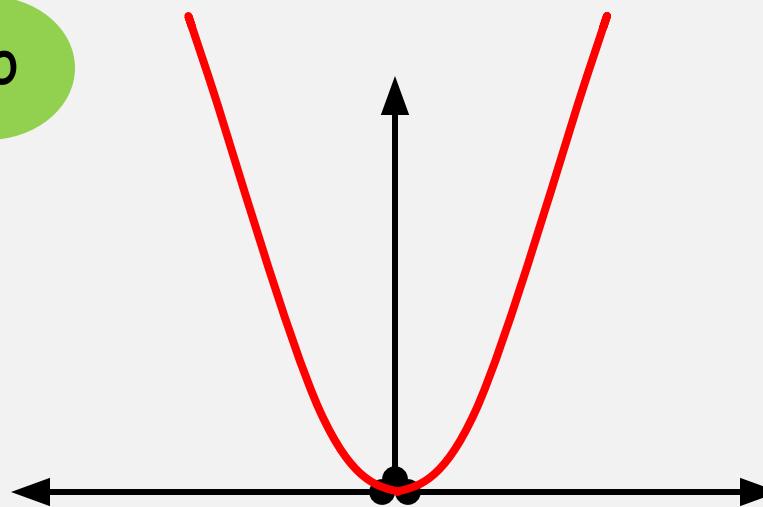
no

Is  $f$  onto?

yes

Is  $f$  bijective?

no



# Functions - examples

Suppose  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x^2$ .

Is  $f$  one-to-one?

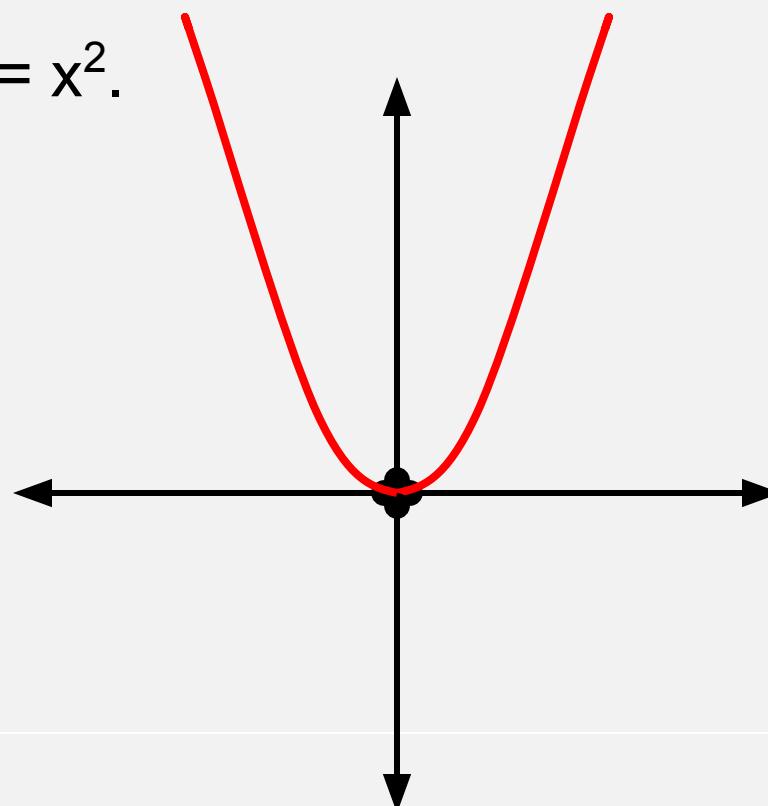
no

Is  $f$  onto?

no

Is  $f$  bijective?

no



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**Thank You**