

Hidden Beauty in Penrose Tiling: Weavings & Lace

Douglas G. Burkholder
 Lenoir-Rhyne University
 625 7th Avenue NE
 Hickory, North Carolina, 28601
 Douglas.Burkholder@LR.edu

Abstract

We explore Penrose's tiling using kites and darts in search for hidden beauty. We focus on the iterative subdivision process that can be used to create Penrose tilings by subdividing half kites and half darts into smaller half kites and half darts. By selectively coloring the half darts and half kites, based only on their relative position in the subdivision process, we create 15 unexpected and distinctive patterns hidden within Penrose tiling. These patterns tend to have the appearance of a weaving. Alternately, by selectively discarding tiles as we recursively subdivide, we can obtain fractal patterns that appear lacelike. We finish by filling the negative spaces in the fractal patterns with pursuit curves.

Introduction

In 1974, Roger Penrose discovered a non-periodic tiling of the plane using only two tiles [3]. While there are three versions of Penrose tilings, we shall focus on the Penrose tiling using *kites* and *darts* as shown in Figure 1. Penrose tiling is quite familiar to mathematicians and artists and has been used in architecture around the world. See the "Beautiful Penrose Tile in Architecture" [1] for just a few of these.

Rather than just considering the final Penrose tiling, in this paper we create and explore three different rules for the painting the tiles based upon a *process of subdivision* that can be used to create the tiling. This subdivision was discovered by Raphael Robinson in 1975 [3]. This method, called *inflation*, starts with an initial group of tiles and uses a substitution rule to replace each tile with smaller tiles. Any finite region of the plane can be tiled by recursively applying this process a finite number of times to a small group of tiles and then stretching the resulting image to cover the desired region.

This subdivision process is central to our painting rules for this paper. For the first method, we determine whether or not to color a tile based upon the last step of the subdivision process. In the second method, we use a sequence of subdivision steps to determine which tiles to discard in a process similar to discarding triangles in the construction of the Sierpiński Triangle. In a previous paper by the author, it was observed that when these two methods are applied to Radin-Conway's Pinwheel Tiling, the first method created images that appeared to be woven and the second method produced images that appeared to be lace [2]. Our third method, motivated by artwork by David Reimann, fills in the negative space of the second method with pursuit curves.

Before we begin painting kites and darts, we need to understand Robinson's subdivision process.

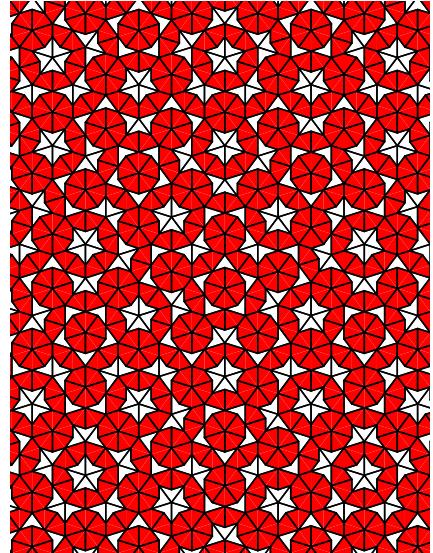


Figure 1: A Penrose tiling with dark kites and white darts.

Penrose Subdivision and our Labeling of the Five Tile Types

Kites, darts, and subdivision. We explore Penrose's tiling that uses the kites and darts shown in Figure 2. All angles are multiples of 36° , that is, $1/10^{\text{th}}$ of a circle. Specifically, the kite consist of the angles $2/5^{\text{th}}$, $1/5^{\text{th}}$, $1/5^{\text{th}}$, and $1/5^{\text{th}}$ of the circle and the dart consist of the angles $1/5^{\text{th}}$, $1/10^{\text{th}}$, $3/5^{\text{th}}$, and $1/10^{\text{th}}$ of the circle. Robinson's subdivision process replaces each tile with smaller tiles. Three generations are shown in Figure 3.

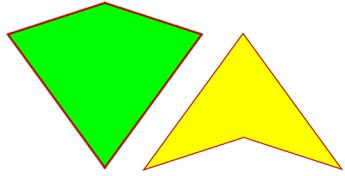


Figure 2: A kite and a dart.

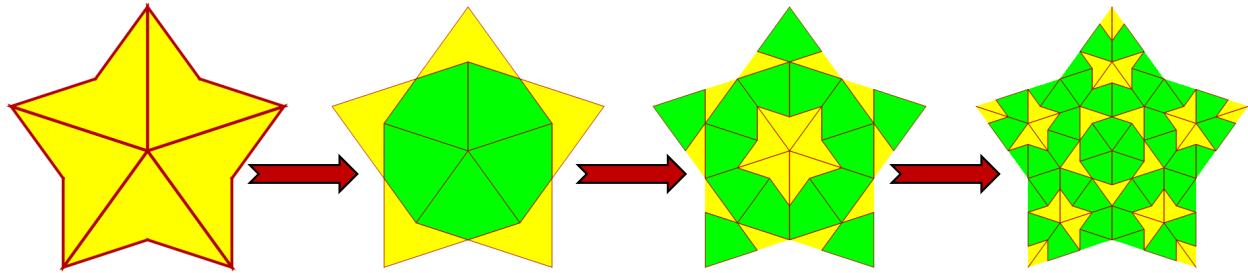


Figure 3: An initial group of tiles, called generation 0, followed by generations 1, 2, and 3.

