# Linear Classifiers: Expressiveness

Machine Learning

THE UNIVERSITY OF UTAH

# Lecture outline

- Linear models: Introduction

- What functions do linear classifiers express?

# Where are we?

- Linear models: Introduction
- What functions do linear classifiers express?
  - Conjunctions and disjunctions
  - m-of-n functions
  - Not all functions are linearly separable
  - Feature space transformations
  - Exercises

# Which Boolean functions can linear classifiers represent?

- Linear classifiers are an expressive hypothesis class

- Many Boolean functions are linearly separable
  - Not all though
  - Recall: In comparison, decision trees can represent any Boolean function

# Conjunctions and disjunctions

Consider this truth table of a conjunction

| $x_1$ | $x_2$ | $x_3$ | y |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

$y = 1$ if and only if *all* the $x$'s are 1

# Conjunctions and disjunctions

$y = x_1 \land x_2 \land x_3$ is equivalent to "$y = 1$ whenever $x_1 + x_2 + x_3 \geq 3$"

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| $x_1$ | $x_2$ | $x_3$ | y | $x_1 + x_2 + x_3 - 3$ | sign |
|---|---|---|---|---|---|
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Negations are okay too.
In general, use $1 - x$ in the linear threshold unit if $x$ is negated

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corresponds to

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Questions?

# m-of-n functions

m-of-n rules

- There is a fixed set of n variables
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- All other variables are ignored

Suppose there are five Boolean variables: $x_1$, $x_2$, $x_{3,}$ $x_4$, $x_5$

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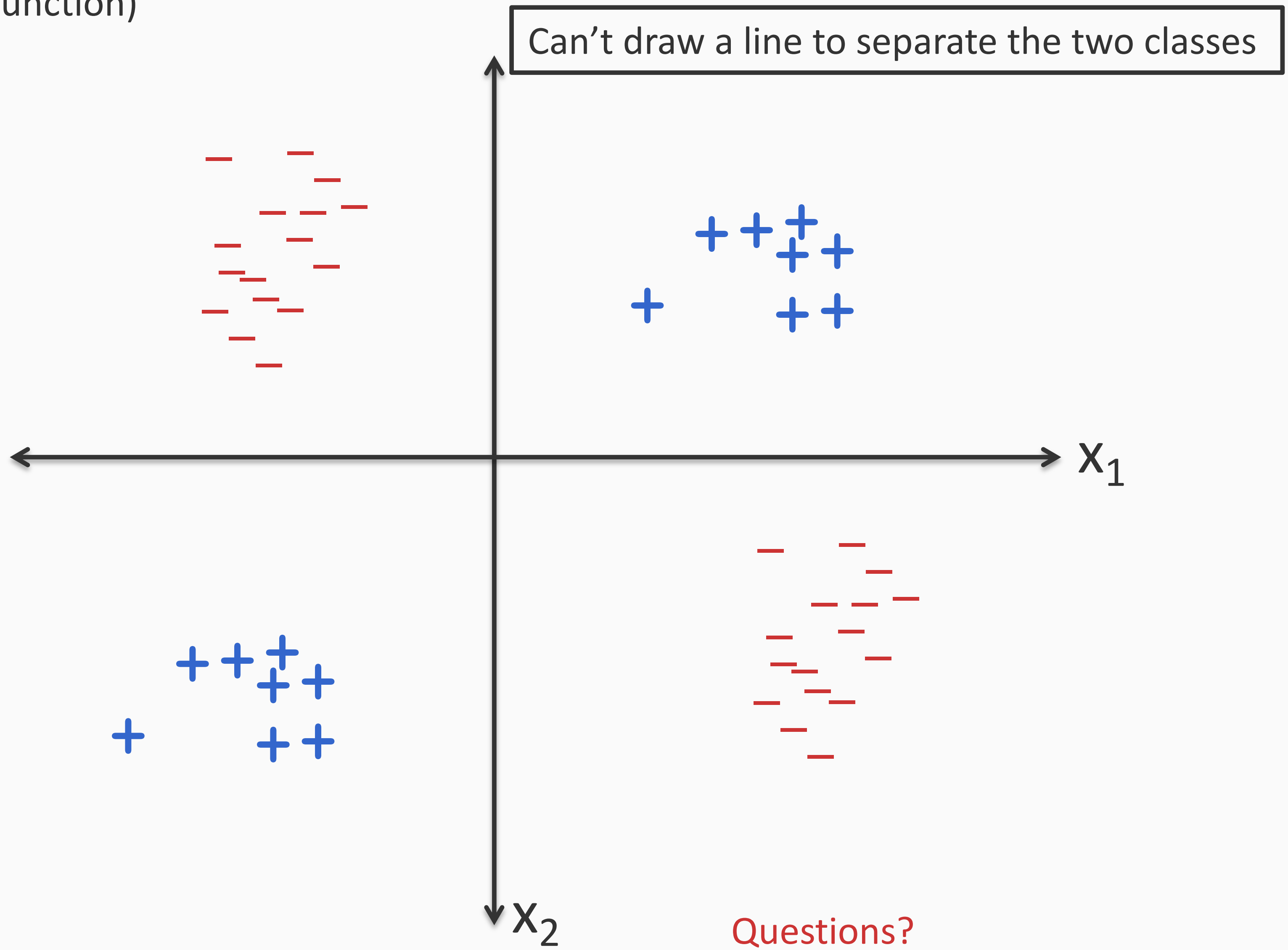
Questions?

