Unit3.Solid analytic geometry

Introduction

In this Unit, we will study the main parts of solid analytical geometry. We will start by elementary problems of solid analytic geometry and then we will establish the different equations of straight lines and planes. The relative positions between point-straight line - plane are examined in details. We close the unit with the study of second order surfaces by considering their canonical equations,

This Unit is in 4 sections:

Section 3.1 Elementary problems

Section 3.2 The plane

Section 3.3 The straight line

Section 3.4 Quadric surfaces

Section 3.1: Elementary problems

Let A^3 be a three dimension affine euclidean space endowed with an orthonormal frame $F(O;\vec{e}_1,\vec{e}_2,\vec{e}_3)$ and P be an arbitrary point of A^3 such that: $\overrightarrow{OP} = x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3$

3.1.1 Distance formula

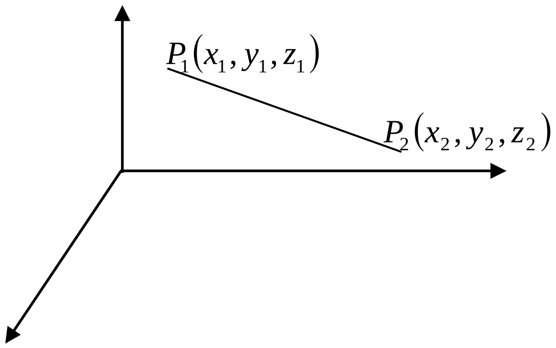


Figure 3.1

The distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by $d(P_1,P_2) = \left\|\overrightarrow{P_1P_2}\right\| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

(3.1)

The following properties are satisfied:

i. $d(P_1,P_2) \geq 0 (=0, P_1 = P_2)$ *ii.* $d(P_1,P_2) = d(P_2,P_1)$

iii. $d(P_1,P_3) \leq d(P_1,P_2) + d(P_2,P_3)$

3.1.2 Dividing of a line segment in a given ratio

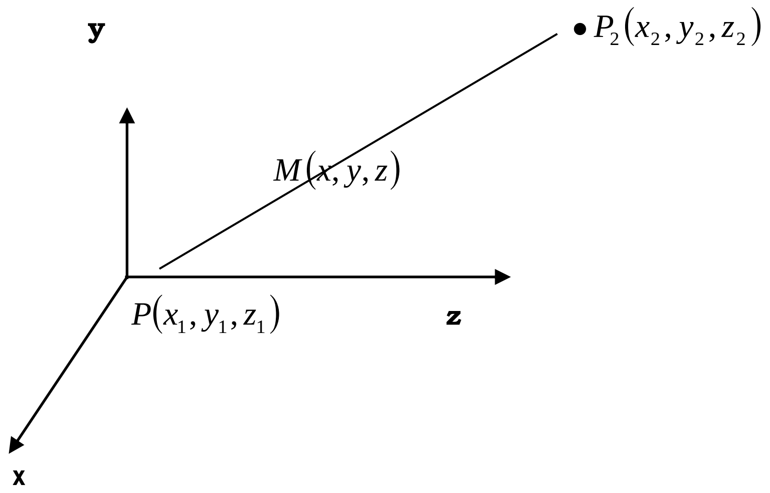


Figure 3.2

If the point P (x, y, z) divides the segment P_1P_2 in the ratio λ, where $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, then:

$$x = \frac{x_1 + \lambda x_2}{1+\lambda}; y = \frac{y_1 + \lambda y_2}{1+\lambda}; z = \frac{z_1 + \lambda z_2}{1+\lambda} \qquad (3.2)$$

If P is the midpoint of P_1P_2, then:

$$x = \frac{x_1 + x_2}{2}; y = \frac{y_1 + y_2}{2}; z = \frac{z_1 + z_2}{2} \qquad (3.3)$$

The proof of the formulas (3.2) and (3.3) is done in similar way as it was done for the case of the plane analyticl geometry. The co-ordinates of the centroid, of a system of material points $P_i(x_i, y_i, z_i)$ with respective masses $m_i (i = 1, 2, ..., n)$ are given by:

$$\overline{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}; \quad \overline{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}; \quad \overline{z} = \frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i} \qquad (3.4)$$

