SPECTRAL DOMAIN METHODS

Robert York
UC Santa Barbara

1 PLANAR LAYERED OR STRATEFIED STRUCTURES

Antennas, transmission lines, and other electromagnetic structures for high frequency microelectronic systems are frequently fabricated by photolithographic techniques on dielectric or semiconducting substrates. Although many of these structures have simple geometric shapes, analysis is complicated by the inhomogeneous nature of the problem (sources radiating in the presence of material interfaces). In those special cases where the material boundaries conform to some canonical surface such as a planar, cylindrical, or spherical surface, Green's functions can be obtained by an eigenmode expansion technique. For planar stratified media (figure x), an eigenmode expansion takes the form of a plane wave spectrum. Using this "spectral decomposition" technique and the properties of the Fourier transform, a very straightforward and powerful method for constructing Green's functions in planar layered media can be developed. We will apply this technique to some common configurations encountered in planar antenna work, and develop the necessary modifications for use in the Method of Moments computation of antenna parameters.

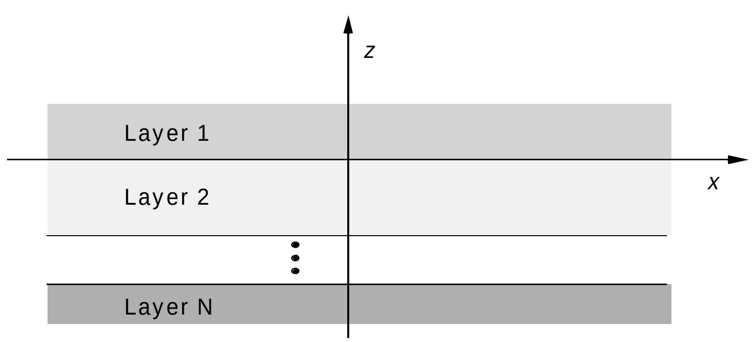

Figure 1 Planar layered substrate encountered in monolithic circuits.

1.1 Spectral representation of fields in stratefied media

Eigenfunctions of the homogeneous Hemholtz equation are elementary solutions to Maxwell's equations in a source-free region, and form a complete set. The Green's function therefore can be represented by a superposition of these eigenfunctions, and the coefficients of the expansion determined by matching the boundary conditions imposed by a delta function source. Maxwell's equations in a source-free region for time-harmonic fields take the form

$$\begin{aligned}
\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right) &= -\jmath\omega\mu H_x & \left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right) &= \jmath\omega\epsilon E_x \\
\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right) &= -\jmath\omega\mu H_y & \left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}\right) &= \jmath\omega\epsilon E_y \\
\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right) &= -\jmath\omega\mu H_z & \left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right) &= \jmath\omega\epsilon E_z
\end{aligned} \tag{1}$$

and can be combined to form the Helmholtz equations

$$\nabla^2\overline{E}+k^2\overline{E}=0 \qquad \nabla^2\overline{H}+k^2\overline{H}=0$$

In a general inhomogeneous problem these equations are difficult to solve. For planar stratified or layered media as shown above, there is only one direction (taken as the $\hat{z}$ direction) along which the material parameters vary inhomogeneously. In the $\hat{x}$ and $\hat{y}$ directions, we assume each layer is homogeneous and infinite in extent.

The Helmholtz equation is separable in rectangular coordinates. For the planar geometry above, the eigenfunctions in the $\hat{x}$ and $\hat{y}$ directions form a continuous spectrum of plane waves, so the field can be expressed as a two-dimensional Fourier integral

$$\overline{E}(x,y,z) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\tilde{\overline{E}}(k_x,k_y,z)e^{\jmath k_x x}e^{\jmath k_y y}\,dk_x\,dk_y \tag{2a}$$

$$\tilde{\overline{E}}(k_x,k_y,z) = \iint_{-\infty}^{\infty}\overline{E}(x,y,z)e^{-\jmath k_x x}e^{-\jmath k_y y}\,dx\,dy \tag{2b}$$

and similarly for the magnetic field. The function $\tilde{\overline{E}}(k_x,k_y,z)$ is the spectrum function for the field. Using (2) transforms Maxwell's equations (1) into a set of differential equations in only one dependent variable, since $\partial\overline{E}/\partial x \Rightarrow \jmath k_x\tilde{\overline{E}}$ and $\partial\overline{E}/\partial y \Rightarrow \jmath k_y\tilde{\overline{E}}$. As is done in cylindrical waveguide theory, we can express the transverse ($\hat{x}$ and $\hat{y}$) fields in terms of the $\hat{z}$ components. After some manipulation we obtain from (1) the following