*Not all functions are linearly separable*

# Parity is not linearly separable

(The XOR function)




Questions?

# Not all functions are linearly separable

- XOR is not linear
  - $y = x \text{ XOR } y = (x \land \neg y) \lor (\neg x \land y)$
  - *Parity* cannot be represented as a linear classifier
    - f(**x**) = 1 if the number of 1's is even

- Many non-trivial Boolean functions
  - Example: $y = (x_1 \land x_2) \lor (x_3 \land \neg x_4)$
  - The function is not linear in the four variables

# Even these functions can be *made* linear

These points are not separable in 1-dimension by a line

What is a one-dimensional line, by the way?

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The trick: Change the representation

# The blown up feature space

## The trick: Use feature *conjunctions*

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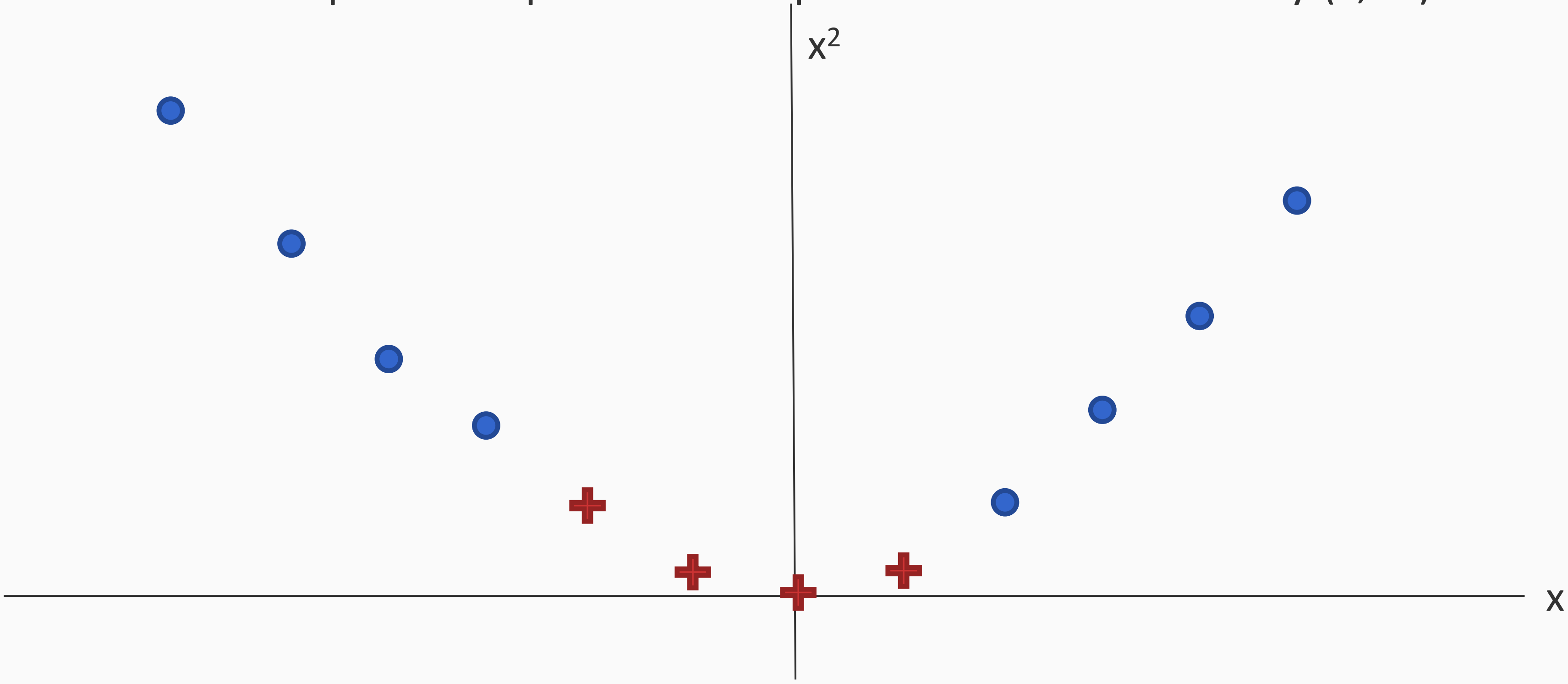

