ON MILMAN'S INEQUALITY AND RANDOM SUBSPACES
WHICH ESCAPE THROUGH A MESH IN $I\!R^n$

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Abstract

Let S be a subset in the Euclidean space $I\!R^n$ and $1 \le k < n$. We find sufficient conditions which guarantee the existence and even with probability close to 1, of k-codimensional subspaces which miss S. As a consequence we derive a sharp form of Milman's inequality and discuss some applications to Banach spaces.

Introduction

Consider the following theorem which we label as the "escape" phenomenon of random k-codimensional subspaces through a mesh S.

Theorem. *Let S be a subset of the unit sphere S_2^{n-1} of R^n and associate to S the number*
$s = a_n \int\limits_{S_2^{n-1}} \sup\limits_{x \in S} \langle x, u \rangle m_{n-1}(du)$, *where m_{n-1} is the normalized rotation invariant measure on the sphere and*

$$a_n = \sqrt{2}\,\Gamma\left(\frac{n+1}{2}\right) \bigg/ \Gamma\left(\frac{n}{2}\right) = \sqrt{n}\left(1 - \frac{1}{4n} + 0(n^{-2})\right) \ .$$

If $s < a_k$ $(1 \le k < n)$ then there is a k-codimensional subspace which does not intersect S.

This theorem and other results developed here, are based on two theorems A and B on Gaussian processes, which were originally proved in [G1] and generalizations proved in [G3]. Using Theorem A, we obtain in section 1, as a simple corollary to the "escape" phenomenon, a sharp new proof of an inequality of V. Milman and its refinements due to Pajor and Tomczak. The inequality, stated here in Theorem 1.5, has some important applications to Banach space theory. One of them is the theorem due to Milman [M1, M2]:

Theorem. *For any $0 < \lambda < 1$, any Banach space X of sufficiently large dimension n has a subspace of a quotient of X of dimension $\geq \lambda n$, which is $f(\lambda)$ isomorphic to a Hilbert space.*

As we shall see in section 1, this and other results can easily be derived from the inequality.

In section 2 we combine the escape phenomenon and Lévy's isoperimetric inequality to prove the following sharper form of a fundamental result stated in section 2 of [MS] and originally proved in [M5].

Theorem. *Given a function $f(x) \in C(S_2^{n-1})$ let M_f and $\omega_f(\varepsilon)$ denote the median and modulus of continuity of f. If $0 < \varepsilon \leq 1$ and $\ell = \lceil n\varepsilon^2/4 \rceil - 1 \geq 0$, then there is a subspace E of dimension ℓ such that $|f(x) - M_f| \leq \omega_f(2\varepsilon)$ for every $x \in E \cap S_2^{n-1}$.*

This eliminates the log term in the estimate $\ell = \lceil n\varepsilon^2/(2\log 4/\varepsilon) \rceil$ of Theorem 2.4 of [MS].

In order to obtain measure estimates as well, it is necessary to replace Theorem A by its quantitative analogue Theorem B, and combine it, with Lévy's theorem or Pisier's theorem, or as we shall do here, with Theorem 3.2 due to Maurey on the tail distribution of a real Lipschitz function defined on the Gaussian probability space $(\mathbb{R}^n, P)$. Thus we are able in section 3 to estimate the measure P of the subset of all k-codimensional subspaces of the Grassman manifold $\mathcal{G}_{n,n-k}$ which miss the set $S + K$, S being an arbitrary closed subset of S_2^{n-1} and K a closed convex subset in $\mathbb{R}^n$ which contains the origin in its relative interior. A special case is $K = rB_2^n$, the Euclidean ball of radius r. P is then the probability that a random k-codimensional subspace will be at Euclidean distance greater than r from S; if $r = 0$ this simply means that ξ will miss S. We shall see for example that if $r = 0$, P is very close to 1 if $s < \sqrt{k} - c$ for large k and suitable large constant c.

In section 4 we consider a finite sequence of N arbitrary convex sets $\{S_i\}_{i=1}^N$ situated at various locations in $\mathbb{R}^n$. We are interested in estimating the probability that a randomly chosen k-codimensional subspace will for all $i = 1, \ldots, N$ pass at a distance greater than d_i (≥ 0) from S_i. This probability can be estimated by an expression which involves only the numbers d_i, t_i, T_i and s_i ($i = 1, \ldots, N$), where t_i and T_i are the smallest and largest distance of the points of S_i to the origin and s_i is the number s we associate above with the set $S = S_i$.

We shall see in section 5 that some of these results admit generalizations. In order to do that we shall also need to extend Theorems A and B. For example, we consider a finite sequence of L arbitrary subsets of the unit sphere S_2^{n-1} and we find conditions which imply that there exists a subspace of codimension k which misses at least one of these sets and estimate the probability of this event.

1. The Escape Phenomenon and Milman's Inequality

Theorem A below is the key to the existence theorems contained in sections 1 and 2 and was originally proved in [G1]. We shall see in section 5 that Theorem A is a particular case of Theorem C, which was proved in [G3] by a different and simpler method.

Theorem A. *Let $\{X_{ij}\}$ and $\{Y_{ij}\}$, $i = 1, 2, , \ldots, n$, $j = 1, 2, , \ldots, m$, be two centered Gaussian processes which satisfy the following inequalities for all choices of indices:*

(i) $\quad E|X_{ij} - X_{ik}|^2 \le E|Y_{ij} - Y_{ik}|^2,$

(ii) $\quad E|X_{ij} - X_{\ell k}|^2 \ge E|Y_{ij} - Y_{\ell k}|^2,$ *if $i \ne \ell$.*

Then, $E \min_i \max_j X_{ij} \le E \min_i \max_j Y_{ij}.$

Taking $n = 1$ in Theorem A we obviously obtain the Fernique, Sudakov inequality [F1,F2].

Corollary 1.1. *Let $\{X_j\}$ and $\{Y_j\}$, $1 \le j \le m$, be two centered Gaussian processes such that $E|X_j - X_k|^2 \le E|Y_j - Y_k|^2$ for all j, k. Then, $E \max_j X_j \le E \max_j Y_j$.*

Throughout, we shall denote by $\{g_{ij}\}, \{h_i\}, \{g_j\}, \{g\}$ independent sets of orthonormal Gaussian r.v.'s and

$$G = G(\omega) = \sum_{i=1}^{n} \sum_{j=1}^{k} g_{ij}(\omega) e_i \otimes e_j$$

will denote the random Gaussian operator from $I\!R^n$ to $I\!R^k$, where $\{e_i\}_{i=1}^n$ is the usual unit vector basis of $I\!R^n$, and $\| \bullet \|_2$ stands for the Euclidean norm on $I\!R^n$. The letter c will denote constants.

Corollary 1.2. *Let $S \subset I\!R^n$ be a closed subset, $1 \le k \le n$. Then*

(1.1)
$$E\big(\min_{x \in S} \|G(\omega)(x)\|_2 \big) \ge E\big(\min_{x \in S} \big\{ \|x\|_2 \big(\sum_1^k g_j^2 \big)^{1/2} + \sum_1^n h_i x_i \big\} \big)$$

$$\ge a_k \min_{x \in S} \|x\|_2 - E\big(\max_{x \in S} \sum_1^n h_i x_i \big) ,$$

(1.2)
$$E\big(\max_{x \in S} \|G(\omega)(x)\|_2 \big) \le E\big(\max_{x \in S} \big\{ \|x\|_2 \big(\sum_1^k g_j^2 \big)^{1/2} + \sum_1^n h_i x_i \big\} \big)$$

$$\le a_k \max_{x \in S} \|x\|_2 + E\big(\max_{x \in S} \sum_1^n h_i x_i \big) .$$

where $a_k = E\big(\sum_1^k g_j^2 \big)^{1/2} = \sqrt{2}\Gamma\left(\frac{k+1}{2} \right) / \Gamma\left(\frac{k}{2} \right)$ satisfies $k/\sqrt{k+1} < a_k < \sqrt{k}$.

Proof: For $x \in S$ and $y \in S_2^{k-1} = \{y \in I\!\!R^k \ ; \ \|y\|_2 = 1\}$, we define the two Gaussian processes

$$X_{x,y} = \|x\|_2 \sum_{j=1}^{k} g_j y_j + \sum_{i=1}^{n} h_i x_i$$

and

$$Y_{x,y} = \langle G(\omega)(x), y \rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} g_{ij} x_i y_j \ .$$

It follows then that for all $x, x' \in S$ and $y, y' \in S_2^{k-1}$, we have

$$I\!\!E|X_{x,y} - X_{x',y'}|^2 - I\!\!E|Y_{x,y} - Y_{x',y'}|^2$$

$$= \|x\|_2^2 + \|x'\|_2^2 - 2\|x\|_2\|x'\|_2 \langle y, y'\rangle - 2\langle x, x'\rangle(1 - \langle y, y'\rangle)$$

$$\geq \|x\|_2^2 + \|x'\|_2^2 - 2\|x\|_2\|x'\|_2 \langle y, y'\rangle - 2\|x\|_2\|x'\|_2(1 - \langle y, y'\rangle)$$

$$\geq 0 \ .$$

with equality if $x = x'$. By Theorem A this implies that

$$I\!\!E\big(\min_{x\in S} \max_{y\in S_2^{k-1}} X_{x,y} \big) \leq I\!\!E\big(\min_{x\in S} \max_{y\in S_2^{k-1}} Y_{x,y} \big)$$

and by Corollary 1.1

$$I\!\!E\big(\max_{x,y} X_{x,y} \big) \geq I\!\!E\big(\max_{x,y} Y_{x,y} \big) \ .$$

The last two inequalities imply inequalities (1.1) and (1.2). The estimate for a_k follows from $a_k^2 \leq I\!\!E\big(\sum_1^k g_j^2 \big) = k$ and $a_k \sqrt{k+1} \geq a_k a_{k+1} = k$. $\quad\square$

