Machine Learning



Outline

The Perceptron Algorithm

Variants of Perceptron

Perceptron Mistake Bound

Where are we?

The Perceptron Algorithm

Variants of Perceptron

Perceptron Mistake Bound

Recall: Linear Classifiers

Inputs are d dimensional vectors, denoted by \mathbf{x} Output is a label $y \in \{-1, 1\}$

Linear Threshold Units classify an example \mathbf{x} using parameters \mathbf{w} (a d dimensional vector) and \mathbf{b} (a real number) according the following classification rule

Output =
$$sign(\mathbf{w}^T\mathbf{x} + b) = sign(\sum_i w_i x_i + b)$$

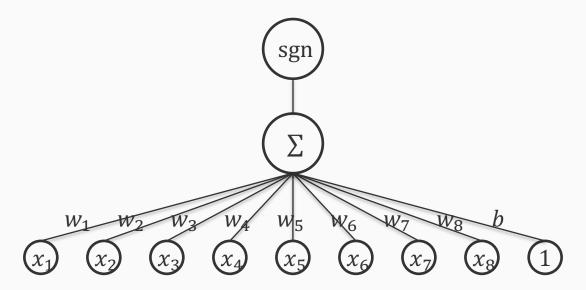
$$\mathbf{w}^T\mathbf{x} + b \ge 0 \Rightarrow y = +1$$

$$\mathbf{w}^T\mathbf{x} + b < 0 \Rightarrow y = -1$$

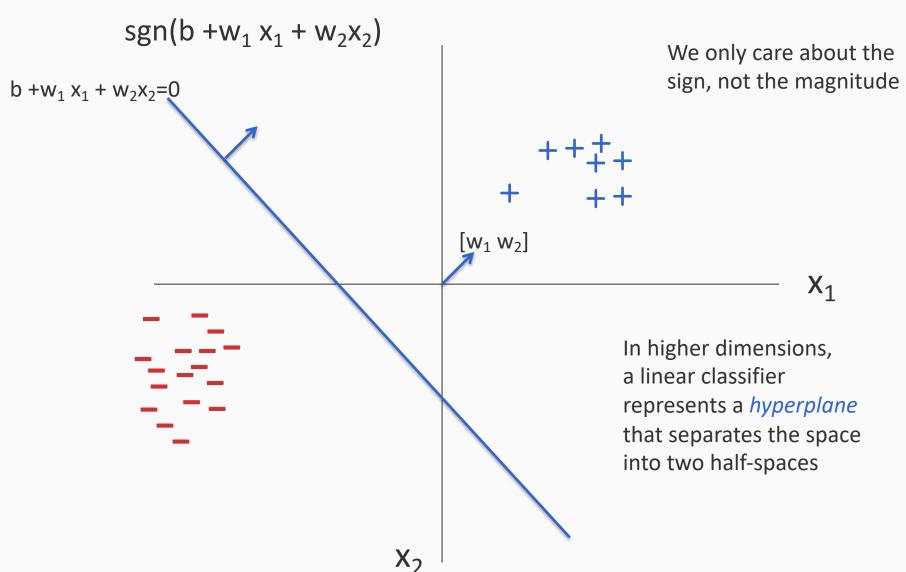
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The geometry of a linear classifier



The Perceptron

REPORT NO. 85-460-1

THE PERCEPTRON

A PERCEIVING AND RECOGNIZING AUTOMATON

(PROJECT PARA)

January, 1957

Prepared by: Frank Roserblatt

Frank Rosenblatt, Project Engineer

Psychological Review Vol. 65, No. 6, 1958

THE PERCEPTRON: A PROBABILISTIC MODEL FOR INFORMATION STORAGE AND ORGANIZATION IN THE BRAIN ¹

F. ROSENBLATT

Cornell Aeronautical Laboratory

- Introduced by Rosenblatt (1958)
 - Though there were some hints of a similar idea earlier Agmon (1954), Motzkin and Schonberg (1954)
- The goal is to find a separating hyperplane
 For separable data, guaranteed to find one
- An online algorithm
 That is, it processes one example at a time
- A mistake-driven algorithm
 That is, it makes updates if, and only if, it makes a mistake on an example
- Several variants exist
 We will see these briefly at towards the end