3.1.3 Volume of a tetrahedron

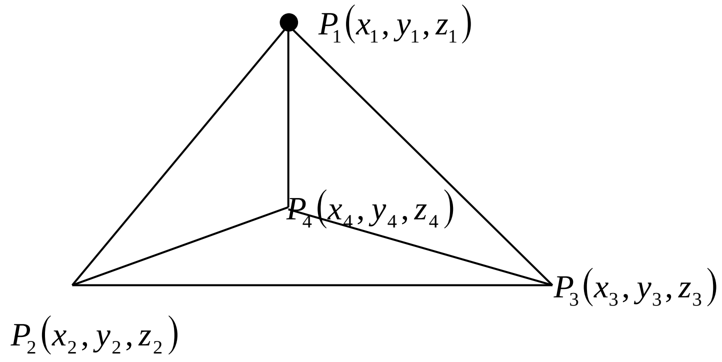


Figure 3.3

The volume of the tetrahedron whose vertices are, $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$ and $P_4(x_4, y_4, z_4)$ is given by:

$$V = \frac{1}{6} A.V \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = \frac{1}{6} A.V \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} \qquad (3.5)$$

where A.V means absolute value. The points P_1, P_2, P_3, P_4 are coplanar if and only if :

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Section 3.2: The plane

3.2.1 Equations of a plane in a space

Proposition 2.1: Let in the plane π, be given a point P_0 and two non co-linear vectors $\vec{u}$ and $\vec{v}$. The point P belongs to the plane π if and only if there are two real numbers α and β such that $\overrightarrow{P_0P} = \alpha\vec{u} + \beta\vec{v}$

Proof: The point P belongs to the plane π if and only if the vectors $\vec{u}$, $\vec{v}$ and $\overrightarrow{P_0P}$ are coplanar. The co-planarity of these three vectors implies that the vector $\overrightarrow{P_0P}$ is a linear combination of $\vec{u}$ and $\vec{v}$. Thus, we can find two real numbers α and β , such that

$$\overrightarrow{P_0P} = \alpha\vec{u} + \beta\vec{v} \qquad (3.6)$$

The equation (3.6) is called the **vector equation** of the plane π .

Denoting $\vec{r}_0$ and $\vec{r}$ the radii vectors of the points P_0 and P respectively, the equation (3.6) is written as

$$\pi : \vec{r} = \vec{r}_0 + \alpha\vec{u} + \beta\vec{v} \qquad (3.7)$$

Using coordinates by putting: $P_0(x_0, y_0, z_0)$, $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ and $P(x, y, z)$, the matrix form of the equation (3.7) is written:

$$\pi : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \alpha \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \beta \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad (3.8)$$

The equation (3.8) is called **parametric equation** of the plane π

In terms of determinants the equation (3.8) is written:

$$\pi : \begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0 \qquad (3.9)$$

Expanding the determinant we get:

$: \pi : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ or

$$\pi : Ax + By + Cz + D = 0 \qquad (3.10)$$

where , $A = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}; B = \begin{vmatrix} u_3 & u_1 \\ v_3 & v_2 \end{vmatrix}; C = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}; D = -Ax_0 - By_0 - Cz_0$

The equation (3.10) is called the **general equation of the plane**.

Remark 2.1: The vector $\vec{N}(A, B, C)$ is perpendicular to the plane π and is called "**normal vector** to the plan π". In vector form the equation (2.5) can be written as:

$$\pi : < \vec{r} - \vec{r}_0, \vec{N} >= 0 \qquad (3.11)$$

The direction cosines of the vector $\vec{N}$ are given by:

$$cos\alpha = \frac{A}{\sqrt{A^2+B^2+C^2}}\ ; cos\beta = \frac{B}{\sqrt{A^2+B^2+C^2}}\ ; cos\gamma = \frac{C}{\sqrt{A^2+B^2+C^2}}$$

- If $D=0$, the plane π passes through the origin of co-ordinates.

$If\ A=0\left(0\ or\ B=0,\ or\ C=0\right)$, the plane is parallel to axes Ox (Oy, or Oz) respectively

$If\ A=B=0\left(or\ A=C=0, or\ B=C=0\right)$, the plane is parallel to plane Oxy (Oxz, or Oyz) respectively

The normal equation of the plane π is given by:

$$\pi : x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0 \qquad (3.12)$$

where p is the distance from the origin of the axes of the co-ordinates to the plan π.

