An Introduction to the Apollonian Fractal

Paul Bourke
Email: pdb@swin.edu.au
Swinburne University of Technology
P. O. Box 218, Hawthorn
Melbourne, Vic 3122, Australia.

Abstract

This paper provides an introduction to the Apollonian fractal, also known by some as the curvilinear Sierpinski gasket. This fractal is not particularly well known, perhaps because it is not as straightforward to construct as many other fractals such as the related Sierpinski gasket or the Menger sponge. The brief history and a general description of the fractal will be given, including the geometric construction in two dimensions and an IFS (Iterated Function System) formulation.

The Apollonian fractal is named after Apollonius of Perga [circa 262BC to 190BC] who is normally associated with his work on conic sections [1] and his most well known book titled simply "Conics". His name has been given to this fractal because of his work on tangents to circles and spheres published in "Tangencies" where he proved that for any three circles on the plane (not necessarily touching) there are at most 8 different circles that are tangent to all three.

To construct the Apollonian fractal, consider three identical circles arranged in a triangle (not collinear) so that each circle is tangent to the other two, thus forming a closed region, namely a curvilinear triangle (Figure 1a). Two circles are tangent to these three initial circles; they are referred to as the inner and outer Soddy circles (Figure 1b). These circles are named after Frederick Soddy, a Nobel Laureate in Chemistry, who studied matters relating to tangent circles around 1936. He derived the expression [2] for the radius r of the inner and outer Soddy circles, given by

$$r = r_a r_b r_c / [r_a r_b + r_b r_c + r_c r_a \pm 2 \sqrt{(r_a r_b r_c (r_a + r_b + r_c))}]$$

where r_a, r_b, and r_c are the radii of the three tangent circles. If the last term in the denominator is negative, the expression gives the radius of the outer Soddy circle. When the last term in the denominator is positive, the solution is the radius of the inner Soddy circle. The Apollonian fractal is created by repeatedly placing inner Soddy circles into the gaps between the circles already making up the fractal (Figure 2). An alternative [3] but equivalent way of describing this fractal is to consider the three initial circles as holes in a solid surface. The fractal is created by iteratively drilling out the largest possible circular hole from the remaining solid. This is an example of a fractal that is not self-similar; the fractal dimension [4][5] has been estimated to be 1.305684.

Surrounding the three initial circles by an outer Soddy circle and filling the additional gaps with inner Soddy circles forms the Apollonian gasket. An IFS (Iterated Function system) for generating the Apollonian gasket, attributed to Kravchenko Alexei and Mekhontsev Dmitriy, is given by a complex series z_0, z_1, z_2, ... z_n, ... The next term z_{n+1} in the series is given by choosing one of the following three complex valued functions randomly with equal probability

$$z_{n+1} = f(z_n)$$

$$z_{n+1} = f(z_n)\ (-1 + s \times i) / 2$$

$$z_{n+1} = f(z_n)\ (-1 - s \times i) / 2$$

where

$$f(z) = 3 / (1 + s - z_n) - (1 + s) / (2 + s)$$

given $i^2 = -1$ and s^2=3. Each point in the complex series is plotted with the real value along the horizontal axis and the imaginary value along the vertical axis (Figure 3).

The Apollonian fractal can be extended into 3 dimensions by replacing each circle with a sphere as shown in Figure 4 -- this is perhaps more correctly referred to as 2.5 dimensions. A true 3-dimensional equivalent is created by starting with 4 spheres positioned at the vertices of a tetrahedron. The equivalent to the inner and outer Soddy circles are an inner and outer Soddy sphere that is mutually tangential to the neighboring spheres. While should be clear to the reader how to extend the Apollonian fractal into 3 dimensions based upon four touching spheres, there is a significant challenge in the graphical representation. A further variation is to consider initial circles or spheres that are not all the same radius. This fractal has a relationship to many other branches of mathematics [6] including packing theory, group theory, and hyperbolic geometry. Armed with the methods of construction given here, the reader is encouraged to explore this unusual fractal geometry further.