Input: A sequence of training examples $(\mathbf{x}_1, y_1), (\mathbf{x}_2, y_2), \cdots$ where all $\mathbf{x}_i \in \mathbb{R}^d, y_i \in \{-1, 1\}$

- 1. Initialize $\mathbf{w}_0 = 0 \in \Re^d$
- 2. For each training example (\mathbf{x}_i, y_i) :
 - 1. Predict $y' = \operatorname{sgn}(\mathbf{w}_t^T \mathbf{x}_i)$
 - 2. If $y' \neq y_i$:
 - Update $\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t + r(y_i \mathbf{x}_i)$
- 3. Return final weight vector

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Remember:

Prediction = $sgn(\mathbf{w}^T\mathbf{x})$

There is typically a bias term also $(\mathbf{w}^T\mathbf{x} + \mathbf{b})$, but the bias may be treated as a constant feature and folded into \mathbf{w}

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r is the learning rate, a small positive number less than 1

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Update only on error. A mistake-driven algorithm

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Update only on error. A mistake-driven algorithm

Mistake can be written as $y_i \mathbf{w}_t^T \mathbf{x}_i \leq 0$

This is the simplest version. We will see more robust versions shortly

```
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The new dot product is $\mathbf{w}_{t+1}^T\mathbf{x} = (\mathbf{w}_t + \mathbf{x})^T\mathbf{x} = \mathbf{w}_t^T\mathbf{x} + \mathbf{x}^T\mathbf{x} \geq \mathbf{w}_t^T\mathbf{x}$

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For a positive example, the Perceptron update will increase the score assigned to the same input

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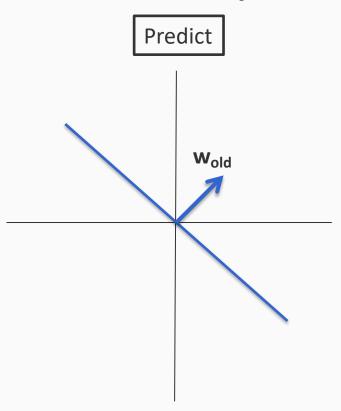
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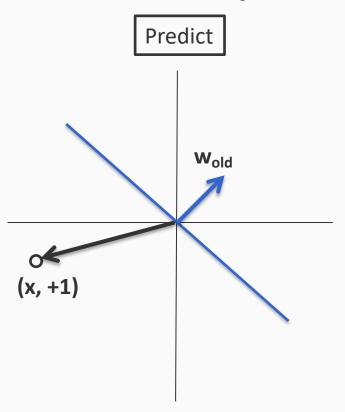
