

Basic Discrete Structure : **Set**

Lecture 4

Basic Discrete Structure

Discrete math =

– study of the **discrete structures** used to **represent discrete objects**

Many **discrete structures** are built using **sets**

– **Sets = collection of objects**

Examples of discrete structures built with the help of sets:

- **Combinations**
- **Relations**
- **Graphs**

Set

Definition:

A set is a (unordered) collection of objects.

These objects are sometimes called **elements or members of the set.**

- **Examples:**

- **Vowels in the English alphabet**

$V = \{ a, e, i, o, u \}$

- **First seven prime numbers.**

$X = \{ 2, 3, 5, 7, 11, 13, 17 \}$

Representing Set

Representing a set by:

- 1) Listing (enumerating) the members of the set.
- 2) Definition by property, using the set builder notation $\{x \mid x \text{ has property } P\}$.

Example:

- Even integers between 50 and 63.

- 1) $E = \{50, 52, 54, 56, 58, 60, 62\}$
- 2) $E = \{x \mid 50 \leq x < 63, x \text{ is an even integer}\}$

If enumeration of the members is hard we often use ellipses.

Example: a set of integers between 1 and 100

- $A = \{1, 2, 3, \dots, 100\}$

Important set in discrete math

- **Natural numbers:**

- $\mathbb{N} = \{0, 1, 2, 3, \dots\}$

- **Integers**

- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

- **Positive integers**

- $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$

- **Rational numbers**

- $\mathbb{Q} = \{p / q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$

- **Real numbers**

- \mathbb{R}

Equality of Set

Definition: Two sets are equal if and only if they have the same elements.

Example:

- $\{1,2,3\} = \{3,1,2\} = \{1,2,1,3,2\}$

Note: **Duplicates** don't contribute anything new to a set, so remove them. The **order** of the elements in a set doesn't contribute anything new.

Example: Are $\{1,2,3,4\}$ and $\{1,2,2,4\}$ equal?

No!

Universal set

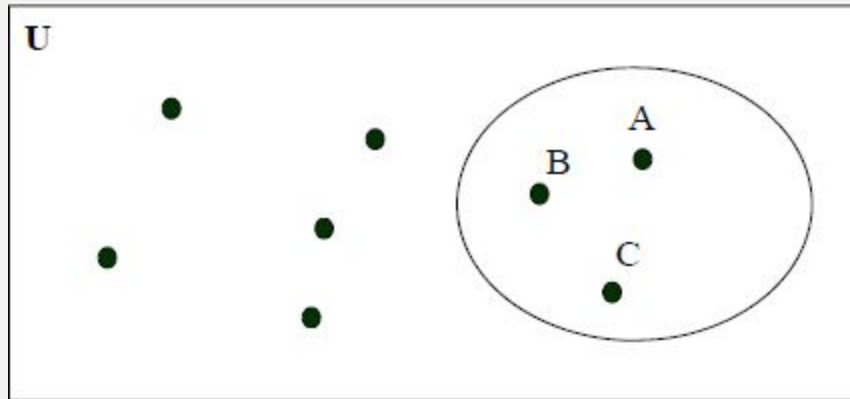
Special sets:

- The **universal set** is denoted by **U**: the set of all objects under the consideration.
- The **empty set** is denoted as \emptyset or $\{ \}$.

Venn Diagram

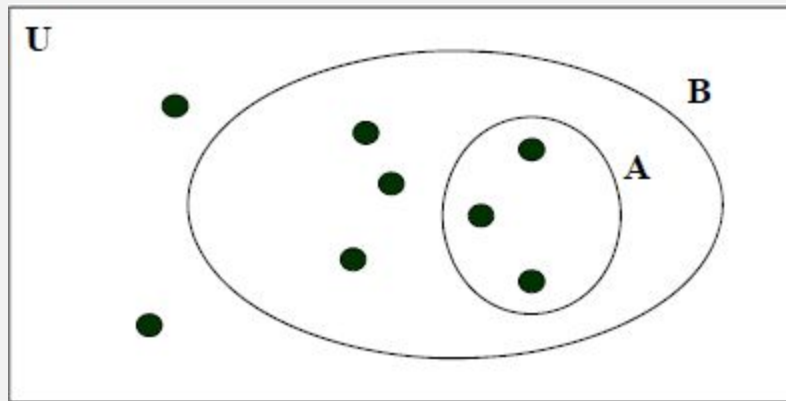
A set can be visualized using **Venn Diagrams**:

– $V = \{ A, B, C \}$



A subset

Definition: A set **A** is said to be a subset of **B** if and only if **every element of A is also an element of B**. We use $A \subseteq B$ to indicate **A is a subset of B**.



