Multi-Period Capital Allocation Under Uncertainty

A Mixed-Integer Stochastic Programming Formulation

Capital Allocation Framework v1.0

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Abstract

We present a mixed-integer stochastic programming formulation for optimal time and resource allocation over a 30-day planning horizon. The model incorporates uncertainty in emotional recovery requirements, tracks progress on critical tasks (immigration documentation), and provides daily decision support while maintaining computational tractability. The formulation ensures feasibility through careful constraint design and provides a rolling horizon framework for continuous re-optimization.

Contents

1 Introduction

1.1 Problem Context

This formulation addresses the multi-period resource allocation problem for a researcher managing competing priorities: immigration documentation, job search, research obligations, and personal well-being. The key innovation is treating time allocation as a capital budgeting problem under uncertainty.

1.2 Model Philosophy

The model operates on three key principles:

1. **Rolling Horizon**: Optimize for 30 days, implement today's decisions
2. **State Tracking**: Immigration progress affects future allocations
3. **Stochastic Recovery**: Emotional exhaustion requires uncertain recovery time

2 Sets and Indices

Definition 1 (Time Periods)**.** *Let $\mathcal{T} = \{1, 2, \ldots, T\}$ denote the set of days in the planning horizon, where $T = 30$.*

Definition 2 (Task Categories)**.** *Let $\mathcal{I}$ denote the set of task categories:*

$$\mathcal{I} = \{GC, JOB, SCH, REST, RES, SOC, EX\} \tag{1}$$

where:

- *GC: Green card preparation*
- *JOB: Job searching activities*
- *SCH: School/teaching obligations*
- *REST: Rest and recovery*
- *RES: PhD research*
- *SOC: Social activities*
- *EX: Exercise*

Definition 3 (Uncertainty Scenarios)**.** *Let $\mathcal{S}$ denote the set of emotional recovery scenarios with associated probabilities p_s for $s \in \mathcal{S}$.*

3 Parameters

3.1 Resource Parameters

3.2 Task Parameters

For each task $i \in \mathcal{I}$ and day $t \in \mathcal{T}$:

$$e_i : \text{Energy consumption rate (units/hour)} \tag{2}$$
$$m_i : \text{Emotional cost rate (units/hour)} \tag{3}$$
$$h_i^{\min} : \text{Minimum hours if undertaken} \tag{4}$$
$$h_i^{\max} : \text{Maximum hours per day} \tag{5}$$

Table 1: Daily Resource Budgets

Parameter	Description	Value
$H^{\max}$	Maximum usable hours per day	16
$E^{\max}$	Maximum energy units per day	100
$M^{\max}$	Maximum emotional capacity	100

3.3 Return Parameters

Each task i generates returns in four dimensions:

$$\alpha_i^I : \text{Immigration progress per hour} \tag{6}$$
$$\alpha_i^Y : \text{Income/employability gain per hour} \tag{7}$$
$$\alpha_i^K : \text{Long-term capital accumulation per hour} \tag{8}$$
$$\alpha_i^O : \text{Optionality improvement per hour} \tag{9}$$

3.4 State-Dependent Parameters

$$G^{\text{total}} : \text{Total green card work required (hours)} \tag{10}$$
$$\gamma : \text{Decay rate for uncompleted GC work stress} \tag{11}$$
$$\beta_s : \text{Recovery hours needed in scenario } s \in \mathcal{S} \tag{12}$$

4 Decision Variables

4.1 Primary Decision Variables

For each task $i \in \mathcal{I}$ and day $t \in \mathcal{T}$:

$$x_{it} \in \{0, 1\} : \text{Binary decision to undertake task } i \text{ on day } t \tag{13}$$
$$h_{it} \in \mathbb{R}_+ : \text{Hours allocated to task } i \text{ on day } t \tag{14}$$

4.2 State Variables

$$g_t \in [0, 1] : \text{Proportion of green card work completed by end of day } t \tag{15}$$
$$G_t \in \mathbb{R}_+ : \text{Cumulative GC hours completed by end of day } t \tag{16}$$
$$\sigma_t \in [0, 1] : \text{Immigration security level on day } t \tag{17}$$

