Unit1.Affine spaces

Introduction

In this unit we will study affine coordinate transformations and establish the connection between polar, cylindrical, spherical and Cartesian coordinate system. The unit is ended by recalling the concepts of cross and box product in 3 D-space and their geometrical interpretations.There are 3 sections in this unit:

Section 1.1: Affine spaces

1.1.1 Preliminaries

The geometry of affine space is built on two basic concepts which are the point and the vector and on three fundamental relationships which are the relation between three vectors $\vec{u},\vec{v},\vec{w}$ such that $\vec{w}=\vec{u}+\vec{v}$; the relation between two vectors $\vec{u},\vec{v}$ and the scalar λ such that $\vec{v}=\lambda\vec{u}$ and the relation between two points P,Q and the vector $\vec{u}$ such that $\overrightarrow{PQ}=\vec{u}$.

These relationships should respect the axioms related to the structure of a vector space and two axioms of an affine space.

Definition 1.0

A real affine space (A,V,φ), is given by a set A whose elements are called points and, a real vector space V and a bijective mapping $\varphi:\ A\times A\to V;(P,Q)\to\varphi(P,Q)=\overrightarrow{PQ}$ satisfying the following conditions:

$$\forall P\in A,\forall\vec{u}\in V\exists!Q\in A:\overrightarrow{PQ}=\vec{u} \qquad (1.1)$$

$$\forall P,Q,T\in A:\overrightarrow{PQ}+\overrightarrow{QT}=\overrightarrow{PT} \qquad (1.2)$$

If in (1.2) $P=Q=T,$ we have $\overrightarrow{PP}=\vec{0}$ and if in (1.2) $P=T,$ we obtain: $\overrightarrow{QP}=-\overrightarrow{PQ}$

The vector space V is associated with the affine space A and the dimension of V is by definition the dimension of A i.e. $\dim V=\dim A.$

Définition 1.1

An **isomorphism** of an affine space (A,V) onto the affine space (A',V') is a bijective mapping $f:A\to A'$ such that there is an isomorphism

$g: V \to V'$ of the associated vector spaces and such that

$\forall P, Q \in A: \overrightarrow{f(P)f(Q)} = g(\overrightarrow{PQ})$

Any affine space A is isomorphic to its associated vector space V considered as an affine space.

This isomorphism is defined by choosing in A an arbitrary point O and putting:

$$f(P) = \overrightarrow{OP} \qquad (1.3)$$

The vector $\overrightarrow{OP}$ defined by (1.3) is called **the radius vector** of the point A.
Any affine space A^n of dimension n is isomorphic to the affine space $\mathbb{R}^n$.
The isomorphism of A^n on $\mathbb{R}^n$ is given by an arbitrary point $O \in A$ and any basis $\vec{e}_1, ..., \vec{e}_n$ of V.

Definition 1.2

A **frame** or **a coordinate system** in A^n is a pair $\{O; B\}$, where $O \in A$ and a $B = \{\vec{e}_1, ..., \vec{e}_n\}$ is any basis in V. This frame is denoted $F = \{O, \vec{e}_1, ..., \vec{e}_n\}$. The point O is called the origin of the frame. Each frame of an affine space $F = \{O, \vec{e}_1, ..., \vec{e}_n\}$ defines an isomorphism

$$f: A \to \mathbb{R}^n : f(P) = (x_1, ..., x_n). \qquad (1.4)$$

Definition 1.3

The numbers $(x_1, ..., x_n)$ defined by (1.4) are called **affine coordinates** of the point P with respect to the frame $F = \{O, \vec{e}_1, ..., \vec{e}_n\}$. The coordinates of the point P are simply the components of its radius vector $\overrightarrow{OP}$ referred to the basis $B = \{\vec{e}_1, ..., \vec{e}_n\}$ i.e. $\overrightarrow{OP} = x_1\vec{e}_1 + ... + x_n\vec{e}_n$.