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Similar reasoning for negative examples

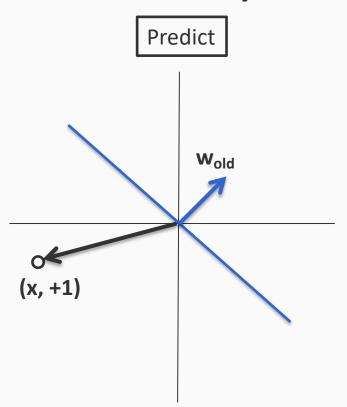
Geometry of the perceptron update



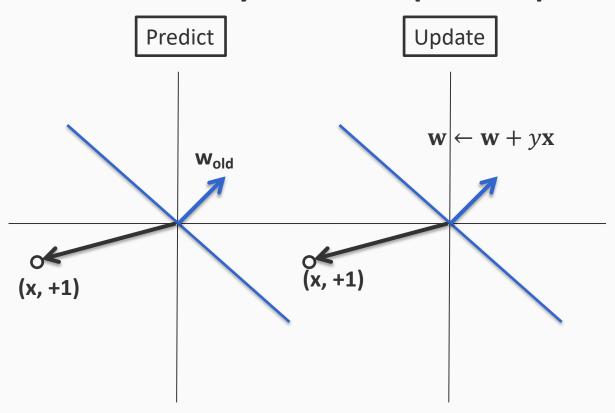
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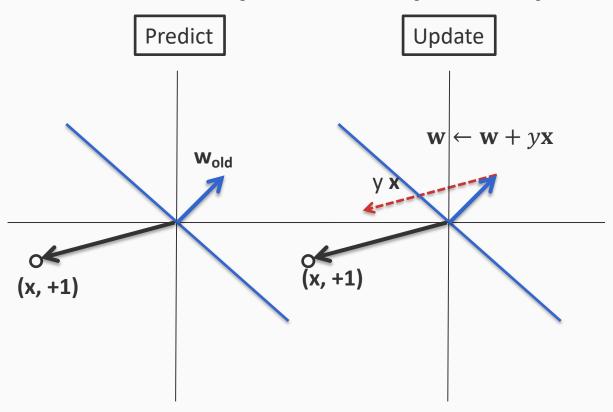


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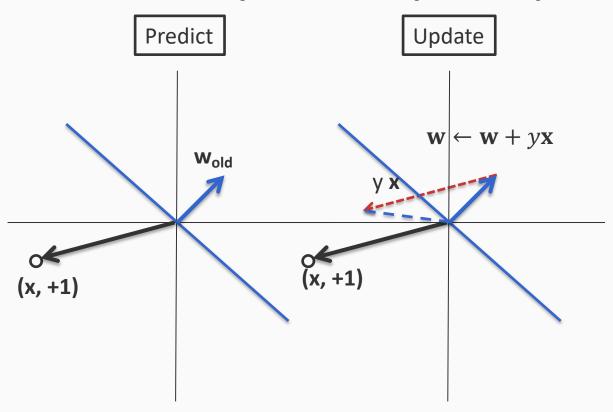


For a mistake on a positive example

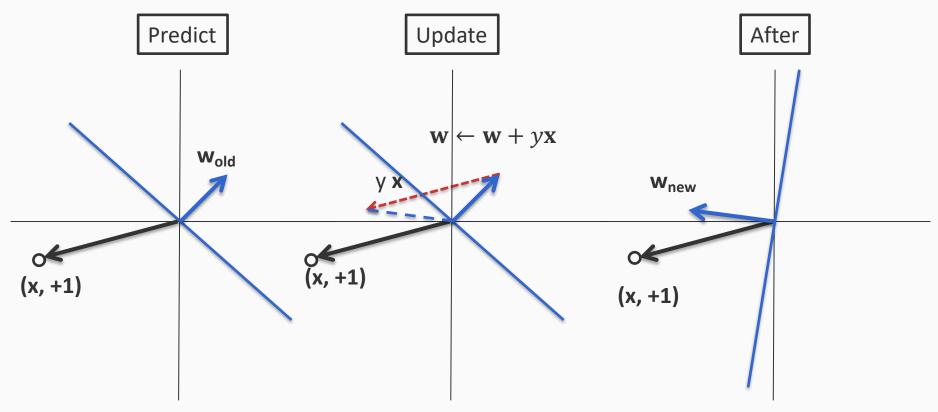
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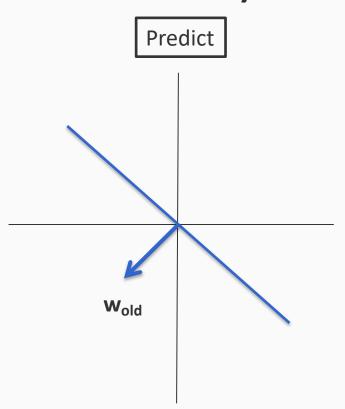


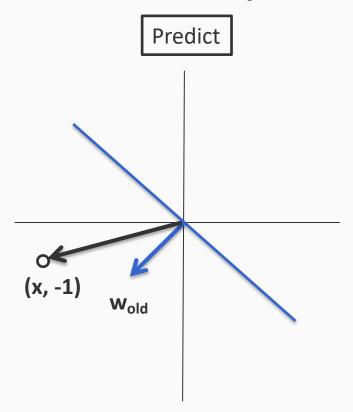
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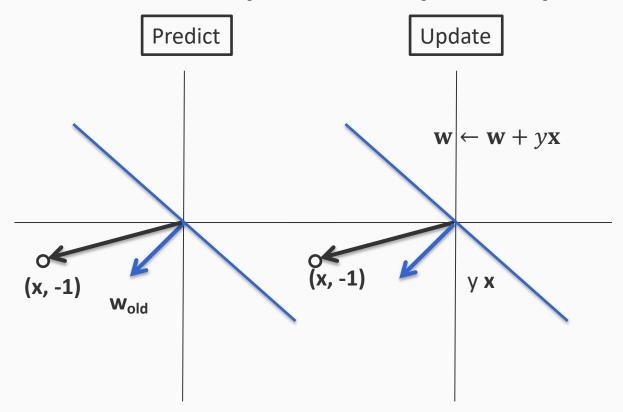


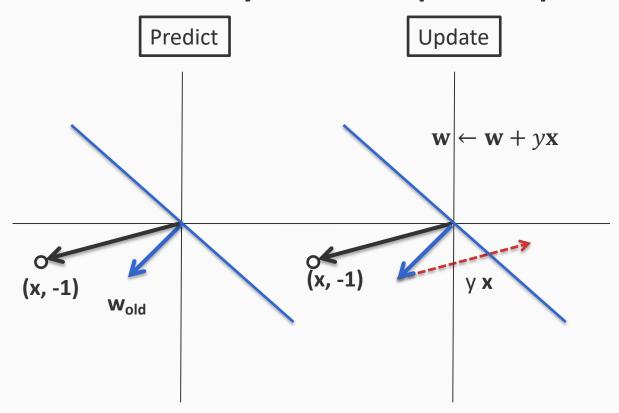
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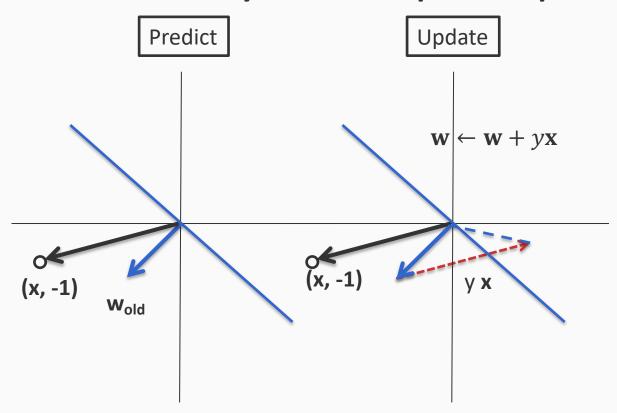


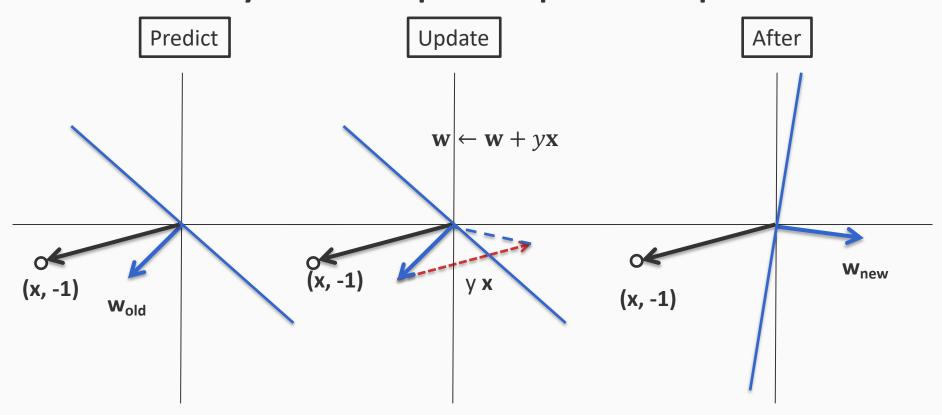












Where are we?

The Perceptron Algorithm

Variants of Perceptron

Perceptron Mistake Bound

Practical use of the Perceptron algorithm

1. Using the Perceptron algorithm with a finite dataset

2. Voting and Averaging

3. Margin Perceptron

1. The "standard" algorithm

Given a training set $D = \{(\mathbf{x}_i, y_i)\}$ where $\mathbf{x}_i \in \mathbb{R}^d$, $y_i \in \{-1, 1\}$

- 1. Initialize $\mathbf{w} = \mathbf{0} \in \mathbb{R}^d$
- 2. For epoch in $1 \cdots T$:
 - 1. Shuffle the data
 - 2. For each training example $(\mathbf{x}_i, y_i) \in D$:
 - If $y_i \mathbf{w}^T \mathbf{x}_i \leq 0$, then:
 - update $\mathbf{w} \leftarrow \mathbf{w} + ry_i \mathbf{x}_i$
- 3. Return w

Prediction on a new example with features \mathbf{x} : $\operatorname{sgn}(\mathbf{w}^T\mathbf{x})$

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Given a training set $D = \{(\mathbf{x}_i, y_i)\}$ where $\mathbf{x}_i \in \mathbb{R}^d$, $y_i \in \{-1, 1\}$

