ELEMENTARY RADIATORS AND FAR-FIELD SOLUTIONS

In previous work we found the general solutions for the vector potentials for 3D problems:

$$\overline{A}(\overline{r}) = \frac{\mu}{4\pi}\iiint \overline{J}(\overline{r}')\frac{e^{-jkR}}{R}\,dV' \tag{1a}$$

$$\overline{F}(\overline{r}) = \frac{\epsilon}{4\pi}\iiint \overline{M}(\overline{r}')\frac{e^{-jkR}}{R}\,dV' \tag{1b}$$

where $R = |\overline{r} - \overline{r}'|$. For 2D problems,

$$A(\bar{\rho}) = \frac{\mu}{4j}\iint \overline{J}(\bar{\rho}')H_0^{(2)}(kR)\,dS' \tag{2a}$$

$$F(\bar{\rho}) = \frac{\epsilon}{4j}\iint \overline{M}(\bar{\rho}')H_0^{(2)}(kR)\,dS' \tag{2b}$$

where

$$R \equiv |\bar{\rho} - \bar{\rho}'| = \left[\rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi - \phi')\right]^{1/2}$$

The fields are then found as

$$\overline{E} = -j\omega\left[\overline{A} + \frac{1}{k^2}\nabla\nabla\cdot\overline{A}\right] - \frac{1}{\epsilon}\nabla\times\overline{F} \tag{3a}$$

$$\overline{H} = \frac{1}{\mu}\nabla\times\overline{A} - j\omega\left[\overline{F} + \frac{1}{k^2}\nabla\nabla\cdot\overline{F}\right] \tag{3b}$$

1 ELEMENTARY RADIATORS

The Green's function gives a solution for the fields produced by an infinitesimal point source, and we have shown that the fields from more general source distributions can be constructed by a superposition of these point-source fields. There are other elementary sources in antenna theory, such as the short current element and short current loop, that play a similar but more physical role in the sense of representing elementary distributions that can be (approximately) realized in practice. These elementary sources are of finite spatial extent, but with dimensions that are small compared to a wavelength.

Consider the arbitrary source distribution shown in figure 1, which has a characteristic dimension D. We wish to consider the case when the source distribution is small compared to a wavelength, $D \ll \lambda$. In this case, $kr' \ll 1$ in the exponential term of the expressions

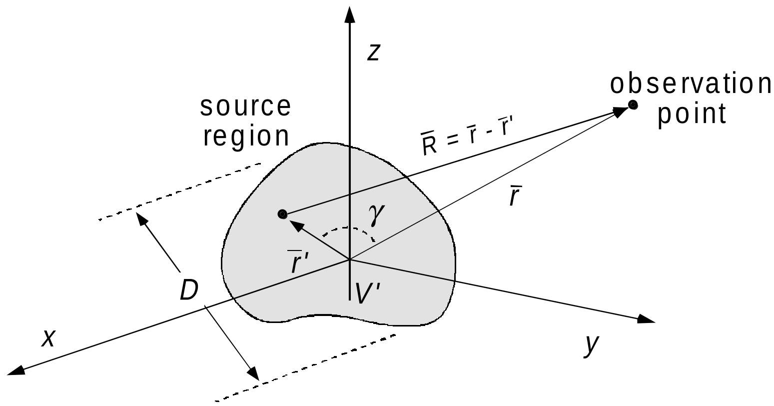

Figure 1 Source region with a characteristic dimension D centered at the origin. For elementary sources, $D \ll \lambda$.

(1). Assuming $r > D$ so that $|\overline{r} - \overline{r}'| = (\overline{r} - \overline{r}')$, and using the vector form of the Taylor expansion

$$f(\overline{r} + \bar{a}) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\bar{a} \cdot \nabla\right)^n f(\overline{r})$$

gives

$$\frac{e^{-\jmath k(\overline{r}-\overline{r}')}}{\overline{r} - \overline{r}'} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\overline{r}' \cdot \nabla\right)^n \frac{e^{-\jmath kr}}{r}$$

Substituting into (1a) and keeping only the first two terms of the expansion gives

$$\overline{A}(\overline{r}) = \frac{\mu}{4\pi} \left[\iiint \overline{J}(\overline{r}')\, dV' \right] \frac{e^{-\jmath kr}}{r} - \frac{\mu}{4\pi} \left[\iiint \overline{J}(\overline{r}')\overline{r}'\, dV' \right] \cdot \nabla \frac{e^{-\jmath kr}}{r} + \dots \tag{4}$$

with a similar expression resulting for $\overline{F}$. This is a "multipole" expansion valid for electrically small source distributions where $kD \ll 1$. The terms in brackets are called "moments" of the current distribution. Successive terms in the expansion become rapidly negligible in the region of validity, and the first non-zero term is usually a sufficiently accurate approximation. We will apply this result to some important canonical current sources.