$$\begin{aligned}
\frac{\partial^2\tilde{E}_x}{\partial z^2}+k^2\tilde{E}_x &= \jmath k_x\frac{\partial\tilde{E}_z}{\partial z}+\omega\mu k_y\tilde{H}_z & \frac{\partial^2\tilde{H}_x}{\partial z^2}+k^2\tilde{H}_x &= \jmath k_x\frac{\partial\tilde{H}_z}{\partial z}-\omega\epsilon k_y\tilde{E}_z \\
\frac{\partial^2\tilde{E}_y}{\partial z^2}+k^2\tilde{E}_y &= \jmath k_y\frac{\partial\tilde{E}_z}{\partial z}-\omega\mu k_x\tilde{H}_z & \frac{\partial^2\tilde{H}_y}{\partial z^2}+k^2\tilde{H}_y &= \jmath k_y\frac{\partial\tilde{H}_z}{\partial z}+\omega\epsilon k_x y\tilde{E}_z
\end{aligned} \tag{3}$$

From the Helmholtz equation, we have

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k^2\right)\left\{\begin{matrix}\overline{E}\\ \overline{H}\end{matrix}\right\}=0 \tag{4}$$

which transforms to

$$\left(\frac{\partial^2}{\partial z^2}+k_z^2\right)\left\{\begin{matrix}\overline{E}\\ \overline{H}\end{matrix}\right\}=0 \tag{5}$$

where $k_z^2 = k^2 - k_x^2 - k_y^2$. This allows the LHS in each of of (3) to be written in terms of k_x and k_y. Defining $\beta^2 = k_x^2 + k_y^2$ gives

$$\begin{aligned}
\tilde{E}_x &= \frac{\jmath k_x}{\beta^2}\frac{\partial \tilde{E}_z}{\partial z} + \frac{\omega\mu k_y}{\beta^2}\tilde{H}_z & \qquad \tilde{H}_x &= \frac{\jmath k_x}{\beta^2}\frac{\partial \tilde{H}_z}{\partial z} - \frac{\omega\epsilon k_y}{\beta^2}\tilde{E}_z \\
\tilde{E}_y &= \frac{\jmath k_y}{\beta^2}\frac{\partial \tilde{E}_z}{\partial z} - \frac{\omega\mu k_x}{\beta^2}\tilde{H}_z & \qquad \tilde{H}_y &= \frac{\jmath k_y}{\beta^2}\frac{\partial \tilde{H}_z}{\partial z} + \frac{\omega\epsilon k_x}{\beta^2}\tilde{E}_z
\end{aligned} \tag{6}$$

The functions $\tilde{E}_z$ and $\tilde{H}_z$ are solutions to the simple 1D wave equation (5), which we know are combinations of $\exp(\pm\jmath k_z z)$, or $\sin(k_z z)$ and $\cos(k_z z)$.

1.2 Spectral Domain Green's functions in stratified media

The Green's function describes the fields produced by a unit point dipole at $\overline{r}'$, given by $\hat{p}\delta(\overline{r}-\overline{r}')$, where $\hat{p}$ describes the orientation of the point source. To find a Green's function using the Fourier transform technique we write appropriate expressions for $\tilde{E}_z$ and $\tilde{H}_z$ in each layer, and enforce boundary conditions on the transverse components calculated from (5), including the appropriate jump discontinuity in the fields at the source point. A set of algebraic equations results for the unknown coefficients of the $\hat{z}$ components. Once these are solved all of the field components in the spectral domain are known, hence we have a "spectral domain Green's function". Note that in this context the spectral representation of a point source at location $\overline{r}'$ is just

$$\hat{p}\delta(\overline{r}-\overline{r}') \quad \Rightarrow \quad \hat{p}e^{-\jmath k_x x'}e^{-\jmath k_y y'}\delta(z-z') \tag{7}$$

This is the well-known shifting property of the Fourier transform. We will exploit this property by defining the following transform pair for the spectral domain Green's function

$$\overline{\overline{G}}(\overline{r}|\overline{r}') = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\tilde{\overline{\overline{G}}}(\overline{r}|k_x,k_y,z')e^{\jmath k_x(x-x')}e^{\jmath k_y(y-y')}\,dk_x\,dk_y \tag{8a}$$

$$\tilde{\overline{\overline{G}}}(\overline{r}|k_x,k_y,z) = \iint_{-\infty}^{\infty}\overline{\overline{G}}(\overline{r}|\overline{r}')e^{-\jmath k_x(x-x')}e^{-\jmath k_y(y-y')}\,dx\,dy \tag{8b}$$

