Unit2: Plane analytic geometry

Introduction

In this unit, we will study the main parts of plane and analytic geometry. We will start by elementary problems of plane analytic geometry, then, the relative position of two straight lines will be examined and we will spend enough time in studying quadric curves by reducing the general equation to canonical form by means of orthogonal transformations and orthogonal invariants. The complete classification of quadric curves will be given.

This unit is in 4 sections

Section 2.1 Elementary problems

Section 2.2 Straight line

Section 2.3 Quadric curves in canonical form

Section 2.4 General equation of a quadric curve

Section 2.1 Elementary problems

In this section, we will consider a two dimensional affine Euclidean plane endowed with orthonormal frame $F(O;\vec{e}_1,\vec{e}_2)$

2.1.1 Distance between two points

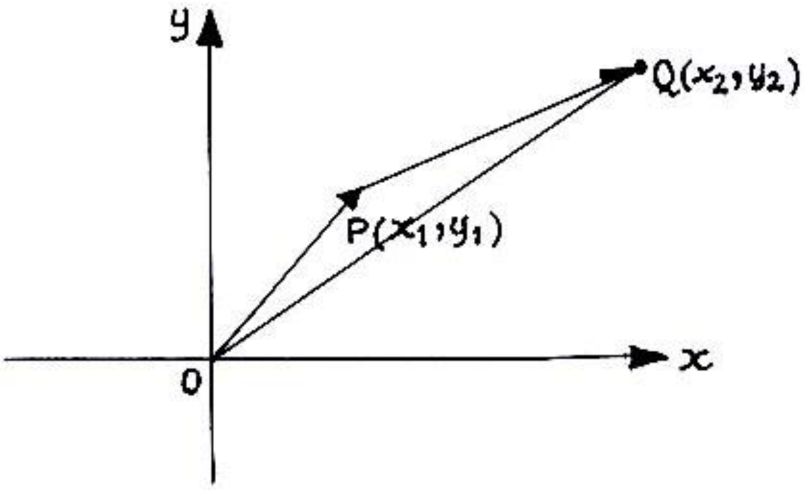

Figure2.1

Proposition1.1

The distance d between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is expressed by the formula:

$$d(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad (2.1)$$

Proof

The distance between P and Q is the length of the vector $\overrightarrow{PQ}$ i.e.

$d(P,Q) = \left\|\overrightarrow{PQ}\right\|$.

From figure1.1, we have $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$, what gives $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$.

The components of $\overrightarrow{PQ}$ are given by $\overrightarrow{PQ} = (x_2 - x_1, y_2 - y_1)$.

Thus $d(P,Q) = \left\|\overrightarrow{PQ}\right\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Example1.1

Determine the distance between the points $P(-1,3)$ and $Q(3,6)$

Solution

According to the formula (2.1) we have:

$d(P,Q) = \sqrt{(3+1)^2 + (6-3)^2} = \sqrt{25 = 5}.$

Proposition1.2

The distance d between the points $P(\rho_1, \varphi_1)$ and $Q(\rho_2, \varphi_2)$ in polar coordinates system is expressed by the formula:

$$d(P,Q) = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos(\varphi_2 - \varphi_1)} \qquad (2.2)$$

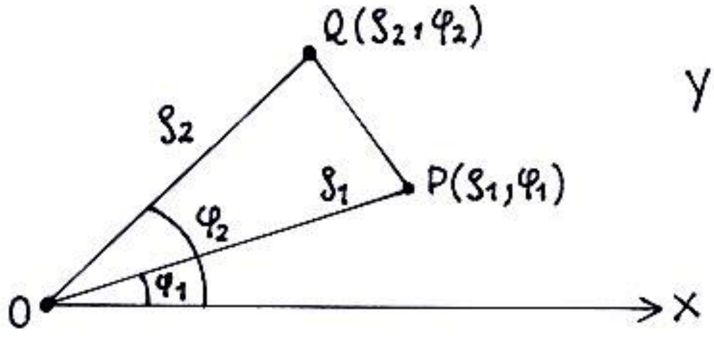

Figure 2.2

Proof. The formula (2.2) is directly derived from the law of the cosines in plane trigonometry. Indeed, let us consider the triangle OPQ (figure1.2)

$$\left\|\overrightarrow{PQ}\right\|^2 = \left\|\overrightarrow{OP}\right\|^2 + \left\|\overrightarrow{OQ}\right\|^2 - 2\left\|\overrightarrow{OP}\right\|\left\|\overrightarrow{OQ}\right\|\cos(\overrightarrow{OP},\overrightarrow{OQ}).$$

But $\left\|\overrightarrow{OP}\right\| = \rho_1; \left\|\overrightarrow{OQ}\right\| = \rho_2; \angle(\overrightarrow{OP},\overrightarrow{OQ}) = \varphi_2 - \varphi_1.$

By the cosines law we have:

$$d(P,Q) = \left\|P\vec{Q}\right\| = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos(\varphi_2 - \varphi_1)}$$

Example1.2

Determine the distance between $P\left(3,\frac{\pi}{3}\right)$ and $Q\left(4,\frac{\pi}{6}\right)$.

Solution

Applying the formula (2.2), we get:

$$d(P,Q) = \sqrt{3^2 + 4^2 - 2.3.3\cos\left(\frac{\pi}{3} - \frac{\pi}{6}\right)} = \sqrt{25 - 12\sqrt{3}}$$

2.1.2 Dividing a line segment in a given ratio

We say that the point M divides the line segment PQ in a given ratio λ if and only if $\overrightarrow{PM} = \lambda\overrightarrow{MQ}.$

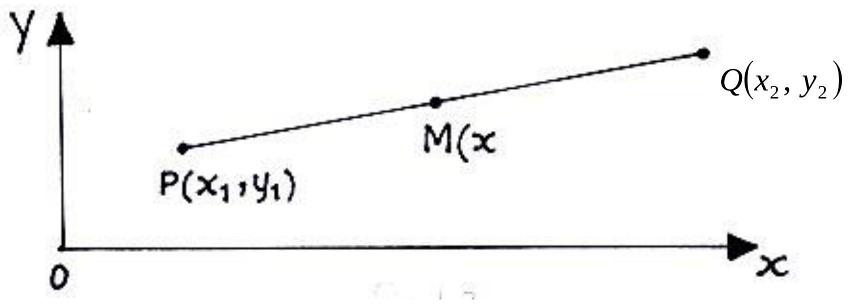

Figure2.3

Proposition1.3:

Let (x_2, y_2) and (x_1, y_1) be the coordinates of the points P and Q respectively and (x, y) be the co-ordinates of the point M which divides the segment PQ in the ratio λ .The following formulas hold:

$$x = \frac{x_1 + \lambda x_2}{1+\lambda}, y = \frac{y_1 + \lambda y_2}{1+\lambda} \qquad (2.3)$$

Proof:

The components of the vectors $\overrightarrow{PM}$ and $\overrightarrow{MQ}$ are $(x - x_1, y - y_1)$ and $(x_2 - x, y_2 - y)$ respectively.

In co-ordinates form, the relation $\overrightarrow{PM} = \lambda\overrightarrow{MQ}$ can be written:

$(x - x_1, y_1 - y_1) = \lambda(x_2 - x, y_2 - y);$

Since $\lambda(x_2 - x, y_2 - y) = (\lambda x_2 - \lambda x, \lambda y_2 - \lambda),$

We have $(x - x_1, y_1 - y_1) = (\lambda x_2 - \lambda x, \lambda y_2 - \lambda).$

The equality of two couples leads to:

$$x = \frac{x_1 + \lambda x_2}{1+\lambda}; y = \frac{y_1 + \lambda y_2}{1+\lambda}.$$

Remark 1.1

- If $\lambda > 0, M$ is inside the segment PQ
- If $\lambda < 0$, M is outside of the segment PQ
- If $\lambda = 0$, then $M = P$
- If $M \to Q$, then $\lambda \to \infty$,
- If $\lambda = -1$, the point M moves away indefinitely on the line carrying PQ
- If M is the midpoint of $PQ, \lambda = 1$, we have $x = \frac{x_1 + x_2}{2}; y = \frac{y_1 + y_2}{2}$

Remark 1.2

The coordinates of the centroid of a system of material points $P_i(x_i, y_i)$ with masses $m_i (i = 1,...,n)$ are given by the formulas:

$$x = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}; y = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i} \qquad (2.4)$$

Example 1.3

$P(1,1)$ and $Q(7,4)$ are the initial and terminal points of the line segment PQ. Find the coordinates of the point M which is twice more closer to P than to Q and located between these two points.

Solution

The point M divides the segment PQ in $\lambda = \frac{1}{2}$

Applying the formulas (2.3), we have: $x = \dfrac{1+\frac{1}{2}.7}{1+\frac{1}{2}} = 3; y = \dfrac{1+\frac{1}{2}.4}{1+\frac{1}{2}} = 2$.

Example 1.4

Let $A(x_1, y_1), B = (x_2, y_2)$ and $C = (x_3, y_3)$ be the vertices of the triangle ABC . Find the co-ordinates of the centroid of the triangle.

Solution. The co-ordinates of the centroid of this triangle are given by

$G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.