1.1 Electric Hertzian dipole

Consider the short linear current element of length $d\ell$ along the z-axis, shown in figure 2 below. This current is described by

$$\overline{J}(\overline{r}') = \begin{cases} \hat{z} I_0 \delta(x')\delta(y') & |z'| < d\ell/2 \\ 0 & \text{elsewhere} \end{cases}$$

Assuming $d\ell \ll \lambda$ we can use (4). The dominant term in the expansion is (in this case) the first term term, which integrates to

$$\overline{A}(\overline{r}) = \hat{z} I_0\, d\ell \frac{\mu}{4\pi} \frac{e^{-\jmath kr}}{r} \tag{5}$$

or, in spherical coordinates,

$$\overline{A}(\overline{r}) = \left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right) I_0 d\ell \frac{\mu}{4\pi}\frac{e^{-\jmath kr}}{r} \tag{6}$$

Note that the same result can be obtained using the more concise representation for the current element

$$\overline{J}(\overline{r}') = I_0 \overline{d\ell}\delta(\overline{r}') \tag{7}$$

which is commonly used in the literature. Using this form, the vector potential is clearly just given by $\overline{A}(\overline{r}) = I_0 \overline{d\ell} g(\overline{r}, \overline{r}')$. In other words, the finite linear current element can be viewed as a building block for more complicated distributions, provided it has a spatial extent much smaller than a wavelength.

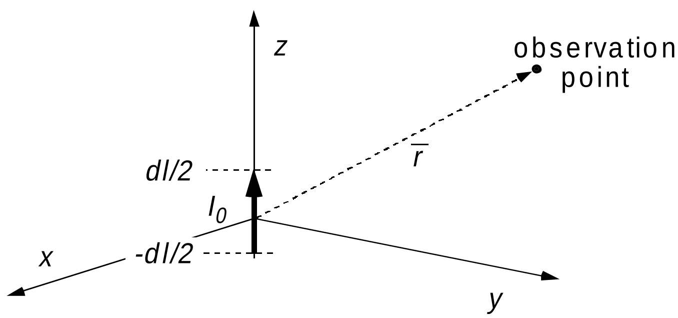

Figure 2 Short linear current element (electric Hertzian dipole) with length $d\ell$ such that $k\,dl \ll 1$.

Substituting into (3) gives

$$\overline{H} = \hat{\phi}\,\jmath k I_0 d\ell \sin\theta \left(1 + \frac{1}{\jmath kr}\right)\frac{e^{-\jmath kr}}{4\pi r} \tag{8a}$$

$$\begin{aligned}\overline{E} = &-\hat{r}\,2\jmath\omega\mu I_0 d\ell\cos\theta\left[\frac{1}{\jmath kr} + \frac{1}{(\jmath kr)^2}\right]\frac{e^{-\jmath kr}}{4\pi r}\\ &+\hat{\theta}\,\jmath\omega\mu I_0 d\ell\sin\theta\left[1 + \frac{1}{\jmath kr} + \frac{1}{(\jmath kr)^2}\right]\frac{e^{-\jmath kr}}{4\pi r}\end{aligned} \tag{8b}$$

There are three contributions to the fields: 1) an electrostatic term, varying as $1/r^3$, which is associated with an electric dipole moment; 2) a quasi-static “induction” field, the magnetic component of which is given by the Biot-Savart law; and 3) a radiation field varying as $1/r$. The radiation field becomes the leading term at at critical distance of $kr > 1$, or $r > \lambda/2\pi$. At distances significant larger than a wavelength, the radiation field dominates and (8) reduce to

$$\text{far-fields:} \qquad \overline{E} = \hat{\theta}\,\jmath\omega\mu I_0 d\ell \sin\theta \frac{e^{-\jmath kr}}{4\pi r} \tag{9a}$$

