1

Elements of Electromagnetic Theory

Ring the bells that still can ring.
Forget your perfect offering.
There is a crack in everything.
That is how the light gets in.
—*Leonard Cohen*

Essential elements of engineering electrodynamics and relevant terminology for antenna analysis are presented here for convenient reference. The reader should already have been exposed to electromagnetic theory at the intermediate level, such that much of the material to follow should be familiar. However even well-prepared readers should consider reviewing the following: source concepts, including fictitious magnetic sources; field energy and power flow concepts (Poynting's theorem); the reciprocity theorem and consequences; and the field equivalence principles. Although field equivalence principles are not strictly necessary to treat radiation problems, they have proved to be an invaluable analytic tool for the antenna engineer, and are widely used in the engineering literature; *i.e.* master them! The chapter concludes with a discussion of the behavior of fields at material edges, which has ramifications in the numerical evaluation of fields and currents discussed in later chapters.

1.1 BASIC LAWS AND FIELD QUANTITIES

1.1.1 Maxwell's equations

The system of units used in this book is the rationalized MKS system. In these units, Maxwell's equations in differential, time-dependent form are

$$\nabla \cdot \overline{\mathcal{D}} = \rho_e \qquad \text{(Gauss' law)} \tag{1.1a}$$

$$\nabla \cdot \overline{\mathcal{B}} = 0 \tag{1.1b}$$

$$\nabla \times \overline{\mathcal{E}} = -\frac{\partial \overline{\mathcal{B}}}{\partial t} \qquad \text{(Faraday's law)} \tag{1.1c}$$

$$\nabla \times \overline{\mathcal{H}} = \overline{\mathcal{J}} + \frac{\partial \overline{\mathcal{D}}}{\partial t} \qquad \text{(Maxwell/Ampère law)} \tag{1.1d}$$

where

$\overline{\mathcal{E}}$	$\equiv$	Electric field intensity [V/m]	$\overline{\mathcal{H}}$	$\equiv$	Magnetic field intensity [H/m]
$\overline{\mathcal{D}}$	$\equiv$	Electric flux density [C/m^2]	$\overline{\mathcal{B}}$	$\equiv$	Magnetic flux density [W/m^2]
ρ_e	$\equiv$	Electric charge density [C/m^3]	$\overline{\mathcal{J}}$	$\equiv$	Electric current density [A/m^2]

By convention, time varying quantities are written with script letters, reserving roman letters for phasor quantities. The equations are consistent with the conservation of charge, expressed by the continuity equation for current

$$\nabla \cdot \overline{\mathcal{J}} + \frac{\partial \rho_e}{\partial t} = 0 \tag{1.2}$$

which follows by taking the divergence of (1.1d) and inserting (1.1a).

As commonly discussed in texts, equation (1.1b) follows from the apparent absence of magnetic charges or monopoles. However, even in the absence of physically real magnetic charges it is often convenient to introduce magnetic charges and currents into equations (1.1), which will be appreciated in our later discussion of field equivalence principles. This is done by analogy with (1.1a); if magnetic charges did exist, (1.1b) would become $\nabla \cdot \overline{\mathcal{B}} = \rho_m$, where ρ_m is the magnetic charge density. Assuming such magnetic charges would also be conserved leads to a continuity equation for magnetic current,

$$\nabla \cdot \overline{\mathcal{M}} + \frac{\partial \rho_m}{\partial t} = 0 \tag{1.3}$$

where $\overline{\mathcal{M}}$ is the magnetic current density. In addition, (1.1c) must also be augmented by a "magnetic displacement current" in order to satisfy (1.3), giving a modified set of Maxwell's equations

$$\nabla \cdot \overline{\mathcal{D}} = \rho_e \tag{1.4a}$$

$$\nabla \cdot \overline{\mathcal{B}} = \rho_m \tag{1.4b}$$

$$\nabla \times \overline{\mathcal{E}} = -\overline{\mathcal{M}} - \frac{\partial \overline{\mathcal{B}}}{\partial t} \tag{1.4c}$$

$$\nabla \times \overline{\mathcal{H}} = \overline{\mathcal{J}} + \frac{\partial \overline{\mathcal{D}}}{\partial t} \tag{1.4d}$$

The symmetry of (1.4) with respect to electric and magnetic quantities leads directly to the duality and complementarity principles of Chapter 3.

The linearity of (1.1) can be exploited using Fourier transform theory for both time and space dependences; this will be done throughout the book. Using the Fourier transform pair in time with an assumed $e^{\jmath\omega t}$ field dependence,

$$\overline{A}(\overline{r}, \omega) = \int_{\infty}^{\infty} \overline{\mathcal{A}}(\overline{r}, t) e^{\jmath\omega t}\, dt \quad \leftrightarrow \quad \overline{\mathcal{A}}(\overline{r}, t) = \frac{1}{2\pi} \int_{\infty}^{\infty} \overline{A}(\overline{r}, \omega) e^{-\jmath\omega t}\, d\omega$$

and operating on Maxwell's equations gives the time-harmonic form

$$\nabla \cdot \overline{D} = \rho_e \tag{1.5a}$$

$$\nabla \cdot \overline{B} = \rho_m \tag{1.5b}$$

$$\nabla \times \overline{E} = -\overline{M} - \jmath\omega \overline{B} \tag{1.5c}$$

$$\nabla \times \overline{H} = \overline{J} + \jmath\omega \overline{D} \tag{1.5d}$$

and similarly the continuity equations become

$$\nabla \cdot \overline{J} = -\jmath\omega\rho_e \tag{1.6a}$$
$$\nabla \cdot \overline{M} = -\jmath\omega\rho_m \tag{1.6b}$$

where the factor of $e^{\jmath\omega t}$ is dropped so that field quantities are now complex phasor functions of position only. The connection between the complex phasor fields and the physically meaningful, time-varying quantities, is given by

$$\overline{\mathcal{E}} = \mathrm{Re}\left\{\overline{E}e^{\jmath\omega t}\right\} \tag{1.7}$$

and similarly for the other field quantities. Some books define the phasors to be root-mean-square (rms) quantities so that factors of $\frac{1}{2}$ do not appear after time-averaging operations. In this book the phasor represents *peak* field quantities, and so we retain the factors of $\frac{1}{2}$.

1.1.2 Charges and Currents

Electromagnetic radiation is fundamentally a result of accelerating charges, or equivalently a time-varying current. The current $\overline{\mathcal{J}}$ is related to the motion of electric charges according to

$$\overline{\mathcal{J}} = \rho_e\overline{v} \qquad \text{(convection current density)} \tag{1.8}$$

where $\overline{v}$ is the velocity of the charges, and ρ_e is the volume density associated with charges that are actually moving and thus contributing to the current flow. The mechanical motion of charges is in turn related to the fields through the Lorentz force law,

$$\overline{\mathcal{F}} = q\left[\overline{\mathcal{E}} + \overline{v} \times \overline{\mathcal{B}}\right] \tag{1.9}$$

For electric charge distributions we can also define a Lorentz force density as

$$\overline{f}_e = \rho_e\overline{\mathcal{E}} + \overline{\mathcal{J}} \times \overline{\mathcal{B}} \tag{1.10}$$

and similarly for magnetic charges and currents we would have $\overline{f}_m = \rho_m\overline{\mathcal{H}} - \overline{\mathcal{M}} \times \overline{\mathcal{D}}$. In the presence of applied fields, a charged particle will be accelerated in the direction of the Lorentz force. If the charge is able to move through matter in response to an applied field (as in a conductor) it may experience a variety of scattering events which collectively impede this motion. The net result is that the charges acquire an average (drift) velocity which is directly proportional to the applied electric field, so (1.8) becomes

$$\overline{\mathcal{J}} = \sigma_e\overline{\mathcal{E}} \qquad \text{(Ohm's law)} \tag{1.11}$$

where the proportionality constant σ_e is the *electrical conductivity* of the medium, with units of [S/m]. Such currents are called *conduction currents*. For magnetic sources we can similarly postulate a "magnetic conductivity" σ_m and corresponding magnetic Ohm's law $\overline{\mathcal{M}} = \sigma_m\overline{\mathcal{H}}$; this has proved useful in the development of fictitious absorbers for numerical solution of radiation problems.

If the charges are not free to move in the material, but rather are bound closely to a constituent atom as in an insulating material, they can still give rise to an AC current since an applied harmonic field will induce some oscillatory motion of the charge about an equilibrium point. This *polarization current* is usually accounted for indirectly using the dielectric permittivity relating $\overline{D}$ to $\overline{E}$, but can also be useful explicitly in the formulation of integral equations for radiation or scattering problems involving fields in matter. This will be discussed again in connection with the *volume equivalence theorem*.

1.1.3 Constitutive relationships

When the sources are known, (1.1) represents a system of six independent scalar equations (the two vector curl equations) in twelve unknowns (four field vectors with three scalar components each). This is true because (1.1a) and (1.1b) are not independent of (1.1c) and (1.1d); they can be derived from these equations using the conservation laws (1.2) and (1.3). Therefore, we need six more scalar equations to completely determine the fields. These could be obtained by expressing two of the field vectors as functions of the other two, such as

$$\overline{\mathcal{D}} \Rightarrow \overline{\mathcal{D}}(\overline{\mathcal{E}}, \overline{\mathcal{H}}) \qquad \overline{\mathcal{B}} \Rightarrow \overline{\mathcal{B}}(\overline{\mathcal{E}}, \overline{\mathcal{H}}) \tag{1.12}$$

in which case Maxwell's equations can be expressed entirely in terms of the vectors $\overline{\mathcal{E}}$ and $\overline{\mathcal{H}}$. From a purely mathematical point of view, the choice of which electric and magnetic field variables to eliminate is arbitrary. Physically it can be argued that the set $\{\overline{\mathcal{E}}, \overline{\mathcal{B}}\}$ are most fundamental, $\{\overline{\mathcal{D}}, \overline{\mathcal{H}}\}$ being derived quantities that incorporate macroscopic polarization effects in materials. However, most engineering texts choose to eliminate $\overline{\mathcal{D}}$ and $\overline{\mathcal{B}}$ as above, since the set $\{\overline{\mathcal{E}}, \overline{\mathcal{H}}\}$ are more directly related to important circuit quantities in the MKS system. We will stick with that convention.

