Module 9 Edge colorings

Contents

9.1 Introduction and Basics

An obvious variation of vertex-coloring is edge-coloring where the edges are colored rather than vertices. The concept is useful to model many scheduling problems. It also arises in many circuit board problems where the wires connecting a device have to be of different color.

Throughout this chapter, graphs do not have loops (reason will be clear soon) but may have multiple edges.

Definition. *Any function $C : E(G) \to \{1, 2, \ldots, k\}$ is called a k-**edge-coloring** of G, if $C(x) \neq C(y)$ for any two adjacent edges x and y.*

Equivalently, a k-edge-coloring of G is a partition $(M_1, M_2, \ldots, M_k)$ of $E(G)$ such that each M_i is a matching in G. Throughout, we assume that the edges in M_i are colored i, $1 \leq i \leq k$. See Figures 9.1 and 9.2 for examples of edge-colorings.

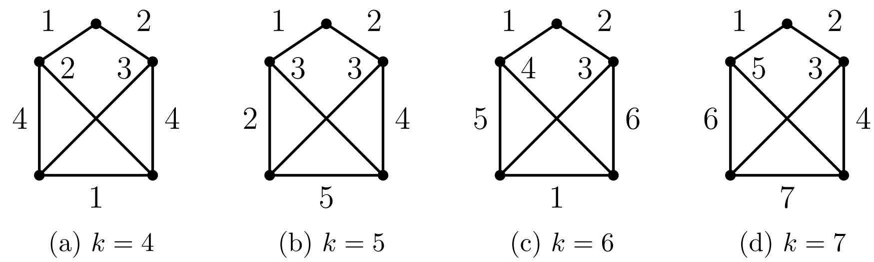


Figure 9.1: A k-edge-coloring of a graph G for $k = 4, 5, 6$ or 7. Notice that there is no k-edge-coloring of G if $k = 1, 2$ or 3.

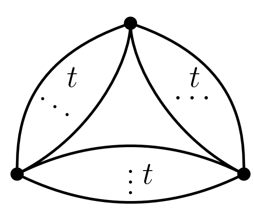


Figure 9.2: An interesting graph for which only edge-coloring is the m-edge-coloring ($m = 3t$).

Definitions. *Let $C : E(G) \to \{1, 2, \dots, k\}$ be an edge-coloring of G.*

- *An edge e colored β (that is, $C(e) = \beta$) is called a β-**edge.***
- *Let v be a vertex in G. If there is no β-edge incident with v, then we say that the color β is **missing** (or is **absent**) at v.*
- *If there is a β-edge incident with v, then we say that the color β **appears** (or is **present**) at v.*

Remarks.

- Every graph G admits a m-edge-coloring; color all the edges differently.
- Given k, a graph may not admit a k-edge-coloring.

The above remarks motivate the following concept.

Definition. *The minimum integer k such that G admits a k-edge-coloring is called the **edge-chromatic-number** of G. It is denoted by $\chi_1(G)$.*

- *Thus, $\chi_1(G)$ is the minimum number of colors required to color the edges of G such that no two adjacent edges receive the same color.*
- *The edge-chromatic-number is also called the **chromatic index**.*

Remarks.

- $\chi_1(G) = \chi(L(G))$, if $E(G) \neq \phi$.
- $\chi_1(G) \geq \Delta(G)$, since in any edge-coloring, the colors appearing at any vertex are all distinct. More generally, if C is an edge-coloring and $C(v)$ denotes the number of colors appearing at v, then $C(v) = deg(v)$.
- A natural question is to ask how much $\chi_1(G)$ is bigger than $\Delta(G)$. A remarkable theorem, independently proved by V.G. Vizing (1964) and R. P. Gupta (1966) states that $\chi_1(G) \leq \Delta(G) + 1$, for any simple graph G. We shall prove this theorem soon.

The following table shows the edge-chromatic-number of some elementary graphs. Proofs for the last two entries require some efforts.

G	P_n	C_n, n even	C_n, n odd	Petersen graph	$K_{r,s}$	K_n, n even	K_n, n odd
$\chi_1(G)$	2	2	3	4	$\max\{r,s\}$	$n-1$	n

Table 9.1: A table of edge-chromatic numbers, where $n \geq 3$.

9.2 Gupta-Vizing theorem

Theorem 9.1 (V. G. Vizing (1964), R. P. Gupta (1966))**.** *For any simple graph G,* $\Delta(G) \leq \chi_1(G) \leq \Delta(G)+1$.

Proof. (Indirect Induction/Contradiction method) As observed before, the lower bound is obvious. So, we have to only prove the upper bound. Assume the contrary and let G be a graph having the minimum number of edges among all the graphs with $\chi_1 \geq \Delta+2$. So, $\chi_1(G) \geq \Delta(G)+2$ but $\chi_1(G-e) \leq \Delta(G-e)+1 \leq \Delta(G)+1$, for every edge e.