If the plane π intersects the axes of coordinates $0x, 0y, 0z$ at the point $P_1(a,0,0), P_2(0,b,0)$ and $P_3(0,0,c)$ respectively, then the equation of π is given by:

$$\pi : \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad (3.13)$$

The equation (2.8) is called the equation of the plane **in intercept form**.

The equation of the plane π passing through non collinear 3 points $P_1(x_1,y_1,z_1), P_2(x_2,y_2,z_2)$ and $P_3(x_3,y_3,z_3)$ is given by:

$$\pi \equiv \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0 \qquad (3.14)$$

or in the compact vector form:

$$\pi : (\vec{r}-\vec{r}_1, \vec{r}_2-\vec{r}_1, \vec{r}_3-\vec{r}_1) = 0 \qquad (3.15)$$

Let $\pi_1 \equiv A_1x+B_1y+C_1z+D_1=0$ and $\pi_2 \equiv A_2x+B_2y+C_2z+D_2=0$ be two intersecting planes. A pencil of planes generated by the plans π_1 and π_2 is a family of the planes passing through the intersection points of the plans π_1 and π_2 .It is given by the equation:

$$A_1x+B_2y+C_1z+D_1+\lambda(A_2x+B_2y+C_2z+D_2)=0 \qquad (3.16)$$

All planes of the pencil can be obtained by varying the parameter λ in the interval $]-\infty,+\infty[$.

If $\lambda=\pm 1$ and π_1 ; π_2 are given by their normal equations, the equation of the pencil gives the equations of the bisecting planes of the plans π_1 and π_2

3.2.2 Relative positions plane-plane and point-plane

The angle between two planes
$\pi_1 : A_1x+B_1y+C_1z+D_1=0$ and $\pi_2 : A_2x+B_2y+C_2z+D_2=0$ is given by the angle between their corresponding normal vectors

$$\cos(\pi_1,\pi_2)=\frac{\langle \vec{N}_1,\vec{N}_2\rangle}{\|\vec{N}_1\|\cdot\|\vec{N}_2\|} \qquad (3.17)$$

where $\overrightarrow{N_1}=(A_1,B_1,C_1)$ and $\overrightarrow{N_2}=(A_2,B_2,C_2)$ .Two planes π_1 and π_2 are intersecting if and only if $\left[\overrightarrow{N_1},\overrightarrow{N_2}\right]\neq 0$, parallel if and only

if $\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}\neq\frac{D_1}{D_2}$ *and coincident if and only if* $\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}=\frac{D_1}{D_2}$

The distance between two parallel planes $\pi_1 : Ax+By+Cz+D_1=0$ and $\pi_2 : Ax+By+Cz+D_2=0$ is given by:

$$d(\pi_1,\pi_2)=\frac{|D_1-D_2|}{\sqrt{A^2+B^2+C^2}} \qquad (3.18)$$

The distance from the point $P(x_0,y_0,z_0)$ to the plane
$\pi : Ax+By+Cz+D=0$ is given by

$$d(P,\pi)=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}} \qquad (3.19)$$

Section 3.3: The straight line

3.3.1 Equation of a straight line

In a three dimensional space, a straight line (d) can be defined as intersection of two planes.

Thus the equation of that straight line can be written as follows:

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases} \qquad (3.20)$$

In vector form, (3.1) can be written

$$(d): \begin{cases} r\overrightarrow{N_1} + D_1 = 0 \\ r\overrightarrow{N_2} + D_2 = 0 \end{cases} \qquad (3.2)$$

The vector $\left[\overrightarrow{N_1}, \overrightarrow{N_2}\right] = \left(\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}, \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \right)$ is the direction vector of the line (d).

The vector equation of the straight line (d) with direction vector $\vec{u} = (u_1, u_2, u_3)$ and passing through the point $P(x_0, y_0, z_0)$ with radius vector $\vec{r}_0$ is written:

$$(d): \vec{r} = \vec{r}_0 + \lambda \vec{u} \qquad (3.21)$$

(3.21) is In co-ordinates (3.21) is written:

$$(d): \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \lambda \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad (3.22)$$

(3.4) represents the **parametric equations** of the line (d)

Excluding the parameter λ, (3.22) can also be written in the form:

$$(d): \frac{x - x_0}{u_1} = \frac{y - y_0}{u_2} = \frac{z - z_0}{u_3} \qquad (3.23)$$

(3.23) is the **canonical equation** line (d)

