IIT Madras

ONLINE DEGREE

Statistics for Data Science – 1
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Lecture 9.4
Variance of a Random Variable

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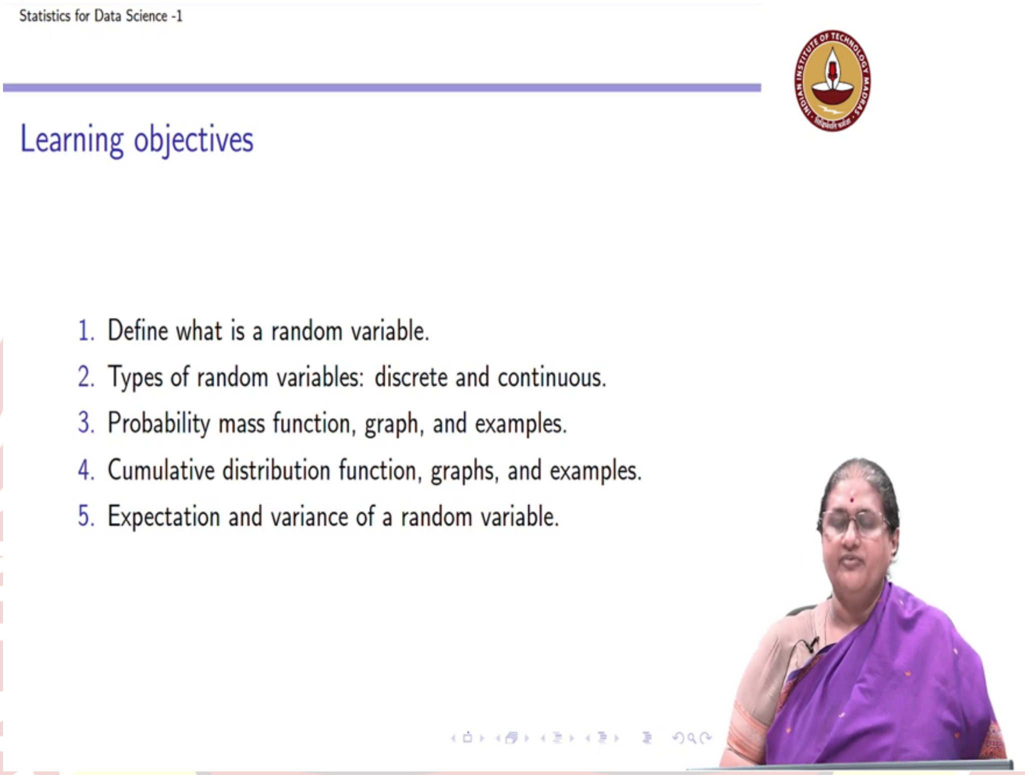

So what we have seen so far is the following we have looked at what was a random variable, we defined what is a random variable and then we looked at the types of random variable namely we said that there are two main types of a random variable we can classify them as discrete random variable and continuous random variables.

Then we looked at when we focus our, we restricted our focus only to discrete random variables, in discrete random variables we defined what was a probability mass function in the sense that you are interested in knowing about a distribution of a random variable, hence we introduced the notion of a probability mass function.

The probability mass function tells us about the probability distribution of a random variable and the cumulative distribution function helps us to know what is the overall distribution of the random variable given a real line. So, we also looked at the graphs of both the probability mass function and the cumulative distribution function.

Then we went out to introduce an extremely important notion of a random variable namely the expectation of a random variable, we focused on properties of a random variable and then after this we looked at how we you can compute the expectation of functions of a random variable. Today we are going to focus on another important summary of a random variable which we refer to as a variance of a random variable.

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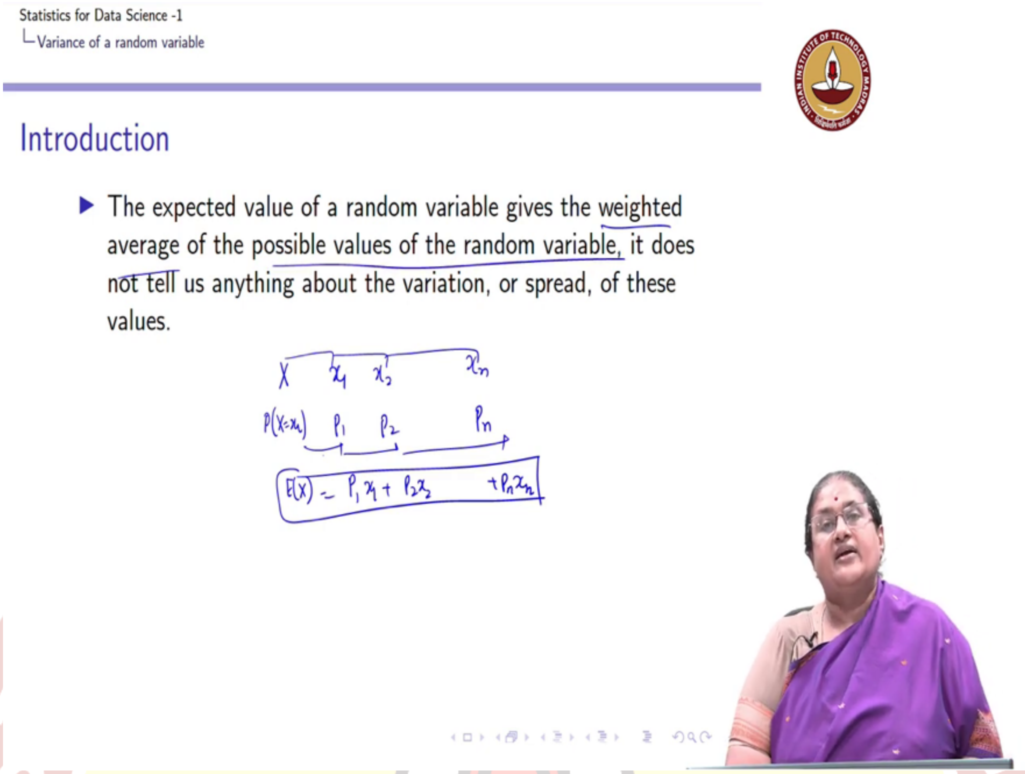

So, let us first of all motivate the need for such a measure. So we saw that we also realize that we can interpret the expected value as a long-run average, so in other words the way we introduced the expected value of a random variable was as a weighted average of possible values the random variable can take, in other words if X can take values $x_1, x_2, \dots, x_n$. I am assuming finitely many with $P(X = x_i)$ equal to P_1, $P_2, \dots, P_n$. These are I assume P_1, $P_2, \dots, P_n$ are the weights given to the values, then I know that the expectation of X can be written as $P_1x_1 + P_2x_2 + \cdots + P_nx_n$. In other words, it is X is taking possible values $x_1, x_2, \dots, x_n$ with weights P_1, $P_2, \dots, P_n$, then expectation of X can be expressed as a weighted average of the possible values of random variable.

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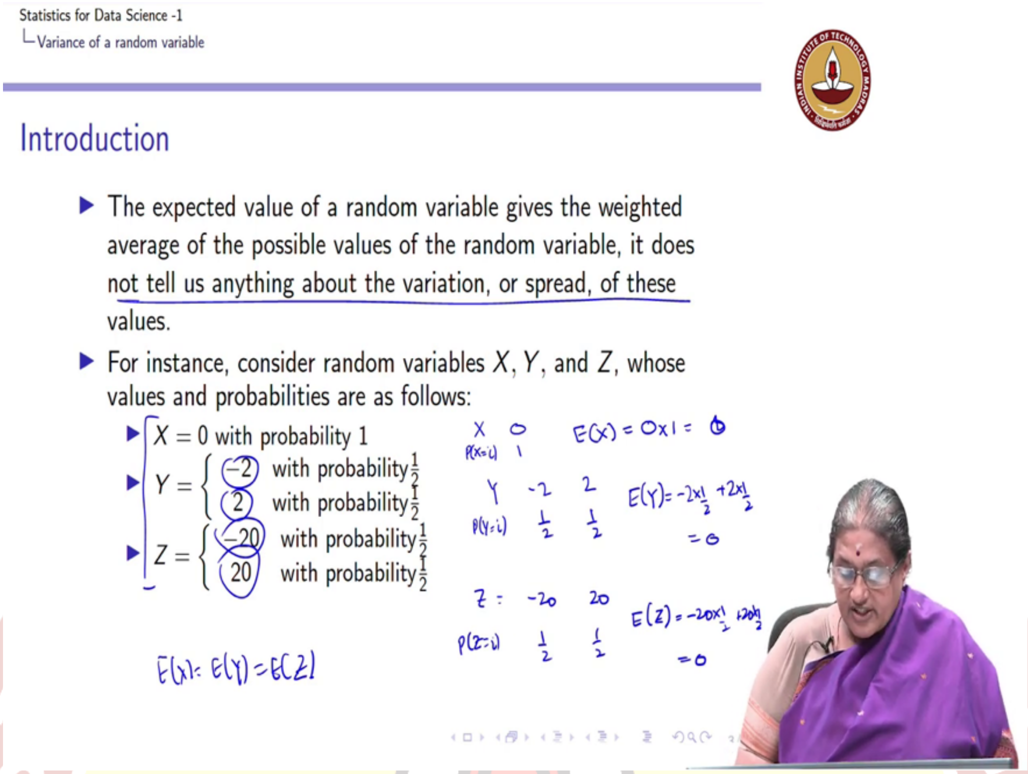

Now, let us look at an example where I have three random variables. So, what are these three random variables, X is a random variable with probability 1, Y is a random variable which takes the values -2 with probability $1/2$ and $+2$ with probability $1/2$ and Z is a random variable which takes the value -20 with probability $1/2$ and $+20$ with probability $1/2$. Now if you compute X, I can write X takes the value 0 with $P(X = i) = 1$ which gives me the $E(X) = 0 \times 1 = 0$.

Similarly, I know Y takes the value -2 and $+2$ with $P(Y = i) = 1/2$ and 1/2 giving me $E(Y) = -2 \times \frac{1}{2} + 2 \times 1/2 = 0$. Z takes the value -20 and 20 with $P(Z = i)$ is again 1/2 and 1/2 which gives me $E(Y)$, which is equal to $= -20 \times \frac{1}{2} + 20 \times 1/2$ which is again 0.

So, I can see that $E(X) = E(Y) = E(Z)$, so this expected value however what we notice is though the expectation is same the distribution of these random variables with the terms of the values they take is different. In other words, the expected value of a random variable does not tell us anything about the spread or the variation. Hence, I need a measure for the spread.

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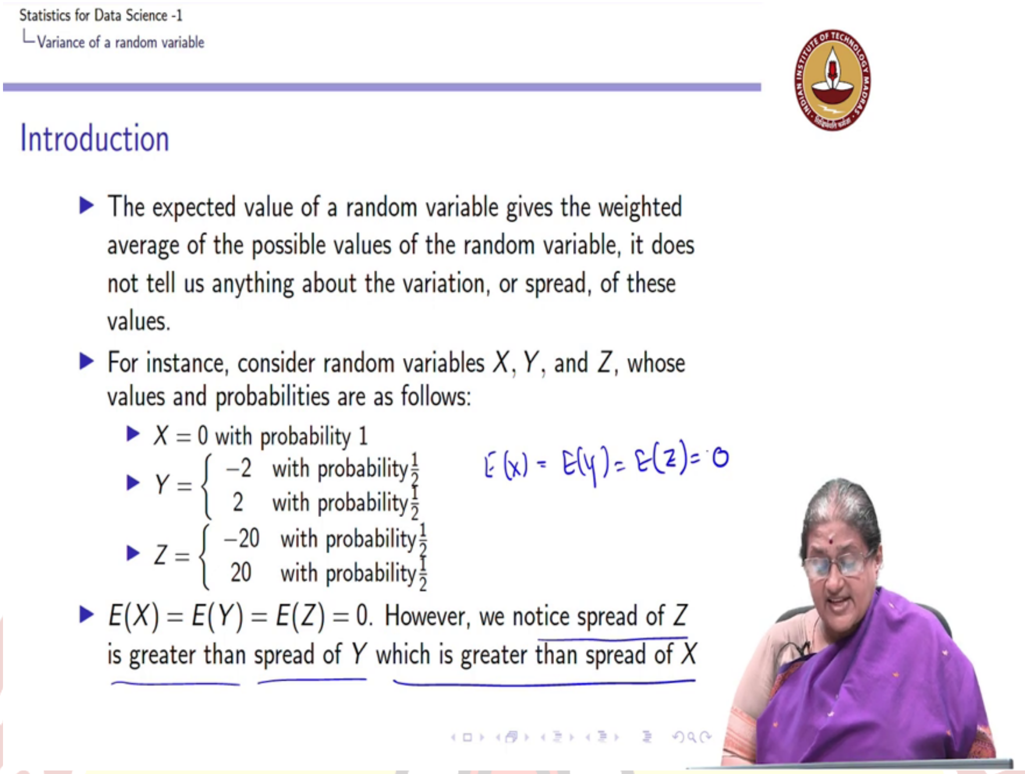