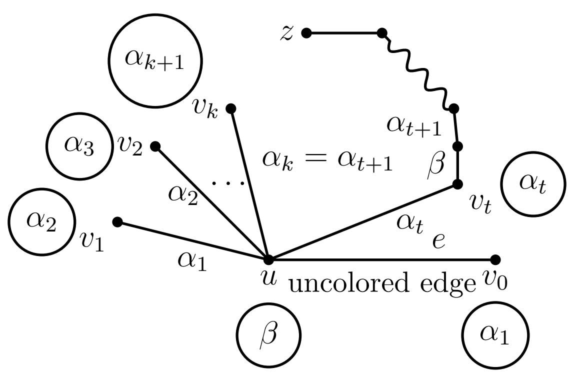


Figure 9.3: Missing colors are shown inside the circles.

Let C be a $(\Delta+1)$-edge-coloring of $G-e$, where e is (u, v_0). Before proceeding further, we remark that:

- At any vertex x at least one color is absent, since there are at most Δ edges incident with x and we have $\Delta + 1$ colors.

- If there exists a $(\Delta + 1)$-edge-coloring C' of $G - e$ such that some color β is absent at u and at v_0, then e can be colored with β to get a $(\Delta+1)$-edge-coloring of G, which provides a contradiction to our choice of G.

Using a recoloring technique, we show that it is always possible to get an edge-coloring C' as described above.

Let α_1 be a color missing at v_0. There is an α_1-edge incident with u; else, by coloring (u, v_0) with α_1 we get a $(\Delta + 1)$-edge-coloring of G. Let (u_1, v_1) be the edge colored α_1; see Figure 9.3.

Let α_2 be a color missing at v_1. There is an α_2-edge incident with u; else, by recoloring (u, v_1) with α_2 and (u, v_0) with α_1 we get a $(\Delta+1)$-edge-coloring of G. So, there must exist an edge say (u, v_2) colored α_2.

Let α_3 be a color missing at v_2. There is an α_3-edge incident with u; else, by recoloring the edges (u, v_2), (u, v_1) and (u, v_0) with colors α_3, α_2 and α_1, respectively we get a $(\Delta + 1)$-edge-coloring of G.

We continue this process to generate:

- a sequence of distinct vertices $v_0, v_1, v_2, \ldots$ incident with u, and

- a sequence of colors $\alpha_1, \alpha_2, \alpha_3, \ldots$ such that the edges $(u, v_1), (u, v_2), \ldots$ are colored $\alpha_1, \alpha_2, \ldots$ respectively, and α_{i+1} is the color missing at $v_i, i \geq 0$.

Since the degree of u is finite, the sequence $v_0, v_1, v_2, \ldots$ is finite. Therefore, there exists some v_t such that its missing color $\alpha_{t+1} \in \{\alpha_1, \ldots, \alpha_t\}$. Let t be the smallest integer such that $\alpha_{t+1} = \alpha_k$, where $\alpha_k \in \{\alpha_1, \ldots, \alpha_t\}$. This is equivalent to saying that there exists an integer k such that (u, v_k) is colored $\alpha_{t+1}(= \alpha_k)$, where $1 \leq k < t$.

Let β be a color missing at u but appearing at v_0. Such a color exists; else we can color (u, v_0) with β to get a $(\Delta + 1)$-edge-coloring of G.

Claim: β appears at every v_i, $0 \leq i \leq t$.

To prove the claim, assume the contrary and let p be the smallest integer such that β is absent at v_p $(1 \leq p \leq t)$. Recolor (u, v_p) with β and edges (u, v_{p-1}), $(u, v_{p-2}), \ldots, (u, v_0)$ with colors $\alpha_p, \alpha_{p-1}, \ldots, \alpha_1$ to get a $(\Delta + 1)$-edge-coloring of G. So the claim holds.

We define a path P as follows:

P is a maximal path with origin v_t and its edges colored alternately β and α_{t+1}. Suppose P terminates at a vertex z. At the outset, observe that the first edge and the last edge of P are colored β; see the claim above. Also $v_t \neq v_k$.

We consider three different cases depending on the location of z, and in each case describe a technique to recolor the edges $G - e$ which leads to a $(\Delta + 1)$-edge-coloring of G.

Case 1: $z \in V(G) - \{v_0, v_1, \ldots, v_t, u\}$.

Interchange the colors β and α_{t+1} of P. After this interchange, there is no β-edge incident with v_t. Recolor (u, v_t) with β, and edges $(u, v_{t-1}), \ldots, (u, v_0)$ with colors $\alpha_t, \ldots, \alpha_1$, respectively. This yields a $(\Delta + 1)$-edge-coloring of G.

Case 2: $z = v_j$, for some j, $0 \leq j \leq t - 1$.

Interchange the colors β and α_{t+1} of P. After this interchange, there is no β-edge incident with v_j. Recolor (u, v_j) with β, and recolor the edges $(u, v_{j-1}), \ldots, (u, v_0)$ with colors $\alpha_j, \ldots, \alpha_1$, respectively. Again we obtain a $(\Delta + 1)$-edge-coloring of G.

