STATIONARY PHASE METHOD

1 ONE-DIMENSIONAL INTEGRALS

The stationary phase method is a procedure for evaluation of integrals of the form

$$I = \int_{-\infty}^{\infty} F(x) e^{-\jmath\phi(x)} dx \tag{1}$$

where $\phi(x)$ is a rapidly-varying function of x over most of the range of integration, and $F(x)$ is slowly-varying (by comparison). Such integrals frequently arise in radiation or scattering problems. Rapid oscillations of the exponential term mean that I is approximately zero over those regions of the integrand; the only significant non-zero contributions to the integral occur in regions of the integration range where $d\phi/dx = 0$, *i.e.* at points of *stationary phase*. Points of stationary phase are labeled x_s and defined by

$$\phi'(x_s) = 0 \tag{2}$$

Note that $F(x) \approx F(x_s)$ in the vicinity of the statinary phase points, since $F(x)$ is assumed to be slowly varying, and hence this term can be pulled outside the integral. Expanding $\phi(x)$ in a Taylor series near the point x_s and keeping only the first two non-zero terms gives

$$\phi(x) \approx \phi(x_s) + \frac{1}{2}\phi''(x_s)\,(x - x_s)^2 \tag{3}$$

Substituting this into the integral (1) gives

$$\begin{aligned} I &\approx F(x_s) e^{-\jmath\phi(x_s)} \int_{-\infty}^{\infty} e^{-\jmath\phi''(x_s)(x-x_s)^2/2} dx \\ &\approx \sqrt{\frac{2\pi}{j\phi''(x_s)}} F(x_s) e^{-\jmath\phi(x_s)} \end{aligned} \tag{4}$$

2 TWO-DIMENSIONAL INTEGRALS

The same ideas can be applied to higher-dimensional integrals. In two-dimensions,

$$I = \iint_{-\infty}^{\infty} F(x,y) e^{-\jmath\phi(x,y)} dx\, dy \tag{5}$$

In this case, points of stationary phase (x_s, y_s) are defined such that

$$\phi_x(x_s, y_s) = 0 \qquad \phi_y(x_s, y_s) = 0 \tag{6}$$

where ϕ_x is shorthand for $\partial\phi/\partial x$. Expanding $\phi(x, y)$ in a 2D Taylor series about the stationary phase point, and keeping terms to second order gives

$$\phi(x,y) \approx \phi(x_s,y_s) + \frac{1}{2}\left[a(x-x_s)^2 + 2b(x-x_s)(y-y_s) + c(y-y_s)^2\right] \tag{7}$$

where

$$a = \left.\frac{\partial^2\phi}{\partial x^2}\right|_{x_s,y_s} \qquad b = \left.\frac{\partial^2\phi}{\partial x\partial y}\right|_{x_s,y_s} \qquad c = \left.\frac{\partial^2\phi}{\partial y^2}\right|_{x_s,y_s}$$

Substituting this into (5) gives

$$I \approx F(x_s,y_s)e^{-\jmath\phi(x_s,y_s)} \iint_{-\infty}^{\infty} e^{-\jmath\frac{1}{2}\left[au^2+2buv+cv^2\right]} du\, dv \tag{8}$$

where $u = (x - x_s)$ and $v = (y - y_s)$. The main contribution to the integral occurs near $u = v = 0$. Note that

$$\left[au^2 + 2buv + cv^2\right] = a\left(u + \frac{b}{a}v\right)^2 + \left(c - \frac{b^2}{a}\right)v^2$$

so

$$\int_{-\infty}^{\infty} e^{-\jmath\frac{1}{2}a(u+bv/a)^2} du = \sqrt{\frac{2\pi}{ja}}$$

$$\int_{-\infty}^{\infty} e^{-\jmath\frac{1}{2}a\left(c-b^2/a\right)v^2} dv = \sqrt{\frac{2\pi}{j(c-b^2/a)}}$$

Putting it all together, we have

$$I \approx \frac{2\pi}{\jmath\sqrt{ac-b^2}} F(x_s,y_s)e^{\jmath\phi(x_s,y_s)} \tag{9}$$