Subdivision and tile types. The key to creating the above sequence is to subdivide *half kites* and *half darts* into smaller *half kites* and *half darts* as shown in Figure 4. For our purposes, we need to keep track of and label all half kites and half darts as shown in the same figure. Hence, after every subdivision, every tile has a label. Figure 5 shows the resulting labeling for the third image in Figure 3. We shall call these the tile *types*. Hence, this paper explores painting algorithms for the five tile types rather than just exploring painting algorithms for two tiles, kites and darts.

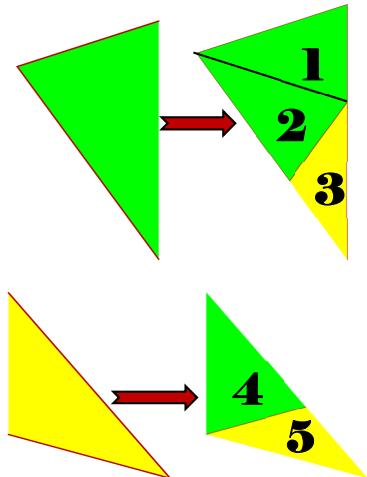


Figure 4:
Robinson's subdivision of
Penrose tiles and our labeling of
the five tile types.

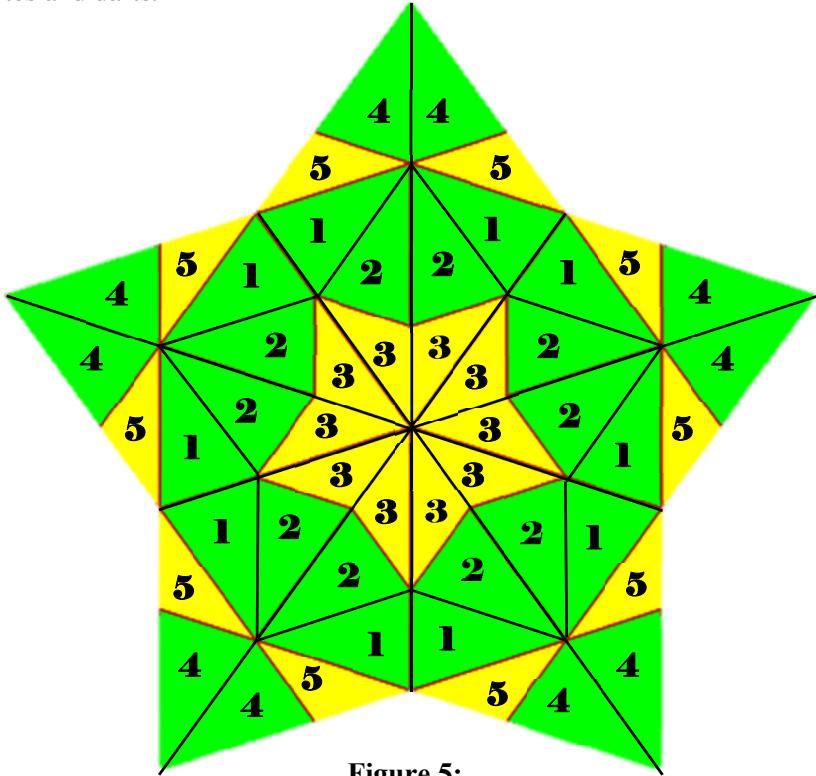


Figure 5:
The tile types labeled at the second generation.

Using Paint-by-Number to Create a Penrose Weaving

To create a Penrose weaving, we simply use *paint-by-number*. We first iterate our region to the desired size of tiny tiles. Both Figure 6 and Figure 7 show unpainted tiles at the 8th generation but the actual artwork is iterated to the 11th generation. Next, divide the image into regions and decide which tile types to paint within each region. In Figure 6, we created five spirals and numbered the regions 1 through 5. We then flip the spirals and the numbers across the central vertical axis to produce the pinecone pattern of regions shown in Figure 7.