What we have done is just explicitly factor out the term $e^{-\jmath k_x x'}e^{-\jmath k_y y'}$ from the spectral Green's function. This means that the function $\tilde{\overline{\overline{G}}}$ in (8) is the spectral representation for the fields produced by a point source at location $(0,0,z')$; the factor $e^{-\jmath k_x x'}e^{-\jmath k_y y'}$ then shifts to the actual location to (x',y',z'). The reason for writing the Green's function in this way can be appreciated by the following example. In the spatial domain the field due to some surface current distribution on a z'=constant surface is given by

$$\overline{E}(\overline{r}) = \iint_S \overline{\overline{G}}(\overline{r}|\overline{r}') \cdot \overline{J}_s(\overline{r}')\, dS' \tag{9}$$

where $dS' = dx'dy'$. Substituting the spectral representation for $\overline{\overline{G}}$ from (8a) gives

$$\overline{E}(\overline{r}) = \iint_S \iint_{-\infty}^{\infty} \tilde{\overline{\overline{G}}} \cdot \overline{J}_s(\overline{r}') e^{\jmath k_x(x-x')} e^{\jmath k_y(y-y')}\, dk_x\, dk_y\; dS' \tag{10}$$

Since $\tilde{\overline{\overline{G}}}$ does not depend on x' or y' this can be rearranged as

$$\overline{E}(\overline{r}) = \iint_{-\infty}^{\infty} \tilde{\overline{\overline{G}}} \cdot \left[\iint_S \overline{J}_s(\overline{r}') e^{-\jmath k_x x'} e^{-\jmath k_y y'}\; dS' \right] e^{\jmath k_x x} e^{\jmath k_y y}\, dk_x\, dk_y \tag{11}$$

From (2) we identify the term in brackets as the spectral representation of the current distribution, $\tilde{\overline{J}}_s$, so

$$\overline{E}(\overline{r}) = \iint_{-\infty}^{\infty} \tilde{\overline{\overline{G}}} \cdot \tilde{J}_s e^{\jmath k_x x} e^{\jmath k_y y}\, dk_x\, dk_y \tag{12}$$

Comparing with (2a) we get the nice result $\tilde{\overline{E}} = \tilde{\overline{\overline{G}}} \cdot \tilde{\overline{J}}_s$; this would not have been as easily obtained without writing the transform pair for $\overline{\overline{G}}$ in the form of (8).

1.3 Electric Green's function for a semi-infinite substrate

As a first example we can find the spectral domain Green's function for an $\hat{x}$-directed dipole on a semi-infinite substrate, as shown in figure fig. 2 below. This is sometimes useful for approximate analysis of some antennas on electrically thick substrates, but also provides a simple illustration of the technique for constructing Green's functions. The unit source can be represented as a current $\overline{J}_s = \hat{x}\delta(x-x')\delta(y-y')$ on the $z=0$ plane. In the spectral domain this current becomes

$$\tilde{\overline{J}}_s = \hat{x} e^{-\jmath k_x x'} e^{-\jmath k_y y'} \tag{13}$$

Using the notation $\tilde{\delta} = e^{-\jmath k_x x'} e^{-\jmath k_y y'}$ we can write the boundary conditions at $z=0$ in the spectral domain as

$$\hat{n} \times (\tilde{\overline{E}}_1 - \tilde{\overline{E}}_2) \;=\; 0 \tag{14}$$

$$\hat{n} \times (\tilde{\overline{H}}_1 - \tilde{\overline{H}}_2) \;=\; \hat{x}\tilde{\delta} \tag{15}$$

From the geometry of the problem we anticipate travelling waves in the $\pm\hat{z}$ direction, so the appropriate forms for $\tilde{E}_z$ and $\tilde{H}_z$ in each region are

$$\tilde{E}_z = \begin{cases} Ae^{+\jmath k_1 z} & \text{for } z<0 \\ Ce^{-\jmath k_2 z} & \text{for } z>0 \end{cases} \qquad \text{and} \qquad \tilde{H}_z = \begin{cases} Be^{+\jmath k_1 z} & \text{for } z<0 \\ De^{-\jmath k_2 z} & \text{for } z>0 \end{cases} \tag{16}$$