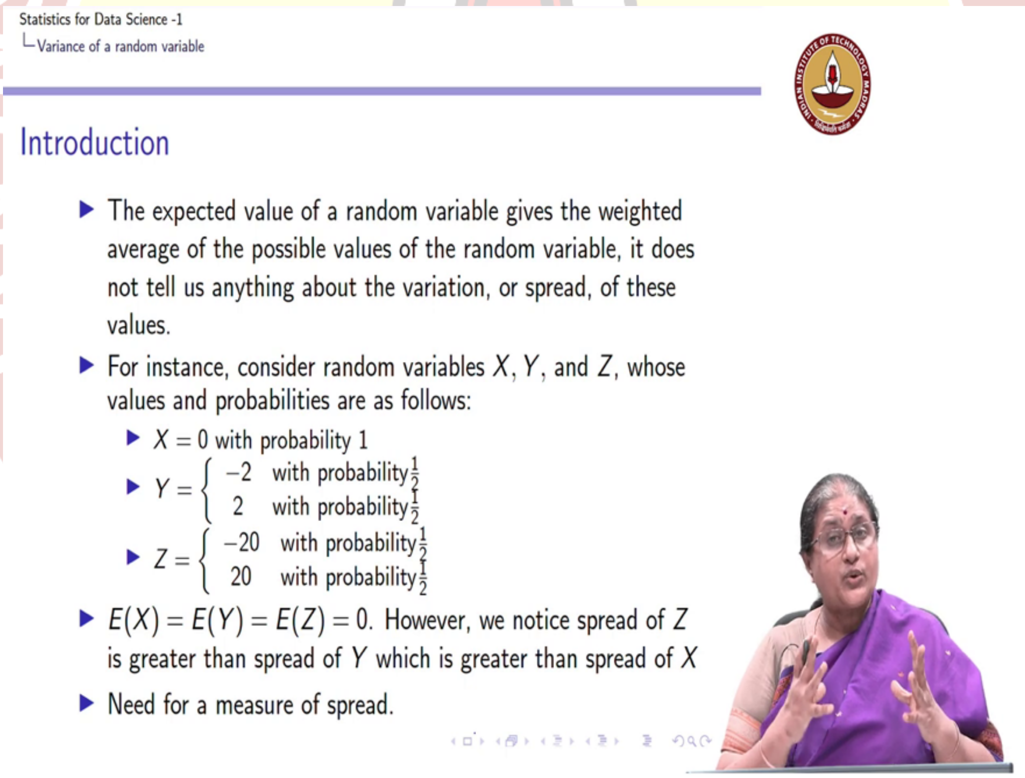

What I can notice here although is the spread of Z is greater than the spread of Y which is greater than the spread of X. The expected value although was the same $E(X) = E(Y)= E(Z)$ which is equal to 0. Hence, I need a measure of spread and just as what we did in a descriptive statistics module mean gave us a measure of central tendency variance gave us a measure of spread.

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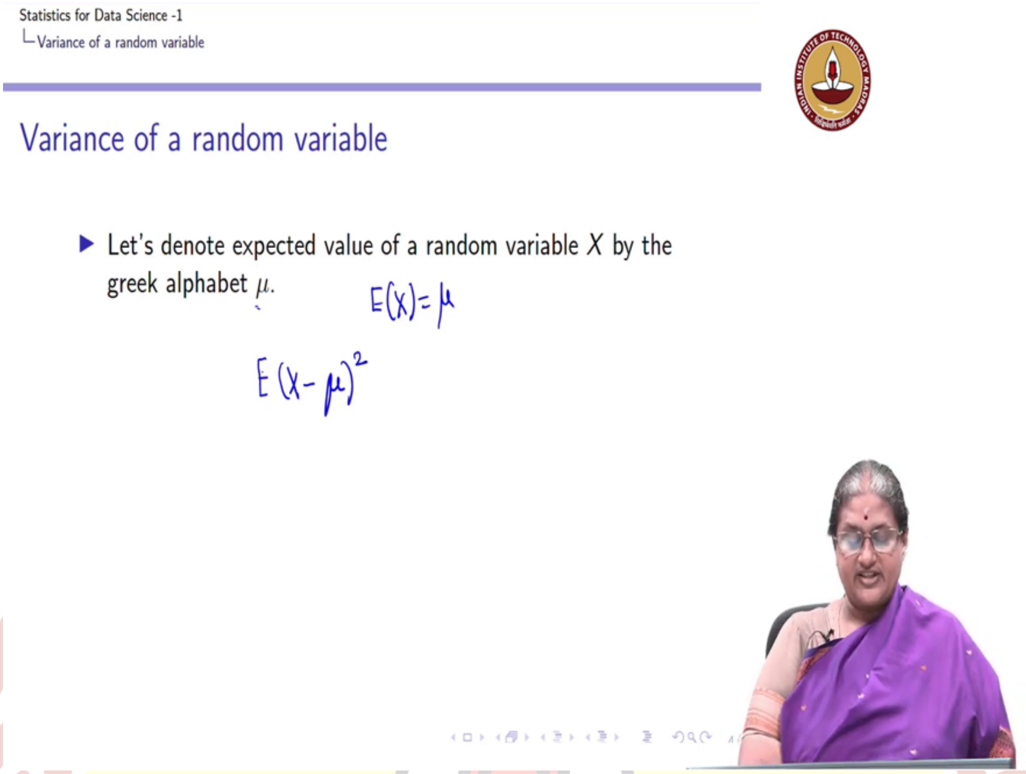

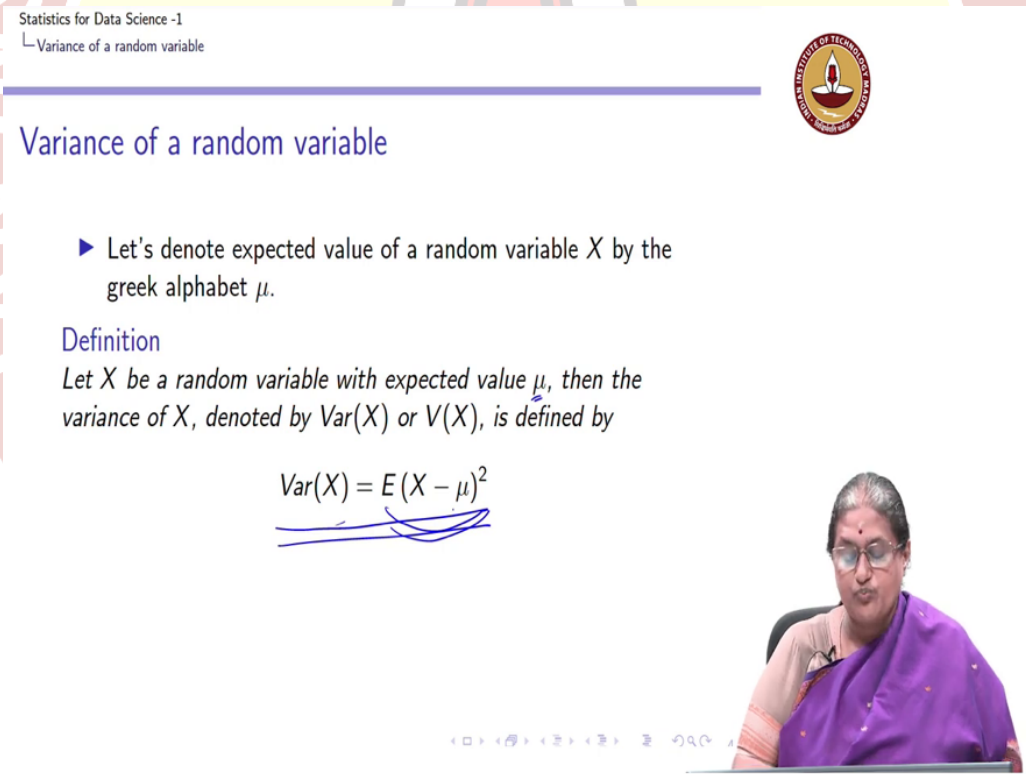

So, we are going to define the variance of a random variable, which is a measure of spread. So, what how do I define the variance of a random variable? So, let me denote the expected value of a random variable by the Greek alphabet μ that is $E(X) = \mu$, X is my random variable I see how far is X from μ, I take the square of the difference and the expected value of the squared difference and that is what I am going to define as my variance of X. So, the if X is a random variable with expected value μ then the $Var(X)$ is defined by $E(X-\mu)^2$.

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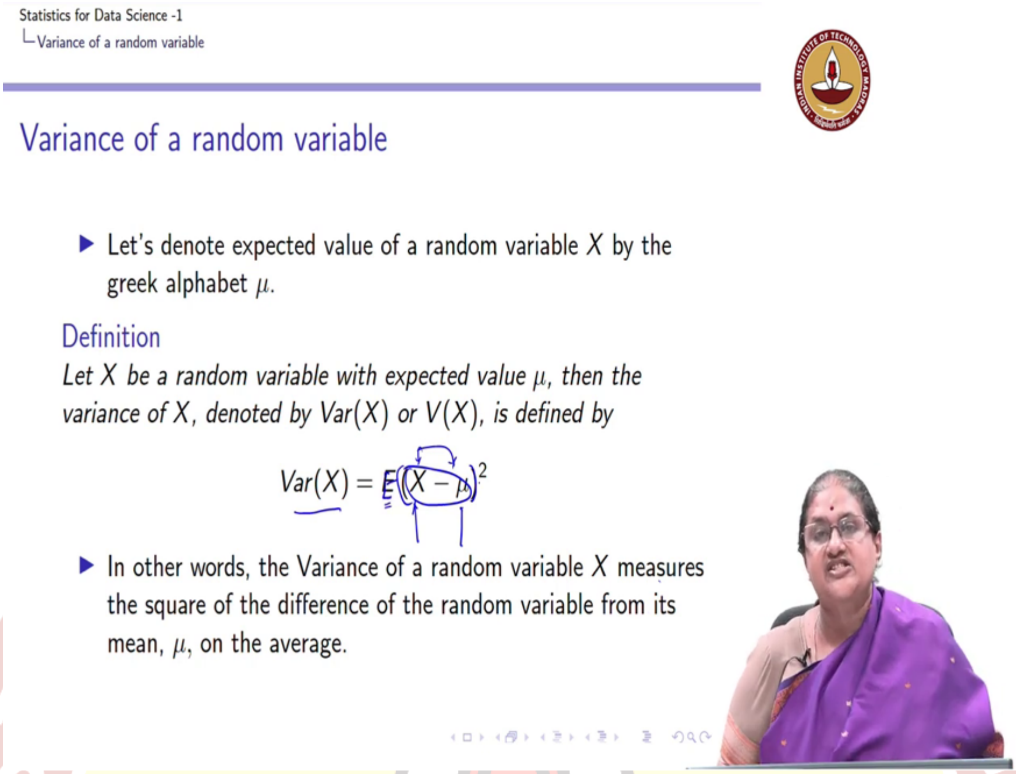

In other words, the variance of a random variable defines the square of the difference of the random variable from its mean on an average. I repeat, the variance of a random variable measures the difference of the random variable from its mean, so that is $(X - \mu)$, it measures the square of the difference which is $(X - \mu)^2$ on an average is $E(X - \mu)^2$.

