# DIFFERENTIAL EQUATIONS

# LECTURES NOTES

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## CHAPTER1. PRELIMINARIES

### 1.0 Introduction

Differential equations (D.E) are fundamental importance in many branches of science, engineering and technology because many laws appearing in these branches can be expressed mathematically in the form of differential equation. A.D.E is an equation which involves derivatives or differentials.

If only derivatives of a function of one variable occur, then a differential equation is **called ordinary**. A **partial differential equation** contains partial derivatives. In our course, we will only focus on ordinary Differential Equations (D.E)

### 1.1. Basic concepts

**Order of an O.D.E**

An ODE of **order** $n$ is an equation of the form $F(x, y, y', y'', ..., y^{(n)}) = 0$ (1.1)

Where $F$ is a function of $n+2$ variables, $y$ is function of $x$, and $y^{(k)}$ denotes the $k^{th}$ derivatives of $y$ with respect to $x$.

**Example1.** The following are ODE of orders 1, 2, 3 and 4 respectively

$y' = x^4; y'' + x^2 y'^3 - 15y = 0; (y''')^4 - x^3 (y'')^5 + 4xy = xe^{2x}; (y^{(4)})^2 - 1 = x^4 y''$

**Degree of ODE**

The **degree** of the DE is the greatest exponent associated with the highest order derivative $y^{(n)}$.

**Example1.2** The degrees of the preceding DE are 1,1,4, and 2, respectively

**Solution of an ODE**

$f(x)$ is called a **solution** of the DE or integral of the DE, if $f$ has the property that when $f(x)$ is substituted for $y$ in a DE, the resulting expression is an identity for all $x$ in some interval .

**Example1.3.** $f(x) = x^2 + c$ is a solution of the ODE $y' = 2x$ because substitution of $f(x)$ for $y$ leads to the identity $2x = 2x$.

$f(x) = x^2 + c$ is called the **general solution** of $y' = 2x$, since every solution has this form.

The general solution of $n^{th}$ order D.E contains $x$ **independent parameters** $c_1, c_2, ..., c_n$.

A **particular solution** is obtained by assigning specific values to the parameters. Some DE have solutions which are not special cases of the general solution. These solutions are called **singular**.

To **solve** or as we frequently say, to **integrate** a DE means:

(i) To find its **general solution** or **complete integral** (if the initial conditional are not specified) or

(ii) To find **a particular solution** of the equation that will satisfy the given initial conditions (if such exist)

**Example 1.4.** Let $y'' - 25y = 0$ be a D.E

$y(x) = c_1e^{5x} + c_2e^{-5x}$ is general solution and ; $y(x) = e^{5x} - 2e^{-5x}$ is particular solution.

**Boundary conditions**

Let us find the particular solution of $y' = 2x$ which satisfies the condition that $y = 5$ if $x = 2$. I.e. solve the following DE: $y' = 2x; y(2) = 5$. The general solution is $y = x^2 + c$. If the given condition is to be satisfied, then necessarily $5 = 4 + c$ or $c = 1$. Hence the desired particular solution is $y = x^2 + 1$. Conditions $y(2) = 5$ for $y' = 2x$ are called **boundary conditions**

If the general solution contains one parameter, one boundary condition is sufficient to find a particular solution.

If the general solution contains two parameters, then two boundary conditions are needed for particular solutions. Similar statements hold if more than two parameters are involved .

The solutions of some differential equation express $y$ explicitly in terms of $x$ and others are stated implicitly.

**1.3. EXERCISES**

**I. Prove that $y$ is a solution of the indicated differential equation**

**1.** $y = c_1 e^{x} + c_2 e^{2x}; y'' - 3y' + 2y = 0$

**2.** $y = ce^{-3x}; y' + 3y = 0$

**3.** $y = e^{x}(c_1 \cos x + c_2 \sin x); y'' - 2y' + 2y = 0$

**4.** $y = (c_1 + c_2 x)e^{-2x}; y'' + 4y' + 4y = 0$

**5.** $y = cx^{-\frac{2}{3}}; 2xy^3 dx + 3x^2 y^2 dy = 0$

**6.** $x^2 - y^2 = c; yy' = x$

**7.** $y^2 - x^2 - xy = c; (x - 2y)y' + 2x + y = 0$

**8.** $y = c_1 x + \frac{c_2}{x} + c_3; y''' + \frac{3}{x} y'' = 0$

**9.** $y = c_1 e^{\arcsin x} + c_2 e^{-\arcsin x}; (1 - x^2)y'' - xy' - y = 0$

**10.** $y = (c_1 + c_2 x)e^{kx} + \frac{e^{x}}{(k-1)^2}; y'' - 2ky' + k^2 y = e^{x}$

**II.** Form the differential equation whose general solution is:

**1.** $y = cx^2 - x$ ;

2. $y = c_1 x^3 + c_2 x + c_3$;

3. $y = e^{cx}$;

4. $y = c_1 e^{x} + c_2 x e^{x}$

**5.** $y = c_1 x^3 + c_2 x$;

6. $x^2 + y^2 = c^2$;

7. $y = c_1 e^{x} + c_2 x e^{x} + c_3 e^{-x}$

8. $y = c_1 e^{x} + c_2 x e^{-x} + c_3 e^{-x}$

**CHAPTER2. ODE OF TH FIRST ORDER**

**General concepts**

An ODE of first order is of the form $F(x, y(x), y'(x)) = 0$ (2.1.)

If this equation can be solved for $y'$, it can be written in the form

$y' = f(x, y)$ $(2.1)'$

In this case we say that the differential equation is solved for the derivative. For such equation, the following theorem, called the theorem of existence and uniqueness of solution of a differential equation holds.

**Theorem 2.1.** If in the equation $y' = f(x, y)$ the function $y' = f(x, y)$ and its partial derivative with respect to $y$ , $\frac{\partial f}{\partial y}$, are continuous in some domain $D$, in an $xy$ plane, containing some point $(x_0, y_0)$, then there is a unique solution to this equation, $y = \varphi(x)$ which satisfies the conditions $y(x_0) = y_0$.

