Key Concepts and Conditional Probability

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Key Concepts

Definition

1. If two events A and B are such that $A \cup B = S$, then $P(A \cup B) = 1$. Hence, A and B are called **Exhaustive events**.
2. If $A \cap B = \emptyset$. A and B are mutually exclusive events in S. Then $P(A \cup B) = P(A) + P(B)$.
3. **Independent event:** We say that two events are independent if the occurrence of one event does not affect the occurrence of an other.
4. **Dependent events:** If the occurrence of one event affects the occurrence of the the other, then two events are called dependent events.

Conditional probability

The conditional probability $P(A/B)$ is defined as the probability of an event A occurring given that B has already occurred.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \, P(B) \neq 0 \Rightarrow P(A \cap B) = P(A/B) * P(B)$$

$$P(B|A) = \frac{P(A \cap b)}{P(A)}, \, p(A) \neq 0, \Rightarrow P(A \cap B) = P(B/A) * P(A)$$

Example: Smoking by Gender

Gender \ Smoke	Yes	No	Total
Male	19	41	60
Female	12	28	40
Total	31	69	100

1. What is the probability of a randomly selected individual being a male who smokes?
2. What is the probability of a randomly selected individual being a male?
3. What is the probability of a randomly selected individual smoking?
4. What is the probability that the male smokes?
5. What is the probability that a randomly selected smoker is male?

Solution

1. The number of Male and Smoke divided by the total is $19/100$.
2. This is the total for male divided by the total is $60/100$. Since no mention is made of smoking or not smoking, it includes all the cases.
3. Again, since no mention is made of gender, this is a marginal probability, the total who smoke divided by the total is $31/100$.
4. Well, 19 males smoke out of 60 males, so $19/60$.
5. This time, you're told that you have a smoker and asked to find the probability that the smoker is also male. There are 19 male smokers out of 31 total smokers, so is $19/31$.

Example 2:

Problem

Given that a heart is picked at random from a standard pack of playing cards, find the probability that it is a **picture card**.

- A standard deck has 52 cards: 4 suits $\times$ 13 cards.
- The suit of **hearts** contains 13 cards:

$$\text{Ace}, \ 2, 3, \ldots, 10, \ \text{Jack}, \ \text{Queen}, \ \text{King}.$$

- **Picture cards** (face cards) in each suit: Jack, Queen, King $\rightarrow$ **3 cards**.
- We are given the card is a heart: our sample space is reduced to 13 cards.

Solution

We seek the conditional probability:

$$P(\text{Picture card} \mid \text{Heart}) = \frac{\text{Number of heart picture cards}}{\text{Total number of hearts}}$$

- Heart picture cards: Jack, Queen, King $\rightarrow$ 3 cards.
- Total hearts: 13.

$$P(\text{Picture card} \mid \text{Heart}) = \frac{3}{13}$$

Final Answer

$$\boxed{\frac{3}{13}}$$

So, given that a card is a heart, the probability it is a picture card is $\frac{3}{13}$.

(iv) Independent events

- If either A or B can occur without being affected by the other, then the 2 events are said to be independent and $P(A \cap B) = P(A) \times P(B)$.
- **Dependent events:** two events A and B are dependent if the first event affects the outcome or occurrence of the second event in a way the probability is changed.
- If A and B are independent events, then

$$
\begin{aligned}
P(A|B) &= \frac{P(A \cap B)}{P(B)} = \frac{P(A) * P(B)}{P(B)} = P(A) \\
P(B|A) &= \frac{P(A \cap b)}{P(A)} = \frac{P(A) * P(B)}{P(A)} = P(B)
\end{aligned}
$$

Exercises

1. Events A and B are such that $P(A) = \frac{1}{3}$, $P(A \cap B) = \frac{1}{12}$. If A and B are independent, find
   a) $P(B)$
   b) $P(A \cup B)$
2. Events A and B are such that $P(A) = \frac{1}{4}$, $P(A|B) = \frac{1}{2}$ and $P(B|A) = \frac{2}{3}$.
   a) Are A and B independent?
   b) Are A and B mutually exclusive events?
   c) Find $P(A \cap B)$ d) Find $P(B)$

