2

Radiation from Unbounded Time-Harmonic Sources

"Science is the captain, practice the soldier."
—Leonardo da Vinci

The general problem of antenna analysis is twofold: 1) determination of the source distribution on the antenna given some specified excitation, and 2) subsequent derivation of the important descriptive parameters (radiation pattern, input impedance, *etc.*) from this source distribution. The sources in this context could be the true currents on the antenna or a set of equivalent currents prescribed by field equivalence principles. In both cases it is necessary to have general solutions of Maxwell's equations—explicit relationships between the fields and the sources—that can be applied to a variety of situations in radiation theory.

There are several representations for the fields that will prove useful. In many cases of interest, an application of field equivalence principles can reduce the problem to that of sources flowing in unbounded-space, and auxiliary potential functions can be employed to compute the fields. This is the focus of the present chapter. We will assume the currents are time-harmonic, and derive general solutions of Maxwell's for unbounded currents. We will use these expressions to show how elementary current distributions and the corresponding radiation can be considered as essential buildign blocks of more complicated sources and fields. We will then specialize the expressions for computation of the "far-fields" of a current distribution, from which can be derived many of the important descriptive parameters of antennas developed in subsequent chapters.

2.1 AUXILIARY POTENTIALS

The time-harmonic Maxwell's equations are

$$\nabla \cdot \overline{D} = \rho_e \tag{2.1a}$$

$$\nabla \cdot \overline{B} = \rho_m \tag{2.1b}$$

$$\nabla \times \overline{E} = -\overline{M} - \jmath\omega\overline{B} \tag{2.1c}$$
$$\nabla \times \overline{H} = \overline{J} + \jmath\omega\overline{D} \tag{2.1d}$$

which are consistent with charge conservation, expressed through the continuity equations

$$\nabla \cdot \overline{J} + \jmath\omega\rho_e = 0 \tag{2.2a}$$
$$\nabla \cdot \overline{M} + \jmath\omega\rho_m = 0 \tag{2.2b}$$

The two curl equations in (2.1) can be decoupled by taking the curl of both sides and making some obvious substitutions, resulting in two inhomogeneous vector wave equations, which (for simple media) are

$$\nabla \times \nabla \times \overline{E} - k^2\overline{E} = -\jmath\omega\mu\overline{J} - \nabla \times \overline{M} \tag{2.3a}$$
$$\nabla \times \nabla \times \overline{H} - k^2\overline{H} = \nabla \times \overline{J} - \jmath\omega\epsilon\overline{M} \tag{2.3b}$$

where $k^2 = \omega^2\mu\epsilon$. Using the identity $\nabla \times \nabla \times \overline{A} = \nabla\nabla \cdot \overline{A} - \nabla^2\overline{A}$, substituting the remaining two divergence equations from (2.1), and rewriting the charge densities in terms of the currents through (2.2), we get

$$\nabla^2\overline{E} + k^2\overline{E} = \jmath\omega\mu\left[\overline{J} + \frac{1}{k^2}\nabla(\nabla \cdot \overline{J})\right] + \nabla \times \overline{M} \tag{2.4a}$$
$$\nabla^2\overline{H} + k^2\overline{H} = -\nabla \times \overline{J} + \jmath\omega\epsilon\left[\overline{M} + \frac{1}{k^2}\nabla(\nabla \cdot \overline{M})\right] \tag{2.4b}$$

These equations are just a restatement of Maxwell's equations in simple media (all four Maxwell's equations were used in the derivation), but in a more convenient form for analysis. Both are in the same form, referred to as the inhomogeneous Helmholtz equation. The right hand side of each—the "forcing function"—involves only the current densities, which are assumed known. The equations are decoupled, each involving only one field variable. In rectangular coordinates, each equation reduces to three scalar equations of exactly the same form.

2.1.1 Vector Potentials in the Lorentz Gauge

In subsequent sections we will discuss the general solution of equations like (2.4) using the formalism of Green's functions. Before doing that, we will examine alternative expressions of Maxwell's equations through the introduction of auxiliary potential functions, which is sometimes helpful in simplifying the analysis for certain types of problems.

Consider electric and magnetic sources separately. When only electric sources are present ($\rho_m = \overline{M} = 0$) then $\nabla \cdot \overline{B} = 0$, and since the divergence of any curl is identically zero we can express $\overline{B}$ as the curl of another vector function

$$\overline{B} = \nabla \times \overline{A} \tag{2.5}$$

Substituting (2.5) into (2.1c) gives

$$\nabla \times \overline{E} = -\jmath\omega\nabla \times \overline{A} \quad \Rightarrow \quad \nabla \times (\overline{E} + \jmath\omega\overline{A}) = 0$$

Since the curl of any gradient is identically zero then we can write

$$\overline{E} + \jmath\omega\overline{A} = -\nabla\phi_e \tag{2.6}$$

where ϕ_e is an arbitrary scalar function. The negative sign in (2.6) is chosen so that the scalar function ϕ_e reduces to the electrostatic potential in the limit of $\omega \to 0$. Note that $\overline{A}$ and ϕ_e are not uniquely determined by (2.5) and (2.6), since we can define a new set of potentials $(\overline{A}', \phi_e')$ which will produce the same fields via the *gauge transformation*

$$\overline{A}' \to \overline{A} + \nabla\xi \qquad \phi_e' \to \phi_e - \jmath\omega\xi$$

The scalar function ξ is arbitrary, so there are an infinite number of potentials $(\overline{A}, \phi_e)$ which will produce a given field distribution; for this reason, $\overline{E}$ and $\overline{B}$ are said to be *gauge invariant*. This is an indirect consequence of the *Helmholtz theorem*, which states that a vector function is uniquely determined only by specifying *both* it's divergence and curl (see Appendix A). So far we have only specified the curl of $\overline{A}$ via (2.5). We can specify $\nabla \cdot \overline{A}$ freely, which then fixes value of ξ. For simple dielectrics, substitution of (2.5) and (2.6) into (2.1a) and (2.1d), and using the identity $\nabla \times \nabla \times \overline{A} = \nabla\nabla \cdot \overline{A} - \nabla^2\overline{A}$ gives

$$\nabla^2\overline{A} + k^2\overline{A} - \nabla(\nabla \cdot \overline{A} + \jmath\omega\mu\epsilon\phi_e) = -\mu\overline{J} \tag{2.7a}$$

$$\nabla^2\phi_e + \jmath\omega\nabla \cdot \overline{A} = -\rho_e/\epsilon \tag{2.7b}$$

This form suggests the following choice for the divergence of $\overline{A}$, called the *Lorentz condition*,

$$\nabla \cdot \overline{A} \equiv -\jmath\omega\mu\epsilon\phi_e \tag{2.8}$$

which converts (2.7) into a pair of inhomogeneous Helmholtz equations

$$\nabla^2\overline{A} + k^2\overline{A} = -\mu\overline{J} \tag{2.9a}$$

$$\nabla^2\phi_e + k^2\phi_e = -\rho_e/\epsilon \tag{2.9b}$$

The Lorentz condition (2.8) is not the only choice of gauge that is useful, but it is the one most commonly encountered.

Equations (2.9a) and (2.9b) are a restatement of Maxwell's equations in terms of a single vector function $\overline{A}$ and the scalar function ϕ_e, and represent four scalar equations of *exactly the same form*. In fact, it is only necessary to solve (2.9a), since both $\overline{E}$ and $\overline{H}$ can be expressed in terms of $\overline{A}$ using the Lorentz condition (2.8),

$$\overline{H} = \frac{1}{\mu}\nabla \times \overline{A} \tag{2.10a}$$

$$\overline{E} = -\jmath\omega\left[\overline{A} + \frac{1}{k^2}\nabla\nabla \cdot \overline{A}\right] \tag{2.10b}$$

Another set of potentials can be derived for the fields produced by magnetic sources. In the absence of electric sources ($\rho_e = \overline{J} = 0$), we can write $\overline{D} = \nabla \times \overline{F}$, where $\overline{F}$ is an electric vector potential. Following exactly the same steps that lead to (2.9a), we find a $\overline{F}$ is governed by a similar inhomogeneous Helmholtz equation

$$\nabla^2\overline{F} + k^2\overline{F} = -\epsilon\overline{M} \tag{2.11}$$

The fields produced by these magnetic currents are then calculated from $\overline{F}$ as follows

$$\overline{E} = -\frac{1}{\epsilon}\nabla \times \overline{F} \tag{2.12a}$$

$$\overline{H} = -\jmath\omega\left[\overline{F} + \frac{1}{k^2}\nabla\nabla \cdot \overline{F}\right] \tag{2.12b}$$

In the general case where both electric and magnetic sources are present, the principle of superposition allows us to write the solution as a the sum of fields produced by electric and magnetic sources separately, so that in general, the fields in the Lorentz gauge are computed as

$$\overline{E} = -\jmath\omega\left[\overline{A} + \frac{1}{k^2}\nabla\nabla\cdot\overline{A}\right] - \frac{1}{\epsilon}\nabla\times\overline{F} \tag{2.13a}$$

$$\overline{H} = \frac{1}{\mu}\nabla\times\overline{A} - \jmath\omega\left[\overline{F} + \frac{1}{k^2}\nabla\nabla\cdot\overline{F}\right] \tag{2.13b}$$