The geometric meaning of the theorem consists in the fact that there exists one and only one such function $y = f(x)$, the graph of which passes through the function $(x_0, y_0)$. The condition that for $x = x_0$ the function must be equal to the given number $y_0$ is called the **initial condition**.

- The **general solution** of a first ODE is a function $y = \varphi(x, c)$ which depends on a single arbitrary constant $C$ and satisfies the following conditions

(i) It satisfies the differential equation for any specific value the constant $C$

(ii) No matter what the initial condition $y = y_0$ for $x = x_0$, that is $y(x_0) = y_0$, it is possible to find a value $c = c_0$, such that function $y = \varphi(x, c_0)$ satisfies the given initial condition

- In searching for the general solution of a differential equation we often arrive at a relation like $\Phi(x, y, c) = 0$ which is not solved for $y$. Solving this relationship for $y$, we get the general solution.
- However, it is not always possible to express $y$ in terms of elementary functions, in such cases, the general solution is left in implicit form.

- An equation of the form $\Phi(x, y, c) = 0$ which gives an implicit general solution is called the **complete integral** of the differential equation
- A **particular solution** is any function $y = \varphi(x, c_0)$ which is obtained from the general solution $y = \varphi(x, c)$, if in the latter we assign to the arbitrary constant $C$ a definitive $C = C_0$ in this case, the relation $\varphi(x, y, c_0) = 0$ is called a **particular integral** of the equation
- From the geometric view point, the general solution (**complete integral** ) is a **family of curves** in a coordinate plane, which family depends on constant $C$ (a single parameter $C$ ) .
- These curves are called **integral curves** of the given differential equation.
- **A particular integral** is associated with **one curve** of the family that passes through a certain given point of the plane.

**Example 2.0.** For the first-order equation : $y' = \frac{-y}{x}$

The general solution is a family of function $y = \frac{c}{x}$.

This can be checked by simple substitution in the equation.

Let us find a particular solution that will satisfy $y(2) = 1$.

Putting these values into the formula $y = \frac{c}{x}$, we have $1 = \frac{c}{2}$ or $c = 2$.

Consequently, the function $y = \frac{2}{x}$ will be the particular solution we are seeking

Fig2.1: $y = \frac{c}{x}$ a family of hyperbolas

### 2.1. Types of first ODE

**1. Simple equation**

If $f(x,y)$ is a function of $x$ alone, the solution of the problem simply involves integration

If $\frac{dy}{dx} = f(x)$ (2.2)

Then $y = \int f(x)dx + C$, where $C$ is an arbitrary constant

**Example 2.1.** Solve the following equation: $y' = e^x$

**Solution.** $y = \int e^x dx = e^x + C$

**2. Equations in which variables are separable**

Consider $a$ DE of the form

$y' = \frac{f(x)}{g(y)}$ (2.3);

where $g(y) \neq 0$.

In such equation, the variables $x$ and $y$ can be separated in the sense that the equation can be written as

$g(y)dy = f(x)dx$.

Integrating both sides of this equation, we get the general solution of the form

$\int g(y)dy = \int f(x)dx + C$

**Example 2.2.** Find the general solution of the equation:

$$\frac{dy}{dx} = \frac{x^2+1}{y+2}$$

**Solution:** Separating the variables, we can rewrite the given equation as:

$(y+2)dy = (x^2+1)dx$ Integrating both sides of equation, we get

$$\int (y+2)dy = \int (x^2+1)dx + C$$

$$\frac{1}{2}y^2 + 2y = \frac{x^3}{3} + x + C$$

### 3. Equations reducible to variable separable form

The form of certain first order DE are not separable but can be made by a simple change of variables. The DE

$$\frac{dy}{dx} = f(\frac{y}{x}) \qquad (2.4.)$$

where $f$ is any function of is $\frac{y}{x}$ is not the variable separable form. But can be reduced to separable form by a simple substitution.

The form of this equation suggests that we set : $\frac{y}{x} = z$,

where $y$ and $z$ are functions of $x$, then $y = xz$ and differentiating this once we get $\frac{dy}{dx} = z + x\frac{dz}{dx}$.

Then equation (2.4) becomes $z + x\frac{dz}{dx} = f(z)$, that is $\frac{dz}{dx} = \frac{f(z) - z}{x}$.

This variable separable form.

Integrating this and replacing $z$ by $\frac{y}{x}$ we obtain the general solution of equation (2.4.)

**Example 2.3.** Solve $\frac{dy}{dx} = (4x + y + 1)^2$

**Solution** Put $4x + y + 1 = z$ so that $4 + \frac{dy}{dx} = \frac{dz}{dx}$.

The given equation becomes

$$(\frac{dz}{dx} - 4) = z^2 \text{ or } \frac{dz}{dx} = z^2 + 4$$

$$\int \frac{d_z}{z^2 + 4} = \int dx;$$

$$\frac{1}{2}\arctan\frac{z}{2} = x + C \Leftrightarrow \arctan\frac{z}{2} = 2(x + C)$$

$$z = 2\tan 2(x + C) \Leftrightarrow 4x + y + 1 = 2\tan 2(x + C)$$

This is the general solution

### 4. Homogeneous equations

A function $f(x,y)$ is said to be homogeneous of degree $n$ if $f(\lambda x,\lambda y)=\lambda^n f(x,y)$

- For example the function $f(x,y)=x^4-x^2y^2$ is homogeneous and of degree 4 since

  $f(\lambda x,\lambda y)=(\lambda x)^4-(\lambda y)^2=\lambda^4(x^4-x^2y^2)=\lambda^4 f(x,y)$

- Similarly, if $f(x,y)=\dfrac{1}{x^2+y^2}e^{\frac{x}{y}}$, then $f$ is homogeneous of degree -2, since

  $$f(\lambda x,\lambda y)=\frac{1}{\lambda^2x^2+\lambda^2y^2}e^{\frac{\lambda x}{\lambda y}}=\lambda^{-2}f(x,y).$$

A homogeneous first order DE is of the form $P(x,y)dx+Q(x,y)dy=0$ (2.5) where $P$ and $Q$ are homogeneous functions of the same degree.