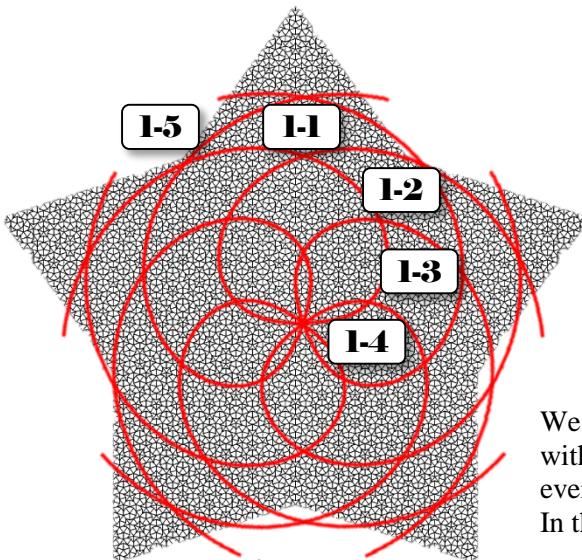


Figure 7:
The painting scheme unpainted.

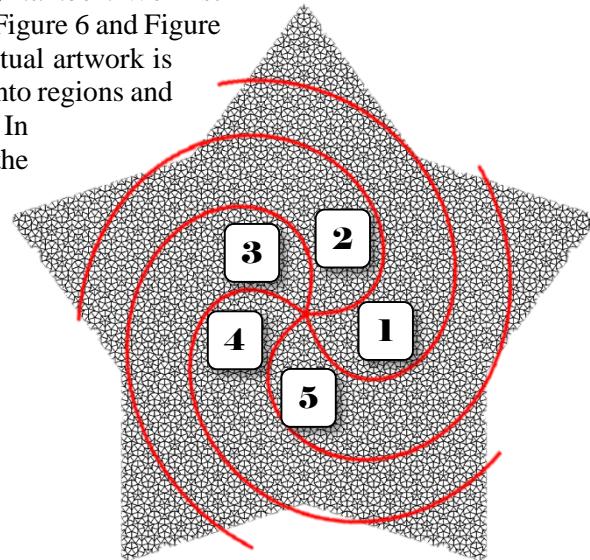


Figure 6: Five regions
and a partial painting scheme.

Every region is now labeled with a pair of numbers. We finish by simply painting all of the tiles of that tile type within each region. For example, in the region labeled 1-2, every tile of type 1 and every tile of type 2 is painted black. In the 2-2 region, only the tiles of type 2 are painted.

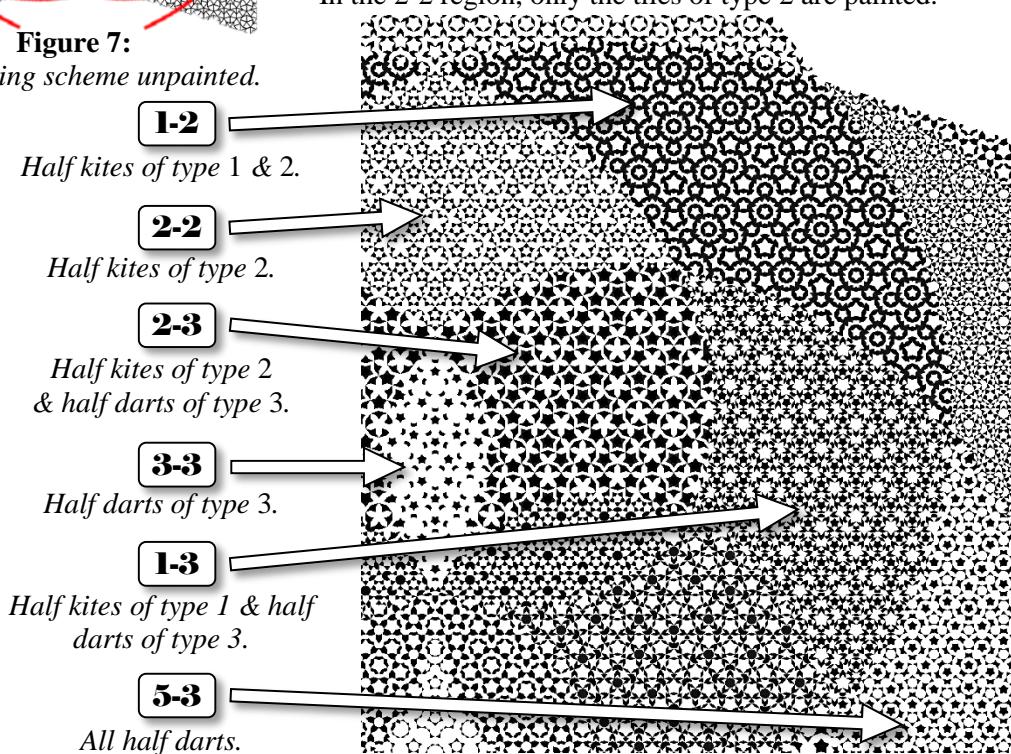


Figure 8: Selected regions with the tile types painted black.

Notice that the central regions contain only a single tile type painted, whereas, off of the central axis, a pair of different tile types are painted. In Figure 9, the central axis is similar to the diagonal of a square matrix. The tile types for the regions along the central axis starting from the top and descending to the center of the image are type 4, type 5, type 1, type 2 and type 3, respectively. The two regions starting from the center of the image to the bottom are type 5 followed by a region of type 4.