Definition 1.4

A vector space V endowed with the dot product $\langle , \rangle$ is called the **Euclidean space**

Definition 1.5

An affine Euclidean space is an affine space such that it is associated vector space is a vector Euclidean space. The dot product of the vectors $\vec{u}$ and $\vec{v}$ will be denoted $\langle \vec{u}, \vec{v} \rangle$

Definition 1.6

An **orthonormal frame** in an affine Euclidean space A^n, is a frame containing an orthonormal basis.

Definition 1.7

The coordinate system for which the frame is orthonormal is called the **Cartesian system** or the **rectangular coordinate system**

The expression of the scalar product of vectors $\vec{u}=\sum_{i=1}^{n} x_i\vec{e}_i$ and $\vec{v}=y_i\vec{e}_i$ in an orthonormal basis is given by:

$$\langle \vec{u},\vec{v}\rangle=\sum_{i=1}^{n} x_i y_i \tag{1.5}$$

The length of $\vec{u}$ is given by

$$\|\vec{u}\|=\sum_{i=i}^{n}\left(x_i\right)^{2} \tag{1.6}$$

The angle θ between two vectors $\vec{u}$ and $\vec{v}$ is given by

$$\cos\theta=\frac{\sum_{i=1}^{n} x_i y_i}{\sqrt{\sum_{0=1}^{n}\left(x_i\right)^2}\sqrt{\sum_{i=1}^{n}\left(y_i\right)^2}} \tag{1.7}$$

1.1.2 Transformation of coordinates

Change of coordinates in two distinct bases

Let $(x_1,\dots,x_n)$, $\left(x_1',\cdots x_n'\right)$ be the components of the vector $\vec{v}$ of the vector space V referred to bases

$B=\left\{\vec{e}_1,\dots,\vec{e}_n\right\}$ and $B'=\left\{\vec{e}_1{}',\dots,\vec{e}_n{}'\right\}$ respectively. With respect to the basis B, the vectors $\vec{e}_1{}',\dots,\vec{e}_n{}'$ of B' are written:

$$\begin{array}{ccccccccc} \vec{e}_1{}' & = & p_{11}\vec{e}_1 & + & p_{21}\vec{e}_2 & + & \dots & + & p_{n1}\vec{e}_n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \vec{e}_n{}' & = & p_{1n}\vec{e}_1 & + & p_{2n}\vec{e}_2 & + & \dots & + & p_{nn}\vec{e}_n \end{array} \tag{1.8}$$

In matrix notations (1.8) is written:

$$\left[\vec{e}_1{}' \quad \dots \quad \vec{e}_n{}'\right] = \left[\vec{e}_1 \quad \dots \quad \vec{e}_n\right] \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \dots & \dots & \dots & \dots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \tag{1.9}$$

The matrix $P=\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$; $\det(P)\neq 0$ is called the **transition matrix** from B to B'.

The columns of this matrix are formed by the components of the vectors of B' with respect to $B.$ Denoting: $\left[\vec{e}\right]=\left[\vec{e}_1 \quad \ldots \quad \vec{e}_n\right]; \left[\vec{e}'\right]=\left[\vec{e}_1' \quad \ldots \quad \vec{e}_n'\right],$ (1.9)

is compactly written $\left[\vec{e}'\right]=\left[\vec{e}\right]P.$
(1.10)

Proposition 1.1

If P is the transition matrix from $B=\left[\vec{e}\right]$ to $B'=\left[\vec{e}'\right]$ and Q the transition matrix from $B'=\left[\vec{e}'\right]$ to $B''=\left[\vec{e}''\right]$, then PQ is the transition matrix from $\left[\vec{e}\right]$ to $\left[\vec{e}''\right]$

Proof: The proposition 1.1 is derived from the associative law of matrix multiplication: $\left[e''\right]=\left[\vec{e}'\right]Q=\left[\vec{e}P\right]Q=\left[\vec{e}\right]PQ$.

