# 4

# Noise and Radiating Systems

*"The greatest enemy of a good plan is the dream of a perfect plan"*
*—Prussian General Clausewitz*

Every communications system will suffer some limitation on the maximum distance for information transmission. This is due to a combination of unavoidable signal attenuation (due to absorption, scattering, and/or diffraction), and the presence of electrical noise which ultimately limits the detection of weak signals. The choice and design of antennas is frequently influenced by the effects of noise, so we will develop some important tools commonly used to model noise in high frequency electronic systems. Once this description is complete, the influence of antenna properties on a few representative communications and radar systems will be explored.

## 4.1 THERMAL NOISE IN COMMUNICATIONS SYSTEMS

Noise can come from a variety of natural and man-made processes. In the case of man-made noise—such as interference from power lines, car ignitions, or another communications system operating at the same frequency—we can take steps to remove or minimize the disturbance by proper filtering, shielding, or perhaps choosing a new frequency band for transmission. Noise from "natural" sources (thermal or quantum fluctuations) are always present to some extent at all frequencies, and therefore represent a fundamental limitation on the transmission of information.

Thermal agitation of charge carriers characterized by an essentially uniform frequency spectrum ("white noise") up to near optical frequencies. There are many other "noisy" electrical processes that have different spectral characteristics; common examples are shot noise, flicker or $1/f$ noise, avalanche noise, etc.. Noise is a complicated subject. Our analysis will be confined to a discussion of thermal noise in dissipative electrical networks, which can be a limiting factor for sensitivity in a well-designed microwave and millimeter-wave receiver, assuming steps have been taken to minimize other sources of noise through proper choice of device technology, circuit design, and receiver topology.

### 4.1.1 Blackbody Radiation

Electromagnetic energy comes in discrete packets of $hf$ where $h$ is Planck's constant

$$h = 6.6262 \times 10^{-34} \quad [\text{J·s}], \tag{4.1}$$

and $f$ is the frequency. Each packet of energy $hf$ is associated with a massless particle called a photon. The field in any finite-sized electromagnetic system (resonant cavity, electronic circuit, etc.) can be thought of as a number of photons distributed over the characteristic modes of the system, with the photons in each mode having a fixed energy corresponding to the resonant frequency of the mode.

When an electromagnetic system is in thermal equilibrium with its environment at some temperature $T$, the number of photons in each mode is constantly fluctuating as energy is transferred back and forth between the system and the environment to maintain the equilibrium. Statistical mechanics [**?**] tells us that the average number of photons per mode at temperature $T$ is given by the Bose-Einstein factor

$$\langle n \rangle = \frac{1}{e^{hf/kT} - 1} \tag{4.2}$$

where $k = 1.38 \times 10^{-23}$ [J/K] is the Boltzmann constant. The average energy per mode is then

$$\langle \mathcal{E} \rangle = \langle n \rangle hf = \frac{hf}{e^{hf/kT} - 1} \tag{4.3}$$

All that remains in order to find the total energy is to find the mode spectrum for the system.

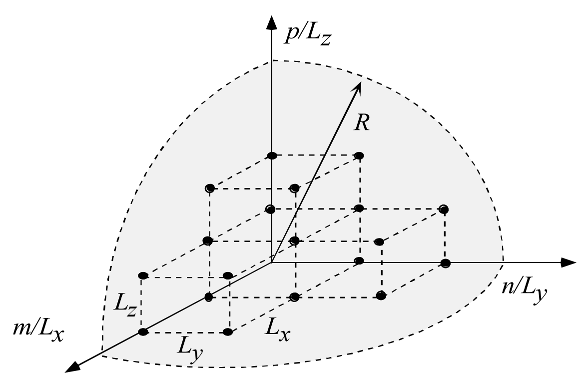


**Figure 4.1** Counting resonant modes of a 3D cavity.

This can be found by solving the Helmholtz equation for the given system geometry. The simplest example is a large rectangular conducting cavity with side dimensions $L_x, L_y, L_z$; the resonant frequencies are given by the well known result

$$f_{mnp} = \frac{c}{2}\left[\left(\frac{m}{L_x}\right)^2 + \left(\frac{n}{L_y}\right)^2 + \left(\frac{p}{L_z}\right)^2\right]^{\frac{1}{2}} \tag{4.4}$$

The mode indices $(m, n, p)$ are positive integers. If for each mode we identify $(m/L_x, n/L_y, p/L_z)$ as a point in "reciprocal" space, as in figure 4.1, then the distance from each point to the origin is proportional to the resonant frequency of the mode. For a given frequency $f$, the number of modes below $f$ (*i.e.* the allowed modes of the system) are the number of points within an octant of radius $R = 2f/c$. For large cavities an approximate result is

$$\begin{aligned} N(f) = \text{number of modes below} f \;&\approx\; 2 \cdot \frac{\text{volume of octant with } R = 2f/c}{\text{volume of unit cell}} \\ &=\; 2 \cdot \frac{(1/8)(4\pi/3)(2f/c)^3}{1/L_xL_yL_z} = \frac{8\pi f^3}{3c^3} V \end{aligned} \tag{4.5}$$

where $V$ is the physical volume of the cavity, $V = L_xL_yL_z$. The extra factor of 2 accounts for the degeneracy of TE and TM modes. The number of modes per unit volume between the frequencies $f$ and $f + df$ is then

$$\frac{N(f+df) - N(f)}{V} = \frac{1}{V}\frac{dN}{df}df = \frac{8\pi f^2}{c^3}df \tag{4.6}$$

So the energy density (energy per unit volume) in a band of frequencies $df$ at $f$ in a 3D cavity is

$$d\mathcal{U}(f) = \frac{8\pi f^2}{c^3}\frac{hf}{e^{hf/kT} - 1}df \qquad [\mathrm{J/m^3}]. \tag{4.7}$$

The total energy density integrated over all frequencies is

$$\begin{aligned} \mathcal{U}_T = \int_0^\infty d\mathcal{U}(f) \;&=\; \frac{8h\pi}{c^3}\int_0^\infty \frac{f^3}{e^{hf/kT} - 1}\,df \\ &=\; \frac{8h\pi}{c^3}\left(\frac{kT}{h}\right)^4 \underbrace{\int_0^\infty \frac{x^3\,dx}{e^x - 1}}_{\pi^4/15} = \left(\frac{8k^4\pi^5}{15h^3c^3}\right)T^4 \end{aligned} \tag{4.8}$$