Case 3: $z = u$. In this case, the last edge of P is (v_k, u). Interchange the colors β and α_{t+1} of P. After this interchange, (v_k, u) is colored β. Recolor the edges $(u, v_{k-1}), \ldots, (u, v_0)$ with colors $\alpha_k, \ldots, \alpha_1$, respectively. This yields a $(\Delta + 1)$-edge-coloring of G. $\square$

The above theorem does not hold for graphs with multiple edges. See Figure 9.4.

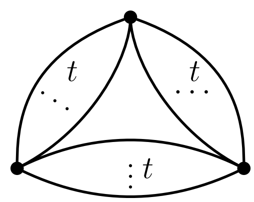


Figure 9.4: $\Delta(G) = 2t, \chi_1(G) = 3t = \Delta(G) + t$.

The following is the generalization of the above theorem for multigraphs.

Theorem 9.2 (V. G. Vizing (1965), R. P. Gupta (1966))**.** *For any loopless graph G,*

$$\chi_1(G) \leq \Delta_1(G) + \mu(G),$$

where $\mu(G)$ denotes the maximum number of edges joining a pair of vertices in G.

9.3 Class-1 and Class-2 graphs

Theorem 9.1 divides the class of simple graphs into two classes.

Definition. *A simple graph G is said to be of **Class-1** if $\chi_1(G) = \Delta(G)$ and it is said to be of **Class-2** if $\chi_1(G) = \Delta(G) + 1$.*

The problem of finding necessary and sufficient conditions for a graph to be of Class-1 (or equivalently Class-2) is a hard unsolved problem. It is called the ***Classification problem***. In the following, we state and prove a few necessary conditions and a few sufficient conditions for a graph to be of Class-1 or Class-2. An alert reader may have noticed that we made similar remarks on the characterization of Hamilton graphs (Chapter 6). It is known that Class-2 graphs are rare. More precisely, it has been shown that if we randomly pick a graph G on n vertices, then the probability that G is Class-1, tends to 1 as $n \to \infty$. For example, among the 143 non-isomorphic simple graphs on at most six vertices only 8 belong to Class-2. See Figure 9.5

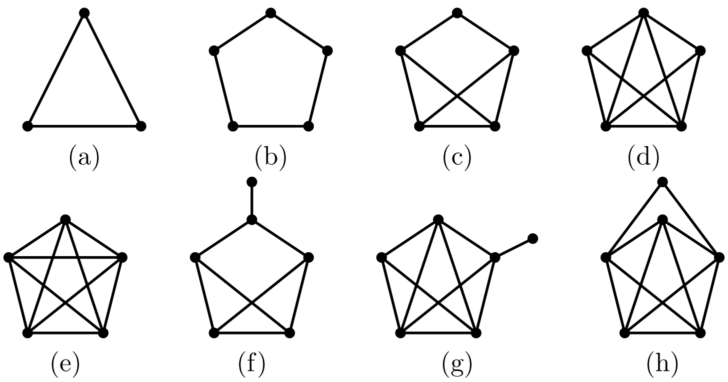


Figure 9.5: All the Class-2 graphs with at most six vertices.

• Edge-coloring of bipartite graphs

The best known theorem on Class-1 graphs is the following.

Theorem 9.3 (König, 1916)**.** *For any bipartite graph G,*

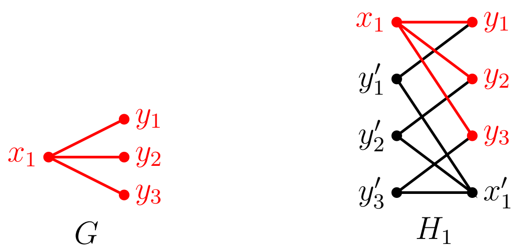
$$\chi_1(G) = \Delta(G).$$

We give two proofs. At the outset, observe that it is enough to prove the theorem for connected graphs, since $\chi_1(G) = \max\{\chi_1(D) : D \text{ is a component of } G\}$.

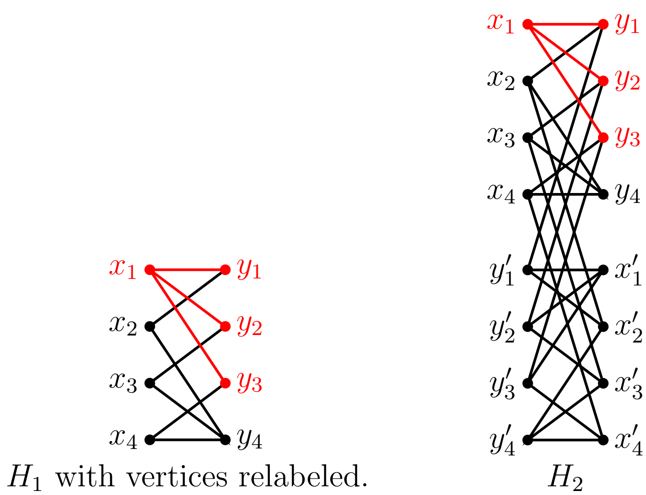
Proof (1). We first construct a $\Delta(G)$-regular bipartite graph H containing G as an induced subgraph. Let $[X, Y]$ be the bipartition of $V(G)$, where $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$. Let $G^1[X^1, Y^1]$ be an isomorphic copy of G, where $X^1 = \{x_1', x_2', \ldots, x_r'\}$ and $Y^1 = \{y_1', y_2', \ldots, y_s'\}$; see Figure 9.6a. Construct the bipartite graph $H_1[X \cup Y^1, X^1 \cup Y]$ by adding the following edges:

- ∘ Join x_i and x_i' $(1 \le i \le r)$, if $\deg_G(x_i) < \Delta(G)$.

◦ Join y_i and y'_i $(1 \leq i \leq s)$, if $\deg_G(y_i) < \Delta(G)$.



(a) $G \to H_1$



(b) $H_1 \to H_2$

Figure 9.6: Construction of a 3-regular bipartite graph which contains G as an induced subgraph.

Then $\Delta(H_1) = \Delta(G)$ and $\delta(H_1) = \delta(G)+1$. If $\delta(G)+1 = \Delta(G)$, then H_1 is a $\Delta(G)$-regular bipartite graph with G as an induced subgraph. If $\delta(G)+1 < \Delta(G)$, we repeat the above process with H_1 to construct a bipartite H_2 which contains H_1 (and hence G) as an induced subgraph such that $\Delta(H_2) = \Delta(G)$ and $\delta(H_2) = \delta(H_1)+1 = \delta(G)+2$. If $\delta(G)+2 = \Delta(G)$, then H_2 is a required graph. We can continue the process enough number of times and construct a sequence of bipartite graphs $H_1, H_2, \ldots, H_t$

such that $G \sqsubseteq H_i \sqsubseteq H_{i+1}$, $\delta(H_{i+1}) = \delta(H_i) + 1$ $(1 \leq i \leq t-1)$, $\Delta(H_i) = \Delta(G)$ $(1 \leq i \leq t)$ and $\delta(H_t) = \Delta(G)$. Then H_t is a required bipartite graph.

By Corollary to Hall's Theorem 7.3, $E(H_t)$ can be partitioned into $\Delta(G)$ perfect matchings $E_1, E_2, \ldots, E_{\Delta(G)}$. Therefore, $\chi_1(H_t) \leq \Delta(G)$. Since $G \sqsubseteq H_t$, we have $\chi_1(G) \leq \chi_1(H_t) \leq \Delta(G) \leq \chi_1(G)$ and so $\chi_1(G) = \Delta(G)$. $\square$

Proof (2). This proof employs the technique which we used to prove Gupta-Vizing theorem. Assume the contrary and let G be a bipartite graph having minimum number of edges among all the graphs with $\chi_1(G) > \Delta(G)$. So, $\chi_1(G) = \Delta(G) + 1$ but $\chi_1(G - e) = \Delta(G - e) \leq \Delta(G)$ $(= \Delta$, say). Let C be a Δ-edge-coloring of $G - e$, where e is (u, v). Let α be a color missing at u and β be a color missing at v; see Figure 9.7. If $\alpha = \beta$, then we can color e with β and get a Δ-coloring of G, a contradiction to our assumption that $\chi_1(G) = \Delta + 1$. Next assume that $\alpha \neq \beta$.

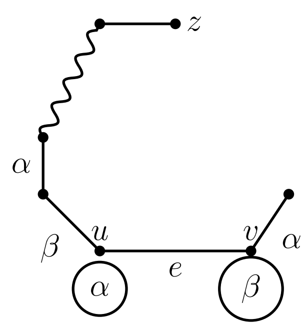


Figure 9.7

There is a β-edge incident with u; else color e with β to get Δ-edge-coloring of G. Similarly, there is a α-edge incident with v.

Let P be a maximal path with origin u and whose edges are colored alternately β and α. Suppose P terminates at a vertex z.

Case 1: $z \neq v$.

Interchange the colors β and α of the edges in P. After this interchange, there is no β-edge incident with u (and with v). Recolor e with β to get a Δ-edge-coloring and thereby a contradiction.

Case 2: $z = v$.

In this case, the first edge of P is a β-edge and the last edge is a α-edge. Therefore, P is a path of even length. Hence, $(u, P(u, v), (v, u))$ is a cycle of odd length in G, which is a contradiction, since G is bipartite. Therefore this case does not arise. $\square$

• Class-2 graphs

In this subsection, we derive a few sufficient conditions for a graph to be of Class-2.