2.1.3 Area of a triangle

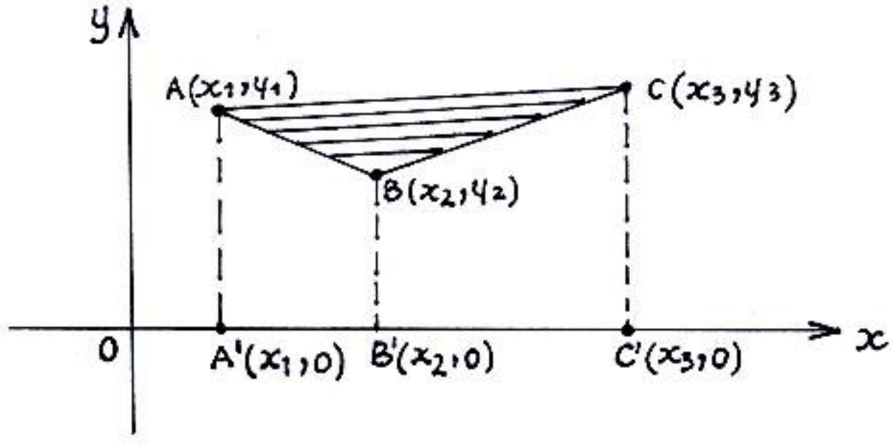

Figure 2.4

Proposition 4: Let $A(x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$ be the vertices of the triangle ABC. Then the area S_{ABC} of the triangle in terms of the coordinates of its vertices is expressed by the formula:

$$S_{ABC} = \frac{1}{2}\left|\left[(x_1 - x_2)(y_1 + y_2) + (x_2 - x_3)(y_2 + y_3) + (x_3 - x_1)(y_3 + y_1)\right]\right| \quad (2.5)$$

Proof: The area of the triangle ABC (figure (2.4) can be found by the relation

$$s_{ABC} = s_{A'ACC'A'} - s_{A'ABB'A'} - s_{B'BCC'} \qquad (2.6)$$

where $S_{A'ACC'A'}, s_{A'ABB'A'}, s_{B'BCC'}$ are the areas of the corresponding trapezoids.

The area A of a trapezoid with bases $B;b$ and altitude h is given by:

$$A = \frac{(B+b)}{2}h\,.$$

Using the above formula we have:

$$s_{A'ACA'A'} = \frac{\|\overrightarrow{A'A}\| + \|\overrightarrow{C'C}\|}{2}\|\overrightarrow{A'C'}\| = \frac{1}{2}(y_1 + y_3)(x_3 - x_1)$$

$$s_{A'ABB'A'} = \frac{\|\overrightarrow{A'A}\| + \|\overrightarrow{B'B}\|}{2}\|\overrightarrow{A'B'}\| = \frac{1}{2}(y_1 + y_2)(x_2 - x_1)$$

$$s_{B'BCC'B'} = \frac{\|\overrightarrow{BB}\| + \|\overrightarrow{C'C}\|}{2}\|\overrightarrow{B'C'}\| = \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

Injecting the expression of the areas of these trapezoids in the relation (2.6), we get:

$$S_{ABC} = \frac{1}{2}\left\|\left[(x_1 - x_2)(y_1 + y_2) + (x_2 - x_3)(y_2 + y_3)(x_3 - x_1)(y_3 + y_1)\right]\right\|$$

The formula (2.5) in terms of determinants can be also expressed in the form

$$S_{ABC} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm\frac{1}{2}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} \qquad (2.7)$$

Remark 1.3

We assume that the area of the triangle is positive. This is why we take the sign ‘plus’ if the value of the determinant is positive and the sign‘minus’ if it is negative.

Example 1.5: Find the area of the triangle whose vertices are $A(1,1); B(6,4); C(8,2)$

Solution: Applying the formula (2.5), we have

$$S_{ABC} = \frac{1}{2}\left|\left[(1-6)(1+4) + (6-8)(4+2) + (8-1).(1+2)\right|\right. = \frac{1}{2}|-16| = 8$$

Remark 1.4: The formula (2.5) can be generalized in the case of a polygon whose the coordinates of its vertices are known.The area of a polygon whose vertices are $P_1(x_1, y_1), P_2(x_2, y_2), \cdots, P_n(x_n, y_n)$ is given by the formula:

$$A = \frac{1}{2}\left\|[(x_1 - x_2)(y_1 + y_2) + (x_2 - x_3)(y_2 + y_3) + \cdots + (x_n - x_1)(y_n + y_1)]\right\|$$

(2.8)

☞ **Remark 1.5**: Three points $A(x_1, y_1), B(x_2, y_2)C(x_3, y_3)$ are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \qquad (2.9)$$

Section 2.2: Straight line

2.2.1 The straight line and its equations

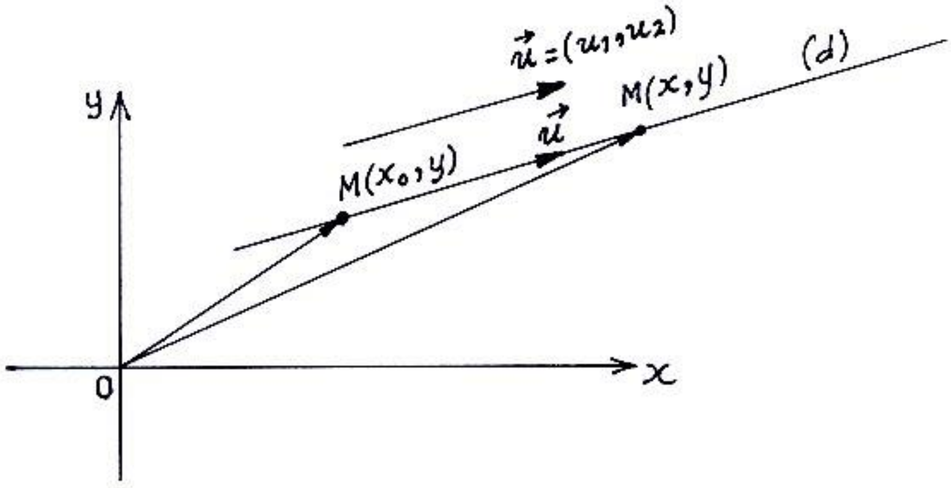

Figure 425

Any non zero vector $\vec{u} \neq \vec{0}$ collinear to a straight line (d) is called the **direction vector** of this straight line. Suppose the straight line (d) passes through the point $M_0(x_0, y_0)$. Let us consider any point $M(x, y)$ lying on the line (d). Since the points M_0, M lie on (d), the vector $\overrightarrow{M_0M}$ is collinear to the vector $\vec{u}$ (figure 2.5). Consequently, there is a real number λ, such that:

$$\overrightarrow{M_0M} = \lambda\vec{u} \qquad (2.10)$$

Conversely, any point M atisfying the relation (2.10) lies on the line (d)

Let $\vec{r}_0$ and $\vec{r}$ be the radii vectors of the points M_0 and M respectively. The equation (2.10) takes the form $\vec{r}-\vec{r}_0=\lambda\vec{u}$, which can be written as

$$(d): \vec{r}=\vec{r}_0+\lambda\vec{u}, \qquad (2.11)$$

The equations (2.10) or (2.11) are called **vector equations** of the straight line (d). If (u_1, u_2) are the co-ordinates of the vector $\vec{u}$, then the equation (2.11) in co-ordinates form is written:

$$(d): \begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}+\lambda\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad (2.12)$$

The equations (2.3) are called **parametric equations** of the line (d) passing through the point $M(x_0, y_0)$ in the direction $\vec{u}$. Excluding the parameter λ in (2.12), we get:

$$(d): \frac{x-x_0}{u_1}=\frac{y-y_0}{u_2} \qquad (2.13)$$

Equation (2.13) is called the **canonical equation** of the straight line (d)

The equation (2.13) can be written in the following form:

$$u_2x-u_1y-u_2x_0+u_1y_0=0$$

Putting $A=u_2, B=-u_1, C=-u_2x_0+u_1y_0$, we obtain

$$(d): Ax+By+C=0 \qquad (2.14)$$

Equation (2.14) is called **the general equation** of the straight line (d)

If in the equation (2.14), $B\neq 0$, then we can write

$$(d): y=-\frac{A}{B}x-\frac{C}{B} \text{ or } (d): y=kx+m \qquad (2.15)$$

The equation (2.15) is studied in secondary school and is called **a slope-intercept form**.

The coefficient k is called the slope of the straight line (d).The vector, parametric and analytic equations of the straight line (d) passing through the points $P(x_1, y_1); Q(x_2, y_2)$ are given respectively by the formulas:

$$(d): \vec{v}=P+\lambda\overrightarrow{PQ} \qquad (2.16)$$

$$(d): \begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}+\lambda\begin{bmatrix} x_2-x_1 \\ y_2-y_1 \end{bmatrix} \qquad (2.17)$$

$$(d):\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = \begin{vmatrix} x-x_1 & y-y_1 \\ x_2-x_1 & y_2-y_1 \end{vmatrix} = 0 \qquad (2.18)$$

The equation of the straight line (d) intersecting the coordinate axes $OX;OY$ at the points $A(a,0);B(0,b)$ respectively Is given by:

$$(d):\frac{x}{a}+\frac{y}{b}=1 \qquad (2.19)$$

This equation is called **the intercept equation of a straight line**.

2.2.2 Relative positions of two straight lines

In the plane, two lines have a common point or do not have a common point or coincide.