$$\overline{H} = \frac{1}{\eta}\hat{r}\times\overline{E} \tag{9b}$$

This is a spherical wave, modified by a $\sin\theta$ dependence to the field intensity. Consequently there is no radiation in the direction parallel to the current flow, and a maximum radiation intensity in the direction normal to the current flow. In fact, radiation fields always result from currents flowing transverse to the observation position vector; this will be proved in a subsequent section. The complex Poynting vector in the far-field is given by

$$\begin{aligned} \overline{\mathcal{P}} &= \frac{1}{2}\overline{E}\times\overline{H}^* \\ &= \frac{\eta(I_0\,d\ell)^2k^2}{32\pi^2}\frac{\sin^2\theta}{r^2}\,\hat{r} \end{aligned} \tag{10}$$

Integrating the Poynting vector over a sphere, the total radiated power is

$$\begin{aligned} P_{\text{rad}} &= \oiint \overline{\mathcal{P}}\cdot d\overline{S} = \int_0^{2\pi}\int_0^{\pi}\mathcal{P}r^2\sin\theta\,d\theta\,d\phi \\ &= \frac{\eta(Id\ell)^2k^2}{32\pi^2}\;2\pi\underbrace{\int_0^{\pi}\sin^3\theta\,d\theta}_{4/3} = \frac{\eta(Id\ell)^2k^2}{12\pi} \end{aligned} \tag{11}$$

It can be shown that this result is also obtained using the full expression for the dipole fields (8). That is, the near-field terms do not contribute to real power flow away from the antenna. Instead, these terms represent a dynamic *stored* energy in the vicinity of the antenna, flowing back and forth between the feed circuit and the fields, and hence contributing to a reactive component of the impedance as seen by the generator. The radiation resistance as seen by the current source is then calculated from $P_{\text{rad}} = \frac{1}{2}I^2R_{\text{rad}}$, giving

$$R_{\text{rad}} = \frac{\eta(kd\ell)^2}{6\pi} \tag{12}$$

In free space where $\eta = 120\pi$,

$$R_{\text{rad}} = 80\pi^2\left(\frac{d\ell}{\lambda_0}\right)^2.$$

It should be remembered that this result was derived under the assumption of $d\ell \ll \lambda$.

1.2 Magnetic Hertzian dipole

Consider the closed current loop of radius a shown in figure 3. We assume that the current is uniform around the loop, which implies that the circumference is a small fraction of a wavelength, $2\pi a \ll \lambda$; this implies $ka \ll 1$, which is a condition for the validity of (4).

For any closed loop of current, the first term in (4) is zero. Therefore the dominant term in the expansion is (at best) the second term. Evaluating the gradient term,

$$\nabla\frac{e^{-jkr}}{r} = -\hat{r}\frac{e^{-jkr}}{r}\left[\frac{1}{r}+jk\right]$$

and denoting the angle between $\overline{r}$ and $\overline{r}'$ as γ so that $\hat{r}\cdot\overline{r}' = r'\cos\gamma$, we can write the second term of the multipole expansion (4) as

$$\overline{A}(\overline{r}) = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\left[\frac{1}{r}+jk\right]\iiint\overline{J}(\overline{r}')r'\cos\gamma\,dV' \tag{13}$$

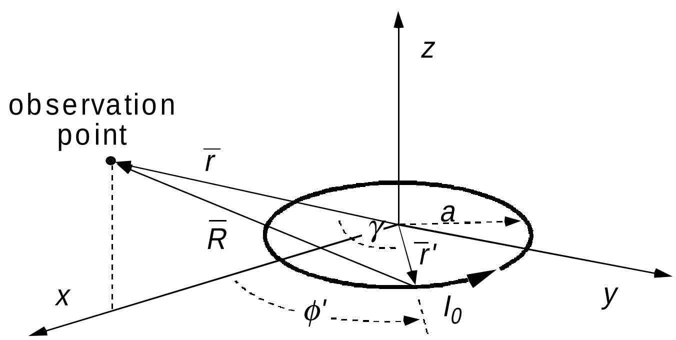

Figure 3 Small uniform current loop (magnetic Hertzian dipole) in the $x-y$ plane at $z=0$, with radius a such that $ka \ll 1$.

where

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

From the azimuthal symmetry of the problem, we expect the vector potential to have only a $\hat{\phi}$ component and to be independent of ϕ. Therefore we can choose the observation point $\phi = 0$ for convenience, where $A_\phi = A_y$. The current loop can be represented as

$$\overline{J}(\overline{r}') = \hat{\phi} I_0 a \frac{\delta(\overline{r}'-a)\delta(\theta'-\pi/2)}{r'^2 \sin\theta'} \tag{14}$$