Theorem 9.4. *For any graph G,*

$$\chi_1(G) \geq \left\lceil \frac{m(G)}{\alpha_1(G)} \right\rceil$$

Proof. Let $C = (M_1, \ldots, M_{\chi_1})$ be a χ_1-edge-coloring of G. Then

$$\begin{aligned} m(G) &= |M_1| + \cdots + |M_{\chi_1}|, \\ &\leq \alpha_1(G) + \cdots + \alpha_1(G), \text{ since each } M_i \text{ is a matching in } G, \\ &= \chi_1(G) \cdot \alpha_1(G). \end{aligned}$$

Therefore, $\chi_1(G) \geq \frac{m(G)}{\alpha_1(G)}$. Since χ_1 is an integer, the theorem follows. $\square$

Corollary. *If G is a simple graph with $m(G) > \Delta(G) \cdot \alpha_1(G)$, then G is a Class-2 graph.*

Proof.

$$\begin{aligned}
\chi_1(G) &\geq \left\lceil \frac{m(G)}{\alpha_1(G)} \right\rceil, \text{ by Theorem 9.4} \\
&> \left\lceil \frac{\Delta(G) \cdot \alpha_1(G)}{\alpha_1(G)} \right\rceil, \text{ by the hypothesis,} \\
&= \Delta(G).
\end{aligned}$$

Therefore, by Gupta-Vizing theorem, $\chi_1(G) = \Delta(G) + 1$. $\square$

Corollary. *If $m(G) > \Delta(G)\lceil \frac{n}{2} \rceil$, then G is Class-2.*

Proof. Exercise. $\square$

Before stating the next result we define a concept which is useful in many other contexts.

Definitions.

- *If v is a vertex in G, then the integer $\Delta(G) - deg(v)$ is called the* ***deficiency*** *of v.*
- *The integer $\sum_{v \in V(G)} (\Delta(G) - deg(v))$ is called the* ***total deficiency*** *of G. We denote it by* ***td(G)***.

Corollary. *If G is simple, $n(G)$ is odd and $td(G) < \Delta(G)$, then G is Class-2.*

Proof.

$$\Delta(G) \;\; > \;\; td(G) = \sum_{v \in V(G)} (\Delta(G) - deg(v)) = n\Delta(G) - 2m(G).$$

Therefore,

$$\begin{aligned} m(G) &> \left(\frac{n-1}{2}\right)\Delta(G), \\ &\geq \alpha_1(G)\Delta(G), \text{ since } n \text{ is odd, we have } \alpha_1(G) \leq \frac{n-1}{2}. \end{aligned}$$

Hence, the result follows by the corollary to Theorem 9.4. □

Corollary. *Let $m(G) > 0$. If G is a regular simple graph with $n(G)$ odd, then G is Class-2.*

Proof. $td(G) = \sum_{v \in G}(\Delta(G) - deg(v)) = 0 < \Delta(G)$. Therefore, the result follows by the above Corollary. □

Corollary. *For any $n \geq 2$,*

$$\chi_1(K_n) = \begin{cases} n, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even} \end{cases}$$

Proof. If n is odd, the result follows by the previous corollary. So next assume that n is even. Consider the graph $K_n - v\,(= K_{n-1})$, where v is a vertex of K_n. $\chi_1(K_n - v) = n - 1$, since $n - 1$ is odd. Let $C = (M_1, M_2, \ldots, M_{n-1})$ be a $(n-1)$-edge-coloring of $K_n - v$. Since, $|M_i| \leq \alpha_1(K_n - v) \leq \frac{n-2}{2}$, for every $i, 1 \leq i \leq n-1$, we have $\frac{(n-1)(n-2)}{2} = m(K_n - v) = \sum_{i=1}^{n-1} |M_i| \leq \frac{(n-1)(n-2)}{2}$. Hence, $|M_i| = \frac{n-2}{2}$, for every i, $1 \leq i \leq n-1$. Therefore, there are exactly $n-2$ vertices which are the end vertices of edges in M_1. Hence, there is a vertex (say) v_1 in $K_n - v$ which is not the end-vertex of any edge in M_1, that is, color 1 is missing at v_1. Moreover, every color $\in \{2, 3, \ldots, n-1\}$ is present at v_1, since $deg(v_1) = n - 2$. By repeating the same argument with every M_i, we conclude that there are vertices $v_1, v_2, \ldots, v_{n-1}$ in $K_n - v$ such that the color i is absent at v_i (and every color $\in \{1, 2, 3, \ldots, n-1\} - \{i\}$

is present at v_i). The latter assertion implies that v_i's are all distinct and hence $V(K_n - v) = \{v_1, \ldots v_n\}$. Therefore, coloring the edges $(v, v_1), (v, v_2) \ldots, (v, v_{n-1})$ with colors $1, 2, \ldots, n-1$ respectively, we obtain a $(n-1)$-edge-coloring of K_n. Hence, $\chi_1(K_n) \leq n-1$. Since $\chi_1(K_n) \geq \Delta(K_n) = n-1$, we conclude that $\chi_1(K_n) = n-1$. □

• Hajos union and Class-2 graphs (Optional)

Hajos union can be used to construct a larger Class-2 graph by combining two smaller Class-2 graphs.