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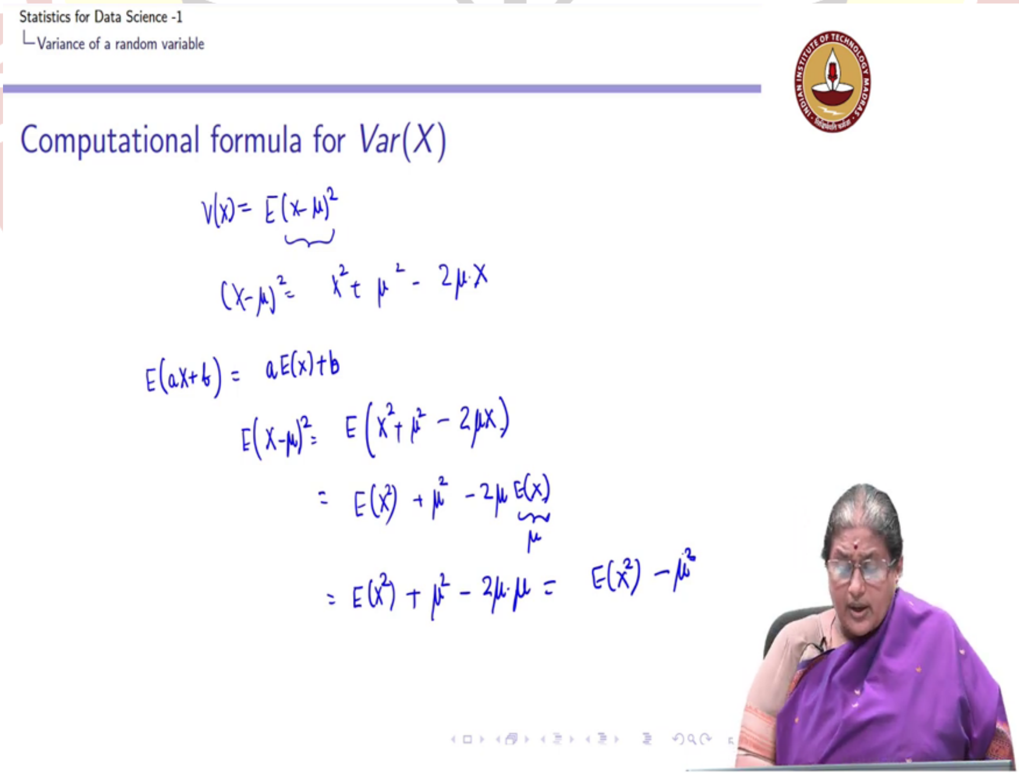
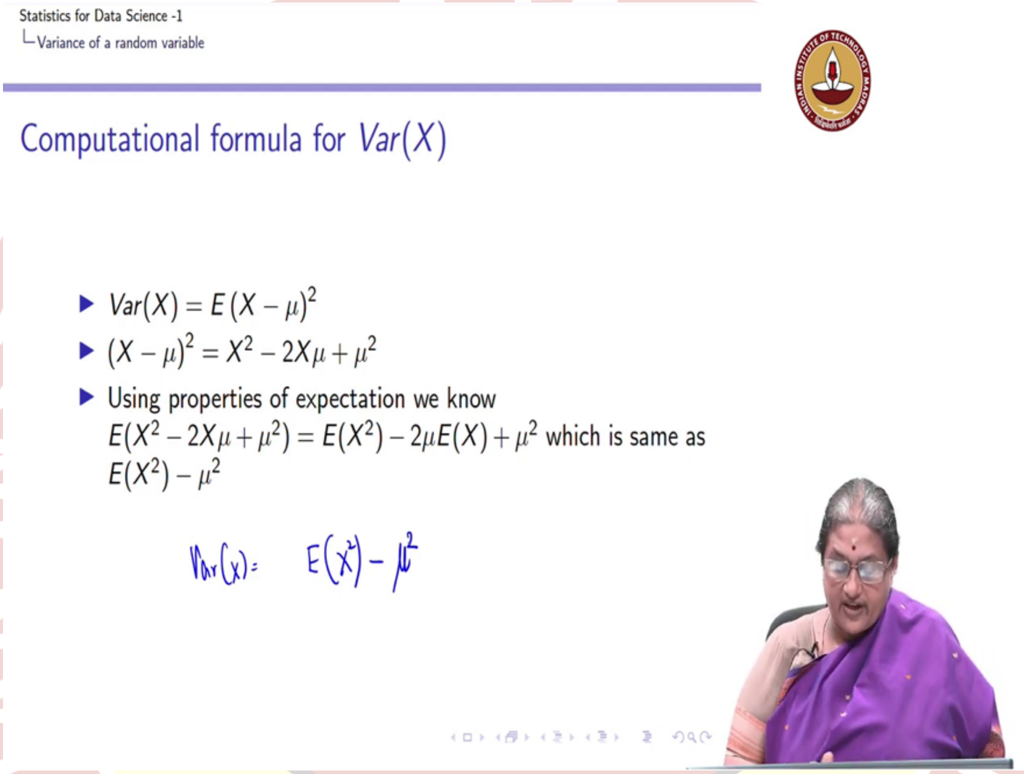
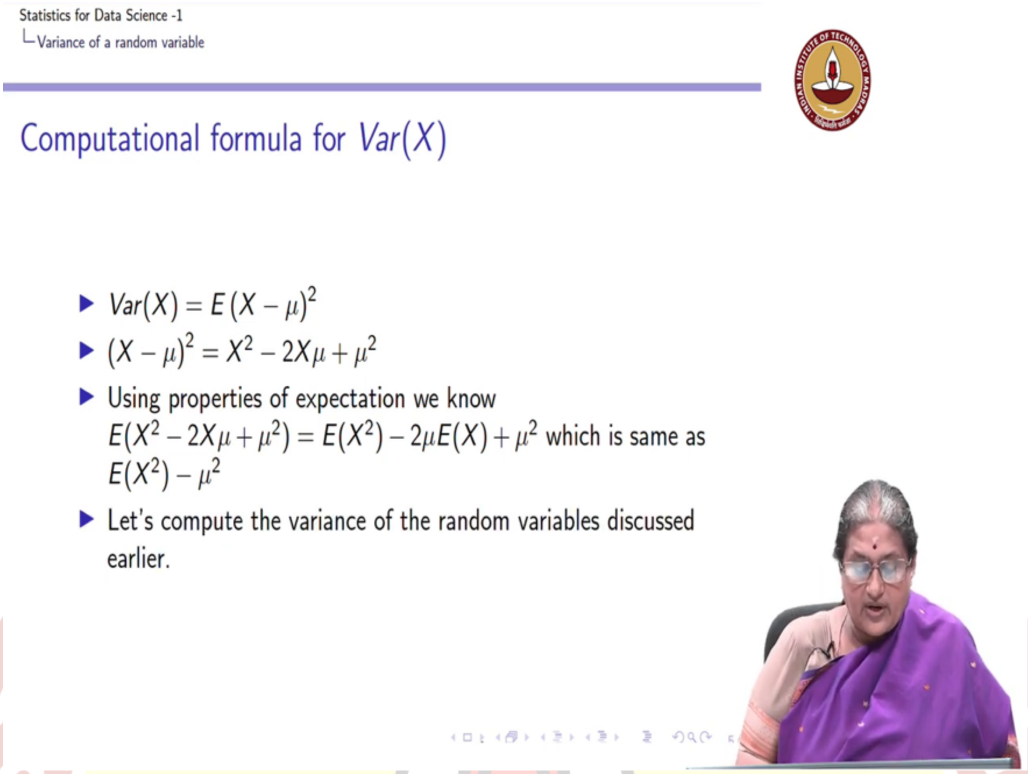


So, when I talk about $E(X-\mu)^2$ the question is can I computationally have a more convenient formula to compute the variance of a random variable? I know variance of a random variable is $E(X-\mu)^2$. Now, let us look at $(X-\mu)^2$, I know $(X-\mu)^2$ is $X^2+\mu^2-2\mu X$. Now, recall from my expectation or properties of expectation, $E(aX+b)$ where a and b are constants as $aE(X)+b$.

So, μ is a constant here, so if I am looking at $E(X-\mu)^2$, I know this is equal to $E(X^2)+\mu^2-2\mu E(X)$ which I can write it as $E(X^2)+E(\mu^2)$ is going to be $\mu^2-2\mu E(X)$, I know $E(X)=\mu$, hence this reduces to $E(X^2)+\mu^2-2\mu.\mu$ which gives me $E(X^2)-\mu^2$ (Refer Slide Time: 08:54)





So, the computational formula for X is a very simple formula which is the same as $Var(X) = E(X-\mu)^2$, where μ is $E(X)$. So, now let us apply this formula to compute the variance of the random variables which we have discussed earlier.

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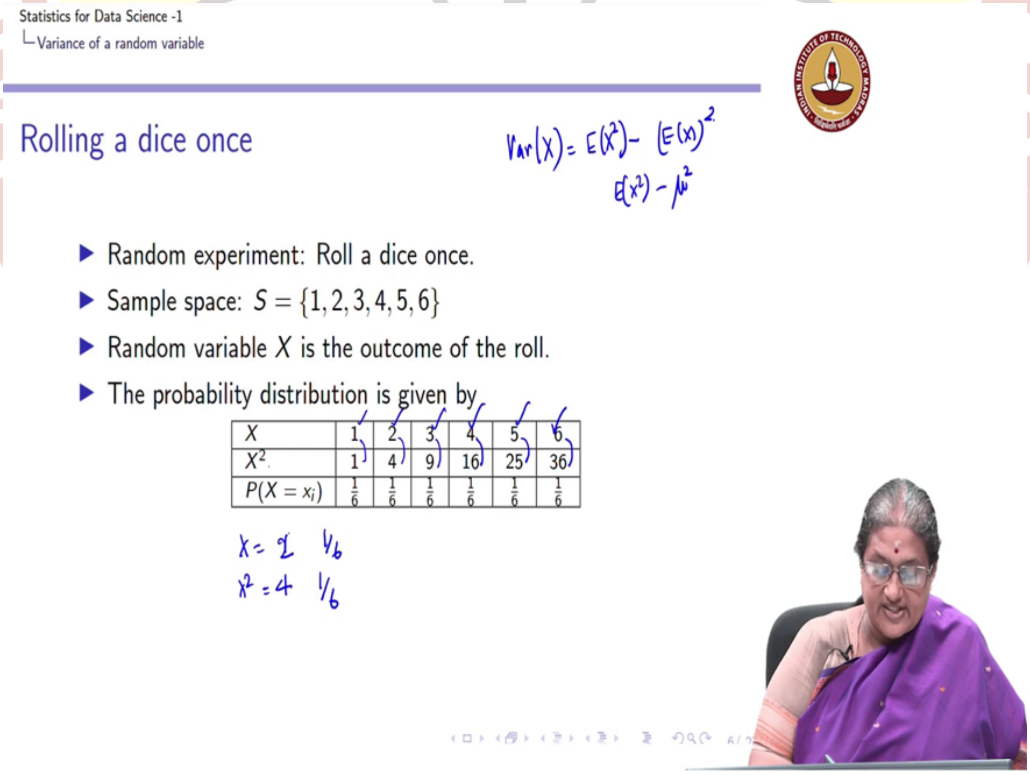