Alternate way to define A is a subset of B:

$$\forall x(x \in A) \rightarrow (x \in B)$$

Empty set/subset property

Theorem : $\phi \in S$

- Empty set is a subset of any set.

Proof:

- Recall the definition of a subset: all elements of a set A must be also elements of B:

$$\forall x(x \in A \rightarrow x \in B)$$

- We must show the following implication holds for any

$$\forall x(x \in \phi \rightarrow x \in S)$$

- Since the empty set does not contain any element, $x \in \phi$ is **always False**

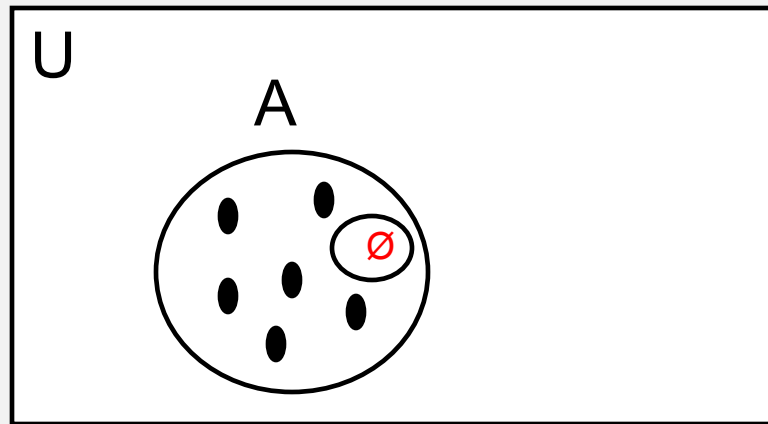
- Then the implication is **always True**. $(F \rightarrow T/F = T)$

End of proof

Venn diagram of Empty set

Theorem : $\phi \in S$

- Empty set is a subset of any set.



Subset property

Theorem: $S \subseteq S$

- Any set S is a subset of itself

Proof:

- the definition of a subset says: all elements of a set A must be also elements of B : $\forall x(x \in A \rightarrow x \in B)$
- Applying this to S we get:
- $\forall x(x \in S \rightarrow x \in S)$ which is trivially **True**
- End of proof

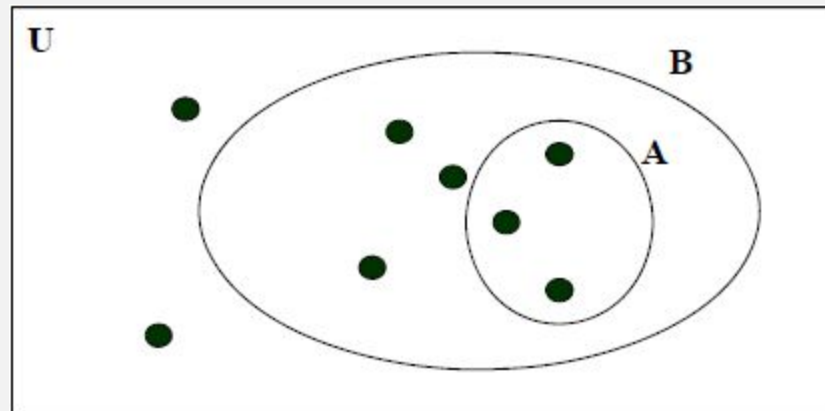
Note on equivalence:

- Two sets are equal if each is a subset of the other set.

A proper Subset

Definition:

A set A is said to be a **proper subset** of B if and only if $A \subseteq B$ **and** $A \neq B$. We denote that A is a proper subset of B with the notation $A \subset B$.



