CSE 260

BRAC University

Theorem and Postulate

- Postulates are assumed to be true and we need not prove them. They provide the starting point for the proof of a theorem.
- A theorem is a proposition that can be deduced from postulates. We make a series of logical arguments using these postulates to prove a theorem.

Binary Logic

- Binary logic consists of binary variables and logical operations.
- Variables are designated by letters such as A, B, C, x, y, z etc. with only 2 possible values: 1 and 0.
- Logic operations: and, or, not etc.

Logic Gates

- The most basic digital devices are called gates.
- A gate has one or more inputs and produces an output that is a function of the current input values.

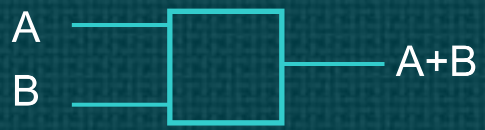
Most Important logic gates

- AND
- OR
- NOT

2-input AND gate

A	B	A.B
0	0	0
0	1	0
1	0	0
1	1	1

2- input OR gate



A	B	A + B
0	0	0
0	1	1
1	0	1
1	1	1

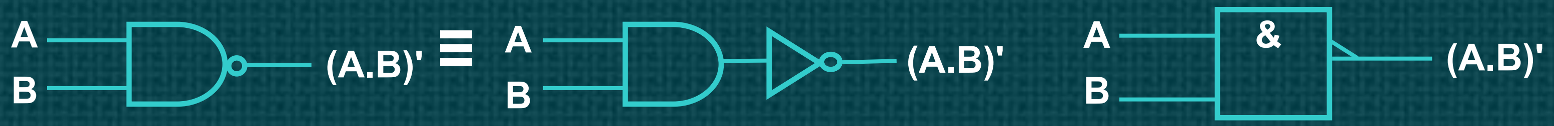
NOT gate (Inverter)

A	A'
0	1
1	0

Some Other Gates

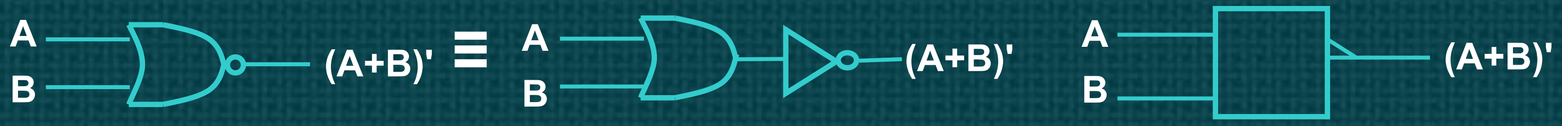
- NAND
- NOR
- XOR
- XNOR (equivalence)