2.1.2 Hertz vectors in the Lorentz Gauge

We noted that (2.9) is simply a restatement of Maxwell's equations in terms of the potentials $\overline{A}$ and ϕ_e (assuming no magnetic sources), or equivalently four scalar functions. In fact, only a single vector function (three scalar functions) is required, since one of the four scalar functions is determined by the gauge condition. For example, if we define $\overline{A}$ and ϕ_e in terms of the vector function $\overline{\Pi}_e$ as

$$\overline{A} \equiv \jmath\omega\mu\epsilon\overline{\Pi}_e \qquad \phi_e = -\nabla\cdot\overline{\Pi}_e \tag{2.14}$$

then the Lorentz gauge is satisfied and the equations (2.9) governing the potentials both reduce to

$$\nabla^2\overline{\Pi}_e + k^2\overline{\Pi}_e = -\frac{\overline{J}}{\jmath\omega\epsilon} \tag{2.15}$$

The vector $\overline{\Pi}_e$ is called the *electric Hertz vector*. We can similarly write the vector potential $\overline{F}$ in terms of a *magnetic Hertz vector*, $\overline{F} = \jmath\omega\mu\epsilon\overline{\Pi}_m$, leading to a correspondingly similar governing equation

$$\nabla^2\overline{\Pi}_m + k^2\overline{\Pi}_m = -\frac{\overline{M}}{\jmath\omega\mu} \tag{2.16}$$

As the reader can observe, for time-harmonic fields there is essentially no difference between the Hertz vectors $(\overline{\Pi}_e, \overline{\Pi}_m)$ and the set $(\overline{A}, \overline{F})$, differing only by a proportionality factor. Both sets of potentials satisfy equations of exactly the same form. However, in the literature the Hertz vector notation has historically been employed to describe fields in *source-free* regions, whereas the set $(\overline{A}, \overline{F})$ is typically associated with a particular solution of Maxwell's equations directly in terms of the currents. In source-free regions, $\overline{J} = \overline{M} = 0$, the Hertz vectors satisfy homogeneous Helmholtz equations, and the source-free fields are expressed as

$$\overline{E} = \nabla\times\nabla\times\overline{\Pi}_e - \jmath\omega\mu\nabla\times\overline{\Pi}_m \tag{2.17a}$$

$$\overline{H} = \jmath\omega\epsilon\nabla\times\overline{\Pi}_e + \nabla\times\nabla\times\overline{\Pi}_m. \tag{2.17b}$$

The conventional use of the various auxiliary potentials is summarized in figure 2.1. We will most frequently use the potentials $\overline{A}$ and $\overline{F}$, but Hertz vectors continue to be used in the literature, and the reader should be aware of these and the relationship with other potential functions. In fact, both of the sets discussed above can be derived from a more general theory of vector eigenfunctions for Maxwell's equations. This can be appreciated by noting that equations (2.4), (2.15), (2.16), (2.9), and (2.11), all involve the same differential operator, $\nabla^2 + k^2$. It can be shown that general solutions to Maxwell's equations can always be represented by complete series expansions of vector functions derived from solutions of scalar equations like $\nabla^2\psi + k^2\psi = 0$. The numerous potential

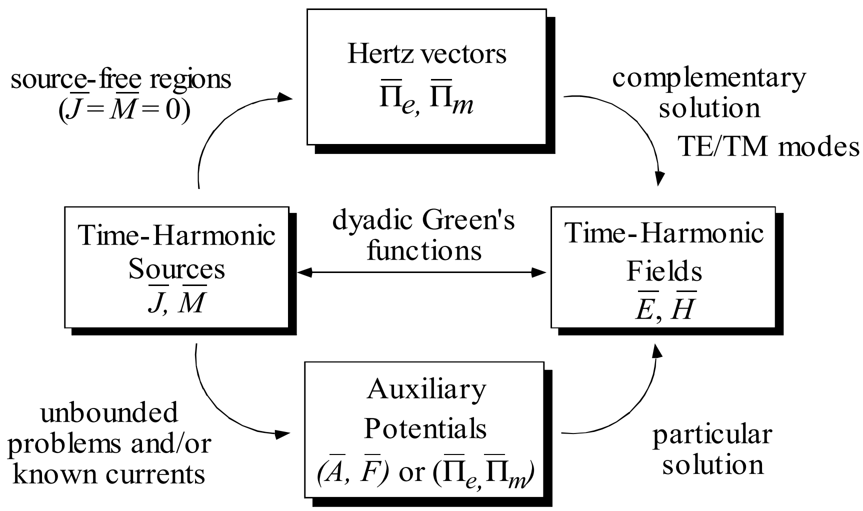

Figure 2.1 Use of auxiliary potentials for problem solving.

functions found in the literature differ only in proportionality factors or the context, such as the choice of coordinate system for the field representation or the boundary conditions.

2.2 GENERAL SOLUTIONS FOR ARBITRARY SOURCES

2.2.1 Scalar Green's Functions

Consider the vector potentials governed by (2.9a) and (2.11). In rectangular coordinates, all cartesian components of the potentials also satisfy the same inhomogeneous scalar wave equation; for example, A_x is governed by

$$\nabla^2 A_x(\overline{r}) + k^2 A_x(\overline{r}) = -\mu J_x(\overline{r}) \tag{2.18}$$

Suppose we have a solution to this equation for the special case of a point source located at $\overline{r}'$; that is, we have a scalar function $g(\overline{r}, \overline{r}')$ which is a known solution to

$$\left(\nabla^2 + k^2\right) g(\overline{r}, \overline{r}') = -\delta(\overline{r} - \overline{r}') \tag{2.19}$$

(the negative sign in front of the delta function is used by convention). The function $g(\overline{r}, \overline{r}')$ is called the "Green's function", and can be used to construct solutions to the Helmholtz equation for arbitrary current distributions as follows. Multiplying both sides of (2.19) by $J_x(\overline{r}')$ and integrating over primed coordinates gives

$$\iiint \left(\nabla^2 + k^2\right) g(\overline{r}, \overline{r}') J_x(\overline{r}')\, dV' = -\iiint \delta(\overline{r} - \overline{r}') J_x(\overline{r}')\, dV'$$

where the limits of integration encompass all space. The right hand side can be recognized as $-J_x(\overline{r})$ by the integral property of the delta function. Since ∇^2 only operates on *unprimed* coordinates, we can pull it outside of the integral on the left hand side, giving

$$\left(\nabla^2 + k^2\right) \iiint g(\overline{r}, \overline{r}') J_x(\overline{r}')\, dV' = -J_x(\overline{r})$$

The Green's function has a singularity at $\overline{r} = \overline{r}'$, and consequently this last step is only easily justifiable when the observation point $\overline{r}$ does not lie within the current-carrying region, so that $\overline{r} \neq \overline{r}'$ over the limits of integration. Let's assume this to be the case for now. Comparing with (2.18) we identify a solution for A_x at points away from the current distribution as

$$A_x(\overline{r}) = \mu \iiint g(\overline{r}, \overline{r}') J_x(\overline{r}')\, dV'$$

Since all cartesian components satisfy the same form of equation, the general vector solution to (2.9a) is

$$\overline{A}(\overline{r}) = \mu \iiint g(\overline{r}, \overline{r}') \overline{J}(\overline{r}')\, dV' \tag{2.20}$$

with a similar result for the vector potential $\overline{F}$. Noe that the vector $\overline{A}$ has the same direction as the current $\overline{J}$, which is a useful feature of the vector potential.

Physically, the Green's function is the response of the linear electromagnetic environment to an impulse forcing function. The total response due to some arbitrary source distribution can be found by representing the source as a collection of impulses and adding up all of the individual responses. The result (2.20) is a mathematical statement of this superposition process.

Equation (2.20) is a *particular* solution of (2.9a). The integration is taken over all regions of space containing currents. This must include not only impressed currents, but *induced currents* as well. In most electromagnetic problems the impressed currents will be known, but the induced currents on material objects are rarely known *a priori*. One way to address this difficulty is to note that induced currents always appear in order to satisfy the relevant boundary conditions in or on the material object. Therefore if we can modify our solution to satisfy the boundary conditions, we implicitly account for the induced currents. This can be done by exploiting the so-called *complementary* solutions $\overline{A}_c$, which are solutions to the homogeneous equation

$$\left(\nabla^2 + k^2\right) \overline{A}_c = 0$$

A general solution for the vector potential can be written as

$$\overline{A}(\overline{r}, t) = \overline{A}_c(\overline{r}, t) + \mu \iiint g(\overline{r}, \overline{r}') \overline{J}(\overline{r}')\, dV' \tag{2.21}$$

The complementary solutions will have unknown constants which can be adjusted to satisfy the boundary conditions. In the present case, this procedure is complicated by the fact that boundary conditions are specified in terms of the electric and magnetic fields, not the vector potential directly.

Alternatively, we could construct a *modified Green's function* by similarly finding a general solution to (2.19) which includes the complementary solutions g_c satisfying $\left(\nabla^2 + k^2\right) g_c = 0$, subject to the appropriate boundary conditions for the problem. It is simple to show that this leads to the same result (2.21), with $\overline{A}_c$ a function of g_c. Physically, a modified Green's function which satisfies the boundary conditions describes the impulse response of the system *including* the response due to secondary sources such as induced conduction or polarization currents. If such a Green's function can be found and inserted into (2.20), then the resulting fields will automatically satisfy the same boundary conditions without explicit knowledge of the induced currents.