Remark (1.1). Using the identity $a_k a_{k+1} = k$, it follows by induction that

$$a_k = \sqrt{k} \left(1 - \frac{1}{4k} + \frac{1}{32k^2} + 0(k^{-3}) \right) \ . \qquad\qquad \square$$

Corollary 1.3. *Let S be a closed subset of the unit sphere*

$$S_2^{n-1} = \{x \in I\!\!R^n \ ; \ \|x\|_2 = 1\} \quad \text{and} \quad s = I\!\!E\big(\max_{x\in S} \sum_1^n h_i x_i \big) \ .$$

If $1 \leq k < n$ satisfies $a_k > s$, then there is an operator $T : \ell_2^n \to \ell_2^k$ such that

(1.3) $$\frac{\|Tx\|_2}{\|Ty\|_2} \leq \frac{a_k + s}{a_k - s} \quad \text{for all} \quad x, y \in S \ .$$

Proof: By Corollary 1.2, $I\!\!E\big(\min_{x\in S} \|G(\omega)(x)\|_2 \big) \geq a_k - s > 0$ and $I\!\!E\big(\max_{x\in S} \|G(\omega)(x)\|_2 \big) \leq a_k + s$. Hence, there exists ω_0 such that $T = G(\omega_0)$ satisfies (1.3). $\quad\square$

There is a natural upper bound for s which is obtained by taking $S = S_2^{n-1}$, thus $s \leq a_n$. Integrating over the unit sphere we get that s can be expressed by $s = a_n \int\limits_{S_2^{n-1}} \max\limits_{x \in S} \langle x, u \rangle m_{n-1}(du)$ where m_{n-1} is the normalized rotation invariant measure defined on S_2^{n-1}.

Another estimate for s can be obtained, for example, by using Dudley's theorem [D]: *Let $N(\varepsilon)$ be the number of elements in a set $S \subset \mathbb{R}^n$ which forms an ε net in the $\| \bullet \|_2$ norm for S. Then, $s \leq c \int\limits_0^\infty \sqrt{\log N(\varepsilon)} d\varepsilon$.* Thus, s can be replaced by $c \int\limits_0^\infty \sqrt{\log N(\varepsilon)} d\varepsilon$ in Corollary 1.3.

The escape phenomenon mentioned in the introduction is summarized in the next theorem.

Theorem 1.4. *Let $S \subset \mathbb{R}^n$ be a compact subset, $1 \leq k < n$ and $G(\omega) = \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{k} g_{ij}(\omega) e_i \otimes e_j$ be the Gaussian map from $\mathbb{R}^n$ to $\mathbb{R}^k$.*

(1) $\mathbb{E} \min\limits_{x \in S} \|G(\omega)(x)\|_2 > 0$ iff there exists a subspace of codimension k which misses S.

(2) If $S \subset S_2^{n-1}$ and $a_k > s = \mathbb{E}\big(\max\limits_{x \in S} \sum\limits_1^n x_i h_i \big)$, then $\mathbb{E} \min\limits_{x \in S} \|G(\omega)(x)\|_2 > 0$.

Proof: The subspace $E_\omega = G(\omega)^{-1}(0)$ has codimension k a.e., hence if $\mathbb{E} \min\limits_{x \in S} \|G(\omega)(x)\|_2 > 0$ then there is an E_{ω_0} of codimension k which misses S. Conversely, every k-codimensional subspace has the form E_ω and if E_{ω_0} misses S, then compactness of S implies that E_ω misses S, i.e., $\min\limits_{x \in S} \|G(\omega)(x)\|_2 > 0$, on a set of positive measure; this proves (1). Statement (2) follows from inequality (1.1) □

Given a closed convex body K in $\mathbb{R}^n$ which contains the origin in its interior, we associate with K the dual body $K^* = \{x \in \mathbb{R}^n \; ; \; \langle x, y \rangle \leq 1 \text{ for all } y \in K\}$ and the "norm" $\|x\|_K = \inf\{t > 0 \; ; \; x \in tK\}$. $\mathbb{E}(K)$ will denote $\mathbb{E}\| \sum\limits_1^n g_i e_i \|_K$ and $\varepsilon_2(K) = \max\{\|x\|_2 \; ; \; x \in K^*\}$. If K is centrally symmetric, then $\| \bullet \|_K$ is a norm and the normed space $X = (\mathbb{R}^n, \| \bullet \|_K)$ has K for its unit ball; we shall then refer to $\mathbb{E}(K)$ by $\mathbb{E}(X)$ and $\varepsilon_2(X) = \varepsilon_2(K)$.

Theorem 1.5. *Let K be a closed convex body in $\mathbb{R}^n$, $0 \in \text{int } K$, $1 \leq k \leq n$ and let $K_r = r^{-1}K \cap B_2^n$ $(r > 0)$. Then there is a subspace E of codimension k such that*

$$\|x\|_2 \leq r_0 \|x\|_K \quad \text{for all} \quad x \in E \ ,$$

where $r_0 = \inf \big\{ r > 0 \; ; \; a_k > \mathbb{E}(K_r^*) \big\} \leq a_k^{-1} \mathbb{E}(K^*)$.

Proof: Let $S = S_2^{n-1} \cap K_r$. By Theorem 1.4, if $a_k > s = \mathbb{E}\big(\sup\limits_{x \in S} \sum x_i h_i \big)$ then there is a k-codimensional subspace E which misses S, hence if $x \in E \cap S_2^{n-1}$ then $x \notin r^{-1}K$, that is $r\|x\|_K \geq \|x\|_2$ for all $x \in E$. Notice that $s \leq \mathbb{E}\big(\sup\limits_{x \in K_r} \sum x_i h_i \big) = \mathbb{E}(K_r^*)$. To prove the inequality for r_0, since $K_r \subseteq r^{-1}K$, $K_r^* \supseteq rK^*$, therefore $r\|x\|_{K_r^*} \leq \|x\|_{K^*}$ and $\mathbb{E}(K_r^*) \leq r^{-1}\mathbb{E}(K^*)$, which implies $r_0 \leq a_k^{-1}\mathbb{E}(K^*)$. □

Remark (1.2).   Originally V. Milman was the first to prove that there is a function ψ : $(0,1) \rightarrow \mathbb{R}_+$ with the property that in every Banach space X of dimension n there is a subspace E of codimension $< \varepsilon n$ such that $\|x\|_2 \leq \psi(\varepsilon) n^{-1/2} \mathbb{E}(X^*)\|x\|$ for all $x \in E$ and the estimate $\psi(\varepsilon) \leq c\varepsilon^{-1}$ was proved in [M1]. Pajor and Tomczak [PT2] improved the estimate to $\psi(\varepsilon) \leq C\varepsilon^{-1/2}$ where C is a universal constant. Their original proof is based on Lévy's isoperimetric inequality [MS] and Sudakov's minoration theorem [F2], an earlier version of which appears in [PT1]. It is worthwhile to recall here some consequences of Theorem 1.5. We first give the operator formulation of the theorem.

Let $u : X \rightarrow Y$ be an operator between the Banach spaces X and Y. The kth Gelfand number of u is defined by $c_k(u) = \inf\{\|u \mid E\| \; ; \; E \subset X \; , \; \text{codim}\, E < k\}$, $k = 1, 2, \ldots$. For $u : \ell_2^n \rightarrow Y$, $\ell(u) = \mathbb{E}\|\sum_1^n g_i u(e_i)\|$ (usually the ℓ norm of u, $\ell(u)$ is defined by $\left(\mathbb{E}\|\sum_1^n g_i u(e_i)\|^2\right)^{1/2}$, but by Kahane's inequality [MS] the two expressions are equivalent). For $u : \ell_2 \rightarrow Y$, $\ell(u)$ is defined as $\sup\{\ell(uv) \; ; \; \|v : \ell_2^n \rightarrow \ell_2\| \leq 1\}$. Theorem 1.5. can be reformulated as follows:

Theorem 1.5'. *Let* $u : X \rightarrow \ell_2^n$ *and* $X_r = (X, \| \bullet \|_r)$ *be the space normed by* $\|x\|_r = \max\{r\|x\| \; , \; \|ux\|_2\}$ *($r > 0$). Let* $1 \leq k \leq n$, *then* $c_{k+1}(u) \leq \tilde{r}_0 = \inf\left\{r > 0 \; ; \; \ell(u^* : \ell_2^n \rightarrow X_r^*) \leq a_k\right\}$. *In particular,* $a_k c_{k+1}(u) \leq \ell(u^* : \ell_2^n \rightarrow X^*)$.

Proof: Without loss of generality we can assume that u is invertible and define on $\mathbb{R}^n$ the norm $|x| = \|u^{-1}x\|$, then $u : X \rightarrow (\mathbb{R}^n, | \bullet |)$ is an isometry. Now let $|x|_r = \max\{r|x| \; , \; \|x\|_2\}$ and apply Theorem 1.5 to the unit ball K_r of the normed space $(\mathbb{R}^n, | \bullet |_r)$, with K equal to the unit ball of $(\mathbb{R}^n, | \bullet |)$. We thus obtain a k-codimensional subspace E for which $\|ux\|_2 \leq \tilde{r}_0\|x\|$ for all $x \in E$, where

$$\tilde{r}_0 = \inf\left\{r > 0 \; ; \; a_k \geq \mathbb{E}(K_r^*)\right\} = \inf\left\{r > 0 \; ; \; \ell(u^* : \ell_2^n \rightarrow X_r^* \leq a_k\right\} . \qquad \square$$

Theorem 1.6. *Let* $X = (\mathbb{R}^n, \| \bullet \|)$ *be an* n*-dimensional Banach space and* $T_2(X^*)$ *and* $C_2(X)$ *be the Gaussian type 2 and cotype 2 constants of* X^* *and* X *respectively. Then for every* $\lambda \in (0, 1)$, *there exists a subspace* E *of* X *of dimension* $[\lambda n]$ *such that*

(i) $d(E, \ell_2^{[\lambda n]}) \leq (1 - \lambda - n^{-1})^{-1/2} T_2(X^*)$, *if* $n^{-1} \leq \lambda \leq 1 - n^{-1}$;

(ii) $d(E, \ell_2^{[\lambda n]}) \leq c(1 - \lambda)^{-1/2} C_2(X) \log\left(1 + c(1 - \lambda)^{-1/2} C_2(X)\right)$, *if* $n \geq N_\lambda$.