2-input NAND gate



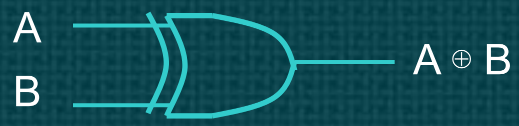
A	B	(A.B)'
0	0	1
0	1	1
1	0	1
1	1	0

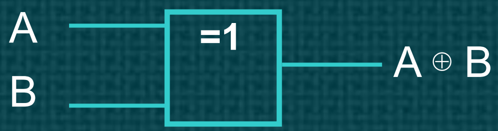
2-input NOR gate



A	B	(A+B)'
0	0	1
0	1	0
1	0	0
1	1	0

2-input XOR gate





A	B	$A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0

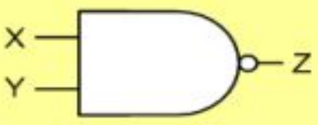
2-input XNOR gate

A	B	A⊙B
0	0	1
0	1	0
1	0	0
1	1	1

Universal gates

NAND

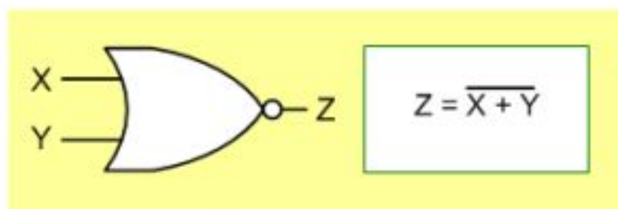
X	Y	NAND
0	0	1
0	1	1
1	0	1
1	1	0



$$Z = \overline{X . Y}$$

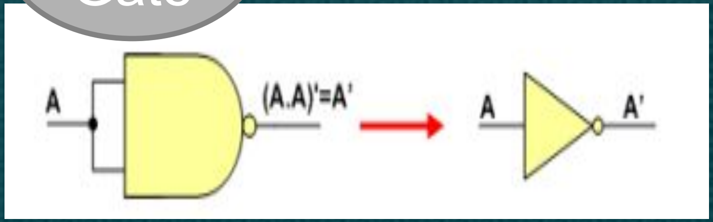
NOR

X	Y	NOR
0	0	1
0	1	0
1	0	0
1	1	0

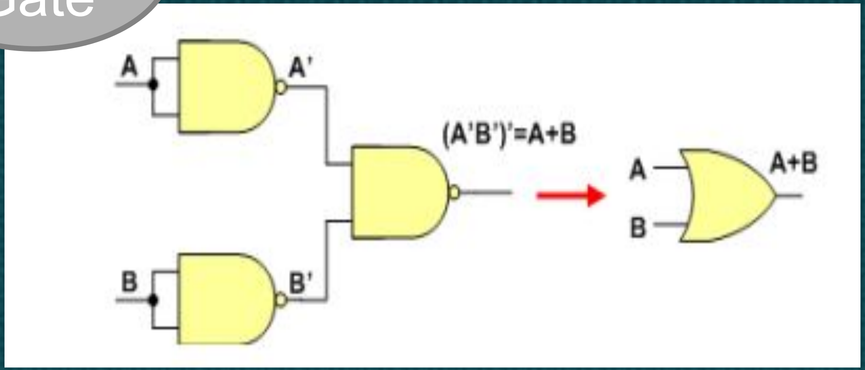


Using NAND

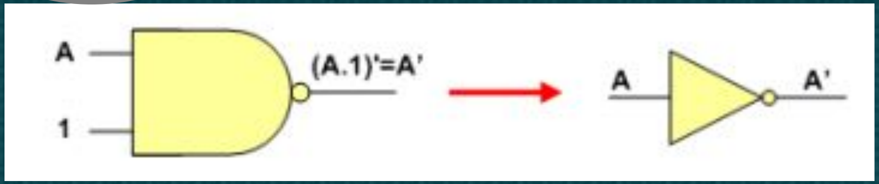
NOT Gate



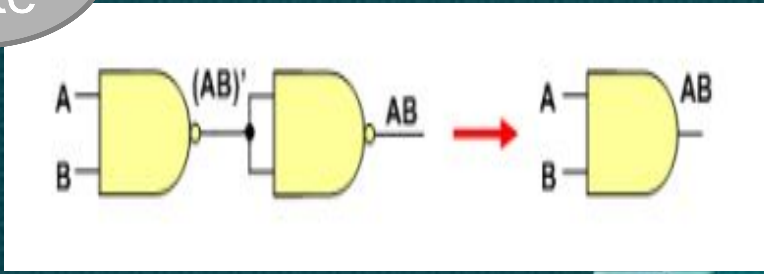
OR Gate



NOT Gate

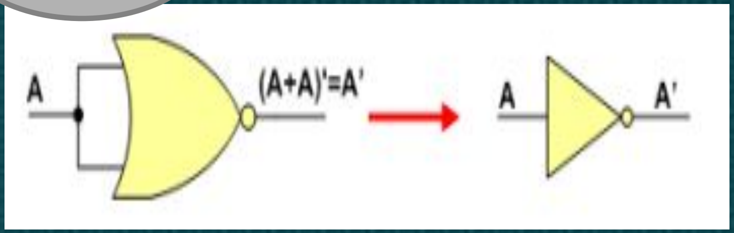


AND Gate

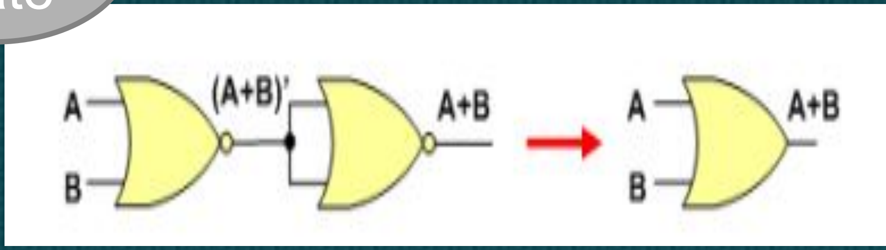


Using NOR

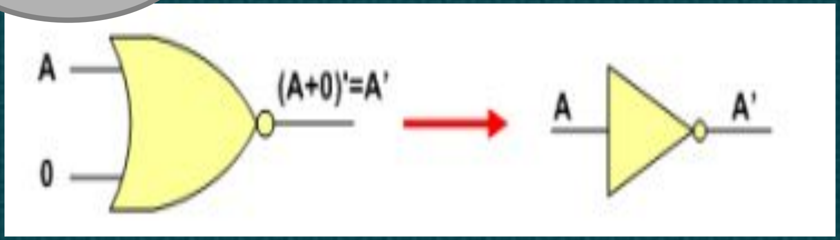
NOT Gate



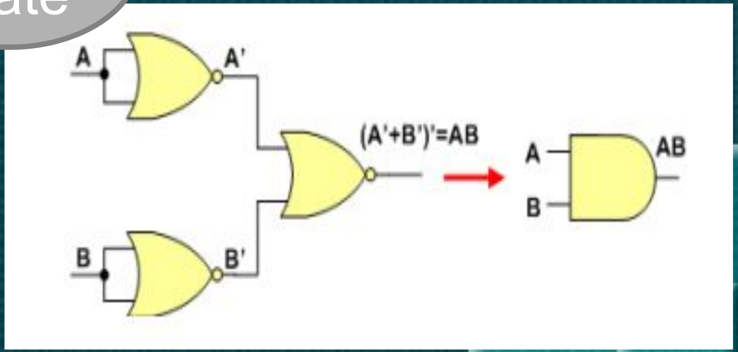
OR Gate



NOT Gate



AND Gate



Truth Table

- Provides a **listing** of every possible combination of inputs and its corresponding outputs.

INPUTS	OUTPUTS
…	…
…	…

- Example (2 inputs, 2 outputs):

x	y	x . y	x + y
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

Proof using Truth Table

- Prove that: x . (y + z) = (x . y) + (x . z)

(i) Construct truth table for LHS & RHS of above equality.

Note: if there are 3 variable, truth table should have 2^n combination of input

x	y	z	y + z	x.(y + z)	x.y	x.z	(x.y)+(x.z)
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

(ii) Check that LHS = RHS

Postulate is SATISFIED because output column 5 & 8 (for LHS & RHS expressions) are equal for all cases.

BOOLEAN ALGEBRA

Boolean Algebra

- Like any other deductive mathematical system, defined with a set of elements, a set of operators and a number of axioms or postulates.

- In Boolean algebra, set consists at least 2 variables say x & y, with 2 binary operations {+} and {.} and 1 unary operation {‘}