4.3 Auxiliary Variables

For scenario-dependent constraints:

$$r_{ts} \in \mathbb{R}_+ : \text{Recovery hours needed on day } t \text{ in scenario } s \tag{18}$$
$$z_{ts} \in \{0, 1\} : \text{Indicator for high emotional load on day } t \text{ in scenario } s \tag{19}$$

5 Objective Function

The objective maximizes expected utility over the planning horizon:

$$\max \quad \mathbb{E}_s \left[\sum_{t=1}^{T} \delta^t \left(\sum_{i \in \mathcal{I}} U_{it}(h_{it}, g_t) \right) \right] \tag{20}$$

where the utility function is:

$$\begin{aligned}
U_{it}(h_{it}, g_t) = & \, w_I(t, g_t) \cdot \alpha_i^I \cdot h_{it} && (21)\\
& + w_Y(t) \cdot \alpha_i^Y \cdot h_{it} && (22)\\
& + w_K(t) \cdot \alpha_i^K \cdot h_{it} && (23)\\
& + w_O(t) \cdot \alpha_i^O \cdot h_{it} && (24)\\
& - \lambda_1 \cdot e_i \cdot h_{it} && (25)\\
& - \lambda_2(g_t) \cdot m_i \cdot h_{it} && (26)
\end{aligned}$$

5.1 Weight Dynamics

The weights evolve based on state and time:

$$w_I(t, g_t) = w_I^0 \cdot (1 - g_t) \cdot \exp(-\rho_I t/T) \tag{27}$$
$$w_Y(t) = w_Y^0 \cdot (1 + \rho_Y t/T) \tag{28}$$
$$w_K(t) = w_K^0 \cdot (1 + \rho_K t/T) \tag{29}$$
$$w_O(t) = w_O^0 \cdot (1 + \rho_O t/T) \tag{30}$$
$$\lambda_2(g_t) = \lambda_2^0 \cdot (1 + \gamma(1 - g_t)) \tag{31}$$

6 Constraints

6.1 Time Constraints

$$\sum_{i \in \mathcal{I}} h_{it} + r_{ts} \leq H^{\max} \quad \forall t \in \mathcal{T}, s \in \mathcal{S} \tag{32}$$

6.2 Logical Constraints

$$h_{it} \geq h_i^{\min} \cdot x_{it} \quad \forall i \in \mathcal{I}, t \in \mathcal{T} \tag{33}$$
$$h_{it} \leq h_i^{\max} \cdot x_{it} \quad \forall i \in \mathcal{I}, t \in \mathcal{T} \tag{34}$$

6.3 Energy Constraints

$$\sum_{i \in \mathcal{I}} e_i \cdot h_{it} \leq E^{\max} \cdot (1 + 0.2 \cdot \sigma_t) \quad \forall t \in \mathcal{T} \tag{35}$$

6.4 Emotional Load Constraints

$$\sum_{i \in \mathcal{I}} m_i \cdot h_{it} \leq M^{\max} \cdot (1 + 0.3 \cdot \sigma_t) \quad \forall t \in \mathcal{T} \tag{36}$$

6.5 Recovery Stochastic Constraints

Emotional exhaustion indicator:

$$\sum_{i \in \mathcal{I}} m_i \cdot h_{it} \geq 0.8 \cdot M^{\max} - M^{\text{big}} \cdot (1 - z_{ts}) \quad \forall t, s \tag{37}$$

Recovery requirement:

$$r_{ts} \geq \beta_s \cdot z_{ts} \quad \forall t \in \mathcal{T}, s \in \mathcal{S} \tag{38}$$

$$r_{ts} \leq 4 \quad \forall t \in \mathcal{T}, s \in \mathcal{S} \tag{39}$$

where $\beta_s \in \{1, 2, 3, 4\}$ with probabilities $p_s = \{0.4, 0.3, 0.2, 0.1\}$.

