Ramsey Number

For any simple graph G with six vertices, G or $\bar{G}$ contains a triangle.

What is the smallest integer $r(m,n)$ such that every graph with $r(m,n)$ vertices contains K_m or $\bar{K}_n$?

Since K_1 does not have any edge, for $m=1$ or $n=1$ $r(m,n)=1$; i.e. $r(1,n) = r(m,1) = 1$

$r(2,2) = 2$, $r(2,3) = 3 = r(3,2)$

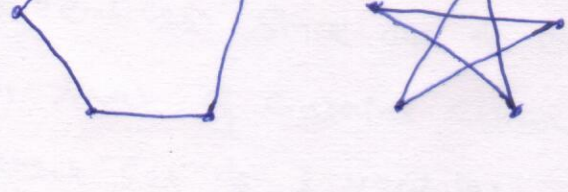
$$\boxed{r(m,n) = r(n,m)}$$

$r(2,n) = n$, $r(m,2) = m$

For $m,n \geq 2$, $r(m,n) \geq \max\{m,n\}$

The numbers $r(m,n)$ are called Ramsey numbers. For $m=n$, they are called diagonal Ramsey number.

$r(3,3) = 6$ as



$\underline{\underline{Th^m}}$ For any two integers $k \geq 2$ and $l \geq 2$,

$$r(k,l) \leq r(k,l-1) + r(k-1,l)$$

Further, if $r(k,l-1)$ and $r(k-1,l)$ are both even, the strict inequality holds.

$\underline{\underline{Proof}}$ Let G be a graph on $r(k,l-1) + r(k-1,l)$ vertices and let $v \in V(G)$.

We distinguish two cases:

(i) v is non adjacent to a set S of at least $r(k,l-1)$ vertices, or

(ii) v is adjacent to a set T of at least $r(k-1,l)$ vertices.

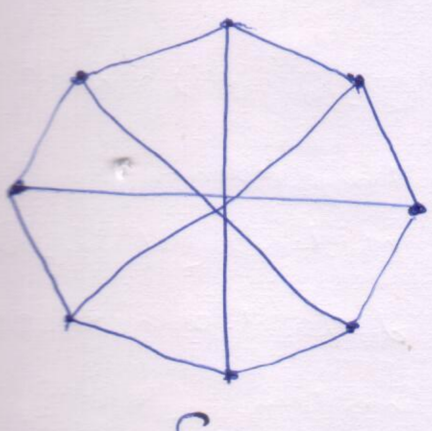
Note that either case(i) or case (ii) must hold because the number of vertices to which v is nonadjacent plus the number of vertices to which v is adjacent is equal to $r(k-1,l) + r(k,l-1) - 1$.

In case (i) $G[S]$ contains either a clique of k vertices or an independent set of $l-1$ vertices, and therefore $G[S \cup \{v\}]$ contains either a clique of k vertices or and independent set of l vertices. Similarly, in case (ii) $G[T \cup \{v\}]$ contains either a clique on k vertices or an independent set of l vertices. Since one of case(i) and case(ii) must hold, it follows that G contains either a clique of k vertices or an independent set of l vertices.

Now suppose that $r(k, l-1)$ and $r(k-1, l)$ are both even, and let G be a graph on $r(k,l-1)+r(k-1,l)-1$ vertices. Since G has odd number of vertices, so ~~some vertex~~ there exist a vertex u of G with $d_G(u)$ is even. ~~ie~~

This implies u can not be adjacent to precisely $r(k-1, l)-1$ vertices. Consequently, either case(i) ~~or~~ case(ii) above holds, and therefore G contains either a clique of k vertices or an independent set of l vertices. Thus $r(k,l) \leq r(k,l-1)+r(k-1,l)-1$.

By the above theorem $r(3,4) \leq 9$.



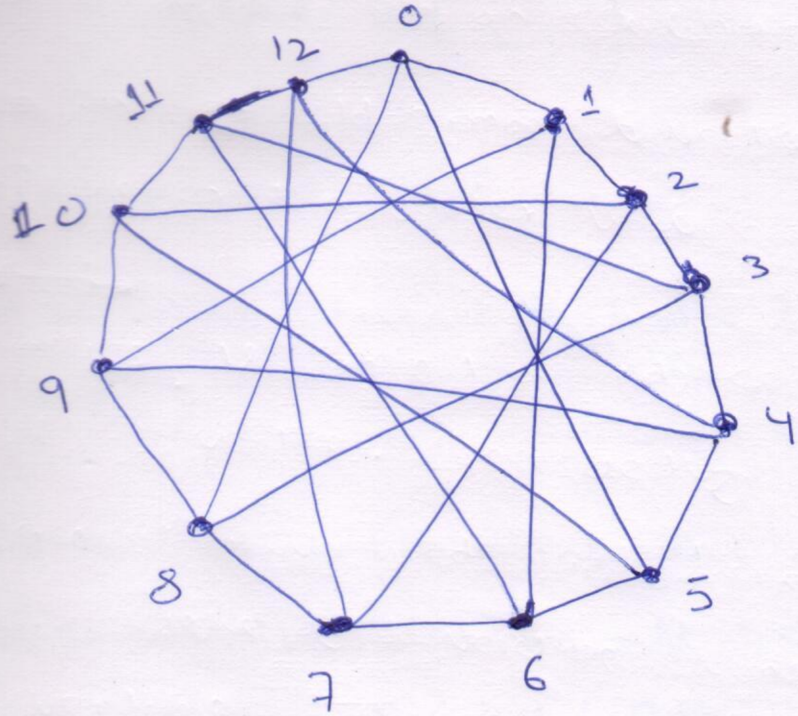
G

G contains neither K_3 or $\overline{K_4}$

Hence $r(3,4)=9$

A (k,l)-Ramsey graph is a graph on $r(k,l)-1$ vertices that contains neither a clique of k vertices nor an independent set of l vertices.