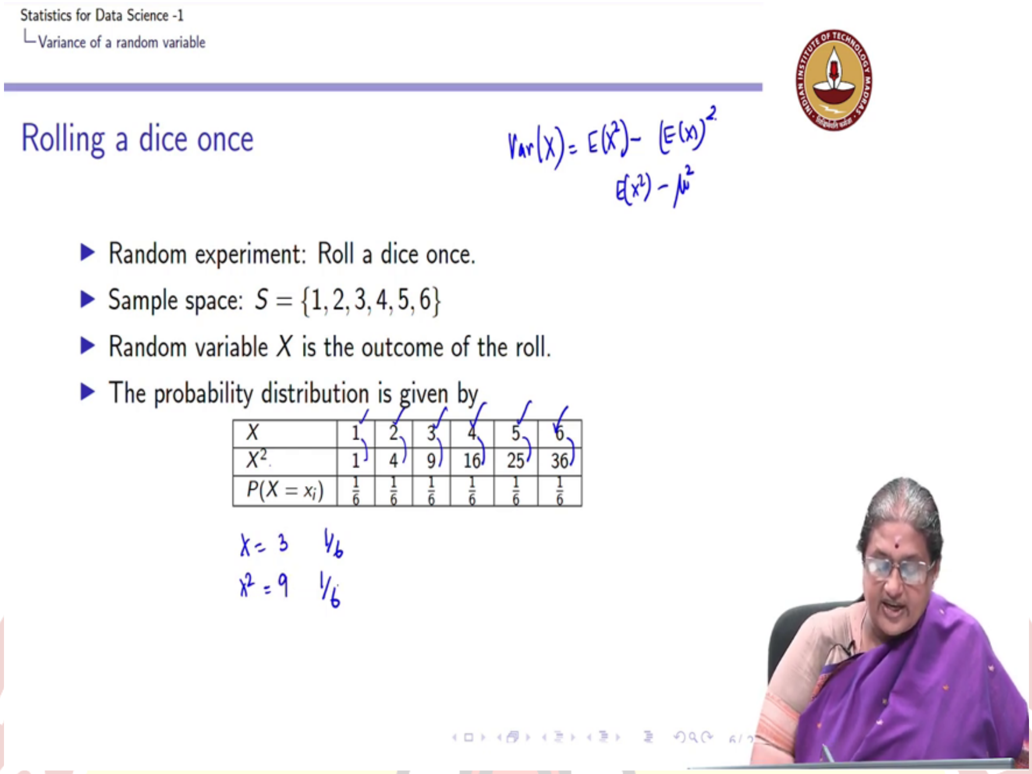

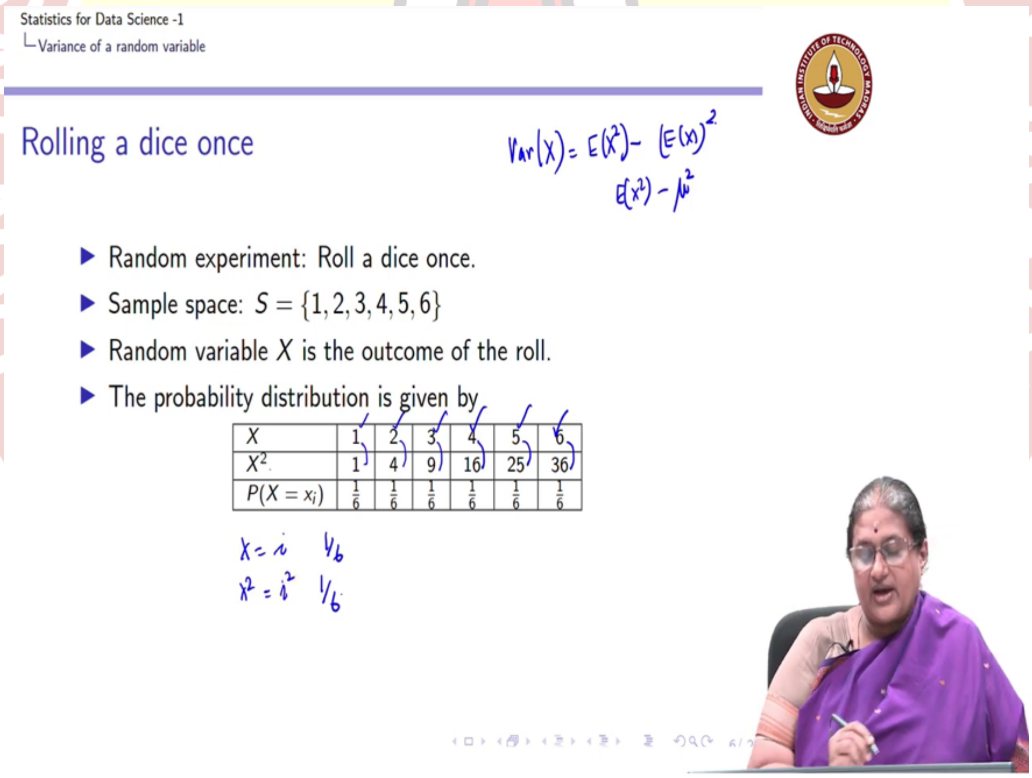

Let us begin with the role of a fair dice and I am rolling it once I know the sample space is $\{1, 2, 3, 4, 5, 6\}$, X is the outcome of the roll, so my X takes values 1, 2, 3, 4, 5, 6, I am going to apply the formula the computational formula $Var(X) = E(X^2) - \left(E(X)\right)^2$ or $E(X^2) - \mu^2$. That is the formula I am going to apply. So, if X takes the value 1, 2, 3, 4, 5, 6, I know X^2 takes the value 1^2, 2^2 is a 4, 3^2 is a 9, 4^2 is a 16, 5^2 is a 25 and 6^2 is a 36.

Now, if X takes the value 1, X^2 takes the value 1 with the same probability. Similarly, if X takes the value 2, X^2takes the value 4 with probability $1/6$, X takes the value 3, X^2 takes the value 9 with probability $1/6$ and so forth X takes the value i, X^2takes the value i^2 with probability $1/6$.

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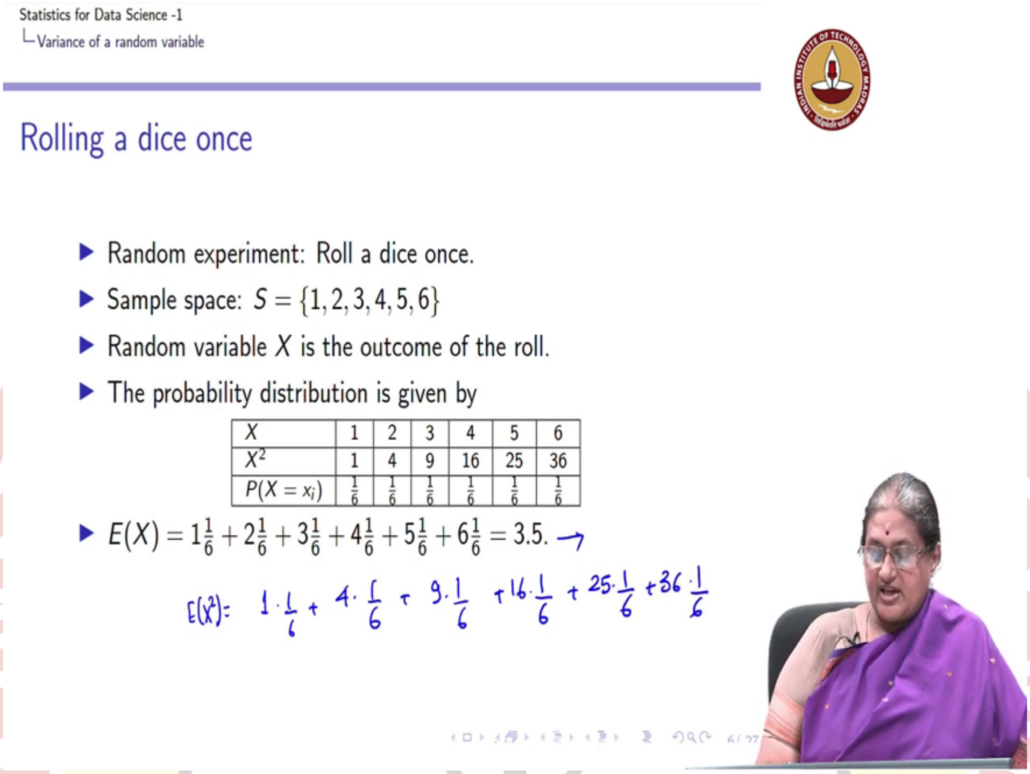

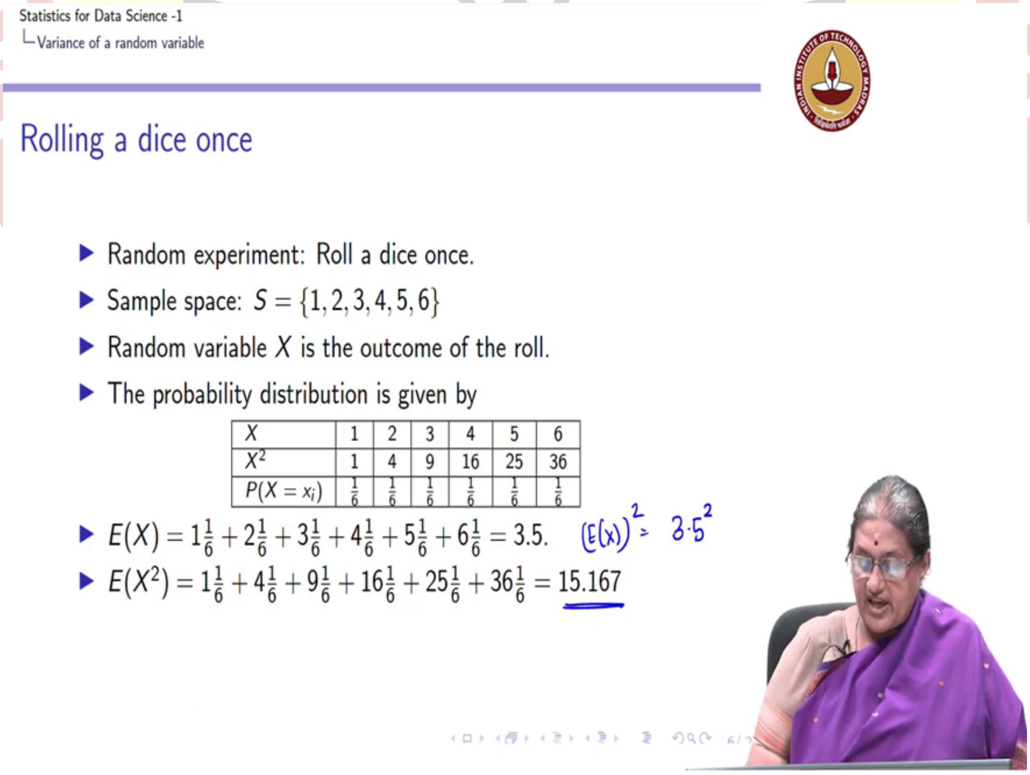

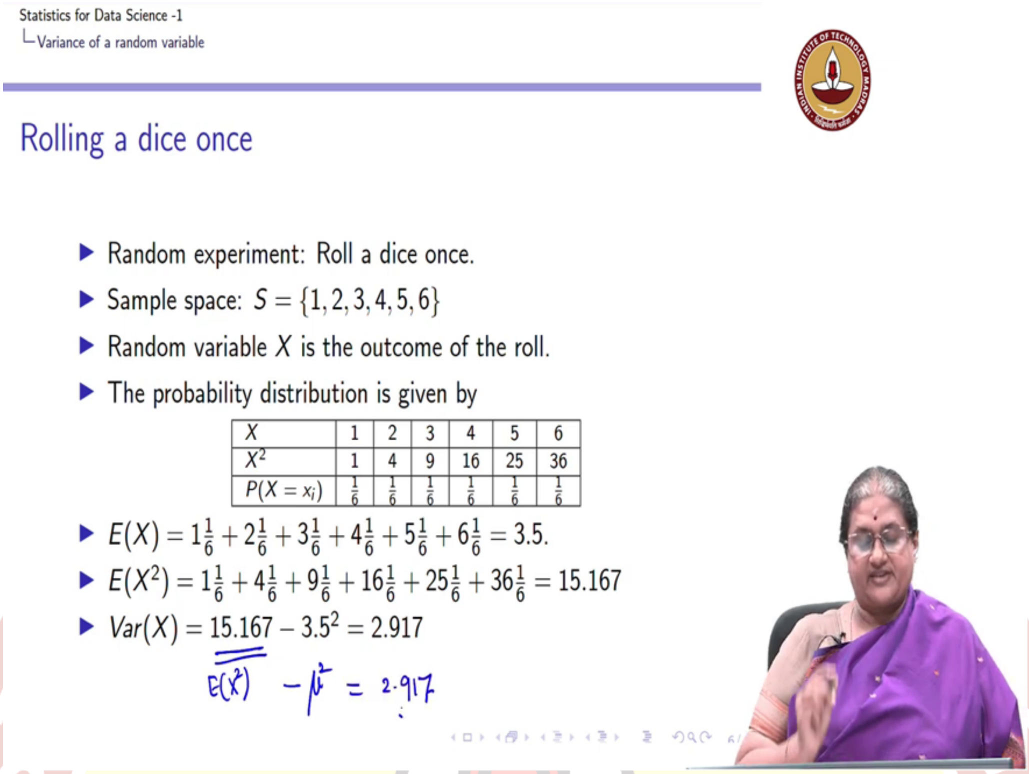

We have already computed expectation of X and we notice that $E(X)$ was 3.5. So, what is $E(X^2)$? X^2 takes the value 1 with probability $1/6$, 4 with probability $1/6$, 9 with probability $1/6$, 16 with probability $1/6$, 25 with probability $1/6$and 36 with probability $1/6$, which gives me $E(X^2)$ is 15.167, if $E(X)$ is 3.5 $E(X)^2$ is 3.5^2 and hence my $Var(X)$ is 15.167 which is $E(X^2) - \mu^2$ which is 2.917.

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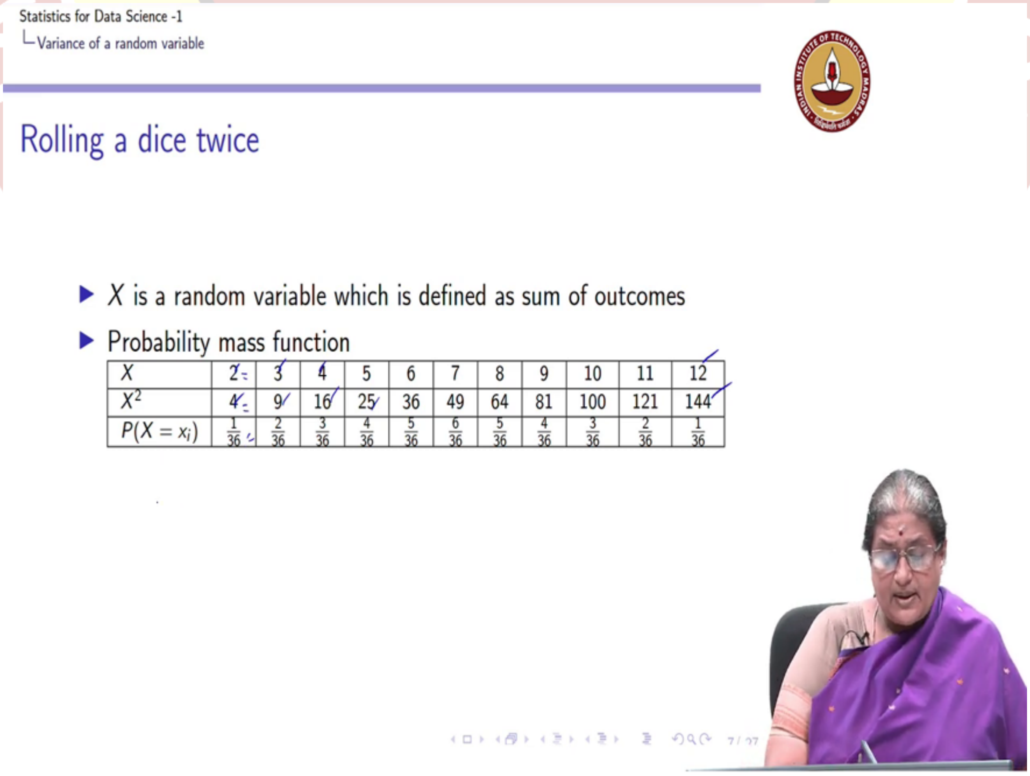