Proposition 1.2

If P is the transition matrix from $B=\left[\vec{e}\right]$ to $B'=\left[\vec{e}'\right]$, then the transition matrix from $B'=\left[\vec{e}'\right]$ to $B=\left[\vec{e}\right]$ is the inverse matrix P^{-1}.

Now let us find the relationship between components $(x_1,\ldots,x_n)$ and $(x_{1'},\ldots,x_{n'})$ of the vector $\vec{v}$ referred to bases $\left[\vec{e}\right]$ and $\left[\vec{e}'\right]$ respectively. The decomposition of the vector $\vec{v}$ in the two bases is written:

$$x=\left[\overrightarrow{e_1} \quad \ldots \quad \overrightarrow{e_n}\right]\begin{bmatrix} x_1 \\ \ldots \\ x_n \end{bmatrix}=\left[\overrightarrow{e_1'} \quad \ldots \quad \overrightarrow{e_n'}\right]\begin{bmatrix} x_1' \\ \ldots \\ x_n' \end{bmatrix} \qquad (1.11)$$

Using(1.10) in (1.11) we get:

$$\left[\overrightarrow{e_1} \quad \ldots \quad \overrightarrow{e_n}\right]\begin{bmatrix} x_1 \\ \ldots \\ x_n \end{bmatrix}=\left[\overrightarrow{e_1'} \quad \ldots \quad \overrightarrow{e_n}\right]P\begin{bmatrix} x_1' \\ \ldots \\ x_n' \end{bmatrix} \qquad (1.12)$$

From (1.12), we have: $$\begin{bmatrix} x_1 \\ \ldots \\ x_n \end{bmatrix}=P\begin{bmatrix} x_1' \\ \ldots \\ x_n' \end{bmatrix} \qquad (1.13)$$

Transformations of coordinates in affine spaces

Let $F=\{O;\vec{e}_1,...,\vec{e}_n\}$ and $F'=\left\{O';\vec{e}_1',...,\vec{e}_n'\right\}$ be two frames in A^n, P the transition matrix from $[\vec{e}]$ to $[\vec{e}']$ and $(a_1,...,a_n)$ the coordinates of the origin O' with respect to the basis $[\vec{e}]$.

Let $(x_1,...,x_n)$ and $(x_1',...,x_n')$ be the coordinates of the point M with respect to the bases $[\vec{e}]$ and $[\vec{e}']$ respectively. Let us find the relationship between $(x_1,...,x_n)$ and. $(x_1',...,x_n')$. In matrix notations, the vector equality $\overrightarrow{OM}=\overrightarrow{OO'}+\overrightarrow{O'M}$ can be written:

$$\begin{bmatrix}\overrightarrow{e_1} & \dots & \overrightarrow{e_n}\end{bmatrix}\begin{bmatrix}x_1\\ \dots\\ x_n\end{bmatrix}=\begin{bmatrix}\overrightarrow{e_1} & \dots & \overrightarrow{e_n}\end{bmatrix}\begin{bmatrix}a_1\\ \dots\\ a_n\end{bmatrix}+\begin{bmatrix}\overrightarrow{e_1'} & \dots & \overrightarrow{e_n'}\end{bmatrix}\begin{bmatrix}x_1'\\ \dots\\ x_n'\end{bmatrix}$$

Using (1.10), we have:

$$\begin{bmatrix}\overrightarrow{e_1} & \dots & \overrightarrow{e_n}\end{bmatrix}\begin{bmatrix}x_1\\ \dots\\ x_n\end{bmatrix}=\begin{bmatrix}\overrightarrow{e_1} & \dots & \overrightarrow{e_n}\end{bmatrix}\begin{bmatrix}a_1\\ \dots\\ a_n\end{bmatrix}+\begin{bmatrix}\overrightarrow{e_1} & \dots & \overrightarrow{e_n}\end{bmatrix}P\begin{bmatrix}x_1'\\ \dots\\ x_n'\end{bmatrix}$$