References

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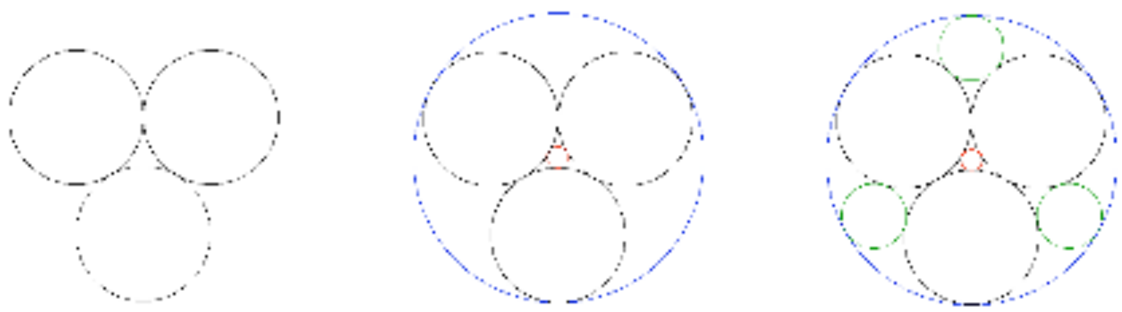

a b c

Figure 1. Inner and outer Soddy circles.

1a. Initial 3 touching circles

1b. Inner Soddy circle (red)

1c. Outer Soddy circle (blue), inner Soddy circles (green)

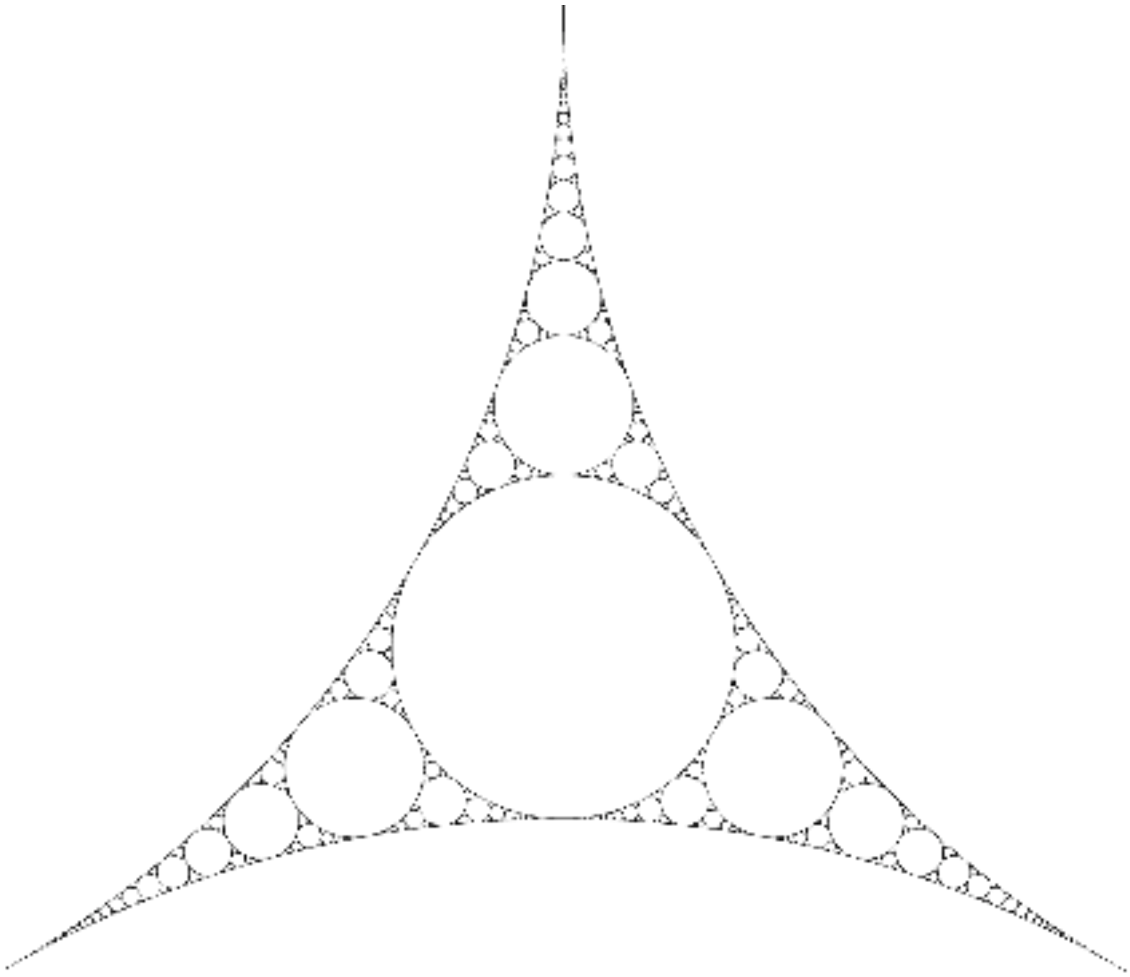

Figure 2. Apollonian fractal made up of the first 1000 circles.