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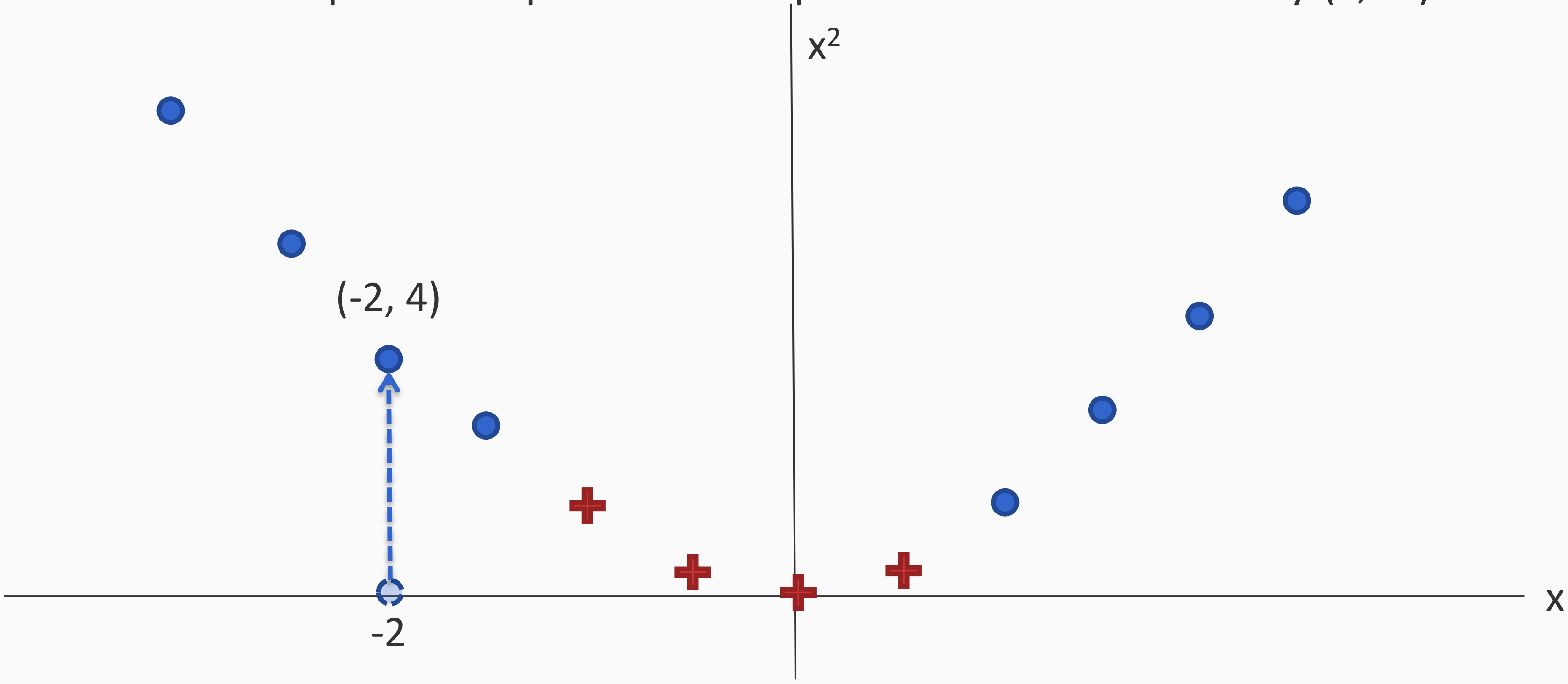