The relations (1.12) are called the *constitutive relationships*, and are completely determined by the material medium occupied by the fields. The functional form of (1.12) can be deduced from the microscopic physics of the material. The material is then classified according to this functional dependence. For example, an *isotropic* material is defined by the time-harmonic constitutive relations

$$\overline{D} = \epsilon \overline{E} \qquad \overline{B} = \mu \overline{H} \tag{1.13}$$

where ϵ is the *permittivity*, μ the *permeability*, and both are scalar quantities. These materials are called isotropic because they respond uniformly to the fields in all directions, that is, the material parameters (ϵ and μ) do not depend on the direction of the fields. Materials which behave according to (1.13) are often loosely referred to as *simple* media. A special case is a vacuum, or "free space", for which

$$\begin{aligned} \epsilon &\equiv \epsilon_0 = 8.85 \times 10^{-12} \text{ [F/m]} \\ \mu &\equiv \mu_0 = 4\pi \times 10^{-7} \text{ [H/m]} \end{aligned}$$

In practice, isotropic material properties are specified relative to the free-space values using

$$\epsilon \equiv \epsilon_r \epsilon_0 \qquad \mu \equiv \mu_r \mu_0$$

where ϵ_r is called the *relative permittivity* or *dielectric constant*, and μ_r is called the *relative permeability*. Materials which obey (1.13) but for which the material parameters vary with frequency (or time) are called *temporally dispersive*. If the material parameters depend on position (as would be the case in a layered or stratified medium like the ionosphere, or a printed circuit board), the medium is termed *inhomogeneous*, or *spatially dispersive*.

There are few *truly* isotropic media in nature. Polycrystalline, ceramic, or amorphous materials—those for which the material constituents are only partially or randomly ordered throughout the medium—are approximately isotropic, and hence many dielectrics and substrates used in commercial antenna work are often made from ceramics or other disordered matter, or mixtures containing such materials. Other materials, and especially crystalline matter, interact with fields in ways that depend to some extent on the *orientation* of the fields; this is most easily appreciated when one considers the atomic structure of the material. Such materials are called *anisotropic*, and can often be described by the relationships

$$\overline{D} = \overline{\overline{\epsilon}} \cdot \overline{E} \qquad \overline{B} = \overline{\overline{\mu}} \cdot \overline{H} \tag{1.14}$$

where $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ are now dyadic quantities. In such materials the direction of $\overline{\mathcal{D}}$, for example, will in general be different than $\overline{\mathcal{E}}$, and each component of $\overline{\mathcal{D}}$ will depend on all of the components of $\overline{\mathcal{E}}$. We can similarly extend Ohm's law for anisotropic materials by writing

$$\overline{J} = \overline{\overline{\sigma}}_e \cdot \overline{E} \tag{1.15}$$

In writing (1.13) we have also made an assumption of linearity. Generally the material parameters could also depend on the strength of the applied field, in which case the material is classified as *nonlinear*, and Maxwell's equations will subsequently include nonlinear terms. All materials exhibit some type of nonlinearity, a common and generally unpleasant example being dielectric breakdown which occurs at large field strengths (typically on the order of 10^6 V/cm). On the other hand, many materials also exhibit approximately linear field dependence over a significant range of applied field strength. When this is true Maxwell's equations are linear differential equations, and the *principle of superposition* can be used. The superposition principle implies that the fields satisfying (1.5) can be expressed as a summation of terms, each of which is a perfectly valid solution to Maxwell's equations. This is very useful, and is often invoked implicitly.

Many interesting phenomena involving the interaction of fields and matter—such as Faraday rotation, optical amplification in lasers, bi-refringence, etc.—can only be described by complicated constitutive relations. However, the associated mathematics is often quite cumbersome and would tend to obscure our treatment of fundamental radiation concepts. In our study of radiation we will mostly confine attention to simple isotropic media and assume the constitutive equations (1.13). Where a derivation is critically dependent on this assumption it will be noted.

For lossy media, the fields can expend energy due to the interaction of moving charges. For free electrons, in conductors, scattering processes leading to energy dissipation are accounted for using a finite conductivity, which through Ohm's law allows us to write

$$\begin{aligned} \nabla \times \overline{H} &= \overline{J}_i + \sigma_e \overline{E} + \jmath\omega\epsilon\overline{E} && (1.16) \\ &= \overline{J}_i + \jmath\omega\epsilon_c\overline{E} && (1.17) \end{aligned}$$

where ϵ_c is a complex permittivity defined by

$$\epsilon_c = \epsilon - \jmath\frac{\sigma_e}{\omega} \tag{1.18}$$

Since the loss associated with free-electrons can be equivalently represented in the form of a complex permittivity, *i.e.* as loss associated with bound charges, the physical origin of the loss is rarely important. In a practical sense, it is impossible to distinguish between the two as far as macroscopic effects are concerned. The complex permittivity is also commonly written in the forms

$$\epsilon_c = \epsilon' - \jmath\epsilon'' = \epsilon(1 - \jmath\tan\delta) \tag{1.19}$$

where $\tan\delta$ is called the *loss tangent* of the material, given by

$$\tan\delta \equiv \frac{\epsilon''}{\epsilon'} = \frac{\sigma_e}{\omega\epsilon}$$

In practice, losses are commonly specified using either an effective conductivity or loss tangent. Phenomenologically we can represent magnetic loss in a similar way.

1.1.4 Boundary conditions

Using the divergence theorem (A.54) and Stokes' theorem (A.59), Maxwell's equations can be expressed in integral form as

$$\oiint \overline{\mathcal{D}} \cdot d\overline{S} = \iiint \rho_e \, dV \tag{1.20a}$$

$$\oiint \overline{\mathcal{B}} \cdot d\overline{S} = \iiint \rho_m \, dV \tag{1.20b}$$

$$\oint \overline{\mathcal{E}} \cdot d\ell = -\iint \left(\overline{\mathcal{M}} + \frac{\partial \overline{\mathcal{B}}}{\partial t} \right) \cdot d\overline{S} \tag{1.20c}$$

$$\oint \overline{\mathcal{H}} \cdot d\ell = \iint \left(\overline{\mathcal{J}} + \frac{\partial \overline{\mathcal{D}}}{\partial t} \right) \cdot d\overline{S} \tag{1.20d}$$

These can be used to determine the behavior of fields at the boundary between two dissimilar materials. Consider first a tiny fictitious "pillbox" of height Δh and cross section ΔA surrounding a point at the interface between two regions as shown in figure 1.1. Evaluating the integrals in (1.20a) by letting Δh and ΔA become infinitesimal gives

$$\lim_{\Delta h \to 0} \oiint \overline{D} \cdot d\overline{S} \approx \hat{n} \cdot (\overline{D}_1 - \overline{D}_2) \Delta A$$

$$\lim_{\Delta h \to 0} \iiint \rho_e \, dV = Q_{\text{enclosed}} = \rho_{se} \Delta A$$

where ρ_{se} is a possible *surface charge density*, with units of [C/m^2], which exists in an infinitesimal layer along the interface (any volume charge density contributes nothing to the integral as $\Delta h \to 0$). The integrals in (1.20b) can be similarly evaluated, which leaves us with the boundary conditions for the normal components of the fields

$$\hat{n} \cdot (\overline{D}_1 - \overline{D}_2) = \rho_{se} \tag{1.21a}$$

$$\hat{n} \cdot (\overline{B}_1 - \overline{B}_2) = \rho_{sm} \tag{1.21b}$$

Boundary conditions for the tangential field components can be derived from (1.20b-c) by constructing a closed rectangular path of length $\Delta \ell$ and height Δh as shown in figure 1.1. Again letting Δh and $\Delta \ell$ shrink to zero gives

$$\lim_{\Delta h \to 0} \oint \overline{H} \cdot d\ell = (\hat{s} \times \hat{n}) \cdot (\overline{H}_1 - \overline{H}_2) \Delta \ell$$

$$\lim_{\Delta h \to 0} \iint \left(\overline{\mathcal{J}} + \frac{\partial \overline{\mathcal{D}}}{\partial t} \right) \cdot d\overline{S} = \overline{J}_s \cdot \hat{s} \Delta \ell$$

where $\overline{J}_s$ is a possible *surface current density*, which has the units of [A/m]. The integrals in (1.20c) can be similarly evaluated, which leaves

$$\hat{n} \times (\overline{H}_1 - \overline{H}_2) = \overline{J}_s \tag{1.22a}$$

$$\hat{n} \times (\overline{E}_1 - \overline{E}_2) = -\overline{M}_s \tag{1.22b}$$

Any set of fields which simultaneously satisfy Maxwell's equations and (1.22) will automatically satisfy (1.21). Boundary-value problems in radiation theory are most frequently formulated in

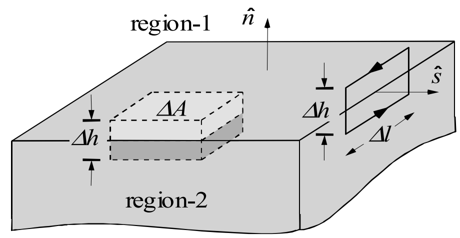


Figure 1.1 Volume and surface elements for determining boundary conditions.