Proof: (i) without loss of generality assume B_2^n, the unit ball of ℓ_2^n, is the ellipsoid of maximal volume contained in B_X, the unit ball of X, and let $k + 1 = n - [\lambda n]$. By Theorem 1.5 there exists a subspace E of codimension $k + 1$ such that

$$\|x\| \leq \|x\|_2 \leq a_{k+1}^{-1} \mathbb{E}(X^*)\|x\| \quad \text{for all} \quad x \in E .$$

Now, $a_{k+1} \geq \sqrt{k} \geq \sqrt{n}(1 - \lambda - n^{-1})^{1/2}$ and by Lemma 4.11 of [BG], $\mathbb{E}(X^*) \leq \sqrt{n}T_2(X^*)$. This proves (i).

(ii) Since $T_2(X^*) \leq C_2(X)K(X) \leq cC_2(X)\log\big(d(X, \ell_2^n) + 1\big)$ where $K(X)$ is the K convexity constant of X (cf. [P1] or [MS] and Remark 2.2 [P2]), the required result follows by applying the iteration procedure of Milman [M3] a finite number of times (with an appropriate sequence of λ_i's in $(0, 1)$) to a subspace of a subspace ..., etc., of X. $\qquad\square$

Remark (1.3). The fact that (ii) has linear dependence on the cotype 2 constant $C_2(X)$, except for a log term, was first proved in [M1, M3]. In the form stated above, (ii) was established in [PT2] by means of the sharp estimate $\psi(\varepsilon) \leq c\varepsilon^{-1/2}$ obtained there. The following theorem is due to Milman and appears in the form stated here in [M4].

Theorem 1.7. Let $f(\lambda) = c_1(1-\lambda)^{-1}\log\big(c_2/(1-\lambda)\big)$, $\lambda \in \big[\frac{1}{2}, 1\big)$. Then, given any $\lambda \in \big(\frac{1}{2}, 1\big)$, every Banach space X of dimension $n(> N_\lambda)$ contains a subspace of a quotient of X, E, of dimension $[\lambda n]$, such that $d(E, \ell_2^{[\lambda n]}) \leq f(\lambda)$.

Proof: Given any $\frac{1}{2} \leq \lambda_1 < 1$, we apply Theorem 1.5 to find a subspace E_1 of X^* of dimension $> [\lambda_1 n]$ such that

$$\|x\|_2 \leq \frac{\mathbb{E}(X)}{\sqrt{n}\sqrt{1 - \lambda_1}}\|x\| \quad \text{for all} \quad x \in E_1 .$$

Dualizing this inequality and applying Theorem 1.5 again, we find a subspace $E_2 \subset E_1^*$ of dimension $> [\lambda_1^2 n]$ such that

$$\|x\|_2 \leq \frac{\mathbb{E}(E_1)}{\sqrt{\lambda_1 n - \lambda_1^2 n}}\|x\| \quad \text{for all} \quad x \in E_2 ,$$

both inequalities imply

$$d(E_2, \ell_2^{\dim E_2}) \leq \frac{\mathbb{E}(E_1)\mathbb{E}(X)}{n\sqrt{\lambda_1}(1 - \lambda_1)} \leq \frac{2\mathbb{E}(X)\mathbb{E}(X^*)}{n(1 - \lambda_1)} .$$

By a result due to Lewis [L], there exists $u : \ell_2^n \to X$ for which $\ell(u)\ell^*(u^{-1}) = n$, and by [FT] $\ell(u^{*-1}) \leq c\ell^*(u^{-1})K(X)$ and now applying the inequality $K(X) \leq c\log\big(d(X, \ell_2^n) + 1\big)$, we obtain by choosing the proper ellipsoid in $\mathbb{R}^n$, namely the Lewis ellipsoid

$$d(E_2, \ell_2^{\dim E_2}) \leq \frac{c\log\big(d(X, \ell_2^n) + 1\big)}{1 - \lambda_1} ,$$

where E_2 is a subspace of a quotient of X of dimension $\geq \lambda_1^2 n$. Replacing X above by E_2 and continuing in this manner, we obtain the result after a finite number of iterations and the proper choice of the sequence λ_i. $\qquad\square$

In order to obtain a subspace of a quotient of X which is $\frac{1+\varepsilon}{1-\varepsilon}$ isomorphic to a Hilbert space for a given $\varepsilon \in (0, 1)$, we shall use the following theorem proved in [G1], see also [G2].

Theorem 1.8. *Let* $Y = (\mathbb{R}^n, \| \bullet \|)$ *be a Banach space and* $\varepsilon_2(Y) = \max_{y \neq 0}(\|y\|/\|y\|_2)$. *If* $1 < m < n$ *satisfies* $\mathbb{E}(Y) > \sqrt{m}\varepsilon_2(Y)$, *then* Y *contains an* m-*dimensional subspace* F *with*

$$d(F, \ell_2^m) \leq \frac{\mathbb{E}(Y) + \sqrt{m}\varepsilon_2(Y)}{\mathbb{E}(Y) - \sqrt{m}\varepsilon_2(Y)} \; .$$

Theorem 1.8 combined with Theorem 1.7 provides immediately and obviously the $\frac{1+\varepsilon}{1-\varepsilon}$ version of Theorem 1.7.

1.9. *Let* $0 < \varepsilon < 1$. *There exists* N *such that for any* $n(\geq N)$ *dimensional Banach space* X, *there exists a subspace of a quotient of* X *of dimension* $> c\varepsilon^2 n$ *which is* $\frac{1+\varepsilon}{1-\varepsilon}$ *isomorphic to a Hilbert space.*

Proof: Take $\lambda = 1/2$ in Theorem 1.7 and apply Theorem 1.8 to the space E of dimension $[n/2]$, to find a subspace $F \subset E$ of dimension m which will be $\frac{1+\varepsilon}{1-\varepsilon}$ isomorphic to ℓ_2^m provided $\sqrt{m}\varepsilon_2(E) < \varepsilon \mathbb{E}(E)$. But since $d(E, \ell_2^{[n/2]}) \leq f\left(\frac{1}{2}\right)$, it follows that $\varepsilon_2(E)/\mathbb{E}(E) \leq f\left(\frac{1}{2}\right)/\mathbb{E}(\ell_2^{[n/2]}) \sim cn^{-1/2}$ and this yields the estimate for m. $\qquad \square$

Remark (1.4). In a similar manner we can obtain $\frac{1+\varepsilon}{1+\varepsilon}$ versions of Theorem 1.6.

2. Applications to Continuous Functions on the Sphere

Given a set $A \subset S_2^{n-1}$ and $\varepsilon > 0$ we denote by A_ε the set $\{x \in S_2^{n-1} \; ; \; \rho(x, A) \leq \varepsilon\}$, where ρ is the geodesic metric on S_2^{n-1}. The median M_f of a function $f \in C(S_2^{n-1})$ is a number such that $m_{n-1}(f \leq M_f) \geq \frac{1}{2}$ and $m_{n-1}(f \geq M_f) \geq \frac{1}{2}$, where m_{n-1} is the normalized rotation invariant measure of S_2^{n-1}. Lévy's classical isoperimetric inequality states (cf. [MS]):

(2.1.) $\quad$ *Let* $f \in C(S_2^{n-1})$ *and* $A = \{x \in S_2^{n-1} \; ; \; f(x) = M_f\}$. *Then* $m_{n-1}(A_\varepsilon) \geq$
$\quad\quad\quad\quad 1 - \sqrt{\pi/2} \exp\left(-\frac{1}{2}(n-1)\varepsilon^2\right)$.

Let $\omega_f(\varepsilon) = \sup\{|f(x) - f(y)| \; ; \; \rho(x, y) \leq \varepsilon\}$ be the modulus of continuity of f. It is clear from the definition of A in (2.1) that $|f(x) - M_f| \leq \omega_f(\varepsilon)$ for every $x \in A_\varepsilon$. Thus by Lévy's isoperimetric inequality, $f(x)$ is close to M_f on the set A_ε which has large measure if $n\varepsilon^2$ is big.

Theorem 2.1. *Let* $A \subset S_2^{n-1}$, $0 < \varepsilon \leq 1$ *and* $\ell = \left[\frac{1}{2}n\varepsilon^2 m_{n-1}(A)\right] - 1 \geq 0$. *Then there exists a subspace* E *of dimension* ℓ *such that* $E \cap S_2^{n-1} \subset A_\varepsilon$.

Proof: We apply Theorem 1.4 to $S = (A_\varepsilon)^C$, the complement of A_ε. By standard integration

$$s = E\big(\sup_{x \in S} \sum_1^n x_i h_i\big) = a_n \int_{S_2^{n-1}} \sup_{x \in S} \langle x, u \rangle m_{n-1}(du) \ .$$

Notice that if $u \in A$ and $x \in S$, then $\langle x, u \rangle \leq \cos \varepsilon$ and for all other $u \in S_2^{n-1}$ $\langle x, u \rangle \leq 1$, hence $s \leq a_n \big(m_{n-1}\big((A)^C\big) + m_{n-1}(A) \cos \varepsilon \big) = a_n \big(1 - (1 - \cos \varepsilon) m_{n-1}(A) \big)$.