The equations of the line (d) passing through two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in canonical and vector forms are given respectively by

$$(d): \frac{x - x_1}{x - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \qquad (3.24)$$

(d): $[\vec{r}-\vec{r}_1, \vec{r}_2-\vec{r}_1]=\vec{0}$ (3.25)

3.3.2 Relative positions of two straight lines

Let $(d_1):\vec{r}=\vec{r}_1+\lambda\vec{u}$ and $(d_2):\vec{r}=\vec{r}_2+\lambda\vec{u}$ be two lines. The following properties are satisfied:

a. (d_1) and (d_2) are parallel if and only if the vectors $\vec{u}$ and $\vec{v}$ are collinear: $[\vec{u},\vec{v}]=\vec{0}$

b. (d_1) and (d_2) coincide if and only if the vectors $\vec{u}$, $\vec{v}$ and $\vec{r}_2-\vec{r}_1$ are collinear

c. (d_1) and (d_2) are concurrent if and only if the vectors $\vec{u},\vec{v}$ and $\vec{r}_2-\vec{r}_1$ are coplanar i.e. $(\vec{r}_2-\vec{r}_1,\vec{u},\vec{v})=0$

d. (d_1) and (d_2) are skew (there is no plane containing both them) if and only if $(\vec{r}_2-\vec{r}_1,\vec{u},\vec{v})\neq 0$

3.3.3 Distance from a point to a line

Proposition.3.1 The distance from a point P_0 of radius vector r_0 to the line $(d):\vec{r}=\vec{r}_1+\lambda\vec{u}$ is given by:

$$d\left(P_0,(d)\right)=\frac{\left\|[\vec{r}_1-\vec{r}_0,\vec{u}]\right\|}{\|\vec{u}\|} \tag{3.26}$$

Proof. The distance from the point P_0 to the line (d)tis equal to the altitude of the parallelogram spanned by vectors $\vec{r}_1-\vec{r}_0$ and $\vec{u}$ .This altitude is equal to the area A of this parallelogram divided by the base with length $\|\vec{u}\|$. Thus $d\left(P_0,(d)\right)=h=\dfrac{A}{\|\vec{u}\|}=\dfrac{\left\|[\vec{r}_1-\vec{r}_0,\vec{u}]\right\|}{\|\vec{u}\|}$

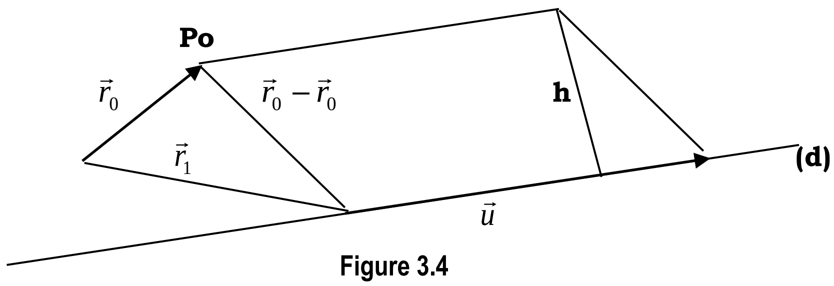


Figure 3.4

In co-ordinates; $\vec{r}_0=(x_0,y_0,z_0)$ $\vec{r}_1=(x_1,y_1,z_1)$ and; $\vec{u}=(u_1,u_2,u_3)$ the formula (3.26) is written:

$$d\left(P_0,(d)\right)=\frac{\sqrt{\begin{vmatrix} y_1-y_0 & z_1-z_0 \\ u_2 & u_3 \end{vmatrix}^2+\begin{vmatrix} z_1-z_0 & x_1-x_0 \\ u_3 & u_1 \end{vmatrix}^2+\begin{vmatrix} x_1-x_0 & y_1-y_0 \\ u_1 & u_2 \end{vmatrix}^2}}{\sqrt{u_1^2+u_2^2+u_3^2}} \tag{3.27}$$

Proposition 3.2: The distance between two skew lines Let $(d_1):\vec{r}=\vec{r}_1+\lambda\vec{u}$ and $(d_2):\vec{r}=\vec{r}_2+\lambda\vec{u}$ (d_1) (i.e, the length of common perpendicular to the both lines) is given by:

$$d\left((d_1),(d_2)\right)=\frac{\left|(\vec{r}_2-\vec{r}_1,\vec{u},\vec{v})\right|}{\left\|[\vec{u},\vec{v}]\right\|} \tag{3.28}$$