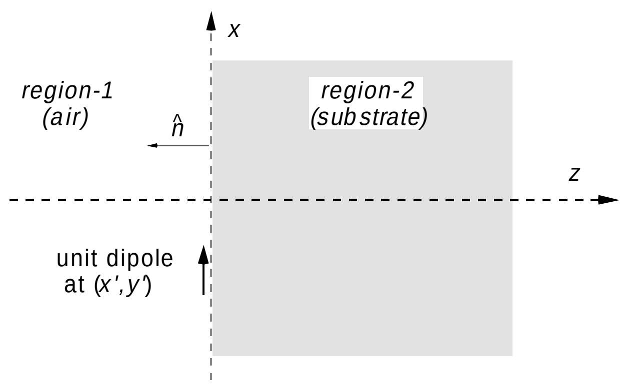

Figure 2 Point dipole on semi-infinite substrate.

where k_1 and k_2 are defined as $k^2 - \beta^2$ in each region, so

$$k_1^2 = k_0^2 - \beta^2 \qquad \mathrm{Im}\{k_1\} < 0 \tag{17}$$

$$k_2^2 = \epsilon_r k_0^2 - \beta^2 \qquad \mathrm{Im}\{k_2\} < 0 \tag{18}$$

and k_0 is the free-space propagation constant. The stipulation that $\mathrm{Im}\{k_z\} < 0$ is required on physical grounds, and is important when performing the inverse transform (**??**a). Substituting (16) into the expressions for the transverse field components (6) and enforcing the boundary conditions (15) gives a set of algebraic equations for the coefficients (A, B, C, D). These can be easily solved to give the following components of the electric field Green's dyadic in the spectral domain

$$\tilde{G}_{xx} = -\frac{\eta_0}{k_0}\left[\frac{k_0^2}{(k_1+k_2)} - \frac{k_x^2}{(\epsilon k_1 + k_2)}\right]\begin{cases} e^{\jmath k_1 z} & z < 0 \\ e^{-\jmath k_2 z} & z > 0 \end{cases} \tag{19a}$$

$$\tilde{G}_{yx} = \frac{\eta_0 k_x k_y}{k_0(\epsilon k_1 + k_2)}\begin{cases} e^{\jmath k_1 z} & z < 0 \\ e^{-\jmath k_2 z} & z > 0 \end{cases} \tag{19b}$$

$$\tilde{G}_{zx} = \frac{k_x \eta_0}{k_0(\epsilon k_1 + k_2)}\begin{cases} k_2 e^{\jmath k_1 z} & z < 0 \\ -k_1 e^{-\jmath k_2 z} & z > 0 \end{cases} \tag{19c}$$

Remember that our spectral Green's function has the term $\tilde{\delta}$ factored out so that it appears explicitly in (8). The fields produced by a $\hat{y}$-directed dipole are obtained from the above through the substitutions $k_x \to k_y$ and $k_y \to -k_x$.

1.4 Electric Green's function for microstrip

Printed dipoles and microstrip antennas such as the patch require the dyadic Green's function for a grounded substrate. The metalization layer is always the top surface of the substrate, and since this is where the integral equations will be enforced, we only need the components of the dyadic Green's function corresponding to tangential fields on the substrate surface. Currents flowing on the top surface can only have an $\hat{x}$ or a $\hat{y}$ component.

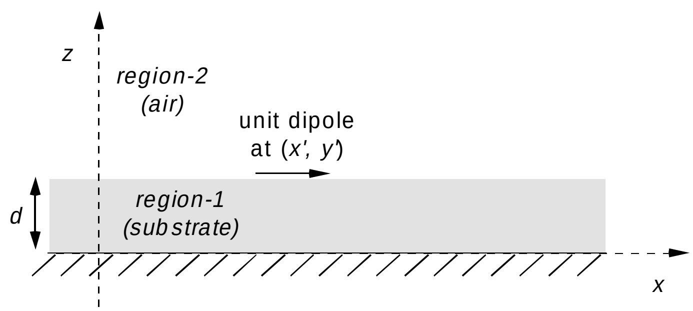

Figure 3 Point dipole on grounded substrate, which is the appropriate problem for finding Green's functions used in microstrip circuit and antenna analysis.