Implicit in our work so far was the assumption that the cavity was in thermal equilibrium with a much larger system at a fixed temperature $T$, which requires exchange of energy (photons). We could imagine that the cavity has a hole in it to accomplish this exchange. In equilibrium, the radiation emitted through the hole must exactly balance the radiation absorbed through the hole, otherwise the cavity would gain or lose energy and hence the temperature would change. Consider an arbitrary element of surface area, $dA$, through which photons can enter of leave the cavity, as shown in figure 4.2. The photons that are available to leave the cavity in a certain time interval $dt$ and in a certain direction $\theta$, are contained within a tube of volume $(c\,dt)(dA\cos\theta)$ near the surface. However, only a fraction of these, $d\Omega/4\pi$ will be traveling in the right direction to actually leave the cavity. Therefore, the actual energy leaving the cavity in time $dt$ through the "flux tube" of solid angle $d\Omega$ will be

$$d\mathcal{E} = d\mathcal{U}(f)\,(c\,dt)(dA\cos\theta)\frac{d\Omega}{4\pi} \tag{4.9}$$

The rate of emission, or power, is then

$$\frac{d\mathcal{E}}{dt} = dP = c\,d\mathcal{U}(f)\;(\cos\theta\,dA)\;\frac{d\Omega}{4\pi} \tag{4.10}$$

The power radiated from the element of area into *all* angles is then found by integrating over all of the possible flux tubes

$$\begin{aligned} dP \;&=\; c\,d\mathcal{U}(f)\,\frac{dA}{4\pi}\int \cos\theta\,d\Omega \\ &=\; c\,d\mathcal{U}(f)\,\frac{dA}{4\pi}\int_0^{2\pi}\int_0^{\pi/2} \cos\theta\sin\theta\,d\theta\,d\phi = \frac{1}{4}c\,d\mathcal{U}(f)\,dA \end{aligned} \tag{4.11}$$

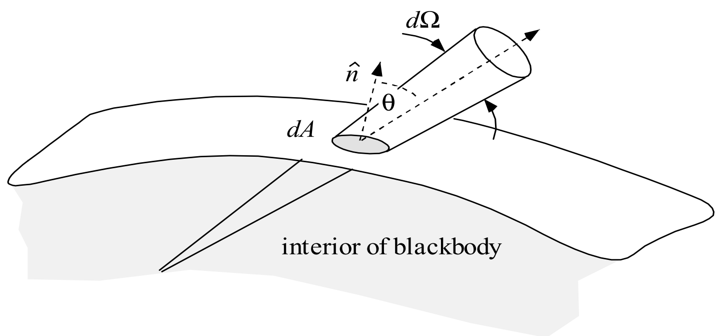


**Figure 4.2** Emission/absorption from a blackbody

Substituting our result for the energy density (4.7) gives

$$dP = \frac{2\pi f^2}{c^2} \frac{hf}{e^{hf/kT} - 1} df\, dA \tag{4.12}$$

This is *Planck's law* for blackbody radiation. In much of our applied work in antennas and radiating systems, $hf/kT \ll 1$, so that (4.12) can be approximated as

$$dP \approx \frac{2\pi f^2}{c^2} kT\, df\, dA \tag{4.13}$$

This is the *Rayleigh-Jeans* law for Blackbody radiation.

Integrating (4.11) over all frequencies gives

$$dP = \frac{1}{4} c\, \mathcal{U}_T\, dA = \left(\frac{2\pi^5 k^4}{15 h^3 c^2}\right) T^4\, dA = \sigma_B T^4\, dA \tag{4.14}$$

The is called the *Stephan-Boltzmann law*, and $\sigma_B$ is called the *Stephan-Boltzmann constant* which has the numerical value

$$\sigma_B = 5.67 \times 10^{-8}\ \left[\mathrm{W/m^2/K^4}\right]$$

A hot body of total surface area $A$ would then radiate a total power of

$$P = \sigma_B T^4 A$$

Bodies that radiate in this way are called "black" bodies. The result is an idealization, but for lack of a more general result, it is frequently used to approximate the emission from real bodies in thermal equilibrium.

### 4.1.2 Johnson Noise

Thermal agitation of electrons (Brownian motion) in any dissipative component is observable as a random current fluctuation. Since the motion is random there is no net current flow, but there can be a non-vanishing RMS current and hence a net flux of energy if the components is connected to an external network. In equilibrium the energy flow out of the dissipative element must be

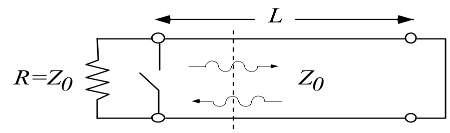


**Figure 4.3** Transmission line resonator with short circuits at both ends (switch closed). When switch is open, one end has a matched load which models an infinitely long short circuited transmission-line.

balanced by an inward flux which maintains the motion of electrons. These observations and results from the preceeding section can be used to derive the Nyquist expression for thermal noise produced by a resistor.

Consider a long transmission-line of length $L$, terminated in short circuits at either end. This is a one-dimensional resonator, with resonant frequencies occurring when

$$L = n\frac{\lambda}{2}, \qquad \text{or} \qquad n = \frac{2Lf}{c}. \tag{4.15}$$

So the number of modes per unit bandwidth per unit length is given by $2/c$ and is independent of frequency. Using (4.3), the average energy density between frequencies $f$ and $f + df$ in this one-dimensional system is given by

$$d\mathcal{U}(f) = \frac{2}{c}\frac{hf}{e^{hf/kT} - 1}df \tag{4.16}$$

Now, in electrical circuits we can almost always make the "classical" assumption that $hf/kT \ll 1$; at room temperature this is satisfied at frequencies well below 6 THz. In this case the energy spectral density is

$$d\mathcal{U}(f) = \frac{2}{c}kT\,df \tag{4.17}$$

In the classical regime, the energy spectral density is a constant with respect to frequency.

Electrical systems will have a well defined passband, or bandwidth, which we will call $\Delta f$. Integrating the energy spectral density in the classical regime over the system bandwidth gives an energy per unit length on the transmission-line of

$$\mathcal{U}_T = \frac{2kT\Delta f}{c} \quad [\mathrm{J/m}]. \tag{4.18}$$

At any point on the line half the energy flows to the right and half to the left. In time $dt$ the energy that crosses any reference plane is

$$d\mathcal{E} = \frac{1}{2}\left(\frac{2kT\Delta f}{c}\right)(c\,dt) = (kT\Delta f)dt \tag{4.19}$$

so the average "noise" power, $P_n$, flowing across any reference plane is

$$P_n = \frac{d\mathcal{E}}{dt} = kT\Delta f \tag{4.20}$$

This is a most useful result! Now if we let one of the ends of the transmission-line extend out to infinity, or equivalently replace one of the shorts by a matched termination, then in thermal equilibrium the termination must supply a noise power of $kT\Delta f$. We can effectively model the termination as an ideal noiseless resistor in series with a noise generator, as suggested in fig. 4.4.

