SAMPLE REVISION EXERCISES AMAT8111

I. Determine if $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is in the span of

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

Solution: We want to find if there are $t_1, t_2, t_3,$ and t_4 such that

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = t_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + t_4 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 & t_1 + t_3 + t_4 \\ t_3 & t_2 + t_4 \end{bmatrix}$$

Since two matrices are equal if and only if their corresponding entries are equal, this gives the system of linear equations

$$\begin{aligned} t_1 + t_2 &= 1 \\ t_1 + t_3 + t_4 &= 2 \\ t_3 &= 3 \\ t_2 + t_4 &= 4 \end{aligned}$$

we have $t_1 = -2$, $t_2 = 3$, $t_3 = 3$, and $t_4 = 1$.

Therefore, $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is in the span of $\mathcal{B}$.

$t_4 = 1$.

II. Prove that the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$ has the same volume as the parallelepiped determined by $(\vec{u} + k\vec{v})$, $\vec{v}$, and $\vec{w}$.

Solution:

The volume of the parallelepiped determined by $\vec{u} + k\vec{v}$, $\vec{v}$, and $\vec{w}$ is

$$\begin{aligned} |(\vec{u} + k\vec{v}) \cdot (\vec{v} \times \vec{w})| &= |\vec{u} \cdot (\vec{v} \times \vec{w}) + k(\vec{v} \cdot (\vec{v} \times \vec{w}))| \\ &= |\vec{u} \cdot (\vec{v} \times \vec{w}) + k(0)| \end{aligned}$$

This equals the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$.

III.

Prove that the similar matrices $B = S^{-1} A S$ have the same determinant: $\det A = \det B$.

Solution:

$$\det B = \det(S^{-1} A S) = \det S^{-1} \det A \det S = \frac{1}{\det S} \det A \ \det S = \det A.$$

IV.

1.9.17. Show that (a) if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is regular, then its pivots are a and $\dfrac{\det A}{a}$;
(b) if $A = \begin{pmatrix} a & b & e \\ c & d & f \\ g & h & j \end{pmatrix}$ is regular, then its pivots are a, $\dfrac{ad - bc}{a}$, and $\dfrac{\det A}{ad - bc}$.
(c) Can you generalize this observation to regular $n \times n$ matrices?

Solution:

(a) Since A is regular, $a \neq 0$ and $ad - bc \neq 0$. Subtracting c/a times the first from from the second row reduces A to the upper triangular matrix $\begin{pmatrix} a & b \\ 0 & d + b(-c/a) \end{pmatrix}$, and its pivots are a and $d - b\,\dfrac{c}{a} = \dfrac{ad - bc}{a} = \dfrac{\det A}{a}$.

(b) As in part (a) we reduce A to an upper triangular form. First, we subtract c/a times the first row from the second row, and g/a times the first row from third row, resulting in the matrix $\begin{pmatrix} a & b & e \\ 0 & \dfrac{ad - bc}{a} & \dfrac{af - ce}{a} \\ 0 & \dfrac{ah - bg}{a} & \dfrac{aj - cg}{a} \end{pmatrix}$. Performing the final row operation reduces the matrix to the upper triangular form $U = \begin{pmatrix} a & b & e \\ 0 & \dfrac{ad - bc}{a} & -\dfrac{af - ce}{a} \\ 0 & 0 & P \end{pmatrix}$, whose pivots are a, $\dfrac{ad - bc}{a}$, and

$$\frac{aj - eg}{a} - \frac{(af - ce)(ah - bg)}{a\,(ad - bc)} = \frac{adj + bfg + ech - afh - bcj - edg}{ad - bc} = \frac{\det A}{ad - bc}.$$

(c) If A is a regular $n \times n$ matrix, then its first pivot is a_{11}, and its k^{th} pivot, for $k = 2, \ldots, n$, is $\det A_k / \det A_{k-1}$, where A_k is the $k \times k$ upper left submatrix of A with entries a_{ij} for $i, j = 1, \ldots, k$. A formal proof is done by induction.

V › 1.9.21. (a) Show that if $D = \begin{pmatrix} A & O \\ O & B \end{pmatrix}$ is a block diagonal matrix, where A and B are square matrices, then $\det D = \det A \det B$. (b) Prove that the same holds for a block upper triangular matrix $\det \begin{pmatrix} A & C \\ O & B \end{pmatrix} = \det A \det B$. (c) Use this method to compute the determinant of the following matrices:

$$(i)\ \begin{pmatrix} 3 & 2 & -2 \\ 0 & 4 & -5 \\ 0 & 3 & 7 \end{pmatrix},\ (ii)\ \begin{pmatrix} 1 & 2 & -2 & 5 \\ -3 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & -2 \end{pmatrix},\ (iii)\ \begin{pmatrix} 1 & 2 & 0 & 4 \\ -3 & 1 & 4 & -1 \\ 0 & 3 & 1 & 8 \\ 0 & 0 & 0 & -3 \end{pmatrix},\ (iv)\ \begin{pmatrix} 5 & -1 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 2 & 4 & 4 & -2 \\ 3 & -2 & 9 & -5 \end{pmatrix}.$$

Solution:

(a) We can individually reduce A and B to upper triangular forms U_1 and U_2 with the determinants equal to the products of their respective diagonal entries. Applying the analogous elementary row operations to D will reduce it to the upper triangular form $\begin{pmatrix} U_1 & O \\ O & U_2 \end{pmatrix}$, and its determinant is equal to the product of its diagonal entries, which are the diagonal entries of both U_1 and U_2, so $\det D = \det U_1 \det U_2 = \det A \det B$.