- 1. Initialize $\mathbf{w} = \mathbf{0} \in \mathbb{R}^d$
- 2. For epoch in $1 \cdots T$: T is a hyper-parameter to the algorithm
 - 1. Shuffle the data
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Another way of writing that there is an error

3. Return w

Prediction on a new example with features \mathbf{x} : $\operatorname{sgn}(\mathbf{w}^T\mathbf{x})$

2. Voting and Averaging

- So far: We return the final weight vector
- Voted perceptron
 - Remember every weight vector in your sequence of updates.
 - At final prediction time, each weight vector gets to vote on the label. The number of votes it gets is the number of iterations it survived before being updated
 - Comes with strong theoretical guarantees about generalization, impractical because of storage issues

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Voted perceptron

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Averaged perceptron

- Instead of using all weight vectors, use the average weight vector (i.e. longer surviving weight vectors get more say)
- More practical alternative and widely used

Averaged Perceptron

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- 3. Return a

Prediction on a new example with features \mathbf{x} : $\operatorname{sgn}(\mathbf{a}^T\mathbf{x})$

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This is the simplest version of the averaged perceptron

There are some easy programming tricks to make sure that **a** is also updated only when there is an error

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If you want to use the Perceptron algorithm, use averaging

Prediction on a new example with features \mathbf{x} : $\operatorname{sgn}(\mathbf{a}^T\mathbf{x})$