To supply a power $kT\Delta f$ to the transmission-line under matched conditions this generator must have an equivalent RMS noise voltage given by

$$P_n = \frac{\langle V_n^2 \rangle}{4R} \qquad \text{or} \qquad V_n = \sqrt{4RkT\Delta f} \tag{4.21}$$

where $\langle\,\rangle$ denotes a time-average. This is *Nyquist's formula*, more commonly called *Johnson noise*. Thermal noise in the classical regime is also called *white-noise*, which refers to fact that the noise power in any bandwidth is independent of frequency. Remember, these results are only valid when $hf/kT \ll 1$, and may be invalid for cryogenic systems at high frequencies.

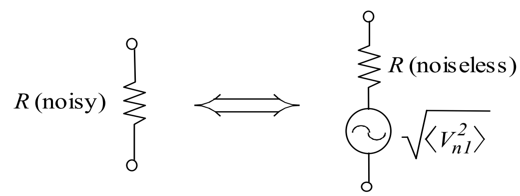


**Figure 4.4** A noisy resistor at temperature $T$ can be modeled as an equivalent noise generator in series with an ideal noiseless resistor.

Expression (4.20) and (4.21) are important results for communications systems analysis. The basic idea is to replace every dissipative element with an equivalent noise generator and a corresponding noiseless circuit element. In doing so, we must remember that the noise fluctuations in each components are typically uncorrelated random processes and hence the noise *powers* from each component, not voltages, are added together. Each component may also be operated at a different temperature. For example, consider two series-connected resistors which are held at different temperatures $T_1$ and $T_2$, as shown in figure 4.5. The equivalent noise generators have

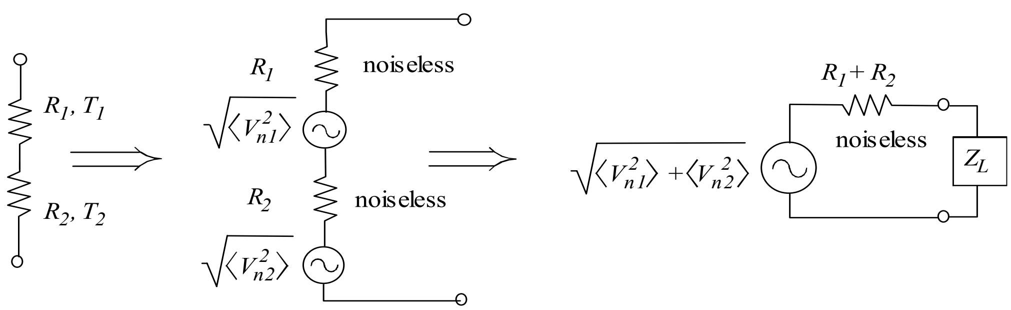


**Figure 4.5** Equivalent noise circuit for two series resistors at different temperatures.

RMS voltages

$$V_{n1} = \sqrt{4kT_1\Delta f R_1} \qquad V_{n2} = \sqrt{4kT_2\Delta f R_2} \tag{4.22}$$

The net RMS voltage of the combined sources is

$$\langle (V_{n1} + V_{n2})^2 \rangle = \langle V_{n1}^2 \rangle + \langle V_{n2}^2 \rangle + 2\langle V_{n1}V_{n2} \rangle$$

Since these voltages are a result of uncorrelated random processes with zero time-average, then $\langle V_{n1}V_{n2} \rangle = \langle V_{n1} \rangle \langle V_{n2} \rangle = 0$, and hence the Thevinin equivalent for the two resistor combinations as shown in the figure. Under matched conditions the noise power delivered to the load is

$$P_n = \frac{\langle V_{n1}^2 \rangle + \langle V_{n2}^2 \rangle}{4R_L} = \frac{k\Delta f (T_1 R_1 + T_2 R_2)}{R_1 + R_2} \tag{4.23}$$

When $T_1 = T_2$ then $P_n = kT\Delta f$ as expected, since the series combination is then equivalent to a single resistor of resistance $(R_1 + R_2)$ at a constant temperature.

When a network contains both reactive and resistive elements, and all components are at the same temperature $T$, the total noise voltage at the terminals can be found as

$$\langle V_n^2 \rangle = 4kT \int_{\Delta f} \mathrm{Re}\left\{Z(f)\right\} df \tag{4.24}$$

### 4.1.3 Effective Area: The Antenna Theorem

We are now in a position to derive one of the most important results in antenna theory: the relationship between the effective area and gain of an antenna. Consider an antenna with a

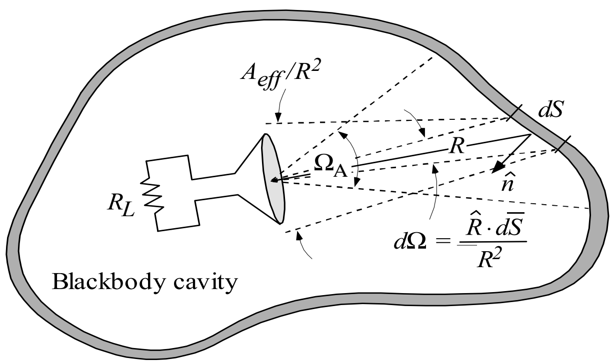


**Figure 4.6** Antenna with matched termination inside a blackbody radiator.

matched termination placed inside a large blackbody and in thermal equilibrium with it, as shown in figure 4.6. Under such conditions there must be a perfect balance between the noise generated by the resistor $R_L$ and radiated into the cavity, and the radiation absorbed by the antenna from the blackbody itself. From (4.7) we have, for $hf/kT \ll 1$,

$$d\mathcal{U} = \frac{8\pi f^2}{c^3} kT\, df \qquad \left[\mathrm{W/m^3}\right] \tag{4.25}$$