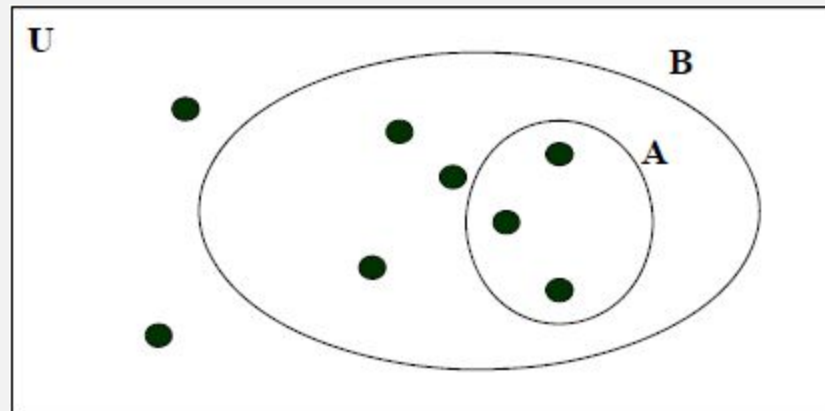
Example: $A = \{1, 2, 3\}$ $B = \{1, 2, 3, 4, 5\}$

Is: $A \subset B$?

A proper Subset

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Example: $A = \{1, 2, 3\}$ $B = \{1, 2, 3, 4, 5\}$

Is: $A \subset B$? **Yes.**

Cardinality

Definition: Let S be a set. If there are **exactly n distinct elements** in S , where **n is a nonnegative integer**, we say S is a finite set and that **n is the cardinality** of S . The cardinality of S is denoted by **$|S|$** .

Examples:

- $V = \{1, 2, 3, 4, 5\}$
 $|V| = 5$
- $A = \{1, 2, 3, 4, \dots, 20\}$
 $|A| = 20$
- $|\emptyset| = 0$

Infinite set

Definition: A set is infinite if it is not finite.

Examples:

- The set of natural numbers is an infinite set.
- $\mathbb{N} = \{1, 2, 3, \dots\}$
- The set of real numbers is an infinite set.

Power set

Definition: Given a set S , the power set of S is the **set of all subsets** of S . The **power set is denoted by $P(S)$** .

Example

- What is the power set of \emptyset ? $P(\emptyset) = \{\emptyset\}$
- What is the cardinality of $P(\emptyset)$? $|P(\emptyset)| = 1$.

Assume $\{1,2,3\}$

- $P(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$
- $|P(\{1,2,3\})| = 8$

If S is a set with $|S| = n$ then $|P(S)| = ?$ **2^n**

N-tuple

Sets are used to represent unordered collections.

- **Ordered-n tuples** are used to **represent an ordered collection**.

Definition: An ordered n -tuple (x_1, x_2, \dots, x_N) is the ordered collection that has x_1 as its first element, x_2 as its second element, ..., and x_N as its N -th element, **$N \geq 2$** .



Example: Coordinates of a point in the 2-D plane $(12, 16)$

Cartesian Product

Definition: Let S and T be sets. The Cartesian product of S and T , denoted by $S \times T$, is the **set of all ordered pairs** (s,t) , where $s \in S$ and $t \in T$. Hence,

- $S \times T = \{ (s,t) \mid s \in S \wedge t \in T \}.$

Examples:

- $S = \{1,2\}$ and $T = \{a,b,c\}$
- $S \times T = \{ (1,a), (1,b), (1,c), (2,a), (2,b), (2,c) \}$
- $T \times S = \{ (a,1), (a,2), (b,1), (b,2), (c,1), (c,2) \}$
- Note: $S \times T \neq T \times S$!!!!

Cardinality of a Cartesian Product

- $|S \times T| = |S| * |T|$.

Example:

- $A = \{\text{John, Peter, Mike}\}$
- $B = \{\text{Jane, Ann, Laura}\}$

- $A \times B = \{(\text{John, Jane}), (\text{John, Ann}), (\text{John, Laura}), (\text{Peter, Jane}), (\text{Peter, Ann}), (\text{Peter, Laura}), (\text{Mike, Jane}), (\text{Mike, Ann}), (\text{Mike, Laura})\}$

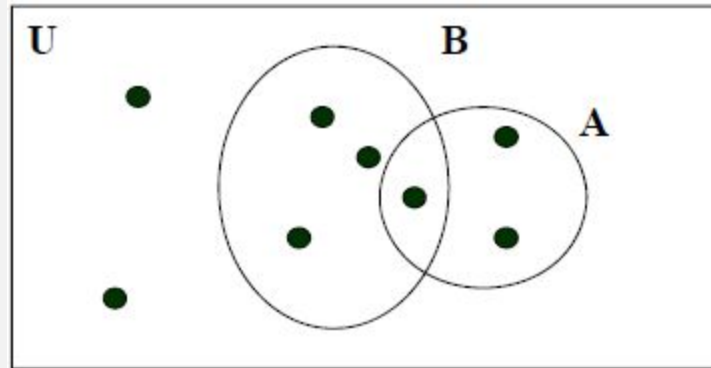
- $|A \times B| = 9$
- $|A|=3, |B|=3 \rightarrow |A| |B|= 9$

Definition: A **subset of the Cartesian product $A \times B$** is called a **relation** from the set A to the set B.