However, the $\hat{\phi}$ direction is in general different at the source and observation points, so it is better to express $\overline{J}$ in terms of its rectangular components as

$$\overline{J}(\overline{r}') = -\hat{x} J_\phi \sin\phi' + \hat{y} J_\phi \cos\phi'$$

The integral in (13) then evaluates to

$$\begin{aligned} \iiint \overline{J}(\overline{r}') r' \cos\gamma \, dV' &= I_0 a^2 \sin\theta \int_0^{2\pi} \cos\phi' \left[-\hat{x}\sin\phi' + \hat{y}\cos\phi'\right] \, d\phi' \\ &= I_0 \pi a^2 \sin\theta \, \hat{y} \end{aligned}$$

and therefore the vector potential is

$$\overline{A}(\overline{r}') = \hat{\phi} \, I_0 \pi a^2 \sin\theta \frac{\jmath k \mu}{4\pi} \frac{e^{-\jmath kr}}{r} \left[1 + \frac{1}{\jmath kr}\right] \tag{15}$$

Substituting into (3) gives

$$\overline{E} = \hat{\phi} \, I_0 \pi a^2 k^2 \eta \sin\theta \left[1 + \frac{1}{\jmath kr}\right] \frac{e^{-\jmath kr}}{4\pi r} \tag{16a}$$

$$\begin{aligned} \overline{H} = &-\hat{r} \, 2 I_0 \pi a^2 k^2 \cos\theta \left[\frac{1}{\jmath kr} + \frac{1}{(\jmath kr)^2}\right] \frac{e^{-\jmath kr}}{4\pi r} \\ &- \hat{\theta} \, I_0 \pi a^2 k^2 \sin\theta \left[1 + \frac{1}{\jmath kr} + \frac{1}{(\jmath kr)^2}\right] \frac{e^{-\jmath kr}}{4\pi r} \end{aligned} \tag{16b}$$

Comparing (16) with (8), we find that they are essentially the same with the roles of $\overline{E}$ and $\overline{H}$ reversed; the short loop is the dual of an electric dipole, and hence is referred to as a magnetic dipole. The term $I_0\pi a^2$ is the magnetic dipole moment. The far fields of a magnetic dipole are similarly

$$\text{far-fields:} \qquad \overline{E} = \hat{\phi}\, I_0\pi a^2 k^2 \eta \sin\theta \frac{e^{-jkr}}{4\pi r} \tag{17a}$$

$$\overline{H} = \frac{1}{\eta}\hat{r}\times\overline{E} \tag{17b}$$

so the radiation pattern and the directivity are the same as the electric Hertzian dipole. The radiation resistance is found as

$$\begin{aligned} R_{rad} &= \frac{2P_{rad}}{|I|^2} = \frac{2}{|I|^2}\oiint \frac{|E_\phi|^2}{2\eta}\,dS \\ &= \frac{\pi\eta}{8}(ka)^4 \underbrace{\int_0^\pi \sin^3\theta d\theta}_{4/3} = \frac{\pi\eta}{6}(ka)^4 \end{aligned} \tag{18}$$

and in free space where $\eta = 120\pi$,

$$R_{rad} = 320\pi^6\left(\frac{a}{\lambda}\right)^4$$

The radiation resistance of the loop can be extremely small, going as $(a/\lambda)^4$, compared with $(d\ell/\lambda)^2$ for the Hertzian dipole. This number can be significantly increased, however, by using a large number of turns in the loop. As long as the total coil length remains electrically small, our previous analysis applies by replacing I with (NI), where N is the number of turns. Therefore

$$R_{rad} \Rightarrow 320\pi^6 N^2 (a/\lambda)^2 \tag{19}$$

This is valid provide that $2\pi aN \ll \lambda$.

