Matrix Multiplication

Madhavan Mukund

https://www.cmi.ac.in/~madhavan

Mathematics for Data Science 1
Week 11

Matrices

- A matrix is a two dimensional table
  - $r \times c$ matrix — r rows, c columns

Matrices

- A matrix is a two dimensional table
  - $r \times c$ matrix — r rows, c columns
  - Assume rows are numbered $\{0, 1, \dots, r-1\}$, columns are numbered $\{0, 1, \dots, c-1\}$

Matrices

- A matrix is a two dimensional table
  - $r \times c$ matrix — r rows, c columns
  - Assume rows are numbered $\{0, 1, \ldots, r-1\}$, columns are numbered $\{0, 1, \ldots, c-1\}$
  - Graph with n nodes, $n \times n$ adjacency matrix, entries are from $\{0, 1\}$

Matrices

- A matrix is a two dimensional table
  - $r \times c$ matrix — r rows, c columns
  - Assume rows are numbered $\{0, 1, \ldots, r-1\}$, columns are numbered $\{0, 1, \ldots, c-1\}$
  - Graph with n nodes, $n \times n$ adjacency matrix, entries are from $\{0, 1\}$
- Example: Freight traffic by rail between major cities: Bangalore, Chennai, Delhi, Hyderabad, Kolkata, Mumbai

Matrices

- A matrix is a two dimensional table
  - $r \times c$ matrix — r rows, c columns
  - Assume rows are numbered $\{0, 1, \ldots, r-1\}$, columns are numbered $\{0, 1, \ldots, c-1\}$
  - Graph with n nodes, $n \times n$ adjacency matrix, entries are from $\{0, 1\}$
- Example: Freight traffic by rail between major cities: Bangalore, Chennai, Delhi, Hyderabad, Kolkata, Mumbai
  - Number the cities:

    0-Bangalore, 1-Chennai, 2-Delhi,
    3-Hyderabad, 4-Kolkata,
    5-Mumbai

Matrices

- A matrix is a two dimensional table
  - $r \times c$ matrix — r rows, c columns
  - Assume rows are numbered $\{0, 1, \dots, r-1\}$, columns are numbered $\{0, 1, \dots, c-1\}$
  - Graph with n nodes, $n \times n$ adjacency matrix, entries are from $\{0, 1\}$
- Example: Freight traffic by rail between major cities: Bangalore, Chennai, Delhi, Hyderabad, Kolkata, Mumbai
  - Number the cities:

    0-Bangalore, 1-Chennai, 2-Delhi, 3-Hyderabad, 4-Kolkata, 5-Mumbai
  - Represent freight volume for a month as a 6×6 matrix

	0	1	2	3	4	5
0	0	694	828	384	247	479
1	642	0	919	575	402	673
2	768	734	0	231	595	540
3	731	606	156	0	351	804
4	825	607	316	490	0	998
5	196	580	339	588	394	0

Adding matrices

- Suppose we have freight volumes for first and second half of financial year.

April–September

	0	1	2	3	4	5
0	0	694	828	384	247	479
1	642	0	919	575	402	673
2	768	734	0	231	595	540
3	731	606	156	0	351	804
4	825	607	316	490	0	998
5	196	580	339	588	394	0

October–March

	0	1	2	3	4	5
0	0	851	626	280	399	365
1	544	0	479	269	432	933
2	867	804	0	681	326	398
3	727	976	418	0	667	294
4	894	390	247	547	0	314
5	914	147	574	859	524	0

Adding matrices

- Suppose we have freight volumes for first and second half of financial year.

April–September

	0	1	2	3	4	5
0	0	694	828	384	247	479
1	642	0	919	575	402	673
2	768	734	0	231	595	540
3	731	606	156	0	351	804
4	825	607	316	490	0	998
5	196	580	339	588	394	0

October–March

	0	1	2	3	4	5
0	0	851	626	280	399	365
1	544	0	479	269	432	933
2	867	804	0	681	326	398
3	727	976	418	0	667	294
4	894	390	247	547	0	314
5	914	147	574	859	524	0

- How do we compute the freight volumes for the entire financial year?

Adding matrices

- Suppose we have freight volumes for first and second half of financial year.