For problems involving radiation from current distributions in unbounded media, *i.e.* with no conductors or dielectric materials present, the only boundary conditions to be satisfied are the boundary conditions at infinity, or radiation conditions (discussed later). These unbounded Green's functions are generally easy to find, as we will see next. Modified Green's functions satisfying problem-specific boundary conditions are much more difficult to find in general.

3D Green's function for unbounded media

For unbounded media the solution to (2.19) is relatively straightforward. For simplicity, first consider the delta function to be centered at the origin. The Green's function will then have spherical symmetry and is a solution to

$$\left[\nabla^2 + k^2\right] g(r) = -\delta(\overline{r}) = -\frac{\delta(r)}{4\pi r^2} \tag{2.22}$$

Note the representation for the three dimensional delta function $\delta(\overline{r})$, which satisfies the essential property$\iiint \delta(\overline{r}) dV = 1$ (see Appedix A for more on the delta function). For observation points away from the source singularity, the Green's function must satisfies the homogeneous wave equation and hence can be represented as

$$g(r) = A\frac{e^{-jkr}}{r}$$

where the "advanced" wave travelling in the $-\hat{r}$ direction is rejected on physical grounds, mathematically expressed using the radiation conditions. The unknown amplitude A is determined by inserting this form for $g(r)$ into (2.22) and integrating over a vanishingly small sphere centered at the origin

$$\lim_{a\to 0} \iiint_{\substack{\text{sphere}\\ \text{of radius } a}} \left[\nabla^2 + k^2\right] A\frac{e^{-jkr}}{r}\, dV = -1$$

The integral over the second term goes to zero since $dV \propto r^2$. Using the divergence theorem the first term is reduced to a surface integral, giving

$$\lim_{a\to 0} 4\pi a^2 \left.\frac{\partial g(r)}{\partial r}\right|_{r=a} = -1$$

which yields $A = 1/4\pi$. Shifting coordinate so that the origin is at $\overline{r}'$, we can generalize the result for a source at $\overline{r}'$ to give

$$g(\overline{r}, \overline{r}') = \frac{e^{-jk|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|} \tag{2.23}$$

The general solutions for the vector potentials are then given by

$$\overline{A}(\overline{r}) = \frac{\mu}{4\pi} \iiint \overline{J}(\overline{r}')\frac{e^{-jkR}}{R}\, dV' \tag{2.24a}$$

$$\overline{F}(\overline{r}) = \frac{\epsilon}{4\pi} \iiint \overline{M}(\overline{r}')\frac{e^{-jkR}}{R}\, dV' \tag{2.24b}$$

where

$$R = |\overline{r} - \overline{r}'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

This is a particular solution to Maxwell's equations in unbounded media, and is an important result in elementary antenna theory.

2D Green's function for unbounded media

In two dimensions the Green's function in cylindrical coordinates satisfies

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+k^2\right]g(\rho)=-\delta(\overline{\rho})=-\frac{\delta(\rho)}{2\pi\rho} \tag{2.25}$$

where the source is positioned at the origin for convenience so that g will possess azimuthal symmetry. Again, the representation for $\delta(\overline{\rho})$ is found from the integral property $\iint\delta(\overline{\rho})dS=1$. For $\rho\neq 0$ this is a form of Bessel's equation with the general solution

$$g(\rho)=AJ_0(k\rho)+BN_0(k\rho)$$

where J_0 and N_0 are zero-order Bessel's functions of the first and second kind, respectively. However, for unbounded wave problems it is more convenient to use Hankel functions, which are linear combinations of of J_0 and N_0, defined by

$$H_0^{(1)}(x) = J_0(x)+\jmath N_0(x) \tag{2.26a}$$
$$H_0^{(2)}(x) = J_0(x)-\jmath N_0(x) \tag{2.26b}$$

These have the following asymptotic expansions for $x\gg 1$,

$$H_0^{(1)}(x) \rightarrow \sqrt{\frac{2}{\pi x}}e^{\jmath(x-\pi/4)} \tag{2.27a}$$
$$H_0^{(2)}(x) \rightarrow \sqrt{\frac{2}{\pi x}}e^{-\jmath(x-\pi/4)} \tag{2.27b}$$

which obviously have the desirable form of propagating waves. In the present case only outward propagating waves are expected physically so

$$g(\rho)=AH_0^{(2)}(k\rho)$$

To find A we again integrate (2.25) around a small cylinder of radius a centered at the origin and let the radius go to zero,

$$\lim_{a\to 0}\int_0^{2\pi}\int_0^{a}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+k^2\right]g(\rho)\,\rho\,d\rho\,d\phi=-1$$

Using the small argument approximation for the Hankel function

$$H_0^{(2)}(k\rho)\approx 1-\jmath\frac{2}{\pi}\left[\ln(k\rho/2)+\gamma\right] \qquad k\rho\ll 1$$

($\gamma=0.5772156$ is Euler's constant) and following the same procedure as used in the derivation of the 3D Green's function, we find that $A=-\jmath/4$, so the Green's function is

$$g(\rho)=-\frac{\jmath}{4}H_0^{(2)}(k\rho)$$

for a source at the origin. Generalizing to a line source at $\overline{\rho}'$ we have the final result

$$g(\overline{\rho},\overline{\rho}')=-\frac{\jmath}{4}H_0^{(2)}(k|\overline{\rho}-\overline{\rho}'|) \tag{2.28}$$

General expressions for the vector potentials are then

$$\overline{A}(\overline{\rho}) = \frac{\mu}{4\jmath} \iint \overline{J}(\overline{\rho}') H_0^{(2)}(kR)\, dS' \tag{2.29a}$$

$$\overline{F}(\overline{\rho}) = \frac{\epsilon}{4\jmath} \iint \overline{M}(\overline{\rho}') H_0^{(2)}(kR)\, dS' \tag{2.29b}$$

where

$$R \equiv |\overline{\rho} - \overline{\rho}'| = \left[\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')\right]^{1/2}$$

An alternate representation for the 2D Green's function can be obtained from our 3D result by noting that sources in two-dimensions are independent of z (of infinite extent in the $\hat{z}$ direction) and consequently produce fields with no z dependence, so taking the observation point at $z = 0$ in (2.24a) gives

$$\overline{A}(\rho) = \frac{\mu}{4\pi} \iint \overline{J}(\rho')\, dS' \int_{-\infty}^{\infty} \frac{e^{-\jmath k\sqrt{R^2+z'^2}}}{\sqrt{R^2 + z'^2}}\, dz'$$

Comparing with (2.29a) we find an integral representation for the Hankel function

$$H_0^{(2)}(kR) = \frac{\jmath}{\pi} \int_{-\infty}^{\infty} \frac{e^{-\jmath k\sqrt{R^2+z'^2}}}{\sqrt{R^2 + z'^2}}\, dz' \tag{2.30}$$

which is sometimes useful.

2.2.2 Dyadic Green's functions

The expression of fields via the vector potentials is convenient for sources radiating in unbounded, isotropic, homogeneous media, for which we have a simple scalar Green's function. The result can be expressed concisely as a direct relationship between the fields and currents through the introduction of the so-called *dyadic Green's functions*. Consider the time-harmonic relationship between $\overline{E}$ and $\overline{A}$:

$$\overline{E} = -\jmath\omega \left[\overline{A} + \frac{\nabla\nabla \cdot \overline{A}}{k^2}\right] \tag{2.31}$$

This expression defines an operation where, due to the $\nabla\nabla \cdot \overline{A}$ term, each individual component of $\overline{A}$ can influence *all* components of $\overline{E}$. We can represent such relationships formally using the notation

$$\overline{E} = \overline{\overline{C}} \cdot \overline{A} \tag{2.32}$$

where the components of $\overline{\overline{C}}$ are defined such that when (2.32) is expanded out we get

$$\begin{aligned} E_x\hat{x} &= [C_{xx}\hat{x}\hat{x} + C_{xy}\hat{x}\hat{y} + C_{xz}\hat{x}\hat{z}] \cdot \overline{A} \\ E_y\hat{y} &= [C_{yx}\hat{x}\hat{x} + C_{yy}\hat{y}\hat{y} + C_{yz}\hat{y}\hat{z}] \cdot \overline{A} \\ E_z\hat{z} &= [C_{zx}\hat{z}\hat{x} + C_{zy}\hat{z}\hat{y} + C_{zz}\hat{z}\hat{z}] \cdot \overline{A} \end{aligned}$$

The components of $\overline{\overline{C}}$ are characterized by pairs of unit vectors, or *dyads* (the word "dyad" means pair; look it up!). For this reason, $\overline{\overline{C}}$ is called a dyadic operator and (2.32) a dyadic relationship (dyads and associated relationships are discussed in more depth in Appendix A). A dyadic operator is expressable as the product of two vector operators

$$\overline{\overline{D}} = \overline{X}\,\overline{Y}$$

Dyadic operators are analogous to matrices in linear algebra, which similarly transform a vector into another vector under the scalar product operation. Note that, in general, dyad-vector multiplications do not obey the familiar vector commutations,

$$\begin{aligned} \overline{\overline{D}} \cdot \overline{A} &\neq \overline{A} \cdot \overline{\overline{D}} \\ \overline{\overline{D}} \times \overline{A} &\neq -\overline{A} \times \overline{\overline{D}} \end{aligned}$$

obeying instead the matrix-like commutation laws

$$\begin{aligned} \overline{\overline{D}} \cdot \overline{A} &= \overline{A} \cdot \overline{\overline{D}}^T \\ \overline{\overline{D}} \times \overline{A} &= -(\overline{A} \times \overline{\overline{D}}^T)^T \end{aligned}$$

Therefore one must exercise caution when applying familiar vector identities of section A.5 to expressions involving dyadic quantities. Some useful identities involving dyadics have been summarized in section A.7 for convenience.