This is the energy density flowing within the cavity between $f$ and $f + df$. As we discussed earlier, the rate of absorption or emission from an element of surface area is given by (4.10). Applying this result to figure 4.6, we can see that one element $dS$ of the cavity wall will produce a received power in the antenna of

$$dP = \frac{1}{2} c\, d\mathcal{U} \left(\hat{R} \cdot \hat{n} dS\right) \frac{A_{eff}/R^2}{4\pi} = A_{eff} \frac{4\pi f^2}{c^2} kT\, df \frac{d\Omega}{4\pi} \tag{4.26}$$

where $A_{eff}/R^2$ is the solid-angle subtended by the antenna's effective area at a distance $R$ from the surface (effective area is a far-field concept, so $R$ must be large), and $d\Omega = \hat{R} \cdot \hat{n} dS/R^2$ is the solid-angle subtended by the surface element at a distance $R$ from the antenna (see Appendix A). The extra factor of $1/2$ is required because the thermal radiation is distributed between both possible polarizations of the field, whereas an antenna is only capable of receiving a single polarization. The total power received by the antenna and delivered to the load is then found by integrating over all elements of the cavity surface which fall within the beam solid-angle of the antenna,

$$P_{\text{rec}} = A_{eff} \frac{kT\,df}{\lambda^2} \iint_{\Omega_A} d\Omega = A_{eff} \frac{kT\,df}{\lambda^2} \Omega_A \tag{4.27}$$

In equilibrium we equate (4.27) with the noise produced by the load resistor (assuming matched conditions) to get

$$\frac{\Omega_A}{\lambda^2} kT\,df\,A_{\text{eff}} = kT\,df \tag{4.28}$$

and therefore

$$A_{\text{eff}} = \frac{\lambda^2}{\Omega_A} = \frac{\lambda^2}{4\pi} G \tag{4.29}$$

The universal constant in (**??**) is therefore $\lambda^2/4\pi$, so in general we have

$$A_{\text{eff}}(\theta, \phi) = \frac{\lambda^2}{4\pi} G(\theta, \phi) \tag{4.30}$$

The is the antenna theorem. From the definition of directivity, integrating both sides of (4.30) gives another form of the antenna theorem

$$\oiint A_{\text{eff}}(\theta, \phi) d\Omega = \lambda^2 \tag{4.31}$$

### 4.1.4 Antenna Noise Temperature

Any physical material at a temperature above absolute zero radiates energy at all frequencies. In a communication system the antenna will receive this energy from all directions. In general the thermal radiation received by the antenna will be coming from a variety of sources at different temperatures. For example, if the antenna is aimed at the sky, incident thermal radiation can be received from clouds, precipitation, ionospheric constituents, radiation from the Sun, the Earth, and other cosmic objects. Naturally this radiation will be a strong function of angle from the antenna, due to both the angular distribution of radiating objects and the angular dependence of the antenna's receiving pattern.

Quantitatively, if the sky is described by an effective temperature of $T_e(\theta, \phi)$, as shown in fig. 4.7, then the noise power received by the antenna in an element of solid angle $d\Omega$ (in the classical regime $hf/kT \ll 1$) is found by substituting (4.30) into (4.26),

$$dP_{na} = kT_e(\theta, \phi) \Delta f G(\theta, \phi) \frac{d\Omega}{4\pi} \tag{4.32}$$

Note that the effective temperature includes contributions from all matter within

the solid angle $d\Omega$, which may be at different physical temperatures. Integrating over all solid angles gives the total noise power received by the antenna

$$P_{na} = \oiint dP_{na} = k\Delta f \frac{1}{4\pi} \oiint T_e(\theta, \phi) G(\theta, \phi)\, d\Omega \tag{4.33}$$

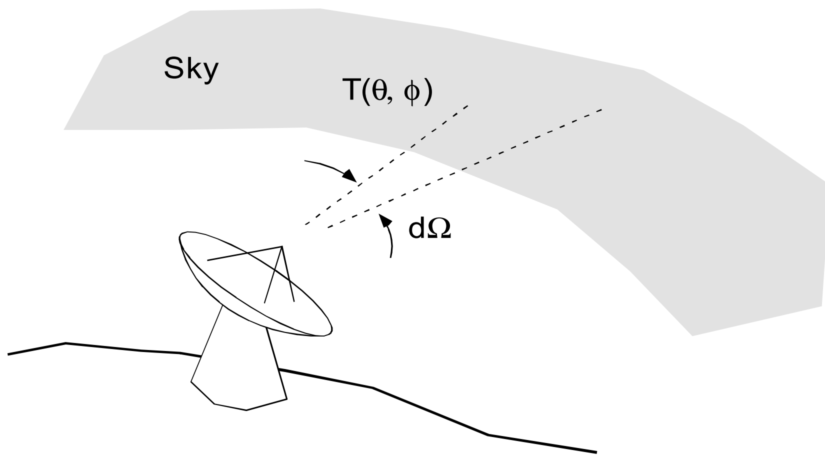


**Figure 4.7** Antenna noise temperature calculation.

From our discussion of Johnson noise we know that the available noise power from a dissipative component (like a resistor) is $kT\Delta f$. The total noise power at the antenna terminals can be described similarly if we define an *effective antenna noise temperature* from (4.33) as

$$T_A \equiv \frac{1}{4\pi} \oiint T_e(\theta,\phi) G(\theta,\phi)\, d\Omega \tag{4.34}$$

so that the noise power received by the antenna can be written as $P_{na} = kT_A\Delta f$. This can be rewritten in terms of the pattern function as

$$T_A = \frac{\oiint T_e(\theta,\phi)|f(\theta,\phi)|^2\, d\Omega}{\oiint |f(\theta,\phi)|^2\, d\Omega} \tag{4.35}$$

The antenna noise temperature is therefore just a weighted average of the observed temperature distribution with respect to the radiation pattern. We might have written this down by inspection! If the sky were at a uniform temperature $T_0$, then $T_A = T_0$, as expected. From the perspective of the receiver electronics, this is the same noise that would originate from the radiation resistance of the antenna if it were at a temperature $T_A$.

For high gain antennas, the antenna noise temperature will usually approximate the background temperature in the direction of the main lobe. An exception would be in situations where the background temperature in the direction of the main lobe is low, and the antenna has a minor sidelobe that happens to point in the direction of a "hot" source like the sun, or perhaps even the earth. This emphasizes that antenna design can play an important role in minimizing background thermal noise, which may be important in sensitive receiver systems.