Such equation can be reduced to the variables separable type by the substitution $y=xu$, where $u=g(x)$ for some function $g$.

Apply the chain rule to $y=xu$ gives $\dfrac{dy}{dx}=u+x\dfrac{du}{dx}$

**Example2.4.** Solve the DE $(y^2-xy)dx+x^2dy=0$

**Solution:** $P(x,y)=y^2-xy$ and $Q(x,y)=x^2$, then the function P and $Q$ are both homogeneous of degree2. Hence the DE is homogeneous and we substitute $y=xu$; $dy=udx+xdu$.

This leads to the followings chain of equations

$$(x^2u^2-x^2u)dx+x^2(udx+xdu)=0; x^2(u^2-u)dx+x^2(udx+xd_u)=0$$

$$(u^2-u)dx+udx+xdu=0; u^2dx=-xdu$$

$$-\frac{1}{u}=-\ell n|x|+\ell n|c|; \ell n|x|-\frac{x}{y}=\ell n|C|$$

$$\frac{x}{C}=e^{\frac{x}{y}}\Rightarrow x=Ce^{\frac{x}{y}};$$

### 5. Equations reducible to homogeneous form

Certain first ODE are not homogenous but can be made homogenous by a simple transformation.

For example an equation of the form : $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ (i)

Provided $a_1b_2-a_2b_1\neq 0$, is not homogeneous but can be reduced to the homogenous form by the transformation of the types $x=u+x_0; y=v+y_0$ (ii),

where $x_0$ and $y_0$ are constants.

Substituting equation (ii) into (i), we get : $\frac{dv}{du}=\frac{a_1u+b_1v+a_1x_0+b_1y_0+c_1}{a_2u+b_2v+a_2x_0+b_2y_0+c_2}$ (iii)

Let us choose $x_0$ and $y_0$ such that ; $\begin{cases} a_1x_0+b_1y_0+c_1=0 \\ a_2x_0+b_2y_0+c_2=0 \end{cases}$

Then equation (iii) becomes : $\frac{dv}{du}=\frac{a_1u+b_1v}{a_2u+b_2v}$

We can rewrite this equation as: $\frac{dv}{du}=\frac{a_1+b_1(\frac{v}{u})}{a_2+b_2(\frac{v}{u})}$

Which is a homogeneous equation.

General solution of such equations can be obtained using $v=uw$; where $w$ depends of u

**Exceptional case**

This method works only for $a_1b_2-a_2b_1\neq 0$. It fails when $\frac{a_1}{a_2}=\frac{b_1}{b_2}$ because $x_0$ and $y_0$ in equation (ii) become infinite or indeterminate.

To find the solution in that case, use $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{1}{m}$

Equation (i) becomes : $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{m(a_1x+b_1y)+c_2}$.

Let $a_1x+b_1y=z$ ,So that $a_1+b_1\frac{dy}{dx}=\frac{dz}{dx}$.

Then equation $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{m(a_1x+b_1y)+c_2}$, becomes $\frac{dz}{dx}=\frac{a_2(z+c_1)}{mz+c_2}+a_1$ .

The variables now, are separable

**Example 2.5.** Solve $\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$.

Put $x=u+x_0; y=v+y_0$ .Then $\frac{dy}{dx}=\frac{dv}{du}$ and the given equation reduces to

$$\frac{dv}{du}=\frac{u+2v+(x_0+2y_0-3)}{2u+v+(2x_0+y_0-3)}.$$

Let us chose $x_0$ and $y_0$ such that $x_0+2y_0-3=0; 2x_0+y_0-3=0$. We get $x_0=y_0=1$.

Then we have $\frac{dv}{du}=\frac{u+2v}{2u+v}=\frac{1+2\frac{v}{u}}{2+\frac{v}{u}}$, which is homogeneous.

Putting $w=\frac{v}{u}$ so that $w+u\frac{dw}{du}=\frac{dv}{du}$

We have $\frac{dv}{du}=\frac{1+2w}{2+w}=w+u\frac{dw}{du} \quad \Leftrightarrow \quad u\frac{dw}{du}=\frac{1-w^2}{2+w}$ Or $\frac{du}{u}=\frac{2+w}{1-w^2}dw$.

Integrating this expression, we get:

$$\ell nu=\frac{1}{2}\ell n(1+w)-\frac{3}{2}\ell n(1-w)+\ell n(1-w)+\ell nc$$
$$u^2(1-w)^3=c^2(1+w)$$

Putting $w=\frac{v}{u}=\frac{y-x_0}{x-y_0}=\frac{y-1}{x-1}$ in the above equation, we get the general solution of the given equation as $(x-y)^3=c^2(x+y-2)$

## 6. Exact differential equations

The DE $P(x,y)dx+Q(x,y)dy=0$ is an exact differential equation iff $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$

**Example 2.6.** Show that D.E $(3x^2y-2y^3+3)d_x+(x^3-6xy^2+2y)dy=0$ is exact

**Solution:** $P(x,y)=3x^2y-2y^3+3$ and $Q(x,y)=x^3-6xy^2+2y$. We see that

$$\frac{\partial P}{\partial y}=3x^2-6y^2=\frac{\partial Q}{\partial x}.$$

Hence the equation is exact by definition

- If $P(x,y)\ dx+Q(x,y)dy=0$ is an exact DE, then there exists a function $U$ of $x$ and $y$ such that $\frac{\partial U}{\partial x}=P$ and $\frac{\partial U}{\partial y}=Q$

- An exact DE $P(x,y)\ dx+Q(x,y)dy=0$ has solution of the form $U(x,y)=C$, where $C$ is a constant and $U$ is any function of $x$ and $y$ such that

$\frac{\partial U}{\partial x}=P(x,y)$ and $\frac{\partial U}{\partial y}=Q(x,y)$

**Example 2.7.** Solve the D.E. $(3x^2y-2y^3+3)dx+(x^3-6xy^2+2y)dy=0$ .

**Solution.**

It was shown in example 2.6. that the above equation is exact.