terms of currents, so (1.22) will be of most use. Conducting bodies are often idealized as *perfect electric conductors* (PEC), characterized by vanishing tangential electric field at the conducting surface, and zero total electric field inside. In setting up equivalent field 3problems the concept of a *perfect magnetic conductor* (PMC) is useful, characterized by a vanishing tangential magnetic field at the surface. Both are described by an infinite conductivity, and the vanishing fields can be argued on the physical grounds that the current density inside remain finite as $\sigma_e \to \infty$. From Faraday's law, zero electric field in a PEC implies that the time-varying component of the magnetic field also vanish. Similar arguments also hold for a PMC. Thus for radiation problems we assume all fields to be zero everywhere inside of both the PEC and PMC. The boundary conditions (1.22) at the surface of such materials then become

$$\text{(PEC)} \quad \hat{n} \times \overline{E} = 0 \tag{1.23a}$$
$$\hat{n} \times \overline{H} = \overline{J}_s \tag{1.23b}$$

$$\text{(PMC)} \quad \hat{n} \times \overline{E} = -\overline{M}_s \tag{1.24a}$$
$$\hat{n} \times \overline{H} = 0 \tag{1.24b}$$

1.2 ELECTROMAGNETIC ENERGY AND POWER FLOW

We know from practical experience that energy delivered to a transmitting antenna can be faithfully recovered at a distant receiver. This transfer of energy is attributed to fields generated at the transmitter which propagate to the receiver. Our (classical) description of electromagnetic radiation rests entirely on this on this field interpretation, so it is appropriate to carefully review the relationships between electromagnetic fields and energy.

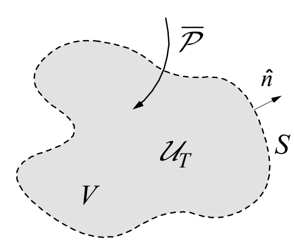


Figure 1.2 Poynting's theorem is an expression of conservation of energy. Any change in the total energy inside a volume must be accompanied by a flow of energy into or out of the volume.

As a starting point we simply assume that conservation of energy holds for electromagnetic fields. Therefore if the total energy in any region of space is observed to increase in time, then there must be a corresponding flow of power into that region to account for the change. Using the notation of fig. 1.2, the total energy in the volume V can be expressed as

$$\iiint_V \mathcal{U}_\mathrm{T}\, dV$$

where $\mathcal{U}_\mathrm{T}$ is the total energy density [J/m^3]. Defining $\overline{\mathcal{P}}$ as a power density vector [W/m^2], then the net power flow *into* the volume is

$$-\oiint_S \overline{\mathcal{P}} \cdot d\overline{S}$$

The conservation of energy can then be written as

$$\oiint \overline{\mathcal{P}} \cdot d\overline{S} = -\frac{d}{dt} \iiint \mathcal{U}_\mathrm{T}\, dV \tag{1.25}$$

Using the divergence theorem (A.54), (1.25) can be written as a continuity equation

$$\nabla \cdot \overline{\mathcal{P}} + \frac{\partial \mathcal{U}_\mathrm{T}}{\partial t} = 0 \tag{1.26}$$

We must now try to relate the power and energy densities to the field variables. If the volume V contains matter (charges), then the total energy $\mathcal{U}_\mathrm{T}$ will include both the energy stored in the fields, and the field energy that is converted to mechanical energy (or vice-versa) through the charge motion. Writing $\mathcal{U}_\mathrm{T} = \mathcal{U}_\mathrm{EM} + \mathcal{U}_\mathrm{d}$, the conservation law (1.26) can be written as

$$-\frac{\partial \mathcal{U}_\mathrm{d}}{\partial t} = \nabla \cdot \overline{\mathcal{P}} + \frac{\partial \mathcal{U}_\mathrm{EM}}{\partial t} \tag{1.27}$$

where $\mathcal{U}_\mathrm{EM}$ is the field energy density, and $\mathcal{U}_\mathrm{d}$ is the energy lost (or gained) by the interaction of fields with matter. The latter can be expressed using the Lorentz force laws. If we describe the matter in the volume using charge density functions ρ_e and ρ_m, then the fields act to displace the charges and hence mechanical energy is consumed. Recall that the incremental energy dW required to move a charge q moved through a distance $d\overline{r}$ by a force $\overline{\mathcal{F}}$ is $dW = \overline{\mathcal{F}} \cdot d\overline{r}$, and therefore the rate of change of energy (power) is $\overline{\mathcal{F}} \cdot \overline{v}$, where $\overline{v} = d\overline{r}/dt$. Similarly, the rate of change of energy density in the volume V due to motion of the charge distributions ρ_e and ρ_m is described by

$$\frac{\partial \mathcal{U}_\mathrm{d}}{\partial t} = \overline{v} \cdot \overline{f} = \overline{v} \cdot \left(\rho_e \overline{\mathcal{E}} + \rho_m \overline{\mathcal{H}}\right) = \overline{\mathcal{J}} \cdot \overline{\mathcal{E}} + \overline{\mathcal{M}} \cdot \overline{\mathcal{H}} \tag{1.28}$$

The currents can in turn be related to the electric and magnetic fields through Maxwell's equations (1.4c-d), giving

$$-\frac{\partial \mathcal{U}_\mathrm{d}}{\partial t} = \nabla \cdot (\overline{\mathcal{E}} \times \overline{\mathcal{H}}) + \overline{\mathcal{H}} \cdot \frac{\partial \overline{\mathcal{B}}}{\partial t} + \overline{\mathcal{E}} \cdot \frac{\partial \overline{\mathcal{D}}}{\partial t} \tag{1.29}$$

where use has been made of vector identity (A.47). So far this equation is generally applicable to any material. Specializing to simple media via (1.13) allows us to write

$$\overline{\mathcal{H}} \cdot \frac{\partial \overline{\mathcal{B}}}{\partial t} = \mu \overline{\mathcal{H}} \cdot \frac{\partial \overline{\mathcal{H}}}{\partial t} = \frac{1}{2}\mu \frac{\partial (\overline{\mathcal{H}} \cdot \overline{\mathcal{H}})}{\partial t}$$

and similarly for the last term in (1.29). Therefore (1.29) becomes, after substituting (1.28),

$$-\frac{\partial \mathcal{U}_\mathrm{d}}{\partial t} = \nabla \cdot (\overline{\mathcal{E}} \times \overline{\mathcal{H}}) + \frac{\partial}{\partial t}\left[\frac{1}{2}\mu \overline{\mathcal{H}} \cdot \overline{\mathcal{H}} + \frac{1}{2}\epsilon \overline{\mathcal{E}} \cdot \overline{\mathcal{E}}\right] \tag{1.30}$$

Comparing (1.30) to (1.27), we can tentatively identify the electromagnetic power density as $\overline{\mathcal{P}} = \overline{\mathcal{E}} \times \overline{\mathcal{H}}$; this is known as *Poynting's vector*, and (1.30) is a statement of *Poynting's theorem*. The term in brackets in (1.30) is the stored field energy.

Integrating (1.30) over a volume V bounded by a surface S and using (1.28) and the divergence theorem (A.54) gives the integral form of the Poynting's theorem

$$-\oiint (\overline{\mathcal{E}} \times \overline{\mathcal{H}}) \cdot d\overline{S} = \iiint \left[\overline{\mathcal{J}} \cdot \overline{\mathcal{E}} + \overline{\mathcal{M}} \cdot \overline{\mathcal{H}}\right] dV + \frac{\partial}{\partial t} \iiint \left[\frac{1}{2}\mu \overline{\mathcal{H}} \cdot \overline{\mathcal{H}} + \frac{1}{2}\epsilon \overline{\mathcal{E}} \cdot \overline{\mathcal{E}}\right] dV \tag{1.31}$$

The left-hand side of (1.31) is interpreted as the total power flowing *into* the volume through the surface S, which is equal to the total power absorbed in the volume (first term on the right) plus the rate of change of field energy in the volume (second term on right).

Although (1.30) is an exact statement of energy conservation for simple media, the identification of $\overline{E} \times \overline{H}$ with the power density is open to question, since the curl of any arbitrary vector field can be added to $\overline{\mathcal{P}}$ without changing (1.30). This mathematical ambiguity is resolved by appealing to experiment: Poynting's vector $\overline{E} \times \overline{H}$ does correctly predict the magnitude and direction of power flow measured in the lab, and so it is accepted as fact. If this makes the reader uncomfortable, remember that Maxwell's equations are also just postulates based on experimental evidence. Further discussions of the ambiguity of the Poynting theorem can be found in [**?**, **?**, **?**].