(1.14)

Passing from the vector equality to the coordinate's equality, we have:

$$\begin{bmatrix}x_1\\ \dots\\ x_n\end{bmatrix}=\begin{bmatrix}a_1\\ \dots\\ a_n\end{bmatrix}+P\begin{bmatrix}x_1'\\ \dots\\ x_n'\end{bmatrix} \qquad (1.15)$$

Denoting $X=[x_1 \dots x_n]^T$; $X'=[x_1' \dots x_n']^T$; $a=[a_1 \dots a_n]^T$, the equality (1.15) is simply written as: $X=PX'+a$ (1.16)

Let us denote by $[P,a]=\begin{bmatrix}p_{11} & p_{12} & \cdots & p_{1n} & a_1\\ p_{21} & p_{22} & \cdots & p_{2n} & a_2\\ \dots & \dots & \dots & \dots & \dots\\ p_{n1} & p_{n2} & \cdots & p_{nn} & a_n\end{bmatrix}$

the transition matrix from the $F=\{O;\vec{e}_1,...,\vec{e}_n\}$ to the frame

$F'=\{O';\vec{e}_1',...,\vec{e}_n'\}$

Proposition 1.3

If $[P,a]$ is the transition matrix from F to F' and $[Q,b]$ the transition matrix from F' to $F''=\{O'';\vec{e}_1'',...,\vec{e}_n''\}$, then the transition matrix from F to F'' is given by: $[PQ, Pb+a]$

Proof: Indeed, relation (1.15) gives:
$X = PX' + a = P(QX''+b)+a = PQX''+Pb+a$

Section 1.2: Polar coordinate system and its generalisations

1.2.1 Polar coordinates

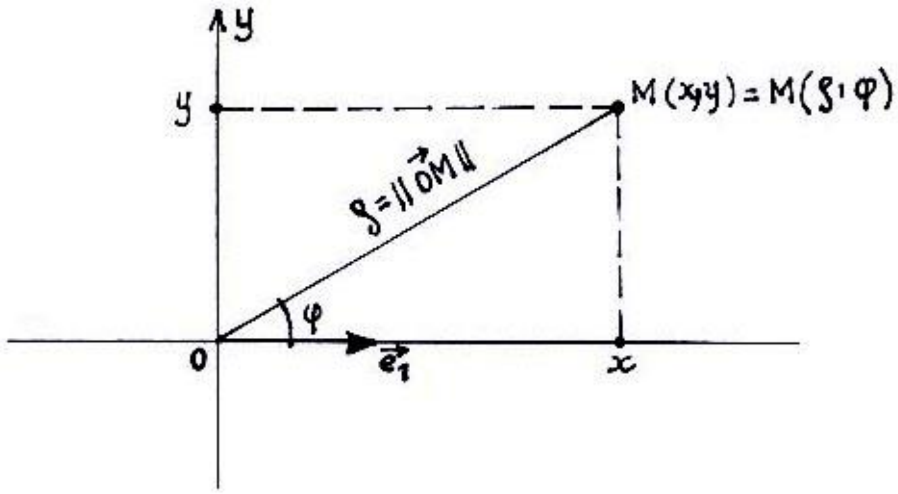

Figure 1.1

In plane geometry, the polar coordinates system is determined by a fixed reference point O called the **pole**, a directed half-straight line OX beginning at O called the **polar axis**, and a **unit of length**. The positive direction is a counterclockwise direction. Let M be any point in the plane different from O.

