MATH8213: M C & DE EXERCISES

MULTIVARIABLE CALCULUS

A. DIFFERENTIATION

I Find the domain of definition of the following functions

1. $z=\sqrt{4-x^2-y^2}$; 2. $z=\ln(x+y)$ 3. $z=\arctan\dfrac{1}{x^2+y^2}$

4. $z=\sqrt{1-x^2}+\sqrt{1-y^2}$ 5. $z=\sqrt{x^2-4}+\sqrt{4-y^2}$ 6. $z=\arctan\dfrac{x-y}{1+x^2y^2}$

7. $\mathrm{u}=\sqrt{1-x^2-y^2-z^2}$ 8. $z=\ln(xyz)$ 9. $z=\sqrt{x}+\sqrt{y}+\sqrt{z}$

10. $z=\arcsin x+\arcsin y+\arcsin z$ 11. $z=\sqrt{x^2+y^2+z^2-4}+\sqrt{4-x^2-y^2-z^2}$

II. Evaluate the following limits

1. $\lim\limits_{(x,y)\to(0,2)}\dfrac{\sin xy}{x}$ 2. $\lim\limits_{(x,y)\to(\infty,k)}\left(1+\dfrac{y}{x}\right)^x$ 3. $\lim\limits_{(x,y)\to(\infty,\infty)}\dfrac{x+y}{x^2+y^2}$

4. $\lim\limits_{(x,y)\to(0,0)}\dfrac{x}{x+y}$ 5. $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2-y^2}{x^2+y^2}$ 6. $\lim\limits_{(x,y)\to(0,0)}(x^2+y^2)\sin\dfrac{1}{xy}$

III. Find the points of discontinuity for the following functions

1. $z=\sqrt{9-x^2-y^2}$ 2. $z=\ln\sqrt{x^2+y^2}$ 3. $z=\dfrac{1}{(x-y)^2}$

4. $z=\dfrac{1}{\sqrt{1-x^2-y^2}}$ 5. $z=\dfrac{1}{\sqrt{1-x^2}+\sqrt{1-y^2}}$ 6. $z=\dfrac{2}{1-x^2-y^2}$

IV. calculate the partial derivatives of the first order for the following functions

1. $z=x^2+y^2-3xy$ 2. $z=x^y$ 3. $z=\arctan\dfrac{y}{x}$ 4. $z=e^{\sin\frac{y}{x}}$ 5. $z=\ln\left(x+\sqrt{x^2+y^2}\right)$

6. $\mathrm{z}=\dfrac{x-y}{x+y}$ 7. $z=\dfrac{x}{\sqrt{x^2+y^2}}$ 8. $z=\arcsin\sqrt{\dfrac{x^2-y^2}{x^2+y^2}}$ 9. $u=(xy)^z$ 10. $\mathrm{u}=\mathrm{z}^{xy}$

11. $u=xz\arctan\dfrac{y}{z}$ 12. $u=x^2y+yz^2+x^2z+xyz$ 13. $u=2y^2+z^2-xy-yz+2x$

Iv. Find the total differential for the following functions

1. $z = x^2 y^2$ 2. $z = \sin^2 x + \cos^2 y$ 3. $z = \ln\tan\frac{y}{x}$ 4. $z = yx^y$

5. $z = \frac{x^2 - y^2}{x^2 + y^2}$ 6. $z = \arctan\frac{y}{x} + \arctan\frac{x}{y}$ 7. $u = \frac{z}{\sqrt{x^2 + y^2}}$ 8. $u = yxz$

v. Find the derivatives of the following compound functions

1. $\frac{dz}{dt}$, if $z = \frac{x}{y}$,where $x = e^t, y = \ln t$ 2. $\frac{dz}{dt}$, if $z = \ln\sin\frac{x}{\sqrt{y}}$,where $x = 3t^2, y = \sqrt{t^2+1}$

3. $\frac{dz}{dt}$, if $z = x^y$,where $x = \sin t, y = \cos t$ 4. $\frac{du}{dt}$, if $u = xyz$,where $x = t+1, y\frac{\partial z}{\partial x} = \ln t, z = \tan t$