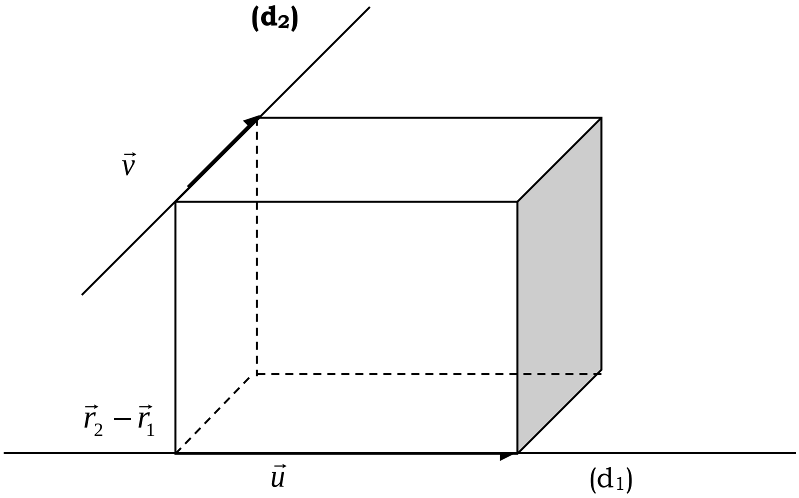


Figure 3.5

Proof: The distance between two skew lines (d_1) and (d_2) is equal to the distance between two parallel plans containing these two lines. This distance equals to the height of the parallelepiped spanned by the vectors $\vec{r}_2-\vec{r}_1$, and $\vec{v}$ (fig.3.5).This height is equal to the quotient of the volume of this parallelepiped and the area of its surface of the base.

Thus: $d\left((d_1),(d_2)\right)=\dfrac{\left|(\vec{r}_2-\vec{r}_1,\vec{u},\vec{v})\right|}{\left\|[\vec{u},\vec{v}]\right\|}$

In co-ordinates form; $\vec{r}_1=(x_1,y_1,z_1)$;

$\vec{r}_2=(x_2,y_2,z_2)$;and $\vec{u}=(u_1,u_2,u_3)$; $\vec{v}=(v_1,v_2,v_3)$, we have:

$$d\left((d_1),(d_2)\right)=\left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}}{\sqrt{\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}^2+\begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}^2+\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2}}\right| \qquad (3.29)$$

3.3.4 Relative position of a straight line and a plane

Proposition 3.3: Let consider the plane $\pi: Ax+By+Cz+D=0$ and the straight line $(d): x=x_0+tu_1; y=y_0+tu_2; z=z_0+tu_3$. Three cases are possible:

a. $(d)\cap\pi=\{\ \}$ iff $Au_1+Bu_2+Cu_3=0$ and $Ax_0+By_0+Cz_0+D\neq 0$

b. (d) and π have a single common point iff $Au_1+Bu_2+Cu_3\neq 0$

c. $(d)\in\pi$ iff $Au_1+Bu_2+Cu_3=0$ and $Ax_0+By_0+Cz_0+D=0$

Proof: Let denote $\overrightarrow{N}(A,B,C)$ the normal vector to π and $\vec{u}=(u_1,u_2,u_3)$, the direction vector of (d)

a. $(d)\cap\pi=\varnothing$ if and only if $\left\langle \vec{N},\vec{u}\right\rangle=0$ and an arbitrary point P_0 of (d) does not belong to π

b. (d) and π have a unique common point if and only if $\left\langle \vec{N},\vec{u}\right\rangle\neq 0$

$(d)\in\pi$ if and only if $\left\langle \vec{N},\vec{u}\right\rangle=0$ and an arbitrary point P of (d) is a point of π. These results are obtained by solving the system:

$$\begin{cases} Ax+By+Cz+D=0 \\ x=x_0+tu_1 \\ y=y_0+tu_2 \\ z=z_0+tu_3 \end{cases}$$

The angle between the line $(d): \vec{r}=\vec{r}_0+\lambda\vec{u}$ and the plan $\pi: \left\langle \vec{r},\vec{N}\right\rangle+D=0$ is given by the formula:

$$\sin\left((d),\pi\right)=\frac{\left\langle \vec{N},\vec{u}\right\rangle}{\left\|\vec{N}\right\|\cdot\left\|\vec{u}\right\|} \qquad (3.30)$$