The right side and the left side of the image are similar to the off-diagonal entries of a matrix. In either Figure 8 above, or in Figure 9 below, can you see that the off diagonal regions are the union of the two corresponding regions along the central axis? For example, the 2-3 region in Figure 8 is the union of the type 2-2 region and the type 3-3 region.

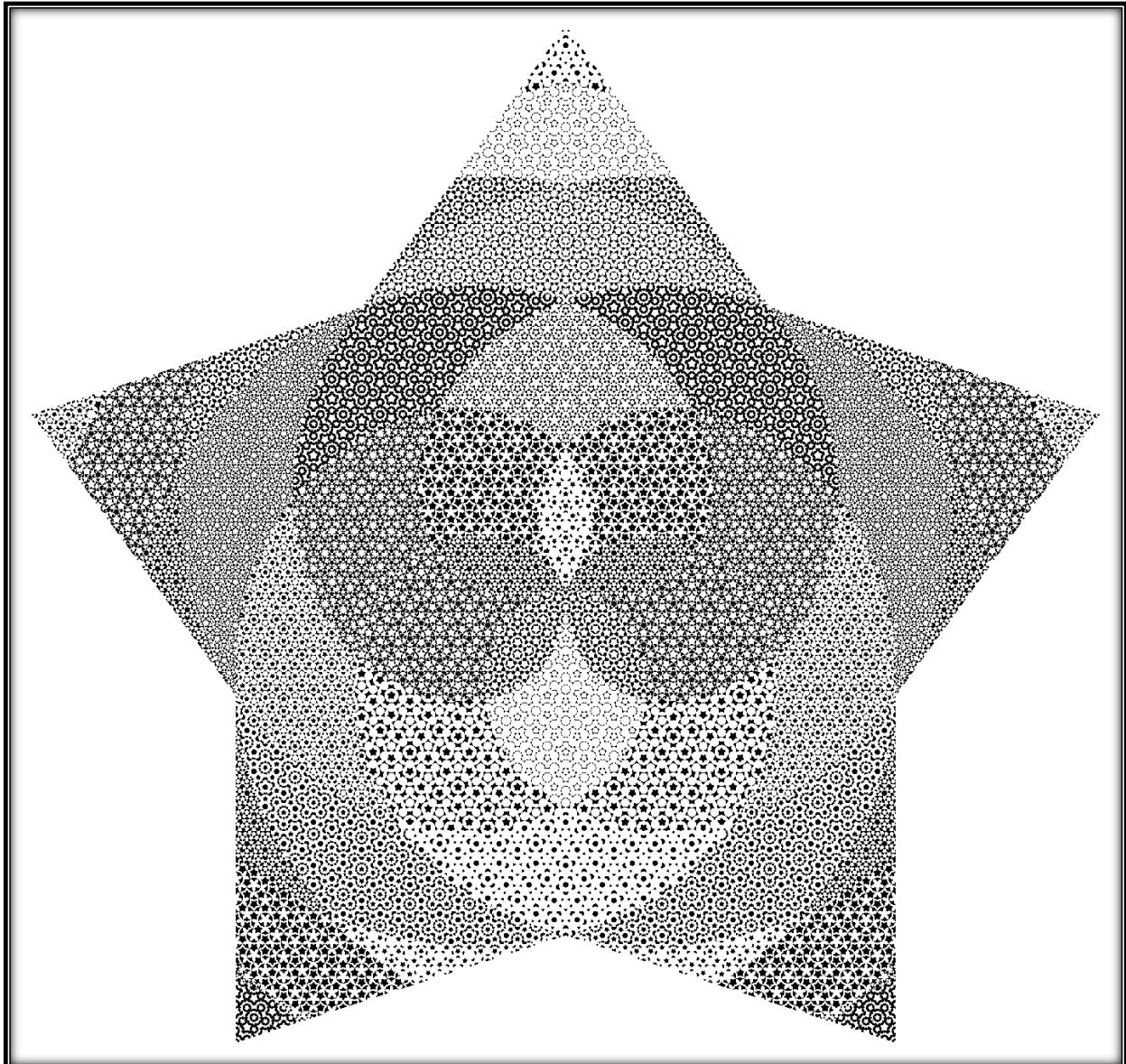


Figure 9: A Penrose Weaving.

Penrose Lace: Applying the Method of Sierpiński

We now apply a Sierpiński type painting process to the subdivision process to create a *lacelike* image. Consider the Sierpiński Triangle shown in Figure 10.

One method for creating this triangle is to start with an equilateral triangle and subdivide the triangle into four equivalent triangles, as shown in Figure 11. Then discard region 4. By repeating the process of subdividing and discarding we obtain the Sierpiński Triangle shown in Figure 10.

