Controlling Chaos: Koopman LQR with Nyström Kernel Approximation

Full Lecture Script

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This is the full script for the Master's degree lecture, organized by slide number, followed by the answers to the quizzes.

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Presentation Script

I. Introduction: Why Can't a Robot Fold a T-Shirt?

Slide 1: Title
Speaker: Good morning, everyone. The title of the paper we are covering today is: *Linear quadratic control of nonlinear systems with Koopman operator learning and the Nyström method*. This work was published in *Automatica* by Caldarelli, Chatalic, Colomé, and colleagues.

But before we dive into the math, let's start with a simple question: Have you ever thought why a robot can build an entire car with incredible precision, but it can't fold a t-shirt?

Slide 2: Defining the Challenge: Nonlinear Chaos
Speaker: This question gets right to the heart of robotics. It has nothing to do with the robot's strength or precision. The problem is the object itself. A piece of cloth is an unpredictable floppy mess: a nonlinear dynamical system.

Linear systems are predictable—like pushing a ball down a ramp. Nonlinear systems are chaotic: a tiny input can lead to a massive unexpected output. Standard control methods fail in this context.

Slide 3: The Breakthrough Solution
Speaker: The researchers propose combining:

1. The Koopman Operator (linearizing dynamics).
2. Kernel Methods (universal approximators).
3. The Nyström Approximation (efficient shortcut).

Main result: the Riccati operator converges at rate $m^{-1/2}$, while the regulator objective converges faster at m^{-1}.

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II. Prerequisite Knowledge

Slide 4: Systems Notation
Speaker: We study $x_{t+1} = f(x_t, u_t)$, focusing on systems affine in input: $f(x,u) = g(x) + Bu$. Augmented state: $w = [x, u]$.

Slide 5: Koopman Operator
Speaker: The Koopman operator transforms our view of the system using nonlinear observables ξ. In lifted space, dynamics are linear: $(K\xi)(w_t) = \xi(w_{t+1})$.

Slide 6: RKHS
Observables live in a Reproducing Kernel Hilbert Space H. Infinite-dimensional and universal, but computationally infeasible.

Slide 7: LQR
Once lifted, apply LQR: minimize quadratic cost. Solution: state-feedback gain K via the Discrete Algebraic Riccati Equation (DARE).

Slide 8: Quiz 1

1. How does the Koopman operator turn nonlinear dynamics into linear ones?
2. What is the main property of the control input u?

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III. System Identification and Nyström Shortcut

Slide 9: RKHS Challenge
We define a lift $\phi(w)$ and seek G_γ with $z_{t+1} = G_\gamma \phi(w_t)$. Infinite-dimensional, intractable.

Slide 10: Nyström Approximation
Sample m landmarks, project onto finite subspaces, build $\widetilde{G}_\gamma$.

Slide 11: Vector Dynamics
Nyström reduces to $m \times m$ matrix inversion. State reconstruction maps z back to x.

Slide 15: Quiz 2

1. What computational issue does Nyström solve?
2. What is the size of the matrix inversion required?

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IV. LQR Formulation

Slide 12: Ideal vs Real
Exact dynamics LQR (infinite) vs approximated Nyström dynamics (finite, feasible).

Slide 13: Final Control Input
Control law: $u_k = \widetilde{K} \Pi_{out,x} \psi(x_k)$.

Slide 14: Full Pipeline
1. Sample landmarks.
2. Compute map.
3. Solve LQR.

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V. Theoretical Analysis

Slide 16: Assumptions
Bounded kernel, stabilizability, detectability, Riccati condition ($\sigma_{min}(P) \geq 1$).

Slide 17: Transition Operator Accuracy
Error $\|G_\gamma - \widetilde{G}_\gamma\|$ converges at $m^{-1/2}$.

Slide 18: Riccati Operator Convergence
Error $\|P - \widetilde{P}\|$ bounded by $O(m^{-1/2})$.

Slide 19: LQR Objective Convergence
Error $J - \hat{J}$ converges at m^{-1}.

Slide 20: Quiz 3

1. If m is quadrupled, how does the LQR cost converge?
2. What adjustment must be made to Q, R?

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VI. Numerical Evaluation

Slides 21–25: Case studies: nonlinear system, Duffing oscillator, cloth manipulation. Nyström shows robustness and efficiency over splines.

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VII. Conclusion and Future Frontiers

Slide 26: Summary
Nyström method + Koopman operator = efficient control of nonlinear systems.

Slide 27: Future Work
Applications: manufacturing, surgery, creative robotics.

Slide 28: Discussion
Open questions about regularization, extending theory, spline instability.

Slide 29: Final Quiz

1. Convergence rates of $\widetilde{G}_\gamma$ and J?
2. Why splines unstable in high dimensions?

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Quiz Answers

Quiz 1:

1. Koopman lifts nonlinear dynamics into a linear space.
2. Systems are affine in control input u.

Quiz 2:

1. Nyström reduces infinite kernel computations to finite approximations.
2. Requires $m \times m$ matrix inversion.

Quiz 3:

1. Error decreases by factor of 4 (rate m^{-1}).
2. Rescale Q and R to ensure $\sigma_{min}(P) \geq 1$.

Final Quiz:

1. $\widetilde{G}_\gamma$: $m^{-1/2}$; LQR cost J: m^{-1}.
2. Splines unstable in high dimensions when centers sampled outside training data.