Boolean algebra Postulates

- **Closure**: For every x, y in *B [let, B is the set]*,
  - x + y is in *B*
  - x . y is in *B*

- **Commutative laws**: For every x, y in *B*,
  - x + y = y + x
  - x . y = y . x
- **Complement**: For every x in *B*, there exists an element x' in *B* such that
  - x + x' = 1
  - x . x' = 0

Boolean algebra Postulates

- **Associative laws**: For every x, y, z in *B*,
  - $(x + y) + z = x + (y + z) = x + y + z$
  - $(x \, . \, y) \, . \, z = x \, .(y \, . \, z) = x \, . \, y \, . \, z$
- **Identities** (0 and 1):
  - $0 + x = x + 0 = x$ for every x in *B*
  - $1 \, . \, x = x \, . \, 1 = x$ for every x in *B*
- **Distributive laws**: For every x, y, z in *B*,
  - $x \, . \, (y + z) = (x \, . \, y) + (x \, . \, z)$
  - $x + (y \, . \, z) = (x + y) \, . \, (x + z)$

Duality

- **Duality Principle** – every valid Boolean expression (equality) remains valid if the operators and identity elements are interchanged, as follows:

  $+ \leftrightarrow .$

  $1 \leftrightarrow 0$

- Example: Given the expression

  $$a + (b.c) = (a+b).(a+c)$$

  then its dual expression is

  $$a \,.\, (b+c) = (a.b) + (a.c)$$

Duality

- Duality gives free theorems – “two for the price of one”. You prove one theorem and the other comes for free!
- If (x+y+z)' = x'.y.'z' is valid, then its dual is also valid:

  (x.y.z)' = x'+y'+z’
- If x + 1 = 1 is valid, then its dual is also valid:

  x . 0 = 0

Basic Theorems of Boolean Algebra

- Postulate 5 (a) $x+0=x$ (b) $x.1=x$ identity
- Postulate 3 (a) $x+x'=1$ (b) $x.x'=0$ complement
- Th 1 (a) $x+x=x$ (b) $x.x=x$
- Th 2 (a) $x+1=1$ (b) $x.0=0$
- Th 3, involution $(x')'=x$
- Pos 2 (a) $x+y=y+x$ (b) $xy=yx$ commutative
- Th 4 (a) $x(yz)=(xy)z$ (b)$x+(y+z)=(x+y)+z$
- Pos 6 (a) $x(y+z)=xy+xz$ (b) $x+yz=(x+y)(x+z)$ Distributi- -ve
- Th 5, DeMorgan (a) $(x+y)'=x'y'$ (b) $(xy)'=x'+y'$
- Th 6, Absorption (a) $x+xy=x$ (b) $x(x+y)=x$

All are very very important!