Consequently, there is a function $U(x,y)$ such that

$\frac{\partial U}{\partial x}=3x^2y-2y^3+3;\frac{\partial U}{\partial y}=x^3-6xy^2+2y$ .

Integrating $\frac{\partial U}{\partial x}$ with respect to $x$ give us $U(x,y)=x^3y-2xy^3+3x+\varphi(y)$

where $\varphi$ is a function of $y$ .It follows that $\frac{\partial U(x,y)}{\partial y}=x^3-6xy^2+\varphi'(y)$ .

Comparing this equation with the previous formula for $\frac{\partial U(x,y)}{\partial y}$, we see that

$\varphi'(y)=2y$, and hence $\varphi(y)=y^2+C_1$. Substituting for $\varphi(y)$ in the formula for $U(x,y)$ give us : $U(x,y)=x^3y-2xy^3+3x+y^2+C_1$ .

Hence, a solution of the DE is $x^3y-2xy^3+3x+y^2=C$,

where the constant $C_1$ has been absorbed the constant $C$

**7. Linear equation**

A first- order linear DE is an equation of the form:

$y'+p(x)y=q(x)$ (2.6),

where $p(x)$ and $q(x)$ are continuous functions.

If $q(x)=0$ for all $x$, then (2.6) is separable and we may write:

$y^{-1}\cdot y'=-p(x)$ provided $y\neq 0$ .

Integrating we obtain:

$\ln|y|=-\int p(x)dx+\ln|C|$ ; $y=Ce^{-\int p(x)dx}$ .

To solve an equation of type (2.6), we multiply throughout by $e^{\int p(x)dx}$ to get

$$e^{\int p(x)dx}\cdot\frac{dy}{dx}+e^{\int p(x)dx}\cdot p(x)y=q(x)\cdot e^{\int p(x)dx},$$

which can be rewritten as: $\frac{d}{dx}\left(ye^{\int p(x)dx}\right)=q(x)e^{\int p(x)dx}$.

On integration, this yields: $ye^{\int p(x)dx}=\int q(x)e^{\int p(x)dx}dx+C$, where $C$ is a constant.

This is a general solution of the equation (2.6).

The factor with which equation (2.6) was multiplied is called an **integrating factor** for equation (2.6).

**Example2.8.** Solve $y'-\frac{1}{(x-2)}y=2(x-2)^2$

**Solution.** In this case the integrating factor is

$e^{-\int\frac{dx}{x-2}}=e^{-\ln(x-2)}=\frac{1}{x-2}$, and the equation becomes

$\left(\frac{1}{x-2}\right)\frac{dy}{dx}-\frac{y}{(x-2)^2}=2(x-2)$ .That is $\frac{d}{dx}\left(\frac{y}{x-2}\right)=2(x-2)$

Therefore, $\frac{y}{x-2}=(x-2)^2+C$ and, $y=(x-2)^3+C(x-2)$.

**8. Bernoulli's equation**

The Bernoulli equation is the generalization of the linear equation. It has the following form: $y'+p(x)y=q(x)y^n$ (2.7)where $n\neq\{0,1\}$.

Evidently $y=0$ is a solution.If $y\neq 0$ we may divide both sides by $y^n$, obtaining

$$y^{-n}y'+p(x)y^{1-n}(x)=q(x) \qquad (2.8)$$

If we let $z=y^{1-n}$ then, $z'=(1-n)y^{-n}y'$,

and hence, $y^{-n}y'=\frac{1}{1-n}z'$.

Replacing $y^{1-n}y'$ in (2.8), by the last expression gives us

$\frac{1}{1-n}z'+p(x)z=q(x)$ ,which equivalent to $z'+(1-n)p(x)z=(1-n)q(x)$.

Which is a linear DE.

**Example2.9.** Solve $x\dfrac{dy}{dx}+(1-x'y)=x^2y^2$

The equation can be written as:

$\dfrac{dy}{dx}+\left(\dfrac{1}{x}-1\right)y=xy^2$, which is a Bernoulli's equation with $n=2$.

Putting $z=y^{1-2}=y^{-1}$, and using

$z'+(1-n)p(x)z=(1-n)q(x)$, we get $z'+\left(1-\dfrac{1}{x}\right)z=-x$, which is a linear DE.

The integrating factor of this equation is $e^{\int\left(1-\frac{1}{x}\right)dx}=e^{x-\ln x}=x^{-1}e^x$.

Therefore, the general solution is, $z\dfrac{e^x}{x}\int -x\dfrac{e^x}{x}dx=e^{-x}+C$ .Substituting $z=\dfrac{1}{y}$,

