Planar Graph

A plane graph is a graph drawn in the plane in such a way that any pair of edges meet only at their end vertices (if they meet at all).

A planar graph is a graph which can be redrawn as a plane graph (i.e. isomorphic to a plane graph).

A plane graph G partition the plane into a number of regions called faces of G. In a plane graph G, the face which is not bounded by any cycle is called the exterior face. Otherwise it is known as an interior face.

Theorem (Euler's Formula)
Let G be a connected plane graph and let n, m and f denote the number of vertices, edges and faces of G, respectively. Then $n-m+f=2$

Proof Induction on the number of faces.

If $f=1$, then G has only one face, the exterior face. If G contains any cycle C, there there is atleast one interior face, which is imposible. Thus G has no cycle. Since it is connected so G is a tree. Hence, G has $n-1$ edges and therefore $n-m+f=2$.

Now suppose that $f > 1$ and the theorem is true for all connected plane graphs with less than f faces. Since $f > 1$, so G has a cycle. Hence G has an edge e which is not a bridge. Then the subgraph $G-e$ is still connected and also a plane graph. Since the edge e must be part of a cycle, it separates two faces of G from each other and so in $G-e$, these two faces combine to form one face of $G-e$. Thus by induction hypothesis,

$$n(G-e) - m(G-e) + f(G-e) = 2$$

So $\quad n - (m-1) + f - 1 = 2 \quad \Rightarrow n - m + f = 2$

<u>Corollary</u>: Let G be a plane graph with n vertices, m edges, f faces and k connected components. Then $\quad n - m + f = k+1$.

Let G be a plane graph. A triangulation of G is a plane graph H that contains G as a spanning subgraph and to which no new edge can be added without crossing some existing edge.

Cor: If G is a simple planar graph with $n \geq 3$ vertices and m edges then $m \leq 3n-6$.

Proof: Suppos H is a triangulation of G having n vertices and $m+l$ edges. Clearly H is connected. If H has total of f faces, then $3f = 2(m+l)$. By substituting it into Euler's equation, we have

$$n + \frac{2(m+l)}{3} = (m+l) + 2$$

$$\Rightarrow m+l = 3n-6 \Rightarrow m \leq 3n-6.$$

Example: The complete graph K_5 is not planar.

Cor Suppose G is a bipartite planar graph with m edges and $n \geq 3$ vertices. Then $m \leq 2n-4$.

Proof Since G is a bipartite planar graph, so G has no odd cycles. It may happen that some non adjacent pair of vertices of G can be joined by a new edge that ~~cross~~ cross non of the existing edges and does not create an odd cycle. Suppose a maximum number ~~of~~ l of such edges have ~~been~~ been added to G. Call the resulting biparctite plane graph H.

If H has total f faces, then $4f = 2(l+m)$.

Then by substituting it into Euler's equation, we have $\frac{2(m+l)}{4} + n = (m+l) + 2$. i.e. $m+l = 2n-4$

so $m \leq 2n-4$.

Example: The complete bipartite graph $K_{3,3}$ is not planar.

Cor: If G is a simple planar graph with n-vertices and m edges then $\delta(G) \leq 5$.

Proof: Suppose the minimal degree of G is atleast 6. Then $2m \leq 6n$. so $m \leq 3n$, a contradiction.