Specializing to harmonic time variations is complicated by the products of field quantities appearing in (1.30). According to the prescription (1.7), the time-dependent Poynting's vector should be expressed in terms of phasor fields as

$$\begin{aligned}
\overline{\mathcal{P}} &= \mathrm{Re}\left\{\overline{E}e^{\jmath\omega t}\right\} \times \mathrm{Re}\left\{\overline{H}e^{\jmath\omega t}\right\} \\
&= \frac{1}{4}\left[\overline{E} \times \overline{H}^* + \overline{E}^* \times \overline{H} + \overline{E} \times \overline{H}e^{2\jmath\omega t} + \overline{E}^* \times \overline{H}^* e^{-2\jmath\omega t}\right] \\
&= \frac{1}{2}\mathrm{Re}\left\{\overline{E} \times \overline{H}^*\right\} + \frac{1}{2}\mathrm{Re}\left\{\overline{E} \times \overline{H}e^{2\jmath\omega t}\right\}
\end{aligned}$$

since $\mathrm{Re}\{z\} = \frac{1}{2}(z + z^*)$. The instantaneous power density thus consists of a time-independent contribution (average power) and a periodic fluctuation at frequency 2ω. Considering just the average power density gives

$$P_{\text{ave}} = \frac{1}{2}\mathrm{Re}\left\{\overline{E} \times \overline{H}^*\right\} \tag{1.32}$$

In view of (1.32), we define a complex Poynting vector as $\overline{P} = \overline{E} \times \overline{H}^*$. Taking the divergence of this expression, using (A.47), and substituting Maxwell's equations leads to the phasor form of the Poynting theorem, analogous to (1.30),

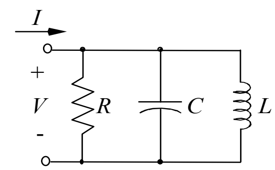
$$-\frac{1}{2}\nabla \cdot (\overline{E} \times \overline{H}^*) = \frac{1}{2}\left(\overline{E} \cdot \overline{J}^* + \overline{H}^* \cdot \overline{M}\right) + 2\jmath\omega\frac{1}{4}\left(\overline{H}^* \cdot \overline{B} - \overline{E} \cdot \overline{D}^*\right) \tag{1.33}$$

Integrating over a volume V bounded by a surface S and using the divergence theorem (A.54) gives the integral form of the phasor Poynting's theorem

$$\begin{aligned}
&- \frac{1}{2}\oiint \left(\overline{E} \times \overline{H}^*\right) \cdot d\overline{S} \qquad (1.34)\\
&= \frac{1}{2}\iiint \left(\overline{E} \cdot \overline{J}^* + \overline{H}^* \cdot \overline{M}\right) dV + 2\jmath\omega \iiint \frac{1}{4}\left(\overline{H}^* \cdot \overline{B} - \overline{E} \cdot \overline{D}^*\right) dV
\end{aligned}$$

The left-hand side of (1.34) is interpreted as the average total complex power flowing *into* the volume through the surface S, which is equal to the power absorbed in the volume plus the rate of change of field energy in the volume. In simple media where the volume integrals in (1.34) are all real, the first term on the right represents the real time-averaged power, and the last term represents the net reactive power in the volume. This interpretation of (1.34) may be more acceptable when

Figure 1.3 RLC circuit for interpreting Poynting's theorem.



compared to a corresponding problem in circuit theory. Consider the RLC circuit in figure 1.3a. The complex power delivered to this circuit, P_{in}, can be written as

$$P_{in} = \frac{1}{2} V I^* = \frac{1}{2} I I^* \left(R + \jmath\omega L + \frac{1}{\jmath\omega C} \right) \tag{1.35}$$

We know from circuit theory that the real power delivered to the circuit is $P_{loss} = \frac{1}{2} R I I^*$, and the energy stored in the inductor and capacitor is given by $U_m = \frac{1}{4} L I I^*$ and $U_e = \frac{1}{4} C V V^*$, respectively, which enables us to write (1.35) as

$$P_{in} = P_{loss} + 2\jmath\omega \left(U_m - U_e \right) \tag{1.36}$$

which is exactly the same form as (1.34). Furthermore, this suggests the following expressions for stored magnetic energy and stored electric energy in terms of the fields:

$$U_m = \frac{1}{4} \mathrm{Re} \iiint \overline{H}^* \cdot \overline{B} \, dV \qquad U_e = \frac{1}{4} \mathrm{Re} \iiint \overline{E} \cdot \overline{D}^* dV \tag{1.37}$$

It should be noted that these expressions for energy density are not valid for dispersive media, but can be suitably modified (see p.94 of [**?**]).

1.3 SOURCES AND GENERATORS

In formulating electromagnetic problems, we may postulate a set of charges or currents as known *sources* of fields, and subsequently attempt to formulate more direct solutions for the field quantities in terms of these sources. However, we must remember that the fields so produced are also capable of inducing surface charges and currents in neighboring matter. These induced currents and charges will then give rise to another set of fields which are superimposed on the first, and so on. This phenomenon is called *scattering*, and the secondary fields produced by the induced currents are the *scattered fields*. Although the induced charges and currents also act as sources of the scattered fields, they are clearly different than the original set of charges and currents, which were assumed to exist independent of the presence of any fields. Using the superposition principle, we therefore express the *total* currents, $\overline{J}$ and $\overline{M}$ in Maxwell's equations, as

$$\overline{J} = \overline{J}_i + \overline{J}_f \qquad \overline{M} = \overline{M}_i + \overline{M}_f \tag{1.38}$$

where $(\overline{J}_i, \overline{M}_i)$ are the *impressed currents*, and $(\overline{J}_f, \overline{M}_f)$ are the currents *induced* by the fields. Impressed currents are assumed to be fixed in some way that is not affected by the fields; these are analogous to the ideal current generators used in circuit theory. Induced currents are those that flow only in response to the fields, and arise physically from motion of either free or bound electrons. The motion of free electrons is described classically through Ohm's law and is called a *conduction* current, whereas the oscillatory motion of bound charges is called a *polarization current*.

The linearity of Maxwell's equations (assuming *linear* media) means that the fields can be similarly decomposed into a component due only to the impressed currents (the "applied" or "incident" field), and a component produced by the induced currents (the "scattered" field),

$$\overline{E} = \overline{E}_{\mathrm{inc}} + \overline{E}_{\mathrm{scatt}} \qquad \overline{H} = \overline{H}_{\mathrm{inc}} + \overline{H}_{\mathrm{scatt}} \tag{1.39}$$

As a example of these ideas, consider the reflection of an incident field from a perfectly conducting plane, as illustrated in figure 1.4. Mathematically, Maxwell's equations describe a self-consistent relationship between the *total* currents and *total* fields for the problem, but physically it is more appealing to consider the situation as a chain of events: the incident field is produced by an impressed source distribution $\overline{J}_i$; this incident field *induces* a conduction current on the surface of the conductor; the induced current then acts as a source, radiating fields which must exactly cancel the incident fields on and within the conductor, as required by the boundary conditions. In later chapters we will develop quite general methods for attacking such scattering problems based on this causal viewpoint, which can be applied to almost any problem, at least in principle.

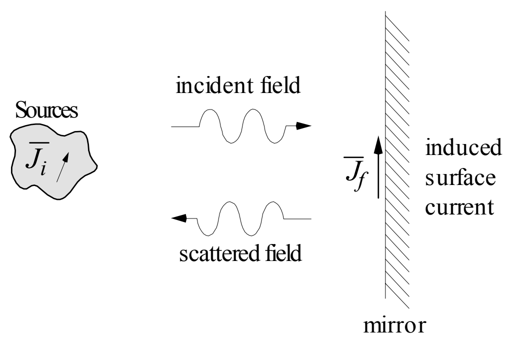


Figure 1.4 Reflection from a mirror as a simple example of scattering processes, described by impressed and induced currents.

The distinction between impressed and induced currents is therefore a natural breakdown in terms of "cause and effect", but it can sometimes lead to confusion in analysis. The trouble starts when making statements such as "currents are induced, which in turn radiate...". In order to calculate the scattered fields we may temporarily view the induced currents as fixed generators, that is, like impressed currents. In this way, the same mathematical relationship between incident fields and impressed currents can be used to relate the scattered fields to the induced currents. But it must be remembered that the currents in question are, in fact, induced currents when making such calculations. They produce only part of the field and hence must be related back to the impressed currents in such a way that all boundary conditions are satisfied. This discussion may seem rather pedantic, but a clear understanding of the differences is especially helpful in our application of field equivalence principles.

Impressed currents can be used to represent "circuit" generators as shown in fig. 1.5. A *current source* is modeled as a short filament of impressed current $\overline{J}_i$ in series with a perfectly conducting wire, as shown in fig. 1.5a. Assuming the dimensions of the circuit are small enough so that Kirchoff's circuit laws apply, then the impressed current will induce a current in the external circuit of the same magnitude, irrespective of the load impedance. If we compute the complex

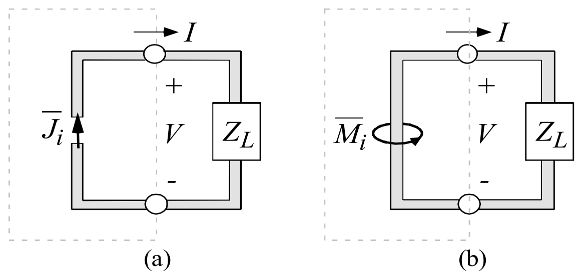


Figure 1.5 Electromagnetic representation of independent circuit sources. (a) Current generator (impressed electric current filament); (b) Voltage generator (impressed magnetic current loop).

power flow out of a volume surrounding the generator (the dashed box in fig. 1.5), we find

$$-\frac{1}{2}\iiint \overline{E}\cdot\overline{J}_i^*\,dV = -\frac{1}{2}I^*\int_{gap}\overline{E}\cdot d\overline{\ell} = \frac{1}{2}I^*V \tag{1.40}$$

which is in accordance with our expectations of a current generator. Note that the internal impedance of the source is infinite, since removal of the impressed current leaves an open circuit in the gap. Similarly, a *voltage source* in circuit theory can be represented as in fig. 1.5b, using a filamentary loop of magnetic current around a perfectly conducting wire. From Maxwell's equations, and neglecting the magnetic flux linked by the circuit,

$$\oint_C \overline{E}\cdot d\overline{\ell} = \iint \overline{M}\cdot d\overline{S}$$

If the path C is coincident with the wire leads and closes across the terminals, then we find the magnitude of the magnetic current filament is just $-V$, and therefore the complex power flowing out of the generator (through the dashed box in fig. 1.5b) is

$$-\frac{1}{2}\iiint \overline{H}^*\cdot\overline{M}\,dV = \frac{1}{2}V\oint_{\text{loop}}\overline{H}^*\cdot d\overline{\ell} = \frac{1}{2}VI^* \tag{1.41}$$

The internal impedance in this case is zero since removal of the current loop leaves a short circuit.