2 FAR-FIELD SOLUTIONS

Returning to the general result for the potentials (1), we now focus on possible simplifications for arbitrarily large source distributions when only the radiation fields (varying as $1/r$) are of interest. We start by making a Taylor expansion of $R = |\overline{r}-\overline{r}'|$ for the case of $|\overline{r}| \gg |\overline{r}'|$, giving

$$\begin{aligned} R = |\overline{r}-\overline{r}'| &= [(\overline{r}-\overline{r}')\cdot(\overline{r}-\overline{r}')]^{\frac{1}{2}} = \left[r^2 - 2\overline{r}\cdot\overline{r}' + r'^2\right]^{\frac{1}{2}} \\ &= \left[r^2\left(1 - \frac{2\overline{r}\cdot\overline{r}'}{r^2} + \frac{r'^2}{r^2}\right)\right]^{\frac{1}{2}} \\ &= r\left[1 - \frac{\overline{r}\cdot\overline{r}'}{r^2} + \frac{1}{2}\frac{r'^2}{r^2} - \frac{1}{2}\frac{(\overline{r}\cdot\overline{r}')^2}{r^4} + \ldots\right] \end{aligned} \tag{20}$$

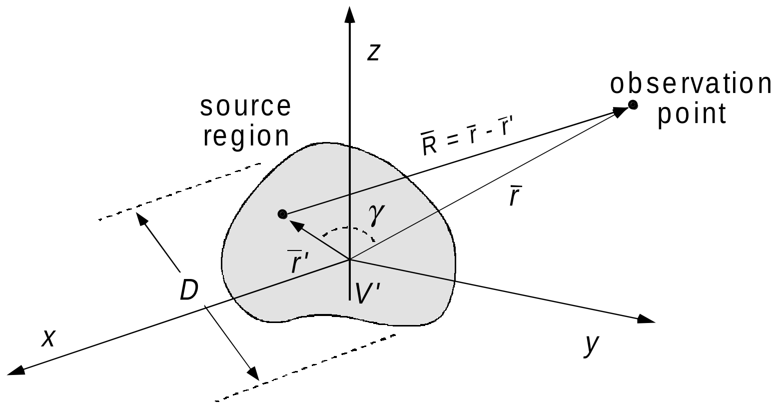

Figure 4 Source region with a characteristic dimension D for delimiting far-field region.

Denoting the angle between $\overline{r}$ and $\overline{r}'$ as γ (see figure 4) we can write $\overline{r} \cdot \overline{r}' = rr' \cos\gamma$ where

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi') \tag{21}$$

and so the first three terms of the expansion in R become

$$R \approx r - \hat{r} \cdot \overline{r}' + \frac{1}{2}\frac{r'^2}{r}\sin^2\gamma + \ldots$$

where $\hat{r} = \overline{r}/r$. When substituted into the exponential term of the Green's function e^{-jkR} then we must keep at least the first two terms, since though $\overline{r}'$ is small compared with $\overline{r}$ it can still be large relative to a wavelength, and consequently kr' can represent a non-negligible phase. The third term contributes a phase of

$$\frac{1}{2}\frac{kr'^2}{r}\sin^2\gamma$$

We can always choose r far enough away so that this term is negligible. A useful (though somewhat arbitrary) rule-of-thumb seen in the literature defines "negligible" to mean a phase contribution of $\pi/8$ or less. Then, if the source distribution is centered on the origin and has a characteristic length of D so that $|r'| \leq D/2$, the critical observation distance should be such that

$$\frac{1}{2}\frac{k\left(\frac{D}{2}\right)^2}{r} < \frac{\pi}{8}$$

or

$$r > \frac{2D^2}{\lambda} \qquad \text{(condition for far-field)} \tag{22}$$

When this condition holds we are in the "far-field" of the source distribution. Then

$$R \approx r - \hat{r} \cdot \overline{r}' \qquad \text{(far-field approximation)} \tag{23}$$

and the Green's function becomes

$$\frac{e^{-jkR}}{R} \approx e^{-jkr} e^{jk\hat{r}\cdot\overline{r}'} \frac{1}{r}\left[1 + \frac{\hat{r}\cdot\overline{r}'}{r} + \ldots\right]$$