Proposition 2.1

Two lines $(d_1): A_1x+B_1y+C_1=0;\ (d_2): A_2x+B_2y+C_2=0$ coincide if and only if $\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}$.

Proof:

Necessity: The vectors $\vec{u}(-B_1,A_1)$ and $\vec{v}(-B_2,A_2)$ are direction vectors of the straight lines (d_1) and (d_2) respectively. Since the two lines coincide, their associated direction vectors are collinear. Thus there is a real number λ, such that

$$(-B_1,A_1)=\lambda(-B_2,A_2) \qquad (2.20)$$

If the point $M_0(x_0,y_0)\in(d_1)=(d_2)$, then $A_1x_0+B_1y_0+C_1=0;$

$A_2x_0+B_2y_0+C_2=0$

Multiplying the second equation by λ and subtracting it from the first equation and using (2.20), we get:

$$C_1-\lambda C_2=0 \qquad (2.21)$$

The relations (2.20) and (2.21) are equivalent to

$$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2} \qquad (2.22)$$

Sufficiency: From (2.21), it follows that $A_1x+B_1y+C_1=\lambda(A_2x+B_2y+C_2)$ for some λ, and this means that the equations $A_1x+B_1y+C_1=0$ and $A_2x+B_2y+C_2=0$ are equivalent.

Proposition 2.2: Two straight lines $(d_2): A_2x+B_2y+C_2=0$ and $(d_1): A_1x+B_1y+C_1=0$ are parallel if and only if $\frac{A_1}{A_2}=\frac{B_1}{B_2}\neq\frac{C_1}{C_2}$

$$\tag{2.23}$$

Proof

Necessity: The necessity condition results directly from the colinearity of the vectors $\vec{u}(-B_1,A_1)$ and $\vec{v}(-B_2,A_2)$ of the straight lines (d_1) and (d_2) and from the proposition (2.23)

Sufficiency: The first condition of (2.23) implies the colinearity of the vectors $\vec{u}$ and $\vec{v}$ and the second condition of (2.23) joined to proposition 2.10 show that the lines (d_1) and (d_2) do not coincide.

From propositions 2.10 and 2.11, we can conclude that the lines $(d_1): A_1x+B_1y+C_1=0$ and $(d_2): A_2x+B_2y+C_2=0$ intersect if and only if

$$\frac{A_1}{A_2}\neq\frac{B_1}{B_2} \tag{2.24}$$

2.2.3 Angle between two straight lines

The angle between two straight lines $(d_1): A_1x+B_1y+C_1=0$ and $(d_2): A_2x+B_2y+C_2=0$ is simply the angle between their respective direction vectors $\vec{u}(-B_1,A_1)$ and $\vec{v}(-B_2,A_2)$ .We have:

$$\cos(d_1,d_2)==\frac{A_1A_2+B_1B_2}{\sqrt{A_1^2+B_1^2}.\sqrt{A_2^2+B_2^2}} \tag{2.25}$$

From (2.25), it follows that (d_1) and (d_2) are perpendicular iff

$$A_1A_2+B_1B_2=0 \tag{2.26}$$

Remark 2.1: If $(d_1): y=k_1x+m_1$ and $(d_2): y=k_2x+m_2$, Then,

$$\tan(d_1,d_2)=\frac{k_2-k_1}{1+k_1k_2} \tag{2.27}$$

and (d_1) is perpendicular to (d_2) if $k_1k_2=-1$. (2.28)

2.2.4 Pencil of straight lines

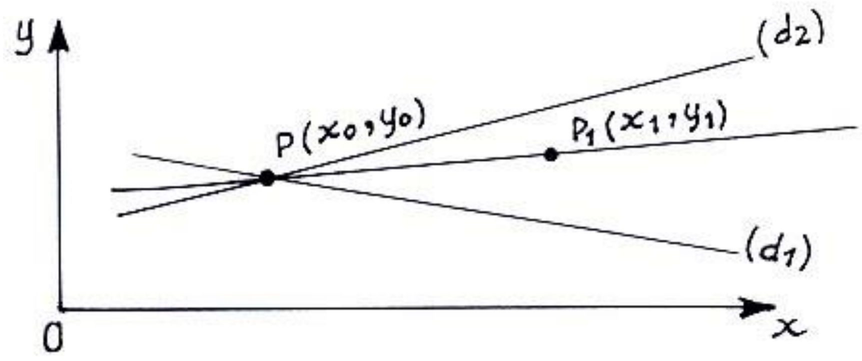

Figure 2.6

The collection of all straight lines in the plane passing through a point $P(x_0, y_0)$ is called the **pencil of all** straight lines centred at the point V called the vertex.

Let $(d_1): A_1x + B_1y + C_1 = 0; (d_2): A_2x + B_2y + C_2 = 0$) be two nonparallel lines and $P_0(x_0, y_0)$ be heir intersecting point. The general equation of the straight lines passing through $P_0(x_0, y_0)$ is given by:

$$A_1x + B_1y + C_1 + \lambda(A_2x + b_2yC_2) = 0 \qquad (2.29)$$

where the parameter λ varies from $-\infty$ to $+\infty$.

Injecting $\lambda = 0$ into (2.29) we get (d_1). Dividing both sides of (2.29) by λ and evaluating the limit of the expressionobtained when $\lambda \to \infty,$ we get (d_2) . The lines (d_1) and (d_2) are fundamental straight lines of pencil centred at (x_0, y_0).

Equation (2.29) is the equation of the firstdegree in x and represents a straight line. Moreover, this straight line passes through $P_0(x_0, y_0)$ since the co-ordinates (x_0, y_0) satisfy both sides of (2.29) by hypothesis. Conversely, let consider an arbitrary point $P_1(x_1, y_1)$ not lying on the fundamental lines of the pencil. It can be shown that the straight line P_0P_1 admits an equation of the type (2.29).

Injecting the co-ordinates (x_1, y_1) in (2.29) and then isolating the corresponding value λ_1 of λ , we have:

$$\lambda_1 = -\frac{Ax_1 + By_1 + C_1}{Ax_2 + By_2 + C_2}$$

The equation $A_1x+B_1y+C_1+\lambda_1(A_2x+B_2y+C_2)=0$ represents a straight line which passes through the points P_0 and P_1 because of λ_1. It is the equation of line P_0P_1. Thus, any straight line passing through the common intersecting point of (d_1) and (d_2) has an equation of the form (2.29)

2.2.5 Distance from a point to a straight line

The distance from the point M_0 to the line (d) is the length of the perpendicular M_0M_1, dropped from this point onto the line (d)

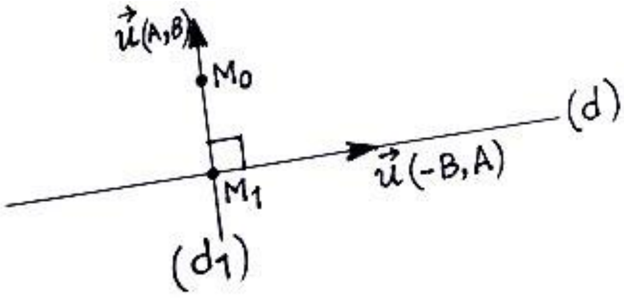

Figure 2.7

Proposition 2.3

The distance from the point $M_0(x_0, y_0)$ to the straight line $(d): Ax+By+C=0$ is expressed by the formula bellow:

$$d(M_0,(d))=\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}} \tag{2.30}$$

Proof

The vector $\vec{u}(-B,A)$ is a direction vector of the straight line (d).The vector $\vec{n}(A,B)$ is orthogonal to the lineh (d) since $\langle\vec{u},\vec{n}\rangle=-BA+AB=0$

Let us consider a straight line (d_1) passing through M_0 and perpendicular to (d).

The common point M_1 of these two straight lines is the foot of the perpendicular dropped from the point M_0 onto the line (d). The parametric equations of (d_1) are given by

$$(d_1):\begin{cases}x=x_0+\lambda A\\ y=y_0+\lambda B\end{cases} \tag{2.31}$$

Let (x_1, y_1) be the co-ordinates of the point M_1. The co-ordinates of this point are obtained from (2.29) for a certain value λ_1 of the parameter λ i.e. $M_1(x_0+\lambda_1 A, y_0+\lambda_1 B)$. As $M_1 \in (d)$, we have:
$A(x_0+\lambda_1 A)+B(y_0+\lambda_1 B)+C=0$.

From this last relation, the value of λ_1 is given by:

$$\lambda_1 = -\frac{Ax_0+By_0+C}{A^2+B^2} \qquad (2.32)$$

The components of the vector $\overrightarrow{M_0M_1}$ are $(\lambda_1 A, \lambda_1 B)$ and we have:

$$d(M_0,(d)) = \left\|\overrightarrow{M_0M_1}\right\| = |\lambda_1|\sqrt{A^2+B^2} = \frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}.$$

Remark 2.2: We say that the general equation $Ax+By+C=0$ of a line is normalized if $A^2+B^2=1$.
From the general equation $Ax+By+C=0$ of a line, we can always obtain its **normal equation** by dividing the both sides of the general equation by $\sqrt{A^2+B^2} \neq 0$: $\frac{A}{\sqrt{A^2+B^2}}x+\frac{B}{\sqrt{A^2+B^2}}y+\frac{C}{\sqrt{A^2+B^2}}=0$,

The above equation can also be written as:

$$x\cos\alpha + y\sin\alpha - p = 0 \qquad (2.33)$$

2.2.6 Equations of a line in polar co-ordinates

Let us introduce in a plane a system of polar coordinates with pole at the point O. Suppose given a straight line (d).