Let $k = n - \ell$, then

$$(2.2) \qquad a_k/a_n > \sqrt{(k-1)/n} \geq \big(1 - \tfrac{1}{2}\varepsilon^2 m_{n-1}(A) \big)^{1/2}$$

$$\geq \left(1 - \frac{2}{3}\varepsilon^2 m_{n-1}(A) + \frac{\varepsilon^4}{9} m_{n-1}(A) \right)^{1/2}$$

$$\geq 1 - \frac{1}{3}\varepsilon^2 m_{n-1}(A) > 1 - (1 - \cos \varepsilon) m_{n-1}(A) \geq s/a_n \ ,$$

therefore $a_k > s$. By Theorem 1.4 there exists a subspace E of dimension ℓ which misses S, hence $E \cap S_2^{n-1} \subset A_\varepsilon$. □

Theorem 2.1, combined with the isoperimetric inequality (2.1), sharpens the estimate obtained in Theorem 2.4 of [MS].

Corollary 2.2. *Let $0 < \varepsilon \leq 1$ and $n\varepsilon^2 \geq 4$. Then for any function $f \in C(S_2^{n-1})$, there exists a subspace E of dimension $\geq [n\varepsilon^2/4] - 1$ such that*

$$(2.3) \qquad |f(x) - M_f| < \omega_f(2\varepsilon) \quad \text{for every} \quad x \in E \cap S_2^{n-1} \ .$$

Proof: If $B = f^{-1}(M_f)$, then by the isoperimetric inequality, $m_{n-1}(B_\varepsilon) \geq 1 - \sqrt{2/\pi} \exp\big(-(n-1)\varepsilon^2/2 \big) > \tfrac{1}{2}$ since by our assumption $n\varepsilon^2 \geq 4$. Hence taking $A = B_\varepsilon$ the dimension ℓ of the subspace E in Theorem 2.1 is $\geq [n\varepsilon^2/4] - 1$ and $E \cap S_2^{n-1} \subset B_{2\varepsilon}$. □

3. An Isoperimetric Inequality on the Sphere of $\mathbb{R}^n$

In this section we shall make use of Theorem B proved in [G1]. Theorem B turns out to be essential for obtaining the measure estimates in this section. A simpler proof of this theorem appeared recently also in [K], and other related results, as well as generalizations in various directions, may be found in [G3].

Theorem B. Let $\{X_{ij}\}$ and $\{Y_{ij}\}$, $1 \leq i \leq n$, $1 \leq j \leq m$, be two centered Gaussian processes which satisfy the following inequalities for all choices of indices

(i) $\mathbb{E}(X_{ij}^2) = \mathbb{E}(Y_{ij}^2)$

(ii) $\mathbb{E}(X_{ij}X_{ik}) \geq \mathbb{E}(Y_{ij}Y_{ik})$

(iii) $\mathbb{E}(X_{ij}X_{\ell k}) \leq \mathbb{E}(Y_{ij}Y_{\ell k})$ if $i \neq \ell$.

Then, $P\big(\bigcap_i \bigcup_j [X_{ij} \geq \lambda_{ij}]\big) \leq P\big(\bigcap_i \bigcup_j [Y_{ij} \geq \lambda_{ij}]\big)$ for all choices of $\lambda_{ij} \in \mathbb{R}$.[1]

Lemma 3.1. Let $G(\omega) = \sum_{i=1}^{n} \sum_{j=1}^{k} g_{ij}(\omega) e_i \otimes e_j : \mathbb{R}^n \to \mathbb{R}^k$ and $S \subset \mathbb{R}^n$ be an arbitrary subset. Then for all choices of real λ_x ($x \in S$),

$$P\big(\bigcap_{x \in S} \big[\|G(\omega)(x)\|_2 + \|x\|_2\, g \geq \lambda_x\big]\big) \geq P\big(\bigcap_{x \in S} \big[\|x\|_2 \big(\sum_1^k g_j^2\big)^{1/2} + \sum_1^n x_i h_i \geq \lambda_x\big]\big) \ .$$

Proof: For $x \in S$ and $y \in S_2^{k-1}$ we define the two Gaussian processes,

$$X_{x,y} = \langle G(\omega)(x), y \rangle + \|x\|_2\, g \quad \text{and} \quad Y_{x,y} = \|x\|_2 \sum_1^k g_j y_j + \sum_1^n x_i h_i \ ,$$

where h_i, g_j, g denote orthonormal Gaussian r.v.'s. It is easy to check that $\mathbb{E}(X_{x,y}^2) = \mathbb{E}(Y_{x,y}^2)$ and

$$\mathbb{E}(X_{x,y}X_{x',y'}) - \mathbb{E}(Y_{x,y}Y_{x',y'}) = \langle x, x' \rangle \langle y, y' \rangle + \|x\|_2 \|x'\|_2 - \|x\|_2 \|x'\|_2 \langle y, y' \rangle - \langle x, x' \rangle$$

$$= \big(\|x\|_2 \|x'\|_2 - \langle x, x' \rangle\big)\big(1 - \langle y, y' \rangle\big)$$

which is always nonnegative and equal to zero if $x = x'$.

For each $x \in S$ the set $\bigcup_{y \in S_2^{k-1}} [X_{x,y} \geq \lambda_x] = \big[\|G(x)\|_2 + g\|x\|_2 \geq \lambda_x\big]$ is closed in the probability space $\{\mathbb{R}^{nk+1}, P\}$ where P is the canonical Gaussian measure of $\mathbb{R}^{nk+1}$. Hence $\bigcap_{x \in S} \bigcup_{y \in S_2^{k-1}} [X_{x,y} \geq \lambda_x]$ is closed. The same can be said for the corresponding expression with the $Y_{x,y}$. By Theorem B, for each finite set $\{x_i\}_1^N \subset S$ we have

$$P\big(\bigcap_{i=1}^{N} \bigcup_{y \in S_2^{k-1}} [X_{x_i,y} \geq \lambda_{x_i}]\big) \geq P\big(\bigcap_{i=1}^{N} \bigcup_{y \in S_2^{k-1}} [Y_{x_i,y} \geq \lambda_{x_i}]\big)$$

and so, ordering the collection of finite subsets of S, F, by inclusion, we obtain that the limits exist and satisfy the inequality

$$\lim_F P\big(\bigcap_{i=1}^{N} \bigcup_{y \in S_2^{k-1}} [X_{x_i,y} \geq \lambda_{x_i}]\big) \geq \lim_F P\big(\bigcap_{i=1}^{N} \bigcup_{y \in S_2^{k-1}} [Y_{x_i,y} \geq \lambda_{x_i}]\big)$$

But as the sets $\bigcap_{x \in S} \bigcup_{y \in S_2^{k-1}} [X_{x,y} \geq \lambda_x])$ and $\bigcap_{x \in S} \bigcup_{y \in S_2^{k-1}} [Y_{x,y} \geq \lambda_x]$ are closed and P is a regular measure, it follows easily that the two respective limits over F are equal to and satisfy the inequality

$$P\Big(\bigcap_{x \in S} \bigcup_{y \in S_2^{k-1}} [X_{x,y} \geq \lambda_x]\Big) \geq P\Big(\bigcap_{x \in S} \bigcup_{y \in S_2^{k-1}} [Y_{x,y} \geq \lambda_x]\Big) . \qquad \Box$$

We shall also use the following theorem due to Pisier on the normal tail distribution of a Lipschitz function [P3]. Pisier proved in [P3] a more general result with the constant $c = 2\pi^{-2}$.

Theorem 3.2. *Let $f : I\!\!R^n \to I\!\!R^1$ be a Lipschitz function satisfying $|f(x) - f(y)| \leq \sigma \|x - y\|_2$, let $c = \frac{1}{2}$ and P be the canonical Gaussian measure on $I\!\!R^n$. Then for all $\lambda > 0$*

$$P\big(f(x) - I\!\!E f > \lambda\big) \leq \exp(-c\lambda^2/\sigma^2) .$$

Remarks (3.1). Let $\mathcal{G}_{n,n-k}$ denote the Grassman manifold of k-codimensional subspaces of $I\!\!R^n$ and $\gamma_{n,n-k}$ the normalized Haar measure on $\mathcal{G}_{n,n-k}$. If $S \subset I\!\!R^n$ is a subset, a subspace $\xi \in \mathcal{G}_{n,n-k}$ meets S if and only if it meets S_{rad}, which will denote the symmetric radial projection of $S \cup (-S)$ onto the unit sphere S_2^{n-1}. When $k = n - 1$ and S is a measurable subset of $I\!\!R^n$, $\gamma_{n,1}(\xi \in \mathcal{G}_{n,1}; \xi \cap S \neq \Phi) = m_{n-1}(S_{\text{rad}})$, where m_{n-1} is the normalized surface area of the sphere S_2^{n-1}.

Let Σ denote the set of all symmetric measurable subsets of S_2^{n-1}, in particular $S_{\text{rad}} \in \Sigma$ if S is a measurable subset of $I\!\!R^n$. $\gamma_{n,n-k}$ induces the measure $\gamma_{n,n-k}(\xi \in \mathcal{G}_{n,n-k} ; \xi \cap S \neq \Phi)$ on the elements $S \in \Sigma$. This is a probability measure which is invariant under orthogonal transformations $u \in 0_n$ applied to S.