Set Operation

Definition: Let A and B be sets. The **union of A and B**, denoted by **$A \cup B$** , is the set that contains those elements that are in both A and B.

- Alternate: $A \cup B = \{ x \mid x \in A \vee x \in B \}$.



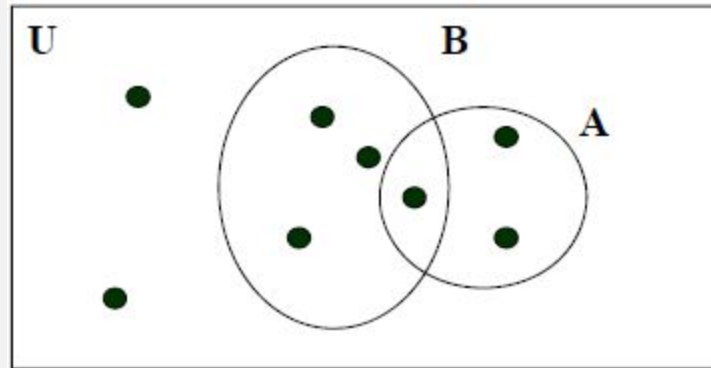
Example:

- $A = \{1, 2, 3, 6\}$ and $B = \{2, 4, 6, 9\}$
- **$A \cup B = \{1, 2, 3, 4, 6, 9\}$**

Set Operation

Definition: Let A and B be sets. The **intersection of A and B**, denoted by **$A \cap B$** , is the set that contains those elements that are in both A and B.

- Alternate: $A \cap B = \{ x \mid x \in A \wedge x \in B \}$.



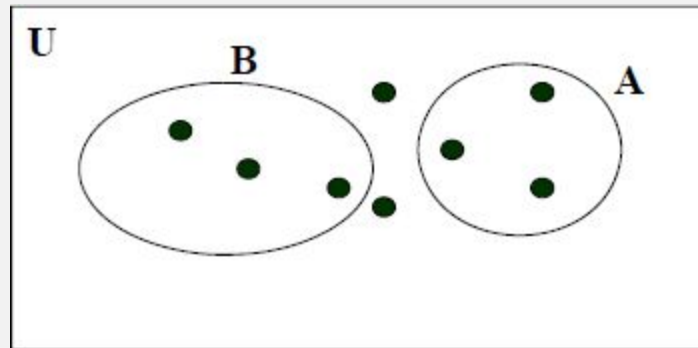
Example:

- $A = \{1, 2, 3\}$ and $B = \{2, 4, 6, 9\}$
- **$A \cap B = \{2\}$**

Disjoin Set

Definition: Two sets are called disjoint if their **intersection is empty**.

- Alternate: A and B are disjoint **if and only if** **$A \cap B = \emptyset$** .



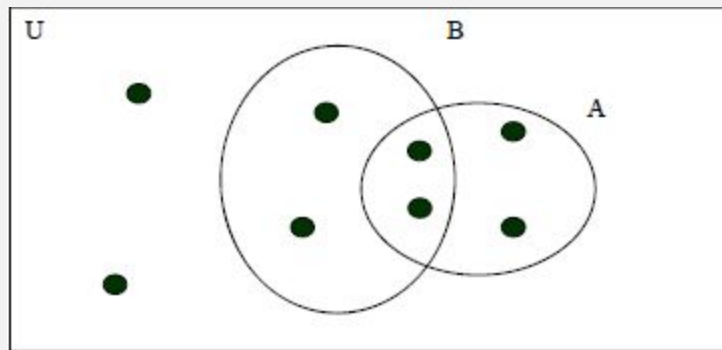
Example:

- $A = \{1, 2, 3, 6\}$ $B = \{4, 7, 8\}$ **Are these disjoint?**
- **Yes.**
- **$A \cap B = \emptyset$**

Cardinality of set union

Cardinality of the set union.