When the antenna has some loss, the input noise power to the receiver will have an added component due to the loss resistance of the antenna. We can use the result derived earlier for a series combination of two noisy resistors, as in figure 4.5. The equivalent circuit for the antenna, shown in figure 4.8, includes the radiation resistance at temperature $T_A$, and the resistance $R_{loss}$ representing the ohmic losses in the antenna, which is at a physical temperature of $T_{phys}$. Under

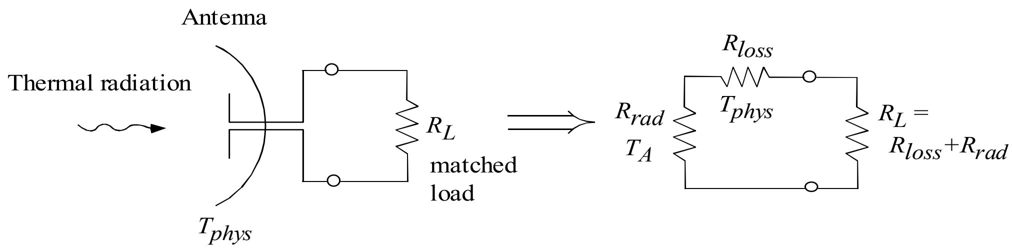


**Figure 4.8** Equivalent circuit for a lossy antenna receiving noise.

matched conditions ($R_L = R_{rad} + R_{loss}$), the noise power delivered to the load (receiver) is given by (4.23,

$$P_n = k\Delta f \frac{(T_A R_{rad} + T_{phys} R_{loss})}{R_{rad} + R_{loss}}$$

We can define an effective antenna noise temperature, $T'_A$, to account for both noise sources as $P_n = kT'_A\Delta f$, and express the result in terms of the radiation efficiency as

$$T'_A = e_r T_A + (1 - e_r)T_{phys} \tag{4.36}$$

In practice, we rarely try to compute the antenna noise temperature from first principles, since we do not often know the temperature distribution of all objects within the purview of the antenna. It should therefore be viewed as an empirical derived quantity. The above analysis merely offers some insight into the dependence of $T_A$ on physical quantities. Note that, as an empirical quantity representing the total noise power incident on the antenna, the antenna noise temperature can effectively represent noise from other non-thermal sources, of natural or man-made origin.

### 4.1.5 Noise Figure, Effective Input Noise Temperature

Receiver systems are constructed from a number of essential building blocks such as amplifiers, mixers, filters, transmission-lines, etc.. As the signal propagates through the receiver components, the signal-to-noise ratio (SNR) will progressively decrease as each component contributes additional noise power of its own. The degradation in SNR due to the noise contributed by each component is typically characterized by a parameter called the *noise figure*, $F$, which is defined as

$$F = \left. \frac{S/N \text{ ratio at input}}{S/N \text{ ratio at output}} \right|_{\substack{\text{under matched conditions,} \\ \text{source at } T_0 = 290\,K}} \tag{4.37}$$

where $S/N$ is the signal-to-noise ratio in terms of *power*. An important aspect of this definition is the stipulation of matched conditions and a well-defined temperature (room temp). This caveat reminds us that the relative degradation in SNR depends on the amount of noise power and signal power transferred *into* the component from external sources, as well as the noise contributed by the component itself.

The noise properties of any two-port can be described by a noise figure. Consider the amplifier shown in figure 4.9 with power gain $A$. The amplifier is assumed matched at both the input and output, and the source is held at standard temp $T_0$. Under matched conditions, the generator is capable of delivering a signal power of $P_{sig} = V_g^2/8R_g$ to the circuit, while the

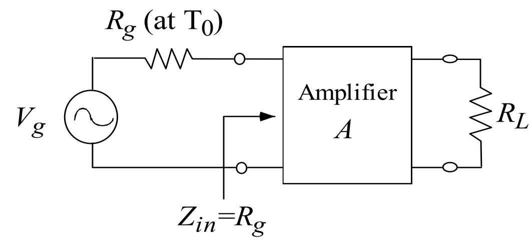


**Figure 4.9** Illustration for deriving amplifier noise figure.

resistor can exchange a noise power of $kT_0\Delta f$ with the circuit. The signal-to-noise ratio at the input is then

$$S/N|_{\text{input}} = \frac{P_{\text{sig}}}{kT_0\Delta f}. \tag{4.38}$$

The amplifier amplifies both signal and noise identically, but also adds some noise of its own due to internal dissipative components, which we call $P_{na}$. The signal-to-noise ratio at the output is then

$$S/N|_{\text{output}} = \frac{AP_{\text{sig}}}{AkT_0\Delta f + P_{na}} \tag{4.39}$$

Therefore the noise figure of an amplifier can be written as

$$F = \frac{P_{\text{sig}}/kT_0\Delta f}{AP_{\text{sig}}/(AkT_0\Delta f + P_{na})} = 1 + \frac{P_{na}}{AkT_0\Delta f} \tag{4.40}$$

A noiseless amplifier would have a noise figure of $F = 1$, or 0 dB, which means there is no degradation in SNR through the component. Equation (4.40) can be rearranged to give the noise power added by the amplifier as

$$P_{na} = A(F-1)kT_0\Delta f \tag{4.41}$$

An alternative way to characterize the noise power added by the amplifier is through an *effective input noise temperature*. This is defined as the temperature, $T_e$, of a fictitious thermal noise source at the input that would produce the same thermal noise power at the output when the amplifier is treated as ideal (noiseless). We can relate this to the noise figure as follows. In order to generate the noise power (4.41) at the output of an ideal amplifier with gain $A$, a noise source at the input would have to supply a noise power of $k(F-1)T_0\Delta f$, so we can define an effective input noise temperature of

$$T_e = (F-1)T_0 \tag{4.42}$$

This just describes the noise contributed by the amplifier alone. The *total* noise at the output of the amplifier, $P_{no}$, includes additional contributions from the input source impedance $R_g$, and is given by

$$P_{no} = AkT_0\Delta f + AkT_e\Delta_f = Ak(T_0 + T_e)\Delta f \tag{4.43}$$

Therefore, under standard operating conditions the total noise can also be described by an effective input noise temperature given by $T_s = T_0 + T_e$, which is referred to as the *system noise temperature*. The total output noise is then $P_{no} = AkT_s\Delta f$, which is thought of as coming from an input noise $kT_s\Delta f$ amplified by a noiseless amplifier of gain $A$. The system noise temperature will prove convenient in later calculations, since it allows us to replace the entire receiver system by an equivalent noiseless system.

**Example:** *SNR Calculation*

The first stage of a receiver system is shown in figure 4.10. An antenna with an effective noise temperature of $T_A$=100 K is connected directly to an amplifier with a 10 dB gain and a 3 dB noise figure. The antenna has an effective area of $A_{eff}$=1 m$^2$, and a radiation efficiency of $e_r = 0.8$. Assuming the amplifier has a bandwidth of 100 MHz, and both the amplifier and antenna are physically at a temperature of 290 K, what incident power density is required to maintain a SNR of 10 at the amplifier output?