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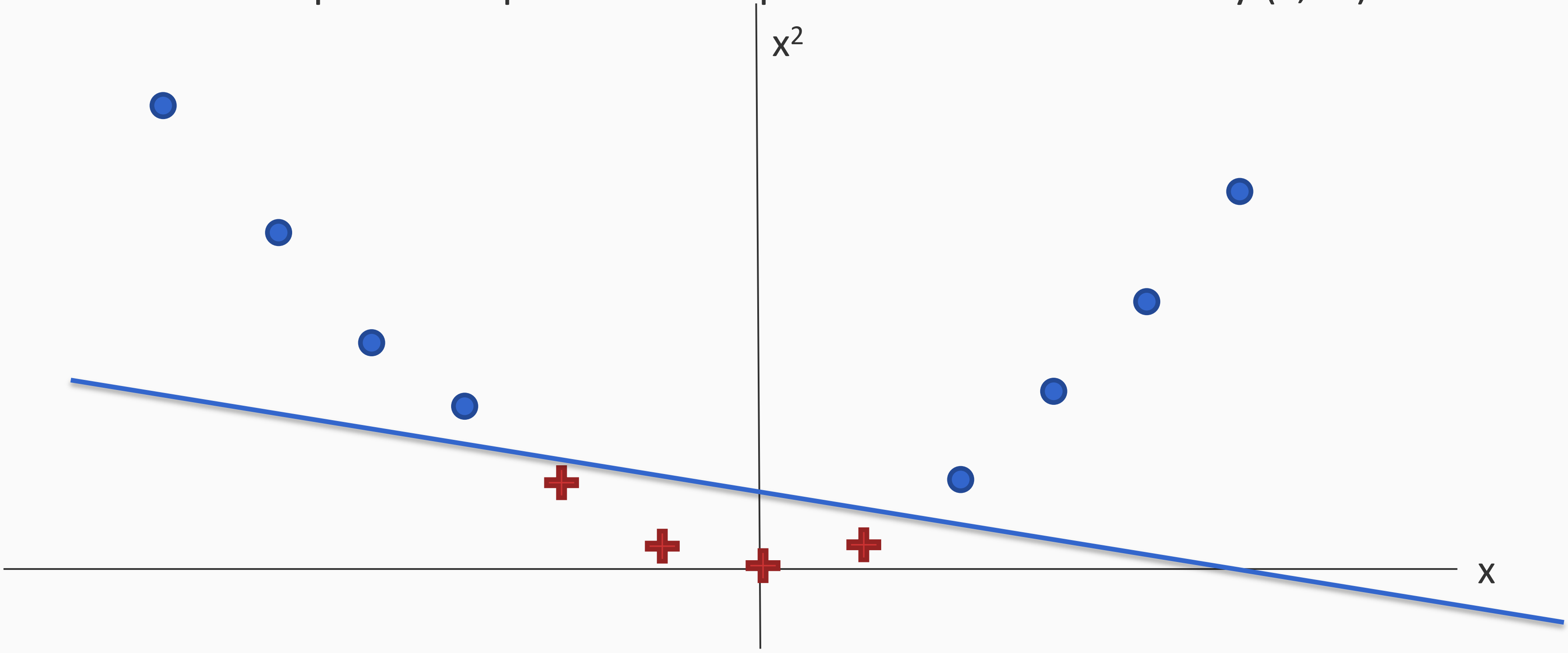



Now the data is linearly separable in this space!