- $|A \cup B| = |A| + |B| - |A \cap B|$

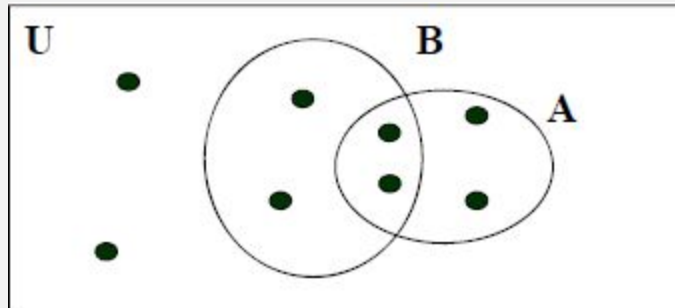


Why this formula? Correct for an over-count.

Set Difference

Definition: Let A and B be sets. The difference of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

- Alternate: $A - B = \{x \mid x \in A \wedge x \notin B\}$



Example: $A = \{1, 2, 3, 5, 7\}$ $B = \{1, 5, 6, 8\}$

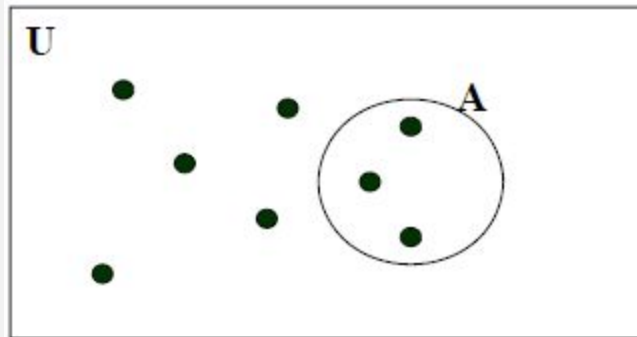
- $A - B = \{2, 3, 7\}$

Complement of a Set

Definition: Let U be the universal set: the set of all objects under the consideration.

Definition: The complement of the set A , denoted by \tilde{A} , is the complement of A with respect to U .

- Alternate: Alternate: $\overline{A} = \{x \mid x \notin A\}$



Example: $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ $A = \{1, 3, 5\}$

- $\tilde{A} = \{2, 4, 6, 7, 8\}$

Generalized union

Definition: The union of a collection of sets is the set that contains those elements that are **members of at least one set** in the collection.

$$\bigcup_{i=1}^n A_i = \{A_1 \cup A_2 \cup \dots \cup A_n\}$$

Example:

- Let $A_i = \{1, 2, \dots, i\}$ $i = 1, 2, \dots, n$

$$\bigcup_{i=1}^n A_i = \{1, 2, \dots, n\}$$

$$\begin{aligned} A_1 &= \{1\} \\ A_2 &= \{1, 2\} \\ A_3 &= \{1, 2, 3\} \\ &\dots\dots\dots \\ &\dots\dots\dots \\ A_n &= \{1, 2, 3, 4, \dots, n\} \end{aligned}$$

Generalized intersection

Definition: The intersection of a collection of sets is the set that contains those elements **that are members of all sets in the collection.**

$$\bigcap_{i=1}^n A_i = \{A_1 \cap A_2 \cap \dots \cap A_n\}$$

Example:

- Let $A_i = \{1, 2, \dots, i\}$ $i = 1, 2, \dots, n$

$$\bigcap_{i=1}^n A_i = \{1\}$$

$$A_1 = \{1\}$$

$$A_2 = \{1, 2\}$$

$$A_3 = \{1, 2, 3\}$$

.....

.....

$$A_n = \{1, 2, 3, 4, \dots, n\}$$

Computer representation of set

How to represent sets in the computer?

- One solution: Data structures like a list
- A better solution: Assign a bit in a bit string to each element in the universal set and set the bit to 1 if the element is present otherwise use 0

Example:

All possible elements: $U = \{1\ 2\ 3\ 4\ 5\}$

- Assume $A = \{2, 5\}$

– Computer representation: $A = 01001$

- Assume $B = \{1, 5\}$

– Computer representation: $B = 10001$

Computer representation of set

Example:

- $A = 01001$
- $B = 10001$
- The **union** is modeled with a bitwise or
- $A \cup B = 11001$
- The **intersection** is modeled with a bitwise and
- $A \cap B = 00001$
- The **complement** is modeled with a bitwise negation
- $\tilde{A} = 10110$

Basic Discrete Structure :

Set and Function

Quiz Problem 1

1. Suppose the following two statements are true.

I love Dad or I love Mum

If I love Dad then I love Mum

Does it necessarily follow that I love Dad? Does it necessarily follow that I love Mum? Use propositional logic to answer the questions.