1.4 RECIPROCITY THEOREMS; RUMSEY'S REACTION

Suppose there are two separate source distributions, $(\overline{J}_1, \overline{M}_1)$ and $(\overline{J}_2, \overline{M}_2)$, in a certain localized region defined by volume V, as shown in figure 1.6. Physically this situation is representative of a general two-port electrical network, such as an antenna link. Characterization of this electrical network involves examining the interaction of fields and sources between the ports. We assume the volume is filled with a simple isotropic media described by (1.13) and (1.11). These sources produce the fields $(\overline{E}_1, \overline{H}_1)$ and $(\overline{E}_2, \overline{H}_2)$, respectively, in accordance with Maxwell's equations

$$\begin{aligned} \nabla\times\overline{E}_1 &= -\jmath\omega\mu\overline{H}_1 - \overline{M}_1 \\ \nabla\times\overline{H}_1 &= \jmath\omega\epsilon\overline{E}_1 + \overline{J}_1 \end{aligned} \quad \text{and} \quad \begin{aligned} \nabla\times\overline{E}_2 &= -\jmath\omega\mu\overline{H}_2 - \overline{M}_2 \\ \nabla\times\overline{H}_2 &= \jmath\omega\epsilon\overline{E}_2 + \overline{J}_2 \end{aligned} \tag{1.42}$$

Using the vector identity (A.47)

$$\nabla\cdot(\overline{A}\times\overline{B}) = (\nabla\times\overline{A})\cdot\overline{B} - (\nabla\times\overline{B})\cdot\overline{A}$$

we find that

$$\nabla \cdot \left(\overline{E}_1 \times \overline{H}_2 - \overline{E}_2 \times \overline{H}_1\right) = \overline{J}_1 \cdot \overline{E}_2 - \overline{J}_2 \cdot \overline{E}_1 + \overline{M}_2 \cdot \overline{H}_1 - \overline{M}_1 \cdot \overline{H}_2 \tag{1.43}$$

Integrating (1.43) over the volume V and using the divergence theorem (A.54) gives

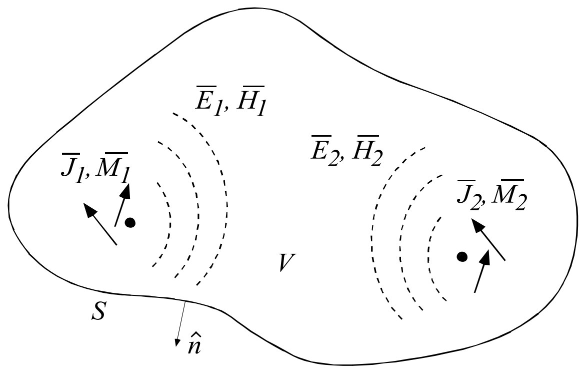


Figure 1.6 Two source distributions and corresponding fields within a volume V.

$$\oint\!\!\!\oint_S \left[\overline{E}_1 \times \overline{H}_2 - \overline{E}_2 \times \overline{H}_1\right] \cdot d\overline{S} = \iiint_V \left[\overline{J}_1 \cdot \overline{E}_2 - \overline{J}_2 \cdot \overline{E}_1 + \overline{M}_2 \cdot \overline{H}_1 - \overline{M}_1 \cdot \overline{H}_2\right] dV \tag{1.44}$$

Note that the only currents on the right hand side which contribute to the integral are the *impressed* sources; induced current terms cancel by virtue of (1.11). This result is usually applied (and more easily interpreted) for certain special cases where either the surface integral or volume integral vanishes. For example, if the surface is chosen to exclude any impressed sources so that $\overline{J}_1 = \overline{J}_2 = \overline{M}_1 = \overline{M}_2 = 0$, then (1.44) reduces to

$$\oint\!\!\!\oint_S \left[\overline{E}_1 \times \overline{H}_2 - \overline{E}_2 \times \overline{H}_1\right] \cdot d\overline{S} = 0 \tag{1.45}$$

Note conditions of validity!

which is called the *Lorentz reciprocity theorem*. In this case the fields are due to sources external to S. We will later use this result to establish the reciprocal properties of an antenna link.

Alternatively, if the surface S coincides with a PEC or PMC boundary, then the surface integral vanishes since, using (A.38),

$$\begin{aligned}\left(\overline{E}_1 \times \overline{H}_2\right) \cdot \hat{n} - \left(\overline{E}_2 \times \overline{H}_1\right) \cdot \hat{n} &= \left(\hat{n} \times \overline{E}_1\right) \cdot \overline{H}_2 - \left(\hat{n} \times \overline{E}_2\right) \cdot \overline{H}_1 \\ &= -\left(\hat{n} \times \overline{H}_2\right) \cdot \overline{E}_1 + \left(\hat{n} \times \overline{H}_1\right) \cdot \overline{E}_2\end{aligned}$$

and either $\hat{n} \times \overline{E} = 0$ for a PEC boundary, or $\hat{n} \times \overline{H} = 0$ on a PMC boundary. Then (1.44) reduces to

$$\iiint \left(\overline{E}_1 \cdot \overline{J}_2 - \overline{H}_1 \cdot \overline{M}_2\right) dV = \iiint \left(\overline{E}_2 \cdot \overline{J}_1 - \overline{H}_2 \cdot \overline{M}_1\right) dV \tag{1.46}$$

Note conditions of validity!

This is a more familiar statement of reciprocity for those knowledgeable in circuit theory, and is often simply referred to as *the reciprocity theorem*. This last result can also be obtained if S is taken as a sphere at infinity. Then the Sommerfeld radiation condition (see Chapter 2) insures that the fields produced by localized currents in V will be spherical outward waves at infinity, so that

$$\overline{H} = \frac{\hat{n} \times \overline{E}}{\eta} \qquad \Rightarrow \qquad (\hat{n} \times \overline{E}_1) \cdot \overline{H}_2 - (\hat{n} \times \overline{E}_2) \cdot \overline{H}_1 = 0.$$

Interestingly, many of the "physical observables" important in applied electromagnetics—that is, quantities that can be measured directly—can be expressed in terms of integrals like those in (1.46). Rumsey [**?**] has argued for the physical significance of these integrals, which he called *reaction integrals*. The left hand side of (1.46) is then thought of as the *reaction* of source #2 on the fields from source #1. This refers to the fact that, in order to keep flowing, the sources must "react" to the fields in their vicinity by supplying/absorbing energy. Reaction integrals are commonly abbreviated as

$$< i, j > = \iiint \left(\overline{E}_i \cdot \overline{J}_j - \overline{H}_i \cdot \overline{M}_j \right) dV \tag{1.47}$$

The reciprocity theorem (1.46) can then be represented concisely as

$$< i, j > = < j, i > \tag{1.48}$$

An important problem for later work that can be described in terms of reaction integrals is the determination of equivalent circuit parameters representing a multiport electromagnetic network, as shown in figure 1.7. Using fig. 1.7a, the impedance matrix is defined by

$$\bar{V} = \overline{\overline{Z}} \cdot \bar{I} \qquad \text{or} \qquad V_i = \sum_{j=1}^{N} Z_{ij} I_j \tag{1.49}$$

Each term in the summation, $Z_{ij}I_j$, gives the contribution to the terminal voltage—the induced EMF—at port i due to currents impressed at port j, with all other ports open-circuited. Assuming that the independent current sources of fig. 1.7a are implemented in the sense of fig. 1.5a, we can compute the reaction $< j, i >$ as

$$< j, i > = \iiint_{\text{port } i} \overline{E}_j \cdot \overline{J}_i \, dV = I_i \int_{\text{port } i} \overline{E}_j \cdot d\bar{\ell} = -I_i \left(Z_{ij} I_j \right) \tag{1.50}$$

where the last equality follows since the path integral or $\overline{E}_j$ over port i is just the voltage induced at port i due to the current source at port j, or $Z_{ij}I_j$. Therefore,

$$Z_{ij} = -\frac{< j, i >}{I_i I_j} = -\frac{1}{I_i I_j} \iiint \overline{E}_j \cdot \overline{J}_i \, dV \tag{1.51}$$

Using the reciprocity theorem (1.46) we find that

$$Z_{ij} = Z_{ji}$$

which is the familiar result from circuit theory. If we had alternatively chosen to express the system in terms of admittance parameters as shown in figure 1.7b, a similar analysis gives

$$Y_{ij} = \frac{< j, i >}{V_i V_j} = \frac{1}{V_i V_j} \iiint \overline{H}_j \cdot \overline{M}_i \, dV \tag{1.52}$$

with a similar consequence of reciprocity,

$$Y_{ij} = Y_{ji}$$

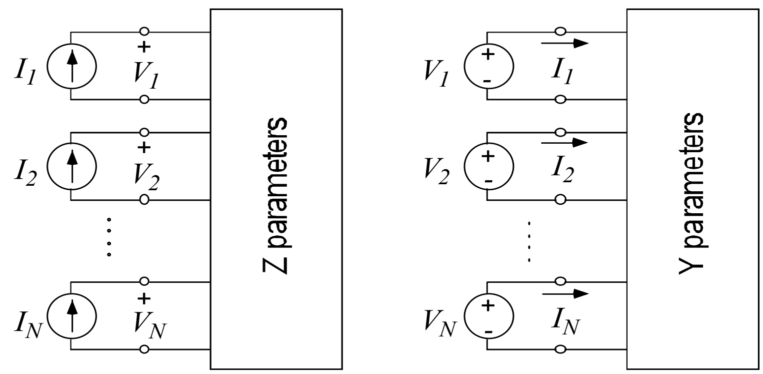


Figure 1.7 Multiport circuit representation of an electromagnetic system. (a) Configuration for characterization in terms of impedance parameters; (b) Configuration for characterization in terms of admittance parameters.