we get the general solution of equation as $xy\left(Ce^{-x}-1\right)=1$

### 1.3 EXERCISES

**1. Solve the following DE**

a) $tgx\sin^2 ydx + cox.\cot gydy = 0$

b) $xy' - y = y^3$  c) $xyy' = 1 - x^3$

d) $y - xy' = 1 + x^2 y'$  e) $y'tgx = y$

f) $3e^x tgydx + (1 - e^x)\sec^2 ydy = 0$

**2. Find the solutions of the following DE satisfying the given initial conditions**

a) $(1+e^x)yy' = e^x, y(0) = 1$  b) $(xy^2 + x)dx + (x^2 y - y)dy = 0, y(0) = 1$

c) $y'\sin x = ylny, y\left(\frac{\pi}{2}\right) = 1$

**3. Integrate the following DE using a suitable substitution**

a) $y' = (x+y)^2$  b) $y' = (8x + 2y + 1)^2$

c) $(2x + 3y - 1)dx + (4x + 6y - 5)dy = 0$

d) $(2y - y)dx + (4x - 2y + 3)dy = 0$

**4. Integrate the following homogeneous DE**

a) $(y - x)dx + (y + x)dy = 0$  b) $(x + y)dx + xdy = 0$

c) $(x + y)dx + (y - x)dy = 0$  d) $xdy - ydx = \sqrt{x^2 + y^2 dx}$

e) $(8y + 10x)dx + (5y + 7x)dy = 0$  f) $xy^2 dy = (x^3 + y^3)dx$

g) $ydx + \left(2\sqrt{xy - x}\right)dy = 0$  h) $(x^2 - 3y^2)dx + 2xydy = 0$

**5. Solve the following DE reducible to homogeneous DE**

a) $(2x0y - 4)dy + (x - 2y + 5)dx = 0$  b) $y' = \frac{1 - 3x - 3y}{1 + x + y}$

c) $y' = \frac{x + 2y + 1}{2x + 4y + 3}$  c) $(x + 2y + 1)dx - (2x + 4y + 3)dy = 0$

e) $(x + 2y + 1)dx - (2x - 3)dy = 0$

**6. Solve the following DE:**

a) $x+y-2+(1-x)y'=0$

b) $(x+y)dx+(x+y-1)dy=0$

c) $x+y-2+(1-x)y'=0$

d) $8x+4y+1+(4x+2y+1)y'=0$

e) $(x+y-2)dx+(x-y+4)dy=0$

**7. Solve the following linear DE**

a) $y'-\dfrac{2y}{x+1}=(x+1)^3$

b) $y'-\dfrac{y}{x}=\dfrac{x+1}{x}$

c) $(x-x^3)y'+(2x^2-1)y=x^3$

d) $x'\cos t+x\sin t=-1$

e) $y'-\dfrac{n}{x}y=e^x x^n$

f) $y'+y=e^{-x}$

g)y $y'+\dfrac{1-2x}{x^2}y-1=0$

h) $y'+2\dfrac{y}{x}=x^3$

**8. Integrate the following Bernoulli's equations**

a) $y'+xy=x^3y^3$

b) $(1-x^2)y'-xy-xy^2=0$

c) $3y^2y'-y^3-x-1=0$

d) $(y\ell nx-2)ydx=xdy$

e) $y-y'\cos x=y^2\cos x(1-\sin x)$

f) $y'+\dfrac{y}{x}=-xy^2$

g) $2xyy'-y^2+x=0$

**9. Solve the following DE**

a) $xy' + y - e^{x} = 0, y(0) = 1$ ; b) $y' - \frac{y}{1-x^2} - 1 - x = 0, y(0) = 0$

c) $y' - ytgx = \frac{1}{\cos x}, y(0) = 0$ d) $xy' + y - e^{x} = 0, y(1) = 2$

**10. Integrate the followig exact DE**

a) $\frac{2x}{y^3}dx + \frac{y^2 - 2x^2}{y^4}dy = 0$ b) $(\sin xy + xy\cos xy)dx + x^2\cos xydy = 0$

c) $(x+y)dx + (x+2y)dy = 0$ d) $(x^2 + y^2 + 2x)dx + 2xydy = 0$

e) $(x^3 - 3xy^2 + 2)dx - (3x^2y - y^2)dy = 0$ f) $xdx + ydy = \frac{xdy - ydx}{x^2 + y^2}$

g) $\frac{2xdx}{y^3} + \frac{y^2 3x^2}{y^4}dy = 0$ h) $(x^2 + y)dx + (x - 2y)dy = 0$

i) $(y - 3x^2)dx - (4y - x)dy = 0$ j) $\frac{x^2dy - y^2dx}{(x-y)^2}$

k) $\left(x + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$

## CHAPTER3. ODE OF THE SECOND ORDER

### 3.0. Basic definition

If $p_1(x), p_2(x), ..., p_n(x)$ and $f(x)$ are functions of one variable which have the same domain, then an equation of the form:

$y^{(n)} + p_1(x)y^{(n-1)} + ... + p_{n-1}(x)y' + p_n(x)y = f(x)$ (3.0),

is called a **linear differential equation**.

If $f(x) = 0$, then the equation (3.0) is called **homogeneous,** otherwise, it is a **non-homogeneous equation.** We shall restrict to second- order equation in which $p_1$ and $p_2$ are constant functions.

### 3.1. Homogeneous equation of second order with constant coefficients

The second-order homogeneous linear differential equation with constant coefficients has form, $y'' + py' + qy = 0$ (3.1)

Where $p$ and $q$ are constants.

Before attempting to find particular solution let us establish the following result:

**Proposition3.1.** If $y_1(x)$ and $y_2(x)$ are solution of $y'' + py' + qy = 0$,

then $C_1y_1(x) + C_2y_2(x)$ is a solution for all real numbers $C_1$ and $C_2$.

**Proof.** By hypothesis, $y_1'' + py_1' + qy_1 = 0$

$y_2'' + py_2' + qy_2 = 0$

If we multiply the first of these equations by $C_1$, the second by $C_2$, and add, the result is $[C_1y_1'' + C_2y_2''] + p[C_1y_1' + C_2y_2'] + q[C_1y_1 + C_2y_2] = 0$.

Thus $C_1y_1 + C_2y_2$ is a solution.

In our search for a solution of (3.1) we shall use $y = e^{\lambda x}$ as a trial solution.

Since $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$, it follows that $y = e^{\lambda x}$ is a solution of $y'' + py' + qy = 0$ iff

$\lambda^2 e^{\lambda x} + p\lambda e^{\lambda x} + qe^{\lambda x} = 0$

Or, since $e^{\lambda x} \neq 0$, if $\lambda^2 + p\lambda + q = 0$ (3.2)

The equation (3.2) is called the **auxiliary equation** of $y'' + py' + qy = 0$.

It can be obtained from this DE by replacing $y''$ by $\lambda^2$, $y'$ by $\lambda$, and $y$ by $1$.

The roots of the equation (3.2) are given by $\lambda_{1,2}=\dfrac{-p\pm\sqrt{p^2-4q}}{2}$ (3.3).