P (I love dad)	Q (I love mum)	$P \vee Q$	$P \rightarrow Q$
F	F	F	T
F	T	T	T
T	F	T	F
T	T	T	T

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Quiz Problem 2

- 2.** Consider the following specification with
 two predicate symbols $p_1 \equiv (<)$, $p_2 \equiv (=)$
 two function symbols $f_1 \equiv (+)$, $f_2 \equiv (\times)$
 two constant symbols $c_1 \equiv (0)$, $c_2 \equiv (1)$

Let domain of discourse be $\langle \mathbb{Z}_+ \cup \{0\} \rangle$ where $\mathbb{Z}_+ = \{1, 2, \dots\}$

What are the truth values of the following statements?

- i.** $\forall x p_1(c_1, x)$
- ii.** $\forall x \forall y \exists z (p_1(x, z) \wedge p_1(z, y))$
- iii.** $\exists x \forall y p_1(x, y)$
- iv.** $\forall x \forall y p_2(f_2(f_1(x, c_2), y), f_1(f_2(x, y), y))$

i.	False	$0 < x$

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i.	False	$0 < x$
ii.	False	$x < z \wedge z < y$

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i.	False	$0 < x$
ii.	False	$x < z \wedge z < y$
iii.	False	$x < y$

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- iv.** $\forall x \forall y p_2(f_2(f_1(x, c_2), y), f_1(f_2(x, y), y))$

i.	False	$0 < x$
ii.	False	$x < z \wedge z < y$
iii.	False	$x < y$
iv	True	$xy + y = xy + y$

Quiz Problem 3

- 3.** Let A be the set of English words that contains x , and B be the set of English words that contain the letter q . Express each of these sets as a combination of A and B .
- i.** The set of English words that do not contain the letter x .
 - ii.** The set of English words that contain an x but not a q .
 - iii.** The set of English words that do not contain either an x or a q .

i.	$U - A$

Quiz Problem 3

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i.	$U - A$
ii.	$A - B$

Quiz Problem 3

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- i.** The set of English words that do not contain the letter x .
 - ii.** The set of English words that contain an x but not a q .
 - iii.** The set of English words that do not contain either an x or a q .

i.	$U - A$
ii.	$A - B$
iii.	$U - (A \cap B)$

Quiz Problem 4

4. Let $A = \{a, b, c\}$, $B = \{x, y\}$ and $C = \{0, 1\}$. Find $C \times B \times A$.

$C \times B$	$\{\{0,x\} \{0,y\}, \{1,x\} \{1,y\}\}$
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Quiz Problem 4

4. Let $A = \{a, b, c\}$, $B = \{x, y\}$ and $C = \{0, 1\}$. Find $C \times B \times A$.

$C \times B$	$\{\{0,x\} \{0,y\}, \{1,x\} \{1,y\}\}$
$C \times B \times A$	$\{\{0,x,a\}, \{0,x,b\}, \{0,x,c\}, \{0,y,a\}, \{0,y,b\}, \{0,y,c\}, \{1,x,a\}, \{1,x,b\}, \{1,x,c\}, \{1,y,a\}, \{1,y,b\}, \{1,y,c\}\}$

Quiz Problem 5

5. Let f be the function from \mathbf{R} to \mathbf{R} defined by $y = f(m, n) = 2m - n$. Write a method signature in C with appropriate return type and parameter list that could be used to realize the function.

```
double f(double m, double n);
```

Definitions and notation

Check these:

Is $\{x\} \subseteq \{x\}$? Yes

Is $\{x\} \in \{x, \{x\}\}$? Yes

Is $\{x\} \subseteq \{x, \{x\}\}$? Yes

Is $\{x\} \in \{x\}$? No

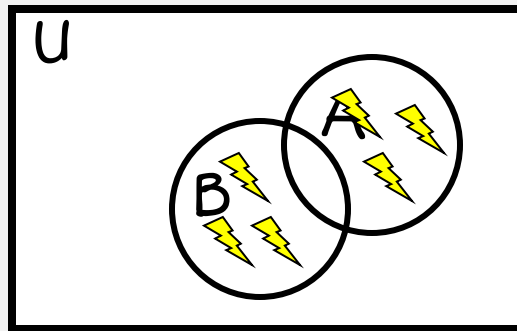
Operators

like
"exclusive or"

