Real Analysis Chapter 7 Study Guide (for "Real Analysis, A First Course", 2nd Edition, Russell A. Gordon)

Number of Starred Exercises: **2**; Number of Notes: 6; Number of Other (non-starred) Exercises: 41; Minimum Number of Other (non-starred) Exercises to Do (to do at least 25% of them): **11**

The most important things to get out of this chapter: (1) The differences between pointwise and uniform convergence of a sequence of functions. (2) Counterexamples that show things can "go wrong" for pointwise convergence. (3) Theorems that guarantee that things "go right" for uniform convergence. (4) Application of these ideas to power series and Taylor series.

Other matters of importance:

1. Weierstrass M-test
2. Abel's Theorem
3. Weierstrass Approximation Theorem
4. Existence of an everywhere continuous nowhere differentiable function

Reading Guide:

1. **Note:** Do you remember seeing calculations like those on page 241 and the top of page 242 in Calculus? These are some "fun" but "naïve" power series calculations. Newton, the Bernoulli's, Lagrange, Laplace, and Euler were great believers in these kinds of calculations. They used them to do powerful things.
2. Take note of the conceptual distinction between f and $f(x)$. Is this confusing to you? Take time to try to describe the distinction in your own words if it is confusing. Ask me questions about it to clarify if you need to…maybe your confusion will be cleared away by reading further in this section (even right away…see the bottom of page 242 and the top of page 243).
3. After reading the definition of pointwise convergence, try to restate it on your own without looking.
4. *Spend lots of time sketching graphs and doing calculations to verify many (though not necessarily all) of the things described in the examples 1 – 7 on pages 243 to 244.
5. Try to come up with a sequence of bounded functions which converge pointwise to an unbounded function.
6. Try to define what it should mean for a series (infinite sum) of functions to converge pointwise to a function on an interval without looking at Definition 7.2. After you are done, compare your definition with Definition 7.2.
7. In the example from the bottom of page 245 to the top of page 246, why doesn't the series converge for other values of x? After all, the function $f(x) = (x - 1)/(30 - 5x)$ is defined for other values of x, isn't it? Is this a paradox? Can you resolve it?

8. Draw some pictures to help you understand what the book is trying to get at in explaining why the inequality near the bottom of page 247 is not sufficient to prove that the continuity of f follows from the continuity of the f_n's.
9. *After reading and thinking about the definition of uniform convergence, wait one minute, and then rewrite it on your own. Rewrite it again later without looking and see if you get a match.
10. The picture in Figure 7.1 and the comments at the bottom of page 248 are key to understanding the definition of uniform convergence. Draw pictures of particular examples (similar to those on pages 243 and 244) that help you understand this. Draw at least one example where the convergence is uniform and at least one example where the convergence is not uniform.
11. Draw a picture similar to Figure 7.1 (with the f_n's added) to illustrate the proof in the first paragraph on page 249.
12. Negate Definition 7.1 before looking at the second paragraph on page 249. Then compare your negation with the one in that paragraph.
13. Make sure you understand the proof in the third paragraph on page 249. Why is what the book has done sufficient to prove non-uniform convergence?
14. Prove Theorem 7.4. Draw a picture similar to Figure 7.1 to illustrate ideas in your proof.
15. **Note:** Take the example in the paragraph after Theorem 7.4 to heart. Especially the part about it being easier than using the negation of Definition 7.3!
16. Draw pictures to illustrate the ideas in the example from the bottom of page 249 to the top of page 250.
17. Prove the converse of the statement already proved in the proof of Theorem 7.5.
18. Prove the Weierstrass *M*-test before reading it's proof (Hint: use the Cauchy Criterion you just read about).
19. **Note:** the Weierstrass *M*-test is very useful!
20. Try to come up with an example where you could use the Weierstrass *M*-test. Outline how you would use it.
21. Fill in the minor details needed to verify that the equality $\lim_{x\to\chi} \phi(\xi) = \phi(\chi)$ is equivalent to the equality $\lim_{x\to\chi} \lambda\mu_{\nu\to\infty} \phi_\nu(\xi) = \lambda\mu_{\nu\to\infty} \lambda\mu_{\xi\to\chi} \phi_\nu(\xi)$ at the bottom of page 252.
22. Try to prove Theorem 7.8 before you look at its proof.
23. After you study the proof of Theorem 7.8, wait 15 minutes, and then prove it without looking.
24. **Note:** carefully think about the comments in the two paragraphs after Corollary 7.9 on page 253.
25. Prove Dini's Theorem (7.10).
26. Explain why the two equations near the top of page 254 are equivalent.
27. What theorem is a reference you could give as a reason for the inequality $\left|\int_\alpha^\beta (f_n - \phi_\mu)\right| \le \int_\alpha^\beta |\phi_\nu - \phi_\mu|$ in the proof of Theorem 7.11.
28. Fill in a few missing details to explain how what the book has done in the first paragraph of the proof of Theorem 7.11 proves that the sequence of numbers $\{\int_a^b f_n\}$ is Cauchy.

