$\underline{\underline{\text{Th}}}^m$ (Vizing - Gupta)

For any simple graph G, $\Delta(G) \leq \chi'(G) \leq 1 + \Delta(G)$.

Proof In a proper edge coloring of G, $\Delta(G)$ colors are to be used for the edges incident at a vertex of maximum degree in G. Hence $\chi'(G) \geq \Delta(G)$.

We now prove that $\chi'(G) \leq 1 + \Delta(G)$.

If G is not $(1+\Delta)$-edge colorable, choose a subgraph H of G with maximum possible number of edges such that H is $(1+\Delta)$-edge colorable. We derive a contradiction by showing that there exist a subgraph H_0 of G that is $(1+\Delta)$-edge colorable and has one edge more than H.

By our assumption, G has an edge $uv_1 \notin E(H)$. Since $\deg u \leq \Delta$ and $1+\Delta$ colors are being used in H, there is a color c that is not represented at u. For the same reason, there is a color c_1 not represented at v_1.

There must be an edge, say uv_2, colored c_1; otherwise, uv_1 can be assigned the color c_1, and $H \cup \{uv_1\}$, which has one edge more than H, would have a proper $(1+\Delta)$-edge coloring

Again there is a color, say c_2 not represented at v_2. Then as above, there is an edge uv_3 colored c_2 and there is a color, say c_3, not represented at v_3.

In this way, we construct a sequence of edges $\{uv_1, \ldots, uv_k\}$ such that color c_i is not represented at the vertex v_i, $1 \leq i \leq k$ and the edge uv_{j+1} ~~receives~~ receives the color c_j, $1 \leq j \leq k-1$

Suppose at some stage, say the mth stage, where $1 \leq m \leq k$, c (the missing color at u) is not represented at v_m. We then "cascade" (ie. shift in order) the colors $c_1, \ldots, c_{m-1}$ from $uv_2, uv_3, \ldots, uv_m$ to $uv_1, uv_2, \ldots, uv_m$. Under this new coloring, c is not represented both at u and v_m and therefore we can color uv_m with c. This yields a proper $(1+\Delta)$-edge coloring to $H \cup \{uv_1\}$, contradicting the choice of H.

Hence, we may assume that c is represented at each of the vertices $v_1, v_2, \ldots, v_k$.

Now we need to know why the sequence of edges, $uv_i, 1 \le i \le k$ had stopped. There are two possible reasons. Either there is no edge incident to u that is colored c_k, or the color $c_k = c_j$ for some $j < k-1$ and so has already been represented at u. Note that the sequence must stop at some finite stage since $d(u)$ is finite; however it may as well stop before all the edges incident to u are exhausted.

If c_k is not represented at u in H, then we can cascade as before so that uv_i gets color c_i, $1 \le i \le k-1$, and then the color uv_k with color c_k. One again we have a contradiction to our assumption on H.

Thus, we must have $c_k = c_j$ for some $j < k-1$. In this case, cascade the colors $c_1, \ldots, c_j$ so that uv_i has color c_i, $1 \le i \le j$ and leave uv_{j+1} uncolored. Let $S = (H \cup \{u, v_1\}) \setminus \{u, v_{j+1}\}$. Then S and H have same number of edges.

Now consider S_{cc_j}, the subgraph of S defined by the edges of S with colors c and c_j. Clearly each component of S_{cc_j} is either an even cycle or a path in which the adjacent edges alternate with colors c and c_j.

Now, C is represented at each of the vertices $v_1, v_2, \ldots v_k$, and inparticular at v_{j+1} and v_k. But C_j is not represented at v_{j+1} and v_k, since we have just moved C_j to uv_j, and $C_j = C_k$ is not represented at v_k. Hence in S_{CC_j}, the degrees of v_{j+1} and v_k are both equal to one. Moreover C_j is represented at u but C is not. Therefore, u also has degree one in S_{CC_j}. As each component of S_{CC_j} is either a path or an evencycle, not all of u, v_{j+1} and v_k can be in the same component of S_{CC_j} (since a path can contain only two vertices of degree 1)

If u and v_{j+1} are in different components of S_{CC_j}, then interchange the colors C and C_{j+1} in the component containing v_{j+1}. Then C is not represented at both u and v_{j+1}, and so we can color the edge uv_{j+1} with C. This gives a $(1+\Delta)$-edge coloring to the graph $S \cup \{u, v_{j+1}\}$

Suppose then u and v_{j+1} are in the same component of S_{CC_j}. Then necessarily, v_k is not in this component. Interchange C and C_j in the component containing v_k.

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In this case, further cascade the colors so that uu_i has color c_i, $1 \le i \le k-1$. Now color uu_k with color c. Thus, we have extended our edge coloring of S with $1+\Delta$ colors to one more edge of G. This contradiction proves that $H = G$, and hence $\chi'(G) \le 1+\Delta$. $\blacksquare$