Complexity of BFS and DFS

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Mathematics for Data Science 1
Week 10

BFS and DFS

Breadth first search

- Explore graph level by level
- Keep track of
  - visited : $V \to \{\text{True},\text{False}\}$
  - Queue of unexplored vertices
- BFS from vertex j
  - Set $\text{visited}(j) = \text{True}$
  - Add j to the queue
- Explore vertex i at head of queue
  - For edge (i, j), if $\text{visited}(j)$ is False,
    - Set $\text{visited}(j)$ to True
    - Append j to the queue
- Stop when queue is empty

BFS and DFS

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Depth first search

- Start from i, visit an unexplored neighbour j
- Suspend the exploration of i and explore j instead
- Continue till you reach a vertex with no unexplored neighbours
- Backtrack to nearest suspended vertex that still has an unexplored neighbour
- Keep track of
  - visited : $V \rightarrow \{\text{True},\text{False}\}$
  - Stack of suspended vertices

Complexity BFS and DFS

- $G = (V, E)$
  - $|V| = n$
  - $|E| = m$
  - If G is connected, m can vary from $n - 1$ to $n(n-1)/2$

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Adjacency list

- To explore i, scan list of neighbours of i
- Time to explore i is degree of i
- Degree varies across vertices
- Estimate overall time?

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If m is proportional to n, big saving by using adjacency list representation

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- Sum of indegrees $= m =$ Sum of outdegrees

Summary

- BFS and DFS with adjacency matrix — time proportional to n^2
- BFS and DFS with adjacency list — time proportional to $n + m$
  - Exploring vertices examines each edge twice, sum of the degrees, $2m$ steps
- For a connected graph, m varies from $n - 1$ to $n(n - 1)/2$
- Considerable saving with small m
  - All degrees bounded by k, at most $kn/2$ edges