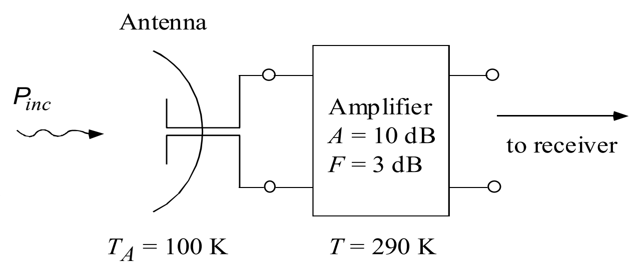


**Figure 4.10** Illustration for example SNR calculation.

Solution: the amplifier noise temperature is $T_e = (2-1)290\,\mathrm{K} = 290\,\mathrm{K}$. The input noise temperature is given by (4.36), and is $T'_A = 0.8(100) + 0.2(290) = 138\,\mathrm{K}$. The system noise temperature at the input of the amplifier is then $T_s = T'_A + T_e = 428\,\mathrm{K}$. In order to maintain a SNR of 10, we must have

$$\frac{\mathcal{P}_{inc}A_{eff}}{kT_s\Delta f} \geq 10$$

Inserting the appropriate numerical values we find that $\mathcal{P}_{inc} \geq 5.9\,\mathrm{pW}$. Note that by defining the system noise temperature as the temperature of a fictitious noise source at the *input* of a noiseless system, we do not have to compute the actual signal and noise power at the *output* to find the output SNR, since the signal and noise are amplified or attenuated by the noiseless system identically.

A system is not always operated at the standard operating conditions under which the noise figure is defined and measured. For example, in many cases the input noise temperature associated with the source impedance is at $T_{in} \neq 290\,K$. In this particular case, assuming the amplifier electronics are at standard conditions, the amplifier noise figure is still useful for characterizing $P_{na}$ as in (4.41), but the noise power supplied by the source impedance is now $kT_{in}\Delta f$. Therefore, using (4.40) and (4.41), the degradation in SNR under these non-standard conditions is described by the *achievable* noise figure of

$$F' = \frac{P_{\mathrm{sig}}/kT_{in}\Delta f}{AP_{\mathrm{sig}}/(AkT_{in}\Delta f + AkT_e\Delta f)} = 1 + \frac{T_e}{T_{in}} = 1 + (F-1)\frac{T_0}{T_{in}} \tag{4.44}$$

where (4.42) was used in the last step. If $T_{in} = T_0$, we recover $F' = F$ as expected. This relationship tells us that if the input noise source is "colder" than 290 K, the achievable noise figure will increase, because the noise power added by the amplifier (unchanged in an absolute sense) is now a larger fraction of the total noise power. We can again identify a system noise temperature referred to the input as

$$T_s = T_{in} + T_e \tag{4.45}$$

If the amplifier itself is not operated at standard temperature, then the amplifier noise $P_{na}$ is not described by the standard noise figure, and must therefore be characterized at the new operating conditions.

Similarly, if the system is not impedance matched, the relative change in SNR through the component will change. It should be noted that the noise figure defined under matched conditions is not necessarily the lowest achievable noise figure. In fact, there is always an optimum source impedance that will minimize the achievable noise figure, and this is usually not a matched impedance. Therefore, it is unlikely to simultaneously achieve minimum noise figure and maximum available gain.

As another useful example, consider a lossy transmission line connected to a source and load as shown in figure 4.11. Since the transmission-line is dissipative it will contribute some noise which can be described by a noise figure. We assume matched conditions ($R_g = Z_0 = R_L$) and standard temperature. In equilibrium the noise power coming from three sources—the source resistance, transmission line, and load resistance—must exactly balance at any reference plane. If

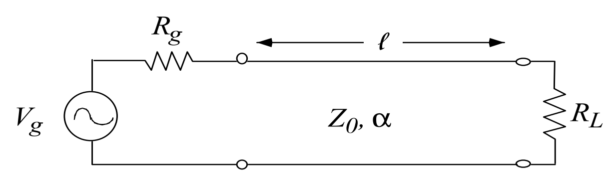


**Figure 4.11** Lossy transmission line.

the transmission line has an attenuation constant $\alpha$, then at the load terminals we have

$$(kT_0\Delta f)e^{-2\alpha\ell} + P_{na} = kT_0\Delta f \tag{4.46}$$

where $P_{na}$ is the noise power contributed by the transmission line. Defining the power loss factor $L = e^{2\alpha\ell}$, then

$$\begin{aligned} P_{na} &= kT_0\Delta f\left(1 - \frac{1}{L}\right) \\ &= \frac{1}{L}(L-1)kT_0\Delta f. \end{aligned} \tag{4.47}$$

Comparing this with (4.41) we see that the lossy line can be thought of as an amplifier with a gain of $1/L$ and noise figure $F = L$.

### 4.1.6 Noise in Cascaded Networks

The receiver front-end will consist a number of noisy components through which the desired signal must pass before the information content can be recovered. The system must be designed so that the SNR at the output of the last component is sufficient for reliable detection of the information.

For a cascade of circuits as shown in figure 4.12 the total noise at the output is the sum of noise contributions from each stage. Again, we assume matched conditions throughout and a standard operating temperature of $T_0 = 290°K$. Describing each stage by an effective power gain

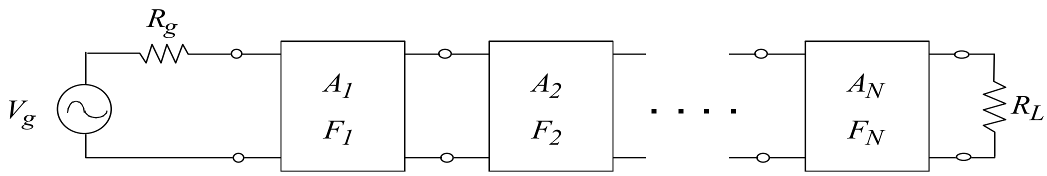


**Figure 4.12** Cascade of noisy two-ports.

and noise figure, the output noise due to each stage is:

$$\text{noise from } R_g: \quad kT_0 \Delta f (A_1 A_2 \dots A_N) = kT_0 \Delta f \prod_{j=1}^{N} A_j \tag{4.48}$$

$$\text{noise from stage 1:} \quad kT \Delta f A_1 (F_1 - 1)(A_2 \dots A_N) \tag{4.49}$$

$$\text{noise from stage i:} \quad kT \Delta f A_i (F_i - 1)(A_{i+1} \dots A_N) \tag{4.50}$$

