Matching

A matching in a graph G is a set M of edges such that no two edges of M are adjacent.

If M is a matching then the two ends of each edge of M are said to be matched under M.

~~Each vertex~~

Each vertex incident with an edge of M is said to be covered by M.

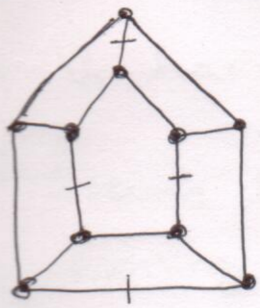
A Perfect matching is the one which covers every vertex of G. A graph G is called matchable if it has a perfect matching.

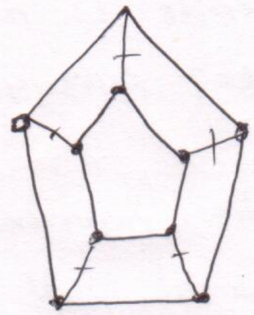
A matching M in G is a maximum matching if G contains no matching more than |M| edges.

The number of edges in a maximum matching of G is called the matching number and we denote it by $\alpha'(G)$.

A maximal matching is a matching of G which can not be extended to a larger matching.

Ex





Perfect matching

Let M_1 and M_2 be matchings of G. Then the symmetric difference of M_1 and M_2, denoted by $M_1 \Delta M_2$, defined as a subgraph H of G whose components are paths or even cycles of G in which the edges alternate between M_1 and M_2

Let M be a matching in G. An M-alternating path or cycle in G is a path or cycle whose edges are alternating in M and $E \setminus M$.

An M-alternating path might or might not start or end with edges from M. If neither its origin nor its terminus is covered by M, then the path is called an M-augmenting path.

$\underline{\underline{Th^m}}$ A matching M of G is maximum if and only if G has no M-augmenting path.

$\underline{\underline{\text{Proof}}}$ Assume that M is maximum. If G has an M-augmenting path $P: v_0 v_1 \cdots v_{2n+1}$ in which the edges alternate between $E \setminus M$ and M, then P has one edge of $E \setminus M$ more than that of M. Define

$$M' = M \cup \{v_0v_1, v_2v_3, \quad \ldots \quad , v_{2n}v_{2n+1}\} \setminus \{v_1v_2, \ldots, v_{2n-1}v_{2n}\}$$

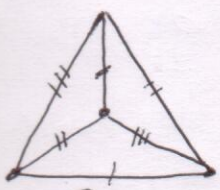
Clearly M' is a matching of G with $|M'| = |M|+1$, which is a contradiction, since M is a maximum matching of G.

Conversely assume that G has no M-augmenting path. Suppose M is not a maximum matching. Then there exists a matching M' of G with $|M'| > |M|$. Let H be the edge subgraph $G[M \Delta M']$. Then the components of H are paths or even cycles in which the edges alternate between M and M'. Since $|M'| > |M|$, atleast one of the components of H must start and end with edges of M'. But then such a path is an M-augmenting path of G, contradicting the assumption.

A factor of G is a spanning subgraph of G.
A K-factor of G is a factor of G that is K-regular. Thus a 1-factor of G is a perfect matching of G. A 2-factor of G is a factor of G that is a disjoint union of cycles of G.

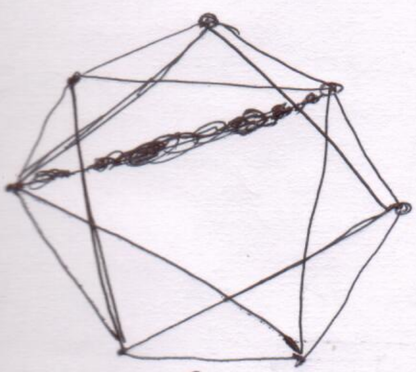
A graph G is K-factorable if G is an edge-disjoint union of k-factors of G.

Ex



G_1

– 1-factorable



G_2

– 2-factorable

G_3

Neither a 1-factor nor a 2-factor.

Matching in a bipartite graph

For a subset S of $V(G)$, $N(S)$ denotes the neighbour set of S, ie. the set of all vertices each of which is adjacent to at least one vertex.

Th^m (Hall)

Let G be a bipartite graph with bipartition (X, Y). Then G has a matching that covers all the vertices of X if and only if $|N(S)| \geq |S|$ for every subset S of X.

Proof If G has a matching that saturates all the vertices of X, then each vertex of X is matched to a distinct vertex of Y. Hence, $|N(S)| \geq |S|$ for every $S \subseteq X$.

Conversely, assume that $|N(S)| \geq |S|$, for all $S \subseteq X$. Suppose that G has no matching that saturates all the vertices of X. Let M be a maximum matching of G. As M does not ~~saturates~~ saturate all the vertices of X, there exists a vertex $x_0 \in X$ that is M-unsaturated. Let Z denote the set of all vertices of G connected to x_0 by M-alternating path. Since M is a maximum matching, so G has no M-augmenting path.

As x_0 is M-unsaturated, this implies that x_0 is the only vertex of Z that is M-unsaturated.

Let $A = Z \cap X$ and $B = Z \cap Y$. Then the vertices of $A \setminus \{x_0\}$ get matched under M to the vertices of B and $N(A) = B$. Thus since $|B| = |A| - 1$, $|N(A)| = |B| = |A| - 1 < |A|$, a contradiction to our assumption.

$\underline{\underline{Th^m}}$ A $K\ (K \geq 1)$-regular bipartite graph is 1-factorable.

$\underline{\underline{Proof}}$ Let G be K-regular with bipartition (X, Y). The $E(G)$ = the set of edges incident to the vertices of X = the set of edges incident to the vertices of Y. Hence $K|X| = E(G) = K|Y| \Rightarrow |X| = |Y|$.

If $S \subseteq X$, then $N(S) \subseteq Y$ and $N(N(S)) \supseteq S$.

Let E_1 and E_2 be the set of edges of G incident to S and $N(S)$, respectively. Then $E_1 \subseteq E_2$, $|E_1| = K|S|$ and $E_2 = K|N(S)|$. Hence $|N(S)| \geq |S|$ as $|E_2| \geq |E_1|$. So by Hall's theorem G has a matching that saturates all the vertices of X; i.e. G has a perfect matching. Deletion of the edges of M from G results in a $(K-1)$ regular bipartite graph. Repeated application of the above argument shows that G is 1-factorable.