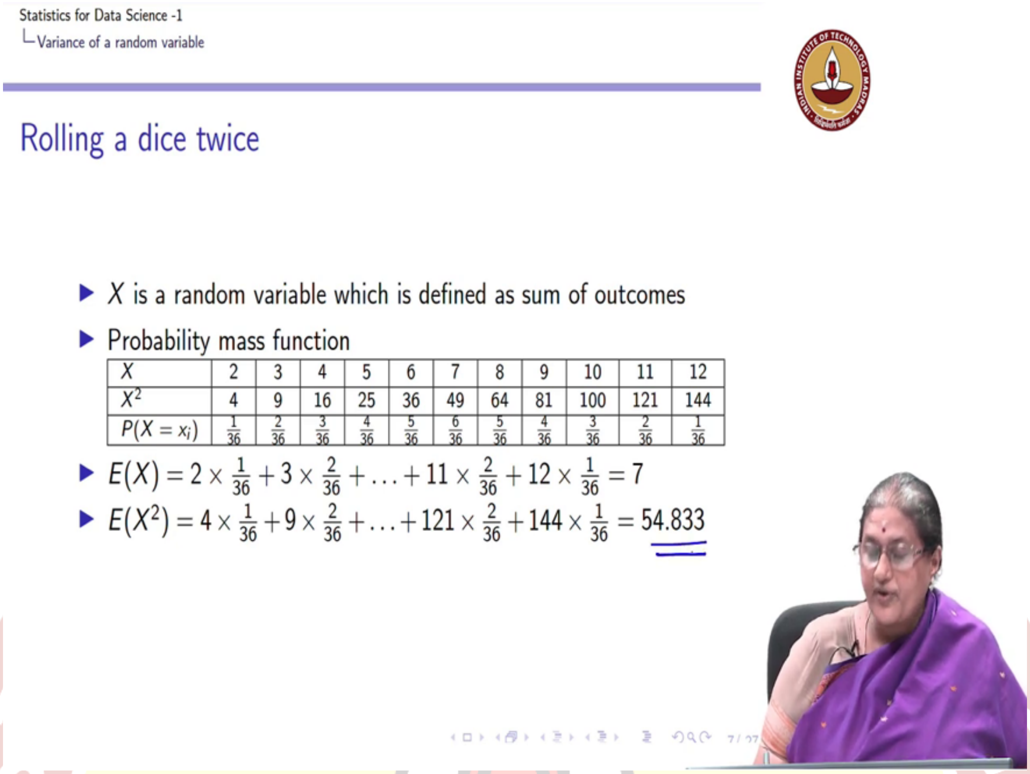

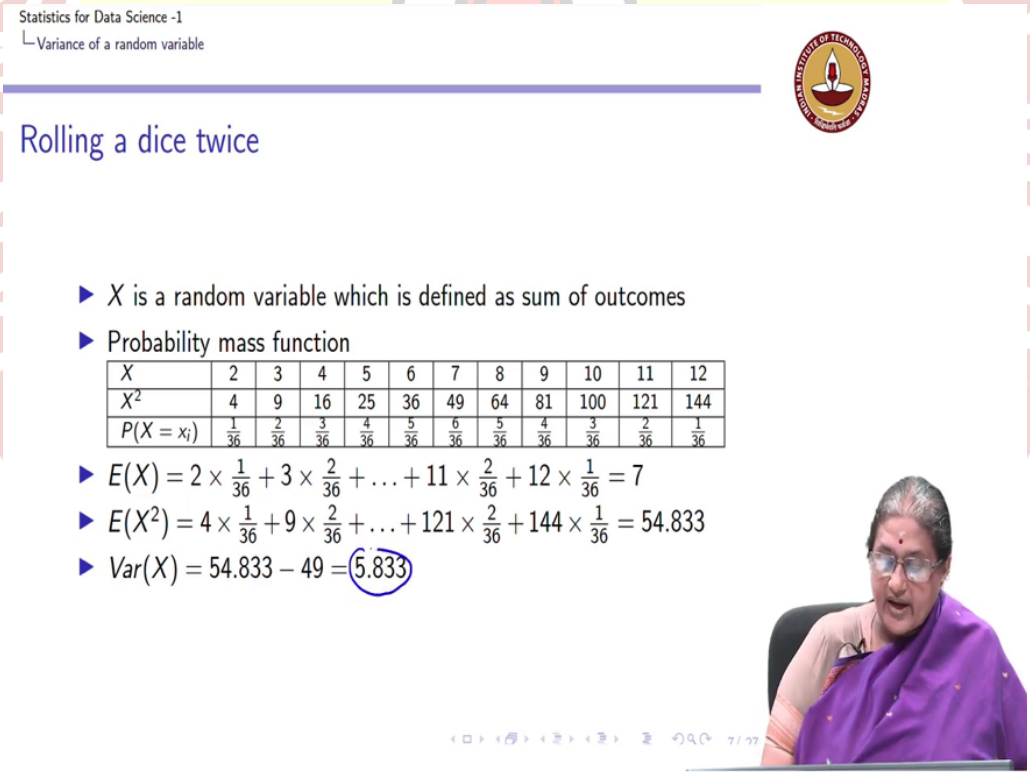

Similarly, let us look at the example of rolling a dice twice, again we know that X takes values 2, 3, 4 up to 12, hence X^2 takes the values 4, 9, 16, 25 up to 144, again the probability with which X takes the value 2 is same as the probability with which X takes the value X^2 takes the value 4 which is equal to $1/6$ in this case and hence my $E(X)$ we have already computed was 7, I can verify that $E(X^2)$ is 54.833 giving me $Var(X)$ is 5.833.

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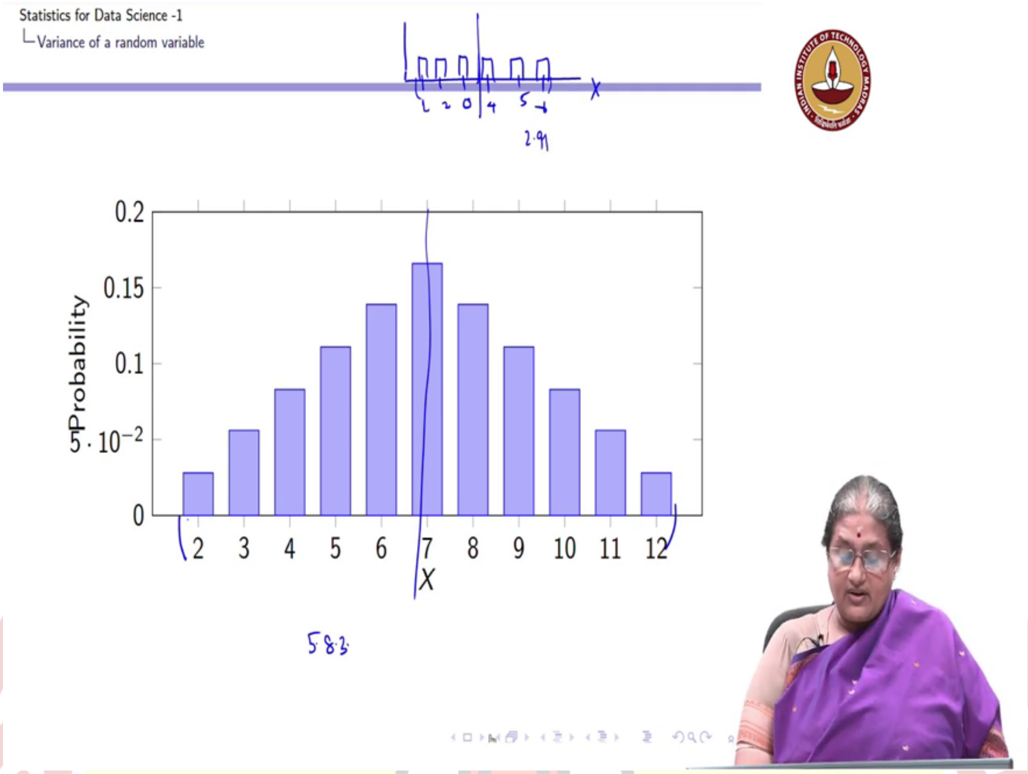

So, we notice the following that in the earlier case when X was a roll of a dice it took the value 1, 2, 3, 4, 5, 6, it had equal probability and now you can see that the mean was around 3.5 this was somehow balancing it, here the mean is at 7 and I can see that the variance in this case is 5.833 because it is taking a value from 2 to 12, so the variability is more, here it is taking the value 1 to 6 and it had a variance of about 2.91.

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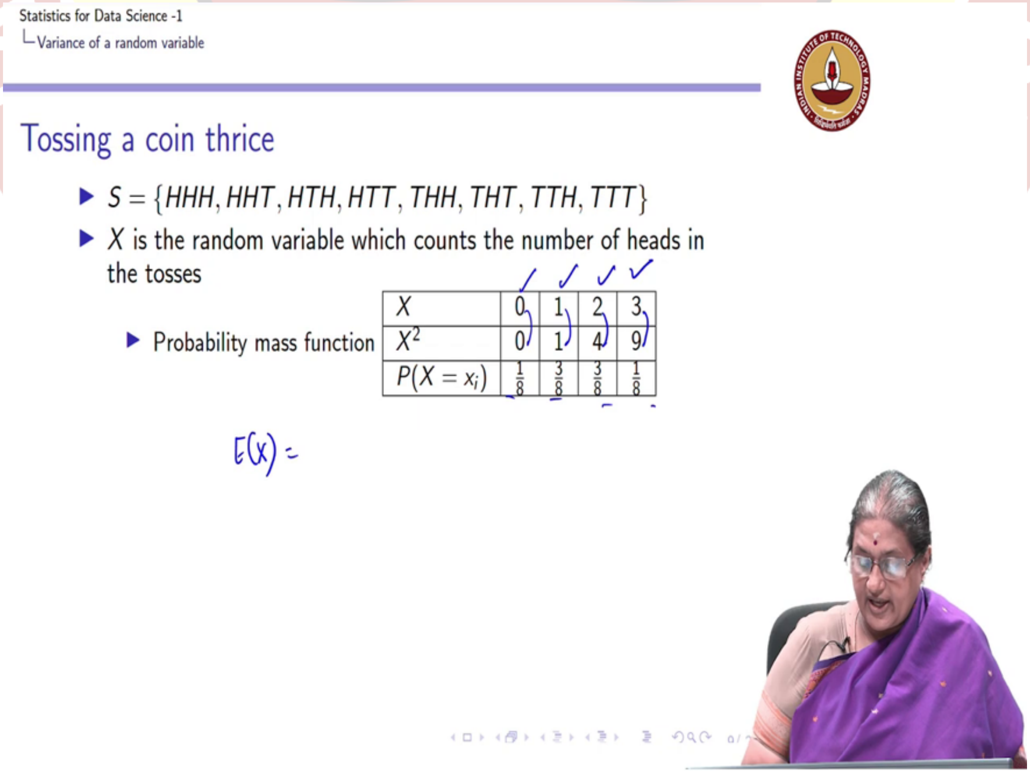

Now, let us apply the formula to tossing a coin thrice, again I know that X is a random variable which takes the values, again I know X is a random variable which takes the values 0, 1, 2, 3, hence X^2 is equal to 0, 1, 4 and 9 with the same probabilities, so I can verify that $E(X)$, I already have checked.

