Real Analysis Chapter 4 Study Guide (for “Real Analysis, A First Course”, 2nd Edition, Russell A. Gordon)

Number of Starred Exercises: **3**; Number of Notes: 2; Number of Other (non-starred) Exercises: 46; Minimum Number of Other (non-starred) Exercises to Do (to do at least 25% of them): **12**

The most important things to get out of this chapter: an understanding of the meaning and use of the definition of the derivative; an understanding of the meaning, use, and importance of the mean value theorem; an understanding of how everything in the chapter fits together; experience with typical counter-examples.

Other matters of importance:

- In the proofs encountered in this chapter, take note of how often a new function is “created” as an aid to complete a proof. This is done so that the previous theorems may more easily be applied to the new situations. Make a note of this each time you see it.
- Recall how to take derivatives.
- Recalling how to use derivatives in optimization problems and L’Hopital’s Rule problems.

Reading Guide:

1. Our author is not very careful about the conceptual difference between velocity and speed or between position and distance. What extra assumption can you make about the particle in the first paragraph that makes these concepts equivalent?
2. Think of other applied situations (besides velocity, acceleration, decay rates, and growth rates) where the limit of a difference quotient gives you the quantity of interest in the application. Write them down.
3. Make a *Mathematica* animation (you can write your code by hand in your journal) that illustrates how a smooth curve looks more and more like a straight line as you zoom in near any point on the smooth curve.
4. Why are the limits near the top of page 131 equivalent?
5. Fill in any steps which help you understand the calculation of $f'(4)$ and $g'(c)$ underneath Definition 4.1.
6. Use the definition of the derivative and one-sided limits to show that $f(x)=|\xi|$ is not differentiable at $x = 0$. Next, show this same result using Theorem 4.2.
7. Write the general form of the equation of the tangent line to the graph of an arbitrary differentiable function $y = \phi(\xi)$ at an arbitrary point $(c, f(c))$ where c is in the domain of f.
8. *Prove Theorem 4.3. This is ultimately an “S” exercise anyway (see Exercise #8).

9. *Draw a picture (of rectangles) that illustrates the ideas and calculations in the paragraph before the product rule. Can you "see" the product rule visually in this picture?

10. Prove the product rule without looking at the book's proof. Here's the key trick: note that $f(v)g(v) - \phi(\xi)\gamma(\xi) = \phi(\varpi)\gamma(\varpi) - \phi(\varpi)\gamma(\xi) + \phi(\varpi)\gamma(\xi) - \phi(\xi)\gamma(\xi)$.

11. Prove the quotient rule from the definition of the derivative. Here's one key trick: note that $f(v)g(x) - \phi(\xi)\gamma(\varpi) = \phi(\varpi)\gamma(\xi) - \phi(\xi)\gamma(\xi) + \phi(\xi)\gamma(\xi) - \phi(\xi)\gamma(\varpi)$.

12. Note: the proof of the Chain Rule is the first instance where we "create" a new function to aid us in completing a proof. This will be done a number of times in the future.

13. Fill in any details in the proof of the Chain Rule that confuse you. For example, why is F continuous at $g(c)$? Why is $F \circ g$ continuous at c? Why are the equalities at the top of page 135 true?

14. Write the "Leibniz notation" form of the Chain Rule (look it up if you need to). Describe how to think of it in terms of rates of change.

15. Make sure you understand the notation at the bottom of page 135 and in Theorem 4.8. Think about examples if that helps. Work through the proof of Theorem 4.8 and try to understand the tricky notation there. Carefully think about the example after this proof.

16. Does the statement of Theorem 4.8 contain a redundancy? If f is strictly monotone on I and differentiable at c, does it automatically follow that $f'(c) \neq 0$?

17. Do you remember all the derivative rules in Theorem 4.9? Do you think you could write them all down without looking? What about the derivatives of arccos(x), arccsc(x), and arccot(x)?

18. Prove Theorem 4.10.

19. Draw a picture that accurately illustrates the example in the middle of page 140.

20. Does it seem like a philosophical problem that a special case of a theorem (like Rolle's Theorem) can be used to prove the theorem (like the Mean Value Theorem)? Did you know the Pythagorean Theorem can actually be used to prove its converse? That is, the truth of the statement "Given a right triangle of side lengths a and b and hypoteneuse of length c, it follows that $c^2 = \alpha^2 + \beta^2$" can be used to prove the statement "Given a triangle with sides of lengths a, b, and c satisfying the relation $c^2 = \alpha^2 + \beta^2$, it follows that the triangle is a right triangle with the side of length c opposite the right angle". The proof of this would make a good in-class presentation.

21. In the hypotheses of Rolle's Theorem and the Mean Value Theorem (MVT), couldn't we just assume that $f:[a,b] \to$ ° is differentiable on [a, b] and leave it at that? What's the point of saying that f is continuous on [a, b] and differentiable on (a, b)? After all, we defined differentiability at endpoints by saying to just consider the appropriate one-sided limits. Wouldn't Theorem 4.3 play a role in justifying the simpler hypothesis?