In (2.31) we can now identify $\nabla\nabla$ as a dyadic operator, and write

$$\overline{E} = -\jmath\omega \left[\overline{\overline{I}} + \frac{\nabla\nabla}{k^2} \right] \cdot \overline{A} \tag{2.33}$$

where $\overline{\overline{I}}$ is called the unit dyad, defined such that

$$\overline{A} \cdot \overline{\overline{I}} = \overline{\overline{I}} \cdot \overline{A} = \overline{A}$$

In rectangular coordinates,

$$\overline{\overline{I}} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$$

Comparing (2.33) and (2.32) we see that the term in square brackets can be regarded as the dyadic operator that relates the vector potential to the electric field. Substituting (2.20) into (2.33) gives

$$\overline{E} = -\jmath\omega\mu \left[\overline{\overline{I}} + \frac{\nabla\nabla}{k^2} \right] \cdot \iiint g(\overline{r}, \overline{r}') \overline{J}(\overline{r}') \, dV'$$

Since the operator $\nabla\nabla$ operates only on *unprimed* coordinates, we are tempted to bring it inside the integral and write

$$\overline{E}(\overline{r}) = -\jmath\omega\mu \iiint \overline{\overline{G}}_e(\overline{r}, \overline{r}') \cdot \overline{J}(\overline{r}') \, dV' \tag{2.34}$$

where $\overline{\overline{G}}_e(\overline{r}, \overline{r}')$ is defined by

$$\overline{\overline{G}}_e(\overline{r}, \overline{r}') \equiv \left[\overline{\overline{I}} + \frac{\nabla\nabla}{k^2} \right] g(\overline{r}, \overline{r}') \tag{2.35}$$

The form of (2.34) suggests interpreting $\overline{\overline{G}}_e$ as a "dyadic Green's function" describing the electric field produced by a unit dyadic impulse of electric current. To verify this interpretation, we substitute (2.34) into the wave equation

$$\nabla \times \nabla \times \overline{E} - k^2 \overline{E} = -\jmath\omega\mu \overline{J}(\overline{r})$$

and use the integral property of the delta function to express the current density as

$$\overline{J}(\overline{r}) = \iiint \overline{\overline{I}} \cdot \overline{J}(\overline{r}') \delta(\overline{r} - \overline{r}') \, dV'$$

Again assuming we can interchange derivative and integral operations gives

$$-\jmath\omega\mu \iiint \left[\nabla \times \nabla \times \overline{\overline{G}}_e - k^2\overline{\overline{G}}_e\right] \cdot \overline{J}(\overline{r}')\, dV' = -\jmath\omega\mu \iiint \overline{\overline{I}} \cdot \overline{J}(\overline{r}')\delta(\overline{r} - \overline{r}')\, dV'$$

Since this must hold for any volume V, the integrands must be equal and

$$\nabla \times \nabla \times \overline{\overline{G}}_e - k^2\overline{\overline{G}}_e = \overline{\overline{I}}\delta(\overline{r} - \overline{r}') \tag{2.36}$$

Since $\overline{\overline{G}}_e$ is a solution to the same vector wave equation as $\overline{E}$ with an impulse forcing function, it can be viewed as Green's function. The subscript 'e' reminds us that it is a Green's function for the electric field.

The step involving interchange of the integral and $\nabla\nabla$ operators leading to (2.34) should be examined more closely. This step is certainly justifiable when the observation point is outside of the source regions, since for $\overline{r} \neq \overline{r}'$ the integrand has well defined and continuous second derivatives. However, if $\overline{r} = \overline{r}'$ in the integral, the integrand is singular and the integral is undefined. A similar problem occurs with the derivation of (2.20), which also has a singular integrand when $\overline{r} = \overline{r}'$. In the present case the problem is exacerbated by the $\nabla\nabla$ operation, which makes the integrand more strongly singular. We will resolve this issue in the next section, but for now simply assume that the observation point is kept away from the source region to avoid the problem.

An expression for the electric field due to magnetic currents can also be developed in terms of a dyadic Green's function. The electric field is given by

$$\overline{E}(\overline{r}) = -\nabla \times \iiint \overline{M}(\overline{r}')g(\overline{r}, \overline{r}')\, dV'$$

Under the same assumptions noted above, the $\nabla\times$ operator can be brought under the integral. Using the vector identity (A.46) we can write

$$\nabla \times \left[\overline{M}(\overline{r}')g(\overline{r}, \overline{r}')\right] = \nabla g(\overline{r}, \overline{r}') \times \overline{M}(\overline{r}')$$

since $\overline{M}(\overline{r}')$ is a function of primed coordinates only. Using (2.35),

$$\nabla \times \overline{\overline{G}}_e = \nabla \times \left(\overline{\overline{I}}g\right) = \nabla g \times \overline{\overline{I}}$$

and since $\overline{M} = \overline{\overline{I}} \cdot \overline{M}$, we can write

$$\overline{E}(\overline{r}) = -\iiint \nabla \times \overline{\overline{G}}_e(\overline{r}, \overline{r}') \cdot \overline{M}(\overline{r}')\, dV' \tag{2.37}$$

In the general case, we combine the above results to find

$$\overline{E}(\overline{r}) = -\iiint \left[\jmath\omega\mu\overline{\overline{G}}_e(\overline{r}, \overline{r}') \cdot \overline{J}(\overline{r}') + \nabla \times \overline{\overline{G}}_e(\overline{r}, \overline{r}') \cdot \overline{M}(\overline{r}')\right] dV' \tag{2.38}$$

and from Maxwell's equations, the corresponding $\overline{H}$ field is given by

$$\overline{H}(\overline{r}) = \iiint \left[\nabla \times \overline{\overline{G}}_e(\overline{r}, \overline{r}') \cdot \overline{J}(\overline{r}') - \jmath\omega\epsilon\overline{\overline{G}}_e(\overline{r}, \overline{r}') \cdot \overline{M}(\overline{r}')\right] dV' \tag{2.39}$$

Again, these expressions must be modified for observation points within the source regions, which will be discussed in the next section.

Note that the scalar Green's function in (2.35) can be any solution of (2.19). When the free-space Green's function (2.23) is used in (2.35), we have

$$\overline{\overline{G}}_{e0} \equiv \left[\overline{\overline{I}} + \frac{\nabla\nabla}{k^2}\right] \frac{e^{-jk|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|} \tag{2.40}$$

This is called the *"free-space Green's dyadic"*. The scalar Green's function for isotropic, homogeneous media is symmetrical with respect to the source and observation points. This reflects the assumed reciprocal properties of the medium. The dyadic Green's functions possess qualitatively similar symmetries, but these are expressed somewhat differently. For the free-space Green's dyadic, these are shown in Problem X to be

$$\overline{\overline{G}}_{e0}(\overline{r},\overline{r}') = \left[\overline{\overline{G}}_{e0}(\overline{r}',r)\right]^{\mathrm{T}} \tag{2.41a}$$

$$\nabla \times \overline{\overline{G}}_{e0}(\overline{r},\overline{r}') = \left[\nabla' \times \overline{\overline{G}}_{e0}(\overline{r}',r)\right]^{\mathrm{T}} \tag{2.41b}$$

2.3 ELEMENTARY DIPOLE RADIATORS

The Green's function gives a solution for the fields produced by an infinitesimal point-source, and we have shown that the fields from more general source distributions can be constructed by a superposition of these point-source fields. There are other elementary sources in antenna theory, such as the short current element and short current loop, that play a similar but more physical role in the sense of representing elementary distributions that can be more closely realized in practice. These elementary sources are of finite spatial extent, but with dimensions that are small compared to a wavelength.

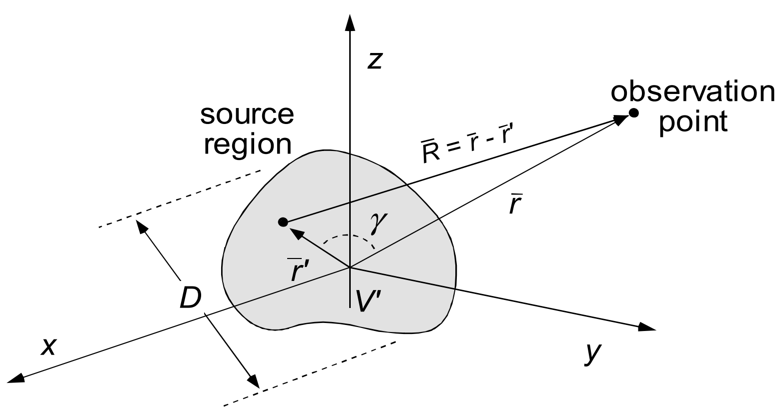

Figure 2.2 Source region with a characteristic dimension D centered at the origin. For elementary sources, $D \ll \lambda$.