April–September

	0	1	2	3	4	5
0	0	694	828	384	247	479
1	642	0	919	575	402	673
2	768	734	0	231	595	540
3	731	606	156	0	351	804
4	825	607	316	490	0	998
5	196	580	339	588	394	0

October–March

	0	1	2	3	4	5
0	0	851	626	280	399	365
1	544	0	479	269	432	933
2	867	804	0	681	326	398
3	727	976	418	0	667	294
4	894	390	247	547	0	314
5	914	147	574	859	524	0

- How do we compute the freight volumes for the entire financial year?
- Add the corresponding entries in the two tables
  - Total freight volume from 2 (Delhi) to 4 (Kolkata) is $595 + 326 = 921$

Adding matrices

- For a matrix M, $M[i,j]$ is the entry in row i, column j

Adding matrices

- For a matrix M, $M[i,j]$ is the entry in row i, column j
- Let A and B represent the volumes in the two half-years

$$A \quad \begin{array}{c|rrrrrr} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 694 & 828 & 384 & 247 & 479 \\ 1 & 642 & 0 & 919 & 575 & 402 & 673 \\ 2 & 768 & 734 & 0 & 231 & 595 & 540 \\ 3 & 731 & 606 & 156 & 0 & 351 & 804 \\ 4 & 825 & 607 & 316 & 490 & 0 & 998 \\ 5 & 196 & 580 & 339 & 588 & 394 & 0 \end{array}$$

$$B \quad \begin{array}{c|rrrrrr} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 851 & 626 & 280 & 399 & 365 \\ 1 & 544 & 0 & 479 & 269 & 432 & 933 \\ 2 & 867 & 804 & 0 & 681 & 326 & 398 \\ 3 & 727 & 976 & 418 & 0 & 667 & 294 \\ 4 & 894 & 390 & 247 & 547 & 0 & 314 \\ 5 & 914 & 147 & 574 & 859 & 524 & 0 \end{array}$$

Adding matrices

- For a matrix M, $M[i,j]$ is the entry in row i, column j
- Let A and B represent the volumes in the two half-years
- Let C represent the annual volume

$$A \quad \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 694 & 828 & 384 & 247 & 479 \\ 1 & 642 & 0 & 919 & 575 & 402 & 673 \\ 2 & 768 & 734 & 0 & 231 & 595 & 540 \\ 3 & 731 & 606 & 156 & 0 & 351 & 804 \\ 4 & 825 & 607 & 316 & 490 & 0 & 998 \\ 5 & 196 & 580 & 339 & 588 & 394 & 0 \end{array}$$

$$B \quad \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 851 & 626 & 280 & 399 & 365 \\ 1 & 544 & 0 & 479 & 269 & 432 & 933 \\ 2 & 867 & 804 & 0 & 681 & 326 & 398 \\ 3 & 727 & 976 & 418 & 0 & 667 & 294 \\ 4 & 894 & 390 & 247 & 547 & 0 & 314 \\ 5 & 914 & 147 & 574 & 859 & 524 & 0 \end{array}$$

Adding matrices

- For a matrix M, $M[i,j]$ is the entry in row i, column j
- Let A and B represent the volumes in the two half-years
- Let C represent the annual volume
- For each i,j, $C[i,j] = A[i,j] + B[i,j]$

A

	0	1	2	3	4	5
0	0	694	828	384	247	479
1	642	0	919	575	402	673
2	768	734	0	231	595	540
3	731	606	156	0	351	804
4	825	607	316	490	0	998
5	196	580	339	588	394	0

B

	0	1	2	3	4	5
0	0	851	626	280	399	365
1	544	0	479	269	432	933
2	867	804	0	681	326	398
3	727	976	418	0	667	294
4	894	390	247	547	0	314
5	914	147	574	859	524	0

C

	0	1	2	3	4	5
0	0	1545	1454	664	646	844
1	1186	0	1398	844	834	1606
2	1635	1538	0	912	921	938
3	1458	1582	574	0	1018	1098
4	1719	997	563	1037	0	1312
5	1110	727	913	1447	918	0

Adding matrices

- For a matrix M, $M[i,j]$ is the entry in row i, column j
- Let A and B represent the volumes in the two half-years
- Let C represent the annual volume
- For each i,j, $C[i,j] = A[i,j] + B[i,j]$
- More concisely, $C = A + B$ — matrix addition

$$A \quad \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 694 & 828 & 384 & 247 & 479 \\ 1 & 642 & 0 & 919 & 575 & 402 & 673 \\ 2 & 768 & 734 & 0 & 231 & 595 & 540 \\ 3 & 731 & 606 & 156 & 0 & 351 & 804 \\ 4 & 825 & 607 & 316 & 490 & 0 & 998 \\ 5 & 196 & 580 & 339 & 588 & 394 & 0 \end{array}$$