If, as illustrated in fig 2.1, $\rho = d(O,M) = \|\overrightarrow{OM}\|$ and φ denotes any angle determined by OP i.e. $\varphi = \angle(OX, OM)$, then the couple (ρ,φ) is called **the polar coordinates of the point** M and the symbol $M(\rho,\varphi)$ is used to denote M . From Fig1.1 the Cartesian coordinates (x,y) of the point M and the polar coordinates (ρ,φ) of the same point are related by:

$$x = \rho\cos\varphi;\ y = \rho\sin\varphi \tag{1.17}$$

$$\rho = \sqrt{x^2 + y^2}; \varphi = \text{arct}\, an \frac{y}{x} \qquad (1.18)$$

The system of polar coordinates can be extended to three dimensional in several ways. The systems most used are cylindrical and spherical.

1.2.2 Cylindrical co-ordinates

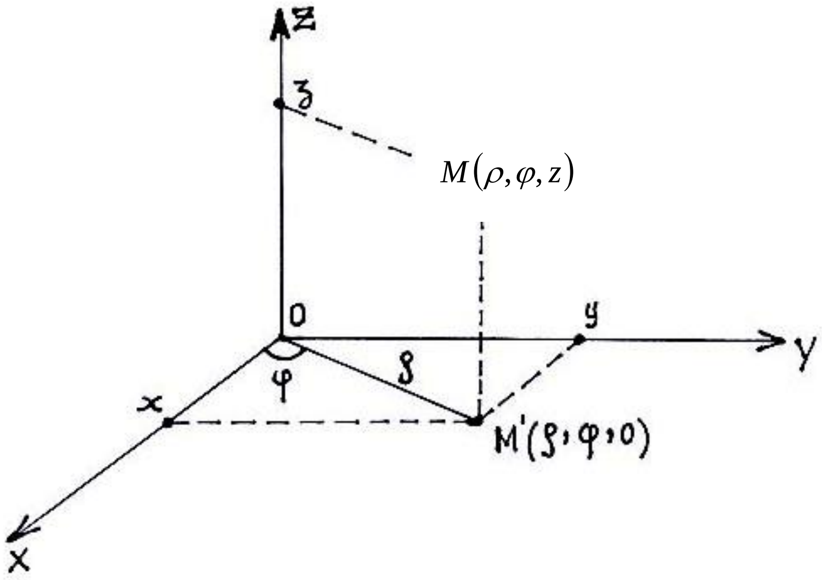

Figure 1.2

In cylindrical coordinates, a point M of space is represented by an ordered triple (ρ, φ, z), where (ρ, φ) are polar co-ordinates of the projection M' of P onto the xy-plane and z is the usual third rectangular coordinate of M. From fig. 1.2, the transition formulas of from cylindrical coordinates to the Cartesian coordinates are given by:

$$x = \rho\cos\varphi; \;\; y = \rho\sin\varphi; \;\; z = z \qquad (1.19)$$

$$\rho = \sqrt{x^2 + y^2}; \quad \varphi = \text{arct}\, an \frac{y}{x}; \quad z = z \qquad (1.20)$$

Example 2.1

a. If the cylindrical co-ordinates of the point M are $(2, \frac{2\pi}{3}, 1)$, determine its Cartesian co-ordinates

b. Determine the cylindrical co-ordinates of the point M if its Cartesian coordinates are (3, -3, -7)

Solution:

a. If $(2, \frac{2\pi}{3}, 1)$ are the cylindrical of M, then using the relation (1.19) the Cartesian co-ordinates of the point M are given by:

b. $x = 2\cos\frac{2\pi}{3} = -1; y = 2\sin\frac{2\pi}{3} = \sqrt{3}; z = 1;$

c. Using the relation (1.20), we have:

$$\rho = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}; \arctan(\frac{-3}{3}) = \frac{7\pi}{4} + 2k\pi;\ z = -7$$