If $\vec{u}=(u_1,u_2,u_3)$ and $\overrightarrow{N}=(A,B,C)$, then:

$$\sin\left((d),\pi\right)=\frac{Au_1+Bu_2+Cu_3}{\sqrt{A^2+B^2+C^2}\cdot\sqrt{u_1^2+u_2^2+u_3^2}} \tag{3.31}$$

(d) is parallel to π iff $\left\langle \vec{N},\vec{u}\right\rangle=0$ and (d) is perpendicular to π iff $\left[\vec{N},\vec{u}\right]=0$

Section 3.4: Quadric surfaces

3.4.1 Sphere

A sphere is the set of all points in a space equidistant from a given point called the centre. The distance from the center to an arbitrary point of the sphere is called the radius.

A sphere (S) of radius R with center at point $C(x_0,y_0,z_0)$ is the set of all points $P(x,y,z)$ in space satisfying the condition: $d(C,P)=\left\|\overrightarrow{CP}\right\|=R$

Applying the distance formulae, we have:

$R=\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$. Squaring both sides of the equality we obtain:

$$(S):(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2 \tag{3.32}$$

(3.32) is the standard equation of the sphere (S) of radius R with center $C(x_0,y_0,z_0)$.

In particular, a sphere (S) of radius R with center at the origin has the equation: $x^2+y^2+z^2=R^2$

Mutual position of a plane and a sphere

The relative position of the plane $(\pi): Ax+By+Cz+D=0$ and the sphere $(S):(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2$ is obtained by solving the system of equation

$$\begin{cases} Ax+By+Cz+D=0 \\ (x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2 \end{cases}$$

Three cases can occur:

1. The system admits a single solution. In this case, the plane is tangent to the sphere and the single solution is the intersection point or the point of tangency.
2. The system has no solution. In this case, the plane passes out of the sphere i.e he plane and the sphere do not intersect.

3. The system has infinity of solutions. In this case, the intersection of this plane and the sphere describes a circle.

3.4.2 Quadric surfaces in canonical form

a. Ellipsoid

The ellipsoid is a quadric surface given by the equation:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,(a,b,c>0) \qquad (3.33)$$

The points $(\pm a,0,0);(0,\pm b,0);(0,0,\pm c)$ are called **vertices** of the ellipsoid. The quantities a,b,c are called **semi axes** of the ellipsoid. If $a=b=c$, the ellipsoid is transformed into a sphere with radius a

The ellipsoid is a symmetric about the plans of equations

$x=0; y=0; z=0$ and o the origin of co-ordinates. The part of ellipsoid

contained in the first octant is defined by the equation:

$$z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}; x\geq 0; y\geq 0; \frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1.$$

If $a\geq b\geq c$, then the ellipsoid is a bounded surface. It is containing in a –ball with center at O .The co-ordinates (x,y,z) of an arbitrary point of the ellipsoid satisfies the inequality:

$$x^2+y^2+z^2\leq a^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=a^2.1=a^2$$

The intersection of the ellipsoid and the plane $z=h(-c\leq h\leq c)$ is an ellipse defined by:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{h^2}{c^2}.$$

If two semi- axes are equal, the ellipsoid (4.2) is an ellipsoid of revolution generated by revolving the ellipse around the third axis.

b. Hyperboloids

i. Hyperboloid with one sheet

The hyperboloid of one sheet is a quadric surface given by:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,(a,b,c>0) \qquad (3.34)$$

The hyperboloid of one sheet is symmetric about the planes of equations $x=0; y=0; z=0$ and the origin of the coordinates. The points $(\pm a,0,0);(0,\pm b,0)$ are vertices.

The section made by the plan $z=h$ is an ellipse of equation: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{h^2}{c^2}$.

The sections made by the plane $x=h\,(y=h)$ are hyperbola s of the equations:

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1-\frac{h^2}{a^2} \quad (\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{h^2}{b^2}).$$

If $h=\pm a$, those parabolas are transforming into two lines $y=\pm\frac{b}{c}z$

If $|h|\le a$, the transverse axis of the corresponding hyperbola is a straight line parallel to the Oy -axis.

If $|h|\succ a$, we have a straight line parallel to the Oz -axis.