Closer examination of the derivation of (1.44) shows that it is critically dependent on the assumption of a simple isotropic media in the volume V. That is, we have only proved reciprocal properties for electrical systems comprised on isotropic media. Using more general constitutive relations for anisotropic media (1.14) and (1.15), we find (Problem **??**) that the result (1.44) is only obtained when the material properties in the volume are described by symmetric dyads

$$\overline{\overline{\epsilon}} = \overline{\overline{\epsilon}}^T \qquad \overline{\overline{\mu}} = \overline{\overline{\mu}}^T \qquad \overline{\overline{\sigma}} = \overline{\overline{\sigma}}^T \tag{1.53}$$

Such materials are therefore called reciprocal materials. An important example of a material that is not reciprocal is a magnetically-biased plasma, such as the Earth's ionosphere. Antenna links involving propagation through the Earth's ionosphere are therefore not reciprocal. Alternatively, antennas themselves may be constructed from non-reciprocal media, such as magnetically-biased ferrites. The resulting non-reciprocal antenna may serve a useful function; for example, simultaneously transmitting and receiving different polarizations. Propagation in non-reciprocal media and analysis of non-reciprocal antennas is, however, a relatively specialized topic that will not be dealt with in this work.

The reciprocity theorem will prove a useful tool in other contexts. For example, consider the situation in fig. 1.8, where there are two sets of impressed currents, denoted as $\overline{J}_1$ and J_2, within a volume V. Current $\overline{J}_1$ is impressed directly adjacent to a PEC object. Current $\overline{J}_2$ is a test source that can be oriented in any arbitrary direction. According to (1.46), these currents and the fields they produce are related by

$$\iiint_v \overline{E}_1 \cdot \overline{J}_2 \, dV = \iiint_V \overline{E}_2 \cdot \overline{J}_1 \, dV \tag{1.54}$$

Now, the field $\overline{E}_2$ is the total field produced by the test source $\overline{J}_2$, which must vanish everywhere along the surface of the PEC object. Therefore the integral on the right in (1.54) is zero and we have

$$\iiint_V \overline{E}_1 \cdot \overline{J}_2 \, dV = 0$$

Since $\overline{J}_2$ can be anything we choose, this must mean that $\overline{E}_1 = 0$ *everywhere* inside of V. This proves that impressed electric currents on PEC surfaces do not radiate. Physically this is because

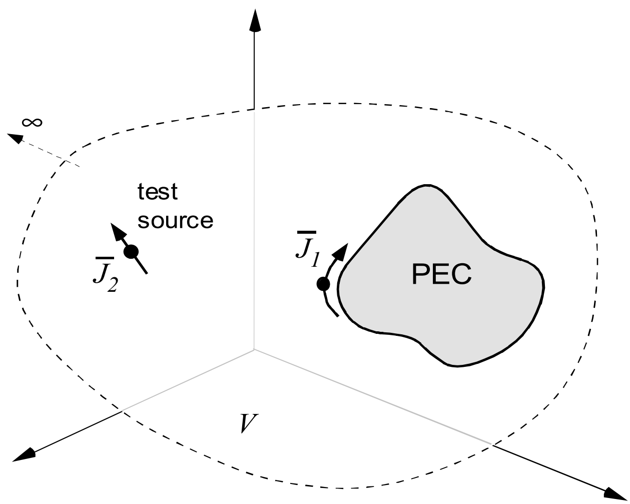


Figure 1.8 Impressed currents above conductors do not radiate, as can be shown by aplying the reciprocity theorem to this example.

the induced currents on the object radiate fields which exactly cancel the fields of the impressed current.

The method just employed is quite powerful. Think of what we have just done—we've solved for the fields produced by an arbitrary current distribution $\overline{J}_1$ radiating in the presence of a conductor, an otherwise difficult boundary-value problem. All that was necessary was a knowledge of the fields produced by our "testing" source, which can be anythign we choose.

1.5 UNIQUENESS OF SOLUTIONS

After going to the trouble of finding a solution to Maxwell's equations for a particular problem, one may wonder if it is the only possible solution. This is guaranteed, under certain conditions, by the uniqueness theorem. To prove the theorem, we assume the existence of two possible solutions, and derive the conditions required to insure they are identical.

Let $(\overline{E}_1, \overline{H}_1)$ and $(\overline{E}_2, \overline{H}_2)$ be two possible solutions to (1.5) for a given set of sources,

$$\begin{array}{ll} \nabla \times \overline{E}_1 = -\overline{M} - \jmath\omega\mu\overline{H}_1 & \nabla \times \overline{H}_1 = \overline{J} + \jmath\omega\epsilon\overline{E}_1 \\ \nabla \times \overline{E}_2 = -\overline{M} - \jmath\omega\mu\overline{H}_2 & \nabla \times \overline{H}_2 = \overline{J} + \jmath\omega\epsilon\overline{E}_2 \end{array} \tag{1.55}$$

Subtracting these equations and defining the difference fields $\delta\overline{E} = \overline{E}_1 - \overline{E}_2$ and $\delta\overline{H} = \overline{H}_1 - \overline{H}_2$ gives

$$\nabla \times \delta\overline{E} = -\jmath\omega\mu\delta\overline{H} \qquad \nabla \times \delta\overline{H} = \jmath\omega\epsilon\delta\overline{E} \tag{1.56}$$

which are just the source-free Maxwell equations. Therefore, the difference fields must satisfy Poynting's theorem (1.34),

$$\oiint_S (\delta\overline{E} \times \delta\overline{H}^*) \cdot d\overline{S} = \jmath\omega \iiint \left[\mu|\delta\overline{H}|^2 - \epsilon^*|\delta\overline{E}|^2\right] dV \tag{1.57}$$

If the solution were indeed unique, then this would imply that $\delta\overline{E} = \delta\overline{H} = 0$ everywhere within the volume of interest, so that both sides of (1.57) vanishes. Suppose we now reverse the problem:

if we can somehow prove that the surface integral in (1.57) vanishes, under what conditions does this imply that the solution is unique? Expanding the volume integral in terms of its real and imaginary components, a vanishing surface integral would require that

$$\iiint_V \left[\mu'|\delta\overline{H}|^2 - \epsilon'|\delta\overline{E}|^2\right]\, dV = 0 \tag{1.58a}$$

$$\iiint_V \left[\mu''|\delta\overline{H}|^2 + \epsilon''|\delta\overline{E}|^2\right]\, dV = 0 \tag{1.58b}$$

In lossy media, μ'' and ϵ'' are always positive. As long as there is some finite (though perhaps infinitesimal) loss in the system, the second of (1.58) can only be satisfied if $\delta\overline{E} = \delta\overline{H} = 0$ everywhere in the volume. Since there is always *some* loss in practice, uniqueness is therefore guaranteed provided we can make the surface integral vanish.

Using (A.38) and $d\overline{S} = \hat{n}\, dS$, the integrand of the surface integral in (**??**) can be written as

$$(\delta\overline{E} \times \delta\overline{H}^*) \cdot d\overline{S} = (\hat{n} \times \delta\overline{E}) \cdot \delta\overline{H}^*\, dS = -(\hat{n} \times \delta\overline{H}^*) \cdot \delta\overline{E}\, dS \tag{1.59}$$

If the tangential electric fields are specified on the bounding surface—for example, if the problem statement fixes the value of $\hat{n} \times \overline{E}$ on S—then this boundary condition must be incorporated into every possible solution, hence $\hat{n} \times \delta\overline{E} = 0$ over S, and the surface integral vanishes. Similarly, if the tangential magnetic fields, $\hat{n} \times \overline{H}$), are specified on the surface, then $\hat{n} \times \delta\overline{H} = 0$ over S, and the surface integral vanishes. Therefore, *the fields produced by sources within a lossy region are unique as long as the tangential components satisfy prescribed conditions at the bounding surface.* To obtain uniqueness in an ideal lossless region, we consider the fields to be the limit of a corresponding field in a lossy region as the loss goes to zero [**?**].

1.6 FIELD EQUIVALENCE PRINCIPLES

It many problems, a knowledge of the fields is required not everywhere in space, but rather in a certain well-defined region that is separate from the sources of the field (for example, the radiation fields of an antenna). In such cases it may be possible to simplify the problem by replacing the actual sources with fictitious sources that produce the same fields in the region of interest. These provide powerful tools for analysis.

1.6.1 Image Theory

The simplest and most familiar equivalence is the method of images. This technique is really just a catalog of certain electromagnetic problems that produce identical field distributions. These are usually identified by noting that conducting surfaces are surfaces of constant potential, and therefore can be placed along equipotential lines in *any* field distribution without altering the fields. For example, in fig. 1.9a, the fields produced by a positive and negative charge separated by a distance $2x$ produce an equipotential surface midway between the two charges. If we place a conducting object along this equipotential surface as shown in fig. 1.9b, then the fields above the surface are unchanged.

This equivalence is usually applied in reverse. Given a situation where there are charges a distance x a conducting plane, the conductor can be replaced by a set of image charges that have the opposite sign as the original charge, spaced a distance x below the original conducting surface. This eliminates the conducting matter, leaving only charges in unbounded space, a considerably

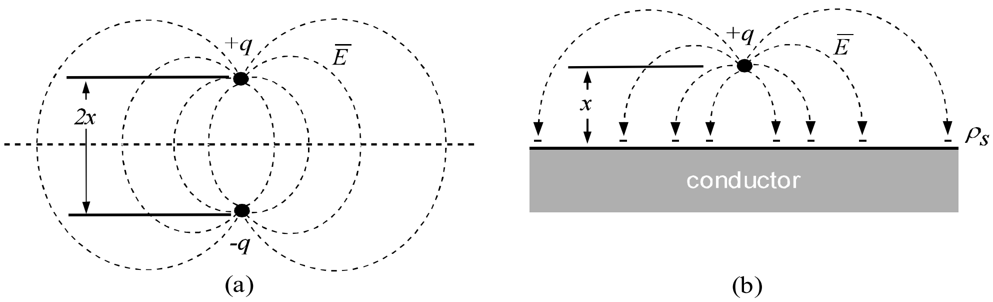


Figure 1.9 (a) Positive and negative charges separated by $2x$. (b) Positive charge $+q$ a distance x above a ground plane. These two situations are identical as far as the fields above the ground plane are concerned.

easier problem to solve. Note, however, that this equivalence applies only to the fields above the original conductor.