6.6 State Evolution Constraints

Green card progress tracking:

$$G_t = G_{t-1} + h_{\text{GC},t} \quad \forall t \in \mathcal{T} \tag{40}$$

$$g_t = \min\left(1, \frac{G_t}{G^{\text{total}}}\right) \quad \forall t \in \mathcal{T} \tag{41}$$

$$G_0 = G^{\text{initial}} \tag{42}$$

Immigration security evolution:

$$\sigma_t = \min(1, g_t + 0.2 \cdot \mathbb{I}[g_t > 0.5]) \quad \forall t \in \mathcal{T} \tag{43}$$

6.7 Minimum Requirements

Critical task minimums:

$$\sum_{t=1}^{7} h_{\text{GC},t} \geq 10 \quad \text{(Weekly GC minimum)} \tag{44}$$

$$\sum_{t=1}^{7} h_{\text{SCH},t} \geq 20 \quad \text{(Weekly school obligations)} \tag{45}$$

$$h_{\text{REST},t} \geq 1 \quad \forall t \in \mathcal{T} \tag{46}$$

6.8 Non-anticipativity Constraints

For the rolling horizon implementation:

$$h_{i,1} = \hat{h}_{i,1} \quad \text{(Implement today's decision)} \tag{47}$$

7 Solution Algorithm

7.1 Rolling Horizon Procedure

7.2 Scenario Reduction

To maintain tractability, we use:

- Sample Average Approximation (SAA) with $|\mathcal{S}| = 4$ scenarios
- Probability-weighted expected value formulation
- Warm-start from previous day's solution

Algorithm 1 Rolling Horizon Implementation

Initialize $G_0 \leftarrow$ current GC completion
for each decision day d **do**
  Update parameters based on completed work
  Solve optimization model for days $[d, d+29]$
  Extract first-day decisions $\{h^*_{i,1}\}_{i\in\mathcal{I}}$
  Implement decisions for day d
  Observe actual recovery requirement
  Update $G_0 \leftarrow G_0 + h^*_{\mathrm{GC},1}$
end for

8 Model Properties

8.1 Feasibility Analysis

Proposition 1 (Feasibility Conditions)**.** *The model is feasible if and only if:*

$$\sum_{i\in\mathcal{I}} h_i^{\min} + \max_s \beta_s \le H^{\max} \tag{48}$$

Proof. The binding constraints are time availability and minimum task requirements. In the worst case, all minimum hours must be satisfied while accommodating maximum recovery. □

8.2 Computational Complexity

Theorem 1 (Problem Class)**.** *The formulation is a Mixed-Integer Linear Program (MILP) with:*

- *Binary variables:* $O(|\mathcal{I}| \cdot T \cdot |\mathcal{S}|)$
- *Continuous variables:* $O(|\mathcal{I}| \cdot T \cdot |\mathcal{S}|)$
- *Constraints:* $O(|\mathcal{I}| \cdot T \cdot |\mathcal{S}|)$

For typical parameters ($|\mathcal{I}| = 7$, $T = 30$, $|\mathcal{S}| = 4$), this yields approximately:

- 840 binary variables
- 840 continuous variables
- 1,680 constraints

This is well within the capability of modern MILP solvers (Gurobi, CPLEX).

9 Implementation Considerations

9.1 Parameter Estimation

9.2 Gurobi Implementation

Key solver settings:

- MIPGap: 0.01 (1% optimality gap)
- TimeLimit: 60 seconds
- Threads: 4
- MIPFocus: 1 (emphasize feasibility)

Table 2: Recommended Parameter Values

Task	Energy/hr	Emotion/hr	Min hrs	Max hrs
GC	15	12	1	6
JOB	12	10	1	4
SCH	10	5	2	8
REST	0	-5	1	4
RES	12	8	1	6
SOC	5	-3	0	3
EX	8	-4	0	2

9.3 Sensitivity Analysis

Critical parameters for calibration:

