REVISION EXERCISES
(AMAT8111)

I.

(a) Show that the set W of all vectors of the form

$$\begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix}$$

is a subspace of $\mathbb{R}^4$.

(b) Show that the set W of all polynomials of the form $a + bx - bx^2 + ax^3$ is a subspace of $\mathcal{P}_3$.

(c) Show that the set W of all matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ is a subspace of M_{22}.

Solution

(a) W is nonempty because it contains the zero vector $\mathbf{0}$. (Take $a = b = 0$.) Let $\mathbf{u}$ and $\mathbf{v}$ be in W—say,

$$\mathbf{u} = \begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} c \\ d \\ -d \\ c \end{bmatrix}$$

Then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} a + c \\ b + d \\ -b - d \\ a + c \end{bmatrix} = \begin{bmatrix} a + c \\ b + d \\ -(b + d) \\ a + c \end{bmatrix}$$

so $\mathbf{u} + \mathbf{v}$ is also in W (because it has the right *form*).

Similarly, if k is a scalar, then

$$k\mathbf{u} = \begin{bmatrix} ka \\ kb \\ -kb \\ ka \end{bmatrix}$$

so $k\mathbf{u}$ is in W.

Thus, W is a nonempty subset of $\mathbb{R}^4$ that is closed under addition and scalar multiplication. Therefore, W is a subspace of $\mathbb{R}^4$, by Theorem 6.2.

(b) W is nonempty because it contains the zero polynomial. (Take $a = b = 0$.) Let $p(x)$ and $q(x)$ be in W—say,

$$p(x) = a + bx - bx^2 + ax^3$$

and

$$q(x) = c + dx - dx^2 + cx^3$$

Then

$$p(x) + q(x) = (a + c) + (b + d)x - (b + d)x^2 + (a + c)x^3$$

so $p(x) + q(x)$ is also in W (because it has the right *form*).

Similarly, if k is a scalar, then

$$kp(x) = ka + kbx - kbx^2 + kax^3$$

so $kp(x)$ is in W.

Thus, W is a nonempty subset of $\mathcal{P}_3$ that is closed under addition and scalar multiplication. Therefore, W is a subspace of $\mathcal{P}_3$ by Theorem 6.2.

(c) W is nonempty because it contains the zero matrix O. (Take $a = b = 0$.) Let A and B be in W—say,

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

and

$$B = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} a + c & b + d \\ -(b + d) & a + c \end{bmatrix}$$

so $A + B$ is also in W (because it has the right *form*).

Similarly, if k is a scalar, then

$$kA = \begin{bmatrix} ka & kb \\ -kb & ka \end{bmatrix}$$

so kA is in W.

Thus, W is a nonempty subset of M_{22} that is closed under addition and scalar multiplication. Therefore, W is a subspace of M_{22}, by Theorem 6.2.

II.

The Cauchy-Schwarz Inequality $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$ is equivalent to the inequality we get by squaring both sides: $(\mathbf{u}\cdot\mathbf{v})^2 \le \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2$.

(a) In $\mathbb{R}^2$, with $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, this becomes

$$(u_1v_1 + u_2v_2)^2 \le (u_1^2 + u_2^2)(v_1^2 + v_2^2)$$

Prove this algebraically. [*Hint:* Subtract the left-hand side from the right-hand side and show that the difference must necessarily be nonnegative.]

(b) Prove the analogue of (a) in $\mathbb{R}^3$.

III.

Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

(a) Show that $A^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$.

(b) Prove, by mathematical induction, that

$$A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix} \text{ for } n \ge 1$$

IV.

a) Determine whether the polynomials $f_1(x)=x^3+2x^2$, $f_2(x)=2x^3+3x-1$, $f_3(x)=(x-3)^3$ are linear independent or linear dependent.

b) Do they span the space of polynomials of degree 3?

V.

Find the values of x so that $\det\begin{pmatrix} 1 & 0 & -3 \\ 5 & x & -7 \\ 3 & 9 & x-1 \end{pmatrix} = 0.$

VI.

Find the LU factorization of the matrix below

$$A=\begin{bmatrix} 1 & -5 & 1 \\ 4 & 7 & -3 \\ 2 & -4 & 0 \end{bmatrix}$$

VII.

Let $J_1(x_1,x_2,x_3)=(2x_1-x_2+x_3, x_1+5x_2+10x_3, x_2-4x_3)$,

$J_2(x_1,x_2,x_3)=(2x_1+x_2+5x_2, -x_2+10x_3, 5x_3)$ and

$J_3(x_1, x_2, x_3) = (-x_1 - x_2 + 2x_3, 2x_2 + x_3, x_1 + 2x_3)$

a) Find the standard matrices corresponding to those transformations.

b) Compare $J_1 \circ J_2 \circ J_3$ and $J_3 \circ J_2 \circ J_1$

VIII.

a) Find the limit of following functions:

$$\lim_{x \to 0} (\sqrt{2} - \sqrt{1 + \cos 2x}) / \sin^2 x.$$

$$\lim_{x \to 1} \frac{1 - x^{-1/3}}{1 - x^{-2/3}}$$

If $\lim\limits_{x \to -a} \dfrac{x^9 + a^9}{x + a} = 9$, find the value of a.

$$\lim_{x \to 0} \left(\frac{3 + 2x}{3 - x} \right)^{(1/x)}$$

b) Evaluate the derivative of

$$x^3 + y^3 = 2xy$$

$$y = e^{5x} \sin 2x \cos x$$

$$y = x^x$$

$$\left. \begin{aligned} x &= a(t - \sin t) \\ y &= a(1 - \cos t) \end{aligned} \right\} \quad 0 \le t \le 2\pi$$

c)

If $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, then prove that

$$\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$$

d)

Find dy/dx, if $x=\sin(\log_e t)$, $y=\cos(\log_e t)$

IX.

a) Evaluate the anti-derivative below

$$\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}\,dx.$$

$$\int (x+1)\sqrt{x^2-x+1}\,dx$$

$$\int x\sqrt{\frac{1+x}{1-x}}\,dx$$

$$\int e^x\left(\frac{x-1}{x^2}\right)dx$$

$$\int \frac{x^2+2}{(x^2+1)(x^2+4)}\,dx.$$

X.

a) Make a rough sketch of $y = \sin 2x$ between $x = 0$ and $x = \pi/4$ and
calculate the area under that curve in the above range.

b) Find the mean and root mean square of $y^2 = 4a^2(x-3)$. What is
the is the area bounded by that curve and the lines x = 3,
y = 4a.

c) Find the volume generated by the semi ellipse of equation

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

XI.

1) Write a method in python to do the following:
   a) Find the length of two vectors in IR3
   b) Verify if two vectors are linear dependent or linear independent.
   c) Find the angle between two vectors in IR3

2) Use python to solve the system of equations below:

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 2 \\ 2 & -10 & -2 & -3 \\ -3 & 1 & 1 & 2 \end{array}\right)$$