REVISION EXERCISES

*AMAT 8111 MID-TERM EXAM (!**NOT EXCLUSIVE!**)*

I.

Let $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$, and $\vec{z} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 2 \end{bmatrix}$. Show that $\vec{v}$ is orthogonal to $\vec{w}$ but $\vec{v}$ is not orthogonal to $\vec{z}$.

II.

(a) Show that the set W of all vectors of the form

$$\begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix}$$

is a subspace of $\mathbb{R}^4$.

(b) Show that the set W of all polynomials of the form $a + bx - bx^2 + ax^3$ is a subspace of $\mathscr{P}_3$.

(c) Show that the set W of all matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ is a subspace of M_{22}.

Solution

(a) W is nonempty because it contains the zero vector $\mathbf{0}$. (Take $a = b = 0$.) Let $\mathbf{u}$ and $\mathbf{v}$ be in W—say,

$$\mathbf{u} = \begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} c \\ d \\ -d \\ c \end{bmatrix}$$

Then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} a + c \\ b + d \\ -b - d \\ a + c \end{bmatrix} = \begin{bmatrix} a + c \\ b + d \\ -(b + d) \\ a + c \end{bmatrix}$$

so $\mathbf{u} + \mathbf{v}$ is also in W (because it has the right *form*).

Similarly, if k is a scalar, then

$$k\mathbf{u} = \begin{bmatrix} ka \\ kb \\ -kb \\ ka \end{bmatrix}$$

so $k\mathbf{u}$ is in W.

Thus, W is a nonempty subset of $\mathbb{R}^4$ that is closed under addition and scalar multiplication. Therefore, W is a subspace of $\mathbb{R}^4$, by Theorem 6.2.

(b) W is nonempty because it contains the zero polynomial. (Take $a = b = 0$.) Let $p(x)$ and $q(x)$ be in W—say,

$$p(x) = a + bx - bx^2 + ax^3$$

and

$$q(x) = c + dx - dx^2 + cx^3$$

Then

$$p(x) + q(x) = (a + c) + (b + d)x - (b + d)x^2 + (a + c)x^3$$

so $p(x) + q(x)$ is also in W (because it has the right *form*).

Similarly, if k is a scalar, then

$$kp(x) = ka + kbx - kbx^2 + kax^3$$

so $kp(x)$ is in W.

Thus, W is a nonempty subset of $\mathscr{P}_3$ that is closed under addition and scalar multiplication. Therefore, W is a subspace of $\mathscr{P}_3$ by Theorem 6.2.

(c) W is nonempty because it contains the zero matrix O. (Take $a = b = 0$.) Let A and B be in W—say,

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

and

$$B = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} a + c & b + d \\ -(b + d) & a + c \end{bmatrix}$$

so $A + B$ is also in W (because it has the right *form*).

Similarly, if k is a scalar, then

$$kA = \begin{bmatrix} ka & kb \\ -kb & ka \end{bmatrix}$$

so kA is in W.

Thus, W is a nonempty subset of M_{22} that is closed under addition and scalar multiplication. Therefore, W is a subspace of M_{22}, by Theorem 6.2. ▲

III.

The Cauchy-Schwarz Inequality $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$ is equivalent to the inequality we get by squaring both sides: $(\mathbf{u}\cdot\mathbf{v})^2 \le \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2$.

(a) In $\mathbb{R}^2$, with $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, this becomes

$$(u_1v_1 + u_2v_2)^2 \le (u_1^2 + u_2^2)(v_1^2 + v_2^2)$$

Prove this algebraically. [*Hint:* Subtract the left-hand side from the right-hand side and show that the difference must necessarily be nonnegative.]

(b) Prove the analogue of (a) in $\mathbb{R}^3$.

IV

Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

(a) Show that $A^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$.

(b) Prove, by mathematical induction, that

$$A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix} \text{ for } n \ge 1$$

V

Find the values of x so that $\det\begin{pmatrix} 1 & 0 & -3 \\ 5 & x & -7 \\ 3 & 9 & x-1 \end{pmatrix} = 0.$

VI

The volume of a parallelepiped (three-dimensional parallelogram) which is spanned by the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is given by $\left|\det\begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix}\right|$.