Basic Theorems of Boolean Algebra

- Theorems can be proved using the truth table method. (Exercise: Prove De-Morgan’s theorem using the truth table.)
- They can also be proved by algebraic manipulation using axioms/postulates or other basic theorems.

- Theorem 2a can be proved by:

$$
\begin{aligned}
x + 1 &= x+(x+x') \quad \text{(complement)} \\
&= (x+x)+x' \quad \text{(Th. 4)} \\
&= x+x' \quad \text{(complement)} \\
&= 1
\end{aligned}
$$

- By duality, theorem 2b:

$$x.(0)= 0$$

- *Note: There can be other ways of making this proof. See Morris Mano*

Basic Theorems of Boolean Algebra

- Theorem 6a (absorption) can be proved by:

$$
\begin{aligned}
x + x.y &= x.1 + x.y && \text{(identity)} \\
&= x.(1 + y) && \text{(distributivity)} \\
&= x.(y + 1) && \text{(commutativity)} \\
&= x.1 && \text{(Theorem 2a)} \\
&= x && \text{(identity)}
\end{aligned}
$$

- By duality, theorem 6b:

$$x.(x+y) = x$$

- Try prove this by algebraic manipulation.

Operator Precedence

- Parenthesis
- NOT
- AND
- OR

Highest

Lowest

Ven Diagram

- ven diagram for boolean algebra

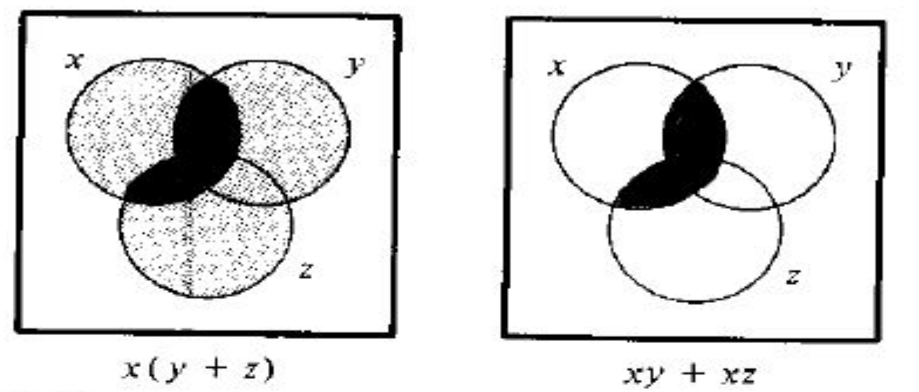


FIGURE 2-3
Venn diagram illustration of the distributive law

Boolean Functions (Solve ?)

- Examples:

  $F1 = xyz'$

  $F2 = x + y'z$

  $F3 = (x'y'z) + (x'yz) + (xy')$

  $F4 = xy' + x'z$

x	y	z	F1	F2	F3	F4
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	0	0	0	0
0	1	1	0	0	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	1	0	0
1	1	1	0	1	0	0

From the truth table, F3=F4.
Can you also prove by algebraic manipulation that F3=F4?

- $F3=(x'y'z)+(x'yz)+(xy')$

$$
\begin{aligned}
&= x'y'z+x'yz+xy' \\
&=x'z(y'+y)+xy' \\
&=x'z(1)+xy' \\
&=x'z+xy' \\
&=F4
\end{aligned}
$$

Try it yourself

a)Simplify to minimum literals: xy+xy'

b)Reduce to 4 literals(variables) : BC+AC'+AB+BCD

TABLE 2-1
Postulates and Theorems of Boolean Algebra

Postulate 2	(a) $x + 0 = x$	(b) $x \cdot 1 = x$
Postulate 5	(a) $x + x' = 1$	(b) $x \cdot x' = 0$
Theorem 1	(a) $x + x = x$	(b) $x \cdot x = x$
Theorem 2	(a) $x + 1 = 1$	(b) $x \cdot 0 = 0$
Theorem 3, involution	$(x')' = x$	
Postulate 3, commutative	(a) $x + y = y + x$	(b) $xy = yx$
Theorem 4, associative	(a) $x + (y + z) = (x + y) + z$	(b) $x(yz) = (xy)z$
Postulate 4, distributive	(a) $x(y + z) = xy + xz$	(b) $x + yz = (x + y)(x + z)$
Theorem 5, DeMorgan	(a) $(x + y)' = x'y'$	(b) $(xy)' = x' + y'$
Theorem 6, absorption	(a) $x + xy = x$	(b) $x(x + y) = x$

