REVISION EXERCISES:

A) **MATRICES, DETERMINANTS, SYSTEM OF LINEAR EQUATIONS**

I. Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & -1 & 4 \\ -1 & 0 & 1 \end{bmatrix}$ and consider $A\vec{x}$ as a linear combination of columns of A to determine $\vec{x}$ if $A\vec{x} = \vec{b}$ where

(a) $\vec{b} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ (b) $\vec{b} = \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix}$

(c) $\vec{b} = \begin{bmatrix} 3 \\ 12 \\ 3 \end{bmatrix}$ (d) $\vec{b} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$

II. Determine if $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 5 \\ 5 & 3 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix} \right\}$ is linearly dependent or linearly independent. Is $X = \begin{bmatrix} 1 & 5 \\ -5 & 1 \end{bmatrix}$ in the span of $\mathcal{B}$?

III. **. Show that if**

$$\mathbf{A}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

then

$$\mathbf{A}(\theta_1)\mathbf{A}(\theta_2) = \mathbf{A}(\theta_2)\mathbf{A}(\theta_1) = \mathbf{A}(\theta_1 + \theta_2). \quad (1)$$

Consider the two-dimensional Euclidean vector represented as

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Examine $\mathbf{A}(\theta)\,\mathbf{x}$ and give a geometric interpretation of the result in Eq. (1).

IV. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and let B be the matrix obtained from A by multiplying the third row of A by the real number r. Show that $\det B = r \det A$.

Solution: We have $B = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ra_{31} & ra_{32} & ra_{33} \end{vmatrix}$. We wish to expand the determinant of B along its third row. The cofactors for this row are

$$C_{31} = (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \quad C_{32} = (-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad C_{33} = (-1)^{3+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Observe that these are also the cofactors for the third row of A. Hence,

$$\det B = ra_{31}C_{31} + ra_{32}C_{32} + ra_{33}C_{33} = r(a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}) = r \det A$$

V.

Find the general solution of the homogeneous system

$$\begin{aligned} 2x_1 + x_2 \quad\quad &= 0 \\ x_1 + x_2 - x_3 &= 0 \\ -x_2 + 2x_3 &= 0 \end{aligned}$$

Solution: We row reduce the coefficient matrix of the system to RREF:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} R_1 \updownarrow R_2 \sim \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} R_2 - 2R_1 \sim$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \begin{matrix} R_1 + R_2 \\ \\ R_3 - R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad (-1)R_2 \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

This corresponds to the homogeneous system

$$\begin{aligned} x_1 \quad + x_3 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned}$$

Hence, x_3 is a free variable, so we let $x_3 = t \in \mathbb{R}$. Then $x_1 = -x_3 = -t$, $x_2 = 2x_3 = 2t$, and the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

VI

Find the determinant and all nine cofactors C_{ij} of this triangular matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Form C^{T} and verify that $AC^{\mathrm{T}} = (\det A)I$. What is A^{-1}?

VII

Solve the following determinants by elementary operations

(a)
$$\begin{vmatrix} 6 & 3 & 2 & 9 \\ 5 & 3 & 2 & 9 \\ 4 & 4 & 8 & 9 \\ 4 & 3 & 2 & 9 \end{vmatrix}.$$

(b)
$$\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} = 4abc.$$

B) LIMITS, DERIVATIVES AND INTEGRATION

VIII. a)

Evaluate $\lim_{x \to 4} \frac{x^3 - 8x^2 + 16x}{x^3 - x - 60}$

Solution: Consider $x^3 - 8x^2 + 16x$

$$= x(x^2 - 8x + 16), \quad \{16 = (-4)(-4)\}$$
$$= x(x^2 - 4x - 4x + 16)$$
$$= x[x(x-4) - 4(x-4)]$$
$$= x[(x-4)(x-4)]$$

Now consider $x^3 - x - 60$

$$= x^3 - 4x^2 + 4x^2 - 16x + 15x - 60$$
$$= x^2(x-4) + 4x(x-4) + 15(x-4)$$
$$= (x-4)(x^2 + 4x + 15)$$

$$\lim_{x \to 4} \frac{x^3 - 8x^2 + 16x}{x^3 - x - 60} = \lim_{x \to 4} \frac{x(x-4)(x-4)}{(x-4)(x^2 + 4x + 15)}$$

$$\lim_{x \to 4} \frac{x(x-4)}{(x^2 + 4x + 15)} = \frac{4(4-4)}{(4^2 + 4(4) + 15)} = 0 \quad \text{Ans.}$$

b)

Let $f(x) = x^2/(1+x^2)$. Determine the numbers at which "f" is continuous.