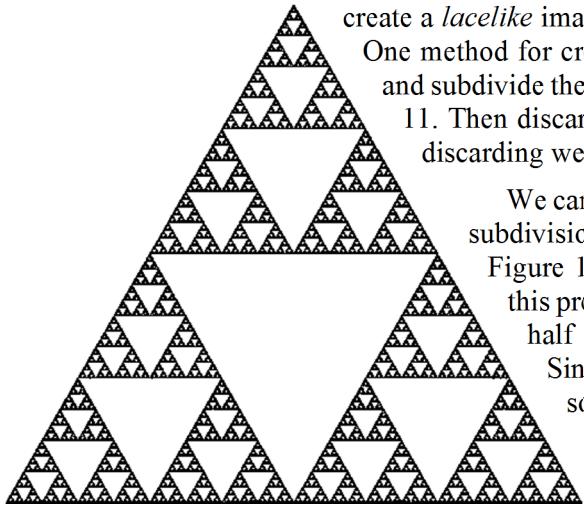


Figure 10: The Sierpiński Triangle.

We can apply a similar method to our Penrose subdivision. By discarding all tiles of type 2 in Figure 12 we obtain Figure 13. Notice that this process may leave either half kites or half darts in the resulting image. Since we will iterate this process so that the individual tiles are very tiny, half tiles won't matter.

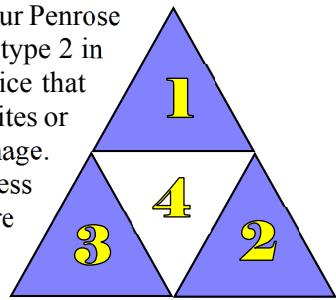


Figure 11:
Subdividing Sierpiński.

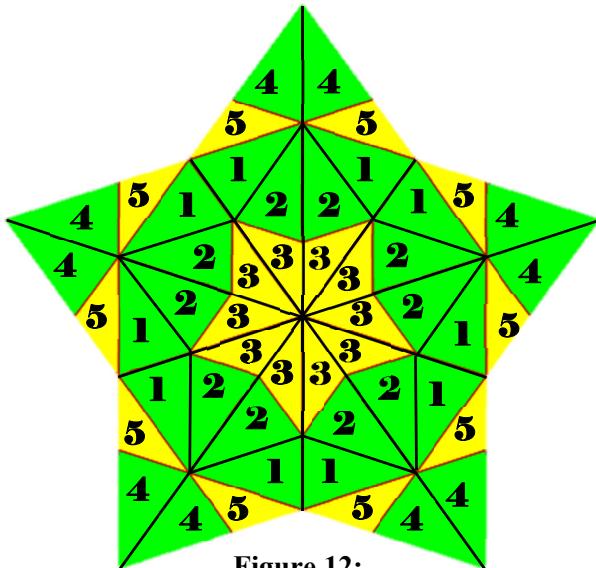


Figure 12:
Subdivision labeled by type.



Figure 13:
Type 2 tiles discarded.

To create a collage containing *variations on the theme*, we shall apply different discarding rules to different regions of our artwork. Also, unlike Sierpiński, we don't discard immediately. For my artwork, I chose to iterate to the 7th generation before discarding any tiles. I then use the clockwise spiral from the previous section and the same numbering system to divide the image into five regions as shown in Figure 14. We now iterate the next three generations using the additional discard step discarding the tile type based upon the labeling of each region. This results in the image shown in Figure 14. We then flip the spiral to counterclockwise and iterate two more times. The resulting image, shown in Figure 16, contains 25 distinct *lacelike* patterns. The patterns on the clockwise arms are similar at the macro level and the patterns on the counterclockwise arms are similar at the micro level. Figure 15 shows an enlargement of a portion of the image and includes the two sets of spirals showing the region boundaries.

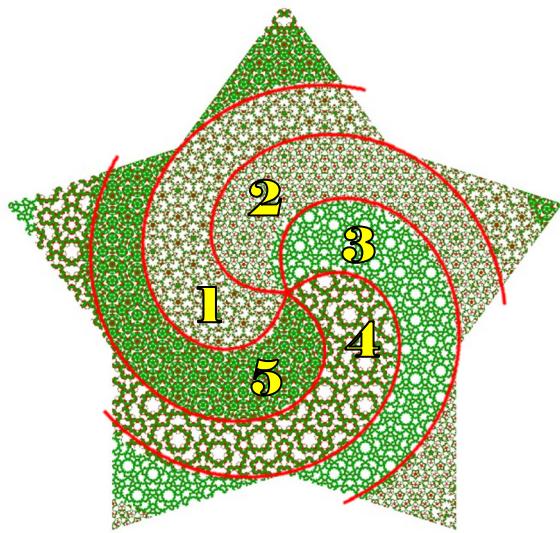


Figure 14: The tiling after the macro tiles have been discarded in generations 8-10.