$$B \quad \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 851 & 626 & 280 & 399 & 365 \\ 1 & 544 & 0 & 479 & 269 & 432 & 933 \\ 2 & 867 & 804 & 0 & 681 & 326 & 398 \\ 3 & 727 & 976 & 418 & 0 & 667 & 294 \\ 4 & 894 & 390 & 247 & 547 & 0 & 314 \\ 5 & 914 & 147 & 574 & 859 & 524 & 0 \end{array}$$

$$C \quad \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 1545 & 1454 & 664 & 646 & 844 \\ 1 & 1186 & 0 & 1398 & 844 & 834 & 1606 \\ 2 & 1635 & 1538 & 0 & 912 & 921 & 938 \\ 3 & 1458 & 1582 & 574 & 0 & 1018 & 1098 \\ 4 & 1719 & 997 & 563 & 1037 & 0 & 1312 \\ 5 & 1110 & 727 & 913 & 1447 & 918 & 0 \end{array}$$

Multiplying matrices

- Can we multiply matrices the same way?

Multiplying matrices

- Can we multiply matrices the same way?
- For each i, j, $C[i,j] = A[i,j] \times B[i,j]$

$$A \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 6 & 8 & 3 \\ 1 & 4 & 0 & 9 & 7 \\ 2 & 6 & 3 & 0 & 1 \\ 3 & 1 & 0 & 5 & 0 \end{array}$$

$$B \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 2 & 8 \\ 1 & 5 & 0 & 4 & 2 \\ 2 & 6 & 0 & 0 & 1 \\ 3 & 2 & 6 & 4 & 0 \end{array}$$

$$C \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 6 & 16 & 24 \\ 1 & 20 & 0 & 36 & 14 \\ 2 & 36 & 0 & 0 & 1 \\ 3 & 2 & 0 & 20 & 0 \end{array}$$

Multiplying matrices

- Can we multiply matrices the same way?
- For each i,j, $C[i,j] = A[i,j] \times B[i,j]$
- This turns out to be not very useful

$$A \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 6 & 8 & 3 \\ 1 & 4 & 0 & 9 & 7 \\ 2 & 6 & 3 & 0 & 1 \\ 3 & 1 & 0 & 5 & 0 \end{array}$$

$$B \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 2 & 8 \\ 1 & 5 & 0 & 4 & 2 \\ 2 & 6 & 0 & 0 & 1 \\ 3 & 2 & 6 & 4 & 0 \end{array}$$

$$C \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 6 & 16 & 24 \\ 1 & 20 & 0 & 36 & 14 \\ 2 & 36 & 0 & 0 & 1 \\ 3 & 2 & 0 & 20 & 0 \end{array}$$

Multiplying matrices

- Can we multiply matrices the same way?
- For each i, j, $C[i,j] = A[i,j] \times B[i,j]$
- This turns out to be not very useful
- Instead, we compute $C[i,j]$ in a more complicated way

$$A \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 6 & 8 & 3 \\ 1 & 4 & 0 & 9 & 7 \\ 2 & 6 & 3 & 0 & 1 \\ 3 & 1 & 0 & 5 & 0 \end{array}$$

$$B \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 2 & 8 \\ 1 & 5 & 0 & 4 & 2 \\ 2 & 6 & 0 & 0 & 1 \\ 3 & 2 & 6 & 4 & 0 \end{array}$$

Multiplying matrices

- Can we multiply matrices the same way?
- For each i,j, $C[i,j] = A[i,j] \times B[i,j]$
- This turns out to be not very useful
- Instead, we compute $C[i,j]$ in a more complicated way
- Assume $r = c = n$. Let

$$\begin{aligned} C[i,j] &= A[i,0] \cdot B[0,j] + \\ &\quad A[i,1] \cdot B[1,j] + \\ &\quad \cdots + \\ &\quad A[i,n-1] \cdot B[n-1,j] \end{aligned}$$

$$A \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 6 & 8 & 3 \\ 1 & 4 & 0 & 9 & 7 \\ 2 & 6 & 3 & 0 & 1 \\ 3 & 1 & 0 & 5 & 0 \end{array}$$

$$B \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 2 & 8 \\ 1 & 5 & 0 & 4 & 2 \\ 2 & 6 & 0 & 0 & 1 \\ 3 & 2 & 6 & 4 & 0 \end{array}$$

Multiplying matrices

- Can we multiply matrices the same way?
- For each i, j, $C[i,j] = A[i,j] \times B[i,j]$
- This turns out to be not very useful
- Instead, we compute $C[i,j]$ in a more complicated way
- Assume $r = c = n$. Let

$$
\begin{aligned}
C[i,j] \quad = \quad & A[i,0] \cdot B[0,j] + \\
& A[i,1] \cdot B[1,j] + \\
& \cdots + \\
& A[i,n-1] \cdot B[n-1,j]
\end{aligned}
$$