The **symmetric difference**, $A \oplus B$, is:

$$A \oplus B = \{ x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A) \}$$

$$= (A - B) \cup (B - A)$$



Operators

$$\begin{aligned}\text{Proof: } \{ x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A) \} \\ = (A - B) \cup (B - A)\end{aligned}$$

$$\begin{aligned}A \oplus B &= \{ x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A) \} \\ &= \{ x : (x \in A - B) \vee (x \in B - A) \} \\ &= \{ x : x \in ((A - B) \cup (B - A)) \} \\ &= (A - B) \cup (B - A)\end{aligned}$$

Famous Identities

- Identity

$$A \cap U = A$$

$$A \cup \emptyset = A$$

- Domination

$$A \cup U = U$$

$$A \cap \emptyset = \emptyset$$

- Idempotent

$$A \cup A = A$$

$$A \cap A = A$$

Famous Identities

- Excluded Middle $A \cup \bar{A} = U$
- Uniqueness $A \cap \bar{A} = \emptyset$
- Double complement $\overline{\bar{A}} = A$

Famous Identities

- Commutativity

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

- Associativity

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

- Distributivity

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Famous Identities

- DeMorgan's I $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$
- DeMorgan's II $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$

Inclusion/Exclusion

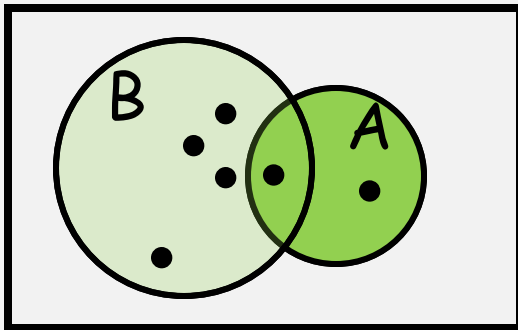
Example:

How many people are wearing a watch?

How many people are wearing sneakers?

How many people are wearing
a watch OR sneakers?

What's wrong?



$$|A \cup B| = |A| + |B| \quad 7$$

$$|A \cup B| = |A| + |B| - |A \cap B| \quad 6$$

Inclusion/Exclusion

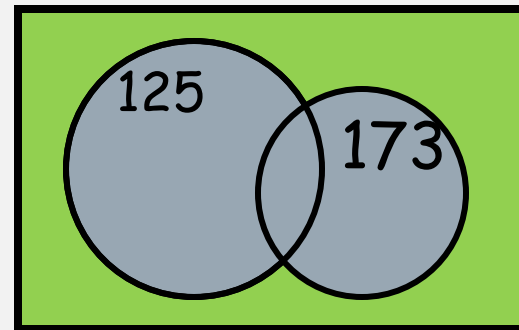
Example:

There are 217 cs majors.

157 are taking cs125.

145 are taking cs173.

98 are taking both.

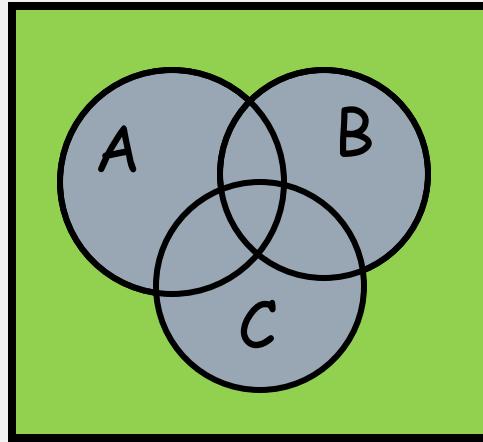


How many are taking neither?

$$217 - (157 + 145 - 98) = 13$$

Generalized Inclusion/Exclusion

Suppose we have:



$$A = \{0, 2, 4, 6, 8\},$$

$$B = \{0, 1, 2, 3, 4\},$$

$$C = \{0, 3, 6, 9\}.$$

And I want to know $|A \cup B \cup C|$

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad - |A \cap B| - |A \cap C| - |B \cap C| \\ &\quad + |A \cap B \cap C| \end{aligned}$$

$$|A \cup B \cup C| = 5+5+4-3-2-2+1 \equiv 8 \equiv \{0, 1, 2, 3, 4, 6, 8, 9\}.$$

Functions - examples

Suppose $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $f(x) = x^2$.