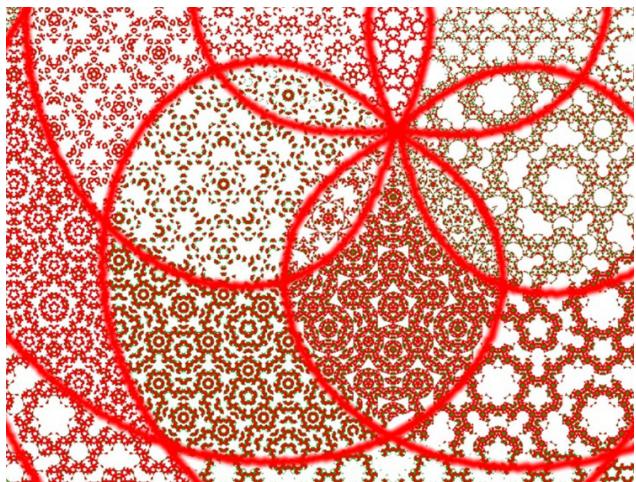


Figure 15: The tiling after the micro tiles have been discarded in generations 11 & 12.

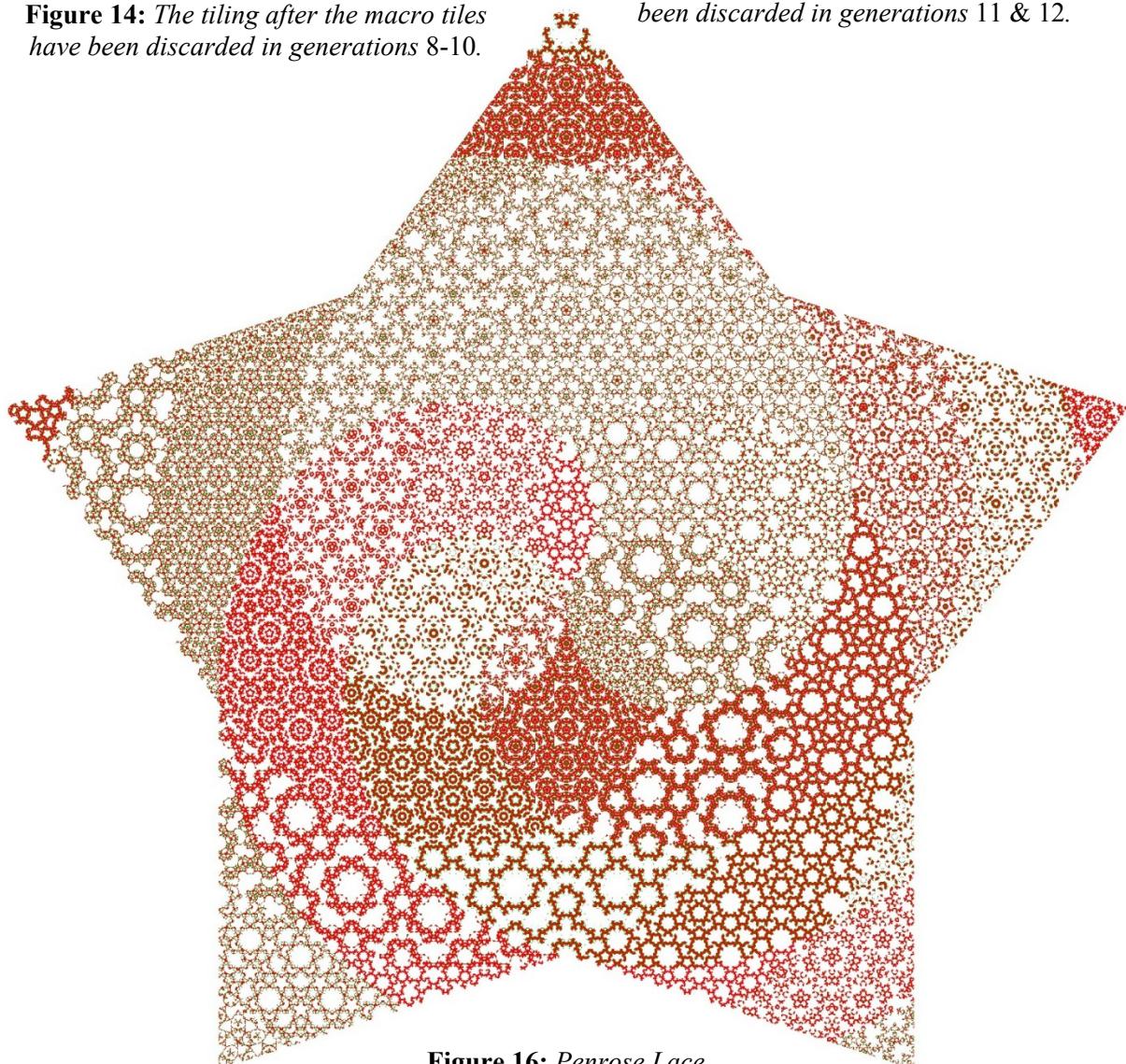


Figure 16: Penrose Lace.