Similarly, if $G = G(\omega) = \sum_{i=1}^{n} \sum_{j=1}^{k} g_{i,j}(\omega) e_i \otimes e_j$ is the Gaussian operator from $I\!\!R^n$ to $I\!\!R^k$, then a.e. $G(\omega)^{-1}(0)$ is a k-codimensional subspace of $I\!\!R^n$ and $P\big(\omega : G(\omega)^{-1}(0) \cap S \neq \Phi\big)$ is also a probability measure on $S \in \Sigma$, which is invariant under orthogonal transformations $u \in 0_n$; because, Gu and G have the same distribution for every $u \in 0_n$. By uniqueness of the Haar measure, the two measures are identical on Σ, i.e., $\gamma_{n,n-k}(\xi \in \mathcal{G}_{n,n-k} ; \xi \cap S \neq \Phi) = P\big(\omega ; G(\omega)^{-1}(0) \cap S \neq \Phi\big)$ for all $S \in \Sigma$, hence also for all measurable subsets S in $I\!\!R^n$.

(3.2) Let $K \subset I\!\!R^n$ be a closed convex subset which contains the origin in its interior. Let $\|G\|$ be the "norm" associated with G as a map from $(I\!\!R^n, \| \bullet \|_K)$ to $\ell_2^k = (R^k, \| \bullet \|_2)$. By Corollary 1.2(2), taking $S = K$, we have

$$I\!\!E \|G\| \leq a_k \varepsilon_2(K^*) + I\!\!E(K^*)$$

and conversely, if $x \in K$, then

$$\mathbb{E}\|G\| \geq \mathbb{E}\|G(x)\|_2 = \big(\sum_1^n x_i^2\big)^{1/2}\mathbb{E}\big(\sum_1^k g_j^2\big)^{1/2}$$

implies

$$\mathbb{E}\|G\| \geq a_k\varepsilon_2(K^*) \ ;$$

moreover,

$$\mathbb{E}\|G\| \geq \mathbb{E}\|G^*(e_1)\|_{K^*} = \mathbb{E}\big\|\sum_1^n g_i e_i\big\|_{K^*} = \mathbb{E}(K^*) \ ,$$

that is,

$$\max\big\{a_k\varepsilon_2(K^*), \mathbb{E}(K^*)\big\} \leq \mathbb{E}\|G\| \leq a_k\varepsilon_2(K^*) + \mathbb{E}(K^*) \ .$$

The following theorem estimates from below the measure of the set of k-codimensional subspaces $\xi \in \mathcal{G}_{n,n-k}$ which miss a given closed subset S in S_2^{n-1} by a distance, with respect to the $\|\bullet\|_K$ "norm", greater than 1:

Theorem 3.3. Let $S \subset S_2^{n-1}$ be a closed subset, $K \subset \mathbb{R}^n$ a closed convex set with $0\varepsilon\,\mathrm{int}\,K$ and $s = \mathbb{E}\big(\max_{x \in S}\sum_1^n x_i h_i\big)$. If $a_k\big(1 - \varepsilon_2(K^*)\big) - \mathbb{E}(K^*) > s$, $1 \leq k < n$, then

$$\gamma_{n,n-k}\big(\xi \in \mathcal{G}_{n,n-k}; \xi \cap (S+K) = \Phi\big) \geq 1 - \frac{7}{2}\exp\left(-\frac{1}{2}\left[\frac{a_k\big(1 - \varepsilon_2(K^*)\big) - \mathbb{E}(K^*) - s}{3 + \varepsilon_2(K^*) + \mathbb{E}(K^*)/a_k}\right]^2\right) \ .$$

Proof: Let $T = \sum_{i=1}^n \sum_{j=1}^k t_{ij}e_i \otimes e_j$ and $\|T\|$ be its norm as a map from $(\mathbb{R}^n, K)$ to ℓ_2^k. It is easily seen that $\big|\|T\| - \|T'\|\big| \leq \varepsilon_2(K^*)\big(\sum_{i,j}(t_{ij} - t'_{ij})^2\big)^{1/2}$, hence $\|T\|$ has Lipschitz constant $\varepsilon_2(K^*)$, so by Theorem 3.2

$$P\big(\|G\| \geq (1+\varepsilon)\mathbb{E}\|G\|\big) \leq \exp\big(-\tfrac{1}{2}(\varepsilon\mathbb{E}\|G\|/\varepsilon_2(K^*))^2\big) \leq \varepsilon(-\varepsilon^2 a_k^2/2) \ ,$$

where $\varepsilon > 0$ is to be chosen later.

Let $\lambda = (1+\varepsilon)\mathbb{E}\|G\|$ and $P = P\big(\omega \ ; \ G(\omega)^{-1}(0) \cap (S+K) = \Phi\big)$.

Setting $Q = P\big(\omega \ ; \ \min_{x \in S}\|G(x)\|_2 \geq \lambda\big)$, we have

$$Q \leq P\big(\omega \ ; \ \min_{x \in G^{-1}(0), y \in S}\|x - y\|_K\|G\| \geq \lambda\big)$$

$$\leq P\big(\|G\| \geq (1+\varepsilon)\mathbb{E}\|G\|\big) + P\big(\min_{x \in G^{-1}(0), y \in S}\|x - y\|_K > 1\big)$$

$$= P\big(\|G\| \geq (1+\varepsilon)\mathbb{E}\|G\|\big) + P \leq \exp(-\varepsilon^2 a_k^2/2) + P \ .$$

On the other hand,

$$Q + \tfrac{1}{2}\exp(-\varepsilon^2 a_k^2/2) \geq Q + P(g \geq \varepsilon a_k) \geq P\big(\bigcap_{x \in S} [\|G(x)\|_2 + g \geq \lambda + \varepsilon a_k]\big)$$

and by Lemma 3.1

$$\geq P\big(\bigcap_{x \in S} [(\sum_1^k g_j^2)^{1/2} + \sum_1^n x_i h_i \geq \lambda + \varepsilon a_k]\big) \equiv R .$$

Moreover, since the Lipschitz constant of the function $(\sum_1^k g_j^2)^{1/2}$ is 1, it follows by Theorem 3.2 that

$$1 - R = P\big(\bigcup_{x \in S} [(\sum_1^k g_j^2)^{1/2} + \sum_1^n x_i h_i < \lambda + \varepsilon a_k]\big)$$

$$\leq P\big((\sum_1^k g_j^2)^{1/2} < (1-\varepsilon)a_k\big) + P\big(\bigcup_{x \in S} [\sum_1^n x_i h_i \leq \lambda - (1-2\varepsilon)a_k]\big)$$

$$\leq \exp(-\varepsilon^2 a_k^2/2) + P\big(\max_{x \in S} \sum_1^n x_i h_i \geq (1-2\varepsilon)a_k - \lambda\big) .$$

Now, the function $f(h) = \max_{x \in S} \sum_1^n x_i h_i$ defined for $h \in I\!\!R^n$, has Lipshitz constant equal to 1, $I\!\!E(f) = s$ and since for the proper choice of $\varepsilon > 0$

$$a_k(1 - 2\varepsilon) - \lambda - s \geq a_k(1 - 2\varepsilon) - s - (1+\varepsilon)\big[a_k \varepsilon_2(K^*) + I\!\!E(K^*)\big] > 0 ,$$

we obtain by Theorem 3.2,

$$P\big(f(h) \geq a_k(1-2\varepsilon) - \lambda\big) \leq \exp\big(-\tfrac{1}{2}\{a_k(1-2\varepsilon) - \lambda - s\}^2\big) .$$

Combining the inequalities we have

$$P = \gamma_{n,n-k}\big(\xi \in \mathcal{G}_{n,n-k} ; \ \xi \cap (K+S) = \Phi\big)$$

$$\geq 1 - \frac{5}{2}\exp(-a_k^2 \varepsilon^2/2) - \exp(-\sigma^2/2) ,$$

where $\sigma = a_k(1 - 2\varepsilon) - s - (1+\varepsilon)\big[a_k \varepsilon_2(K^*) + I\!\!E(K^*)\big]$. We now choose ε so that $\varepsilon a_k = \sigma$, i.e.,

$$\varepsilon a_k = \frac{a_k\big(1 - \varepsilon_2(K^*)\big) - I\!\!E(K^*) - s}{3 + \varepsilon_2(K^*) + I\!\!E(K^*)/a_k} > 0 ,$$

then the required estimate $P \geq 1 - \tfrac{7}{2}\exp(-\varepsilon^2 a_k^2/2)$ follows.   □

Remark (3.3). By continuity Theorem 3.3 remains true if K has dimension lower than n, i.e., when K is a closed convex set which contains the origin (not necessarily in its interior) and $\dim \operatorname{span}(K) < n$. The values $\varepsilon_2(K^*)$ and $I\!E(K^*)$ are then understood as

$$\varepsilon_2(K^*) = \max\{\|x\|_2 \; ; \; x \in K\} \; , \; \text{and}$$

$$I\!E(K^*) = I\!E\big(\max_{x \in K} \sum_1^n x_i g_i\big) \; .$$

Taking for example $K = r B_2^\ell = \big\{x = (x_1, \ldots, x_\ell, 0, \ldots, 0) \; ; \; \sum_1^\ell x_i^2 \le r^2\big\}$, we obtain $\varepsilon_2(K^*) = r$, $I\!E(K^*) = r a_\ell$ and so, if $1 \le k$, $\ell \le n$ and r satisfies $a_k(1 - r) - r a_\ell > s$, then the measure of the k-codimensional subspaces $\xi \in \mathcal{G}_{n,n-k}$ which miss the set $S + K$ is greater than

$$1 - \frac{7}{2} \exp\left(-\frac{1}{2}\left[\frac{a_k(1 - r) - r a_\ell - s}{3 + r + r a_\ell / a_k}\right]^2\right) \; .$$

In particular, taking $r = 0$ we obtain:

Corollary 3.4. *Let $S \subset S_2^{n-1}$ be a closed subset, $s = I\!E\big(\max_{x \in S} \sum_1^n x_i g_i\big)$, $1 \le k < n$ and assume $a_k > s$. Then*

$$\gamma_{n,n-k}(\xi \in \mathcal{G}_{n,n-k} \; ; \; \xi \cap S = \Phi) \ge 1 - \frac{7}{2} \exp\left(-\frac{1}{18}(a_k - s)^2\right) \; .$$

Corollary 3.5. *Let $S \subset S_2^{n-1}$ be a closed set and $s = I\!E\big(\max_{x \in S} \sum_1^n x_i h_i\big)$. If $1 \le k < n$, $\alpha > 0$ and $s < \sqrt{k} - 3\alpha$, then*

$$\gamma_{n,n-k}(\xi \in \mathcal{G}_{n,n-k} \; ; \; \xi \cap S = \Phi) \ge 1 - \frac{7}{2} \exp(-\alpha^2/2) + 0(k^{-1/2}) \; .$$

In particular, if $s < \sqrt{n} - 3\alpha$, then $S \cup (-S)$ has surface area smaller then $\frac{7}{2} \exp(-\alpha^2/2) + 0(n^{-1/2})$.