5. $\frac{du}{dt}$, if $u = \frac{z}{\sqrt{x^2+y^2}} xyz$,where $x = a\cos t, y = a\sin t, z = b$

6. $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$, if $z = \arctan\frac{x}{y}$,where $x = u\sin v, y = u\cos v$

7. $\frac{\partial x}{\partial r}$, $\frac{\partial y}{\partial r}$, $\frac{\partial z}{\partial r}$, $\frac{\partial x}{\partial u}$, $\frac{\partial y}{\partial u}$, $\frac{\partial z}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial v}$, $\frac{\partial z}{\partial v}$, if $x = r\sin u\cos v, y = r\sin u\sin v\ z = r\cos u$

VI. Calculate the partial derivatives of second order for the following functions

1. $z = x^2 + y^2 - 3xy$ 2. $z = x^y$ 3. $z = \arctan\frac{y}{x}$ 4. $z = \sin x\sin y\sin(x+y)$

5. $u = \sqrt{x^2 + y^2 + z^2}$ 6. $u = z^{xy}$ 7. $u = xz\arctan\frac{y}{z}$ 8. $u = x^2y + yz^2 + x^2z + xyz$

VII. Calculate the derivatives of the functions given implicitely

1. $\frac{dy}{dx}; \frac{d^2y}{dx^2}$, if $y = 1 + y^x$ 2. $\frac{dy}{dx}; \frac{d^2y}{dx^2}$, if $e^x\sin y + e^y\sin x = 1$ 3. $\frac{dy}{dx}; \frac{d^2y}{dx^2}$, if $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

4. $\frac{\partial z}{\partial x}; \frac{\partial z}{\partial y}$, if $x\cos y + y\cos z + z\cos x = 1$ 5. $\frac{\partial z}{\partial x}; \frac{\partial z}{\partial y}$, if $x^2 + y^2 - z^2 - xy = 0$

VIII. Find the equations of the tangent plane and normal line to the given surfaces at given points.

1. $z = 3x^2 + 2y^2 - 11; P(2,1,3)$ 2. $z = x^2 + y^2; P(1,-2,5)$ 3. $x^2 + 4y^2 - 4z^2 - 4 = 0\ ; P(2,1,1)$

4. $x^2 + y^2 + z^2 = 169; P(3,4,12)$ 5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1; P(x_0, y_0, z_0)$ 6. $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1; P(x_0, y_0, z_0)$

IX. Examine the following functions for extrema.

1. $z = x^2 + y^2 - 4x + 6y + 25$ 2. $z = x^3 + y^3 + 3xy$ 3. $z = x^2 + 3xy^2 - 15x - 12y$

4. $z = 6 - 4x - 3y; S/C : x^2 + y^2 = 1$ 5. $z = xy; S/C : x + y = 1$ 6. $z = x + y; S/C : x^2 + y^2 = 5$

B. INTEGRATION

I. Evaluate the following double integrals

1. $\iint_D \frac{dxdy}{\sqrt{(1-x^2)(1-y^2)}}; D: 0 \le x \le 1; 0 \le y \le 1.$
2. $\iint_D \frac{dxdy}{x^2 + y^2}; D: 0 \le x \le 2; 0 \le y \le x$
3. $\iint_D (x^2 + y^2) dxdy, D: 0 \le x \le 1; x \le y \le \sqrt{x}$
4. $\iint_D r dr d\theta, D: 0 \le \theta \le 2\pi; a \sin\theta \le r \le a$
5. $\iint_D r^2 \sin^2 \theta dr d\theta; D: \frac{-\pi}{2} \le \theta \le \frac{\pi}{2} 2\pi; 0 \le r \le 3\cos\theta$
6. $\iint_D (x - y) dxdy; D: y = 2 - x^2; y = 2x - 1$
7. $\iint_D \sin(x + y) dxdy; D: x = 0; y = \frac{\pi}{2}; y = x$
8. $\iint_D x dxdy; D: \Delta_{ABC} : A(2,3); B(7,2), C(4,5)$
9. $\iint_D \sqrt{x^2 + y^2} dxdy; D: x^2 + y^2 \le a^2; x \ge 0; y \ge 0$
10. $\iint_D (x + y)^3 (x - y)^2 dxdy; D: x + y = 1; x - y = 1; x + y = 3; x - y = -1$