Solution

- A) $xy+xy'=x(y+y')=x(1)=x$
- B) $BC+AC'+AB+BCD$

  $=BC(1+D)+AC'+AB$

  $=BC(1)+AC'+AB$

  $=BC+AB+AC'$

  $=B(C+A)+AC'$

Try it yourself: simplify the following equations

1. x+x'y
2. x(x'+y)
3. x'y'z+x'yz+xy'

TABLE 2-1
Postulates and Theorems of Boolean Algebra

Postulate 2	(a) $x + 0 = x$	(b) $x \cdot 1 = x$
Postulate 5	(a) $x + x' = 1$	(b) $x \cdot x' = 0$
Theorem 1	(a) $x + x = x$	(b) $x \cdot x = x$
Theorem 2	(a) $x + 1 = 1$	(b) $x \cdot 0 = 0$
Theorem 3, involution	$(x')' = x$	
Postulate 3, commutative	(a) $x + y = y + x$	(b) $xy = yx$
Theorem 4, associative	(a) $x + (y + z) = (x + y) + z$	(b) $x(yz) = (xy)z$
Postulate 4, distributive	(a) $x(y + z) = xy + xz$	(b) $x + yz = (x + y)(x + z)$
Theorem 5, DeMorgan	(a) $(x + y)' = x'y'$	(b) $(xy)' = x' + y'$
Theorem 6, absorption	(a) $x + xy = x$	(b) $x(x + y) = x$

Solution

1. $x+x'y=(x+x').(x+y)=1.(x+y)=x+y$
2. $x(x'+y)=xx'+xy=0+xy=xy$
3. $x'y'z+x'yz+xy' =$
$x'z(y'+y)+xy'=x'z+xy'$

Now Try Proving Using Truth Table!!!

Complementing a function

1. Take dual of the function
2. Complement each literals

<u>Example:</u> $F1 = x'yz' + x'y'z$

1. Dual of the function F1 is $(x'+y+z')(x'+y'+z)$
2. Complement each literal= $(x+y'+z)(x+y+z')$

Therefore, $F1' = (x+y'+z)(x+y+z')$

Same as applying De-Morgan's law on the function

Try it urself

- What is the complement of $F2=x(y'z'+yz)$

Solution

- $F2' = (x(y'z'+yz))'$

- Duality: $x+(y'+z')(y+z)$
- Complement= $x'+(y'+z')(y+z)$

Therefore $F2' = x'+(y+z)(y'+z')$

More Practice:

Simplify the following Boolean expression to a minimum number literals:

- a) $xy + xy'$
- b) $(x + y)(x + y')$
- c) $xyz + x'y + xyz'$
- d) $(A+B)'(A'+B')'$

Solution

a) $xy + xy' = x(y+y') = x.1 = x$

b) $(x+y)(x+y') = xx + xy' + yx + yy' = x + xy' + xy + 0 = x(1 + y' + y) = x.1 = x$

Also $(x+y)(x+y') = x + yy' = x + 0 = x$

c) $xyz + x'y + xyz' = xy(z+z') + x'y = xy + x'y = y(x+x') = y$

d) $(A+B)'(A'+B')' = (A'B').(AB) = 0$

Practice! Practice! Practice!

Find the complement of the following expressions:

- a) $xy'+x'y$
- b) $(AB'+C)D'+E$
- c) $(x+y'+z)(x'+z')(x+y)$

Solution

a) $[xy'+x'y]' = (xy')' . (x'y)' = (x'+y).(x+y') = xx' + yy' + xy+x'y'=xy+x'y'$

b) $[(AB'+C)D'+E]' = [(AB'+C)D']'.E' = [(AB'+C)'+D] . E'= [(A'+B).C' +D].E'$

c) $[(x+y'+z)(x'+z')(x+y)]' = (x+y'+z)'+(x'+z')'+(x+y)'= x'yz' + xz + x'y'$

Practice time

- Solve : 2-5, 2-6, 2-11
  (Morris Mano chapter 2)