Is f one-to-one?

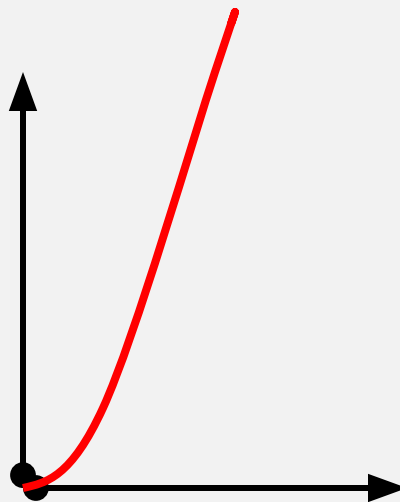
yes

Is f onto?

yes

Is f bijective?

yes



Functions - examples

Suppose $f: \mathbb{R} \rightarrow \mathbb{R}^+$, $f(x) = x^2$.

Is f one-to-one?

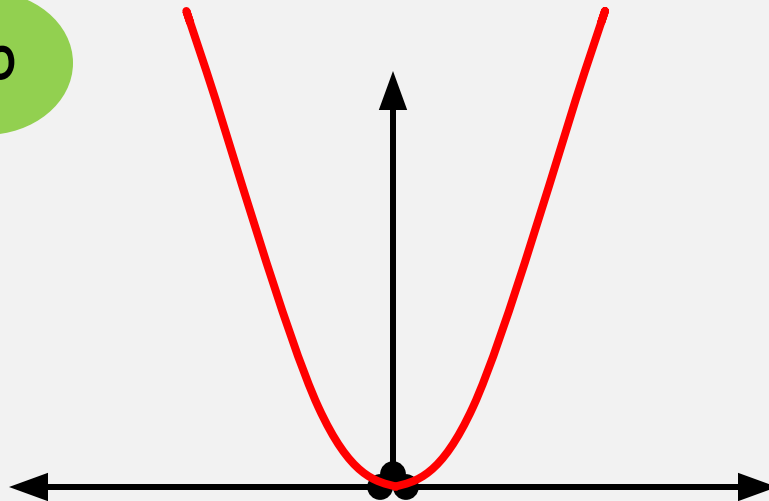
no

Is f onto?

yes

Is f bijective?

no



Functions - examples

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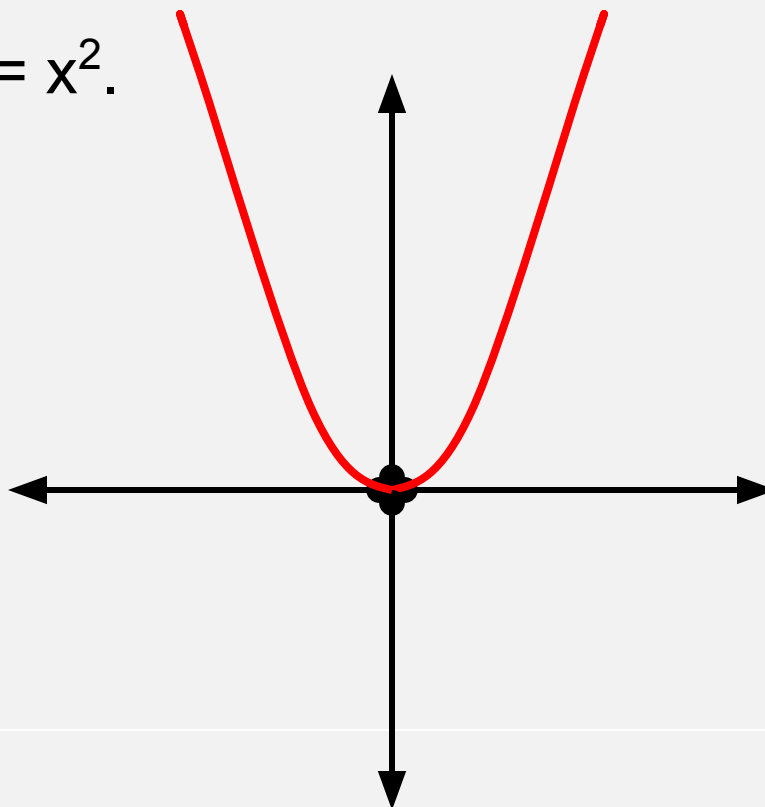
no

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Thank You