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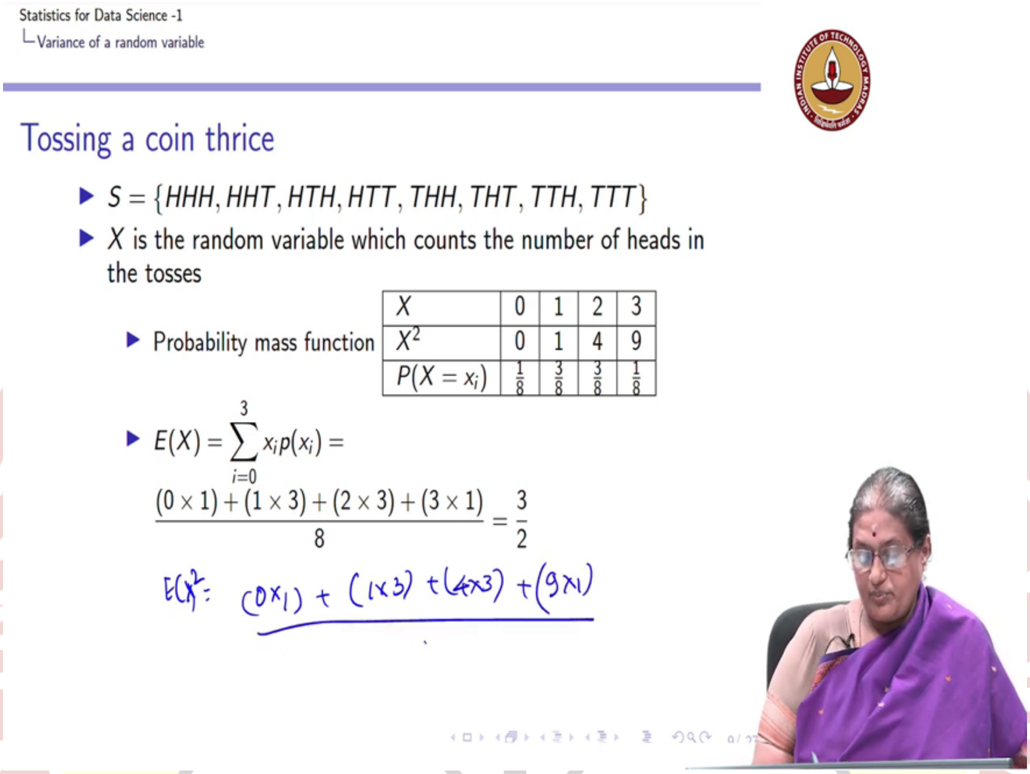

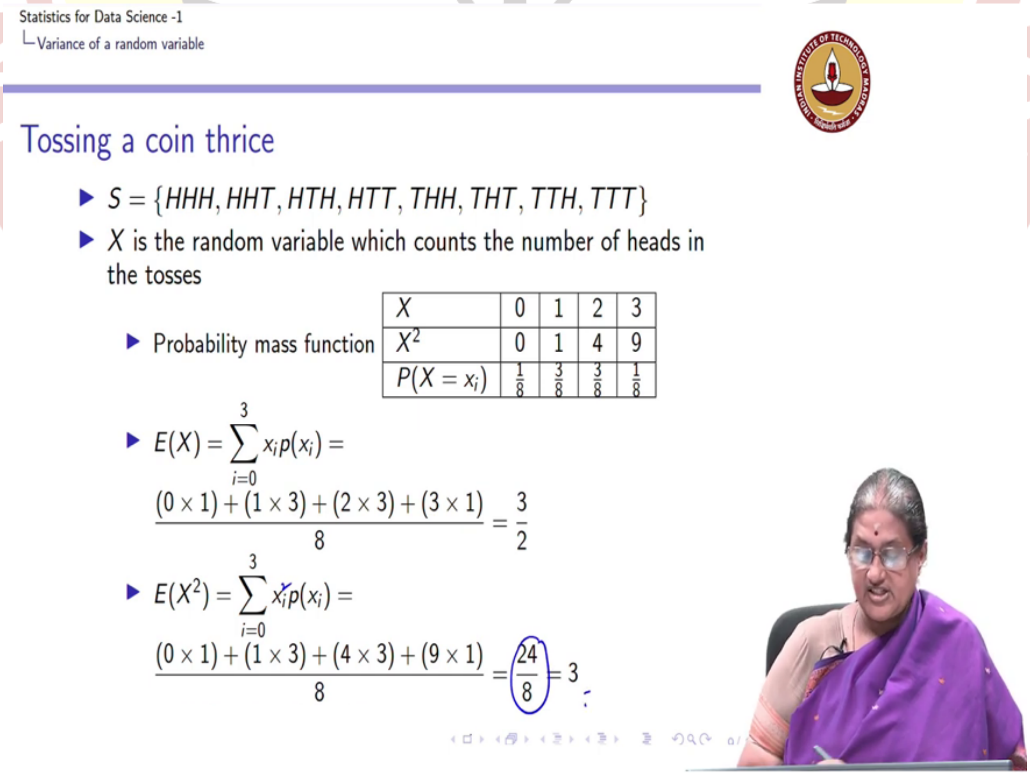

$E(X)$ was $3/2$, $E(X^2)$ is going to be $\frac{(0\times1)+(1\times3)+(4\times3)+(9\times1)}{8}$ that would be my $E(X^2)$ which I can verify that to be $24/8$, which is 3.

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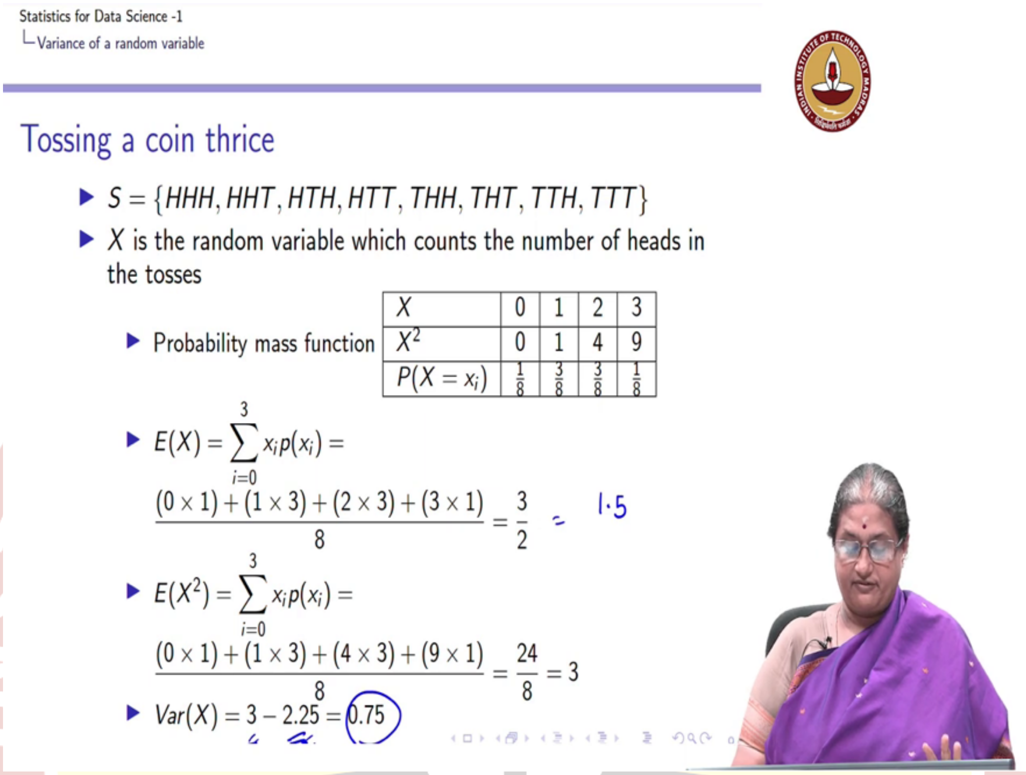

Hence, my $Var(X)$ is $3 - 2.25$ which is 0.75, because this is 1.5, 1.5^2 is 2.25, $Var(X)$ is $3 - 2.25$ which is a 0.75.

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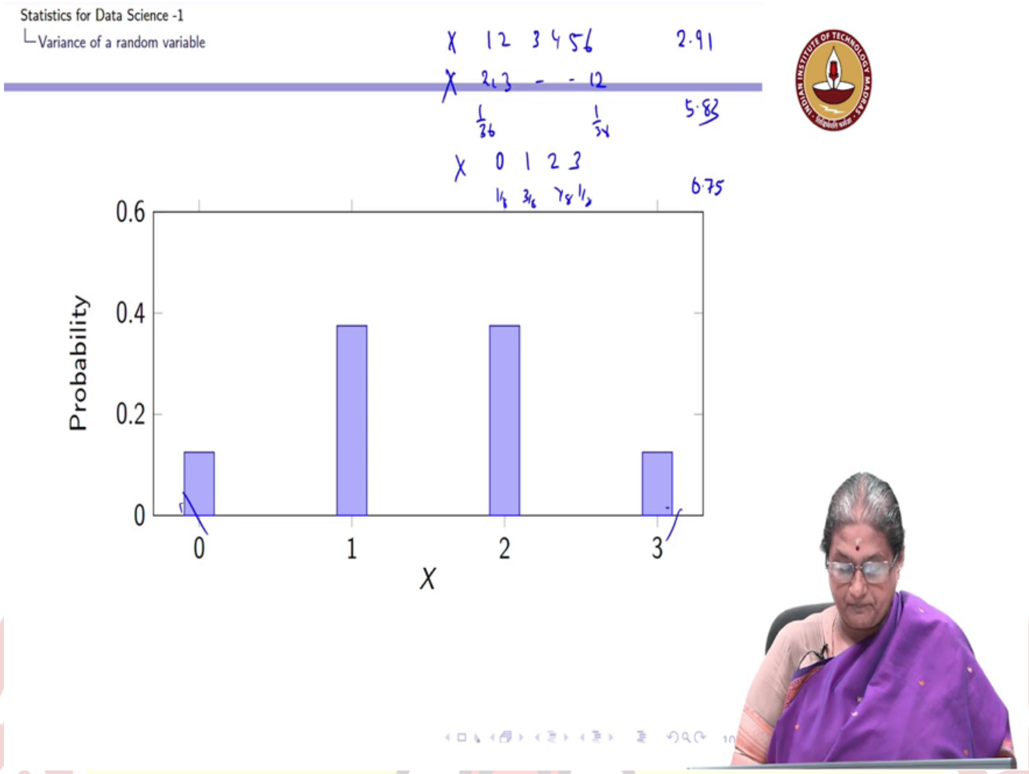

Again, you can see that the value of X is between 0 and 3, so I had when X took the value 1, 2, 3, 4, 5, 6 with equal probability I had a variance of 2.91, X took the value 2, 3, 4 up to 12 with varied probabilities, then I had 5.83, now X is again taking the value 0, 1, 2, 3 with probability $1/8$, $3/8$, $3/8$ and $1/8$, my variance is very less which is 0.75.

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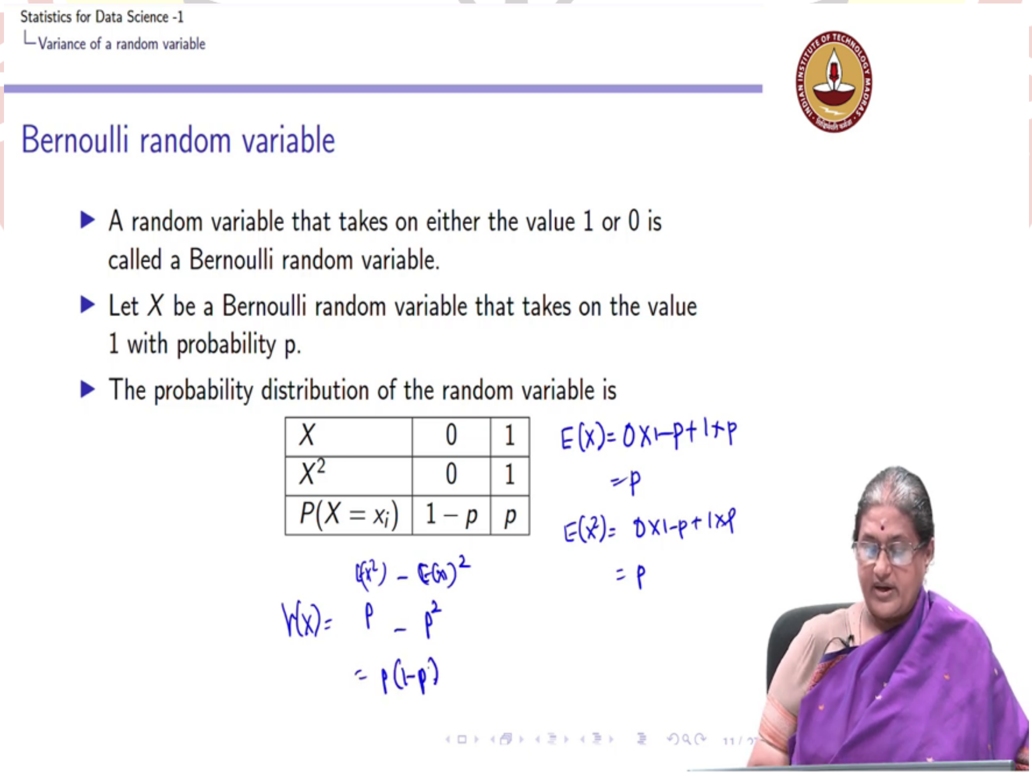