We will consider an $\hat{x}$ directed dipole first as shown below. Again, the unit source can be represented as a current $\overline{J}_s = \hat{x}\delta(x-x')\delta(y-y')$ on the $z=d$ plane. In the spectral domain this current becomes

$$\tilde{\overline{J}}_s = \hat{x}e^{-\jmath k_x x'}e^{-\jmath k_y y'} \tag{20}$$

Using the notation $\tilde{\delta} = e^{-\jmath k_x x'}e^{-\jmath k_y y'}$ we can write the boundary conditions in the spectral domain as

$$\begin{aligned} \hat{z}\times\tilde{\overline{E}}_1 &= 0 \quad \text{at } z=0 & (21)\\ -\hat{z}\times(\tilde{\overline{E}}_1-\tilde{\overline{E}}_2) &= 0 \quad \text{at } z=d & (22)\\ -\hat{z}\times(\tilde{\overline{H}}_1-\tilde{\overline{H}}_2) &= \hat{x}\tilde{\delta} \quad \text{at } z=d & (23) \end{aligned}$$

From the geometry of the problem we anticipate the following forms for the $\hat{z}$ components of the fields

$$\tilde{E}_z = \begin{cases} A\cos k_1 z + B\sin k_1 z & \text{for } 0\le z\le d\\ Ce^{-\jmath k_2 z} & \text{for } z>d \end{cases} \tag{24}$$

$$\tilde{H}_z = \begin{cases} D\cos k_1 z + E\sin k_1 z & \text{for } 0\le z\le d\\ Fe^{-\jmath k_2 z} & \text{for } z>d \end{cases} \tag{25}$$

where k_1 and k_2 are defined as $k^2-\beta^2$ in each region, so

$$k_1^2 = \epsilon_r k_0^2 - \beta^2 \qquad \mathrm{Im}\{k_1\}<0 \tag{26}$$

$$k_2^2 = k_0^2 - \beta^2 \qquad \mathrm{Im}\{k_2\}<0 \tag{27}$$

and k_0 is the free-space propagation constant. As before, the stipulation that $\mathrm{Im}\{k_z\}<0$ is required on physical grounds, and is important when performing the inverse transform (**??**). All the transverse field components are again determined by myeqrefx.

Enforcing the boundary conditions myeqrefx gives a set of algebraic equations for the coefficients (A,B,C,D,E,F). For example, the first of (23) immediately gives $B=E=0$. The remaining equations can be easily solved to give the following $\hat{z}$-components of the fields in the substrate

$$\overline{E}_{z1} = \tilde{\delta}\frac{k_2 k_x \eta_0}{k_0 T_m}\cos k_1 z \tag{28a}$$

$$\overline{H}_{z1} = -\tilde{\delta}\frac{\jmath k_y}{T_e}\sin k_1 z \tag{28b}$$

where $T_e = 0$ and $T_m = 0$ are the characteristic equations for the TE and TM surface wave modes for a grounded substrate, given by

$$T_e = k_1 \cos k_1 d + \jmath k_2 \sin k_1 d \tag{29a}$$

$$T_m = \epsilon_r k_2 \cos k_1 d + \jmath k_1 \sin k_1 d \tag{29b}$$

The spectral domain Green's function will have singularities corresponding to the allowed substrate modes for each problem. There are a finite number of allowed modes for a given frequency. The lowest order TM mode can propagate at all frequencies, but in many cases the substrate thickness is such a small fraction of a wavelength that this is the only mode (and hence singularity) that must be treated. We will discuss how to handle the singularity later.

The remaining field components are computed by substituting (28) into (6). The relevant components of the Green's dyadic are then

$$\tilde{G}_{xx} = -\frac{j\eta_0}{k_0}\frac{(\epsilon_r k_0^2 - k_x^2)k_2 \cos k_1 d + jk_1(k_0^2 - k_x^2)\sin k_1 d}{T_e\ T_m}\sin k_1 z \tag{30a}$$

$$\tilde{G}_{yx} = \frac{j\eta_0}{k_0}\frac{k_x k_y[k_2 \cos k_1 d + jk_1 \sin k_1 d]}{T_e\ T_m}\sin k_1 z \tag{30b}$$

$$\tilde{G}_{zx} = \frac{k_2 k_x \eta_0}{k_0 T_m}\cos k_1 z \tag{30c}$$

The fields produced by a $\hat{y}$-directed dipole are obtained from the above through the substitutions $k_x \to k_y$ and $k_y \to -k_x$. For convenience, these are $\tilde{G}_{xy}$ and $\tilde{G}_{yy}$, given by

$$\tilde{G}_{xy} = -\tilde{G}_{yx} \tag{31a}$$

$$\tilde{G}_{yy} = -\frac{j\eta_0}{k_0}\frac{(\epsilon_r k_0^2 - k_y^2)k_2 \cos k_1 d + jk_1(k_0^2 - k_y^2)\sin k_1 d}{T_e\ T_m}\sin k_1 z \tag{31b}$$

$$\tilde{G}_{zy} = \frac{k_2 k_y \eta_0}{k_0 T_m}\cos k_1 z \tag{31c}$$

2 RIGOROUS ANALYSIS OF PATCH ANTENNAS

Once the dyadic Green's function is known, the formulation of a solution in terms of an integral equation is relatively straightforward. The technique is nicely illustrated by application to the microstrip patch antenna, which was first discussed by Pozar [1]. The goal of the analysis is, as usual, to find the unknown current distribution on the antenna, from which we can determine the driving-point impedance and radiation characteristics. Since