If $a=b$, the sections of surface (4.3) made by the plans $z=h$ are circles with radii $a\sqrt{1+\frac{h^2}{c^2}}$ and the equation (4.3) gives the hyperboloid of revolution of one sheet obtained by revolving the hyperbola $\frac{x^2}{a^2}-\frac{z^2}{c^2}=1, y=0$ about z -axis.

ii. Hyperboloid with two sheets

The hyperboloid of two sheets is a surface defined by the equation:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,(a,b,c>0)\ (a,b,c>0) \qquad (3.35)$$

The hyperboloid of two sheets is symmetric about the planes $x=0, y=0$ and $z=0$ and the origin of the co-ordinates. The equation (4.4) can be written also in the form:

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1+\frac{x^2}{a^2} \qquad (3.35')$$

The section of (3.35) made by the plane $x=h\left(|h|\ge a\right)$ is an ellipse of equation: $\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1+\frac{h^2}{a^2}$

If $|h| < a$, the number $\frac{h^2}{a^2} - 1$ is strictly negative, the surface $(4.4)'$ and the plane $x = h$ have no common point. The sections of (4.4) made by a planes $z = h$ or $y = h$ are hyperbolas of equations:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 + \frac{h^2}{c^2} \text{ or } \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 + \frac{h^2}{c^2} \text{ respectively.}$$

The points $(\pm\ a, 0, 0)$ are the vertices of the surface (4.4).

C. Paraboloid

i. Elliptic paraboloid

The elliptic paraboloid is a quadric surface given by the equation

$$\frac{x^2}{p} + \frac{y^2}{q} = 2z \ (p, q > 0) \qquad (3.36)$$

(3.36) is symmetric about the plans of equations; $y = 0$ $x = 0$

If $p, q > 0$, the quadric surface (3.36) is located in the half space $z \geq 0$

The section of surface (3.36) made by the plan $z = h(h \geq 0)$ is an ellipse of equation:

$\frac{x^2}{p} + \frac{y^2}{q} = 2h$. When h varies from 0 to $+\infty$, this ellipse describes the surface (3.36) .The section of (4.5) made by a plane $x = h$ or $y = h$ are parabolas of equation:

$$y^2 = 2q\left(z - \frac{h^2}{2p}\right) \text{or } x^2 = 2p\left(z - \frac{h^2}{2q}\right) \text{ respectively.}$$

If $p = q$, the surface (4.5) is a surface of revolution generated by revolving the parabola $x^2 = 2pz$ around the Oz-axis. The point (0, 0, 0) is the vertex of the elliptic paraboloid.

ii. Hyperbolic paraboloid

The hyperbolic paraboloid is a quadratic surface given by the equation

$$\frac{x^2}{p} - \frac{y^2}{q} = 2z \ (p, q \geq 0) \qquad (3.37)$$

The surface (3.37) is symmetric about the planes $x = 0$ and. $y = 0$

The section of (3.37) made by the plane $z = h$ is a hyperbole of equation. $\frac{x^2}{p} - \frac{y^2}{q} = 2h$

If $h > 0$, the transverse axis of the hyperbole is parallel to Ox, if $h < 0$ is parallel with the axis Oy and if $h = 0$, the section consists in two intersecting lines. The section of (3.37) made by the planes $x = h$ or $y = h$ are parabolas of equation $\frac{x^2}{p} = 2z + \frac{h^2}{q}$ whose concavity is downward or upward respectively.

d. Quadric cone

The quadric cone is a surface defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, (a,b,c > 0)\ (a,b,c > 0) \qquad (3.38)$$

The surface (3.38) is symmetric about the planes; $y = 0$ $x = 0$; $z = 0$ and the origin of the co-ordinates.

The section of (3.38) made by a plane $z = h$ is an ellipse of equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{h^2}{c^2}$. The sections of (3.38) made by the plane $x = h$ or $y = h$ are hyperbolas of equation: $\frac{z^2}{c^2} - \frac{y^2}{b^2} = \frac{h^2}{a^2}$ or $\frac{z^2}{c^2} - \frac{x^2}{a^2} = \frac{h^2}{b^2}$ respectively.