- For instance, $C[1,3] = 4 \cdot 8 + 0 \cdot 2 + 9 \cdot 1 + 7 \cdot 0 = 41$

$$
A \quad \begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 6 & 8 & 3 \\
1 & 4 & 0 & 9 & 7 \\
2 & 6 & 3 & 0 & 1 \\
3 & 1 & 0 & 5 & 0
\end{array}
$$

$$
B \quad \begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 & 8 \\
1 & 5 & 0 & 4 & 2 \\
2 & 6 & 0 & 0 & 1 \\
3 & 2 & 6 & 4 & 0
\end{array}
$$

Multiplying matrices

- Can we multiply matrices the same way?
- For each i, j, $C[i,j] = A[i,j] \times B[i,j]$
- This turns out to be not very useful
- Instead, we compute $C[i,j]$ in a more complicated way
- Assume $r = c = n$. Let

$$\begin{aligned} C[i,j] \quad = \quad & A[i,0] \cdot B[0,j] + \\ & A[i,1] \cdot B[1,j] + \\ & \cdots + \\ & A[i,n-1] \cdot B[n-1,j] \end{aligned}$$

- For instance, $C[1,3] = 4 \cdot 8 + 0 \cdot 2 + 9 \cdot 1 + 7 \cdot 0 = 41$

$$A \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 6 & 8 & 3 \\ 1 & 4 & 0 & 9 & 7 \\ 2 & 6 & 3 & 0 & 1 \\ 3 & 1 & 0 & 5 & 0 \end{array}$$

$$B \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 2 & 8 \\ 1 & 5 & 0 & 4 & 2 \\ 2 & 6 & 0 & 0 & 1 \\ 3 & 2 & 6 & 4 & 0 \end{array}$$

$$C \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 84 & 18 & 36 & 20 \\ 1 & 68 & 46 & 36 & 41 \\ 2 & 17 & 12 & 28 & 54 \\ 3 & 30 & 1 & 2 & 13 \end{array}$$

Multiplying matrices

- Matrix product: $C = A \times B$

$$
\begin{array}{cc|cccc}
 & & 0 & 1 & 2 & 3 \\
 & 0 & 0 & 6 & 8 & 3 \\
A & 1 & 4 & 0 & 9 & 7 \\
 & 2 & 6 & 3 & 0 & 1 \\
 & 3 & 1 & 0 & 5 & 0
\end{array}
$$

$$
\begin{array}{cc|cccc}
 & & 0 & 1 & 2 & 3 \\
 & 0 & 0 & 1 & 2 & 8 \\
B & 1 & 5 & 0 & 4 & 2 \\
 & 2 & 6 & 0 & 0 & 1 \\
 & 3 & 2 & 6 & 4 & 0
\end{array}
$$

$$
\begin{array}{cc|cccc}
 & & 0 & 1 & 2 & 3 \\
 & 0 & 84 & 18 & 36 & 20 \\
C & 1 & 68 & 46 & 36 & 41 \\
 & 2 & 17 & 12 & 28 & 54 \\
 & 3 & 30 & 1 & 2 & 13
\end{array}
$$

Multiplying matrices

- Matrix product: $C = A \times B$
- For $r = c = n$,

$$
\begin{aligned}
C[i,j] \quad = \quad & A[i,0] \cdot B[0,j] + \\
& A[i,1] \cdot B[1,j] + \\
& \cdots + \\
& A[i,n-1] \cdot B[n-1,j]
\end{aligned}
$$

$$
A \quad \begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 6 & 8 & 3 \\
1 & 4 & 0 & 9 & 7 \\
2 & 6 & 3 & 0 & 1 \\
3 & 1 & 0 & 5 & 0
\end{array}
$$

$$
B \quad \begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 & 8 \\
1 & 5 & 0 & 4 & 2 \\
2 & 6 & 0 & 0 & 1 \\
3 & 2 & 6 & 4 & 0
\end{array}
$$

$$
C \quad \begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & 84 & 18 & 36 & 20 \\
1 & 68 & 46 & 36 & 41 \\
2 & 17 & 12 & 28 & 54 \\
3 & 30 & 1 & 2 & 13
\end{array}
$$

Multiplying matrices

- Matrix product: $C = A \times B$
- For $r = c = n$,

$$\begin{aligned} C[i,j] &= A[i,0] \cdot B[0,j] + \\ & \quad A[i,1] \cdot B[1,j] + \\ & \quad \cdots + \\ & \quad A[i,n-1] \cdot B[n-1,j] \end{aligned}$$