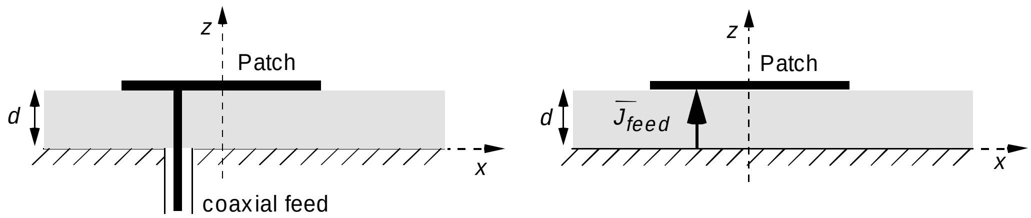

Figure 4 Patch antenna with coaxial feed (left), and an approximate equivalent (right) known as the *idealized probe feed*. The probe feed model can be used for microstrip edge fed patches as well.

our purpose at this point is merely to illustrate the theory, we also make some simplifying assumptions: a) the conductors are perfect and of negligible thickness, b) the substrate has no loss, and c) the probe feed is modelled as a uniform 1 A filament of current extending through the substrate at the location (x_p, y_p), so that

$$\overline{J}_{feed}(x', y', z') = \begin{cases} \hat{z}\delta(x' - x_p)\delta(y' - y_p) & 0 \leq z \leq d \\ 0 & \text{otherwise} \end{cases} \tag{32}$$

This is the so-called "idealized probe feed", and is a rough approximation to a coaxially fed patch when the substrate is electrically thin. Note that each of these assumptions can be relaxed if necessary without significantly changing the analytical procedure, but at a cost of additional computational (and notational) burden.

The problem can be formulated as a surface integral equation. The electric field must satisfy the boundary condition

$$\hat{z} \times (\overline{E}^{inc} + \overline{E}^{scatt}) = 0 \quad \text{on } S \tag{33}$$

where S denotes the patch surface defined by $|x| \leq L/2, |y| \leq W/2, z = d$. The fact that only tangential field components are required allows us to ignore several components of the Green's dyadic. The incident field $\overline{E}^{inc}$ is the field produced by the vertical feed current J_{feed}, and can be written as

$$\overline{E}^{inc} = \iiint \overline{\overline{G}} \cdot \overline{J}_{feed}\, dV' \tag{34}$$

Substituting for the feed current (32) and using the spectral representation of the Green's function gives

$$\overline{E}^{inc} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \int_0^d \tilde{\overline{\overline{G}}} \cdot \hat{z}\, dz'\, e^{jk_x(x-x_p)} e^{jk_y(y-y_p)}\, dk_x\, dk_y \tag{35}$$

Since only tangential components of $\overline{E}^{inc}$ on the patch surface are required in (33), the relevant parts of the dyadic Green's function for (35) are just G_{xz} and G_{yz}, with the observation point on the patch surface ($z = d$). We have not explicitly treated this case in our previous examples, but these can be obtained by reciprocity using the results derived for a horizontal dipole on the substrate surface: the $\hat{x}$ directed field on the surface due to a $\hat{z}$-directed dipole in the substrate, G_{xz}, is equal to the $\hat{z}$-directed field due to an $\hat{x}$-directed source on the substrate surface, G_{zx} from (30). The integral over z' in (35) can then easily be done to give

$$\overline{E}_t^{inc} = E_x^{inc}\hat{x} + E_y^{inc}\hat{y} \tag{36}$$

where

$$\begin{aligned} E_x^{inc}(x,y,d) &= \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \tilde{E}_{xz}^{inc}(k_x,k_y)e^{jk_x x}e^{jk_y y}dk_x dk_y \\ E_y^{inc}(x,y,d) &= \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \tilde{E}_{yz}^{inc}(k_x,k_y)e^{jk_x x}e^{jk_y y}dk_x dk_y \end{aligned}$$

and

$$\begin{aligned} \tilde{E}_{xz}^{inc} &= \frac{-\eta_0 k_x k_2 \sin k_1 d}{k_0 k_1 T_m} e^{-jk_x x_p} e^{-jk_y y_p} \\ \tilde{E}_{yz}^{inc} &= \frac{-\eta_0 k_y k_2 \sin k_1 d}{k_0 k_1 T_m} e^{-jk_x x_p} e^{-jk_y y_p} \end{aligned}$$