II. Applications of double integral: Find the area bounded by

1. $x = 4y - y^2; x + y = 6$ 2. $x = y^2 - 2y; x + y = 0$ 3. $y^2 = 4x; x^2 = 4y$

3. $x = 4 - y^2; x + 2y = 4$ 4. $x^2 + y^2 = 4$ 5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

III. Evaluate the following triple integrals

1. $\iiint\limits_D (x^2+y^2+z^2)dxdy\mathrm{dz}; D: 0 \le x \le 1; 0 \le y \le 1; 0 \le z \le 1$
2. $\iiint\limits_D x^2 dxdy\mathrm{dz};\ D: x^2+y^2+z^2 \le R^2$
3. $\iiint\limits_D (x^2+y^2+z^2)dxdy\mathrm{dz}; D: 0 \le x \le 1; 0 \le y \le 1; 0 \le z \le \sqrt{1-x^2-y^2}$
4. $\iiint\limits_D (x^2+y^2+z^2)dxdy\mathrm{dz}; D: 0 \le x \le 1; 0 \le y \le 1-x; 0 \le z \le 1-x-y$
5. $\iiint\limits_D xyzdxdydz; D: 0 \le x \le 1; 0 \le y \le 1-x; 0 \le z \le 1-x-y$

IV. Applications of triple integral: Find the volume

1. bounded by the Surface: $z = a^2 - x^2$ and the planes: $x = 0$; $y = 0; z = 0;\ y = b$
2. of the sphere: $x^2+y^2+z^2 = a^2$
3. of the tetrahedron bounded by the coordinates planes and thePlane: $x+y+z=1$

DIFFERENTIAL EQUATIONS

I. Prove that y is a solution of the indicated differential equation

1. $y = c_1e^x + c_2e^{2x}; y'' - 3y' + 2y = 0$
2. $y = ce^{-3x}; y' + 3y = 0$
3. $y = e^x(c_1 \cos x + c_2 \sin x); y'' - 2y' + 2y = 0$
4. $y = (c_1 + c_2x)e^{-2x}; y'' + 4y' + 4y = 0$

7. $y^2 - x^2 - xy = c; (x-2y)y' + 2x + y = 0$
8. $y = c_1x + \frac{c_2}{x} + c_3; y''' + \frac{3}{x}y'' = 0$
9. $y = c_1e^{\arcsin x} + c_2e^{-\arcsin x}; (1-x^2)y'' - xy' - y = 0$

II. Form the differential equation whose general solution is:

1. $y = cx^2 - x$
2. $y = c_1x^3 + c_2x + c_3$
3. $y = c_1e^x + c_2xe^x$
4. $y = c_1x^3 + c_2x$
5. $x^2 + y^2 = c^2$

7. $y = c_1e^x + c_2xe^x + c_3e^{-x}$
8. $y = c_1e^x + c_2xe^{-x} + c_3e^{-x}$

III. First order ODE

1. Solve the following DE

a) $\tan x \sin^2 y dx + cox \cot y dy = 0$ b) $xy' - y = y^3$ c) $xyy' = 1 - x^3$

d) $y - xy' = 1 + x^2 y'$ e) $y' tgx = y$ f) $3e^x \tan y dx + (1 - e^x)\sec^2 y dy = 0$