- More concisely, $C[i,j] = \sum_{k=0}^{n-1} A[i,k] \cdot B[k,j]$

$$A \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 6 & 8 & 3 \\ 1 & 4 & 0 & 9 & 7 \\ 2 & 6 & 3 & 0 & 1 \\ 3 & 1 & 0 & 5 & 0 \end{array}$$

$$B \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 2 & 8 \\ 1 & 5 & 0 & 4 & 2 \\ 2 & 6 & 0 & 0 & 1 \\ 3 & 2 & 6 & 4 & 0 \end{array}$$

$$C \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 84 & 18 & 36 & 20 \\ 1 & 68 & 46 & 36 & 41 \\ 2 & 17 & 12 & 28 & 54 \\ 3 & 30 & 1 & 2 & 13 \end{array}$$

Multiplying matrices

- Matrix product: $C = A \times B$
- For $r = c = n$,

$$
\begin{aligned}
C[i,j] \quad = \quad & A[i,0] \cdot B[0,j] + \\
& A[i,1] \cdot B[1,j] + \\
& \cdots + \\
& A[i,n-1] \cdot B[n-1,j]
\end{aligned}
$$

- More concisely, $C[i,j] = \sum_{k=0}^{n-1} A[i,k] \cdot B[k,j]$
- Don't require both A and B to be $n \times n$
    - Each row entry of A must have a matching column entry in B
    - If A is $m \times n$ and B is $n \times p$, $A \times B$ is $m \times p$

$$
A \quad \begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 6 & 8 & 3 \\
1 & 4 & 0 & 9 & 7 \\
2 & 6 & 3 & 0 & 1 \\
3 & 1 & 0 & 5 & 0
\end{array}
$$

$$
B \quad \begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 & 8 \\
1 & 5 & 0 & 4 & 2 \\
2 & 6 & 0 & 0 & 1 \\
3 & 2 & 6 & 4 & 0
\end{array}
$$

$$
C \quad \begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & 84 & 18 & 36 & 20 \\
1 & 68 & 46 & 36 & 41 \\
2 & 17 & 12 & 28 & 54 \\
3 & 30 & 1 & 2 & 13
\end{array}
$$

Multiplying matrices

- Matrix product: $C = A \times B$
- For $r = c = n$,

$$\begin{aligned} C[i,j] &= A[i,0]\cdot B[0,j] + \\ & A[i,1]\cdot B[1,j] + \\ & \cdots + \\ & A[i,n-1]\cdot B[n-1,j] \end{aligned}$$

- More concisely, $C[i,j] = \sum_{k=0}^{n-1} A[i,k]\cdot B[k,j]$
- Don't require both A and B to be $n \times n$
  - Each row entry of A must have a matching column entry in B
  - If A is $m \times n$ and B is $n \times p$, $A \times B$ is $m \times p$

$$A \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 6 & 8 & 3 \\ 1 & 4 & 0 & 9 & 7 \\ 2 & 6 & 3 & 0 & 1 \end{array}$$

$$B \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 5 & 0 & 4 \\ 2 & 6 & 0 & 0 \\ 3 & 2 & 6 & 4 \end{array}$$

$$C \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 84 & 18 & 36 \\ 1 & 68 & 46 & 36 \\ 2 & 17 & 12 & 28 \end{array}$$

Transitive closure

- A is an adjacency matrix
  - $A[i,j] = 1$ if and only if there is a direct edge (path of length 1) from i to j

	0	1	2	3	4	5
0	0	1	0	0	1	0
1	0	0	1	0	0	0
2	1	0	0	0	0	0
3	0	0	0	0	1	0
4	1	0	0	1	0	0
5	0	0	0	1	0	0

Transitive closure

- A is an adjacency matrix
  - $A[i,j] = 1$ if and only if there is a direct edge (path of length 1) from i to j
- A^2 represents paths of length 2
  - $A^2[i,j] = 1$ if, for some k, $A[i,k] = 1$ and $A[k,j] = 1$

	0	1	2	3	4	5
0	0	1	0	0	1	0
1	0	0	1	0	0	0
2	1	0	0	0	0	0
3	0	0	0	0	1	0
4	1	0	0	1	0	0
5	0	0	0	1	0	0

Transitive closure

- A is an adjacency matrix
  - $A[i,j] = 1$ if and only if there is a direct edge (path of length 1) from i to j
- A^2 represents paths of length 2
  - $A^2[i,j] = 1$ if, for some k, $A[i,k] = 1$ and $A[k,j] = 1$
- Algebra of boolean values