Consider the arbitrary source distribution shown in figure 2.2, which has a characteristic dimension D. We wish to consider the case when the source distribution is small compared to a wavelength, $D \ll \lambda$. In this case, $kr' \ll 1$ in the exponential term of (2.24). Using the vector form of the Taylor expansion

$$f(\overline{r}+\bar{a}) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\bar{a}\cdot\nabla\right)^n f(\overline{r})$$

and assuming $r > D$ so that $|\overline{r}| > |\overline{r}'|$, gives

$$\frac{e^{-jkR}}{R} = \frac{e^{-jk|\overline{r}-\overline{r}'|}}{|\overline{r}-\overline{r}'|} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\overline{r}'\cdot\nabla\right)^n \frac{e^{-jkr}}{r}$$

Substituting into (2.24a) and keeping only the first two terms of the expansion gives

$$\overline{A}(\overline{r}) = \frac{\mu}{4\pi}\left[\iiint \overline{J}(\overline{r}')\,dV'\right]\frac{e^{-jkr}}{r} - \frac{\mu}{4\pi}\left[\iiint \overline{J}(\overline{r}')\overline{r}'\,dV'\right]\cdot\nabla\frac{e^{-jkr}}{r} + \ldots \tag{2.42}$$

with a similar expression resulting for $\overline{F}$ (see Problem X). This is a so-called “multipole” expansion, *valid for electrically small source distributions where* $\boxed{kD \ll 1}$. The terms in brackets are called “moments” of the current distribution. Successive terms in the expansion become rapidly negligible in the region of validity, and the first non-zero term is usually a sufficiently accurate approximation. We will apply this result to some important canonical current sources.

2.3.1 Electric Hertzian dipole

Consider the short linear current element of length $d\ell$ along the z-axis, shown in figure 2.3. This current is described by

$$\overline{J}(\overline{r}') = \begin{cases} \hat{z}I_0\delta(x')\delta(y') & |z'| < d\ell/2 \\ 0 & \text{elsewhere} \end{cases}$$

Assuming $d\ell \ll \lambda$ we can use (2.42). The dominant term in the expansion is (in this case) the

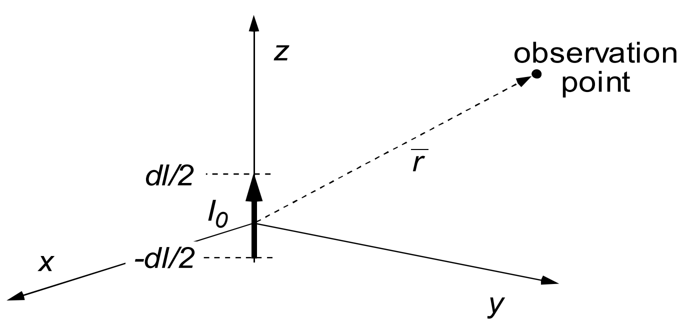

Figure 2.3 Short linear current element (electric Hertzian dipole) of length $d\ell$ ($kd\ell \ll 1$).

first term term, which integrates to

$$\overline{A}(\overline{r}) = \hat{z}I_0 d\ell \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}$$

or, in spherical coordinates,

$$\overline{A}(\overline{r}) = \left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right) I_0 d\ell \frac{\mu}{4\pi}\frac{e^{-\jmath kr}}{r} \tag{2.43}$$

Note that the same result can be obtained from (2.42) using the following concise representation for the current element

$$\overline{J}(\overline{r}') = I_0 \overline{d\ell}\delta(\overline{r}') \tag{2.44}$$

which is commonly used in the literature. Using this form, the vector potential is clearly just given by $\overline{A}(\overline{r}) = \mu I_0 \overline{d\ell} g(\overline{r}, \overline{r}')$. In other words, the finite linear current element can be viewed as a building block for more complicated distributions, provided it has a spatial extent much smaller than a wavelength.

Substituting our solution for $\overline{A}$ into (2.10) gives (Problem X),

$$\overline{H} = \hat{\phi}\,\jmath k I_0 d\ell \sin\theta \left(1 + \frac{1}{\jmath kr}\right)\frac{e^{-\jmath kr}}{4\pi r} \tag{2.45a}$$

$$\begin{aligned}\overline{E} &= \hat{r}\,2\jmath\omega\mu I_0 d\ell\cos\theta\left[\frac{1}{\jmath kr} + \frac{1}{(\jmath kr)^2}\right]\frac{e^{-\jmath kr}}{4\pi r} \\ &\quad + \hat{\theta}\,\jmath\omega\mu I_0 d\ell\sin\theta\left[1 + \frac{1}{\jmath kr} + \frac{1}{(\jmath kr)^2}\right]\frac{e^{-\jmath kr}}{4\pi r}\end{aligned} \tag{2.45b}$$

There are three contributions to the fields: 1) an electrostatic term, varying as $1/r^3$, which is associated with an electric dipole moment; 2) a quasi-static "induction" field, the magnetic component of which is given by the Biot-Savart law; and 3) a radiation field varying as $1/r$. The radiation field becomes the leading term at at critical distance of $kr > 1$, or $r > \lambda/2\pi$. At distances significantly larger than a wavelength, the radiation field dominates and (2.45) reduce to

$$\text{far-fields:} \qquad \overline{E} = \hat{\theta}\,\jmath\omega\mu I_0 d\ell\sin\theta\frac{e^{-\jmath kr}}{4\pi r} \tag{2.46a}$$

$$\overline{H} = \frac{1}{\eta}\hat{r}\times\overline{E} \tag{2.46b}$$

This is a spherical wave, modified by a $\sin\theta$ dependence to the field intensity. Consequently there is no radiation in the direction parallel to the current flow, and a maximum radiation intensity in the direction normal to the current flow. Radiation fields always result from currents flowing transverse to the observation position vector, as we will prove later. Using (2.46), the time-averaged Poynting vector in the far-field is

$$\begin{aligned}\overline{\mathcal{P}}_{\text{ave}} &= \frac{1}{2}\text{Re}\left\{\overline{E}\times\overline{H}^*\right\} \\ &= \frac{\eta(I_0 d\ell)^2 k^2}{32\pi^2}\frac{\sin^2\theta}{r^2}\,\hat{r}\end{aligned} \tag{2.47}$$

As shown in Problem X, this result is also obtained using the full expression for the dipole fields (2.45). That is, the near-field terms do not contribute to real power flow away from the antenna. Instead, these terms represent a dynamic *stored* energy in the vicinity of the antenna, flowing back and forth between the feed circuit and the fields and hence contributing to a reactive component of impedance as seen by the generator. Integrating this Poynting vector over a sphere, the total radiated power is

$$
\begin{aligned}
P_{\text{rad}} &= \oiint \overline{\mathcal{P}} \cdot d\overline{S} = \int_0^{2\pi}\int_0^{\pi} \mathcal{P}_{\text{ave}}(r,\theta)\, r^2 \sin\theta \, d\theta \, d\phi \\
&= \frac{\eta (I d\ell)^2 k^2}{32\pi^2}\, 2\pi \underbrace{\int_0^{\pi} \sin^3\theta \, d\theta}_{4/3} = \frac{\eta (I d\ell)^2 k^2}{12\pi}
\end{aligned}
\tag{2.48}
$$

The radiation resistance as seen by the current source is then calculated from $P_{\text{rad}} = \frac{1}{2} I^2 R_{\text{rad}}$, giving

$$
R_{rad} = \frac{\eta (k d\ell)^2}{6\pi} \tag{2.49}
$$

In free space where $\eta = 120\pi$,

$$
R_{\text{rad}} = 80\pi^2 \left(\frac{d\ell}{\lambda_0} \right)^2
$$

It should be remembered that this result was derived under the assumption of $d\ell \ll \lambda$.

2.3.2 Magnetic Hertzian dipole

Consider the closed current loop of radius a shown in figure 2.4. We assume that the current is uniform around the loop, which implies that the circumference is a small fraction of a wavelength, $2\pi a \ll \lambda$; this implies $ka \ll 1$, which is also a condition for the validity of (2.42).

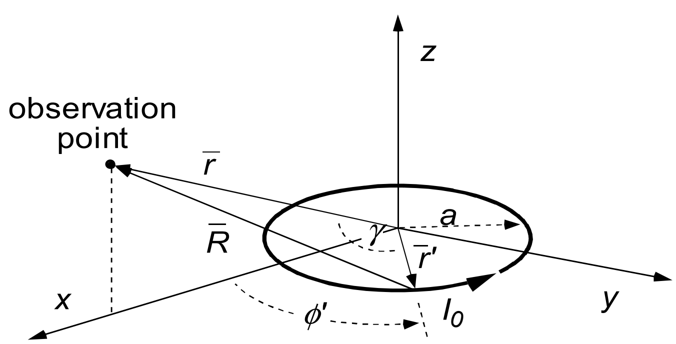

Figure 2.4 Small uniform current loop (magnetic Hertzian dipole) in the x-y plane at $z = 0$, with radius a such that $ka \ll 1$.