$r(3,5) \leq r(3,4) + r(2,5) = 9 + 5 = 14$



(3,5)-Ramsey graph

So $\quad r(3,5) = 14$

k	3	3	3	3	3	4	4
l	3	4	5	6	7	4	5
r(k,l)	6	9	14	18	23	18	25

It has be conjectured that the (k,k)-Ramsey graphs are always self complementary. This is true for $k = 2, 3, 4$.

$\underline{\underline{Th^m}}$ For all the integers k and l,

$$r(k,l) \le \binom{k+l-2}{k-1}.$$

$\underline{\underline{Proof}}$ Induction on $k+l$.

It can be easily checked that the theorem holds for $k+l \le 5$.

Let a and b be the integers and assume that the theorem is valid for all the integer k and l such that $5 \le k+l < a+b$.

Then
$$r(a,b) \le r(a,b-1) + r(a-1,b)$$
$$\le \binom{a+b-3}{a-1} + \binom{a+b-3}{a-2} = \binom{a+b-2}{a-1}$$

Thus the theoreem holds for all values of k and l.

$\underline{\underline{Cor}}$ $r(k,l) \le 2^{k+l-2}$ with equality if and only if $k=l=1$.

$\underline{\underline{Th^m}}$ (Erdös, 1947) $r(k,k) \ge 2^{k/2}$

$\underline{\underline{Proof}}$ Since $r(1,1)=1$ and $r(2,2)=2$, we may assume that $k \geq 3$.

Let Γ_n be the set of simple graphs with vertex set $\{v_1, \dots, v_n\}$, and let Γ_n^k be the set of those graphs in Γ_n that have a clique of k-vertices.

Clearly $|\Gamma_n| = 2^{\binom{n}{2}}$, since each subset of the $\binom{n}{2}$ possible edges $\{v_i, v_j\}$ determines a graph in Γ_n.

Similarly, the number of graphs in Γ_n having a particular set of k vertices as a clique is $2^{\binom{n}{2}-\binom{k}{2}}$. Since there are $\binom{n}{k}$ distinct k-element subsets of $\{v_1, \dots, v_n\}$, we have

$$|\Gamma_n^k| \leq \binom{n}{k} 2^{\binom{n}{2}-\binom{k}{2}}$$

So $$\frac{|\Gamma_n^k|}{|\Gamma_n|} \leq \binom{n}{k} 2^{-\binom{k}{2}} < \frac{n^k 2^{-\binom{k}{2}}}{k!}$$

Suppose now that $n < 2^{k/2}$. Then

$$\frac{|\Gamma_n^k|}{|\Gamma_n|} < \frac{2^{\frac{k^2}{2}} 2^{-\binom{k}{2}}}{k!} = \frac{2^{k/2}}{k!} < \frac{1}{2}$$

Therefore, fewer than half of the graphs in Γ_n contains a clique of k-vertices. Also, $\Gamma_n = \{G \mid \bar{G} \in \Gamma_n\}$, fewer than half of the graphs in Γ_n contains an independent set of k-vertices. Hence some graph in Γ_n contains neither a clique of k-vertices nor an independent set of k-vertices. Because this holds for any $n < 2^{k/2}$, we have $r(k,k) \geq 2^{k/2}$

<u>Cor</u> If $a = \min\{k, l\}$ then $r(k,l) \geq 2^{\frac{a}{2}}$.

~~Different way~~

~~r(k,n) can be thought of as the smallest integer~~

The Ramsey numbers $r(k,l)$ are sometimes defined in a slightly different way. It can be easily seen that $r(k,l)$ can be thought of as the smallest integer n such that every 2-edge coloring (E_1, E_2) of K_n contains either a complete subgraph on k-vertices all of whose edges are in color 1 or a complete subgraph on l-vertices all of whose edges are in color 2.

Expressed in this form, the Ramsey numbers have a natural generalization.

We define $r(k_1, \dots, k_m)$ to be the smallest integer n such that every m ~~edge~~ coloring ~~[illegible]~~ of $E(K_n)$ contains a complete subgraph on k_i vertices for some i, all of whose edges are in color i.

$\underline{\underline{Th^m}}$ $r(k_1, k_2, \dots, k_m) \le r(k_1-1, k_2, \dots, k_m) + r(k_1, k_2-1, \dots, k_m) + \dots + r(k_1, \dots, k_m-1) - m + 2$

$\underline{\underline{Cor}}$ $r(k_1+1, k_2+1, \dots, k_m+1) \le \dfrac{(k_1+k_2+\dots+k_m)!}{k_1!\, k_2! \dots k_m!}$

$\underline{\underline{Def^n}}$ Given simple graphs $G_1, \dots, G_k$, the graph Ramsey number $R(G_1, \dots, G_k)$ is the smallest integer n such that the every k-coloring of $E(K_n)$ contains a copy of G_i in color i for some i. When $G_i = G$ for all i, we ~~don~~ write ~~[illegible]~~ $R_k(G)$ for $R(G_1, \dots, G_k)$.