	0	1	2	3	4	5
0	0	1	0	0	1	0
1	0	0	1	0	0	0
2	1	0	0	0	0	0
3	0	0	0	0	1	0
4	1	0	0	1	0	0
5	0	0	0	1	0	0

Transitive closure

- A is an adjacency matrix
  - $A[i,j] = 1$ if and only if there is a direct edge (path of length 1) from i to j
- A^2 represents paths of length 2
  - $A^2[i,j] = 1$ if, for some k, $A[i,k] = 1$ and $A[k,j] = 1$
- Algebra of boolean values
  - `True` is 1, `False` is 0

	0	1	2	3	4	5
0	0	1	0	0	1	0
1	0	0	1	0	0	0
2	1	0	0	0	0	0
3	0	0	0	0	1	0
4	1	0	0	1	0	0
5	0	0	0	1	0	0

Transitive closure

- A is an adjacency matrix
  - $A[i,j] = 1$ if and only if there is a direct edge (path of length 1) from i to j
- A^2 represents paths of length 2
  - $A^2[i,j] = 1$ if, for some k, $A[i,k] = 1$ and $A[k,j] = 1$
- Algebra of boolean values
  - `True` is 1, `False` is 0
  - Logical `or` is represented by $+$: $0+0=0$, $0+1=1+0=1+1=1$

	0	1	2	3	4	5
0	0	1	0	0	1	0
1	0	0	1	0	0	0
2	1	0	0	0	0	0
3	0	0	0	0	1	0
4	1	0	0	1	0	0
5	0	0	0	1	0	0

Transitive closure

- A is an adjacency matrix
  - $A[i,j] = 1$ if and only if there is a direct edge (path of length 1) from i to j
- A^2 represents paths of length 2
  - $A^2[i,j] = 1$ if, for some k, $A[i,k] = 1$ and $A[k,j] = 1$
- Algebra of boolean values
  - `True` is 1, `False` is 0
  - Logical `or` is represented by $+$: $0+0=0$, $0+1=1+0=1+1=1$
  - Logical `and` is represented by $\times$: $1\times 1 = 1$, $0\times 0 = 0\times 1 = 1\times 0 = 0$

	0	1	2	3	4	5
0	0	1	0	0	1	0
1	0	0	1	0	0	0
2	1	0	0	0	0	0
3	0	0	0	0	1	0
4	1	0	0	1	0	0
5	0	0	0	1	0	0

Transitive closure

- A is an adjacency matrix
  - $A[i,j] = 1$ if and only if there is a direct edge (path of length 1) from i to j
- A^2 represents paths of length 2
  - $A^2[i,j] = 1$ if, for some k, $A[i,k] = 1$ and $A[k,j] = 1$
- Algebra of boolean values
  - `True` is 1, `False` is 0
  - Logical `or` is represented by $+$: $0+0=0$, $0+1=1+0=1+1=1$
  - Logical `and` is represented by $\times$: $1\times 1=1$, $0\times 0=0\times 1=1\times 0=0$

	0	1	2	3	4	5
0	0	1	0	0	1	0
1	0	0	1	0	0	0
2	1	0	0	0	0	0
3	0	0	0	0	1	0
4	1	0	0	1	0	0
5	0	0	0	1	0	0

- $$\begin{aligned} A^2[i,j] = \ & (A[i,0] \texttt{ and } A[0,j]) \texttt{ or} \\ & (A[i,1] \texttt{ and } A[1,j]) \texttt{ or} \\ & \ldots \\ & (A[i,n{-}1] \texttt{ and } A[n{-}1,j]) \end{aligned}$$

Transitive closure

- A is an adjacency matrix
  - $A[i,j] = 1$ if and only if there is a direct edge (path of length 1) from i to j
- A^2 represents paths of length 2
  - $A^2[i,j] = 1$ if, for some k, $A[i,k] = 1$ and $A[k,j] = 1$
- Algebra of boolean values
  - `True` is 1, `False` is 0
  - Logical `or` is represented by $+$: $0+0=0$, $0+1=1+0=1+1=1$
  - Logical `and` is represented by $\times$: $1\times 1=1$, $0\times 0=0\times 1=1\times 0=0$

	0	1	2	3	4	5
0	0	1	0	0	1	0
1	0	0	1	0	0	0
2	1	0	0	0	0	0
3	0	0	0	0	1	0
4	1	0	0	1	0	0
5	0	0	0	1	0	0