Thus the auxiliary equation has unequal roots $\lambda_1$ and $\lambda_2$, a double real root $\lambda$, or two complexes conjugate roots according as $p^2-4q$ is positive, zero, or negative, respectively.

**Proposition 3.2**

- If the roots $\lambda_1,\lambda_2$ of the auxiliary equation are real and unequal, then the general solution of (3.1) is, $y=C_1e^{\lambda_1 x}+C_2e^{\lambda_2 x}$.
- If the auxiliary equation has a double root $\lambda$, then the general solution of (3.1) is, $y=C_1e^{\lambda_1 x}+C_2xe^{\lambda_2 x}=\left(C_1+C_2x\right)e^{\lambda x}$
- If the auxiliary equation has distinct complex roots $\alpha\pm i\beta$, then the general equation of (3.1) is, $y=e^{\alpha x}\left(C_1\cos\beta x+C_2\sin\beta x\right)$.

**Example3.1.** Solve the following DE.

i. $y''+3y'-4y=0$ ; ii) $y''-4y+4y=0$; iii) $y''-4y'+13y=0$

**Solutions**

i. The roots of the auxiliary equation $\lambda^2+3\lambda-4=0$ are $\lambda_1=-4$ and $\lambda_2=1$.

Hence, the general solution of the DE is $y=C_1e^{-4x}+C_2e^{x}$

ii. The auxiliary equation $\lambda^2-4\lambda+4=0$, or equivalently $\left(\lambda-2\right)^2=0$, has a double root 2.Hence, the general solution is

$y=C_1e^{2x}+C_2xe^{x}=\left(C_1+C_2x\right)e^{2x}$

iii. The roots of the auxiliary equation $\lambda^2-4\lambda+13=0$ are $\lambda_1=2+zi$ and $\lambda_2=2-zi$. Hence, the general solution is $y=e^{2x}\left(C_1\cos 3x+C_2\sin 3x\right)$.

**3.2. Non-homogeneous second-order linear DE with constant coefficients.**

Let us consider the DE of the form: $y''+py'+qy=f(x)$ (3.4)

Where $p$ and $q$ are constants and the function $f(x)$ is continuous.

Given the DE(3.4), the corresponding homogenous equation(3.1) is called the **complementary equation .**

**Proposition 3.3** If $y_P$ is a particular solution of the DE $y'' + py' + qy = f(x)$ and if $y_c$ is the general solution of the complementary equation $y'' + py' + qy = 0$, then the general solution of $y'' + py' + qy = f(x)$ is $y = y_p + y_c$

**Example 3.2.** Solve the DE $y'' - 4y = 6x - 4x^3$

**Solution. W**e see by inspection that $y_p = x^3$ is a particular solution of the given equation. The complementary equation is $y'' - 4y = 0$, which has general solution

$y_c = c_1e^{2x} + c_2e^{-2x}$.

Applying proposition 3.3., the general solution of the given non homogenous equation is

$y = y_c + y_p = c_1e^{2x} + c_2e^{-2x} + x^3$.

**Note**. In most case a particular solution (34) can not be found by inspection, some methods are employed for finding a particular solution.

**1.The method of variation of arbitrary constants or variation of parameters**

Let $y_1$ and $y_2$ be expressions that appear in the general solution $y = c_1y_1 + c_2y_2$ of the complementary equation $y'' + py' + qy = 0$.

Let us now attempt to find a particular solution of $y'' + py' + qy = f(x)$ which has the form $y_p = c_1(x)y_1 + c_2(x)y_2$, where $c_1(x)$ and $c_2(x)$ are functions of $x$.

The first and second derivatives of $y_p$ are

$$y_p' = (c_1(x)y_1' + c_2(x)y_2') + (c_1'(x)y_1 + c_2'(x)y_2)$$

$$y_p'' = (c_1(x)y_1' + c_2(x)y_2'') + (c_1'(x)y_1' + c_2(x)y_2') + (c_1'(x)y_1 + c_2'(x)y_2)'$$

Substituting these in $y_p'' + py_p' + qy_p'$ and rearranging terms we obtain

$$(c_1(x)(y_1'' + py_1' + qy_1') + c_2(x)(y_2'' + py_2' + qy_2) +$$
$$q(c_1'(x)y_1' + c_2'(x)y_2) + (c_1'(x)y_1 + c_2'(x)y_2)' + (c_1(x)y_1' + c_2'(x)y_2')$$

Since $y_1$ and $y_2$ are solutions of $y'' + py' + qy = 0$, the first two terms on the right in the above expression are $0$. Hence, in the order to obtain $y'' + py' + qy = f(x)$, it is sufficient to chose $c_1(x)$ and $c_2(x)$ such that

$$\begin{aligned} c_1'(x)y_1 + c_2'(x)y_2 &= 0 \\ c_1'(x)y_1' + c_2'y_2' &= f(x) \end{aligned} \qquad (*)$$

It can be shown that this system of equations always has a unique solution $c_1'(x)$ and $c_2'(x)$ .We may then determine $c_1(x)$ and $c_2(x)$ by integration and use $y_p = c_1(x)y_1 + c_2(x)y_2$ to find $y_p$

**Example 3.3** Solve the DE $y'' + y = \cot x$

**Solution** The complementary equation is $y'' + y = 0$. The general solution of this equation is $y = c_1\cos x + c_2\sin x$. Let $y = c_1(x)\cos x + c_2(x)\sin x$.