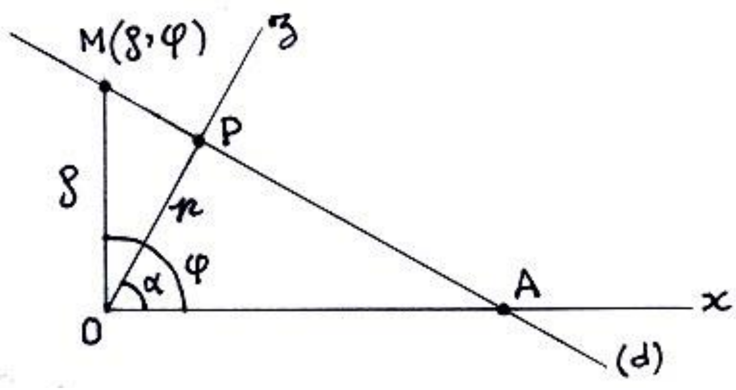

Figure 2.8

Let us draw a perpendicular from the pole to the given line and denote its foot by P **.** Then, we denote the coordinates of the point P by (p,α) and those of the current point M of the given line by (ρ,φ) .

From the triangle OPM , it follows that: $\left\|\overrightarrow{OP}\right\| = \left\|\overrightarrow{OM}\right\|\cos(\varphi-\alpha)$

which is equivalent to

$$\rho = \frac{p}{\cos(\varphi-\alpha)} \qquad (2.34)$$

The converse is also true: any point whose coordinates satisfy equation (2.34) belongs to the given straight line. Equation (2.34) is called the **equation of a straight line in polar coordinates**. This form of the equation of a straight line is readily obtained from its normal equation $x\cos\alpha + y\sin\alpha - p = 0$ by means of the substitution $x = \rho\cos\varphi; y = \rho\sin\varphi$

If the line passes through the pole, it admits the obvious equation

$$\varphi = \alpha \qquad (2.35)$$

Section 2.3: Canonical equations of quadric curves

2.3.1 The circle

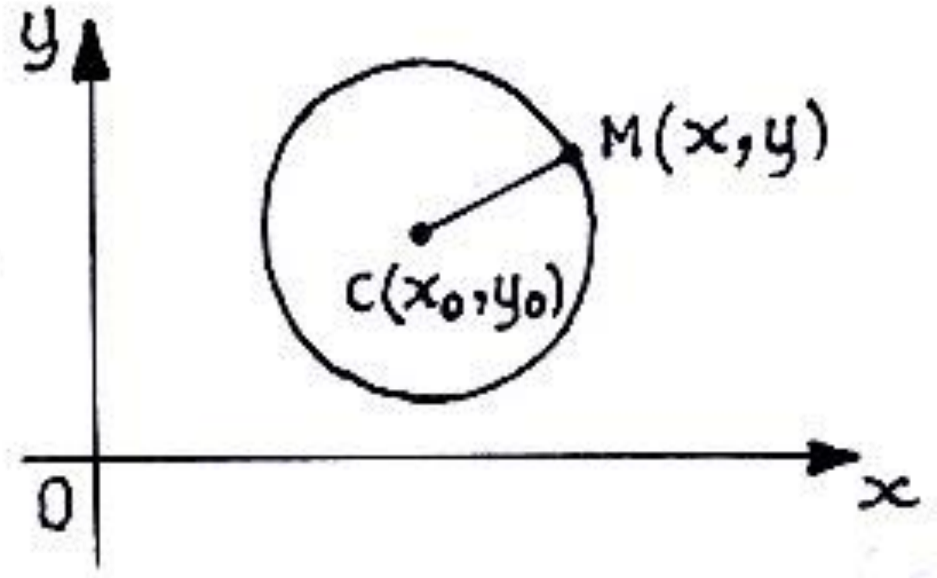

Figure 2.9

Definition 3.1: The circle is the set all points in a plane equidistant from a fixed point called the **centre** (fig 2.9).

Let in the plane the point $C(x_0, y_0)$ be chosen as the centre of the circle.

According to the above definition, the general property of any point $M(x, y)$

of this set can be described as follows: $d(C, M) = \left\|\overline{CM}\right\| = r$.

The number r is called the radius of the circle.Using the formula for the determination of the distance between two points, the equation of the circle of radius R with centre at point $C(x_0, y_0)$ and denoted by: $\Gamma(C(x_0, y_0), r)$ is given by

$$\Gamma(C(x_0, y_0), r) : (x - x_0)^2 + (y - y_0)^2 = r^2 \tag{2.36}$$

(2.36) can also be written in the form:

$$\Gamma(C(x_0, y_0), r) : x^2 + y^2 + 2mx + 2ny + p = 0 \tag{2.37}$$

If the center is at the origin of the axes of the co-ordinates, the equation (2.36) takes the form:

$$\Gamma(C(0,0), R) \equiv x^2 + y^2 = r^2$$

The parametric equations of the circle of radius R with the centre at $C(x_0, y_0)$ are:

$$x = x_0 + r\cos\varphi;\ y = y_0 + r\sin\varphi \tag{2.38}$$

The polar equations of the circle of radius R with the centre at $C\left(\rho_0, \varphi_0\right)$ are:

$$\Gamma\left(C\left(\rho_0, \varphi_0\right), R\right) = \rho^2 + \rho_0^2 - 2\rho\rho_0\cos\left(\varphi - \varphi_0\right) = R^2 \tag{2.39}$$

where $\left(\rho, \varphi\right)$ are the polar co-ordinates of an arbitrary point of the circle.

Intersection of a line and a circle

The intersection of a straight line $(d) : Ax + By + C = 0$ and a circle $\Gamma : x^2 + y^2 + 2mx + 2ny + p = 0$

is obtained by solving the system

$$\begin{cases} Ax + By + C = 0 \\ x^2 + y^2 + 2mx + 2n + p = 0 \end{cases} \tag{2.40}$$

This system leads to a quadratic equation of the form:

$$ax^2 + bx + c = 0 \tag{2.41}$$

obtained by injecting $y = -\frac{A}{B}x - \frac{C}{A}$ into the equation of the circle.

Three scenarii can occur:

1. If (2.41) has two double roots, the line intersectsthe circle in two points
2. If (2.41) has a double root, the line is tangent to the circle.
3. If (2.41) has two conjugated complex roots, the line does not have any real common point with the circle.

Intersection of two circles: common chord

Let $\Gamma_1 : x^2 + y^2 + 2m_1x + 2n_2y + p_1 = 0$ and

$\Gamma_2 : x^2 + y^2 + 2m_2x + 2n_2y + p_2 = 0$

be two non concentric circles. From the equations of the above circles we have:

$$2(m_1 - m_2)x + 2(n_1 - n_2)y + p_1 - p_2 = 0 \qquad (2.42)$$

Equation (2.42) represents a straight line which is the **common chord** to both circles. The common chord always exists even if the two circles are not geometrically secant. In this latter case, the real common chord joins the two complex conjugate intersecting points.

2.3.2 The ellipse

Definition 3.2: An **ellipse** is the set of all points in the plane for each of which the sum of distances from two fixed points in the same plane is constant. The two fixed points are called the **foci**of the ellipse, and the distance between them is called the focal **length**.

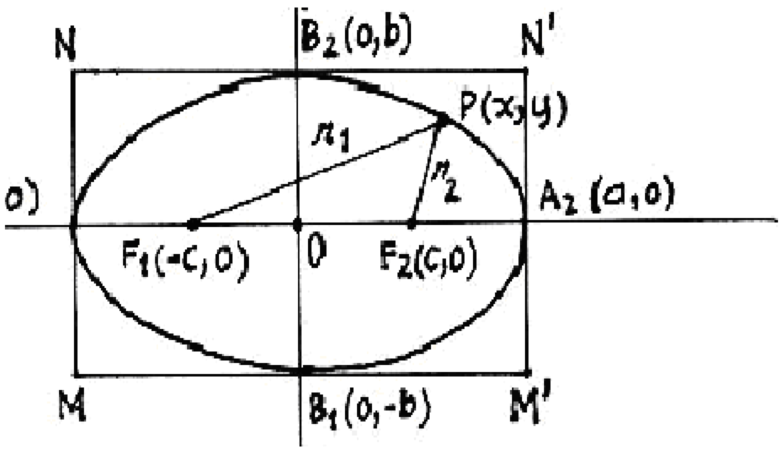

Figure 2.10

Let denote the foci by F_1 and F_2, and the distance between them by $d(F_1, F_2) = \left\|\overrightarrow{F_1F_2}\right\| = 2c$ and let P be an arbitrary point belonging to the ellipse. The distances from point P to the foci F_1 and F_2 are called **focal radii** and are denoted by r_1, r_2 respectively. We have: $r_1 = d(F_1, P); r_2 = d(F_2, P)$.

According to the definition of the ellipse, the sum of focal radii is constant. Denoting it by $2a$, we have:

$$d(F_1, P) + d(F_2, P) = 2a \qquad (2.43)$$

Equality (2.43) is the equation of the ellipse. Let us write this equality in terms of coordinates.