(v) Bayes' theorem

A formula which allows one to find the probability that an event occurred as the result of a particular previous event.
Suppose that $A_i\,(i = 1, 2, 3, \ldots, n)$ are mutually exclusive and exhaustive events so that $U_{i=1}^{n} A_i = S$, the possible space and B an arbitrary event on s.

$$P(A_i|B) = \frac{P(B|A_i) * P(A_i)}{\sum_{i=1}^{n} P(B|A_i) * P(A_i)} \quad for\ i = 1, 2, 3, \ldots, n$$

Exercise

Three machines A, B and C produce $50\%, 30\%$ and 20% of the total number of materials made respectively in a factory. The percentages of defective materials of these machines are respectively $3\%, 4\%$ *and* 5%.

If one takes a material at random.

a) What is the probability so that this material is defective?

b) Calculate the probability so that this material has been produced by the machine A?

Problem Statement

Three machines produce materials in a factory:

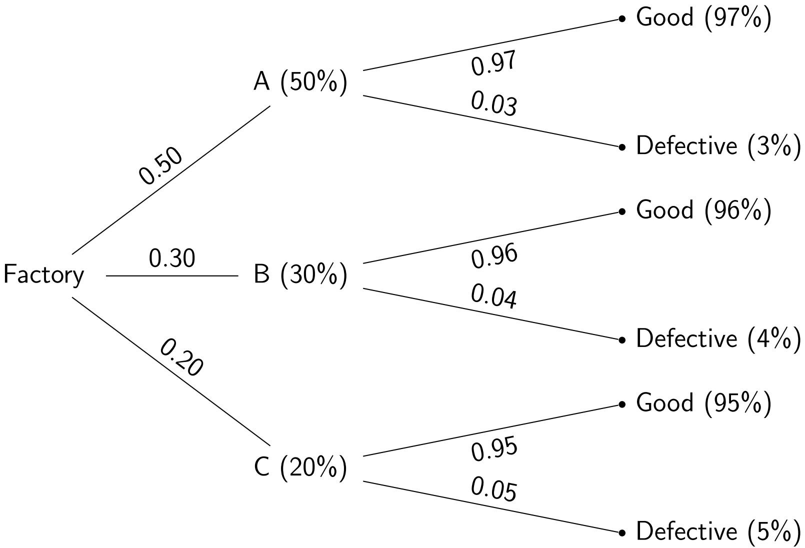
- Machine A: 50% of total output, 3% defective
- Machine B: 30% of total output, 4% defective
- Machine C: 20% of total output, 5% defective

A material is selected at random. Find:

a) The probability that it is defective.

b) The probability it came from Machine A **given** that it is defective.

Tree Diagram



Solution: Part (a)

We use the Law of Total Probability:

$$\begin{aligned} P(D) &= P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C) \\ &= (0.03)(0.50) + (0.04)(0.30) + (0.05)(0.20) \\ &= 0.015 + 0.012 + 0.010 \\ &= 0.037 \end{aligned}$$

$$\boxed{P(\text{Defective}) = 0.037 \text{ (or } 3.7\%)}$$

Solution: Part (b)

We apply Bayes' Theorem:

$$
\begin{aligned}
P(A|D) &= \frac{P(D|A)P(A)}{P(D)} \\
&= \frac{(0.03)(0.50)}{0.037} \\
&= \frac{0.015}{0.037} \\
&\approx 0.4054
\end{aligned}
$$

$$\boxed{P(\text{Machine A} \mid \text{Defective}) \approx 0.405 \text{ (or } 40.5\%)}$$

Conclusion

- Only 3.7% of all materials are defective.
- Given a defective item, there's about a 40.5% chance it came from Machine A—despite A producing half of all items, its lower defect rate reduces its share among defective units.