The radiation fields (varying as $1/r$) become the dominant term at a distance of $r > D/2$. This condition is automatically satisfied in the far-field defined by (22), unless the antenna is very small compared with a wavelength ($D < \lambda/4$), in which case the expansion (4) would be more appropriate. In any case, we can always choose a distance far enough away from the antenna so that only the radiation fields are important, so the far-field Green's function is

$$g_{ff}(\overline{r},\overline{r}') \Rightarrow \frac{e^{-\jmath kr}}{4\pi r} e^{\jmath k\hat{r}\cdot\overline{r}'} \tag{24}$$

and our far-field vector potentials are therefore

$$\overline{A}(\overline{r}) \;\Rightarrow\; \frac{\mu}{4\pi}\frac{e^{-\jmath kr}}{r}\iiint \overline{J}(\overline{r}')e^{+\jmath k\hat{r}\cdot\overline{r}'}\,dV' \tag{25a}$$

$$\overline{F}(\overline{r}) \;\Rightarrow\; \frac{\epsilon}{4\pi}\frac{e^{-\jmath kr}}{r}\iiint \overline{M}(\overline{r}')e^{\jmath k\hat{r}\cdot\overline{r}'}\,dV' \tag{25b}$$

Note that the integrand of these "radiation integrals" does not depend on the observation distance r. Now consider the calculation of the fields from (3). In spherical coordinates

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

Only the $\hat{r}$ component of the ∇ operator is capable of generating fields with $1/r$ dependence; the other components produce terms going as $1/r^2$, which we have so far consistently neglected. Keeping only these terms in $1/r$ gives

$$\begin{aligned}\overline{A} + \frac{1}{k^2}\nabla\nabla\cdot\overline{A} \;&\Rightarrow\; \overline{A} - (\hat{r}\cdot\overline{A})\hat{r} = A_\theta\hat{\theta} + A_\phi\hat{\phi}\\ \nabla\times\overline{F} \;&\Rightarrow\; -\jmath k(\hat{r}\times\overline{F}) = \jmath k(F_\phi\hat{\theta} - F_\theta\hat{\phi})\end{aligned}$$

so from (3) the radiation fields are

$$\overline{E}(\overline{r}) = -\jmath\omega\left[A_\theta\hat{\theta} + A_\phi\hat{\phi}\right] - \frac{\jmath k}{\epsilon}\left[F_\phi\hat{\theta} - F_\theta\hat{\phi}\right] \tag{26a}$$

$$\overline{H}(\overline{r}) = \frac{\hat{r}\times\overline{E}}{\eta} \tag{26b}$$

Writing these out explicitly for later use, we have, for electric sources only,

$$\begin{aligned}\overline{E}(\overline{r}) &= -\jmath\omega\mu\frac{e^{-\jmath kr}}{4\pi r}\iiint\left[\overline{J}(\overline{r}') - \left(\hat{r}\cdot\overline{J}(\overline{r}')\right)\hat{r}\right]e^{+\jmath k\hat{r}\cdot\overline{r}'}\,dV' &(27a)\\ &= -\jmath\omega\mu\frac{e^{-\jmath kr}}{4\pi r}\hat{r}\times\hat{r}\times\iiint\overline{J}(\overline{r}')e^{+\jmath k\hat{r}\cdot\overline{r}'}\,dV'\\ \overline{H}(\overline{r}) &= -\jmath k\frac{e^{-\jmath kr}}{4\pi r}\hat{r}\times\iiint\overline{J}(\overline{r}')\,e^{+\jmath k\hat{r}\cdot\overline{r}'}\,dV' &(27b)\end{aligned}$$

and for magnetic sources only,

$$\begin{aligned}\overline{E}(\overline{r}) &= \jmath k\frac{e^{-\jmath kr}}{4\pi r}\hat{r}\times\iiint\overline{M}(\overline{r}')\,e^{+\jmath k\hat{r}\cdot\overline{r}'}\,dV' &(28a)\\ \overline{H}(\overline{r}) &= -\jmath\omega\epsilon\frac{e^{-\jmath kr}}{4\pi r}\iiint\left[\overline{M}(\overline{r}') - \left(\hat{r}\cdot\overline{M}(\overline{r}')\right)\hat{r}\right]e^{+\jmath k\hat{r}\cdot\overline{r}'}\,dV' &(28b)\\ &= -\jmath\omega\epsilon\frac{e^{-\jmath kr}}{4\pi r}\hat{r}\times\hat{r}\times\iiint\overline{M}(\overline{r}')e^{+\jmath k\hat{r}\cdot\overline{r}'}\,dV'\end{aligned}$$

These are the so-called "radiation equations." It is clear from these equations that for any particular observation point $\overline{r}$, the far-field has the form of a spherical TEM wave propagating in the $\hat{r}$ direction, and is due to only that part of the current distribution that is flowing transverse to $\hat{r}$.