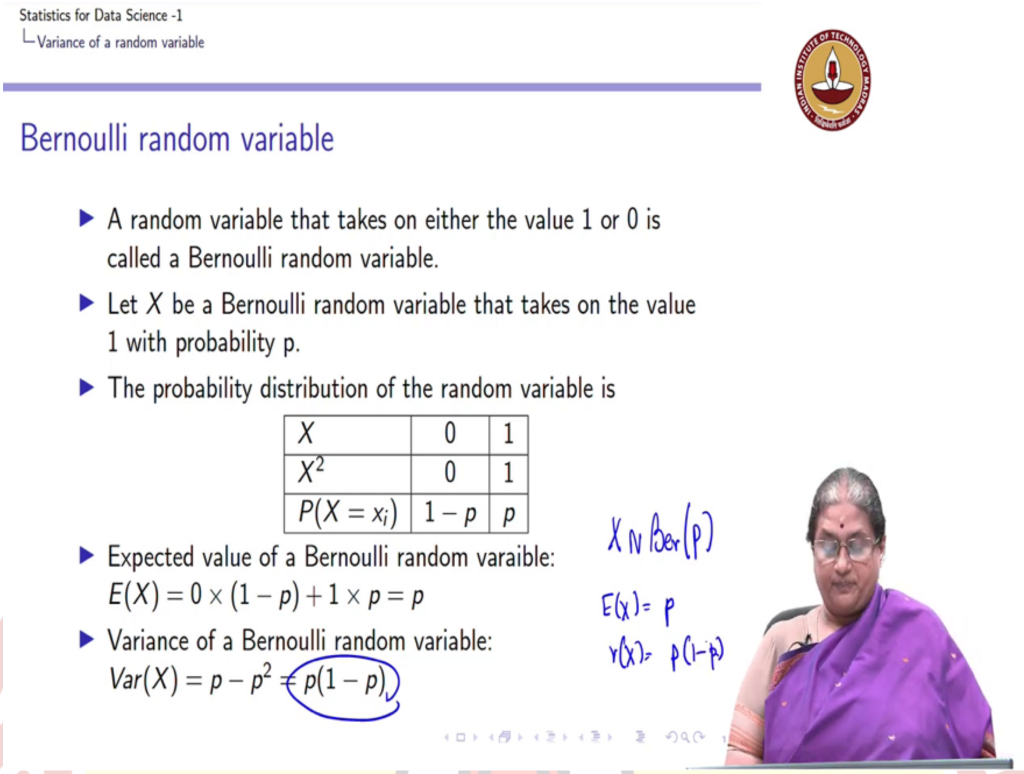

So, now let us look at the specific random variables which we have already defined recall we defined a Bernoulli random variable as that random variable which takes only 2 values for convenience I assumed it took the value 0 and 1 with $P(X)$ taking value 0 as $(1-p)$ and value 1 equal to p, if X takes the value 0 and 1, X^2 also takes the value 0 and 1 with the same probability is $(1-p)$ and p.

We verified that $E(X)$ is $0 \times (1-p) + 1 \times p$, which gave me the $E(X)$ equal to p, $E(X^2)$ is also $0 \times (1-p) + 1 \times p$ which is p, hence the variance equal to $E(X^2)$ which is equal to $p - E(X)^2$ which is $-p^2$, which I can write as $p(1-p)$. Hence the variance of a Bernoulli random variable is $p(1-p)$. So, I can refer to this as if X is a Bernoulli random variable with parameter p, then the E(X) is p, the $Var(X)$ is $p(1-p)$.

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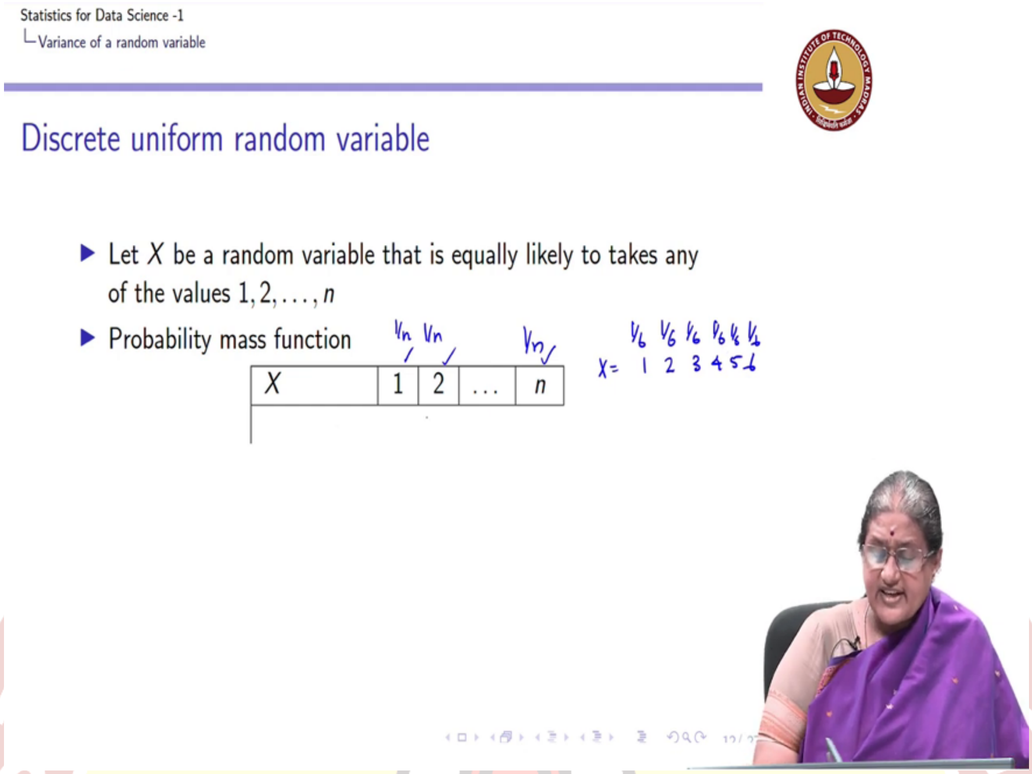

Let us, look at that case of a discrete uniform random variable, again I know I call a random variable to be a uniformly distributed random variable if X takes the value $1, 2, \ldots n$ with probability $1/n$, $1/n$, $1/n$, again an example of this is my roll of a single dice where my n was 6 and I knew that $P(X)$ equal to each value is $1/6$, $1/6$, $1/6$.

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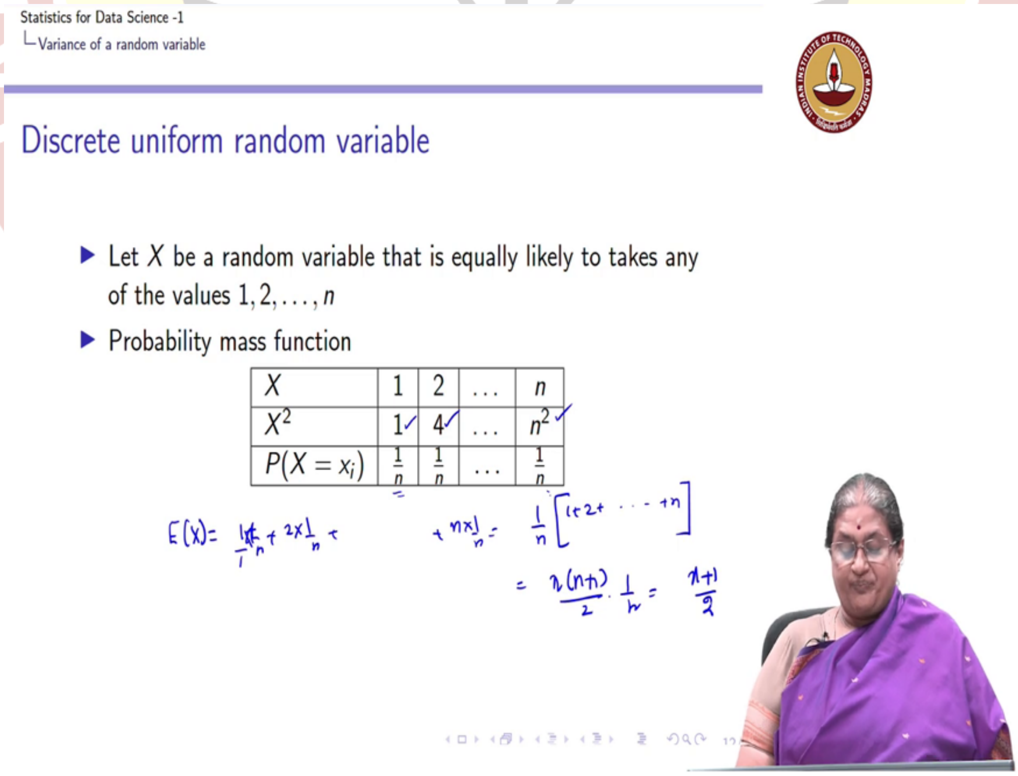

So, if X takes these values then X^2 takes the value $1, 4, \ldots, n^2$, the probabilities remain the same. We have already seen the $E(X) = 1 \times \frac{1}{n} + 2 \times \frac{1}{n} + \cdots + n \times \frac{1}{n}$ which we showed was $\frac{1}{n}(1 + 2 + \cdots + n)$ which we further saw that this was $\frac{n(n+1)}{2n}$ which gave me $\frac{(n+1)}{2}$.

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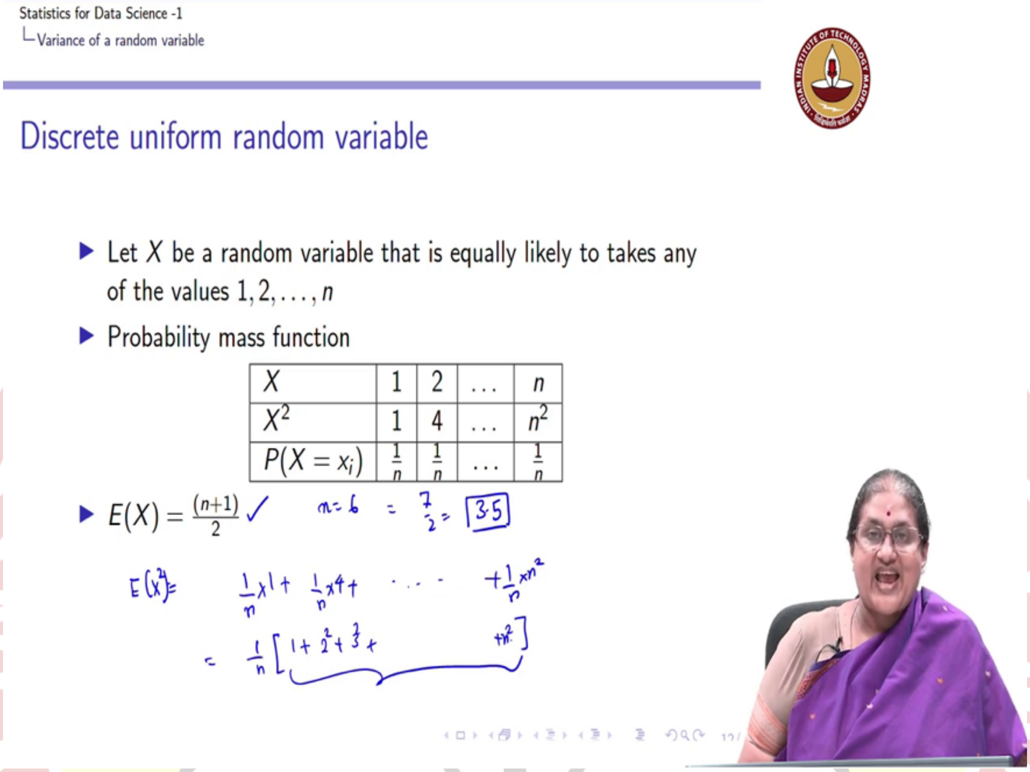