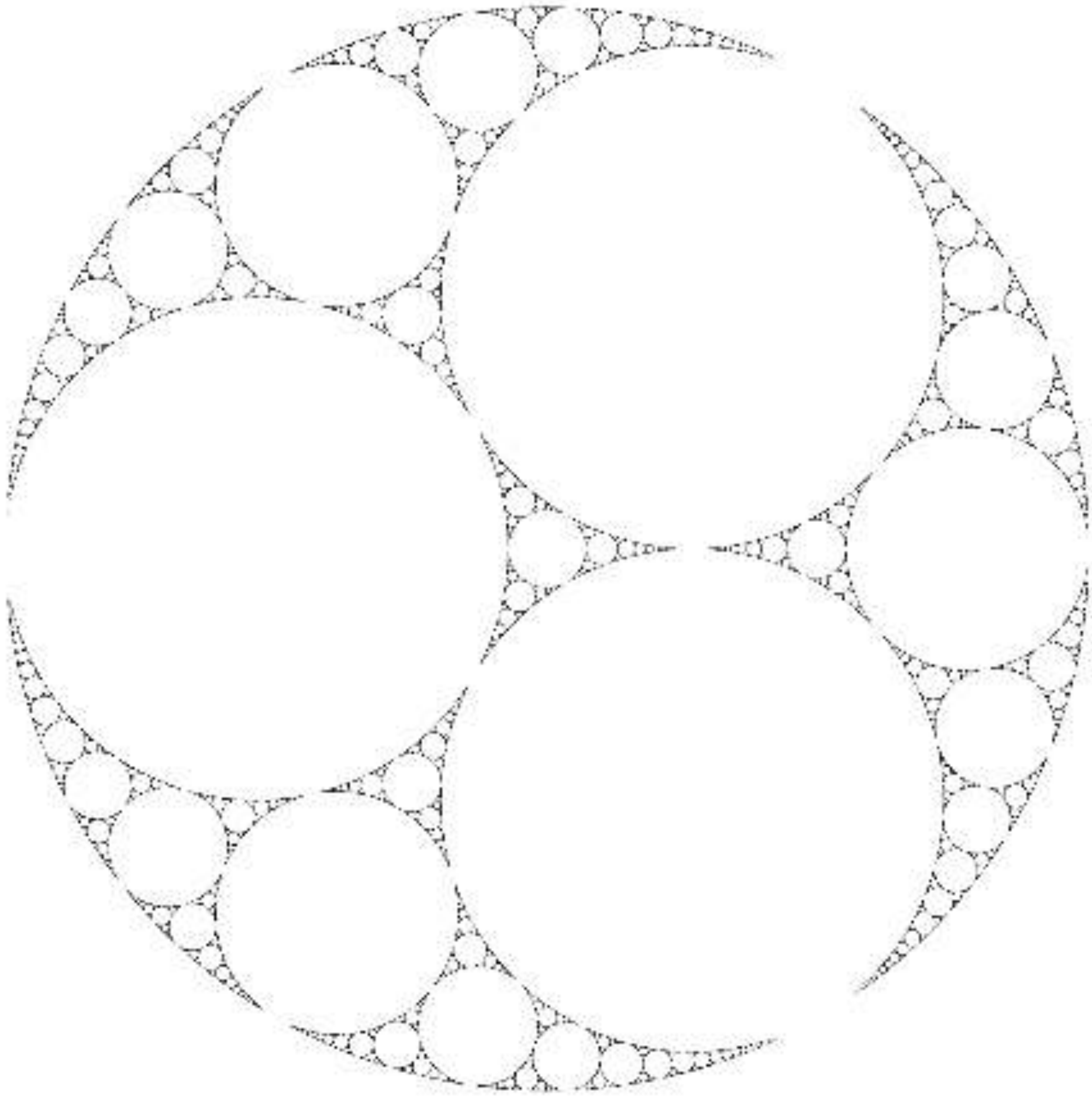

Figure 3. Apollonian gasket IFS (Iterated function system) constructed from 10 million points.