The probe induces surface currents $\overline{J}_s$ on the patch which must satisfy the boundary condition (33). The scattered field due to the currents on the patch are given by

$$\overline{E}^{scatt} = \iint_{S'} \overline{\overline{G}} \cdot \overline{J}_s \, dS' \tag{37}$$

where the integration is over the surface of the patch. Since the current is constrained to the surface and only tangential scattered fields are required, the relevant components of the Green's dyadic for (37) are

$$\overline{\overline{G}}_t \Rightarrow \hat{x}G_{xx}\hat{x} + \hat{y}G_{yx}\hat{x} + \hat{x}G_{xy}\hat{y} + \hat{y}G_{yy}\hat{y} \tag{38}$$

with the observation point again on the surface $z = d$. These have been previously evaluated (30)-(31). The integral equation is

$$\overline{E}_t^{inc} = -\iint_{S'} \overline{\overline{G}}_t \cdot \overline{J}_s \, dS' \quad \text{on patch surface, } S \tag{39}$$

Following the Method of Moments procedure we expand the unknown patch currents in a set of vector basis functions as

$$\overline{J}_s(x',y') = \sum_{j=1}^{N} I_j \overline{\psi}_j(x',y') \tag{40}$$

so that (39) becomes

$$\overline{E}_t^{\text{inc}}(x,y) = -\sum_{j=1}^{N} I_j \iint_{S'} \overline{\overline{G}}_t(x,y|x',y') \cdot \overline{\psi}_j(x',y') \, dS' \tag{41}$$

Now multiply both sides by weighting functions which we will take to be the same as the basis functions (Galerkin's method), and integrate over the patch surface to give

$$\iint_S \overline{\psi}_i(x,y) \cdot \overline{E}_t^{inc}(x,y) \, dS = -\sum_{j=1}^{N} I_j \iint_S \iint_{S'} \overline{\psi}_i(x,y) \cdot \overline{\overline{G}}_t(x,y|x',y') \cdot \overline{\psi}_j(x',y') \, dS' \, dS \tag{42}$$

This can be written in matrix form as

$$\overline{V} = \overline{\overline{Z}} \cdot \overline{I} \tag{43}$$

where

$$Z_{ij} = -\iint_S \iint_{S'} \overline{\psi}_i(x,y) \cdot \overline{\overline{G}}_t(x,y|x',y') \cdot \overline{\psi}_j(x',y')\, dS'\, dS \tag{44a}$$

$$V_i = \iint_S \overline{\psi}_n(x,y) \cdot \overline{E}_t^{inc}(x,y)\, dS \tag{44b}$$

Now we can insert the spectral representation of the incident field from (36), and the Green's function from (8). The impedance matrix elements can then be written in the form

$$Z_{ij} = -\frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \left[\iint_S \overline{\psi}_i(x,y) e^{jk_x x} e^{jk_y y}\, dS \right] \cdot \tilde{\overline{\overline{G}}}_t(k_x,k_y) \tag{45}$$

$$\cdot \left[\iint_{S'} \overline{\psi}_j(x',y') e^{-jk_x x'} e^{-jk_y y'}\, dS' \right] dk_x dk_y \tag{46}$$

Define the Fourier transform of the expansion currents as

$$\overline{F}_i(k_x,k_y) = \int_{x'} \int_{y'} \overline{\psi}_i(x',y') e^{-jk_x x'} e^{-jk_y y'} dx' dy' \tag{47}$$

For simple basis functions these integrals can often be done analytically. Assuming the basis functions are real valued, the matrix elements for (43) become

$$Z_{ij} = -\frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \overline{F}_i^*(k_x,k_y) \cdot \tilde{\overline{\overline{G}}}_t(k_x,k_y) \cdot \overline{F}_j(k_x,k_y)\, dk_x\, dk_y \tag{48}$$

$$V_i = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \overline{F}_i^*(k_x,k_y) \cdot \tilde{\overline{E}}_t^{inc}\, dk_x\, dk_y \tag{49}$$