For any closed loop of current, the first term in (2.42) is zero. Therefore the leading term in the expansion is the second term. Evaluating the gradient gives

$$
\nabla \frac{e^{-jkr}}{r} = -\hat{r} \frac{e^{-jkr}}{r} \left[\frac{1}{r} + jk \right]
$$

and denoting the angle between $\overline{r}$ and $\overline{r}'$ as γ so that $\hat{r} \cdot \overline{r}' = r' \cos\gamma$, we can write the second term of the multipole expansion (2.42) as

$$
\overline{A}(\overline{r}) = \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} \left[\frac{1}{r} + jk \right] \iiint \overline{J}(\overline{r}') r' \cos\gamma \, dV' \tag{2.50}
$$

where

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$$

From the azimuthal symmetry of the problem, we expect the vector potential to have only a $\hat{\phi}$ component and to be independent of ϕ. Therefore we can choose the observation point $\phi = 0$ for convenience, where $A_\phi = A_y$. The current loop can be represented as

$$\overline{J}(\overline{r}') = \hat{\phi}\, I_0 \frac{\delta(\overline{r}' - a)\delta(\theta' - \pi/2)}{r'} \tag{2.51}$$

However, the $\hat{\phi}$ direction is in general different at the source and observation points, so it is better to express $\overline{J}$ in terms of its rectangular components as

$$\overline{J}(\overline{r}') = -\hat{x}J_\phi \sin\phi' + \hat{y}J_\phi\cos\phi'$$

The integral in (2.50) then evaluates to

$$\begin{aligned}\iiint \overline{J}(\overline{r}')r'\cos\gamma\, dV' &= I_0 a^2 \sin\theta \int_0^{2\pi} [-\hat{x}\sin\phi' + \hat{y}\cos\phi']\cos\phi'\, d\phi' \\ &= I_0\pi a^2 \sin\theta\,\hat{y}\end{aligned}$$

and therefore the vector potential is

$$\overline{A}(\overline{r}') = \hat{\phi}\, I_0\pi a^2 \sin\theta \frac{\jmath k\mu}{4\pi}\frac{e^{-\jmath kr}}{r}\left[1 + \frac{1}{\jmath kr}\right]$$

Substituting into (2.10) gives

$$\overline{E} = \hat{\phi}\, I_0\pi a^2 k^2 \eta \sin\theta \left[1 + \frac{1}{\jmath kr}\right]\frac{e^{-\jmath kr}}{4\pi r} \tag{2.52a}$$

$$\begin{aligned}\overline{H} &= -\hat{r}\, 2I_0\pi a^2 k^2 \cos\theta \left[\frac{1}{\jmath kr} + \frac{1}{(\jmath kr)^2}\right]\frac{e^{-\jmath kr}}{4\pi r} \\ &\quad - \hat{\theta}\, I_0\pi a^2 k^2 \sin\theta \left[1 + \frac{1}{\jmath kr} + \frac{1}{(\jmath kr)^2}\right]\frac{e^{-\jmath kr}}{4\pi r}\end{aligned} \tag{2.52b}$$

Comparing (2.52) with (2.45), we find that they are essentially the same with the roles of $\overline{E}$ and $\overline{H}$ reversed; the short loop is the dual of an electric dipole, and is referred to as a magnetic dipole, where $I_0\pi a^2$ is the magnetic dipole moment. The far fields of a magnetic dipole are identified as

$$\text{far-fields:} \qquad \overline{E} = \hat{\phi}\, I_0\pi a^2 k^2 \eta \sin\theta \frac{e^{-\jmath kr}}{4\pi r} \tag{2.53a}$$

$$\overline{H} = \frac{1}{\eta}\hat{r} \times \overline{E} \tag{2.53b}$$

and so the radiation pattern and the directivity are the same as the electric Hertzian dipole. The radiation resistance is found as

$$\begin{aligned}R_{rad} &= \frac{2P_{rad}}{|I|^2} = \frac{2}{|I|^2}\oiint \frac{|E_\phi|^2}{2\eta}\, dS \\ &= \frac{\pi\eta}{8}(ka)^4 \underbrace{\int_0^\pi \sin^3\theta d\theta}_{4/3} = \frac{\pi\eta}{6}(ka)^4\end{aligned} \tag{2.54}$$

and in free space where $\eta = 120\pi$,

$$R_{rad} = 320\pi^6 \left(\frac{a}{\lambda}\right)^4$$

The radiation resistance of the loop can be extremely small, going as $(a/\lambda)^4$, compared with $(d\ell/\lambda)^2$ for the Hertzian dipole. This number can be significantly increased, however, by using a large number of turns in the loop. As long as the total coil length remains electrically small, our previous analysis applies by replacing I with (NI), where N is the number of turns. Therefore

$$R_{rad} \Rightarrow 320\pi^6 N^2 (a/\lambda)^2 \tag{2.55}$$

This is valid provide that $2\pi a N \ll \lambda$.

2.4 FAR-FIELD SOLUTIONS

2.4.1 Far-fields in 3D

We now focus on possible simplifications for arbitrarily large source distributions when only the radiation fields (varying as $1/r$) are of interest. We start by making a Taylor expansion of

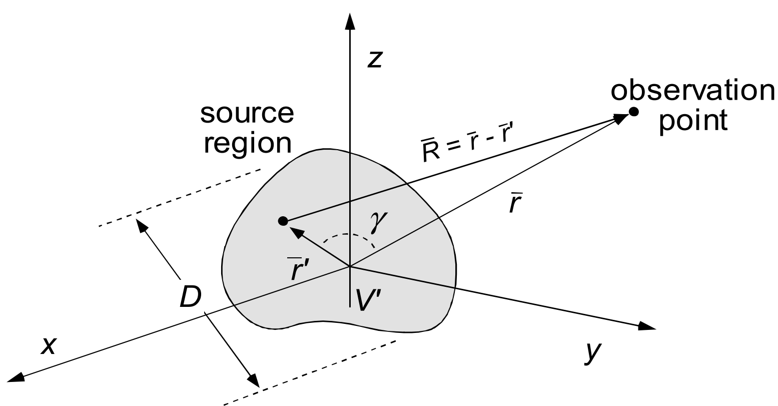

Figure 2.5 Source region with a characteristic dimension D for delimiting far-field region.

$R = |\overline{r} - \overline{r}'|$ for the case of $|\overline{r}| \gg |\overline{r}'|$, giving

$$\begin{aligned} R = |\overline{r} - \overline{r}'| &= \left[(\overline{r} - \overline{r}') \cdot (\overline{r} - \overline{r}')\right]^{\frac{1}{2}} = \left[r^2 - 2\overline{r} \cdot \overline{r}' + r'^2\right]^{\frac{1}{2}} \\ &= \left[r^2 \left(1 - \frac{2\overline{r} \cdot \overline{r}'}{r^2} + \frac{r'^2}{r^2}\right)\right]^{\frac{1}{2}} \\ &= r\left[1 - \frac{\overline{r} \cdot \overline{r}'}{r^2} + \frac{1}{2}\frac{r'^2}{r^2} - \frac{1}{2}\frac{(\overline{r} \cdot \overline{r}')^2}{r^4} + \ldots\right] \end{aligned} \tag{2.56}$$

Denoting the angle between $\overline{r}$ and $\overline{r}'$ as γ (see figure 2.5) we can write $\overline{r} \cdot \overline{r}' = rr' \cos\gamma$ where

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi') \tag{2.57}$$

and so the first three terms of the expansion in R become

$$R \approx r - \hat{r}\cdot\overline{r}' + \frac{1}{2}\frac{r'^2}{r}\sin^2\gamma + \ldots$$

where $\hat{r} = \overline{r}/r$. When substituted into the exponential term of the Green's function e^{-jkR} we must keep at least the first two terms, since though $\overline{r}'$ is small compared with $\overline{r}$ it can still be large relative to a wavelength, and consequently kr' can represent a non-negligible phase. The third term contributes a phase of

$$\frac{1}{2}\frac{kr'^2}{r}\sin^2\gamma$$

We can always choose r far enough away so that this term is negligible. A useful—though somewhat arbitrary—rule-of-thumb defines "negligible" to mean a phase contribution of $\pi/8$ (22.5°) or less. Then, if the source distribution is centered on the origin and has a characteristic length of D so that $|r'| \le D/2$, the critical observation distance should be such that

$$\frac{1}{2}\frac{k\left(\frac{D}{2}\right)^2}{r} < \frac{\pi}{8}$$

or

$$r > \frac{2D^2}{\lambda} \qquad \text{(condition for far-field)} \tag{2.58}$$

When this condition holds we are in the "far-field" of the source distribution. Then

$$R \approx r - \hat{r}\cdot\overline{r}' \qquad \text{(far-field approximation)} \tag{2.59}$$

and the Green's function becomes

$$\frac{e^{-jkR}}{R} \approx e^{-jkr}e^{jk\hat{r}\cdot\overline{r}'}\frac{1}{r}\left[1 + \frac{\hat{r}\cdot\overline{r}'}{r} + \ldots\right]$$