2. Find the solutions of the following DE satisfying the given initial conditions

a) $(1+e^x)yy' = e^x,\ y(0) = 1$ b) $(xy^2 + x)dx + (x^2 y - y)dy = 0, y(0) = 1$

3. Integrate the following homogeneous DE

a) $(y - x)dx + (y + x)dy = 0$ b) $(x + y)dx + xdy = 0$ c) $(x + y)dx + (y - x)dy = 0$

d) $xdy - ydx = \sqrt{x^2 + y^2 dx}$ e) $(8y + 10x)dx + (5y + 7x)dy = 0$ f) $xy^2 dy = (x^3 + y^3)dx$

4. Solve the following DE reducible to homogeneous DE

a) $(2x + y - 4)dy + (x - 2y + 5)dx = 0$ b) $y' = \dfrac{1 - 3x - 3y}{1 + x + y}$

c) $y' = \dfrac{x + 2y + 1}{2x + 4y + 3}$ d) $(x + 2y + 1)dx - (2x + 4y + 3)dy = 0$

5. Solve the following linear DE

a) $y' - \dfrac{2y}{x+1} = (x+1)^3$ b) $y' - \dfrac{y}{x} = \dfrac{x+1}{x}$ c) $(x - x^3)y' + (2x^2 - 1)y = x^3$

d) $y' \cos x + y \sin x = -1$ e) $y' - \dfrac{n}{x} y = e^x x^n$ f) $y' + y = e^{-x}$ g)y $y' + \dfrac{1 - 2x}{x^2} y - 1 = 0$

6. Solve the following DE

a) $y' - \dfrac{y}{1 - x^2} - 1 - x = 0, y(0) = 0$ b) $y' - ytgx = \dfrac{1}{\cos x}, y(0) = 0$

c) $xy' + y - e^x = 0, y(1) = 2$ d) $xy' + y - e^x = 0, y(0) = 1$

8. Integrate the following Bernoulli's equations

a) $y' + xy = x^3 y^3$ b) $(1-x^2)y' - xy - xy^2 = 0$ c) $3y^2 y' - y^3 - x - 1 = 0$ d) $(y\ell nx - 2)ydx = xdy$

e) $y - y'\cos x = y^2 \cos x(1-\sin x)$ f) $y' + \dfrac{y}{x} = -xy^2$ g) $2xyy' - y^2 + x = 0$

10. Integrate the followig exact DE

a) $(x+y)dx + (x+2y)dy = 0$ b) $(x^2 + y^2 + 2x)dx + 2xydy = 0$ c) $(x^2 + y)dx + (x-2y)dy = 0$

d) $(x^3 - 3xy^2 + 2)dx - (3x^2 y - y^2)dy = 0$ e) $(\sin xy + xy\cos xy)dx + x^2 \cos xydy = 0$

ODE OF THE SECOND ORDER

1. Solve the following DE

a) $y'' - 5y' + 6y = 0$ b) $y'' - 9y = 0$ c) $y'' - y' = 0$ d) $y'' - 2y' + 2y = 0$

i) $y'' - 5y' + 4y = 0; y(0) = 5, y'(0) = 8$ j) $y'' + 3y' + 2y = 0; y(0) = 1, y'(0) = -1$

c) $y'' + 3y = 0; y(0) = 0\ ; y(3) = 0$

2. Solve the following DE

a) $y'' - 7y' + 12y = -e^{4x}$ b) $y'' - 2y' = x^2 - 1$ c) $y'' - 2y' = e^{2x} + 5$

g) $y'' + 2y' + y = e^x + e^{-x}$ h) $y'' - 2y' + 10y = \sin 3x + e^x$ i) $y'' - 3y' = x + \cos x$

j) $y'' - 2y' = e^{2x}; y(0) = \dfrac{1}{8}; y'(0) = 1$ k) $y'' + 4y = \sin x; y(0) = y'(0) = 1$

3. Solve the following DE using the method of variation of arbitrary constants

a) $y'' + y = \tan x$ b) $y'' + y = \cot gx$ c) $y'' + 2y' + y = \dfrac{e^{-x}}{x}$

d) $y'' = y + \sec x$ e) $y'' + y = \cos ecx$ f) $y'' - 2y = 4x^2 e^{2x}$