Definition (Hajos union). *Let G_1 and G_2 be any two graphs with $\Delta(G_1) = \Delta(G_2) = \Delta$, $\delta(G_1) > 0$ and $\delta(G_2) > 0$. Suppose there exists vertices $x \in V(G_1)$ and $y \in V(G_2)$ such that $deg_{G_1}(u) + deg_{G_2}(v) \leq \Delta + 2$, combine the graphs G_1 and G_2 by applying the following steps:*

1. *Choose any two vertices $u \in V(G_1)$ and $v \in V(G_2)$ such that $deg_{G_1}(u) + deg_{G_2}(v) \leq \Delta + 2$.*

2. *Delete an edge (u, w_1) from G_1 and an edge (v, w_2) from G_2, chosen arbitrarily.*

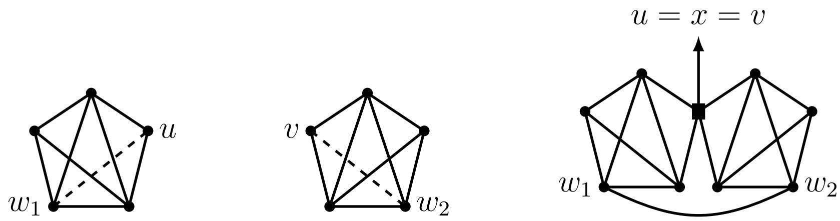
3. *Identify u and v and call the resultant vertex as x.*

4. *Join the vertices w_1 and w_2 by an edge.*

*The resultant graph is called a **Hajos union** of graphs and it is denoted by $G_1 \cup_H G_2$. See Figure 9.8.*

Note that the Hajos union is not a unique graph, since the choice of u, v, (u, w_1) and (v, w_2) are arbitrary.

Theorem 9.5. *Let G_1 and G_2 be Class-2 graphs satisfying the assumptions made in the definition of Hajos union. Then their Hajos union $G = G_1 \cup_H G_2$ is a Class-2 graph.*



(a) G_1 (b) G_2 (c) $G_1 \cup_H G_2$

Figure 9.8: Hajos union of two graphs.

Proof. (Contradiction method) At the outset observe that, for any vertex z in $V(G)$

$$d_G(z) = \begin{cases} d_{G_1}(u) + d_{G_2}(v) - 2, & \text{if } z = x = u = v, \\ d_{G_1}(z), & \text{if } z \neq x \text{ and } z \in V(G_1), \\ d_{G_2}(z), & \text{if } z \neq x \text{ and } z \in V(G_2). \end{cases}$$

Therefore, we conclude that $\Delta(G) = \Delta$.

Next assume that the theorem is false and let $C : E(G) \to \{1, 2, \ldots, \Delta\}$ be a Δ-edge-coloring of G. Without loss of generality let $C(w_1, w_2) = 1$.

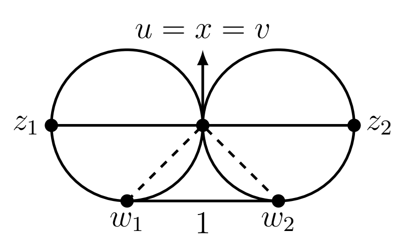


Figure 9.9

Claim: There is a 1-edge, say (u, z_1) in G_1, where $z_1 \in V(G_1)$. Else, by defining $C' : E(G) \to \{1, 2, \ldots, \Delta\}$ by

$$C'(e) = \begin{cases} C(e) & \text{if } e \neq (u, w_1), \\ 1 & \text{if } e = (u, w_1), \end{cases}$$

we obtain a Δ-edge-coloring of G_1, a contradiction. Hence, the claim holds.

Similarly, there is a 1-edge (v, z_2) in G_2, where $z_2 \in V(G_2)$. This implies that there are two 1-edges incident with x, a contradiction to the coloring C. □

9.4 A scheduling problem and equitable edge-coloring (Optional)

An example of a scheduling problem: A conference is planned with five invited speakers S_1, S_2, S_3, S_4, S_5 to handle six short courses $C_1, C_2, C_3, C_4, C_5, C_6$. For the convenience of the participants, on any day a speaker speaks at most once and a course is scheduled at most once.

The following matrix $M = [m_{ij}]_{5\times 6}$ describes the lecture requirements, where m_{ij} is the number of times speaker S_i speaks on the course C_j.

	C_1	C_2	C_3	C_4	C_5	C_6
S_1	0	0	1	0	3	0
S_2	0	2	0	1	0	1
S_3	2	0	0	0	2	0
S_4	0	1	0	2	0	2
S_5	2	0	2	0	0	0

(a) Find the smallest number of days needed to conduct the conference.

(b) Is it possible to schedule the time table where there are only four conference halls available?

Graph theory model and a solution.

Associate the bipartite graph $G[S, C]$ shown in Figure 9.10 with the given matrix M, where S_i and C_j are joined with m_{ij} edges.