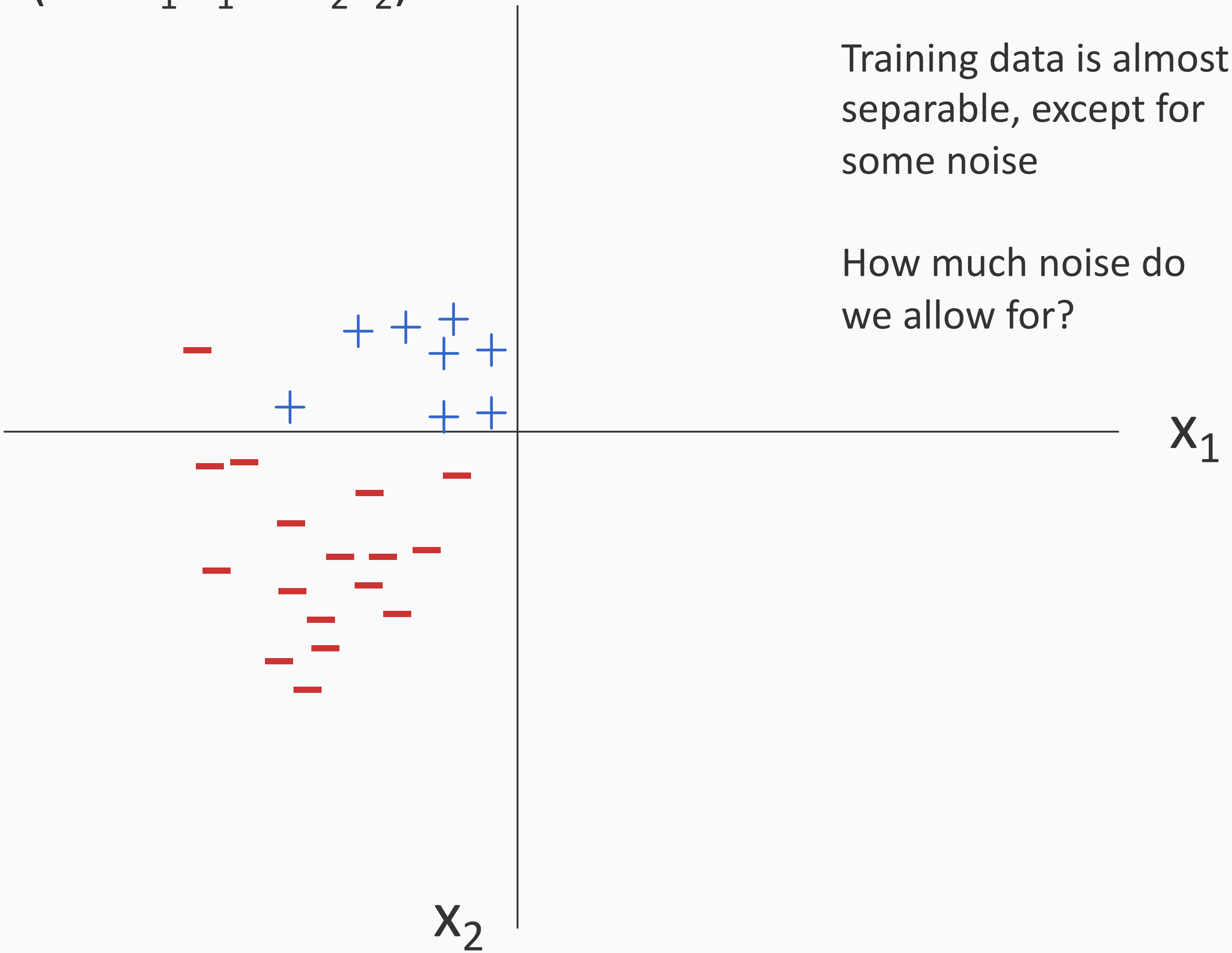
# Exercise

How would you use the feature transformation idea to make XOR  in two dimensions linearly separable in a new space?

To answer this question, you need to think about a function that maps examples from two dimensional space to a higher dimensional space.

