Real Analysis Chapter 6 Study Guide (for "Real Analysis, A First Course", 2nd Edition, Russell A. Gordon)

Number of Starred Exercises: **3**; Number of Notes: 5; Number of Other (non-starred) Exercises: 18; Minimum Number of Other (non-starred) Exercises to Do (to do at least 25% of them): **5**

The most important things to get out of this chapter: (1) The definition of convergence of an infinite series. (2) The realization that the terms of a series converging to zero is ***not*** sufficient to guarantee that the series converges. (3) Knowledge of convergence tests and ability to apply them. (4) Absolute versus nonabsolute (or "conditional") convergence.

Other matters of importance:

1. Thoroughly understanding geometric series
2. Rearrangements and products

Reading Guide:

1. *Before starting to read Chapter 6, write down what you think it should mean for an infinite series $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$ of numbers to converge (that is, what does it mean for it to equal a particular real number?)
2. *Compare and contrast your answer to #1 with Definition 6.1.
3. **Note:** infinite series are not literal sums. By definition, a true sum must terminate at some point. Newton and Leibniz did imagine that they were sums, however, and the notation has stuck. It usually doesn't cause problems to pretend they are sums.
4. **Note:** make sure you are careful to distinguish the sequence of partial sums of a series with the sequence of terms of the series (see the second paragraph on pag 211). Also take note of the first sentence of the second paragraph on page 211. Chapter 2 will be very useful for Chapter 6!
5. How is the main point of the second paragraph on page 212 (the one that starts with the word "Since") similar to something that happens with integrals? (Hint: think in terms of evaluating integrals using antiderivatives.)
6. Write down the contrapositive of the statement in Theorem 6.2. This is a true statement too (see the bottom of page 212).
7. What other "famous" series (which has a name) can be used to verify that the converse of Theorem 6.2 is false?
8. **Note:** an actual closed-form expression for $s_n = \sum_{k=1}^{n} \frac{1}{\sqrt[3]{k}}$ is not found at the bottom of page 212. This does not matter. We can still prove that $\{s_n\}$ diverges.
9. *Prove Theorem 6.4.

10. Prove Theorem 6.5.
11. Prove Theorem 6.6(a).
12. About an hour after reading the proof of the Limit Comparison Test. Come back to it and try to prove it yourself without looking at the book's proof.
13. **Note:** be careful to heed the lessons/warnings in the book after the Comparison and Limit Comparison Tests (about how to use them).
14. Create your own example like that in the book near the bottom of page 218 where you need to do a bit of manipulation with a summation to get the index starting at 0 so that you can apply the formula in Theorem 6.8.
15. **Note:** the formulas at the bottom of page 219 could generate a mini-research project to find out why they are true.
16. Do you understand why the proof of Theorem 6.10 is a simple consequence of Theorem 6.3 and the inequality in the proof? Fill in some details if you are not sure.
17. Recall how the ideas of limsups and liminfs were justified in class by referring to the Ratio Test on page 223. Compare this Theorem with the Ratio Test in your Calculus textbook (hopefully there's a ratio test there). Try to think of an example where the limsup or the liminf is really needed rather than just an ordinary limit. Also note how these ideas compare with the comments in the book at the bottom of page 224.
18. Try to do the same thing as in #17 for the Root Test on page 224.
19. Prove the Root Test.
20. Draw a picture of a number line and the sequence of partial sums for an alternating series to help you understand the statement and proof of the Alternating Series Test (Theorem 6.14).
21. Use *Mathematica* and the bound in the Alternating Series test to estimate the sum of the series at the bottom of page 225 to within 10^{-6}.
22. Consider the sequence of equations
    $0 = (1-1)+(1-1)+(1-1)+\cdots = 1+(-1+1)+(-1+1)+\cdots = 1+0+0+\cdots = 1$.
    Where is the mistake? How does this illustrate what can go "wrong" with infinite series.
23. Compare the definition of a permutation of the positive integers near the bottom of page 228 with the definition of a permutation in your Algebraic Structures book. Are they the same kind of idea? Can you use array and cycle notation to represent the permutation at the bottom of page 228?
24. Verify the details in the proof of Theorem 6.16 near the bottom of page 229.
25. Draw a picture to help you understand the proof of Theorem 6.17.
26. Write and draw anything you need to in order to help you understand Theorem 6.18 and its proof. (though you won't need to know this theorem for the final exam).

Deep Thoughts to Ponder (but not necessarily answer):

- How would you define what it means for an infinite product to converge?

- I have claimed that infinite sums are not true sums. Do you think it's possible to have a philosophical perspective where they are true sums?