For example, if there is only a single basis function in the $\hat{x}$ direction, then the weighting function is also in the $\hat{x}$ direction and we get

$$Z_{11} = -\frac{1}{4\pi^2} \iint_{-\infty}^{\infty} F_1^*(k_x,k_y) \tilde{G}_{xx} F_1(k_x,k_y)\, dk_x\, dk_y \tag{50}$$

$$V_1 = -\frac{1}{4\pi^2} \iint_{-\infty}^{\infty} F_1^*(k_x,k_y) \tilde{E}_{xz}^{inc}\, dk_x\, dk_y \tag{51}$$

where $\tilde{G}_{xx}$ is found in (30) and $\tilde{E}_{xz}^{inc}$ is given in (36). Once the patch currents are known, the input impedance can be calculated as

$$Z_i n = \frac{V_{in}}{I_{in}} = -\int_0^d \overline{E}^{scatt} \cdot \hat{z}\, dz \tag{52}$$

$$= -\sum_{j=1}^{N} I_j V_j \tag{53}$$

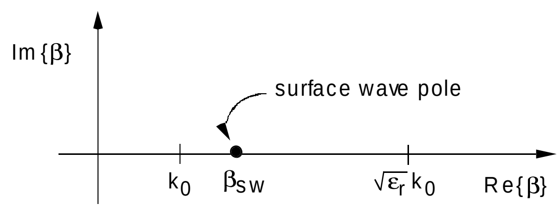

Figure 5 Singularity of the spectral Green's function occurs at surface wave poles in the range $k_0 \le \beta_{\rm sw} \le \sqrt{\epsilon_r} k_0$.

2.1 Computational considerations

All of the matrix elements (49) must be evaluated numerically. This is simplified by conversion to polar coordinates in the spectral domain through

$$k_x = \beta \cos\alpha \qquad k_y = \beta \sin\alpha \tag{54}$$

(this is consistent with our previous notation $\beta^2 = k_x^2 + k_y^2$) so that the integrals become

$$\iint_{-\infty}^{\infty} H(k_x, k_y)\, dk_x\, dk_y = \int_0^{\infty} \int_0^{2\pi} H(k_x, k_y) \beta\, d\beta\, d\alpha \tag{55}$$

The integral in β can usually be truncated at around $200k_0$, but one should experiment with this value for proper convergence of the results. The α integral can be reduced to a range of $0 \to \pi/2$ using symmetry arguments, which also speeds the calculation.

The tricky part of the numericaly evaluation of the integrals involves dealing with the singularities of the Green's function, which correspond to surface wave poles. These occur for critical values of β such that $T_e(\beta_{\rm sw}) = 0$ or $T_m(\beta_{\rm sw}) = 0$ from (29). For lossless substrates these occur in the range $k_0 \le \beta_{\rm sw} \le \sqrt{\epsilon_r} k_0$, and can be found using a root finding procedure such as the Newton-Rhapson method. For lossy substrates the poles move off the real β axis into the lower half plane, so in principal there is no singularity since the integration is over real values of β. However, in practice the loss is usually so small that the argument of the integrand still gets very large as β approaches the pole, so it is advisable to treat this part of the integral carefully even in the lossy case. The singularity term(s) can be integrated analytically by separating out a small region of the integration around the pole(s). For example, if only one surface wave pole is present, corresponding to a zero of T_m, then we can write

$$\iint_{-\infty}^{\infty} H(k_x, k_y)\, dk_x\, dk_y \quad = \tag{56}$$

$$\int_0^{2\pi} \left[\int_0^{\beta_{\rm sw}-\delta} ()\, d\beta + \int_{\beta_{\rm sw}-\delta}^{\beta_{\rm sw}+\delta} \frac{f(\beta,\alpha)}{T_m(\beta)}\, d\beta + \int_{\beta_{\rm sw}+\delta}^{\infty} ()\, d\beta \right] d\alpha \tag{57}$$

where δ is a small number, $\delta \approx 0.001k_0$. Expanding $T_m(\beta)$ in a Taylor series about the pole gives

$$\int_{\beta_{\rm sw}-\delta}^{\beta_{\rm sw}+\delta} \frac{f(\beta,\alpha)}{T_m(\beta)}\, d\beta \simeq \frac{-\jmath\pi f(\beta_{\rm sw},\alpha)}{T'_m(\beta_{\rm sw})} \tag{58}$$

If more than one pole is present this same procedure can be used for each pole. The lossy case can be treated similarly.

The procedures outlined above can be easily extended to other types of antennas, more complicated patch geometries, and phased arrays. Pozar has written a series of papers on the subject using this spectral domain technique; representative papers are listed in the references [1]-[3].

REFERENCES

[1] D.M. Pozar, “Input impedance and mutual coupling of rectangular microstrip antennas”, *IEEE Trans. Antennas & Propagation*, vol. AP-30, pp. 1191-1196, Nov 1982.

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