1.2.3 Spherical co-ordinates

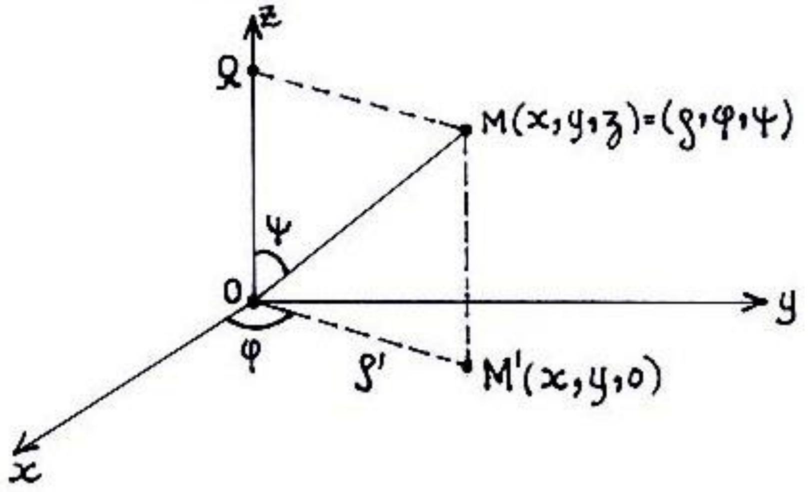

Figure 1.3

In spherical coordinates, a point M of space is represented by an ordered triple (ρ,φ,ψ), where φ is a polar angle associated with the projection M' of M onto the xy - plane , ψ is the angle between the positive z - axis OZ and $\overrightarrow{OM}$ and $\rho = \left\|\overrightarrow{OM}\right\|$

☞ **Remark**: $\rho \geq 0; 0 \leq \psi \prec \pi, 0 \leq \varphi \prec 2\pi$ ☜

The relations between the Cartesian and spherical coordinates are given by:

$$x = \rho\sin\psi\cos\varphi; y = \rho\sin\varphi\sin\psi; z = \rho\cos\psi \qquad (1.21)$$

$$\rho = \sqrt{x^2 + y^2 + z^2}; \varphi = \arctan\frac{y}{x}; \psi = ar\cos\frac{z}{\sqrt{x^2 + y^2 + z^2}} \qquad (1.22)$$

Example 2.2

a. Determine the Cartesian co-ordinates of the point M if its spherical co-ordinates are: $(2, \frac{\pi}{4}, \frac{\pi}{3})$

b. If the Cartesian co-ordinates of a point M are $\left(0, 2\sqrt{3}, -2\right)$, find its corresponding spherical coordinates.

Solution

a. Using (1.21), we have:

$$x = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4} = \sqrt{\frac{3}{2}};\ y = 2\sin\frac{\pi}{3}\sin\frac{\pi}{4} = \sqrt{\frac{3}{2}};\ z = 2\cos\frac{\pi}{3} = 1$$

b. Using (1.22), we have:

$$\rho = \sqrt{0+12+4} = 4 : \varphi = ar\cos\left(\frac{-1}{2}\right) = \frac{2\pi}{3};\ \psi = \arctan\frac{2\sqrt{3}}{0} = \frac{\pi}{2}$$

Section 3.3: Cross and box products

1.3.1 Cross product

Let (A^3, V) be a three dimensional affine Euclidean space endowed with an orthonormal frame $F = \{O; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and let $\vec{u}$ and $\vec{v}$ be two vectors of V.

Definition 3.1

The cross product or vector product of the vector $\vec{u}$ and vector $\vec{v}$ is a new vector of V denoted by $[\vec{u}, \vec{v}]$ and satisfying the following conditions:

a. $[\vec{u}, \vec{v}]$ is orthogonal to vectors $\vec{u}$ and $\vec{v}$

b. $\|[\vec{u}, \vec{v}]\| = \sqrt{\|\vec{u}\|^2 \|\vec{v}\|^2 - \langle \vec{u}, \vec{v}\rangle^2} = \|\vec{u}\|\|\vec{v}\|\sin(\vec{u}, \vec{v})$

c. If $\vec{u}$ and $\vec{v}$ are linearly independent in V, then $\{\vec{u}, \vec{v}, [\vec{u}, \vec{v}]\}$ is a right-handed basis of V i.e $\det(\vec{u}, \vec{v}, [\vec{u}, \vec{v}]) > 0$