22. Where is the Completeness Axiom used in the proof of the Mean Value Theorem? You may have to backtrack in the text. You may find it useful to look at the *Mathematica* notebooks from the first week of class. Describe what you find.

23. Mimic the example on page 142 to prove that $|\sin(a) - \alpha\nu(\beta)| \leq |\alpha - \beta|$ for all *a* and *b* in ° . In fact, $|\sin(x)| < |\xi|$ φορ αλλ $\xi \neq 0$. Why does this mean that *x* = 0 is the only fixed point of the function $f(x) = \alpha\nu(\xi)$ under iteration of this function via the recursive equation $x_{n+1} = \phi(\xi_\nu)$?

24. *Is the converse of Theorem 4.13, part (a) true? How about part (b)? Explain.

25. Prove Corollary 4.14. Hint: create a new function to apply Theorem 4.13 to.

26. See if you can find a statement of "The Racetrack Principle" on the Internet or in a Calculus text. Once you've found it, state it, and prove it with the MVT.

27. Use Theorem 4.13 part c) to prove that if *f* is a function satisfying the property that $f'(x) = f(x)$ for all $x \in$ ° , it follows that $f(x) = X\varepsilon^{\xi}$ for some constant *C*. Hint: create a new function to apply the theorem to.

28. Use Theorem 4.13 part c) to prove that if *f* and *g* are functions satisfying the conditions $f'(x) = g(x)$ for all $x \in$ ° and $g'(x) = -f(x)$ for all $x \in$ ° , it then follows that the function $(f(x))^2 + (\gamma(\xi))^2$ is constant. If $f(0) = 0$ and $g(0) = 1$, what is the value of the constant? What are *f* and *g* in this case?

29. Prove the First Derivative test, part (a). This is also a parenthesized problem… you could do it either here or in another section of your journal.

30. Prove the Second Derivative test, part (a). This is also a parenthesized problem… you could do it either here or in another section of your journal.

31. Which test is more "powerful", the First or Second Derivative test? That is, which test applies to more situations…even when the functions considered are all twice differentiable? Come up with an example where one of the tests "works" and one doesn't "work"…meaning one gives you the correct conclusion about the nature of a critical point but the other one doesn't tell you anything.

32. Carry out more details in the proof of the Cauchy Mean Value Theorem than the book does.

33. How is the statement of Theorem 4.18 different than the assumptions used in the equations in the middle of page 144?

34. Come up with your own examples to apply L'Hopital's Rule to. Work through the use of L'Hoptial's Rule in your examples.

35. Verify the formula for $f'(x)$ in the second paragraph on page 147. Be careful when you verify $f'(0)$ …you should use the definition of the derivative for that one!

36. Verify any confusing details in the proof of Theorem 4.19.

37. Write *Mathematica* code that illustrates the proof of Theorem 4.19. You can write your code by hand in your journal.

38. Some functions that are increasing and concave down have horizontal asymptotes and some do not. Write down two functions, one of which is increasing and concave down with a horizontal asymptote, the other of which is increasing and concave down with no horizontal asymptote. Is there a way to distinguish between these two functions in terms of the behavior of their derivatives? You may want to think about integrals to answer this question.

39. Note: by Definition 4.20, constant functions are both concave up and concave down.

40. Before looking at the proofs of Theorems 4.23 and 4.24, find formulas for $T_c(x)$ and $S_{ab}(x)$ in Definition 4.22.

41. Fill in any details in the proof of Theorem 4.23 that are confusing.

42. Fill in any details in the proof of Theorem 4.24 that are confusing.

43. Why isn't *f* assumed to be twice differentiable on *I* in the statements of Theroems 4.23 and 4.24?

44. Prove that the absolute value function lies below its secant lines on any interval.

45. Look up the definition of a convex *set* and compare and contrast it with the definition of a convex function in Definition 4.25.

46. Prove that Definition 4.25 and 4.22(b) are equivalent.

47. Draw pictures that illustrate how discontinuous functions on open intervals fail to be convex (even when they are "mostly" concave up).

48. Finish off the final details in the proof of Lemma 4.27.

49. Draw an example of a discontinuous convex function on a closed interval.

50. Fill in any details in the proof of Theorem 4.30 that are confusing.

51. Fill in any details in the proof of Theorem 4.31 that are confusing.

Deep Thoughts to Ponder (but not necessarily answer):

- Does the Mean Value Theorem have any physical interpretations? (Hint: the answer is "yes"… ☺). Think of some.
- Does it seem like a contradiction that $f(x) = \xi^3$ has $f'(0) = 0$ yet *f* is strictly increasing on every interval? Can you resolve this seeming paradox visually in your mind?
- Do you think it is possible for a function f to satisfy the conditions that $f(0) = \phi'(0) = \phi''(0) = \phi'''(0) = ... = 0$ yet f is not constant on any interval containing 0? If so, can you come up with such a function?