# Almost linearly separable data

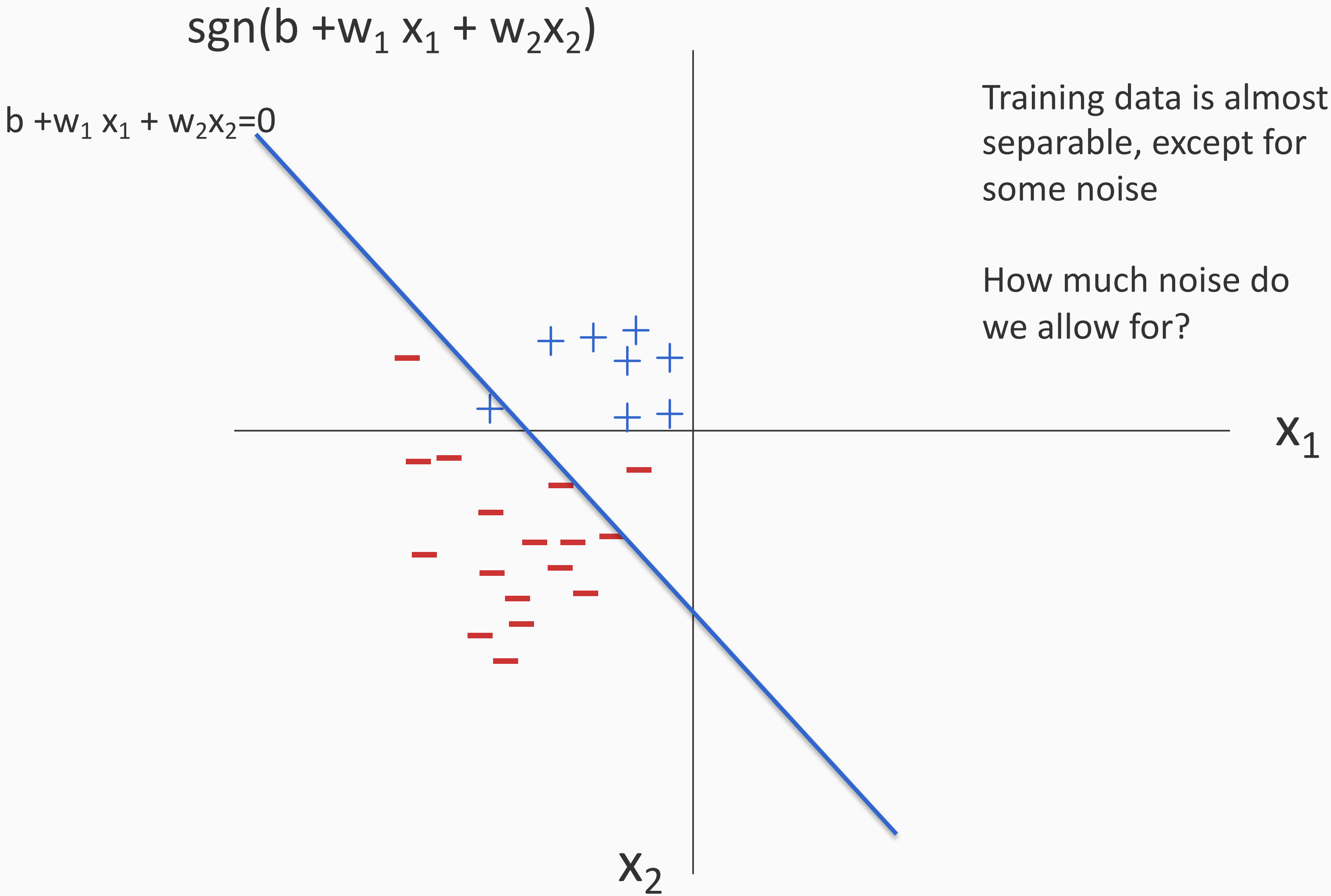
$\mathrm{sgn}(b + w_1 x_1 + w_2 x_2)$




Training data is almost separable, except for some noise

How much noise do we allow for?