So the output signal-to-noise ratio is

$$\left. \frac{S}{N} \right|_{\text{out}} = \frac{P_{\text{sig}} \prod_{j=1}^{N} A_j}{kT\Delta f \left[ \prod_{j=1}^{N} A_j + A_1(F_1 - 1) \prod_{j=2}^{N} A_j + A_2(F_2 - 1) \prod_{j=3}^{N} A_j + \dots A_N(F_N - 1) \right]} \tag{4.51}$$

and hence the total system noise figure is

$$F = 1 + (F_1 - 1) + \frac{(F_2 - 1)}{A_1} + \frac{(F_3 - 1)}{A_1 A_2} + \dots \frac{(F_N - 1)}{\prod_{j=1}^{N-1} A_j} \tag{4.52}$$

$$= F_1 + \frac{F_2 - 1}{A_1} + \frac{F_3 - 1}{A_1 A_2} + \dots \frac{F_N - 1}{\prod_{j=1}^{N} A_j} \tag{4.53}$$

This is an important result. We have shown that if the first stage of the cascade has a significant power gain $A_1$, then the noise figure of the entire cascade is primarily determined by the first stage (assuming, of course, that the noise figures of subsequent stages are not excessively large). For this reason, high gain, low noise amplifiers (LNAs) are often placed very close to the terminals of the receiving antenna . In fact, the demand for very low noise and high gain amplifiers in communications systems has been one of the driving factors behind many advances in semiconductor device technology. However, it is not always possible to reduce the noise figure of a receiver system in this simple way. In some radar systems, for example, cross-talk or leakage between the transmitter and receiver can be orders of magnitude larger than the desired signal to be received, and can easily saturate (and sometimes destroy!) sensitive low-noise amplifier circuitry. Other steps must then be taken to minimize noise in the system, such as reducing the bandwidth, increasing the gain of the antennas, increasing transmitter power, etc.

# 4.2 COMMUNICATION LINKS

## 4.2.1 Friis Transmission Equation

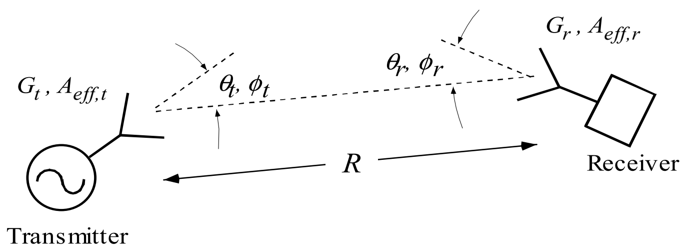


**Figure 4.13** Point-to-point antenna link.

Assuming impedance-matched and polarization matched conditions, we can apply our knowledge of descriptive parameters of antennas to compute the power transfer between two antennas in a microwave link, as shown in fig. 4.13,

$$\begin{aligned} P_{\mathrm{rec}} &= \frac{P_{\mathrm{in}}}{4\pi R^2} G_t(\theta_t, \phi_t)\, A_{\mathrm{eff}r}(\theta_r, \phi_r) \\ &= \frac{P_{\mathrm{in}}}{4\pi R^2} G_t(\theta_t, \phi_t)\, \frac{\lambda^2}{4\pi} G_r(\theta_r, \phi_r) \end{aligned}$$

This is the ideal *Friss transmission equation*. Collecting terms, we can write it as

$$P_{\mathrm{rec}} = P_{\mathrm{in}} \left(\frac{\lambda}{4\pi R}\right)^2 G_t(\theta_t, \phi_t)\, G_r(\theta_r, \phi_r) \tag{4.54}$$

If the antennas are aligned for maximum signal,

$$P_{\mathrm{rec}} = P_{\mathrm{in}} \left(\frac{\lambda}{4\pi R}\right)^2 G_{0t} G_{0r} \tag{4.55}$$

where $G_{0t}$ and $G_{0r}$ are the maximum gains for the transmitter and receiver, respectively. This can also be written in terms of effective areas as $A_{\mathrm{eff}}$

$$P_{\mathrm{rec}} = P_{\mathrm{in}} \left(\frac{\lambda}{4\pi R}\right)^2 \frac{A_{\mathrm{eff}t}}{\lambda^2/4\pi} \frac{A_{\mathrm{eff}r}}{\lambda^2/4\pi} = P_{\mathrm{in}} \left(\frac{1}{\lambda^2 R^2}\right) A_{\mathrm{eff}t} A_{\mathrm{eff}r}$$

If there are impedance mismatches in the system, then we can make the following replacements to account for them

$$\begin{aligned} &\text{Transmitter side:} && P_{\mathrm{in}} \to P_{\mathrm{in}} \left(1 - |\Gamma_t|^2\right) \\ &\text{Receiver side:} && A_{\mathrm{eff}r} \to \left(1 - |\Gamma_r|^2\right) A_{\mathrm{eff}r} \end{aligned}$$

(Remember, radiation efficiency is already included in the antenna gain). Polarization mismatch is then accounted for using the polarization loss factor PLF, giving

$$P_{\mathrm{rec}} = P_{\mathrm{in}} \left(\frac{\lambda}{4\pi R}\right)^2 G_{0t} G_{0r} \left(1 - |\Gamma_t|^2\right) \left(1 - |\Gamma_r|^2\right) \mathrm{PLF} \tag{4.56}$$

This generalized Friis equation is the fundamental relation describing energy transfer in all communication links.

Since the product $P_{\text{in}}G_{0t}$ always occurs in these calculations, it is gives a special name, the *Effective Isotropic Radiated Power*, abbreviated as EIRP, or sometimes just ERP.

$$\text{EIRP} = P_{\text{in}}G_{0t}$$

In communication systems, this is an important number a 10 Watt transmitter and antenna with gain of 100 works the same as a 100 Watt transmitter feeding an antenna with $G = 10$. The EIRP is also fundamentally easier to measure than the individual parameters.

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**Example:** *Microwave Link*

Consider the communications link shown in fig. 4.14 below. The link is operated at 1 GHz, and the transmitting and receiving antennas have a gain of 20 dB and 15 dB, respectively. the antennas are separated by 1 km. Find the maximum received power under impedance-matched conditions for $P_{in}$=150 W, if (a) the antennas are polarization matched, and (b) the transmitting antenna is circularly polarized (either right- or left-handed) and the receiving antenna in linearly polarized.