- $$\begin{aligned} A^2[i,j] = \;& (A[i,0] \texttt{ and } A[0,j]) \texttt{ or} \\ & (A[i,1] \texttt{ and } A[1,j]) \texttt{ or} \\ & \cdots \\ & (A[i,n{-}1] \texttt{ and } A[n{-}1,j]) \end{aligned}$$
- $$\begin{aligned} A^2[i,j] \;=\; & A[i,0]\times A[0,j] + \\ & A[i,1]\times A[1,j] + \\ & \cdots + \\ & A[i,n{-}1]\times A[n{-}1,j] \end{aligned}$$

Transitive closure

- $$\begin{aligned} A^2[i,j] = \; & (A[i,0] \texttt{ and } A[0,j]) \texttt{ or} \\ & (A[i,1] \texttt{ and } A[1,j]) \texttt{ or} \\ & \cdots \\ & (A[i,n{-}1] \texttt{ and } A[n{-}1,j]) \end{aligned}$$
- $$\begin{aligned} A^2[i,j] \; = \; & A[i,0] \times A[0,j] + \\ & A[i,1] \times A[1,j] + \\ & \cdots + \\ & A[i,n{-}1] \times A[n{-}1,j] \end{aligned}$$

Transitive closure

- $$A^2[i,j] = \begin{array}{l} (A[i,0] \texttt{ and } A[0,j]) \texttt{ or} \\ (A[i,1] \texttt{ and } A[1,j]) \texttt{ or} \\ \ldots \\ (A[i,n{-}1] \texttt{ and } A[n{-}1,j]) \end{array}$$
- $$A^2[i,j] = \begin{array}{l} A[i,0] \times A[0,j] + \\ A[i,1] \times A[1,j] + \\ \cdots + \\ A[i,n{-}1] \times A[n{-}1,j] \end{array}$$
- So $A^2[i,j] = \displaystyle\sum_{k=0}^{n-1} A[i,k] \times A[k,j]$

Transitive closure

- $$\begin{aligned} A^2[i,j] = \ & (A[i,0] \texttt{ and } A[0,j]) \texttt{ or} \\ & (A[i,1] \texttt{ and } A[1,j]) \texttt{ or} \\ & \cdots \\ & (A[i,n-1] \texttt{ and } A[n-1,j]) \end{aligned}$$
- $$\begin{aligned} A^2[i,j] \ = \ & A[i,0] \times A[0,j] + \\ & A[i,1] \times A[1,j] + \\ & \cdots + \\ & A[i,n-1] \times A[n-1,j] \end{aligned}$$
- So $A^2[i,j] = \sum_{k=0}^{n-1} A[i,k] \times A[k,j]$
- In other words, $A^2 = A \times A$

Transitive closure

- $$A^2[i,j] = \begin{array}{l}(A[i,0] \texttt{ and } A[0,j]) \texttt{ or} \\ (A[i,1] \texttt{ and } A[1,j]) \texttt{ or} \\ \ldots \\ (A[i,n{-}1] \texttt{ and } A[n{-}1,j])\end{array}$$
- $$A^2[i,j] = \begin{array}{l}A[i,0] \times A[0,j] + \\ A[i,1] \times A[1,j] + \\ \cdots + \\ A[i,n{-}1] \times A[n{-}1,j]\end{array}$$
- So $A^2[i,j] = \sum_{k=0}^{n-1} A[i,k] \times A[k,j]$
- In other words, $A^2 = A \times A$
- Likewise, $A^3[i,j] = 1$ if, for some k, $A^2[i,k] = 1$ and $A[k,j] = 1$
  - Paths of length 3

Transitive closure

- $$\begin{aligned} A^2[i,j] = \; & (A[i,0] \texttt{ and } A[0,j]) \texttt{ or} \\ & (A[i,1] \texttt{ and } A[1,j]) \texttt{ or} \\ & \cdots \\ & (A[i,n{-}1] \texttt{ and } A[n{-}1,j]) \end{aligned}$$
- $$\begin{aligned} A^2[i,j] \;=\; & A[i,0] \times A[0,j] + \\ & A[i,1] \times A[1,j] + \\ & \cdots + \\ & A[i,n{-}1] \times A[n{-}1,j] \end{aligned}$$
- So $A^2[i,j] = \displaystyle\sum_{k=0}^{n-1} A[i,k] \times A[k,j]$
- In other words, $A^2 = A \times A$
- Likewise, $A^3[i,j] = 1$ if, for some k, $A^2[i,k] = 1$ and $A[k,j] = 1$
  - Paths of length 3
- $A^3 = A^2 \times A$