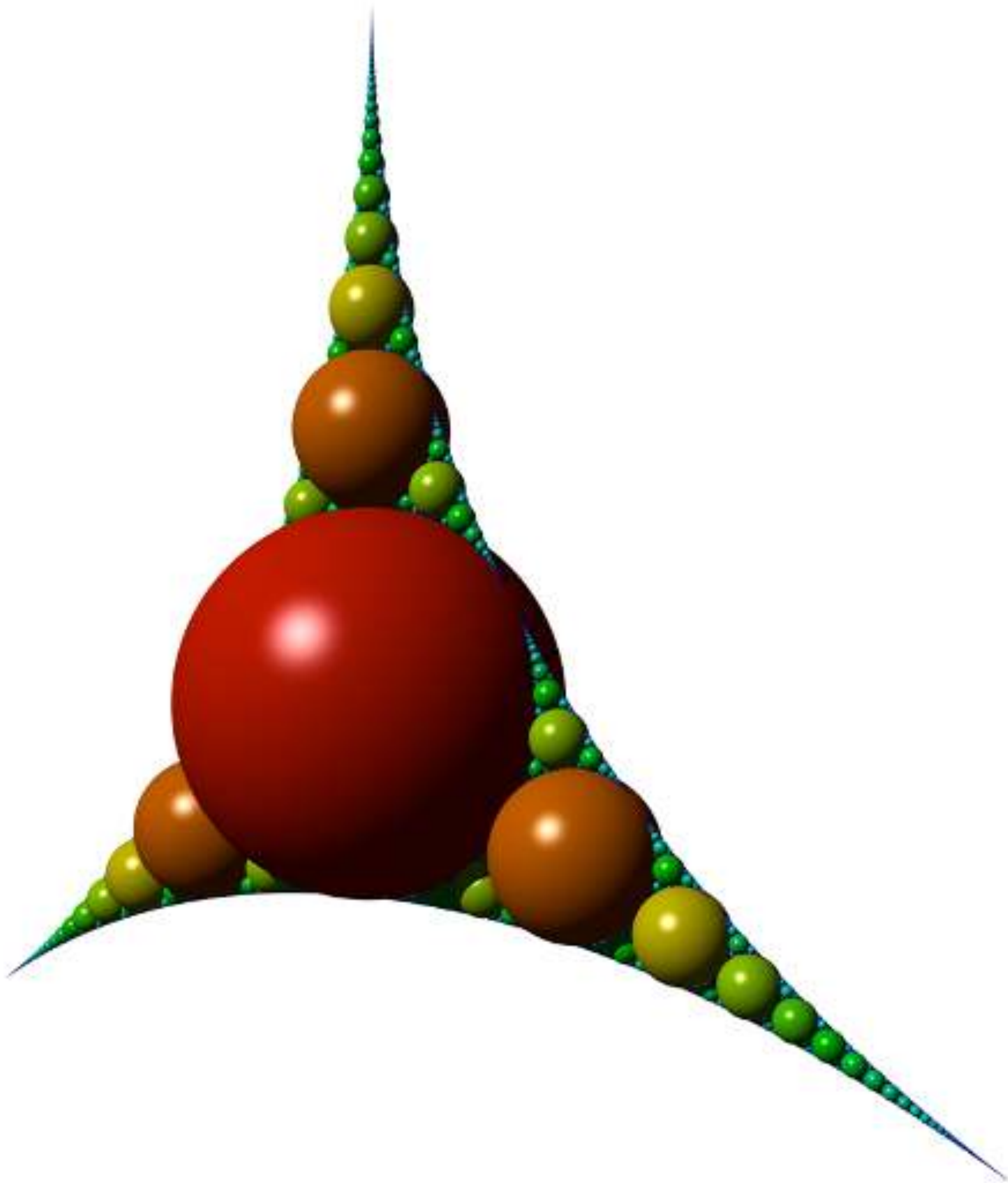

Figure 4. Apollonian fractal rendered with spheres using the PovRay raytracer.