The section of (3.38) made by a plane $y = hx$ consists a couple of intersecting straight lines of equations: $z = \pm cx\sqrt{\frac{1}{a^2} + \frac{h^2}{b^2}}$

e. Cylindrical surfaces

A cylindrical surface is generated by straight line which moves a long fixed curve and remains parallel to a fixed straight line. The fixed curve is called the directrix of the surface and the moving line is the generatrix of the surface.

i. Elliptic cylinder

The elliptic cylinder is a surface defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a,b > 0)\ (a,b > 0) \qquad (3.39)$$

The straight line parallel with the $z-$axis is the generatrix and the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $z = 0$ is the directrix

ii. Hyperbolic cylinder

The hyperbolic cylinder is described by the equation:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad (a,b>0) \tag{3.40}$$

The directrix is here a hyperbola and the generatrix is a straight line parallel to the $z-$axis.

iii. Parabolic cylinder

The parabolic cylinder is described by the equation:

$$y^2=2px \quad (p>0) \tag{3.41}$$

The directrix is a parabola and the generator a straight line parallel to the $z-$axis

f. Planes

i. Real intersecting planes

A couple of real intersecting planes is described by the equation:

$$a^2x^2-b^2y^2=0 \tag{3.42}$$

The directrices of the surface (3.42) are the lines of equation:

$$y=\pm\frac{a}{b}x, z=0$$

ii. Parallel planes

A couple of parallel plans is described by the equation:

$$x^2-a^2=0 \quad (a>0) \tag{3.43}$$

The equation (3.43) defines two lines $x=\pm a$ in the plan $x0y$.

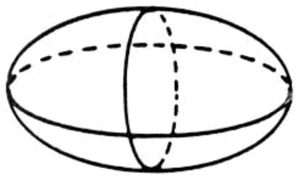
iii. Coincident planes

A couple of coincident planes is described by the equation:

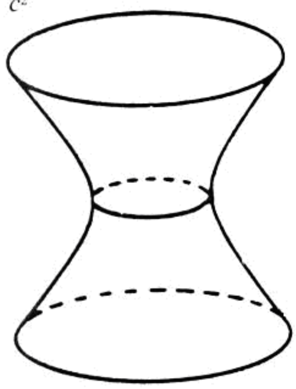
$$z^2=0 \tag{3.44}$$

Sketch of some quadric surfaces

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$ ellipspoid



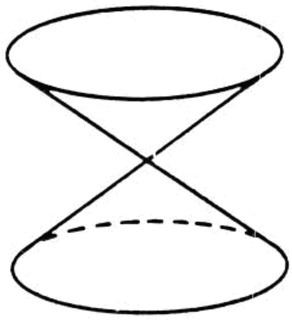
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1;$ hyperboloid of 1sheet



hyperboloi of two sheets

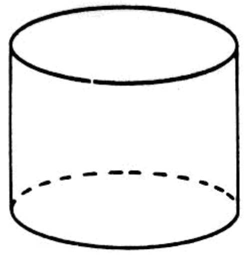
$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1;$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0;$ cone

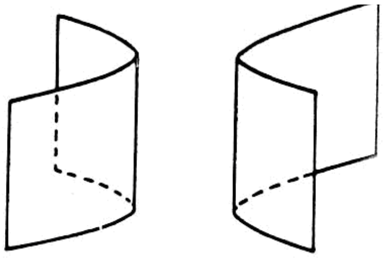


Prepared by Theoneste Hakizimana

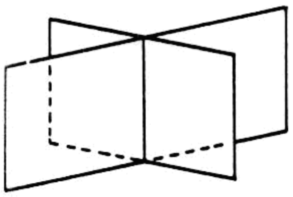
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$ elliptic cylider



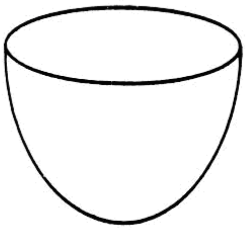
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;$ hyperbolic cylinder



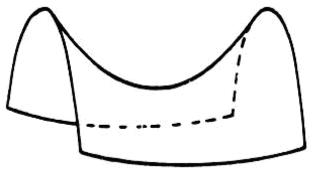
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0;$ two intersecting planes



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z;$ elliptic paraboloid

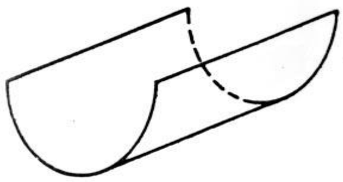


$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z;$ hyperbolic paraboloid



Two eigenvalues are nil or zeros

$\frac{x^2}{a^2} = z$ parabolic cylinder



$\frac{x^2}{a^2} = 1;$ 2 parallel planes