Remark 3.1: In geometrical terms, the length of the vector $[\vec{u}, \vec{v}]$ is numerically equal to the area of the parallelogram constructed on vectors $\vec{u}$ and $\vec{v}$, i.e. $A_{\vec{u},\vec{v}} = \|[\vec{u}, \vec{v}]\| = \|\vec{u}\|\|\vec{v}\|\sin(\vec{u}, \vec{v})$

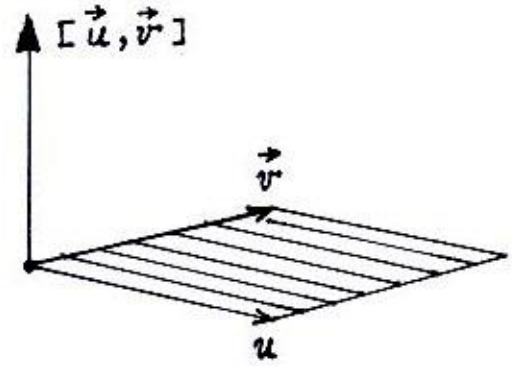

Figure 1.4

From the definition of the vector product, we have the following properties:

$\forall \vec{u}, \vec{v}, \vec{w} \in V, \forall \lambda \in \mathbb{R}$

a. $[\vec{u}, \vec{v}] = -[\vec{v}, \vec{u}]$

b. $[\vec{u}, \lambda\vec{v}] = \lambda[\vec{u}, \vec{v}]$

c. $[\vec{u}, \vec{v} + \vec{w}] = [\vec{u}, \vec{v}] + [\vec{u}, \vec{w}]$

If $\vec{u} = \lambda\vec{v}$, then $[\vec{u}, \vec{v}] = 0.$ More particularly, $[\vec{u}, \vec{u}] = \vec{0}.$

$[\vec{e}_1, \vec{e}_2] = \vec{e}_3 : [\vec{e}_2, \vec{e}_3] = \vec{e}_1; [\vec{e}_3, \vec{e}_1] = \vec{e}_2; [\vec{e}_1, \vec{e}_1] = [\vec{e}_2, \vec{e}_2] = [\vec{e}_3, \vec{e}_3] = \vec{0}$

Proposition 3.1

The expression of the vector product of the vectors: $\vec{u} = x_1\vec{e}_1 + y_1\vec{e}_2 + z_1\vec{e}_3$

and $\vec{v} = x_2\vec{e}_1 + y_2\vec{e}_2 + z_2\vec{e}_3$, in orthonormal basis is given by:

$$[\vec{u}, \vec{v}] = (y_1z_2 - z_1y_2)\vec{e}_1 + (z_1x_2 - x_1z_2)\vec{e}_2 + (x_1y_2 - y_1x_2)\vec{e}_3 \qquad (1.23)$$

Remark 2.2

Relation (1.23) in terms of determinant can formally be written as follow:

$$[\vec{u}, \vec{v}] = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \qquad (1.24)$$

1.3.2 Box product

Definition 2.2

The triple dot product (also called triple scalar product) of three vectors $\vec{u}, \vec{v}$ and $\vec{w}$ is defined by:

$$(\vec{u}, \vec{v}, \vec{w}) = \langle [\vec{u}, \vec{v}], \vec{w} \rangle. \qquad (1.25)$$

Proposition 3.2

The **absolute value** of the triple dot product of three non coplanar vectors $\vec{u}, \vec{v}$ and $\vec{w}$ is the volume of the parallelepiped spanned by these vectors i.e.