Solution: Here again "f" is a rational function, *but its denominator* $(1+x^2)$ *is never 0.* Thus, "f" is defined for all x and therefore "f" is *continuous for every real value of* x.

c)

Calculate $\lim_{x \to 0} \left(\frac{1 - \cos x}{x^2}\right)$

IX.

a)

Differentiate $w = \left(\theta + \frac{1}{\theta}\right)\left(\sqrt{\theta} + \frac{1}{\sqrt{\theta}}\right)$

This may be written

$$w = (\theta + \theta^{-1})(\theta^{\frac{1}{2}} + \theta^{-\frac{1}{2}})$$

$$\frac{dw}{d\theta} = (\theta + \theta^{-1})\frac{d(\theta^{\frac{1}{2}} + \theta^{-\frac{1}{2}})}{d\theta} + (\theta^{\frac{1}{2}} + \theta^{-\frac{1}{2}})\frac{d(\theta + \theta^{-1})}{d\theta}$$

$$= (\theta + \theta^{-1})(\tfrac{1}{2}\theta^{-\frac{1}{2}} - \tfrac{1}{2}\theta^{-\frac{3}{2}}) + (\theta^{\frac{1}{2}} + \theta^{-\frac{1}{2}})(1 - \theta^{-2})$$

$$= \tfrac{1}{2}(\theta^{\frac{1}{2}} + \theta^{-\frac{3}{2}} - \theta^{-\frac{1}{2}} - \theta^{-\frac{5}{2}}) + (\theta^{\frac{1}{2}} + \theta^{-\frac{1}{2}} - \theta^{-\frac{3}{2}} - \theta^{-\frac{5}{2}})$$

$$= \frac{3}{2}\left(\sqrt{\theta} - \frac{1}{\sqrt{\theta^5}}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{\theta}} - \frac{1}{\sqrt{\theta^3}}\right)$$

b)

If $t = \dfrac{1}{5\sqrt{\theta}}$; $x = t^3 + \dfrac{t}{2}$; $v = \dfrac{7x^2}{\sqrt[3]{x-1}}$, find $\dfrac{dv}{d\theta}$

c)

If $y = n^t$, find $\dfrac{d(\ln y)}{dt}$

d)

Find the maximum or minimum of

$$y = x^3 - \ln x$$

X

a)

Find $\int \sqrt{1 - x^2}\,dx$.

Write $u = \sqrt{1 - x^2}, \quad dx = dv$

then $du = -\dfrac{x\,dx}{\sqrt{1-x^2}}$

and $x = v$; so that

$$\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \int \frac{x^2dx}{\sqrt{1-x^2}}$$

Here we may use a little dodge, for we can write

$$\int \sqrt{1-x^2}\,dx = \int \frac{(1-x^2)dx}{\sqrt{1-x^2}} = \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x^2dx}{\sqrt{1-x^2}}$$

Adding these two last equations, we get rid of $\displaystyle\int \frac{x^2dx}{\sqrt{1-x^2}}$, and we have

$$2\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} + \int \frac{dx}{\sqrt{1-x^2}}$$

$$\int \sqrt{1-x^2}\,dx = \frac{x\sqrt{1-x^2}}{2} + \tfrac{1}{2}\text{ arc sin } x + C \qquad \int \frac{dx}{\sqrt{1-x^2}} = \text{arc sin } x + C$$

$\displaystyle\int \frac{dx}{x^2+2x-3}$ if we split $\dfrac{1}{x^2+2x-3}$ into partial fractions, this becomes:

b) $$\frac{1}{4}\left[\int\frac{dx}{x-1}-\int\frac{dx}{x+3}\right]=\frac{1}{4}\,[\ln(x-1)-\ln(x+3)]+C$$

c) $$\int \ln\left|\frac{x^3-1}{x^2+x+1}\right|dx$$ Hint: Use the same technique

d) $$\int_1^2 \frac{1}{x(1+x^2)}\,dx$$ Hint: Use partial fractions.

e) Use integration to find the area between the curve $y= x^3-4x^2-8x$ and the x-axis with equation $y= 4$

C INTRODUCTION TO LINEAR ALGEBRA WITH PYTHON

TRY AND REVIEW ALL HOME WORKS
RELATED TO THIS EXCEPT GAUSS-ELIMINATION.
solutions are already provided!!!