(b) The same argument as in part (a) proves the result. The row operations applied to A are also applied to C, but this doesn't affect the final upper triangular form.

(c) (i) $\det \begin{pmatrix} 3 & 2 & -2 \\ 0 & 4 & -5 \\ 0 & 3 & 7 \end{pmatrix} = \det(3) \det \begin{pmatrix} 4 & -5 \\ 3 & 7 \end{pmatrix} = 3 \cdot 43 = 129,$

(ii) $\det \begin{pmatrix} 1 & 2 & -2 & 5 \\ -3 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & -2 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix} \det \begin{pmatrix} 1 & 3 \\ 2 & -2 \end{pmatrix} = 7 \cdot (-8) = -56,$

(iii) $\det \begin{pmatrix} 1 & 2 & 0 & 4 \\ -3 & 1 & 4 & -1 \\ 0 & 3 & 1 & 8 \\ 0 & 0 & 0 & -3 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 0 \\ -3 & 1 & 4 \\ 0 & 3 & 1 \end{pmatrix} \det(-3) = (-5) \cdot (-3) = 15,$

(iv) $\det \begin{pmatrix} 5 & -1 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 2 & 4 & 4 & -2 \\ 3 & -2 & 9 & -5 \end{pmatrix} = \det \begin{pmatrix} 5 & -1 \\ 2 & 5 \end{pmatrix} \det \begin{pmatrix} 4 & -2 \\ 9 & -5 \end{pmatrix} = 27 \cdot (-2) = -54.$

VI Use the matrix

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

to obtain the Hill cipher for the plaintext message

I AM HIDING

Solution If we group the plaintext into pairs and add the dummy letter G to fill the last pair, we obtain

$$IA \quad MH \quad ID \quad IN \quad GG$$

or, equivalently, from Table 1,

$$9\ 1 \quad 13\ 8 \quad 9\ 4 \quad 9\ 14 \quad 7\ 7$$

To encipher the pair IA, we form the matrix product

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 3 \end{bmatrix}$$

Table 1

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0

which, from Table 1, yields the ciphertext KC.

To encipher the pair MH, we form the product

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 13 \\ 8 \end{bmatrix} = \begin{bmatrix} 29 \\ 24 \end{bmatrix} \tag{1}$$

However, there is a problem here, because the number 29 has no alphabet equivalent (Table 1). To resolve this problem, we make the following agreement:

Whenever an integer greater than 25 occurs, it will be replaced by the remainder that results when this integer is divided by 26.

Because the remainder after division by 26 is one of the integers $0, 1, 2, \ldots, 25$, this procedure will always yield an integer with an alphabet equivalent.

Thus, in (1) we replace 29 by 3, which is the remainder after dividing 29 by 26. It now follows from Table 1 that the ciphertext for the pair MH is CX.

The computations for the remaining ciphertext vectors are

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 17 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 14 \end{bmatrix} = \begin{bmatrix} 37 \\ 42 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 11 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} 21 \\ 21 \end{bmatrix}$$

These correspond to the ciphertext pairs QL, KP, and UU, respectively. In summary, the entire ciphertext message is

$$KC \quad CX \quad QL \quad KP \quad UU$$

which would usually be transmitted as a single string without spaces:

$$KCCXQLKPUU$$ ◀

VII

2.9. Let us consider the following matrix **A**:

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & 3 \end{pmatrix}.$$

1. Verify that **A** is invertible.
2. Invert the matrix.
3. Verify that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

Solution:

1. Since $\det(\mathbf{A}) = 21$ the matrix **A** is invertible.
2.

$$\mathbf{A}^{-1} = \frac{1}{21} \begin{pmatrix} 7 & -1 & -4 \\ 0 & 9 & -6 \\ 0 & 6 & 3 \end{pmatrix}$$

3.

$$\mathbf{A}\mathbf{A}^{-1} = \frac{1}{21} \begin{pmatrix} 3 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & 3 \end{pmatrix} \begin{pmatrix} 7 & -1 & -4 \\ 0 & 9 & -6 \\ 0 & 6 & 3 \end{pmatrix} = \frac{1}{21} \begin{pmatrix} 21 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

VIII

The formula for the area of a triangle with coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3) is given by Area $= \frac{1}{2}\left|\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}\right|$. Determine the areas of triangles connecting:

(a) $(0,\ 0),\ (3,\ 2),\ (7,\ -4)$

(b) $(-3,\ 2),\ (2,\ 6),\ (8,\ -3)$

(c) $(-2,\ -1),\ (1,\ 5)\ (0.5,\ 4)$. What do you notice about this result?