Let take x-axis for the straight lines F_1F_2 (Fig 2.10)

Let $P(x, y)$ be an arbitrary point of the ellipse and $F_1(-c, 0)$; $F_2(c, 0)$ be its foci. From the triangle F_1F_2P we deduce: $a > c$. Analytically, (2.43) is expressed by: $\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$

Writing the above equation as: $\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$

and squaring both sides, we obtain

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

which simplifies to: $a\sqrt{(x+c)^2 + y^2} = a^2 + cx$

Squaring both sides gives: $a^2(x^2 + 2cx + c^2 + y^2) = a^4 + x^2 + 2a^2cx + c^2x^2$

which may be written in the form: $(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$

Since $a > c > 0$, introducing $b^2 = a^2 - c^2$

The above equation becomes: $b^2x^2 + a^2y^2 = a^2b^2$

Dividing both sides by a^2b^2 leads to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (2.44)$$

The equation (2.44) is called **canonical** or **standard equation** of the ellipse.

The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be also been written in the form:

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2} \qquad (2.45)$$

The curve representing this equation is found inside the rectangle $MM'N'N$ with base $2a$ and height $2b$

The segments $A'A$ and $B'B$ are called **major** and minor **axes** respectively**.**

The numbers a and b are called the **semi axes** of the ellipse.

The points $A'(-a,0), A(a,0); B'(0,-b), B(0,b)$ are the **vertices.**

The point $O(0,0)$ is the **centre** of the ellipse. The ellipse is symmetric to both x-axis and y-axis. It is also symmetric with respect to the origin.

The numbers $e=\frac{c}{a}$ and $2p=2\frac{b^2}{a}$ are called **eccentricity** and **latus rectum**respectively.

The eccentricity of ellipse is responsible for its shape.

The lines $(d_1): x=-\frac{a}{e}$ and $(d_2): x=\frac{a}{e}$ are **directrices** of the ellpse.

Remark 3.1: For a circle: $a=b, e=0$, the foci coincide at the centre of the circle, the directrices are at infinity.

Let us determine the rational expressions of the focal radii of the ellipse. The length r_1 can be found using the following relation:

$$\begin{aligned} r_1^2 &= (x+c)^2+y^2 \\ &= (x+c)^2+b^2\left(1-\frac{x^2}{a^2}\right) \\ &= \left(1-\frac{b^2}{a^2}\right)x^2+2cx+c^2+b^2 \\ &= \frac{c^2}{a^2}x^2+2eax+a^2=(ex+a)^2 \end{aligned}$$

Since $|x|\le a$, we have $|ex|<a$ and $r_1=a+ex$

Similarly $r_2=a-ex$. Finally we have

$$\begin{cases} r_1 &= a+ex \\ r_2 &= a-ex \end{cases}$$

This relation leads to: $r_1+r_2=2a$

The **parametric equations** of the ellipse $E:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are given by:

$$\begin{cases} x = a\cos\varphi \\ y = b\sin\varphi \end{cases} \tag{2.46}$$

$$\begin{cases} x = a\dfrac{1-t^2}{1+t^2} \\ y = b\dfrac{2t}{1+t^2} \end{cases} \tag{2.47}$$

The tangent to the ellipse $E:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $T(x_0, y_0)$ is expressed by the equation:

$$\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1 \tag{2.48}$$

2.3.3 Hyperbola

Definition 2.2. A **hyperbola** is the set of all points in the plane for each of which the absolute value of the difference of distances from two fixed points in the same plane is constant.

The two fixed points are called the **foci** of the hyperbola, and the distance between them is called the **focal length**.

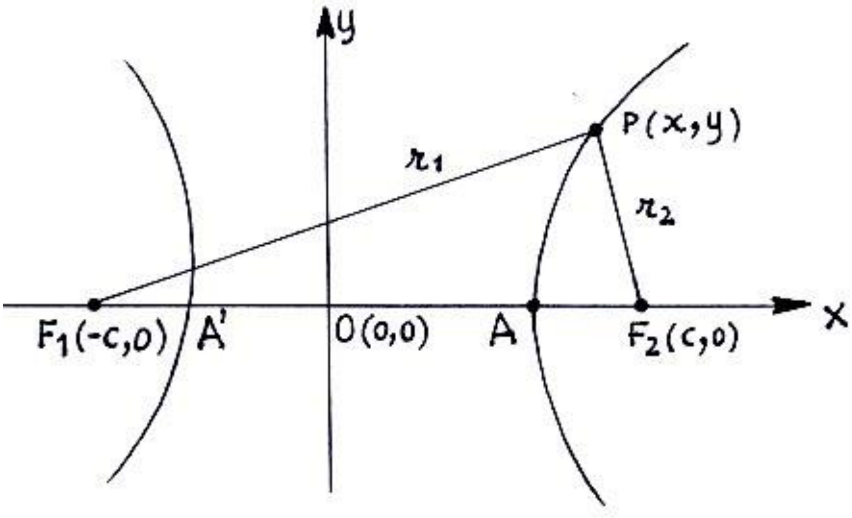

Figure 2.11

Let denote the foci by F_1 and F_2, and the distance between them by $d(F_1, F_2)=\left\|\overrightarrow{F_1F_2}\right\|=2c$ and let P be an arbitrary point belonging to the hyperbola. The distances from the point P to the foci F_1 and F_2 are called **focal radii** and are denoted by r_1, r_2 respectively. We have:

$r_1=d(F_1, P); r_2=d(F_2, P)$.

According to the definition of the hyperbola, the absolute value of their difference is constant.

Denoting it by $2a$, we have:

$$\left| \; d(F_1,P) - d(F_2,P) \; \right| = 2a \qquad (2.49)$$

Equality (2.49) is the equation of the hyperbola. Let us write this equality in terms of coordinates. Let us take x-axis for the straight lines F_1F_2 (Fig 2.11)

Let $P(x,y)$ be an arbitrary point of the hyperbola and $F_1(-c,0)$; $F_2(c,0)$ be its foci. From the triangle F_1F_2P we deduce: $a < c$. Analytically (2.49) is expressed by:

$$\left|\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2}\right| = 2a$$

$$\left| \; \sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} \; \right| = 2a$$

Writing the above equation as: $\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = \pm 2a$

Transposing one radical: $\sqrt{(x+c)^2+y^2} = \sqrt{(x-c)^2+y^2} \pm 2a$

Squaring and collecting terms: $cx - a^2 = a\sqrt{(x-c)^2+y^2}$

Squaring and simplifying: $(c^2-a^2)x^2 - a^2y^2 = a^2(c^2-a^2)$

Since $a > c > 0$, for convenience let $b^2 = a^2 - c^2$

The above equation becomes: $b^2x^2 + a^2y^2 = a^2b^2$

Dividing both sides by a^2b^2 leads to

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (2.50)$$

The equation (2.48) is called the **canonical** or **standard equation** of the hyperbola.

The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be also be written in the form

$$y = \pm\frac{b}{a}\sqrt{x^2-a^2} \Leftrightarrow x = \pm\frac{a}{b}\sqrt{y^2+b^2} \qquad (2.51)$$

From these equations, we can deduce that the hyperbola has two axes of symmetry: one axisintersects the curve and is called the **transverse (real**) axis of symmetry. The other does not intersect the hyperbola and is called non-transverse or as the **conjugate (imaginary**) axis of symmetry.

The points $A_1(-a,0)$ and $A_2(a,0)$ are called **vertices.** The numbers a and b are called semi transverse and semi conjugate axis of the hyperbola.The point $O(0,0)$ is called the **centre** of the hyperbola and it is the point of intersection of the axes of symmetry. The hyperbola has two branches: the points for which $x \geq a$ form the **right-hand branch**, the points for which $x \leq -a$ form the **left-hand branch.**

The lines $A_1O: y = \frac{-b}{a}x$ and $A_2O: y = \frac{b}{a}x$ are called the **asymptotes** of the hyperbola.

The number $e = \frac{c}{a}$ is the **eccentricity** of the hyperbola $(e > 1)$. The number $2p = 2\frac{b^2}{a}$ is called the **lactus rectum**. The straight lines of the equations $(d_1): x = -\frac{a}{e}$ and $(d_2): x = \frac{a}{e}$ are the **directrices**.

Remark 3.2: If $a = b$, the hyperbola is called **equilateral hyperbola** or **the equiangular hyperbola**.

The epressions of focal radii of the hyperbola can be obtained in similar way as the case of the ellipse. For the hyperbola, we have:
$r_1^2 = (ex+a)^2; r_2^2 = (ex-a)^2$. Since $|ex| > |x| \geq a$, we have

$$\begin{cases} r_1 = |a+ex| = \begin{cases} a + ex, & x > 0 \\ -a - ex, & x < 0 \end{cases} \\ r_2 = |ex-a| = \begin{cases} -a + ex, & x > 0 \\ a - ex, & x < 0 \end{cases} \end{cases} \tag{2.52}$$

From (2.52) we have: $|r_1 - r_2| = 2a$

The parametric equations of the hyperbola $H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are given by:

$$\begin{cases} x = a ch\varphi \\ y = b sh\varphi \end{cases} \tag{2.53}$$

or

$$\begin{cases} x = a \sec t \\ y = b tg t \end{cases} \tag{2.54}$$

The tangent to the hyperbola $H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $T(x_0, y_0)$ is given by the equation

$$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1 \qquad (2.55)$$

2.3.4 Parabola

Definition 3.4: The parabola is the set of all points in the plane each of which is equidistant from a given point and a given straight line not passing through the given point .The given point is called **focus** and denoted by F and the given straight line is the **directrix** of the parabola and is denoted by (d).