2.1 Far-field in 2D

For problems in which variation of fields is negligible in one cartesian direction (taken as the $\hat{z}$-axis) then the vector potential is given by (2a),

$$\overline{A}(\overline{\rho}) = \frac{\mu}{4j} \iint \overline{J}(\overline{\rho}') H_0^{(2)}(kR)\, dS'$$

where

$$\begin{aligned} R = |\overline{\rho} - \overline{\rho}'| &= \sqrt{(x-x')^2 + (y-y')^2} \\ &= \sqrt{\rho^2 + {\rho'}^2 - 2\rho\rho' \cos(\phi - \phi')} \end{aligned}$$

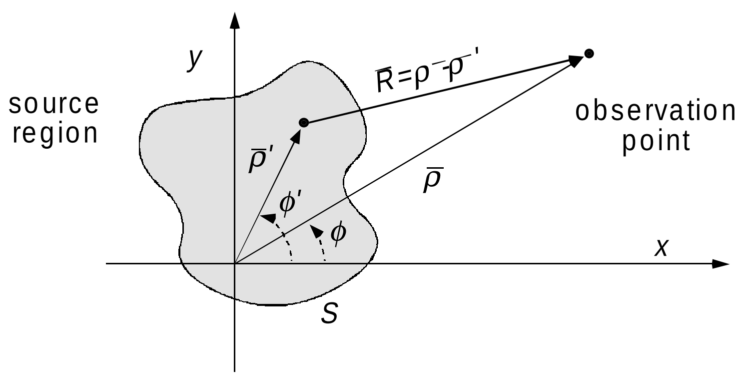

Figure 5 Two-dimensional source distribution (infinite extent in the $\hat{z}$ direction).

In the far-field, $kR \gg 1$, and we can use the asymptotic expansion for the 2D Green's function,

$$H_0^{(2)}(kR) \approx \sqrt{\frac{2}{\pi k R}} e^{-j(kR - \frac{\pi}{4})} \tag{29}$$

For $|\overline{\rho}| \gg |\overline{\rho}'|$, then we can make a Taylor expansion of R much like for the 3D case, giving

$$\begin{aligned} R &\cong \rho - \hat{\rho} \cdot \overline{\rho}' \\ &\cong \rho - \overline{\rho}' \cos(\phi - \phi') \end{aligned}$$

or we could express R as

$$R \cong \rho - x' \cos\phi - y' \sin\phi$$

Keeping only terms in $1/\sqrt{\rho}$ (radiation fields in 2D) gives

$$\overline{A}(\bar{\rho}) = \frac{\mu}{2} \frac{e^{-jk\rho}}{\sqrt{2jk\rho\pi}} \iint_S \overline{J}(\bar{\rho}')e^{jk\hat{\rho}\cdot\overline{\rho}'}\, dS' \tag{30}$$

and the electric field in the far-field is given by

$$\overline{E}(\bar{\rho}) \cong -\jmath\omega(A_\phi\hat{\phi} + A_z\hat{z}) \tag{31}$$

The extension to magnetic currents is straightforward. It is very important to remember when computing A_ϕ that the $\hat{\phi}$ direction at the source and observation points are general-lydifferent and must be handled accordingly. If the current is best described by rectangular components, then

$$\begin{aligned} A_\phi &= \frac{\mu}{2} \frac{e^{-jk\rho}}{\sqrt{2jk\rho\pi}} \iint_S \left[-J_x(\bar{\rho}')\sin\phi + J_y(\bar{\rho}')\cos\phi\right] e^{jk\hat{\rho}\cdot\overline{\rho}'}\, dS' && \text{(32a)} \\ A_z &= \frac{\mu}{2} \frac{e^{-jk\rho}}{\sqrt{2jk\rho\pi}} \iint_S J_z(\bar{\rho}')e^{jk\hat{\rho}\cdot\overline{\rho}'}\, dS' && \text{(32b)} \end{aligned}$$

Note that the explicit $\sin\phi$ and $\cos\phi$ terms in (32a) are functions of the *unprimed* ϕ (at observation point). If $\overline{J}$ is best described in cylindrical coordinates then we can again use (32) but with the substitutions

$$\begin{aligned} J_x &= J_\rho \cos\phi' - J_\phi \sin\phi' && \text{(33a)} \\ J_y &= J_\rho \sin\phi' + J_\phi \cos\phi' && \text{(33b)} \end{aligned}$$

where these terms involve the primed (source) variable ϕ'.