29. Near the bottom of the proof of Theorem 7.11 on page 254, verify that $\left|S(f, {}^tP) - \Sigma(\phi_\theta, {}^\tau\Pi)\right| < \varepsilon(\beta - \alpha)$ for any tagged partition of [a, b] which satisfies $\left\|{}^tP\right\| < \delta$, where δ is chosen so that for all tagged partitions satisfying $\left\|{}^tP\right\| < \delta$, we have $\left|S(f_q, {}^tP) - \int_\alpha^\beta \phi_\theta\right| < \varepsilon$.
30. Explain why the two equations near the top of page 255 are equivalent.
31. Verify that Example (7) of Section 7.1 provides a counterexample to the "anticipated theorem" near the top of page 255.
32. Why can Theorem 7.12 not be applied to Example (7) of Section 7.1?
33. Think about Theorem 7.13 in the context of the examples on pages 241 and 242. Also, look in your calculus book (in the sections about power series and Taylor series) for where this theorem is used. Write down some of those examples. Compare what you've thought about and done here with the statement of Theorem 7.17 in the next section.
34. Draw a picture to illustrate a typical interval of convergence that comes from Theorem 7.15.
35. **Note:** take note of how the Root Test and Weierstrass M-test are used in the proof of Theorem 7.15. Do you think the Ratio Test could also be used instead of the Root Test?
36. Think about Exercise 15 in Section 6.3 to verify the statement in the paragraph at the bottom of page 259.
37. Prove Lemma 7.16. (Hint: The Root and/or Ratio Tests might be helpful)
38. Use induction to give a more rigorous proof of the equation in the proof of Theorem 7.18.
39. **Note:** The main point of the paragraph after Corollary 7.19 is pretty amazing. The derivative of a given function at a given point provides *local* information about the function. However, if the function can be represented by a power series centered at that point, then this local information produces *global* information (on the interval of convergence of the power series). A function that has a power series representation over a certain interval is often called an **analytic** function on that interval. Most of the functions encountered in Calculus are analytic. Analytic functions are even more significant in the context of complex-valued functions of a complex variable, where the concept of analyticity is equivalent to the concept of differentiability (existence of one derivative ends up implying the existence of infinitely many derivatives in that context!). This is not true for real-valued functions of a real variable. The function f defined by $f(x) = \varepsilon^{-1/\xi^2}$ for $x \neq 0$ and $f(0) = 0$ provides a counterexample. This function is infinitely differentiable at 0, yet only equals its power series representation centered at 0 when $x = 0$ (its power series representation there is trivial: $0 + 0x + 0x^2 + 0x^3 + \ldots$)…see Exercise 22 on page 265.
40. Write the equations in the middle of page 263 out in "long form" (without the summation sign).
41. Write the equations at the bottom of page 263 out in "long form".
42. Verify the details in the proof of Theorem 7.20 (Taylor's Formula with Integral Remainder).

43. Referring to the proof of Theorem 7.21, prove that the sequence $\{(x-c)^{\nu}/\nu!\}$ converges to 0 for any fixed *x* and any fixed *c*.
44. Verify that the Maclaurin series for e^x converges for all real numbers *x*.
45. Prove Lemma 7.22.
46. Verify the details in the proof of Theorem 7.23.
47. In the proof of Theorem 7.24, elaborate on how to choose *A* so that $F(c) = \phi(\xi)$.
48. Do the computations to see that $A = \dfrac{\phi^{(\nu+1)}(\zeta)}{(\nu+1)!}$.
49. How is the statement of Theorem 7.24 (Taylor's Formula with Derivative Remainder) a generalization of the statement of the Mean Value Theorem from Chapter 4?

Deep Thoughts to Ponder (but not necessarily answer):

- Take some time to ponder the intricacies of Calculus and Real Analysis: the definitions of various kinds of limits, continuity, derivatives, integrals….various amazing theorems: the IVT, EVT, MVT, FTC…and all the facts you've just been learning about sequences and series of functions. Think about how these things are related. Hopefully you can now see it as an amazing whole. Imagine yourself teaching Calculus in the future. Think about how you might bring in some of the more rigorous ideas you learned in Real Analysis.
- It's also amazing that, though infinite series are not truly sums, we can usually pretend they are and get away with it (same comment with integrals). What about an infinite product? How would you define such a thing?