The distance from the focus to the directrix is the **focal parameter**

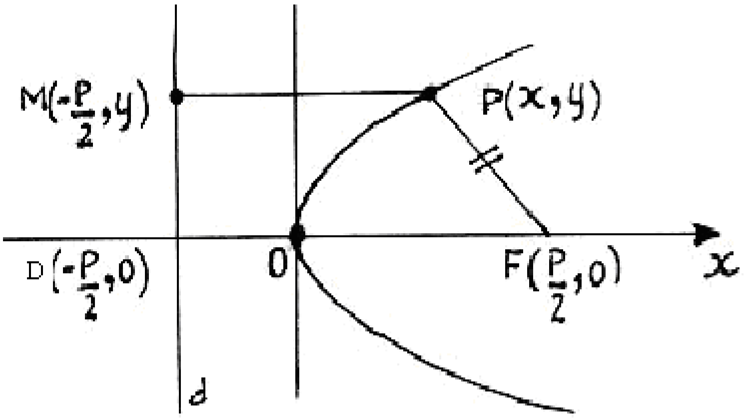

Figure 2.12

In order to obtain the equation of the parabola in a simple form, a system of coordinates should be chosen in the following way: the x-axis is drawn through the focus F perpendicular to the directrix d . The point of intersection of the x-axis and directrix will be denoted by D**.** For the coordinate origin, we take the midpoint O of the line segment DF . In the chosen system, the focus has coordinates $F\left(\frac{p}{2},0\right)$ and the equation of the directix will be $d : x + \frac{p}{2} = 0$.

Let $P(x, y)$ be an arbitrary point of the parabola. From this point, we drop a perpendicular onto the directrix d , and let M be the foot of this perpendicular.Then,

$$d(P,M) = \left|x + \frac{p}{2}\right| \text{and}\, d(P,F) = \sqrt{(x - \frac{p}{2})^2 + y^2}$$

The point $P(x, y)$ belongs to the parabola if $d(P,M) = d(P,F)$

$$\text{or}\left|x + \frac{p}{2}\right| = \sqrt{(x - \frac{p}{2})^2 + y^2}$$

Squaring both sides of the above equality and simplifying, we have

$$y^2 = 2px \tag{2.58}$$

The equation (2.58) is called **the canonical** or **the standard equation** of the parabola. It can also be written as:

$$y = \pm\sqrt{2px} \tag{2.59}$$

All the points of the parabola $y^2 = 2px$ lie on the right of y-axis.

x- axis is the axis of symmetry and is called the **focal axis**.The point $O(0,0)$ is the **vertex** of the parabola. The number p is called the **parameter of the parabola**: it regulates its opening.

The distance $d(O,F) = \dfrac{p}{2}$ is the focal distance.

The number expressed by: $e = \dfrac{d(F,M)}{d(M,d)} = 1$ is called **eccentricity** of the parabola.

The parametric equations of the parabola $y^2 = 2px$,are written as:

$$\begin{cases} x = \dfrac{t^2}{2p} \\ y = t \end{cases} \tag{2.60}$$

or

$$\begin{cases} x = 2p\cot^2\varphi \\ y = 2p\cot\varphi \end{cases} \tag{2.61}$$

The equation of the tangent to the parabola $y^2 = 2px$ at the point $T(x_0, y_0)$ is given by:

$$yy_0 = p(x + x_0) \tag{2.62}$$

2.3.5 Focus-directrix- eccentricity property of conics

Theorem 3.1: Let r_i be the distance from an arbitrary point $P(x, y)$ to one of the foci F_i of the ellipse $E: \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and d_i the distance from the same point to the corresponding directrix. Then, the ratio $\frac{r_i}{d_i}$ is a constant and equals the eccentricity e of the ellipse : $\frac{r_i}{d_i} = e$; $i = 1, 2$.

Proof. Consider the left focus $F_1(-c, 0)$ and its corresponding directrix.

$(d_1): x = -\frac{a}{e}$

Let $P(x, y)$ bean arbitrary point of the ellipse. The distancesfrom the point P to the directrix (d_1) and its associated focus F_1 are given respectively by:

$$d_1 = d(P,(d_1)) = x + \frac{a}{e}; \;\; r_1 = a + ex \qquad (2.63)$$

From (2.63) we obtain : $\frac{r_1}{d_1} = \frac{a+ex}{x+\frac{a}{e}} = e$

If we consider the right focus and the corresponding directrix, we will find the same results.

Theorem 3.2

Let r_i be the distancefrom arbitrary point $P(x, y)$ to one of the foci F_i of the hyperbola $H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and d_i the distance from the same point toits corresponding directrix (d_i) . Then ratio $\frac{r_i}{d_i}$ is a constant and equals to the eccentricity e of the hyperbola: $\frac{r_i}{d_i} = e$; $i = 1, 2$.

Proof. Two cases are considered:

The point P lies on the right branch of the hyperbola.

The distances from P to the directrix (d_2) and its associated focuss $F_2(c, 0)$ are given respectively by:

$$d_2 = d(P,(d_2)) = x - \frac{a}{e}; \; r_2 = ex - a \qquad (2.64)$$

From (2.64) we get: $\frac{r_2}{d_2} = \frac{ex-a}{x-\frac{a}{e}} = e$

The point P lies on the leftt branch of the hyperbola.

The distances from P to the directrix (d_1) and its associated focuss $F_1(-c,0)$

are

given respectively by:

$$d_1 = d(P,(d_1)) = -x + \frac{a}{e};\ r_1 = -ex + a \tag{2.65}$$

From (2.65) we get: $\frac{r_1}{d_1} = \frac{-ex+a}{-x+\frac{a}{e}} = e$

Remark 3.3

The ellipse, hyperbola and the parabola can be defined in terms of focus, directrix and eccentricity as the set all points in the plane for which the ratio of the distance r from any arbitrary point of the set and a certain fixed point (focus) to the distance d from the same point of the locus to a certain fixed line (directrix) is a constant value (eccentricity):

$$e = \frac{r}{d}\ . \tag{2.66}$$

If $0 < e < 1$, we have **an ellipse**; if $e > 1$, we have **a hyperbola**and if $e = 1$, we have**a parabola**.

2.3.6 The equations of conics in polar coordinates

Let denote Γ denote onene of three conics: ellipse, hyperbola and parabola.

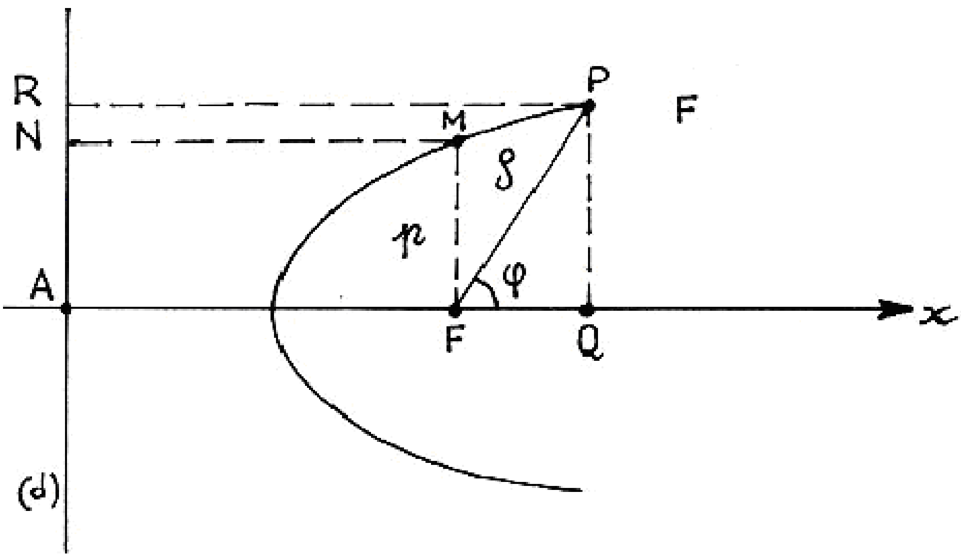

Figure 2.13

In the case of hyperbola, one of its branches will be considered.

Let F be the focus of the conic and (d) its corresponding directrix. In the case of the hyperbola, we will consider the focus and the directrix which are closer to the chosen branch.

Let introduce the polar coordinates (ρ, φ) in such a way the pole coincides with the focus F and the polar axis be perpendicular to the directrix (d) (fig 2.13). To obtain the polar equations of the conic Γ we have to use the relation: $\frac{r}{d} = e$, where r is the focal radius and d the distance from any point of the conic to the directrix. Since the pole coincides with the focus we have:

$$r = \rho \qquad (2.67)$$

From fig (2.13) we have:

$$d = RP = AQ = AF + FQ = AF + \rho\cos\varphi \qquad (2.68)$$

(R is the projection of P on the directrix (d)).

Let M be a point of the curve Γ such that the segment FM be perpendicular to the axis of the curve Γ and p be the length of FM i.e p is the half of the latus rectum Γ (focal parameter). Using (2.68) which is satisfied for any point of Γ, for the point M, we have:

$$\frac{FM}{NM} = e \Leftrightarrow NM = \frac{FM}{e} = \frac{p}{e}$$

(N is the projection of M on the directrix (d)).