The radiation fields (varying as $1/r$) become the dominant term at a distance of $r > D/2$. This condition is automatically satisfied in the far-field defined by (2.58), unless the antenna is very small compared with a wavelength ($D < \lambda/4$), in which case the expansion (2.42) would be more appropriate. In any case, we can always choose a distance far enough away from the antenna so that only the radiation fields are important, so the far-field Green's function is

$$g_{ff}(\overline{r}, \overline{r}') \Rightarrow \frac{e^{-jkr}}{4\pi r}e^{jk\hat{r}\cdot\overline{r}'} \tag{2.60}$$

and our far-field vector potentials are therefore

$$\overline{A}(\overline{r}) \;\Rightarrow\; \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\iiint \overline{J}(\overline{r}')e^{+jk\hat{r}\cdot\overline{r}'}\,dV' \tag{2.61a}$$

$$\overline{F}(\overline{r}) \;\Rightarrow\; \frac{\epsilon}{4\pi}\frac{e^{-jkr}}{r}\iiint \overline{M}(\overline{r}')e^{jk\hat{r}\cdot\overline{r}'}\,dV' \tag{2.61b}$$

Note that the integrand of these "radiation integrals" does not depend on the observation distance r. Now consider the calculation of the fields from (2.13). In spherical coordinates

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

Only the $\hat{r}$ component of the ∇ operator is capable of generating fields with $1/r$ dependence; the other components produce terms going as $1/r^2$, which we have so far consistently neglected. Keeping only these terms in $1/r$ gives

$$\begin{aligned}\overline{A} + \frac{1}{k^2}\nabla\nabla\cdot\overline{A} \;&\Rightarrow\; \overline{A} - (\hat{r}\cdot\overline{A})\hat{r} = A_\theta\hat{\theta} + A_\phi\hat{\phi} \\ \nabla\times\overline{F} \;&\Rightarrow\; -jk(\hat{r}\times\overline{F}) = jk(F_\phi\hat{\theta} - F_\theta\hat{\phi})\end{aligned}$$

so from (2.13) the radiation fields are

$$\overline{E}(\overline{r}) \;=\; -j\omega\left[A_\theta\hat{\theta} + A_\phi\hat{\phi}\right] - \frac{jk}{\epsilon}\left[F_\phi\hat{\theta} - F_\theta\hat{\phi}\right] \tag{2.62a}$$

$$\overline{H}(\overline{r}) \;=\; \frac{\hat{r}\times\overline{E}}{\eta} \tag{2.62b}$$

Writing these out explicitly for later use, we have, for electric sources only,

$$\overline{E}(\overline{r}) \;=\; -j\omega\mu\frac{e^{-jkr}}{4\pi r}\iiint\left[\overline{J}(\overline{r}') - \left(\hat{r}\cdot\overline{J}(\overline{r}')\right)\hat{r}\right]e^{+jk\hat{r}\cdot\overline{r}'}\,dV' \tag{2.63a}$$

$$\overline{H}(\overline{r}) \;=\; -jk\frac{e^{-jkr}}{4\pi r}\iiint\hat{r}\times\overline{J}(\overline{r}')\,e^{+jk\hat{r}\cdot\overline{r}'}\,dV' \tag{2.63b}$$

and for magnetic sources only,

$$\overline{E}(\overline{r}) \;=\; jk\frac{e^{-jkr}}{4\pi r}\iiint\hat{r}\times\overline{M}(\overline{r}')\,e^{+jk\hat{r}\cdot\overline{r}'}\,dV' \tag{2.64a}$$

$$\overline{H}(\overline{r}) \;=\; -j\omega\epsilon\frac{e^{-jkr}}{4\pi r}\iiint\left[\overline{M}(\overline{r}') - \left(\hat{r}\cdot\overline{M}(\overline{r}')\right)\hat{r}\right]e^{+jk\hat{r}\cdot\overline{r}'}\,dV' \tag{2.64b}$$

These are the so-called “radiation equations.” It is clear from these equations that for any particular observation point $\overline{r}$, the far-field has the form of a spherical TEM wave propagating in the $\hat{r}$ direction, and is due to only that part of the current distribution that is flowing transverse to $\hat{r}$.

Far-Fields in Dyadic Notation

Substituting (2.60) into (2.35) we find, in the far-field,

$$\overline{\overline{G}}_{e0}(\overline{r},\overline{r}') \;\Rightarrow\; (\overline{\overline{I}} - \hat{r}\hat{r})\frac{e^{-jkr}}{4\pi r}e^{jk\overline{r}'\cdot\hat{r}} \tag{2.65a}$$

$$\nabla\times\overline{\overline{G}}_{e0}(\overline{r},\overline{r}') \;\Rightarrow\; -jk(\hat{r}\times\overline{\overline{I}})\frac{e^{-jkr}}{4\pi r}e^{jk\overline{r}'\cdot\hat{r}} \tag{2.65b}$$

and so (2.39) reduce to, for electric currents,

$$\overline{E}(\overline{r}) \;=\; -j\omega\mu\frac{e^{-jkr}}{4\pi r}(\overline{\overline{I}} - \hat{r}\hat{r})\cdot\iiint\overline{J}(\overline{r}')e^{jk\overline{r}'\cdot\hat{r}}\,dV \tag{2.66a}$$

$$\overline{H}(\overline{r}) \;=\; -jk\frac{e^{-jkr}}{4\pi r}\hat{r}\times\iiint\overline{J}(\overline{r}')e^{jk\overline{r}'\cdot\hat{r}}\,dV \tag{2.66b}$$

which are, of course, identical to the previously derived result (2.63), but using dyadic notation. A similar result for magnetic currents can be written down by inspection of (2.64).

2.4.2 Far-fields in 2D

For problems in which variation of fields is negligible in one cartesian direction (taken as the $\hat{z}$-axis) then the vector potential is given by (2.29a),

$$\overline{A}(\overline{\rho}) = \frac{\mu}{4j} \iint \overline{J}(\overline{\rho}') H_0^{(2)}(kR)\, dS'$$

where

$$\begin{aligned} R = |\overline{\rho} - \overline{\rho}'| &= \sqrt{(x-x')^2 + (y-y')^2} \\ &= \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')} \end{aligned}$$

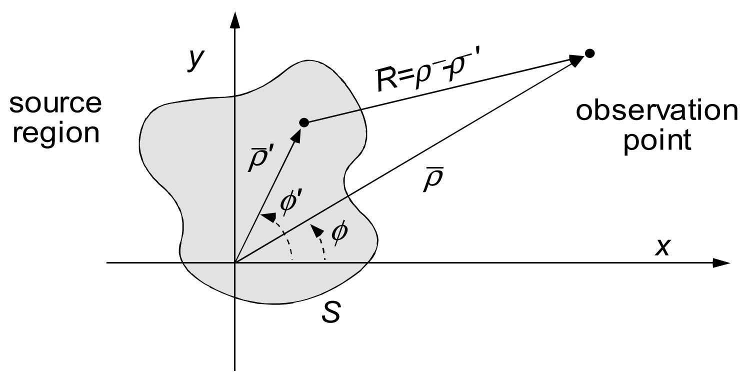

Figure 2.6 Two-dimensional source distribution (infinite extent in the $\hat{z}$ direction).

In the far-field, $kR \gg 1$, and we can use the asymptotic expansion for the 2D Green's function from (2.27),

$$H_0^{(2)}(kR) \approx \sqrt{\frac{2}{\pi k R}} e^{-j(kR - \frac{\pi}{4})}$$

For $|\overline{\rho}| \gg |\overline{\rho}'|$, then we can make a Taylor expansion of R much like for the 3D case, giving

$$\begin{aligned} R &\cong \rho - \hat{\rho} \cdot \overline{\rho}' \\ &\cong \rho - \overline{\rho}' \cos(\phi - \phi') \end{aligned} \tag{2.67}$$

or we could express R as

$$R \cong \rho - x' \cos\phi - y' \sin\phi$$

Keeping only terms in $1/\sqrt{\rho}$ (radiation fields in 2D) gives

$$\overline{A}(\overline{\rho}) = \frac{\mu}{2} \frac{e^{-jk\rho}}{\sqrt{2jk\rho\pi}} \iint_S \overline{J}(\overline{\rho}') e^{jk\hat{\rho}\cdot\overline{\rho}'}\, dS' \tag{2.68}$$

and the electric field in the far-field is given by

$$\overline{E}(\overline{\rho}) \cong -j\omega(A_\phi \hat{\phi} + A_z \hat{z}) \tag{2.69}$$

The extension to magnetic currents is straightforward. It is very important to remember when computing A_ϕ that the $\hat{\phi}$ direction at the source and observation points are generallydifferent and must be handled accordingly. If the current is best described by rectangular components, then

$$A_\phi = \frac{\mu}{2}\frac{e^{-jk\rho}}{\sqrt{2jk\rho\pi}}\iint_S \left[-J_x(\overline{\rho}')\sin\phi + J_y(\overline{\rho}')\cos\phi\right] e^{jk\hat{\rho}\cdot\overline{\rho}'}\,dS' \tag{2.70a}$$

$$A_z = \frac{\mu}{2}\frac{e^{-jk\rho}}{\sqrt{2jk\rho\pi}}\iint_S J_z(\overline{\rho}')e^{jk\hat{\rho}\cdot\overline{\rho}'}\,dS' \tag{2.70b}$$

Note that the explicit $\sin\phi$ and $\cos\phi$ terms in (2.70a) are functions of the *unprimed* angle ϕ at observation point. If $\overline{J}$ is best described in cylindrical coordinates then we can again use (2.70) but with the substitutions

$$J_x = J_\rho\cos\phi' - J_\phi\sin\phi' \tag{2.71a}$$

$$J_y = J_\rho\sin\phi' + J_\phi\cos\phi' \tag{2.71b}$$

where these terms involve the primed (source) variable ϕ'.