Proof: Apply Corollary 3.4 and use the fact that the surface area of $S \cup (-S)$ is

$$\gamma_{n,1}(\xi \in \mathcal{G}_{n,1} \; , \; \xi \cap S \ne \Phi) \; . \qquad \square$$

Remark (3.4.). If $0 < \lambda < 1$ and $0 < r < \frac{1}{2}$ are fixed and $s < \lambda(1 - 2r)\sqrt{n}$, then taking in Theorem 3.3, $k = n - 1$ and $K = r B_2^n$ we obtain

$$\gamma_{n,1}\big(\xi \in \mathcal{G}_{n,1} \; , \; \xi \cap (S + r B_2^n) \ne \Phi\big) \le c_1 \exp\big(-f(\lambda, r)n\big) \; ,$$

where c_1 is a constant and $f(\lambda, r)$ a positive function. This says that the surface area $m_{n-1}\big((S + r B_2^n)_{\text{rad}}\big)$ of the radial projection of $(S + r B_2^n) \cup (-S + r B_2^n)$ onto S_2^{n-1} tends to 0 exponentially fast with n. A more careful analysis of this situation is as follows:

Corollary 3.6. *Let* $S \subset S_2^{n-1}$ *be an arbitrary closed set,* $s = I\!\!E\big(\max\limits_{x \in S} \sum\limits_1^n x_i h_i \big)$ *and* t *be any number such that* $s < t < \sqrt{n}$. *Let* $K_t = \frac{1}{2}(1 - tn^{-1/2})B_2^n$. *Then, there are universal constants* $c_1, c_2 > 0$ *such that,* $m_{n-1}\big((S + K_t)_{\mathrm{rad}} \big) \leq c_1 \exp\big(-c_2(x - s)^2 \big)$.

Proof: We apply Theorem 3.3 with $k = n - 1$. Since $a_n = \sqrt{n}\big(1 - \frac{1}{4n} + 0(n^{-2})\big)$, $a_{n-1} = \sqrt{n-1}\big(1 - \frac{1}{4n} + 0(n^{-2})\big) = \sqrt{n}\big(1 - \frac{3}{4n} + 0(n^{-2})\big)$. We also have $\varepsilon_2(K_t^*) = \frac{1}{2}(1 - tn^{-1/2})$ and $I\!\!E(K_t^*) = \frac{1}{2}(1 - tn^{-1/2})a_n$ and the inequality in Theorem 3.3 yields the result. $\quad\square$

Combining the estimates of Corollary 3.4 and Theorem 2.1 we obtain a quantitative version of Theorem 2.1.

Corollary 3.7. *Let* $0 < \varepsilon \leq 1$ *and* $\ell + 1 \leq \frac{1}{2}n\varepsilon^2 m_{n-1}(A)$, *where* $A \subset S_2^{n-1}$ *is an arbitrary closed subset. Then*

$$\gamma_{n,\ell}(\xi \in \mathcal{G}_{n,\ell} \,;\, \xi \cap S_2^{n-1} \subset A_\varepsilon) \geq 1 - c_1 \exp\big\{ -c_2 n^{-1}\big(\tfrac{1}{2}n\varepsilon^2 m_{n-1}(A) - \ell - 1 \big)^2 \big\},$$

where c_1 *and* c_2 *are positive universal constants.*

Proof: Let $k = n - \ell$. By inequality 2.2 with $S = (A_\varepsilon)^C$ we have

$$a_k - s \geq \sqrt{k-1} - s \geq \sqrt{k-1} - \sqrt{n}\sqrt{1 - \tfrac{1}{2}n\varepsilon^2 m_{n-1}(A)}$$
$$\geq \big\{ k - 1 - n\big(1 - \tfrac{1}{2}\varepsilon^2 m_{n-1}(A)\big) \big\}/2\sqrt{n}$$
$$\geq \big\{ \tfrac{1}{2}n\varepsilon^2 m_{n-1}(A) - \ell - 1 \big\}/2\sqrt{n} > 0$$

and Corollary 3.4 concludes the proof. $\quad\square$

Remark(3.5). Corollary 3.7 and Lévy's isoperimetric inequality imply that if $n\varepsilon^2$ is big, then the set of subspaces E of dimension $\sim n\varepsilon^2$ for which (2.3) is satisfied has probability close to 1.

4. Missing Convex Sets by Random Subspaces

Let $\{K_i\}_1^N$ be a sequence of N closed convex sets in $I\!\!R^n$ which contain the origin 0 in their relative interiors and let $\{z_i\}_1^N$ be arbitrary points in $I\!\!R^n$. We denote by t_i (resp., T_i) the smallest (resp., largest) Euclidean distance of a point in $z_i + K_i$ to the origin and set $I\!\!E(K_i^*) = I\!\!E\big(\max\limits_{x \in K_i} \sum\limits_{j=1}^n x_j h_j \big)$. In the next theorem we estimate from below the measure of the set of all k-codimensional subspaces ξ in $\mathcal{G}_{n,n-k}$ which are distant d_i at least from $z_i + K_i$ for all $i = 1, 2, \ldots, N$.

Theorem 4.1. $\gamma_{n,n-k}\big\{\xi \in \mathcal{G}_{n,k-k} \ ; \ d(\xi, z_i + K_i) \geq d_i \text{ for all } i = 1, 2, \ldots, N\big\}$

$$\geq 1 - \exp(-\delta^2 a_n^2/2) - (3/2)\exp(-\varepsilon^2 a_k^2/2)$$

$$- \sum_{i=1}^{N} \exp\big(-\big\{(1-2\varepsilon)a_k t_i - \mathbb{E}(K_i^*) - (1+\delta)d_i(a_n + a_k)\big\}^2/2T_i^2\big) \ ,$$

where $\varepsilon, \delta > 0$ and where we assume $(1 - 2\varepsilon)a_k t_i > \mathbb{E}(K_i^*) + d_i(1 + \delta)(a_n + a_k)$ for all $i = 1, \ldots, N$.

Proof: Let $\|G\|$ be the norm of $G = \sum_{i=1}^{n}\sum_{j=1}^{k} g_{ij}(\omega)e_i \otimes e_j$ as a map of ℓ_2^n to ℓ_2^k. We have $a_n \leq \mathbb{E}\|G\| \leq a_n + a_k$. Let $\lambda_i = d_i(1+\delta)\mathbb{E}\|G\|$ and

$$P = \gamma_{n,n-k}\big(\xi \in \mathcal{G}_{n,n-k} \ ; \ d(\xi, z_i + K_i) \geq d_i \ , \ 1 = 1, 2, \ldots, N\big)$$

$$= P\big(\omega \ ; \ d\big(G^{-1}(0), z_i + K_i\big) \geq d_i \ , \ i = 1, 2, \ldots, N\big) \ .$$

Observe that

$$Q := P\big(\forall i \ , \ \min_{x \in z_i + K_i} \|G(x)\|_2 \geq \lambda_i\big)$$

$$\leq P\big(\forall i \ , \ \|G\| d\big(G^{-1}(0), z_i + K_i\big) \geq \lambda_i\big)$$

$$\leq P\big(\|G\| \geq (1+\delta)\mathbb{E}\|G\|\big) + P\big(\forall i, d(G^{-1}(0), z_i + K_i) \geq d_i\big)$$

$$\leq \exp\left(-\frac{\delta^2}{2}(\mathbb{E}\|G\|)^2\right) + P \ .$$

Now, apply Lemma 3.1 with $\lambda_x = \lambda_i + \varepsilon a_k \|x\|_2$ whenever $x \in z_i + K_i$ and with $S = \bigcup_{1}^{N}(z_i + K_i)$, to get a lower estimate for Q

$$Q + \tfrac{1}{2}\exp(-\varepsilon^2 a_k^2/2) \geq Q + P(g > a_k\varepsilon)$$

$$\geq P\big(\bigcap_{x \in S}\big[\|G(x)\|_2 + \|x\|_2 g \geq \lambda_x\big]\big)$$

$$\geq P\big(\bigcap_{x \in S}\big[\|x\|_2\big(\sum_{1}^{k} g_j^2\big)^{1/2} + \langle x, h\rangle \geq \lambda_x\big]\big) \equiv R \ .$$

But

$$1 - R = P\big(\bigcup_{x \in S}\big[\|x\|_2\big(\sum_{1}^{k} g_j^2\big)^{1/2} + \langle x, h\rangle \leq \lambda_x\big]\big)$$

$$\leq P\big(\big(\sum_{1}^{k} g_j^2\big)^{1/2} \leq (1-\varepsilon)a_k\big) + P\big(\bigcup_{i=1}^{N}\bigcup_{x \in z_i + K_i}\big[\langle x, h\rangle \leq \lambda_i - (1-2\varepsilon)a_k\|x\|_2\big]\big)$$

$$\leq \exp(-\varepsilon^2 a_k^2/2) + \sum_{i=1}^{N} P\big(\bigcup_{x \in z_i + K^i}\big[\langle x, h\rangle \geq (1-2\varepsilon)a_k\|x\|_2 - \lambda_i\big]\big)$$

$$\leq \exp(-\varepsilon^2 a_k^2/2) + \sum_{i=1}^{N} P\big(\bigcup_{x \in z_i + K_i}\big[\langle x, h\rangle \geq (1-2\varepsilon)a_k t_i - \lambda_i\big]\big) \ .$$

Since the function $f(h) = \max\limits_{x \in z_i + K_i} \langle x, h \rangle$, defined for $h \in I\!\!R^n$, has Lipschitz constant T_i and, since $I\!\!E(f) = I\!\!E(K_i^*)$ and, given the assumption $(1 - 2\varepsilon)a_k t_i > \lambda_i + I\!\!E(K_i^*)$, it follows from Theorem 3.2 that $P\big(f \geq (1 - 2\varepsilon)a_k t_i - \lambda_i\big) \leq \exp\big(-\big((1 - 2\varepsilon)a_k t_i - \lambda_i - I\!\!E(K_i^*)\big)^2 / 2T_i^2\big)$, concludes the proof. □

Taking $\delta \to \infty$, $(1 + \delta)d_i \downarrow 0$ for each i, we obtain the measure of all k-codimensional subspaces which miss $\bigcup\limits_1^N (z_i + K_i)$.