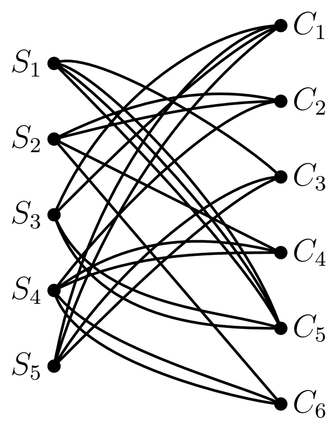


Figure 9.10: The bipartite graph G associated with M.

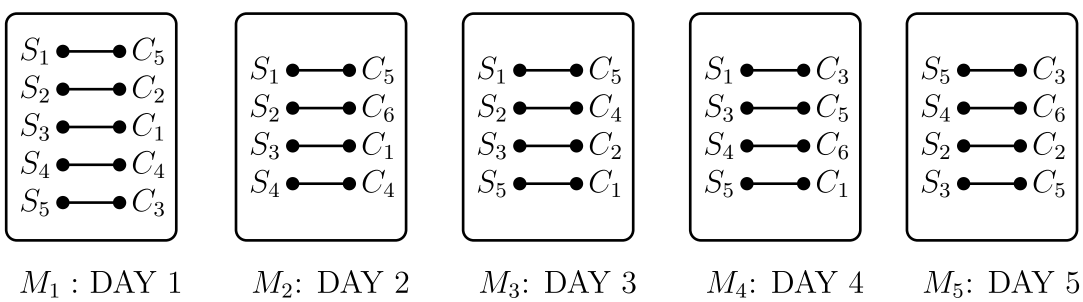


Figure 9.11: Five disjoint matchings of G and the resultant schedule.

Because of the restrictions on the scheduling of speakers and courses, the problem (a), is to partition the edge set of G into subsets $M_1, M_2, \ldots, M_t$ such that each M_i is a matching and t is minimum possible. Obviously, M_i is the schedule for the i-th day. Hence, in the terminology of edge-coloring, we have to find $\chi_1(G)$. Thanks to König's Theorem 9.3, we immediately have that $\chi_1(G) = \Delta(G) = 5$. So, the answer to problem (a) is that five days are required to conduct the conference. An actual schedule is shown in Figure 9.11. And a conference time table derived using this schedule is shown in Table 9.2.

	Course 1	Course 2	Course 3	Course 4	Course 5	Course 6
Day 1	S_3	S_2	S_5	S_4	S_1	–
Day 2	S_3	–	–	S_4	S_1	S_2
Day 3	S_5	S_4	–	S_2	S_1	–
Day 4	S_5	–	S_1	–	S_3	S_4
Day 5	–	S_2	S_5	–	S_3	S_4

Table 9.2: Conference time table derived using the schedule given in Figure 9.11.

We next solve problem (b).

Since there are twenty one edges and there are 5 sets of matchings, at least one matching, say M_1 contains 5 or more edges, by the Pigeon-hole-principle. Since an edge in M_i indicates the requirement of a conference hall on i-th day, 5 conference halls are necessarily required. So, the answer to problem (b) is NO.

If there are only four conference halls available we require six days to conduct the conference. It is easy to modify the table so that every day we require at most four conference halls. For example, we can shift the edge (S_5, C_3) which appears in M_1 to a new matching M_6. But the resultant time table is obviously unbalanced: only one talk on sixth day. So to have a balance, we would like to have a 6-edge-coloring of G such that $||M_i| - |M_j|| \leq 1$, $1 \leq i, j \leq 6$, that is, the number of talks are "nearly equal" on all the days. Next theorem asserts that it is always possible to schedule the time table in a balanced manner.

Definition. *A k-edge-coloring $C = (M_1, M_2, \ldots, M_k)$ is said to be a* ***equitable k-edge-coloring*** *if $||M_i| - |M_j|| \leq 1$, for every $i, j \in \{1, 2, \ldots, k\}$.*

Theorem 9.6. *If there exists a k-edge-coloring of G, then there exists an equitable k-edge-coloring of G.*

Proof. Let $C = (M_1, M_2, \ldots, M_k)$ be a k-edge-coloring of G. Suppose there exist i and j such that $|M_i| \geq |M_j| + 2$.

We describe a procedure to get matchings M_i' and M_j' such that $|M_i'| = |M_i| - 1$, $|M_j'| = |M_j| + 1$ and $M_i \cup M_j = M_i' \cup M_j'$. By repeating such a procedure enough times, we can get an equitable k-edge-coloring of G.

Consider the subgraph H of G induced by the set of edges $M_i \cup M_j$. $\Delta(H) \leq 2$, since there can be at most one i-edge and at most one j-edge incident with any vertex in H. So, every component of H is either a cycle or a path whose edges are alternately colored i and j. Therefore, any cycle component is an even cycle; see Figure 9.12. We conclude that there exists a path component P in H whose first edge and last edge are colored i, since $|M_i| \geq |M_j| + 2$.