2.4.3 Displaced Distributions

In later work with multiple antenna systems we will find it useful to have an expression for the far-fields of a source distribution that is displaced from the origin. If this displacement is denoted by the constant vector $\overline{xi}$, as shown in fig. 2.7, then we can make use of the previous expressions

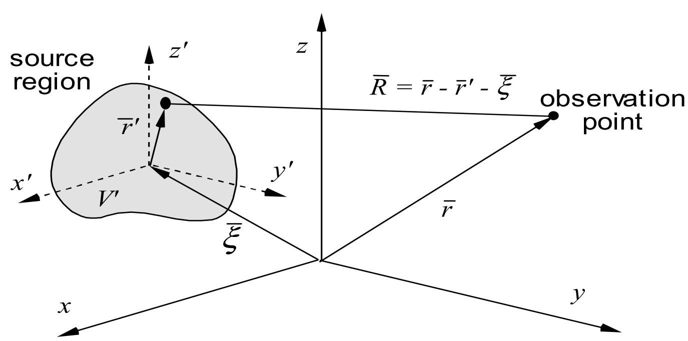

Figure 2.7 Displaced source distribution.

(2.61) with the replacement

$$\overline{r}' \rightarrow \overline{r}' + \overline{\xi}$$

giving

$$\overline{A}(\overline{r}) = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}e^{jk\hat{r}\cdot\overline{\xi}}\iiint \overline{J}(\overline{r}')e^{+jk\hat{r}\cdot\overline{r}'}\,dV' \tag{2.72a}$$

$$\overline{F}(\overline{r}) = \frac{\epsilon}{4\pi}\frac{e^{-jkr}}{r}e^{jk\hat{r}\cdot\overline{\xi}}\iiint \overline{M}(\overline{r}')e^{jk\hat{r}\cdot\overline{r}'}\,dV' \tag{2.72b}$$

where the primed coordinates are now "local" coordinates of the current distribution. Hence the effect of the displacement is simply to multiply the fields by the phase factor

$$e^{\jmath k\hat{r}\cdot\overline{\xi}} \tag{2.73}$$

It is a simple matter to show that a similar result holds for the two-dimensional fields.

2.5 USING FIELD EQUIVALENCE AND RECIPROCITY

2.5.1 Reduction to unbounded currents

We now illustrate how field equivalence principles are used to reduce a typical problem in antenna theory to one involving volume currents flowing in free-space. We start with a generic antenna as shown in fig. **??**a, which is fed by a voltage generator. Using the Schelkunoff field-equivalence concepts described in Chapter 1, we can replace the feed region by a PEC shell, as shown in fig. **??**b, surrounded by an impressed magnetic current loop $\overline{M}$. The conductor is now continuous

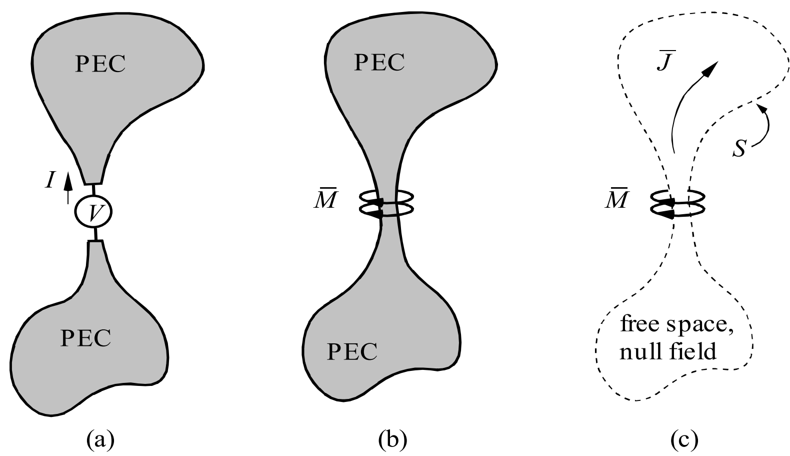

Figure 2.8 (a) Typical antenna problem. (b) Application of Schelkenoff equivalence to replace the generator region by an equivalent magnetic current. (c) Love's equivalent where the conductors are replaced by impressed sources flowing in free space.

through the feed region. This impressed current generates an applied field which induces a current $\overline{J}$ on the conductor. The combination of these currents produces the same fields *outside* of S as exist in the original problem.

If we instead use a Love equivalence, the conductors and feed region are replaced by free space as shown in fig. **??**c. The impressed currents $\overline{M}$ and $\overline{J}$ are again related to the applied field and induced surface currents, respectively, from the original problem. The combination of these currents produces the same fields *outside* of S as exist in the original problem, and produce a null field within V. This equivalent is suitable for the application of the field expressions developed in this Chapter.

Since we can often reduce an antenna problem to currents flowing in free-space in the manner of fig. 2.8, the real challenge in the computation of the fields is in finding or approximating the currents. This will occupy much of our effort throughout this book.

2.5.2 Use of known fields to find unknown fields

We noted in Chapter 1 that the reciprocity theorem provides a powerful tool for field computation. We now show how the reciprocity theorem can be used to compute the unknown fields of a current distribution using the known fields of a simpler current distribution, usually a Hertzian dipole. Consider the arrangement in fig. 2.9a, in which there is a known source distribution $(\overline{J}, \overline{M})$

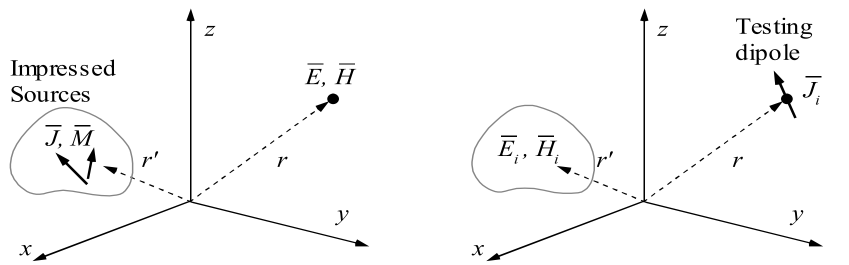

Figure 2.9 (a) Known source distribution, possibly in the presence of matter, generating unknown fields $(\overline{E}, \overline{H})$ at $\overline{r}$. (b) Known testing current $\overline{J}_i$ at $\overline{r}$ generating known fields $(\overline{E}_i, \overline{H}_i)$ in the original source region.

generating fields $(\overline{E}, \overline{H})$. The problem is to determine these fields. We introduce an auxiliary problem in fig. 2.9b which is the same physical environment as the original problem, except that the original sources are removed and a known Hertzian dipole source $\overline{J}_i = Id\ell\hat{\zeta}\delta(\overline{r} - \overline{r}')$ is introduced at $\overline{r}$. This "testing dipole" generates fields $(\overline{E}_i, \overline{H}_i)$ which are known everywhere. Since the physical environment is the same in each case, we can apply the reciprocity theorem () to connect the two sets of fields as

$$\iiint \left[\overline{J} \cdot \overline{E}_i - \overline{M} \cdot \overline{H}_i\right] dV' = \iiint \overline{J}_i \cdot \overline{E}\, dV' = Id\ell\hat{\zeta} \cdot \overline{E}(\overline{r}) \tag{2.74}$$

The left-hand side involves known quantities and can be evaluated to yield the $\hat{\zeta}$ component of $\overline{E}(\overline{r})$ on the right-hand side. Hence the testing dipole effectively samples the $\hat{\zeta}$ component of $\overline{E}$ at $\overline{r}$.

It would appear that this method of evaluating the field is significantly different than our potential integrals (2.24) or (2.63), but we can show that the two approaches are essentially equivalent, as they should be. For simplicity we show correspondence with the far-field expressions (2.63). If the testing dipole is placed in the far-field of the current distribution, and the original problem involves only currents in free-space, then the field $\overline{E}_i$ at $\overline{r}'$ is given by (see Problem X)

$$\overline{E}_i(\overline{r}') = -\jmath\omega\mu\frac{e^{-\jmath kr}}{4\pi r}Id\ell\left(\hat{\zeta} - (\hat{r} \cdot \hat{\zeta})\hat{r}\right)e^{\jmath k\hat{r}\cdot\overline{r}'} \tag{2.75}$$

and $\overline{H}_i = -\hat{r} \times \overline{E}_i/\eta$. Substituting this into (2.74), some minor manipulation yields (2.63)a and (2.64)a.

Although the reciprocity method is formally the same as our potential integral method, the reciprocity approach gives a significantly different perspective that can often simplify the problem. This is particularly true when the sources are radiating in the presence of material boundaries, as we will see later. In this sense it is akin to the field equivalence principles.

As a final example which will prove useful later, consider the application of (2.74) to the current distribution in fig. 2.8c. The currents are flowing in free-space, so the fields from the testing dipole are again the known dipole fields of Problem X. If the magnetic current loop is small, it can easily be shown that the term in (2.74) vanishes, since the incident field $\overline{H}_i$ is essentially constant over the loop. Hence

$$\iiint \overline{J} \cdot \overline{E}_i \, dV' = \iiint \overline{J}_i \cdot \overline{E} \, dV' \tag{2.76}$$