From this simple example we can derive many other image equivalents involving electric currents, and magentic charges and currents, and PMC surfaces. These are summarized in fig. 1.10 below. Note that the images for current elements depend on the direction of the current element, and can be derived from a knowledge of the behavior of the image charges as they are moved relative to the conducting surface. For example, a horizontal current element $\overline{J}_h$ above a PEC ground plane corrsponds to a positive charge movement in the direction of the current flow. The image charge in this case would move in the same direction, but has the opposite sign so that the effective image current direction is reversed.

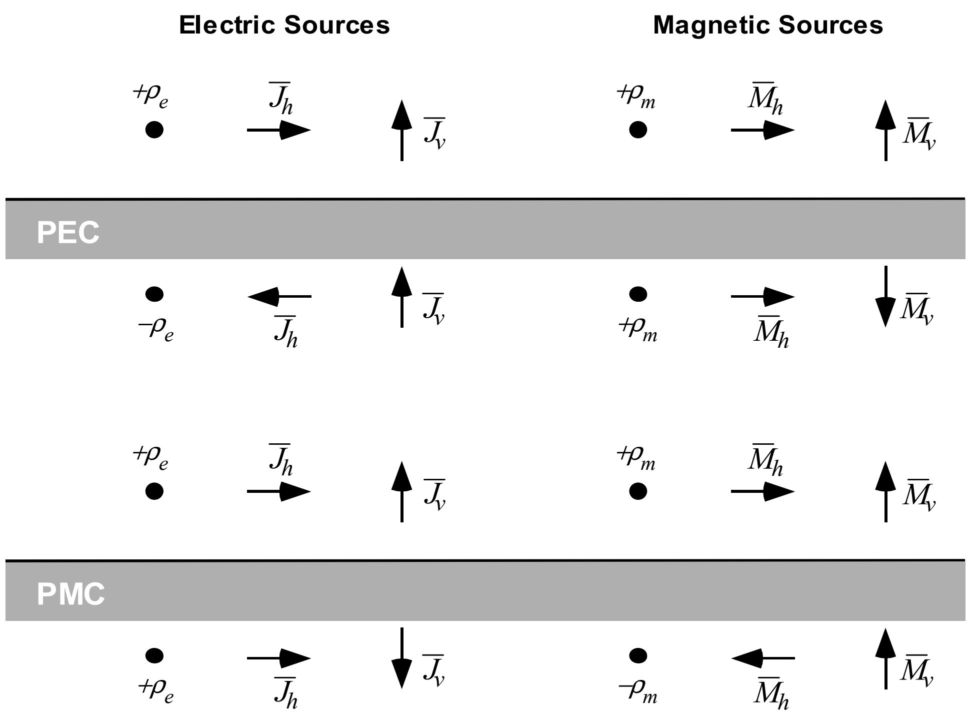


Figure 1.10 Summary of equivalent images for sources near conductors.

1.6.2 Love-Shelkunoff Equivalents

Readers familiar with circuit theory will remember that a network containing sources which drives a passive load network can be replaced by a Thevenin or Norton equivalent. This is illustrated in figure 1.11. Insofar as the calculation of voltage and current in the load network is concerned, the original and equivalent sources behave the same.

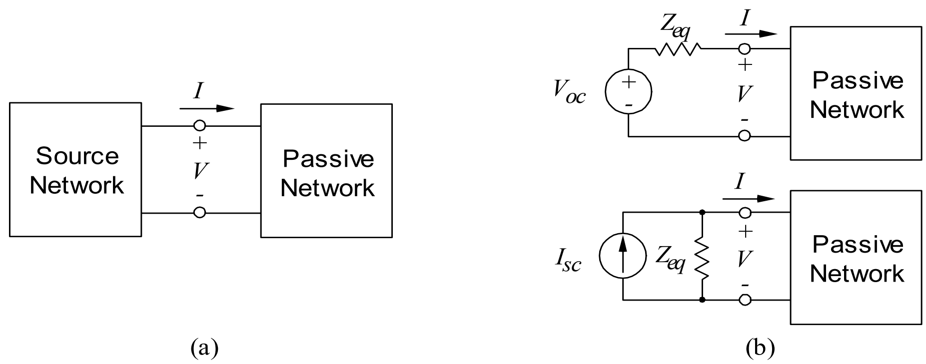


Figure 1.11 Thevenin and Norton equivalence principles from circuit theory.

In the case of the Thevenin and Norton circuits, the equivalent sources are expressed in terms of the open circuit voltage, V_{oc}, and short-circuit current, I_{sc}, which are measured at the terminals of the source network. The equivalent source impedance in each case, Z_{eq}, is the impedance of the original source network with all the sources shut off. However, the Thevenin and Norton sources are not the only possible equivalents. In order to relate the concept more directly to field theory, it is desirable to express the equivalent sources in terms of the actual terminal voltage and current in the original problem of figure 1.11. This is accomplished by the equivalent circuits of figure 1.12. These circuits are equivalent to the original problem because the relationship between V and I is fixed by the passive load network, which remains unchanged. Simple analysis shows that there is no current flowing in the source impedance of figure 1.12a, so it can be specified arbitrarily; figures 1.12b and 1.12c result from the choice $Z_{eq} = 0$ and $Z_{eq} = \infty$, respectively.

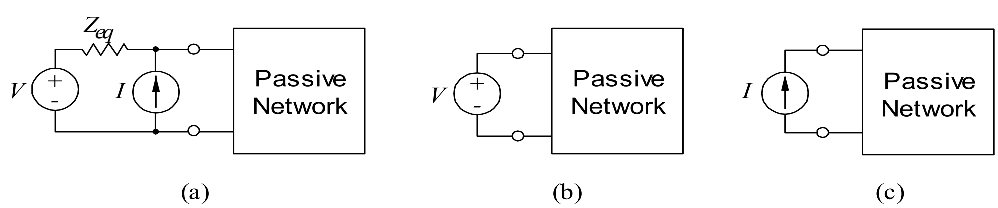


Figure 1.12 Other possible equivalents sources in terms of the actual terminal voltage V and current I from the previous figure.

The same concepts are extended to field theory by considering the situation depicted in figure 1.13a. Sources within some bounded region, possibly containing matter, produce the fields $(\overline{E}, \overline{H})$ outside of that region. To simplify the calculation of these fields, we replace the original sources by *impressed* surface currents J_s and M_s flowing on the boundary of the source region. From (1.22), we know that the magnitude of the surface currents required to produce the same fields outside of the boundary depends on the difference of the tangential fields across the boundary. Since the region within the boundary is of no interest, we can arbitrarily specify that the fields are zero within that region, giving the equivalent of figure 1.13b. This is known as *Love's equivalence principle* [**?**]. Note that this equivalence is only helpful when the tangential fields at the boundary of the original problem are known (or can be approximated). Comparing this situation to the circuit model of figure 1.12a, we see that the magnetic current is analogous to the terminal voltage V, while the electric current is analogous to the terminal current, I.

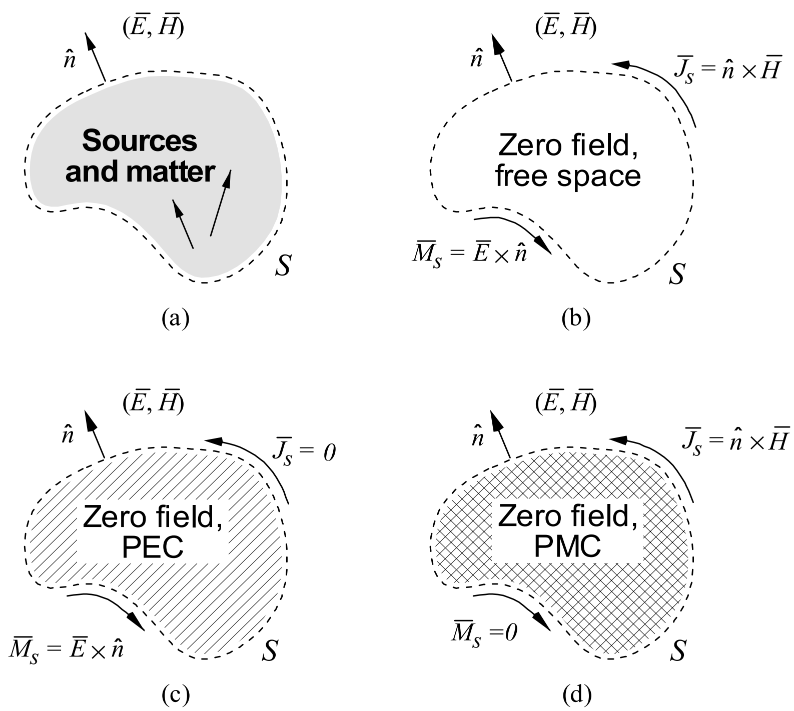


Figure 1.13 Four possible source configurations which produce the same field configuration external to the boundary S. (a) Original problem. (b) Love's equivalent, where the original source region is replaced by free-space, and surface currents are impressed on the bounding surface to produce null field within S. (c) and (d) are Shelkunoff equivalents, where the original source region is replaced by a perfect conductor. In the latter case, the impressed currents induce additional currents on the conductors.