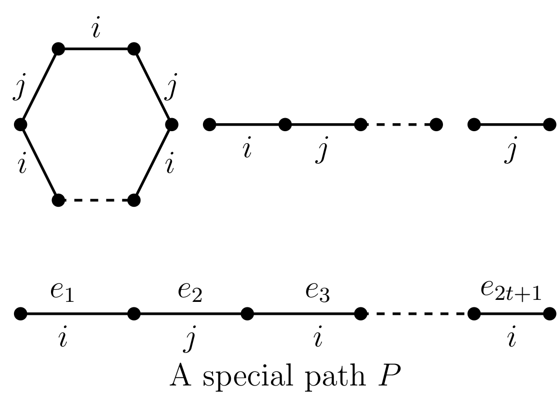


Figure 9.12: Components of H.

Define

$$\begin{aligned} M_i' &= M_i - \{e_1, e_3, \ldots, e_{2t+1}\} \cup \{e_2, e_4, \ldots, e_{2t}\}, \\ M_j' &= M_j - \{e_2, e_4, \ldots, e_{2t}\} \cup \{e_1, e_3, \ldots, e_{2t+1}\}. \end{aligned}$$

That is, interchange the colors of P (and retain the colors of edges not in P). Then $M_i \cup M_j = M_i' \cup M_j'$, $|M_i'| = |M_i| - 1$, and $|M_j'| = |M_j| + 1$. □

Exercises

1. Prove:
   (a) If G is a loopless graph with n odd and at least $n \cdot \Delta(G)$ edges, then $\chi_1(G) = \Delta(G) + 1$.
   (b) If G is simple, Hamilton and 3-regular, then $\chi_1(G) = 3$.
2. What is the number of iterations required to construct the regular graph H in the first proof of Theorem 9.3.
3. Let G be a graph with $\chi_1(G) = \Delta(G) + 1$. Let $e(u, v)$ be an edge such that $\chi_1(G - e) = \Delta(G)$. Show that $deg_G(u) + deg_G(v) \geq \Delta(G) + 2$.
4. Find the edge-chromatic number of the following graphs.
   (a) $T + T$, where T is a tree on $n(\geq 3)$ vertices.
   (b) $T_k + C_{2k+1}$, where T_k is a tree on k vertices.
   (c) $K_5 - e$
   (d) $C_5 + C_5$.
   (e) $C_{2s} + C_{2t}$.
   (f) Petersen graph.
   (g) d-cube, Q_d.
   (h) A k-regular bipartite graph.
   (i) $K_{5,5,5,5,5}$.

5. Show that the graphs shown in Figure 9.5 are of Class-2.

6. Let G be a d-regular simple graph with odd number of vertices, where $d \geq 2$. Let H be any graph obtained from G by deleting at most $(d-1)/2$ edges. Show that $\chi_1(H) = \Delta(H) + 1$.

7. Show that every regular simple graph on odd number of vertices is a Class-2 graph.

8. Show that $\chi_1(G \square K_2) = \Delta(G \square K_2)$, for every simple graph G.

9. Prove or disprove: If G_1 and G_2 are Class-1 graphs, then $G_1 + G_2$ is a Class-1 graph.

10. In a school there are seven vacancies, one each in the departments of Chemistry, English, French, Geograph, History, Mathematics and Physics. There are seven applicants $A_1, \ldots, A_7$ for the vacancies. The applicants and their specializations are listed in the table below. Determine the maximum number of suitably qualified teachers the school can employ. Justify your claim.

Teachers	Specialization
A_1	Mathematics, Physics
A_2	Chemistry, English, Mathematics
A_3	Chemistry, French, History, Physics
A_4	English, French, History, Physics
A_5	Chemistry, Mathematics
A_6	Mathematics, Physics
A_7	English, Geography, History.

11. A mathematics department plans to offer seven courses in the next semester namely, Complex Analysis (C), Numerical Methods (N), Linear Algebra (L), Probability and Statistics (P), Differential Equations (D), Special Functions (S) and Graph Theory (G). Each of the following combination of courses has attracted the students.

C,L,D	C,N,G	N,P	C,L
L,P	C, N	S,P	L,N
C,D	C,G,D	S,G	S, D

   (a) Use graph theory techniques to find the minimum number of slots required for these courses on a day.

(b) If one wants to schedule the timetable with minimum number of slots, what is the minimum number of class rooms required.

12. There are 5 teachers and 6 classes. The following matrix $M = [m_{ij}]$ describes the teaching requirements in a week, where m_{ij} is the number of times a teacher i meets the class j.

$$M = \begin{bmatrix} 0 & 0 & 3 & 0 & 4 & 0 \\ 0 & 2 & 0 & 2 & 0 & 2 \\ 3 & 0 & 0 & 0 & 4 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 3 & 0 & 2 \end{bmatrix}$$

(a) Find the smalles number of periods needed so that each teacher can meet all the required classes.

(b) Is it possible to schedule the time table where there are only 4 classrooms available?