The system (*) is therefore,

$$\begin{cases} c_1'(x)\cos x + c_2'(x)\sin x = 0 \\ -c_1(x)\sin x + c_2(x)\cos x = \cot x \end{cases}$$

Solving for $c_1'(x)$ and $c_2'(x)$ give

$$c_1'(x) = -\cos x$$
$$c_2'(x) = \cos ecx - \sin x$$

If we integrate each of these expressions, we obtain

$$c_1(x) = -\sin x$$
$$c_2(x) = \ell n\left|\cos ect - \cot x\right| + \cos x$$

Applying $y_p = c_1(x)y_1 + c_2(x)y_2$, a particular solution of the given equation is

$y_p = -\sin x\cos x + \sin x\ell n\left|\cos ecx - \cot x\right| + \sin x\cos x$ or

$$y_p = -\sin \ell nx\left|\cos ecx - \cot x\right|$$

Finally, the general solution of $y'' + y = \cot x$ is

$$y_p = c_1\cos x + c_2\sin x + \sin x\ell n\left|\cos ecx - \cot x\right|$$

## 2. Method of undetermined coefficients

Given the DE $y'' + py' + qy = f(x)$, it is sometimes easier to find a particular solution without resorting to integration. In this case it is reasonable to expect that there exists a particular solution $y_p$ with unknown coefficients and we use it as a trial solution in the given equation and attempt to find the value of the coefficients of $y_p$. This technique is called the **method of undetermined coefficients.**

Let us consider several such possibilities for equation $y'' + py' + qy = f(x)$ (3.4)

I. Let $f(x) = p_n(x)e^{\alpha x}$,

where $p_n(x)$ is a polynomial of degree $n$.

Then following particular cases are possible

i) The number $\alpha$ **is not a root** of the auxiliary equation $\lambda^2 + p\lambda + q = 0$.

In this case, the particular solution must be sought for in the form

$y_p = (A_0 + A_1 x + ... + A_n x^n)e^{\alpha x} = Q_n(x)e^{\alpha x}$.

Substituting in(3.4), we get a system of $n+1$equations for determining the unknown coefficients $A_0, A_1, ..., A_n$ of $Q_n(x)$

ii) The number $\alpha$ is a **simple (single)root** of the auxiliary equation.

In this case, we should seek the particular solution the form $y_p = xQ_n(x)e^{\alpha x}$

iii) The number $\alpha$ is **a double root** of the auxiliary equation .

The particular solution may be taken in the form $y_p = x^2 Q_n(x)e^{\alpha x}$

**Example3.4.** Solve the following DE

i) $y'' + 2y' - 8y = e^{3x}$

**Solution. T**he general solution of the complementary equation is $y_c = c_1 e^{2x} + c_2 e^{-4x}$.

We seek a particular solution of the form $y_p = Ae^{3x}$.

Since $y_p' = 3Ae^{3x}$ and $y_p'' = 9Ae^{3x}$ , substitution in the given equation leads to

$9Ae^{3x} + 6Ae^{3x} - 8Ae^{3x} = e^{3x}$

Dividing both sides by $e^{3x}$ we obtain

$9A+6-8A=1$,or $A=\frac{1}{7}$.Thus $y_p=\frac{1}{7}e^{3x}$ and

The general solution is $y_p=y_c+y_p=c_1e^{2x}+c_2e^{-4x}+\frac{1}{7}e^{3x}$

ii) $y''+9y=(x^2+1)e^{3x}$

**Solution. T**he general solution of the complementary equation is $y_c=c_1\cos 3x+c_2\sin 3x$

we seek a particular solution of the form $y_p=(Ax^2+Bx+c)e^{3x}$.

Substituting this expression in the given equation, we have

$[9(Ax^2+Bx+C)+6(2Ax+B)+2A+9(Ax^2+Bx+C)e^{3x}=(x^2+1)e^{3x}$.

Canceling out $e^{3x}$ and equation the coefficient of identical powers, we obtain

$18A=1,12A+18B=0,2A+GB+18C=1$.Whence $18A=\frac{1}{18};B=;\frac{-2}{27};C=\frac{51}{81}$.

Consequently the particular solution is $y_p\left(\frac{1}{18}x^2-\frac{1}{27}x+\frac{51}{81}\right)e^{3x}$.

The general solution is $y=y_c+y_p=c_1\cos 3x+c_2\sin 3x\left(\frac{1}{18}x^2-\frac{1}{27}x+\frac{51}{81}\right)e^{3x}$

iii) $y'''-7y'+6y=(x-2)e^x$

**Solution:** $f(x)$ is of the form $p_1(x)e^{1x}$ and coefficient 1in the exponent is a simple root of the auxiliary equation.

Hence, we seek the particular solution in the form $y_p=xQ_1(x)e^x$ or $y_p=x(Ax+B)e^x$

substitution in the given equation produces

$[(Ax^2+13x)+(4Ax+2B)+2A-7(Ax^2+Bx)-7(2Ax+B)+6(Ax^2+Bx)]e^x=(x-2)e^x$

Or $(-10Ax-5b+A)e^x=(x-2)e^x$

Equating the coefficient of identical powers of $x$, we get $-10A=1;-5B+2A=-2$.

Thus $A=-\frac{1}{10},B=\frac{9}{25}$.Consequently $y_p=x\left(-\frac{1}{10}x+\frac{9}{25}\right)e^x$ and the general solution is

$y=y_c+y_p=c_1e^{6x}+c_2e^xx\left(-\frac{1}{10}x+\frac{9}{25}\right)e^x$

II. Let $f(x)=e^{\alpha x}[P_n(x)\cos\beta x+Q_m(x)\sin\beta x]$

where $P_n(x)$ and $Q_m(x)$ are polynomials whose degree are $n$ and $m$ respectively.