Find the volume of the parallelepiped generated by the vectors

$$\mathbf{u} = (1\ \ 2\ \ 1)^T,\quad \mathbf{v} = (2\ \ 3\ \ 5)^T \text{ and } \mathbf{w} = (7\ \ 10\ \ -1)^T$$

VII

Compute the determinants of the following matrices:

(a) $\mathbf{A} = \begin{pmatrix} \alpha & -\beta & \delta \\ 0 & \beta & -\alpha \\ 0 & 0 & \gamma \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} \sin(\theta) & 0 & 0 \\ 1 & \cos(\theta) & 0 \\ -2 & \sin(\theta) & 2 \end{pmatrix}$ (c) $\mathbf{C} = \begin{pmatrix} x & y & z \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$

VIII

If a square matrix $\mathbf{A}$ satisfies $\mathbf{A}^T\mathbf{A} = \mathbf{I}$ ($\mathbf{A}$ is an orthogonal matrix) where $\mathbf{I}$ is the identity matrix, show that $\det(\mathbf{A}) = \pm 1$.

IX

Use the matrix

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

to obtain the Hill cipher for the plaintext message

I AM HIDING

Solution If we group the plaintext into pairs and add the dummy letter G to fill c last pair, we obtain

IA MH ID IN GG

or, equivalently, from Table 1,

9 1 13 8 9 4 9 14 7 7

To encipher the pair IA, we form the matrix product

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 3 \end{bmatrix}$$

Table 1

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0

which, from Table 1, yields the ciphertext KC.

To encipher the pair MH, we form the product

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 13 \\ 8 \end{bmatrix} = \begin{bmatrix} 29 \\ 24 \end{bmatrix} \qquad (1)$$

However, there is a problem here, because the number 29 has no alphabet equivalent (Table 1). To resolve this problem, we make the following agreement:

Whenever an integer greater than 25 occurs, it will be replaced by the remainder that results when this integer is divided by 26.

Because the remainder after division by 26 is one of the integers $0, 1, 2, \ldots, 25$, this procedure will always yield an integer with an alphabet equivalent.

Thus, in (1) we replace 29 by 3, which is the remainder after dividing 29 by 26. It now follows from Table 1 that the ciphertext for the pair MH is CX.

The computations for the remaining ciphertext vectors are

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 17 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 14 \end{bmatrix} = \begin{bmatrix} 37 \\ 42 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 11 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} 21 \\ 21 \end{bmatrix}$$

These correspond to the ciphertext pairs QL, KP, and UU, respectively. In summary, the entire ciphertext message is

KC CX QL KP UU

which would usually be transmitted as a single string without spaces:

KCCXQLKPUU ◀

Because the plaintext was grouped in pairs and enciphered by a 2×2 matrix, the Hill cipher in Example 1 is referred to as a ***Hill 2-cipher***. It is obviously also possible to group the plaintext in triples and encipher by a 3×3 matrix with integer entries; this is called a ***Hill 3-cipher***. In general, for a ***Hill n-cipher***, plaintext is grouped into sets of n letters and enciphered by an $n \times n$ matrix with integer entries.

X

Consider the statement "If $\vec{u} \neq \vec{0}$, and both $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$ and $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, then $\vec{v} = \vec{w}$." Either prove the statement or give a counterexample.

XI

, (a) Evaluate this determinant by cofactors of row 1:

$$\begin{vmatrix} 4 & 4 & 4 & 4 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 \end{vmatrix}.$$

(b) Check by subtracting column 1 from the other columns and recomputing.

c) Find the determinant and all nine cofactors C_{ij} of this triangular matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

d) Form C^{T} and verify that $AC^{\mathrm{T}} = (\det A)I$.

e) Which of the following matrices are guaranteed to equal $(A+B)^2$?

$$A^2+2AB+B^2, \quad A(A+B)+B(A+B), \quad (A+B)(B+A), \quad A^2+AB+BA+B^2.$$