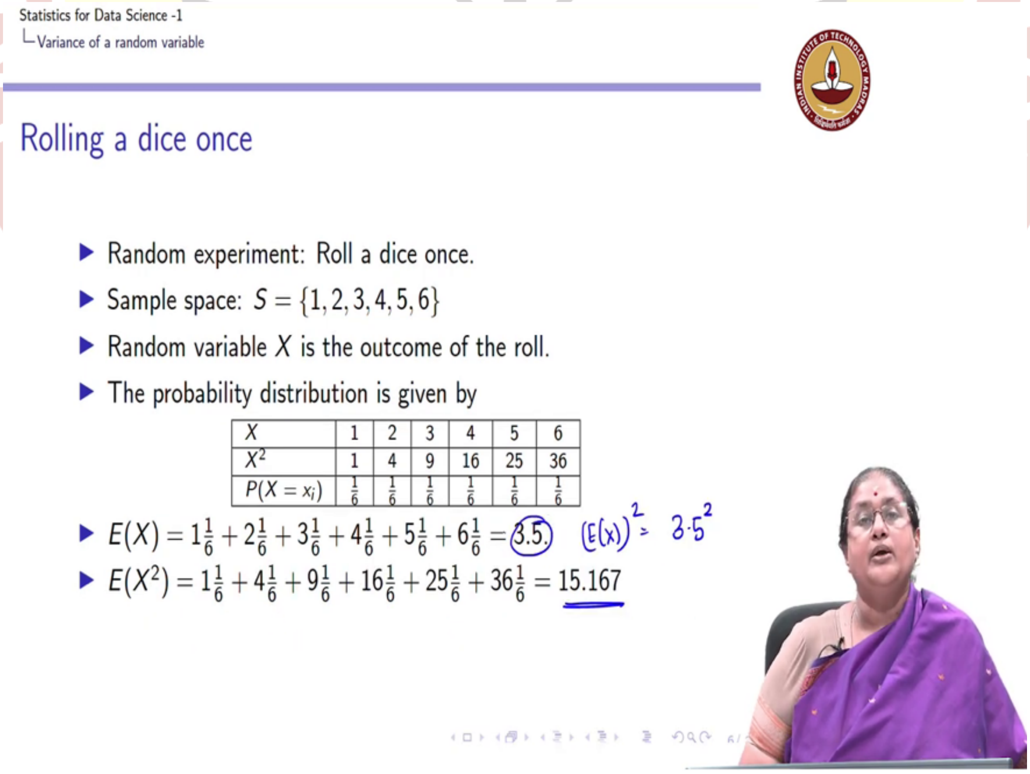

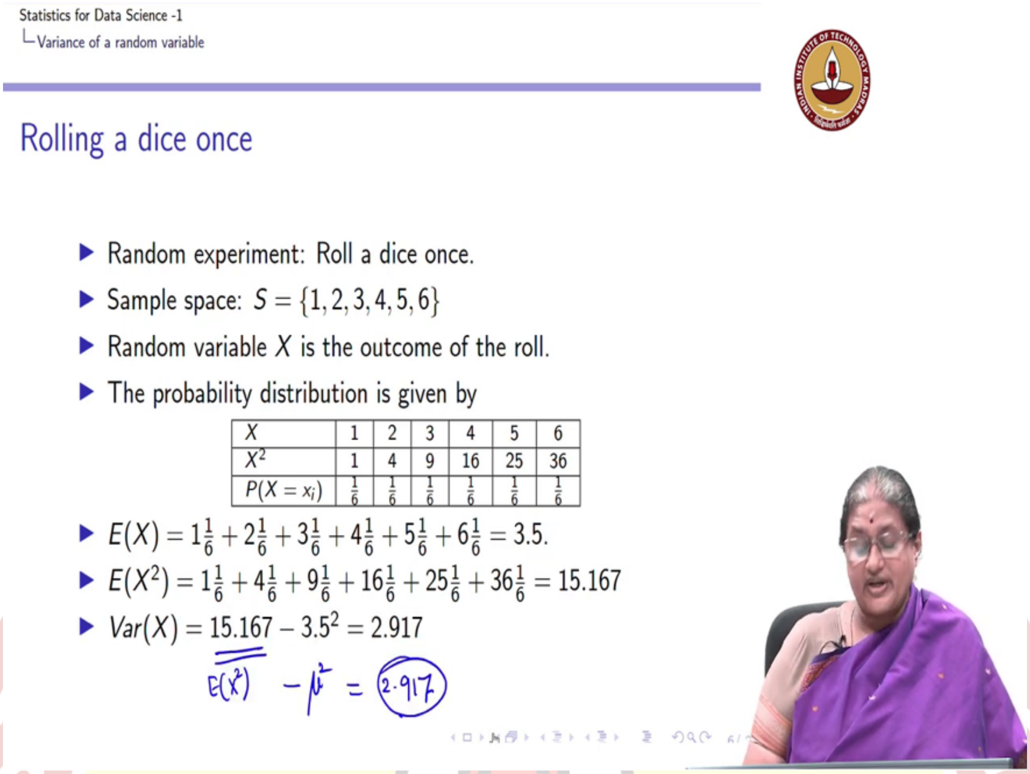

So, I already verified that $E(X) = \frac{(n+1)}{2}$, now when $n = 6$, I know that this is equal to $7/2$, which is 3.5 and this 3.5 actually matches with my expectation of a single roll of a dice that is 3.5, recall, I got a 3.5 when I found out the expectation of rolling a dice 1 was 3.5. Now, from here remember the variance was 2.91, so let us go and check for a discrete random variable. So, again when I go and do my check for a discrete random variable I see that what is $E(X^2)$ in this case.

Now, if I look at X^2, X^2 takes the value $1 + 4 + \cdots + n^2$ so I know $E(X^2)$, I can write it as 1 by so $E(X^2)$ I can write it as $1 \times \frac{1}{n} + 4 \times \frac{1}{n} + \cdots + n^2 \times \frac{1}{n}$, so I get $\frac{1}{n}(1 + 2^2 + 3^2 + \cdots + n^2)$, the term within the brackets is nothing but the sum of the squares of the first n natural numbers.

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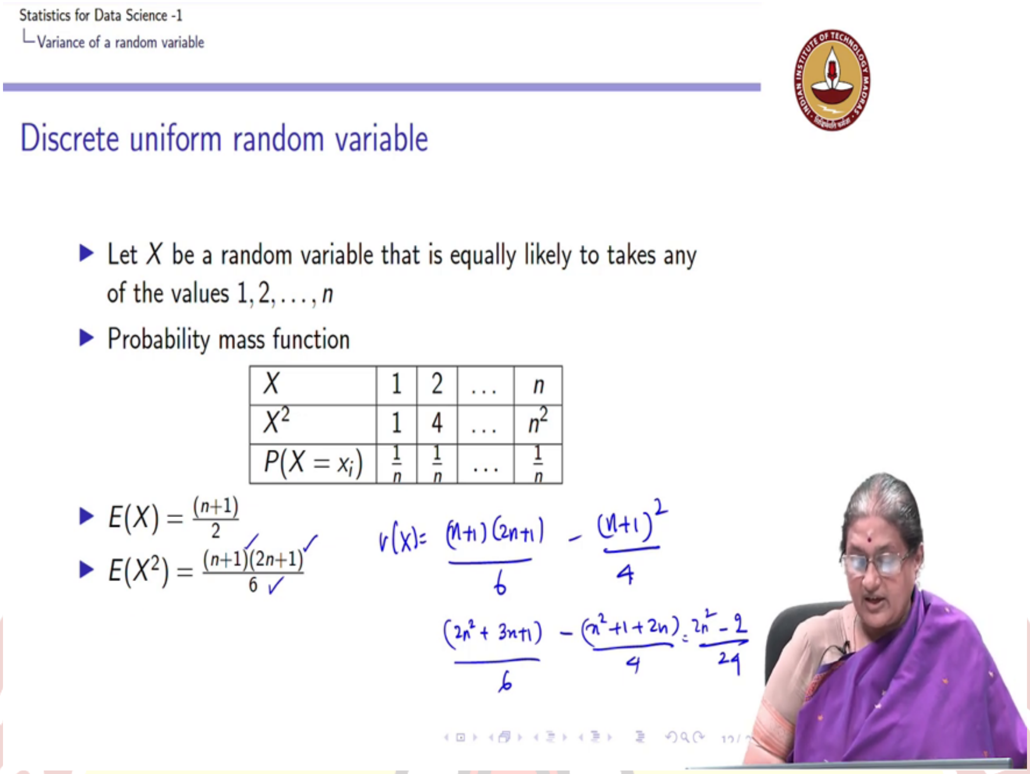

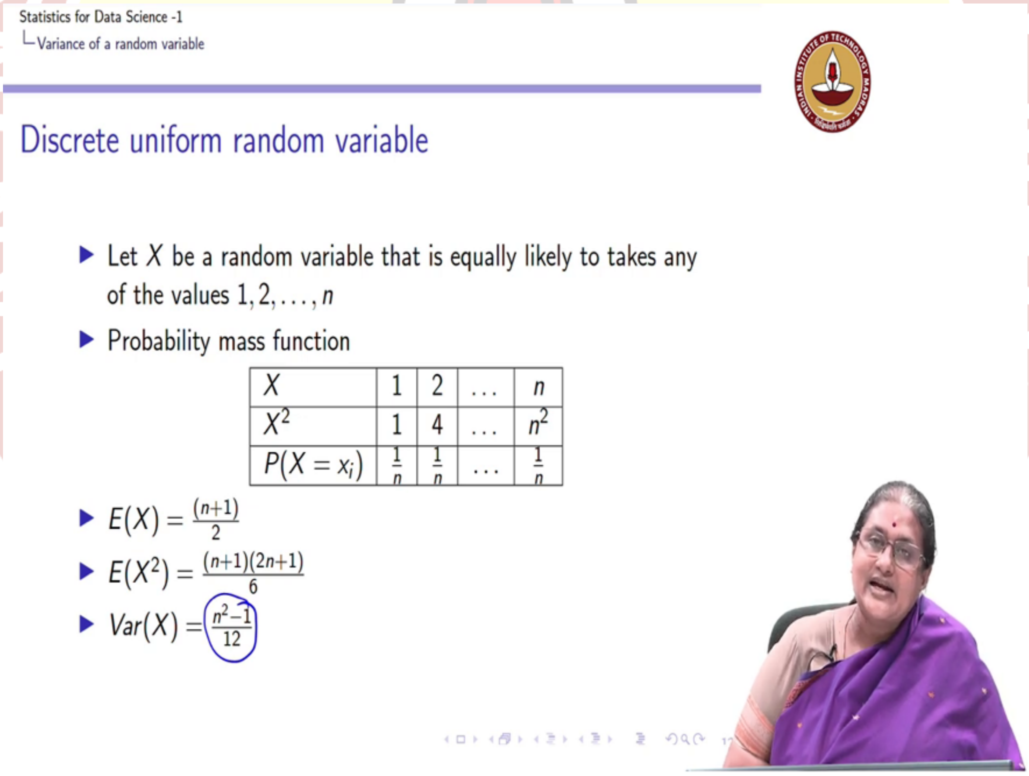

And we can verify and we already know that the $E(X^2)$ is nothing but $\frac{(n+1)(2n+1)}{6}$, because the sum of squares is $\frac{n(n+1)(2n+1)}{6}$. Now, the $Var(X)$ is going to be $[\frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}]$. This is $E(X^2)$, this is $E(X)^2$ and I can show that this would be this is going to be a $\frac{2n^2+3n+1}{6} - \frac{n^2+1+2n}{4}$ which gives me the fact that Var(X)=$\frac{(n^2-1)}{12}$. So, we computed the variance of very well-known distributions namely the Bernoulli distribution and the discrete uniform distribution.

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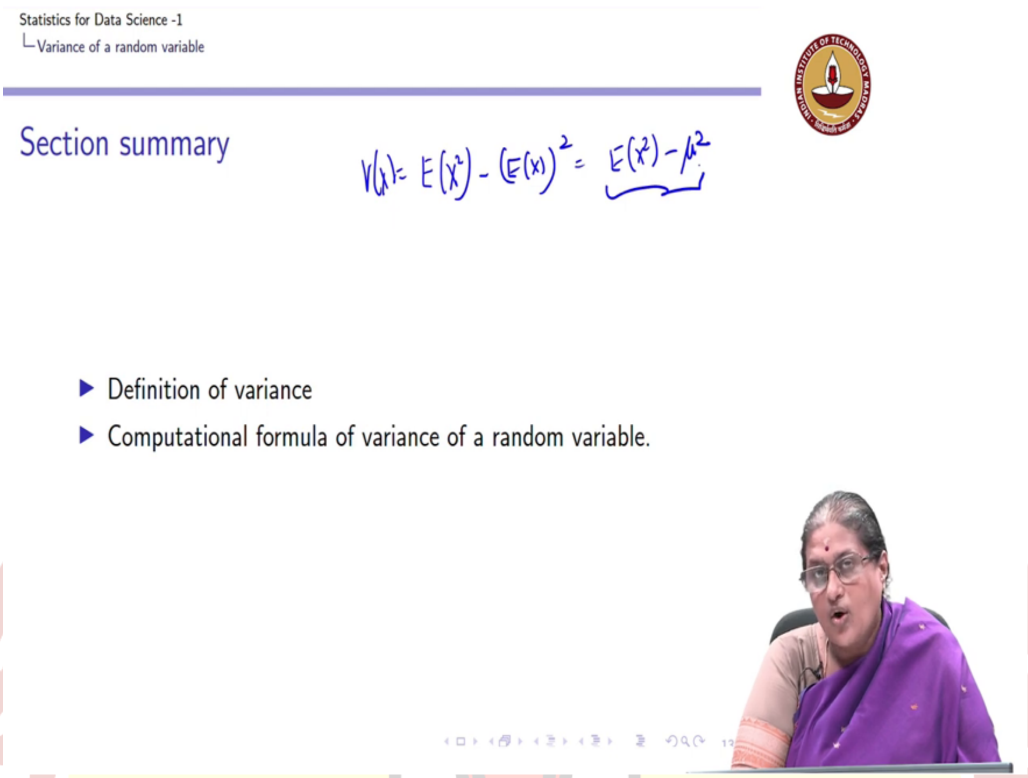

We will be looking at a few more distributions later, but for now in summary, we introduced the notion of a variance, the notion of a variance of a random variable captures the measure of spread, hence it is an important measure and we also computed through the computational formula and the computational formula is variance of X is $E(X^2) - E(X)^2$ which is the same as $E(X^2) - \mu^2$, we applied this formula to compute the variance of some well-known distributions and examples that we have already seen.