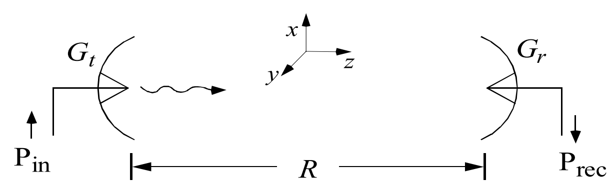


**Figure 4.14** Illustration for example link calculation.

a.) If the antennas are polarization-matched, then the received power will be given by (4.55),

$$P_{\text{rec}} = P_{\text{in}}G_tG_r\left(\frac{\lambda}{4\pi R}\right)^2$$

In this problem, $\lambda = c/f = 30\,\text{cm}$, and

$$G_tG_r = 35\,\text{dB} = 10^{3.5} = 3162.3$$

So

$$P_{\text{rec}} = (150\,\text{W})(3162.3)\left(\frac{3\times 10^{-4}\,\text{km}}{4\pi\cdot 1\,\text{km}}\right)^2 = 0.27\,\text{mW}$$

b.) If the transmitter is circularly polarized, then $\hat{\rho}_t = \frac{1}{\sqrt{2}}(\hat{x} \pm j\hat{y})$. If the receiver is linearly polarized, then assume $\hat{\rho}_r = \hat{x}$. The polarization loss factor is then $\text{PLF} = |\hat{\rho}_t\cdot\hat{\rho}_r|^2 = \frac{1}{2}$. Half the power is lost, so

$$P_{\text{rec}} = \frac{0.27\,\text{mW}}{2} = 0.135\,\text{mW}$$

---

Note that we have left out signal attenuation or fading due to propagation or scattering losses between the antennas. These losses are especially important in a real communication system, and can arise from a number of factors such as: propagation through fog or rain; ground interference effects or multipath distortion; reflection from, or propagation through the ionosphere (shortwave radio, or satellite communications). A good discussion of some of of these issues can be found in Chapter 6 of [**?**].

# 4.3 RADAR AND REMOTE SENSING

## 4.3.1 Radar Range Equation

Radar (Radio Detection and Ranging) systems are also ubiquitous: Doppler radar systems are used extensively for local weather forecasting; airport tracking and landing radars are used to guide airplanes all over the world; police radars are used for traffic control. There are countless military radar systems, and radar is also used for remote earth sensing, terrain mapping, surveillance, and for non-destructive tomographic imaging systems. Some types of motion detectors can even be considered as simple radar systems.

Radar systems are close electromagnetic cousins of the communications links described earlier. A generic model that applies to most radar systems is shown in fig. 4.15. This configuration is called a *bistatic radar system*, using separate antennas for transmitting and receiving at different locations, possibly quite far apart. The more familiar *monostatic* radar system is a special case where the transmit and receive antennas are at the same location, $R_1 = R_2$. In this case, a single antenna is often used for both transmit and receive functions, whereby $G_r = G_t$.

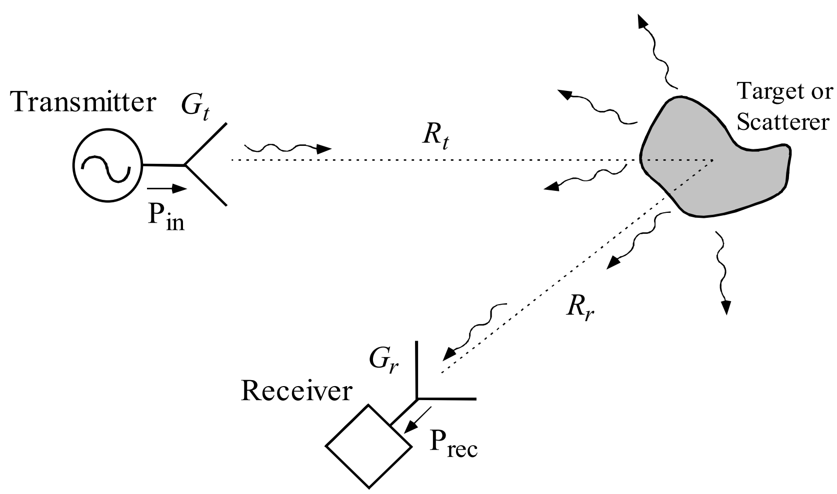


**Figure 4.15** Generic bistatic radar system.

In order to develop an equation like the Friiss formula which predicts power transfer in a radar system, we must decide on a method for characterizing the scattering process from remote targets. The conventional method for doing this is to assign a *Radar Cross Section* to the object, or RCS (usually denoted by $\sigma$, or $\sigma_{\text{RCS}}$, with units of area [m$^2$]), which is an effective area of an equivalent isotropic scatterer which produces the same power at the receiver as the original object. Yes, you read that correctly: we are defining $\sigma$ to be a number which gives us the right answer! Clearly RCS is not usually known *a priori*, but often some reasonable estimates can be made based on the size and shape of the object. We will discuss this more in the next section.

This definition of RCS is actually designed to make things as easy as possible. Here is how it works: the transmitter generates a wave of power density $\overline{\mathcal{P}}_{inc}$ which impinges on the target, given by

$$\overline{\mathcal{P}}_{inc} = \frac{P_{in}}{4\pi R_1^2} G_t$$

The target "intercepts" a certain fraction of this wave and re-radiates it, generally in all directions with a complicated pattern. Since we are interested in the scattering in only *one* particular direction (towards the receiver), the actual pattern of the scattered field in all other directions is not important, and we can imagine that the object scatters equally in all directions, *i.e.* an *isotropic* scatterer. The scattered field at the receiver is then written as

$$\overline{\mathcal{P}}_r = \frac{\overline{\mathcal{P}}_{inc}\sigma}{4\pi R_2^2} \tag{4.57}$$

where $\overline{\mathcal{P}}_{inc}\sigma$ is the total power that must be re-radiated by the target to produce the observed power density. The total received power is then $P_{rec} = \overline{\mathcal{P}}_r A_{\text{eff}r}$. Combining the above results together we find

$$P_{\text{rec}} = \frac{P_{\text{in}} G_t}{4\pi R_1^2} \frac{\sigma}{4\pi R_2^2} A_{\text{eff}r} = P_{\text{in}} \frac{\lambda^2}{(4\pi)^3 R_1^2 R_2^2} \sigma G_r G_t \tag{4.58}$$

This is the *radar range equation*, in its simplest form. For a monostatic radar with $R_1 = R_2$ and $G \equiv G_r = G_t$, this reduces to

$$P_{\text{rec}} = P_{\text{in}} \frac{\lambda^2}{(4\pi)^3 R^4} \sigma G^2 \tag{4.59}$$

As with the Friis equation, we can modify (4.58) to include impedance and polarization mismatch as

$$P_{\text{rec}} = P_{\text{in}} \frac{\lambda^2}{(4\pi)^3 R_1^2 R_2^2} \sigma G_r G_t \left(1 - |\Gamma_t|^2\right) \left(1 - |\Gamma_r|^2\right) \text{PLF} \tag{4.60}$$