3. Margin Perceptron

Perceptron makes updates only when the prediction is incorrect

$$y_i \mathbf{w}^T \mathbf{x}_i \leq 0$$

• What if the prediction is close to being incorrect? That is, Pick a small positive η and update when

$$y_i \mathbf{w}^T \mathbf{x}_i \leq \eta$$

• Can generalize better, but extra hyper-parameter η Exercise: Why is the margin perceptron a good idea?

The Perceptron

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The hype

NEW NAVY DEVICE LEARNS BY DOING

Psychologist Shows Embryo of Computer Designed to Read and Grow Wiser

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able to walk, talk, see, write,
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The embryo—the Weather Bureau's \$2,000,000 "704" computer—learned to differentiate between right and left after fifty attempts in the Navy's demonstration for newsmen.

The service said it would use this principle to build the first of its Perceptron thinking machines that will be able to read and write. It is expected to be finished in about a year at a cost of \$100,000. HAVING told you about the giant digital computer known as I.B.M. 704 and how it has been taught to play a fairly creditable game of chess, we'd like to tell you about an even more remarkable machine, the perceptron, which, as its name implies, is capable of what amounts to original thought. The first perceptron has yet to be built,

The New Yorker, December 6, 1958 P. 44

The hype

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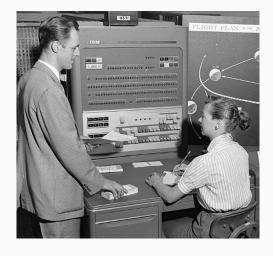
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The IBM 704 computer

What you need to know

The Perceptron algorithm

The geometry of the update

What can it represent

Variants of the Perceptron algorithm