# Almost linearly separable data



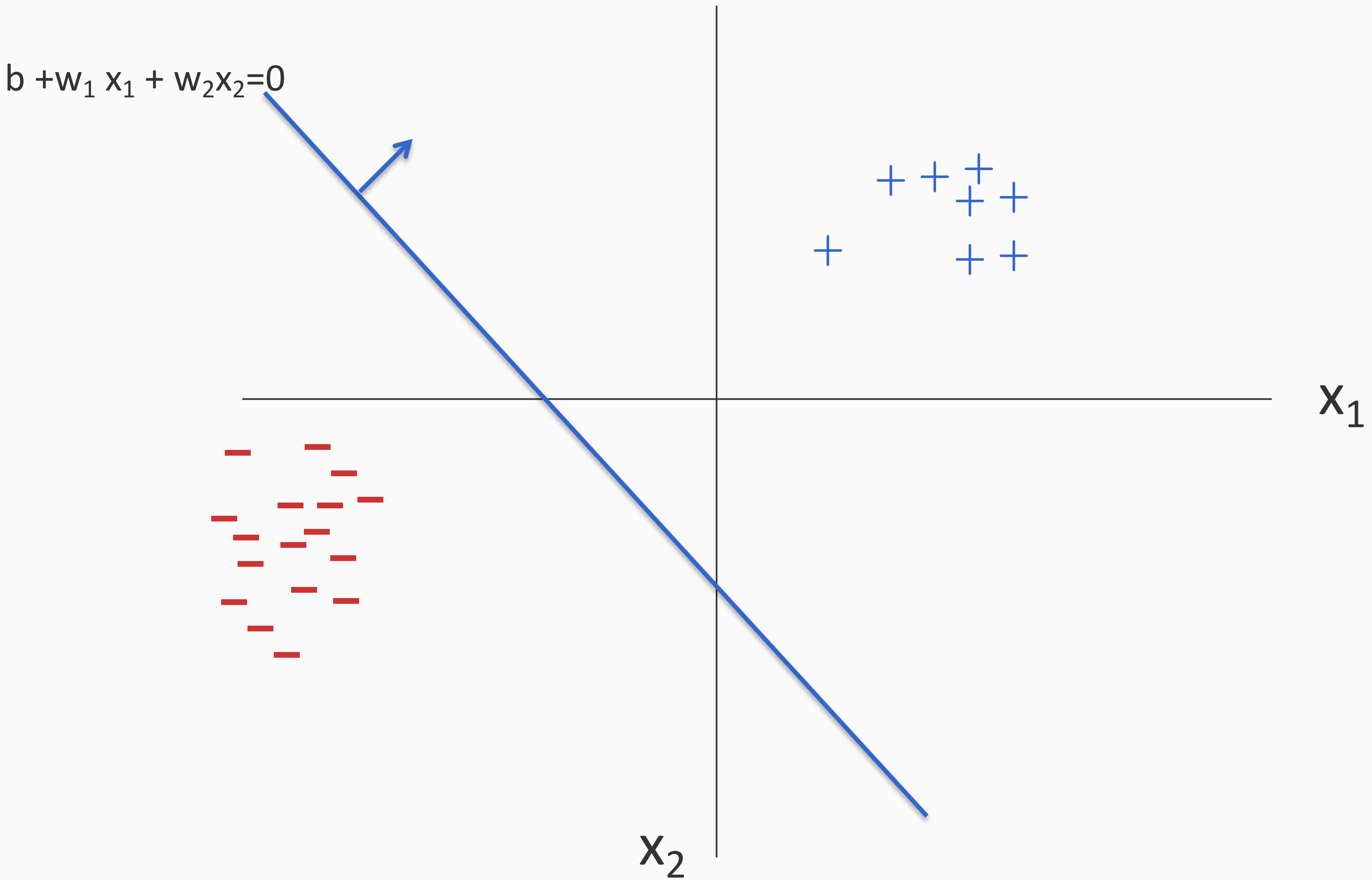

Training data is almost separable, except for some noise

How much noise do we allow for?

# Linear classifiers: An expressive hypothesis class

- Many functions are linear
- Often a good first guess for a hypothesis space
- Some functions are not linear
  - The XOR function
  - Non-trivial Boolean functions
- But there are ways of making them linear in a higher dimensional feature space

# Why is the bias term needed?



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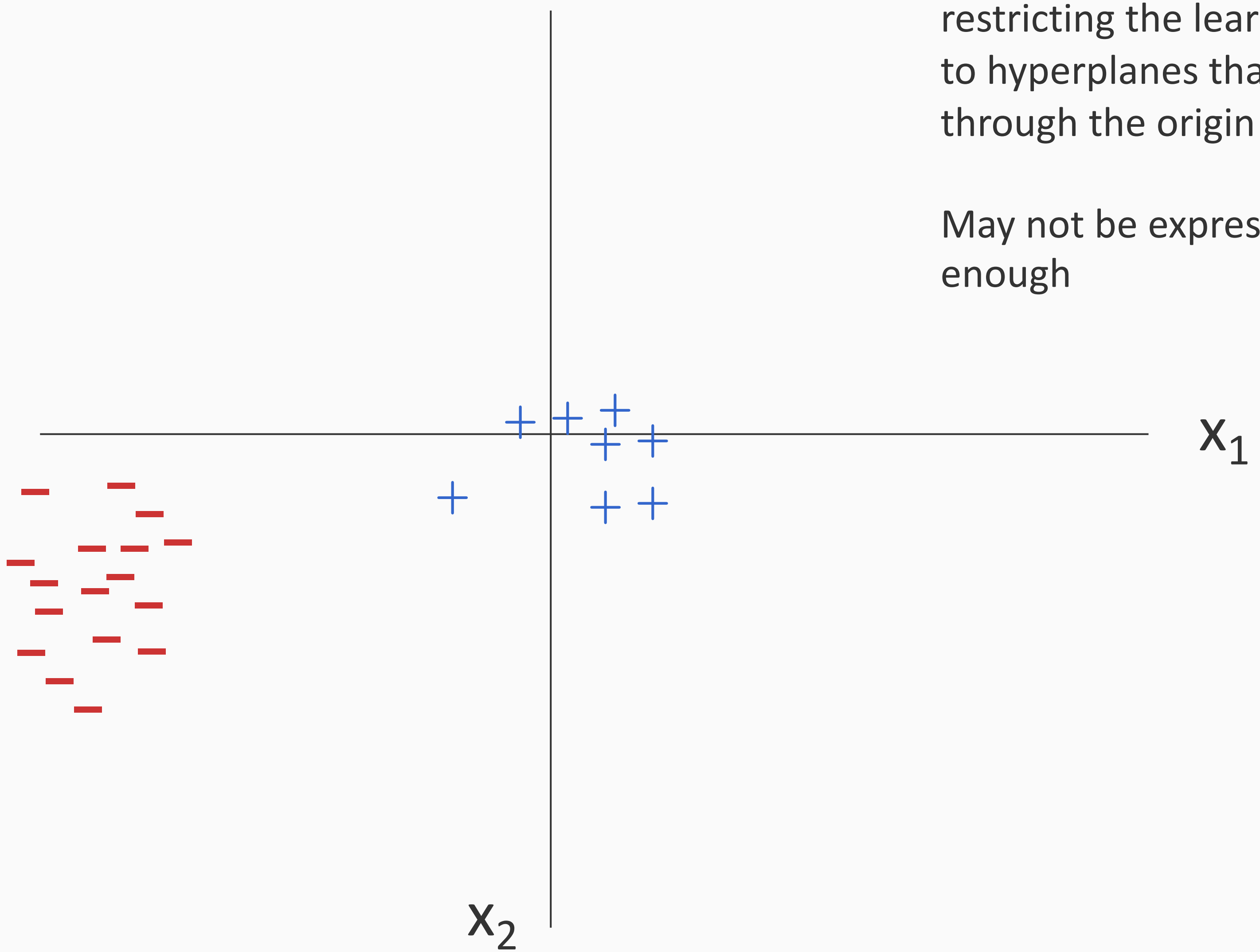