$$V_{\vec{u},\vec{v},\vec{w}} = |(\vec{u}, \vec{v}, \vec{w})|$$

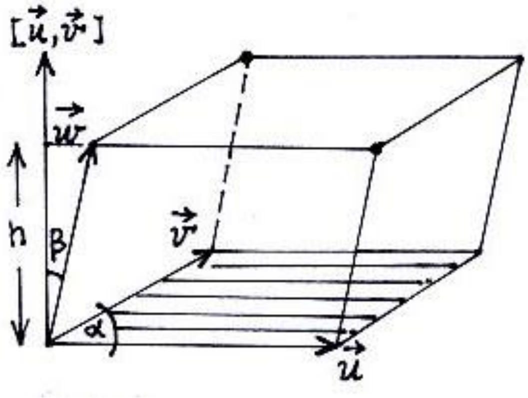

Figure 1.5

Proof

Let h be the height of the parallelepiped. We have:

$h = \|\vec{w}\| \cos\beta$; where $\beta = \angle(\vec{w}, [\vec{u},\vec{v}])$

$(\vec{u},\vec{v},\vec{w}) = \langle [\vec{u},\vec{v}], \vec{w} \rangle = \|[\vec{u},\vec{v}]\| \|\vec{w}\| \cos\beta.$

$\|[\vec{u},\vec{v}]\| = \|\vec{u}\| \|\vec{v}\| \sin\alpha; \alpha = \angle(\vec{u},\vec{v})$

$A = \|[\vec{u},\vec{v}]\|$ is the area of the base of the parallelepiped. The volume of the parallelepiped is the product of the area of its base and altitude i.e. $V = S \times h$. Thus $|(\vec{u},\vec{v},\vec{w})| = V_{\vec{u},\vec{v},\vec{w}}$

Proposition 3.3

In an orthonormal basis $B = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, the box product of the vectors $\vec{u} = x_1\vec{e}_1 + y_1\vec{e}_2 + z_1\vec{e}_3$, $\vec{v} = x_2\vec{e}_1 + y_2\vec{e}_2 + z_2\vec{e}_3$ and $\vec{w} = x_3\vec{e}_1 + y_3\vec{e}_2 + z_3\vec{e}_3$ is given by:

$$(\vec{u},\vec{v},\vec{w}) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}. \qquad (1.26)$$

Proof

$$(\vec{u},\vec{v},\vec{w}) = \langle [\vec{u},\vec{v}], \vec{w} \rangle = \langle (y_1z_2 - z_1y_2)\vec{e}_1 + (z_1z_1x_2 - x_1z_2)\vec{e}_2 + (x_1y_2 - y_1x_2)\vec{e}_3, x_3\vec{e}_1 + y_3\vec{e}_2 + z_3\vec{e}_3 \rangle$$

Using the formula for calculating the dot product, we have

$$(\vec{u},\vec{v},\vec{w}) = \langle [\vec{u},\vec{v}], \vec{w} \rangle = \langle (y_1z_2 - z_1y_2)x_3 + (z_1x_2 - x_1z_2)y_3 + (x_1y_2 - y_1x_2)z_3, x_3\vec{e}_1 + y_3\vec{e}_2 + z_3\vec{e}_3 \rangle$$

This last equality, in terms of determinants is simply relation (1.26)

The following properties of the box product are satisfied:

$$<[\vec{u},\vec{v}],\vec{w}>=<\vec{u},[\vec{v},\vec{w}]>$$

$$(\vec{u},\vec{v},\vec{w})=(\vec{v},\vec{w},\vec{u})=(\vec{w},\vec{u},\vec{v})$$

$$(\alpha\vec{u},\beta\vec{v},\gamma\vec{w})=\alpha\beta\gamma(\vec{u},\vec{v},\vec{w})$$

$$(\vec{u}+\vec{t},\vec{v},\vec{w})=(\vec{u},\vec{v},\vec{w})+(\vec{t},\vec{v},\vec{w})$$