Corollary 4.2. *Under the assumptions $\varepsilon > 0$ and $(1 - 2\varepsilon)a_k t_i > I\!\!E(K_i^*)$ for all $i = 1, \ldots, N$, we have*

$$\gamma_{n,n-k}\big(\xi \in \mathcal{G}_{n,n-k} \; ; \; \xi \cap (z_i + K_i) = \Phi \quad \text{for all} \quad i = 1, \ldots, N\big)$$
$$\geq 1 - \frac{3}{2}\exp(-\varepsilon^2 a_k^2 / 2) - \sum_{i=1}^N \exp\big(-\big\{(1 - 2\varepsilon)a_k t_i - I\!\!E(K_i^*)\big\}^2 / 2T_i^2\big) \; .$$

5. Random Subspaces which Escape through at least One Mesh

Theorem 1.4 gave a sufficient condition for the existence of a subspace of given codimension k which misses a given subset of the sphere. However, it may happen that every subspace of codimension k hits the set and yet there might be a subspace which misses a "piece" of this set. To study such cases we consider a finite collection A_ℓ $(\ell = 1, 2, \ldots, L)$ of closed subsets of S_2^{n-1} and ask for a condition which implies that there exists a subspace of codimension k which misses at least one of the sets A_ℓ. To obtain this condition we shall use the following theorem proved in [G3], of which Theorem A is a special case.

Theorem C. *Let $f(x) = \min\limits_{i_1}\max\limits_{i_2}\min\limits_{i_3}\max \ldots x_{i_1,i_2,\ldots,i_k}$ where for each $\ell = 1, 3, \ldots, k$, the index i_ℓ ranges over some finite non-empty set $C(i_1, i_2, \ldots, i_{\ell-1})$ (which, as indicated, shows that this set may depend on the previous choice of $i_1, i_2, \ldots, i_{\ell-1}$). Given two distinct vector indices $i = (i_1, \ldots, i_k)$ and $j = (j_1, \ldots, j_k)$ let $m = m(i,j)$ be the first coordinate such that $i_m \neq j_m$. If $\{X_i\}$ and $\{Y_i\}$ are two Gaussian processes such that*

(i) $I\!\!E|X_i - X_j|^2 \leq I\!\!E|Y_i - Y_j|^2$ *if m is even*

(ii) $I\!\!E|X_i - X_j|^2 \geq I\!\!E|Y_i - Y_j|^2$ *if m is odd,*

then $I\!\!E f(X) \leq I\!\!E f(Y)$.

Theorem A is obtained from Theorem C by taking $k = 2$ and $1 \leq i_1 \leq n$, $1 \leq i_2 \leq m$.

Corollary 5.1. *Let $\{A_\ell\}_{\ell=1}^{L}$ be a finite collection of closed subsets of the sphere S_2^{n-1} and $G(\omega) = \sum_{i=1}^{n} \sum_{j=1}^{k} g_{ij}(\omega) e_i \otimes e_j$ a Gaussian operator from $\mathbb{R}^n$ to $\mathbb{R}^k$. Then*

$$\mathbb{E}\Big(\max_\ell \big\{ \min_{x \in A_\ell} \|G(\omega)(x)\|_2 + 2g_\ell \big\}\Big) \geq a_k - a_n \int\limits_{S_2^{n-1}} \min_\ell \max_{x \in A_\ell} \langle x, u \rangle m_{n-1}(du) \ .$$

Proof: We apply Theorem C and for each $\ell \in \{1, 2, \ldots, L\}$, $x \in A_\ell$ and $y \in S_2^{k-1}$ we define the two triple indexed Gaussian processes

$$Y_{\ell,x,y} = \sum_{j=1}^{k} g_j y_j + \sum_{i=1}^{n} h_i x_i \quad \text{and} \quad X_{\ell,x,y} = \langle G(\omega)x, y \rangle + 2g_\ell \ .$$

For any two triples $\alpha = (\ell, x, y)$ and $\beta = (\ell', x', y')$ we have

$$\mathbb{E}|Y_\alpha - Y_\beta|^2 = \sum_{j=1}^{k} (y_j - y_j')^2 + \sum_{i=1}^{n} (x_i - x_i')^2 = 4 - 2\langle x, x' \rangle - 2\langle y, y' \rangle$$

and

$$\mathbb{E}|X_\alpha - X_\beta|^2 = \sum_{i=1}^{n} \sum_{j=1}^{k} (x_i y_j - x_i' y_j')^2 + 8(1 - \delta_{\ell,\ell'}) = 10 - 8\delta_{\ell,\ell'} - 2\langle x, x' \rangle \langle y, y' \rangle \ .$$

Therefore if $\ell = \ell'$ and $x \neq x'$, i.e., when $m(\alpha, \beta) = 2$, then

$$\mathbb{E}|Y_\alpha - Y_\beta|^2 - \mathbb{E}|X_\alpha - X_\beta|^2 = 2(1 - \langle x, x' \rangle)(1 - \langle y, y' \rangle) \geq 0 \ .$$

To check condition (ii) for Theorem C, notice that if $\ell = \ell'$ and $x = x'$ then $\mathbb{E}|X_\alpha - X_\beta|^2 = \mathbb{E}|Y_\alpha - Y_\beta|^2$, and if $\ell \neq \ell'$ then

$$\mathbb{E}|Y_\alpha - Y_\beta|^2 - \mathbb{E}|X_\alpha - X_\beta|^2 = -2(3 + \langle x, x' \rangle + \langle y, y' \rangle - \langle x, x' \rangle \langle y, y' \rangle) \leq 0 \ ,$$

hence, by Theorem C, $\mathbb{E} \min_\ell \max_{x \in A_\ell} \min_{y \in S_2^{k-1}} X_{\ell,x,y} \leq \mathbb{E} \min_\ell \max_{x \in A_\ell} \min_{y \in S_2^{k-1}} Y_{\ell,x,y}$ therefore, replacing X by $-X$ and Y by $-Y$ we obtain

$$\mathbb{E} \max_\ell \min_x \max_y X_{\ell,x,y} = \mathbb{E} \max_\ell \big\{ \min_{x \in A_\ell} \|G(\omega)(x)\|_2 + 2g_\ell \big\}$$

$$\geq \mathbb{E} \max_\ell \min_x \max_y Y_{\ell,x,y} = \mathbb{E}\Big(\sum_{1}^{k} g_j^2 \Big)^{1/2} + \mathbb{E} \max_\ell \min_x \sum_{1}^{n} h_i x_i$$

$$= a_k - \mathbb{E} \min_\ell \max_x \sum_{1}^{n} h_i x_i = a_k - a_n \int\limits_{S_2^{n-1}} \min_\ell \max_{x \in A_\ell} \langle x, u \rangle m_{n-1}(du) \ .$$

$\square$

As a conclusion we obtain the following generalization of Theorem 1.4.

Theorem 5.2. *Let $\{A_\ell\}_{\ell=1}^L$ be a finite collection of closed subsets of the sphere S_2^{n-1} and $1 \le k \le n$. If*

$$(5.1) \qquad a_k > a_n \int_{S_2^{n-1}} \min_\ell \max_{x \in A_\ell} \langle x, u \rangle m_{n-1}(du) + c\sqrt{\log L}$$

then there is a subspace of codimension k which misses at least one of the sets A_ℓ.

Proof: Since $I\!E \max_\ell 2g_\ell \le c\sqrt{\log L}$ we obtain from Corollary 5.1

$$I\!E \max_\ell \min_{x \in A_\ell} \|G(\omega)(x)\|_2 \ge a_k - a_n \int_{S_2^{n-1}} \min_\ell \max_{x \in A_\ell} \langle x, u \rangle m_{n-1}(du) - c\sqrt{\log L} > 0 .$$

But, since the subspace $E_\omega = G(\omega)^{-1}\{0\}$ has codimension k a.e., there is a subspace E_{ω_0} of codimension k which misses at least one of the sets A_ℓ. □

We now give two different sufficient conditions on the sets A_ℓ which imply inequality (5.1) and hence the conclusion of Theorem 5.2.