Filling Negative Space with Pursuit Curves

Our last method for painting Penrose tiles was inspired by David Reimann's *Binomial Pursuit* on the cover of the June 2016 issue of the MAA Mathematics Magazine [4]. Here, Reimann replaced the empty spaces of the Sierpiński triangle, Figure 17, with pursuit curves. To understand pursuit curves, look at the center triangle in Reimann's image. Imagine animals placed at each of the three vertices of this center triangle with each animal facing inward toward the animal on its left. The three animals then pursue directly towards the animal on its left all at the same speed. The straight lines are the sight lines for each animal and the gentle curves are the paths that each animal follows as they continually turn toward the moving animal on their left. Eventually the three animals meet at the center.

Since the previous painting method used Sierpiński, we are already prepared for Reimann's method. Naturally, Figure 16 is way too busy to add this extra detail. Instead we apply the pursuit curves at the first or second generation. Each of the first three

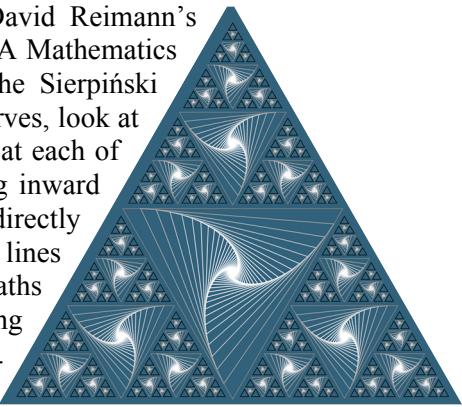


Figure 17:
David Reimann's *Binomial Pursuit*.
(Copyright David Reimann.
Used by permission of the artist.)

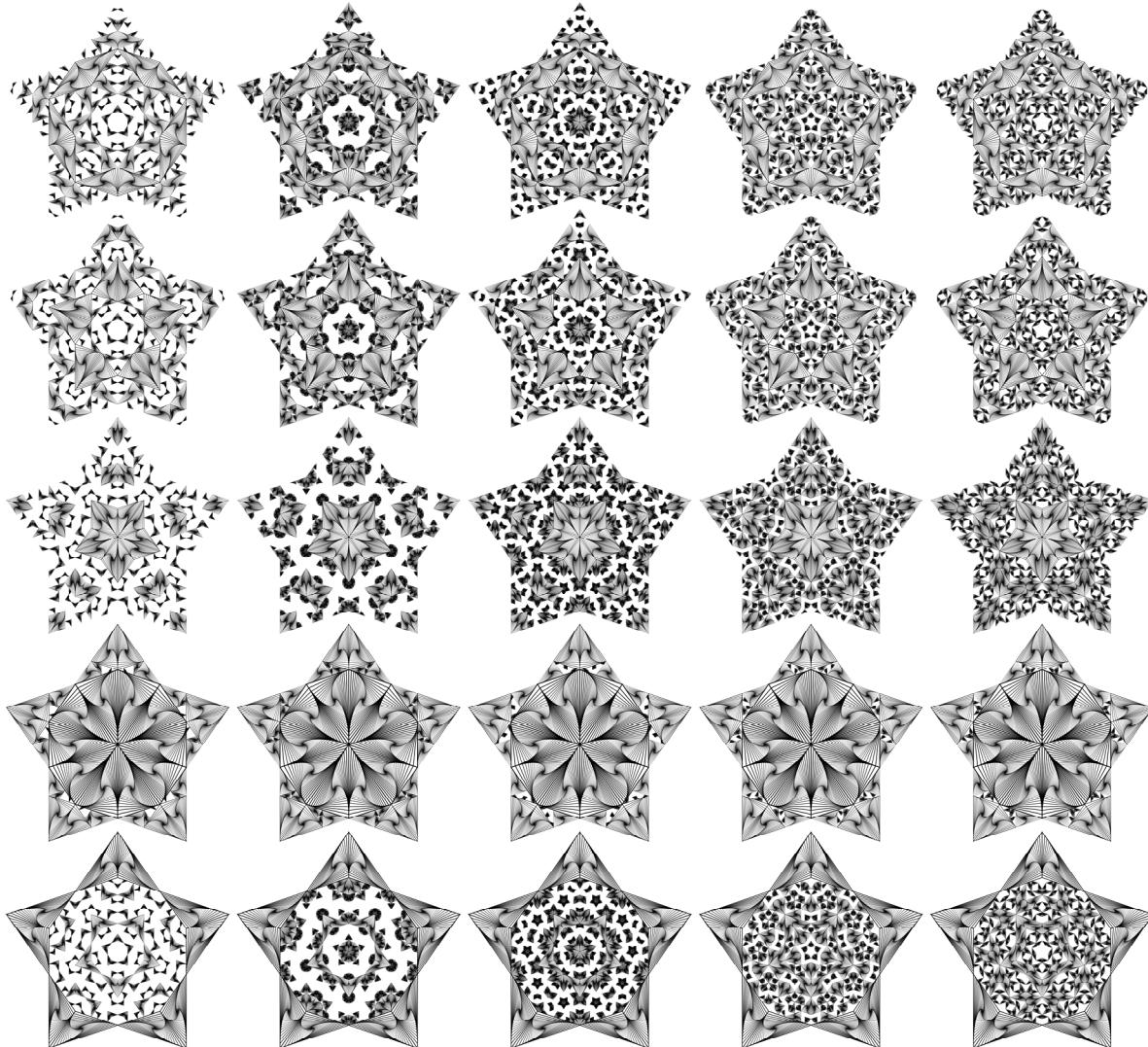


Figure 18: A matrix of Penrose tilings painted with pursuit curves.