1. γ: Stress from incomplete GC work
2. λ_1, λ_2: Energy and emotional cost weights
3. w_I^0: Initial immigration priority weight
4. Recovery probabilities p_s

10 Model Extensions

10.1 Adaptive Learning

Incorporate Bayesian updating of recovery probabilities:

$$p_s^{(t+1)} = \frac{p_s^{(t)} \cdot \mathcal{L}(data_t|\beta_s)}{\sum_{s'} p_{s'}^{(t)} \cdot \mathcal{L}(data_t|\beta_{s'})} \tag{49}$$

10.2 Multi-Stage Stochastic Programming

Extend to multi-stage with recourse decisions:

- Stage 1: Morning allocation decisions
- Observation: Actual energy levels
- Stage 2: Afternoon reallocation

10.3 Risk Measures

Replace expectation with CVaR for risk-averse planning:

$$\max \quad \mathrm{CVaR}_\alpha \left[\sum_{t=1}^{T} \delta^t U_t \right] \tag{50}$$

11 Validation and Testing

11.1 Feasibility Checks

Before solving, verify:

1. $\sum_i h_i^{\min} \leq H^{\max} - 4$ (accommodate max recovery)
2. $\sum_i e_i \cdot h_i^{\min} \leq E^{\max}$
3. Weekly minimums achievable within 7-day windows

11.2 Solution Quality Metrics

Post-optimization validation:

- Green card completion trajectory
- Daily energy utilization rate
- Emotional load variance
- Weekly pattern stability

12 Conclusions

This formulation provides a rigorous mathematical framework for multi-period resource allocation under uncertainty. The model balances immediate priorities (immigration) with long-term objectives while maintaining computational tractability. The rolling horizon approach enables daily re-optimization based on updated state information.

12.1 Key Features

- Explicitly models uncertainty in recovery requirements
- Tracks immigration progress as state variable
- Balances four return dimensions with resource costs
- Provides implementable daily decisions
- Maintains feasibility through careful constraint design

12.2 Implementation Checklist

1. Set initial GC completion percentage
2. Calibrate energy and emotional cost parameters
3. Configure Gurobi solver settings
4. Run optimization for 30-day horizon
5. Extract and implement Day 1 decisions
6. Update state and re-optimize tomorrow

Table 3: Complete Notation Reference

Symbol	Description
Sets	
$\mathcal{T}$	Planning horizon days
$\mathcal{I}$	Task categories
$\mathcal{S}$	Recovery scenarios
Variables	
x_{it}	Binary task selection
h_{it}	Hours allocated
g_t	GC completion proportion
σ_t	Immigration security level
r_{ts}	Recovery hours
Parameters	
$H^{\max}$	Daily hour budget
$E^{\max}$	Daily energy budget
$M^{\max}$	Daily emotional budget
e_i	Energy cost rate
m_i	Emotional cost rate
α_i^j	Return rate in dimension j
$w_j(t)$	Time-varying weights
β_s	Recovery hours in scenario s
p_s	Scenario probability

A Notation Summary

B Parameter Configuration File

Example JSON configuration for Gurobi implementation:

```
{
  "horizon": 30,
  "tasks": {
    "GC": {"energy": 15, "emotion": 12, "min": 1, "max": 6,
           "returns": {"I": 1.0, "Y": 0.1, "K": 0.2, "O": 0.1}},
    "JOB": {"energy": 12, "emotion": 10, "min": 1, "max": 4,
            "returns": {"I": 0.0, "Y": 1.0, "K": 0.3, "O": 0.4}},
    ...
  },
  "scenarios": {
    "recovery_hours": [1, 2, 3, 4],
    "probabilities": [0.4, 0.3, 0.2, 0.1]
  },
  "initial_state": {
    "gc_completed": 0.3,
    "total_gc_hours": 100
  },
  "weights": {
    "immigration": 10.0,
    "income": 5.0,
```

```
    "capital": 3.0,
    "optionality": 2.0,
    "energy_cost": 0.1,
    "emotion_cost": 0.15
  }
}
```