Corollary 5.3. *Let $\{A_\ell\}_{\ell=1}^L$ be closed subsets of S_2^{n-1}, $A = \bigcup_1^L A_\ell$ and $1 \le k < n$. Assume one of the following two conditions are satisfied:*
(a) there exists $0 < \gamma < \pi$ such that $\bigcap_{\ell=1}^L (A_\ell)_\gamma = \phi$ and

$$(5.2) \qquad a_k > a_n \cos \gamma + c\sqrt{\log L} .$$

(b) Let α, β $(0 \le \alpha, \beta \le \pi)$ be defined by

$$\alpha = \max_\ell \operatorname{diam} A_\ell \quad \text{and} \quad \beta = \sup \left\{ t \ge 0 \; ; \; (-A^c) \bigcap_{\ell=1}^L (A_\ell)_t = \phi \right\}$$

(with $\beta = 0$ if $(-A^C) \bigcap_{\ell=1}^L A_\ell \ne \phi$) and assume

$$(5.3) \qquad a_k > a_n \{ -m_{n-1}(A) \cos \alpha + m_{n-1}(A^C) \cos \beta \} + c\sqrt{\log L} .$$

Then inequality (5.1) is satisfied.

Proof: We shall estimate the integral which appears in (5.1). If condition (a) holds, then for every $u \in S_2^{n-1}$, $u \notin (A_{\ell_0})_\gamma$ for some ℓ_0, hence for every $x \in A_{\ell_0}$ the geodesic distance $\rho(x, u) \ge \gamma$, therefore

$$\min_\ell \max_{x \in A_\ell} \langle x, u \rangle \le \max_{x \in A_{\ell_0}} \langle x, u \rangle \le \cos \gamma$$

which shows that (5.2) implies (5.1). If condition (b) holds, let $u \in S_2^{n-1}$. If $u \in -A$, then $-u \in A$, so $-u \in A_{\ell_0}$ for some ℓ_0. But then for all $x \in A_{\ell_0}$ we have

$$\rho(x, u) \geq \rho(-u, u) - \rho(-u, x) \geq \pi - \alpha$$

hence

$$\min_\ell \max_{x \in A_\ell} \langle x, u \rangle \leq \max_{x \in A_{\ell_0}} \langle x, u \rangle \leq \cos(\pi - \alpha) = -\cos\alpha .$$

On the other hand, let $0 \leq t < \beta$ (take $t = 0$ if $\beta = 0$). If $u \in -A^C$, then $u \notin (A_{\ell_0})_t$ for some ℓ_0, so $\rho(u, x) \geq t$ for every $x \in A_{\ell_0}$, hence

$$\min_\ell \max_{x \in A_\ell} \langle x, u \rangle \leq \max_{x \in A_{\ell_0}} \langle x, u \rangle \leq \cos t \xrightarrow[t \to \beta]{} \cos\beta .$$

Therefore,

$$\int_{S_2^{n-1}} = \int_{-A} + \int_{-A^C} \leq -m_{n-1}(-A)\cos\alpha + m_{n-1}(-A^C)\cos\beta$$

$$= -m_{n-1}(A)\cos\alpha + m_{n-1}(A^c)\cos\beta .$$

This proves that (5.3) implies (5.1). □

Remark (5.1). Corollary 5.3(a) is illustrated by the following example. *Let $0 < \varepsilon, \delta < 1$, $\gamma = \sqrt{2\delta} n^{-(1-\varepsilon)/4}$ and $L \leq \exp(c_1 \delta^2 n^\varepsilon)$ be an integer. If $\{A_\ell\}_{\ell=1}^L$ is a collection of closed subsets of S_2^{n-1} such that $\bigcap_{\ell=1}^L (A_\ell)_\gamma = \phi$, then there exists a subspace of dimension $m \geq \delta n^{(\varepsilon+1)/2}$ in $\mathbb{R}^n$ which misses at least one A_ℓ, provided $n \geq N(\varepsilon)$. (c_1 denotes a positive absolute constant.)* The proof of this is done by estimating the quantities which appear in inequality 5.2(a). Take $k \sim n - \delta n^{(\varepsilon+1)/2}$, so $a_k \sim k^{1/2} \sim n^{1/2}(1 - \frac{1}{2}\delta n^{(\varepsilon-1)/2})$, $a_n \sim n^{1/2}$, $\cos\gamma \sim 1 - \gamma^2/2 = 1 - \delta n^{-(1-\varepsilon)/2}$ (since $\gamma \xrightarrow[n \to \infty]{} 0$) and choose c_1 to be a suitable positive constant.

As done in section 3, we can assign the probability measure $\gamma_{n,n-k}$ to the set of all subspaces $\xi \in \mathcal{G}_{n,n-k}$ which miss at least one of the sets A_ℓ ($1 \leq \ell \leq 1$). For this estimate we need the following theorem of [G3] which extends Theorem B above.

Theorem D. *Let $f(x)$ and the processes $\{X_i\}, \{Y_i\}$ be as in Theorem C and in addition to conditions (i) and (ii) we assume also that $\mathbb{E}X_i^2 = \mathbb{E}Y_i^2$ for all i. Then for all choices of the real sequence $\{\lambda_i\}$,*

$$P\left(\bigcap_{i_1} \bigcup_{i_1} \bigcap_{i_3} \cdots [X_i \geq \lambda_i]\right) \leq P\left(\bigcap_{i_1} \bigcup_{i_2} \bigcap_{i_3} \cdots [Y_i \geq \lambda_i]\right) .$$

Applying this together with Theorem 3.2 we obtain

Theorem 5.4. *Let $\{A_\ell\}_{\ell=1}^{L}$ be a finite collection of closed subsets of S_2^{n-1} and $1 \le k < n$. If inequality (5.1) holds then $\gamma_{n,n-k}(\xi \in \mathcal{G}_{n,n-k}$; there exists ℓ such that $\xi \cap A_\ell = \phi) \ge 1 - \frac{7}{2}\exp\left(-(a_k - s - c\sqrt{\log L})^2/2(4 + \sqrt{3})^2\right)$, where $s = a_n \int_{S_2^{n-1}} \min_{\ell} \max_{x \in A_\ell} \langle x, u \rangle m_{n-1}(du)$.*

Proof: Let $X_{\ell,x,y}$ and $Y_{\ell,x,y}$ be defined as in the proof of Corollary 5.1 and $Z_{\ell,x,y} = Y_{\ell,x,y} + \sqrt{3}g$ where g is an $N(0,1)$ normal variable. The two processes $\{-X_{\ell,x,y}\}$ and $\{-Z_{\ell,x,y}\}$ satisfy the conditions of Theorem D and therefore

$$P(\bigcap_{\ell}\bigcup_{x}\bigcap_{y}[-X_{\ell,x,y} \ge -a]) \le P(\bigcap_{\ell}\bigcup_{x}\bigcap_{y}[-Z_{\ell,x,y} \ge -a])$$

for all $a \in \mathbb{R}^1$, hence

$$P(\bigcup_{\ell}\bigcap_{x}\bigcup_{y}[X_{\ell,x,y} \ge a]) \ge P(\bigcup_{\ell}\bigcap_{x}\bigcup_{y}[Z_{\ell,x,y} \ge a]) \ .$$

We set $a = c\sqrt{\log L} + 2\varepsilon a_k$ where $\mathbb{E}\max 2g_\ell \le c\sqrt{\log L}$ and $\varepsilon > 0$ is to be chosen later. Then setting $P = P\left(\max_{\ell}\min_{x \in A_\ell}\|G(\omega)(x)\|_2 > 0\right)$ we have

$$P(\bigcup_{\ell}\bigcap_{x}\bigcup_{y}[X_{\ell,x,y} \ge a]) \le P\left(\max_{\ell}\min_{x \in A_\ell}\|G(\omega)(x)\|_2 > 0\right) +$$
$$+ P(\max_{\ell} 2g_\ell - \mathbb{E}\max_{\ell} 2g_\ell \ge 2\varepsilon a_k) \le P + \exp(-\varepsilon^2 a_k^2/2) \ .$$

The last inequality is due to Theorem 3.2 and the fact that the function $f(x) = \max_{\ell} x_\ell$ has Lipschitz constant 1. By Theorem D and Theorem 3.2 we get

$$P + \exp(-\varepsilon^2 a_k^2/2) \ge P\left[\left(\sum_1^k g_j^2\right)^{1/2} + \max_{\ell}\min_{x \in A_\ell}\sum_1^n h_i x_i + \sqrt{3}g \ge a\right] \equiv R \ .$$

$$1 - R \le P\left(\left(\sum_1^k g_j^2\right)^{1/2} \le (1-\varepsilon)a_k\right) + P(\sqrt{3}g \le -\sqrt{3}\varepsilon a_k)$$

$$+ P\left(\max_{\ell}\min_{x \in A_\ell}\sum_1^n h_i x_i \le a + \sqrt{3}\varepsilon a_k - (1-\varepsilon)a_k\right)$$

$$\le \exp(-\varepsilon^2 a_k^2/2) + \tfrac{1}{2}\exp(-\varepsilon^2 a_k^2/2)$$

$$+ P\left(\min_{\ell}\max_{x \in A_\ell}\sum_1^n h_i x_i \ge (1-\varepsilon)a_k - a - \sqrt{3}\varepsilon a_k\right) \ .$$

Now, the function $f(h) = \min_{\ell}\max_{x \in A_\ell}\sum_1^n x_i h_i$ has Lipschitz constant 1 and $\mathbb{E}(f) = s$, hence we can apply Theorem 3.2 again and obtain

$$P\left(f(h) \ge (1-\varepsilon)a_k - a - \sqrt{3}\varepsilon a_k\right) = P\left(f(h) - s \ge (1-\varepsilon)a_k - a - \sqrt{3}\varepsilon a_k - s\right)$$

$$\le \exp(-\varepsilon^2 a_k^2/2)$$

where we select ε so that $\varepsilon a_k = (1-\varepsilon)a_k - a - \sqrt{3}\varepsilon a_k - s$, i.e., $\varepsilon a_k = \left(a_k - c\sqrt{\log L} - s\right)/(4+\sqrt{3})$.

$\square$

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