Since the null field was specified within the original source region, the material found within the source region is irrelevant to the calculation of fields external to that region. This is analogous to the arbitrariness of the source impedance in the circuit equivalent of figure 1.12a. The most common choices are to fill the volume with free-space, as was tacitly assumed in Love's equivalent, or to surround the region by a perfect electric conductor or perfect magnetic conductor, as shown in figures 1.13c, and 1.13d. The latter two choices are due to Shelkunoff [**?**], and are analogous to the circuit models of figures 1.12b and 1.12c. In the first case (figure 1.13c) only magnetic currents are required, since the impressed current J_s is "short-circuited" by the PEC and does not radiate (proved earlier using the reciprocity theorem). Similarly, the magnetic current M_s is short-circuited in figure 1.13d, and only the electric surface current is required. From the uniqueness theorem we are assured that the fields calculated in each case will be identical to the original problem, as only one of the tangential fields ($\overline{E}$ or $\overline{H}$) is required.

Although it may not be immediately obvious, the equivalence of the four physical situations depicted in figure 1.13 can greatly simplify radiation problems. A complicated boundary-value problem (ie. sources radiating in the presence of nearby objects, such as figures 1.13a, 1.13c, 1.13d) can be reduced to an equivalent set of currents radiating in a homogeneous unbounded medium (figure 1.13b). Alternatively, we will use the equivalence of figures 1.13a, 1.13c, and 1.13d in our formulation of Huygen's principle in Chapter 2, which reduces a complicated source distribution (1.13a) to a (hopefully) simpler surface current on a conductor (1.13c or 1.13d).

1.6.3 Volume Equivalence Theorem

The volume equivalence theorem is based on the following observation: for any material body characterized by a simple scalar permittivity and permeability as in (1.13), we can write Maxwell's curl equations as

$$\nabla \times \overline{H} = \overline{J}_i + \jmath\omega\epsilon\overline{E} = \overline{J}_i + \overline{J}_p + \jmath\omega\epsilon_0\overline{E} \tag{1.60a}$$

$$\nabla \times \overline{E} = -\overline{M}_i - \jmath\omega\mu\overline{H} = -\overline{M}_i - \overline{M}_p - \jmath\omega\mu_0\overline{H} \tag{1.60b}$$

where we have defined the *polarization currents*

$$\overline{J}_p = \jmath\omega(\epsilon - \epsilon_0)\overline{E} \qquad \overline{M}_p = \jmath\omega(\mu - \mu_0)\overline{H} \tag{1.61}$$

In other words, we can replace the material by a volume current distribution (J_p, M_p) flowing in free space. The magnitude of the polarization currents are dependent on the *total* fields, and the total fields include some contribution from the polarization currents, so this approach is typically used to set up a self-consistent integral equation for the currents.

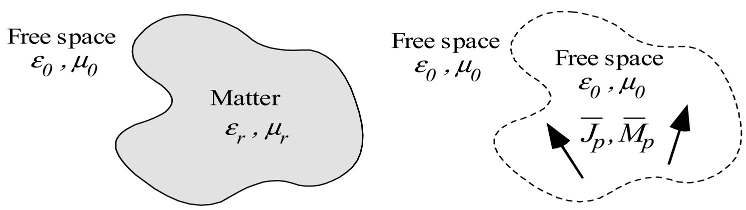


Figure 1.14 Volume equivalence theorem. Simple dielectric or magnetic matter can be replaced by volume polarization currents flowing in free-space.

1.7 FIELD BEHAVIOR NEAR A SHARP EDGE

At sharp edges of material bodies, the charge and/or current density may be highly concentrated, and in fact can become infinite in the limit of a mathematically perfect edge. Consequently, some of the field components may also be highly peaked near an edge. It is important to understand the mathematical nature of this possible singularity, especially in numerical computation where anticipating the correct form of the fields can often greatly speed the convergence to an accurate result.

As a simple and practical example, consider the two-dimensional conducting wedge in fig. 1.15, which is infinite in the $\hat{z}$ direction. A TE-to-z excitation as shown will induce $\hat{z}$-directed currents on this object, and the associated current density function will have a mathematical singularity at the edge. From the boundary conditions, this implies that the $\hat{\rho}$ and $\hat{\phi}$ components of the scattered magnetic field will be singular at the edge. This is the only solution as guaranteed by the uniqueness theorem. Now, infinite field quantities may be acceptable as long as the *physical*

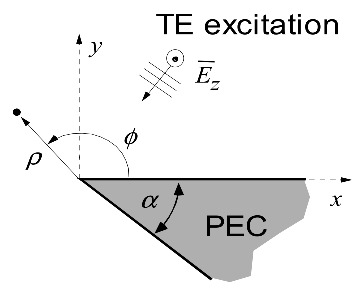


Figure 1.15 Cross section of a PEC wedge, with TE excitation.

observables derived from them remain finite. If Maxwell's equations are correct, the singular solution must therefore behave such that the *field energy* remain finite. This physical constraint gives us the critical information for predicting the behavior of the fields close to an edge. In the present case, the electric stored energy per unit length in a cylinder of radius a surrounding the edge is

$$\mathcal{U}_e = \frac{1}{2}\epsilon_0 \int_0^{2\pi-\alpha} \int_0^a |E_z|^2 \,\rho \, d\rho \, d\phi \tag{1.62}$$

A similar expression describes the magnetic stored energy. The fields within the fictitious cylinder can be expressed as a power series in ρ; the dominant term in the series (for small ρ) will have the form ρ^γ, where γ is the exponent to be determined. For the electric energy in (1.62) to remain finite with $E_z \propto \rho^\gamma$ requires that $\gamma > -1$. From Maxwell's equations, the dominant term in the expansion for the magnetic field can be determined as $\overline{H} \propto \nabla \times \overline{E} \propto \rho^{\gamma-1}$, and therefore for the magnetic stored energy per unit length to remain finite requires that $\gamma > 0$. Clearly then γ must be positive in order to have a finite total field energy within the cylinder. This also agrees with our expectation that $\overline{E}_z$ must vanish as $\rho \to 0$.

Near the edge we can write $\overline{E} = E_z \approx \rho^\gamma f(\phi)$. Substituting into Maxwell's equations gives

$$\frac{\partial^2 f}{\partial \phi^2} + \left(\gamma^2 + k^2\rho^2\right) f = 0. \tag{1.63}$$

Electrically close to the edge where $k\rho \ll \gamma$ we can neglect the term $k^2\rho^2$ and hence

$$f(\phi) \approx A \sin\gamma\phi + B\cos\gamma\phi \tag{1.64}$$

The unknown coefficients and allowed values of γ can be determined by the requirement that E_z vanish at $\phi = 0$ and $\phi = 2\pi - \alpha$, which gives $B = 0$ and $\gamma = n\pi/(2\pi - \alpha)$ where $n = 1, 2, \ldots$. The smallest positive γ describes the dominant term in the power series for the fields near the edge (*i.e.* for small ρ) , so we have

$$\textbf{TE case:} \quad E_z \approx A\rho^\gamma \sin\gamma\phi \qquad \text{where} \qquad \gamma = \frac{\pi}{2\pi - \alpha} \tag{1.65}$$

$$\overline{H} = -\tfrac{1}{\jmath\omega\mu}\nabla \times \overline{E} = -\frac{A}{\jmath\omega\mu}\frac{\gamma}{\rho^{1-\gamma}}\left[\hat{\rho}\cos\gamma\phi - \hat{\phi}\sin\gamma\phi\right]$$

A similar derivation can be carried out for a TM excitation (Problem **??**), with the result

$$\textbf{TM case:} \quad H_z \approx B\rho^\gamma \cos\gamma\phi \tag{1.66}$$

$$\overline{E} = \tfrac{1}{\jmath\omega\epsilon}\nabla \times \overline{H} = \frac{B}{\jmath\omega\epsilon}\frac{\gamma}{\rho^{1-\gamma}}\left[-\hat{\rho}\sin\gamma\phi - \hat{\phi}\cos\gamma\phi\right]$$

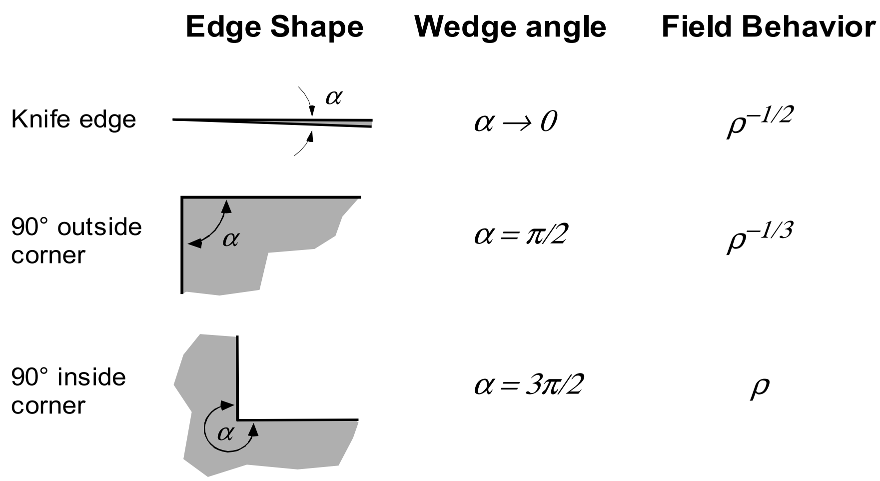


Figure 1.16 Examples of frequently encountered edges and associated singular field behavior.

So in both cases the singular fields vary as $\rho^{\gamma-1}$, where $\gamma = \pi/(2\pi - \alpha)$. Figure 1.16 illustrates three special cases often encountered in problems, and the anticipated field behavior close to the edge. Numerous other cases and treatments of singular fields can be found in the excellent monograph by Van Bladel[**?**].