Transitive closure

- $A^2[i,j] = \ (A[i,0] \texttt{ and } A[0,j]) \texttt{ or }$
  $(A[i,1] \texttt{ and } A[1,j]) \texttt{ or }$
  $\ldots$
  $(A[i,n{-}1] \texttt{ and } A[n{-}1,j])$
- $A^2[i,j] = A[i,0] \times A[0,j] +$
  $A[i,1] \times A[1,j] +$
  $\cdots +$
  $A[i,n{-}1] \times A[n{-}1,j]$
- So $A^2[i,j] = \sum_{k=0}^{n-1} A[i,k] \times A[k,j]$
- In other words, $A^2 = A \times A$
- Likewise, $A^3[i,j] = 1$ if, for some k, $A^2[i,k] = 1$ and $A[k,j] = 1$
  - Paths of length 3
- $A^3 = A^2 \times A = A \times A^2$

Transitive closure

- $$A^2[i,j] = \begin{array}{l}(A[i,0] \texttt{ and } A[0,j]) \texttt{ or}\\ (A[i,1] \texttt{ and } A[1,j]) \texttt{ or}\\ \cdots\\ (A[i,n{-}1] \texttt{ and } A[n{-}1,j])\end{array}$$
- $$A^2[i,j] = \begin{array}{l}A[i,0] \times A[0,j] +\\ A[i,1] \times A[1,j] +\\ \cdots +\\ A[i,n{-}1] \times A[n{-}1,j]\end{array}$$
- So $A^2[i,j] = \displaystyle\sum_{k=0}^{n-1} A[i,k] \times A[k,j]$
- In other words, $A^2 = A \times A$
- Likewise, $A^3[i,j] = 1$ if, for some k, $A^2[i,k] = 1$ and $A[k,j] = 1$
  - Paths of length 3
- $A^3 = A^2 \times A = A \times A^2$
- In general, $A^{\ell+1} = A^\ell \times A = A \times A^\ell$
  - Paths of length $\ell+1$

Transitive closure

- $A^2[i,j] = \begin{array}{l}(A[i,0] \texttt{ and } A[0,j]) \texttt{ or} \\ (A[i,1] \texttt{ and } A[1,j]) \texttt{ or} \\ \dots \\ (A[i,n{-}1] \texttt{ and } A[n{-}1,j])\end{array}$

- $A^2[i,j] = \begin{array}{l} A[i,0] \times A[0,j] + \\ A[i,1] \times A[1,j] + \\ \cdots + \\ A[i,n{-}1] \times A[n{-}1,j]\end{array}$

- So $A^2[i,j] = \sum_{k=0}^{n-1} A[i,k] \times A[k,j]$

- In other words, $A^2 = A \times A$

- Likewise, $A^3[i,j] = 1$ if, for some k, $A^2[i,k] = 1$ and $A[k,j] = 1$
  - Paths of length 3

- $A^3 = A^2 \times A = A \times A^2$

- In general, $A^{\ell+1} = A^\ell \times A = A \times A^\ell$
  - Paths of length $\ell{+}1$

- Finally, $A^+ = A + A^2 + \cdots + A^{n-1}$
  - $A^+[i,j] = 1$ if i,j connected by
    path of length 1 `or`
    path of length 2 `or`
    $\cdots$ `or`
    path of length $n{-}1$

Summary

- Matrix addition is defined elementwise
  - $C[i,j] = A[i,j] + B[i,j]$
  - A, B, C have same dimensions $r \times c$

Summary

- Matrix addition is defined elementwise
  - $C[i,j] = A[i,j] + B[i,j]$
  - A, B, C have same dimensions $r \times c$
- Matrix multiplication is a more complicated operation
  - Let A be $m \times n$ and B be $n \times p$
  - $C[i,j] = \sum_{k=0}^{n-1} A[i,k] \times B[k,j]$
  - C has dimensions $m \times p$

Summary

- Matrix addition is defined elementwise
  - $C[i,j] = A[i,j] + B[i,j]$
  - A, B, C have same dimensions $r \times c$
- Matrix multiplication is a more complicated operation
  - Let A be $m \times n$ and B be $n \times p$
  - $C[i,j] = \sum_{k=0}^{n-1} A[i,k] \times B[k,j]$
  - C has dimensions $m \times p$
- Using Boolean algebra, describe transitive closure using matrix multiplication
  - A, adjacency matrix, paths of length 1
  - $A^{\ell+1} = A^{\ell} \times A$, paths of length ℓ
  - Transitive closure, $A^{+} = A + A^2 + \cdots A^{n-1}$