Considering that $NM = AF$, we have $AF = \frac{p}{e}$.

The relations (2.67) and (2.68) lead to:

$$d = \frac{p}{e} + \rho\cos\varphi \qquad (2.69)$$

Replacing in (2.66) r and d by their expressions given by (2.67) and (2.69) we get:

$$\frac{\rho}{\frac{p}{e} + \rho cos\varphi} = e\,.$$

This last equality can be written in thelowing form

$$\rho = \frac{p}{1 - ecos\varphi} \tag{2.70}$$

Equation (2.70) is called the **equation of conics in polar coordinates**.

p is the focal parameter and e the eccentricity.

Section 2.4: General equation of quadric curves

2.4.1 Simplification of general equation

In a system of rectangular Cartesian coordinates oxy eof the plane endowed with an orthonormal frame $F(O;\overrightarrow{e_1},\overrightarrow{e_2})$, the general equation of a the second order curve is described by

$$\Gamma(x,y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0 \quad (2.71)$$

where, $a_{11}^2 + a_{12}^2 + a_{22}^2 \neq 0$ .In terms of quadratic and linear forms, the equation (2.71) can be written

$$\Gamma(x,y) = q(x,y) + 2l(x,y) + a_{33} = 0 \qquad (2.72)$$

Where, $q(x,y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2$ and $l(x,y) = a_{13}x + a_{22}y$ are quadratic and linear forms associated with the quadric curve. In matrix notations, (2.71) takes the form:

$$\Gamma(x,y) = X^T AX + 2BX + a_{33} = 0 \qquad (2.73)$$

where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$ is the matrix of the quadratic form $q(x,y)$ and $B = \begin{bmatrix} a_{13} & a_{23} \end{bmatrix}$, $X = \begin{bmatrix} x & y \end{bmatrix}^T$ or in compact form

$$\Gamma(x,y) = \hat{X}^T \hat{A}\hat{X} = 0 \qquad (2.74)$$

Where, $\hat{X} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ and $\hat{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

Introducing a new coordinates system $O'x'y'$ associated with the orthonormal frame $F'(O',e_1',e_2')$, the systems x,y and x',y' are connected by:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \qquad \text{or}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_1 \\ p_{21} & p_{22} & p_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \qquad (2.75)$$

where $P=\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ and $\overrightarrow{OO'}=p_1\overrightarrow{e_1}+p_2\overrightarrow{e_2}$. The matrix P is the transition matrix from the basis $\{\vec{e}_1,\vec{e}_2\}$ to the basis $\{e'_1,e'_2\}$ and is orthogonal.
Geometrically it describes a rotation or a rotation followed by a reflection in direction of the vector $\overrightarrow{e_1}$ and can be written in one of two following forms:

$$\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \text{or} \begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix}$$

The transformation $\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} x' \\ y' \end{bmatrix}+\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ represents geometrically the translation of axes of coordinates from the old origin to the new one. We observe that the translation does not affect the linear part of the quadric curve.

Passing from the system Oxy to the system $Ox'y'$ under the rotation through the angle φ

$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$, the general equation (2.71) of the curvein

thenew system is written as

$$\Gamma(x',y'): a'_{11}x'^2+2a'_{12}x'y'+a'_{22}y'^2+2a'_{13}x'+2a'_{23}y'+a'_{33}=0 \qquad (2.76)$$

The matrix $A'=\left[a_{i'j'}\right]_{2\times 2}$ of the quadratic part of $\Gamma(x',y')$ is given by:

$$A'=\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

Therefore, we have:

$$a'_{11}=a_{11}\cos^2\varphi+2a_{12}\cos\varphi\sin\varphi+a_{22}\sin^2\varphi$$

$$a'_{22}=a_{11}\sin^2\varphi-2a_{12}\cos\varphi\sin\varphi+a_{22}\cos^2\varphi$$

$$a'_{12}=(a_{22}-a_{11})\cos\varphi\sin\varphi+a_{12}(\cos^2\varphi-\sin^2\varphi)$$

$$a'_{13}=a_{13}\cos\varphi+a_{23}\sin\varphi\,;\; a'_{13}=a_{13}\sin\varphi+a_{23}\cos\varphi$$

$$a'_{33}=a_{33}$$

Assuming $a_{12}\neq 0$, then choosing the angle φ:

$$tg2\varphi = \frac{2a_{12}}{a_{11} - a_{22}} \tag{2.77}$$

We obtain a system of coordinates in which the matrix A' is diagonal i.e $A' = \begin{bmatrix} a'_{11} & 0 \\ 0 & a'_{22} \end{bmatrix}$ and

$\Gamma(x', y')$ is written as

$$\Gamma(x', y') = a'_{11}x'^2 + a'_{22}y'^2 + 2a'_{13}x' + 2a_{23}y' + a'_{33} = 0 \tag{2.78}$$

2.4.2 Classification of the second order curves

Theorem 4.1

The general equation of a quadric curve can be reduced to the one of the following nine forms:

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$: Real ellipse
2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$: Imaginary ellipse
3. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$: Conjugate complex intersecting lines
4. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$: Hyperbola
5. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$: Real intersecting lines
6. $y^2 = 2px$: Parabola
7. $y^2 - k^2 = 0$: Real distinct parallel lines
8. $y^2 + k^2 = 0$: Conjugate complex parallel lines
9. $y^2 = 0$: Coincident real lines

Proof: Passing from the system of coordinates Oxy to the system $Ox'y'$ under he rotation through the angle φ defined by (2.77), the general equation of the curve (2.71) is transforming into the equation

$$\Gamma(x', y') = a'_{11}x'^2 + a'_{22}y^2 + 2a'_{13}x' + 2a'_{23}y' + a'_{33} = 0 \tag{2.79}$$

Therefore, this rotation eliminates the cross term of the quadratic part of Γ

Two scenarios can ocur:

Scenario 1: $a'_{11} \cdot a'_{22} \neq 0$

The equation $a'_{11}x'^2 + a'_{22}y^2 + 2a'_{13}x' + 2a'_{23}y' + a'_{33} = 0$

can be written: $a'_{11}\left(x' + \dfrac{a'_{13}}{a'_{11}}\right)^2 + a'_{22}\left(y' + \dfrac{a'_{23}}{a'_{22}}\right)^2 + a'_{33} - \dfrac{a'^2_{13}}{a'_{11}} - \dfrac{a'_{23}}{a'_{22}} = 0$

The translations of coordinates axes by: $x'' = x' + \dfrac{a'_{13}}{a'_{11}}$; $y'' = y' + \dfrac{a'_{23}}{a'_{22}}$ leads to:

$$a''_{11}x''^2 + a''^2_{22}y''^2 + a''_{33} = 0 \qquad (2.80)$$

where $a''_{22} = a'_{22}$ $a''_{11} = a'_{11}$; $a''_{33} = a'_{33}$

The equation (2.80) represents one of the equations (1) - (5) of the theorem (4.1).

Scenario 2: $a'_{11} \cdot a'_{22} = 0$; $a'^2_{11} + a'^2_{22} \neq 0$

Without loosing the generality, lett $a'_{11} = 0$

The translation given by: $x'' = x'$; $y'' = y' + \dfrac{a'_{23}}{a'_{22}}$; transform our equation into the form

$$a''^2_{22}y''^2 + 2a''_{13}x'' + a''_{33} = 0 \qquad (2.81)$$

If in (2.81); $a''_{13} \neq 0$, performing the transformations $x''' = x'' + \dfrac{a''_{33}}{2a_{13}}$ we obtain

$$a'''_{22}y'''^2 + a'''_{13}x''' = 0 \qquad (2.82)$$

The equation (2.81) stands for the equation (6) in theorem 4.1

If in (2.22) $a''_{13} = 0$, then (2.81) becomes

$$a''_{22}y''^2 + a''_{33} = 0 \qquad (2.83)$$

The equation (2.83) represents one of the equations (7), (8), and (9) in theorem 4.1.

Remark 4.1

Introducing a new denotation of the axes of the co-ordinates or by changing their orientations, we can assume that: in (1) $-$ (3): $a^2 \geq b^2$**;** in (6): $p > 0$**;** in (7) and (8): $k^2 \neq 0$

The equations (1) – (9) are canonical or standard equations of the quadric curves.

2.4.3 Orthogonal invariants

Let $\Gamma(x,y) = \hat{X}^T \hat{A}\hat{X} = 0$ (4.4) be the general equation of quadric curve in matrix notations, $\Delta = \det \hat{A}$ and $\delta = \det A$.

Passing from the coordinates system Oxy to the s coordinates system $O'x'y'$ via the matrix

$$\hat{P} = \begin{bmatrix} p_{11} & p_{11} & p_1 \\ p_{21} & p_{22} & p_2 \\ 0 & 0 & 1 \end{bmatrix},$$ $\det P = \pm 1$, the equation (4.1) becomes

$\Gamma(x',y') = \hat{X}'^T \hat{A}'\hat{X}' = 0$,and the matrices A' and A' are connected with the matrices A and A by $\hat{A}' = \hat{P}^T \hat{A}\hat{P}$ and $A' = P^T AP$

Proposition 2.1

Under theorthogonal transformation $X' = PX$, the quantities Δ and δ are not modified i.e $\Delta' = \Delta$ and $\delta' = \delta$.