The following particular cases are possible

**i)The number $\alpha+i\beta$ is not the root of the auxiliary equation**

The particular solution should be sought in the form $y_p=e^{\alpha x}[T_k(x)\cos\beta x+R_k(x)\sin\beta x]$

where $T_k(x)$ and $R_k(x)$ are polynomials of degree equal to the highest degree of the polynomials $P_n(x)$ and $Q_m(x)$ i.e. $k=\max\{n,m\}$

ii) **The number $\alpha+i\beta$ is the root of the auxiliary equation**

The particular solution is written in the form $y_p=xe^{\alpha x}[T_k(x)\cos\beta x+R_k(x)\sin\beta x]$

III. Let $f(x)=P\cos\beta x+Q\sin\beta x$ Where $P$ and $Q$ are constants

**i) If $\beta i$ is not root** of the auxiliary equation, the particular equation should be sought in the form $y_p=A\cos\beta x+B\sin\beta x$

ii) **If $\beta_i$ is a root of** the auxiliary equation, the particular equation should be sought in the form $y_p=x(A\cos\beta x+B\sin\beta x)$ .We remark that III is a special case of II

**Example 3.5.** Solve the following DE

i) $y''-y=3e^{2x}\cos x$

**Solution:** The auxiliary equation $\lambda^2-1=0$ has root $\lambda_1=1,\lambda_2=-1$.

Thus $y_c=c_1e^x+c_2e^{-x}$

$f(x)$ has the form $f(x)=e^{2x}(P\cos x+Q\sin x)$ and $p=3$, $Q=0$.

Since the number $\alpha+i\beta=2+i$ is not a root of the auxiliary equation, we seek the particular equation in the form $y_p=e^{2x}(A\cos x+B\sin x)$.

Putting this expression into the equation, after collecting like terms we get

$(2A+4B)e^{2x}\cos x+(-4A+2B)e^{2x}\sin x=3e^{2x}\cos x$.

Equating the coefficients of $\cos x$ and $\sin x$, we obtain $2A+4B=3,-4A+2B=0$.

Whence $A=\frac{3}{10}$ and $B=\frac{3}{5}$.Consequently, $y_p=e^{2x}\left(\frac{3}{10}\text{co}\,x+\frac{3}{5}\sin x\right)$

and the general solution is $y = y_c + y_p = c_1 e^x + c_2 e^{-x} + e^{2x}\left(\frac{3}{10}x + \frac{3}{5}\sin x\right)$

ii) $y'' + 4y = \cos 2x$

**Solution:** The auxiliary equation has roots $\lambda_1 = 2i, \lambda_2 = -2i$.

Therefore $y_c = c_1 \cos 2x + c_2 \sin 2x$.

$\alpha + i\beta = 0 + 2i$ is a root of the auxiliary equation, therefore we seek the particular solution in the form $y_p = x(A\cos 2x + B\sin 2x)$

Then $y_p{}' = 2x(-A\sin 2x + B\cos 2x) + (A\cos 2x + B\sin 2x)$

$y_p{}'' = -4x(-A\cos 2x + B\sin 2x) + 4(-A\sin 2x + B\cos 2x)$

Putting these expressions of the derivatives into the given equation and equating the coefficients of $\cos 2x$ and $\sin 2x$, we get a system of equations for determining $A$ and $B$:

$4B = 1, -4A = 0$. Whence $A = 0; B = \frac{1}{4}$.Thus,

$y = y_c + y_p = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}x\sin 2x$

iii) $y'' - 10y' + 41y = \sin x$

**Solution:** The auxiliary equation $\lambda^2 - 10\lambda + 41 = 0$ has the roots $\lambda_1 = 5 + 4i; \lambda_2 = 5 - 4i$.

Thus $y_c = e^{5x}(c_1 \cos 4x + c_2 \sin 4x)$.We seek a particular solution of the form

$y_p = A\cos x + B\sin x$ .Since $y_p{}' = -A\sin x + B\cos x; y_p{}'' = -A\cos x - B\sin x$, substitution in the given equation produces $(40A - 10B)\cos x + (10A + 40B)\sin x = \sin x$.

Consequently, $y_p$ is a solution provided $40A - 10B = 0; 10A + 40B = 1$.

The solution of this system of equation is $A = \frac{1}{170}$ and $B = \frac{4}{170}$.

Hence $y_p = \frac{1}{170}(\cos x + 4\sin x)$ and the general solution is

$y = y_c + y_p = e^{5x}(c_1 \cos 4x + c_2 \sin 4x) + \frac{1}{170}(\cos x + 4\sin x)$

**EXERCISES**

**1. Solve the following DE**

a) $y''-5y'+6y=0$ b) $y''-9y=0$ c) $y''-y'=0$

d) $y''-2y'+2y=0$ e) $y''+4y+13y=0$ f) $y''+2y'+y=0$

g) $y=y''+y'$ h) $\frac{y'-y}{y''}=3$

**2. Solve the following DE**

a) $y''-5y'+4y=0; y=5, y'=8$ for $x=0$

b) $y''+3y'+2y=0; y=1, y'=-1$ for $x=0$

c) $y''+3y=0; y=0$ for $x=0$ and $y=0$ for $x=3$

**3. Indicate the form of particular solutions of the following nonhomogeneous DE**

a) $y''-4y=x^2e^{2x}$ b) $y''+9y=\cos 2x$ c) $y''-4y'+4y=\sin 2x$

d) $y''-4+4y=e^{2x}$ e) $y''-2y'+5y=xe^x$ f) $y''-2y'+5y=x^2e^x\sin 2x$

**4. Solve the following DE**

a) $y''-7y'+12y=-e^{4x}$ b) $y''-2y'=x^2-1$ c) $y''-2y'=e^{2x}+5$

d) $y''-2y'-8y=e^x-8\cos 2x$ e) $y''+y'=5x+2e^x$ f) $y''-y'=2x-1-3e^x$

g) $y''+2y'+y=e^x+e^{-x}$ h) $y''-2y'+10y=\sin 3x+e^x$ i) $y''-3y'=x+\cos x$

**5. Solve the following DE using the method of variation of arbitrary constants**

a) $y''+y=tgx$ b) $y''+y=\cot gx$ c) $y''+2y'+y=\frac{e^{-x}}{x}$

d) $y''=y+\sec x$ e) $y''+y=\cos ecx$ f) $y''-2y=4x^2e^{x^2}$

**6. Integrate the following DE**

a) $\frac{d^2x}{dt^2}+\omega^2x=A\sin px$ Consider both cases: $p\neq\omega; p=\omega$

b) $y''-2y'=e^{2x}; y(0)=\frac{1}{8}; y'(0)=1$

c) $y''+4y=\sin x; y(0)=y'(0)=1$