rows of Figure 18 correspond to replacing the tile type 1, 2, or 3, respectively, with a pursuit curve for the second and third generations. For example, notice how the large pursuit curve kites in the second row fill the blank regions in Figure 13. We then discard the tile type corresponding to the column value for generations 4, 5, & 6. In the fourth and fifth row, the pursuit curves replace tile types 4 or 5, respectively, starting in the first generation. Looking back at Figure 3, we see that the first generation only contains tiles of type 4 or 5.

We finish with my wife's favorite image in Figure 19, *Penrose Skates II*. This image is from the second row and the third column of Figure 18 with finer details added by iterating two additional generations.

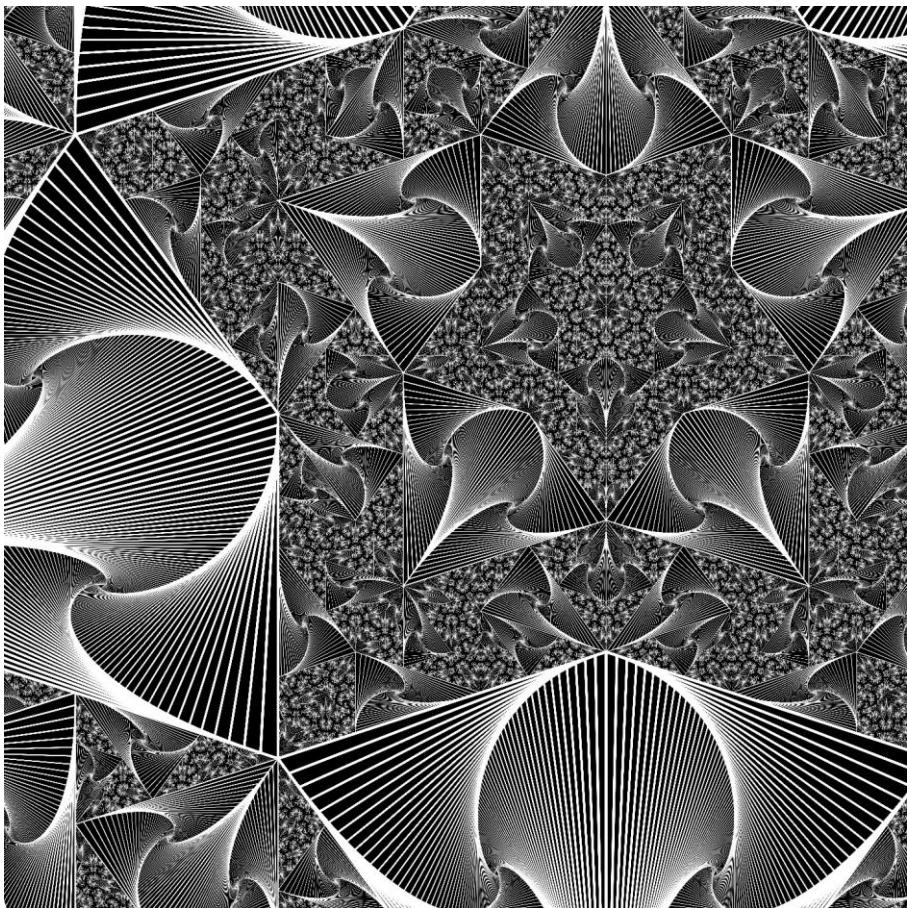


Figure 19: *Penrose Skates II*.

Conclusion

The painting rules created by these three different methods can be modified and applied to any tiling that can be constructed using subdivision. The Tiling Encyclopedia [5] contains a large collection of tilings, many created by subdivision. In this paper we only discussed three painting rules. The author challenges you to create your own paining rule and apply it to your favorite tiling pattern. Have fun exploring!

References

- [1] Beautiful Penrose Tile in Architecture: <http://www.eschertile.com/penrose.htm>
- [2] D.G. Burkholder. Unexpected Beauty Hidden in Radin-Conway's Pinwheel Tiling. In *Bridges Baltimore 2015: Mathematics, Music, Art, Architecture, Culture*: 383-386. 2015.
- [3] B. Grünbaum and G.C. Shephard, *Tilings and Patterns*, W.H. Freeman. 1987.
- [4] D. Reimann. Binomial Pursuit. *MAA Mathematics Magazine*: **89** (3). 2016.
- [5] The Tiling Encyclopedia: <http://tilings.math.uni-bielefeld.de/>