Proof. $\Delta' = \det \hat{A}' = \det(\hat{P}^T \hat{A}\hat{P}) = \det \hat{A} \cdot (\det \hat{P})^2 = \det \hat{A} \cdot (\det P)^2 = \det \hat{A} = \Delta.$

$\delta' = \det A' = \det(P^T AP) = \det A \cdot (\det P)^2 = \det A = \delta.$

Thus the quantities Δ and δ are orthogonal invariants.

Proposition 2.2

The trace of the matrix A, denoted by $S = a_{11} + a_{22}$ is also an orthogonal invariant.

Proof: $A' = \begin{bmatrix} a'_{11} & a'_{12} \\ a'_{12} & a'_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$

$$a'_{22} = p_{12}(p_{12}a_{11} + p_{22}a_{12}) + p_{22}(p_{12}a_{12} + p_{22}a_{22})$$

$$S' = a'_{11} + a'_{22} = a_{11}(p_{11}^2 + p_{12}^2) + a_{22}(p_{21}^2 + p_{22}^2) + 2a_{12}(p_{11}p_{21} + p_{12}p_{22}) = a_{11} + a_{22} = S.$$

Proposition 2.3

The characteristic polynomial of the quadratic part of Γ is an orthogonal invariant.

Proof: Let us denote the characteristic polynomial by $P_A(\lambda)$.

By definition,

$$P_A(\lambda) = \det(A - \lambda I_2) = \lambda^2 - S\lambda + \delta$$

We have,

$$P_{A'}(\lambda) = \det(A' - \lambda I_2) = \det(P^T AP - P^T \lambda P) = \det(P^T (A - \lambda I)P) = \det(A - \lambda I_2) = P_A(\lambda)$$

The invariance of $P_A(\lambda)$ results also from the invariance of the quantities S and δ . The roots λ_1 and λ_2 of the characteristic polynomial $P_A(\lambda)$ are also orthogonal invariants.

Proposition 2.4

Under the rotation, the quantity $K = \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{13} & a_{33} \end{vmatrix}$ is not modified

Proof: The rotation of the axes of the co-ordinates about the origin throughthe angle φ :

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

transforms the coefficients of quadric curve according to the formulas

$$\begin{cases} a'_{13} = a_{13}\cos\varphi + a_{23}\cos\varphi \\ a'_{23} = -a_{13}\sin\varphi + a_{23}\cos\varphi \\ a'_{33} = a_{23} \end{cases} \tag{2.84}$$

Using (4.13) and the invariance of S' , weget:

$$K' = a'_{33}(a'_{11} + a'_{22}) - a'^2_{13} - a'^2_{23} = a_{33}S' - (a_{13}\cos\varphi + a_{23}\sin\varphi)^2 -$$
$$(-a_{13}\sin\varphi + a_{23}\cos\varphi)^2 = a_{33}(a_{11} + a_{22}) - a_{13}^2 - a_{23}^2 = K$$

Remark 2.3

K is not modifiedeither under reflection in one axis wit respect to another. In general under translation K is not invariant and for this reason it is called a semi orthogonal invariant.

Proposition 2.5

If $\Delta = \delta = 0$, then K is an orthogonal invariant.

Proof: We have to prove that passing from the system Oxy to the system $O'x'y'$, K is invariant.Let us introduce two intermediate systems of coordinates $Ox''y''$ and $O'x'''y'''$: The system $Ox''y''$ is obtained from system Oxy under rotation through a certain angle around O and the system $O'x'''y'''$ is obtained from the system $Ox''y''$ translating the origin by the vector $\overrightarrow{OO'}$.From the proposition 4.4, it follows that passing from the system Oxy to the system $Ox''y''$ and from the system $O'x'''y'''$ to the system $O'x'y'$, K remains invariant. We have to prove that during the translation of the system $Ox''y''$ to the system $O'x'''y'''$, K remains invariant. The system $Ox''y''$ is obtained from the system Oxy under the rotation through a certain angle which eliminates the coefficient a_{12} of the general equation of th curve (2.71).In this system, the matrix A of the quadratic part is diagonal i.e. $A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$.Thus, $\delta = \det A = a_{11}a_{12}$. From $\delta = 0$, either $a_{11} = 0$ or $a_{22} = 0$

Assuming $a_{11} = 0$, in the system $Ox''y''$, the matrix $\hat{A}$ has the form:

$$\hat{A} = \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

From $\Delta = \det \hat{A} = 0$, it follows $a_{33} = 0$.Thus, in the system $Ox''y''$, the quadratic part of Γ has the form: $q(x'', y'') = a_{22}y''^2 + 2a_{23}y'' + a_{33}$

Under translation the $x'' = x''' + a; y'' = y''' + b$,

we have: $q(x''', y''') = a_{22}y'''^2 + 2(a_{22}b + a_{22})y''' + a_{22}b^2 + 2a_{23}b + a_{33} = 0$.

In the system $O'x'''y'''$, K has the form

$$K' = \det\begin{bmatrix} a_{22} & a_{23} + a_{22}b \\ a_{23} + a_{22}b & a_{33} + 2a_{23}b + a_{22}b^2 \end{bmatrix} =$$

$$= a_{22}a_{33} + 2a_{22}a_{33}b + a_{22}^2b^2 - a_{23}^2 - 2a_{22}a_{23}b - a_{22}^2b^2 = a_{22}a_{33} - a_{23}^2 = K$$

We conclude that if $\Delta = \delta = 0$, then K is an orthogonal invariant.

2.4.4 Classification of the quadric curves in terms of orthogonal invariants

In simplification of the general equation of the quadric curve by means of rotation and translation we saw that its general equation can take one of three forms:

$$a_{11}x^2 + a_{22}y^2 + a_{33} = 0 \tag{2.85}$$

$$a_{22}y^2 + 2a_{13}x = 0 \tag{2.86}$$

$$a_{22}y^2 + a_{33} = 0 \tag{2.87}$$

The matrix $\hat{A}$ associated with each of the three forms is written as:

$$\text{I. } \hat{A} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}; \text{ II. } \hat{A} = \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{13} & 0 & 0 \end{bmatrix}; \text{ III. } \hat{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

The form I is characterised by: $\delta = a_{11}a_{22} \neq 0$,; $\Delta = a_{11}a_{22}a_{33}$ $\lambda_2 = a_{22}$

The equation (2.85) is written as

$$\Gamma(x, y) : \lambda_1 x^2 + \lambda_2 y^2 + \frac{\Delta}{\delta} = 0 \tag{2.88}$$

The form II is characterised by: $\delta = 0$,; $\Delta = -a_{22}a_{13}^2 \neq 0$; $S = a_{22}$

From the expressions of Δ and S , we get: $a_{13} = \pm\sqrt{\frac{-\Delta}{S}}$

The equation (2.86) can be written as:

$$Sy^2 \pm 2\sqrt{\frac{-\Delta}{S}}x = 0 \Leftrightarrow y^2 = \pm 2\sqrt{\frac{-\Delta}{S^3}}x = 0 \tag{2.89}$$

The form III is characterised by: $\delta = \Delta = 0$.

From the proposition 4.5, we know that if $\delta = \Delta = 0$, and then K is an orthogonal invariant. In our case, $K = a_{22}a_{33}; S = a_{22}$ .Thus $K = a_{33}S$ and the equation (2.87) is written as

$$Sy^2 + \frac{K}{S} = 0 \Leftrightarrow$$

$$y^2 + \frac{K}{S^2} = 0$$

(2.90)

We can summarize all results in the following table:

Various cases	Conditions on the invariants: δ ; Δ ; S ; K			Canonical equation		Cuven0
I $a_{11}x^2 + a_{22}y^2 + a_{33} = 0$	$\delta > 0$	$\Delta \neq 0$	$S\Delta < 0$	$\lambda_1 x^2 + \lambda_2 y^2 + \frac{\Delta}{\delta} = 0$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	(1)
			$S\Delta > 0$	$\lambda_1 x^2 + \lambda_2 y^2 + \frac{\Delta}{\delta} = 0$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$	(2)

		$\Delta = 0$		$\lambda_1 x^2 + \lambda_2 y^2 + \frac{\Delta}{\delta} = 0$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$	(3)
		$\Delta \neq 0$		$\lambda_1 x^2 + \lambda_2 y^2 + \frac{\Delta}{\delta} = 0$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$	(4)
	$\delta < 0$	$\Delta = 0$		$\lambda_1 x^2 + \lambda_2 y^2 + \frac{\Delta}{\delta} = 0$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	(5)
II $a_{22} y^2 + 2a_{13} x = 0$	$\delta = 0$	$\Delta \neq 0$		$y^2 = 2\sqrt{\frac{-\Delta}{S^3}}\, x$	$y^2 = 2px$	(6)
III $a_{22} y^2 + a_{33} = 0$	$\delta = 0$	$\Delta = 0$	$K < 0$	$y^2 + \frac{K}{S^2} = 0$	$y^2 = a^2$	(7)
			$K > 0$	$y^2 + \frac{K}{S^2} = 0$	$y^2 = a^2$	(8)
			$K > 0$	$y^2 + \frac{K}{S^2} = 0$	$y^2 = a^2$	(9)