If b is zero, then we are restricting the learner only to hyperplanes that go through the origin

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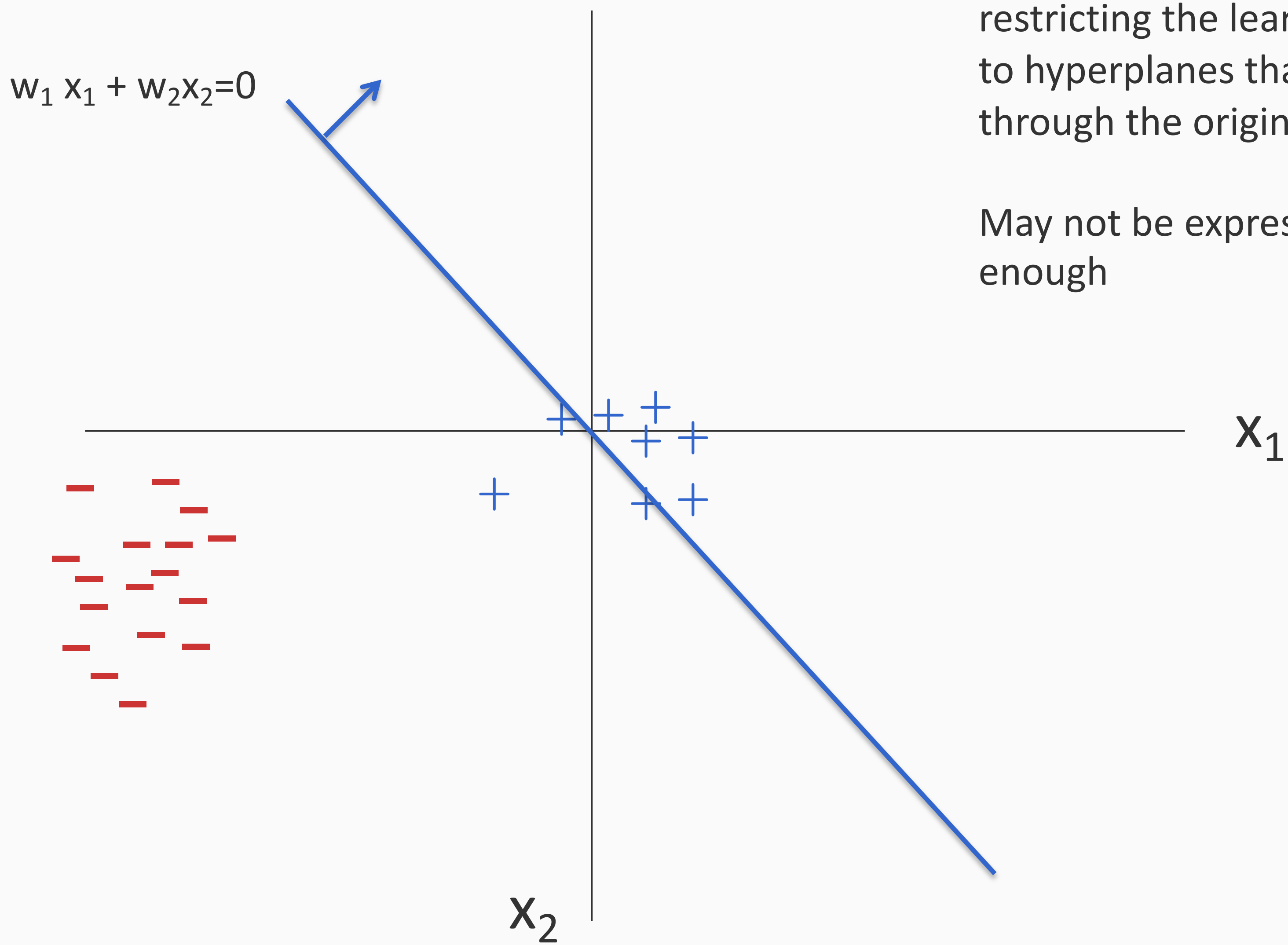


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# Exercises

1. Represent the simple disjunction as a linear classifier.

2. How would you apply the feature space expansion idea for the XOR function?