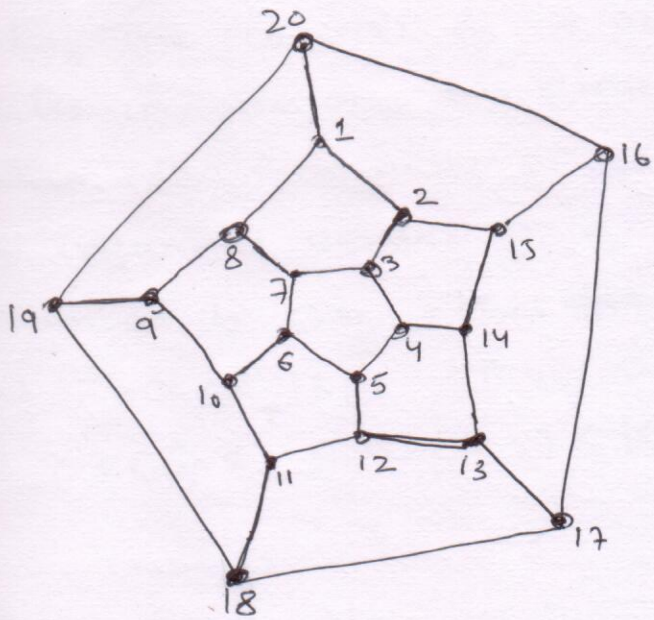
Hamiltonian Graphs



A graph is called Hamiltonian if it has a spanning cycle.

A spanning cycle of a graph G, when it exists is called a Hamiltonian cycle of G.

A graph is called traceable if it has a spanning path of G.

A spanning path of G is called a Hamiltonian path of G.

Th^m If G is Hamiltonian, then for every nonempty proper subset S of V, $\omega(G-S) \le |S|$, where $\omega(G-S)$ is the number of components of G−S.

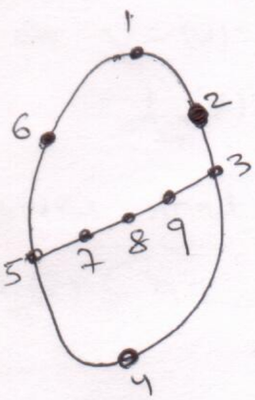
__Proof__ Let C be a Hamiltonian cycle in G. Since C is a spanning subgraph of G, $\omega(G-S) \le \omega(C-S)$, for any ~~set~~ proper subset S of V.

If $|S| = 1$, $C-S$ is a path and therefore, $\omega(C-S) = 1 = |S|$. The removal of a vertex from a path P results in one or two components, according to whether the removed vertex is an end vertex or an internal vertex of P, respectively. Hence the number of components ~~in C-S~~ in $C-S$ cannot exceed $|S|$. This proves that

$$\omega(G-S) \leq \omega(C-S) \leq |S|.$$

Cor If G is Hamiltonian then it has no cut vertices.

Ex



- Theta graph

Th^m Let G be a simple graph with $n \geq 3$ vertices. If for every pair of nonadjacent vertices u, v of G, $d(u) + d(v) \geq n$, then G is Hamiltonian.

Proofs Suppose that G satisfies the cond^n of the theorem but G is not Hamiltonian. Add edges to G (without adding vertices and get a supergraph G^* of G such that G^* is maximal simple graph that satisfies the cond^n of the theorem but G^* is non-Hamiltonian.

Such a graph G^* must exist since G is non-Hamiltonian while the complete graph of $V(G)$ is Hamiltonian.
Hence, for any pair u and v of non adjacent vertices of G^*, G^*+uv must contain a Hamiltonian cycle C.

This cycle C would certainly contain the edge $e=uv$

Then $C-e$ is an Hamiltonian path $u=v_1v_2v_3 \cdots v_n=v$ of G^*

Now if $v_i \in N(u)$ then $v_{i-1} \notin N(v)$; otherwise $v_1 v_2 \; v_{i-1} v_n v_{n-1} \cdots \; v_{i+1} v_i v_1$ would be a Hamiltonian cycle in G^*. Hence, for each vertex adjacent to u, there is a vertex of $V-(v)$ non adjacent to v. But then

$$d_{G^*}(v) \leq (n-1) - d_{G^*}(u)$$

This gives that $d_{G^*}(u) + d_{G^*}(v) \leq n-1$, and therefore $d_G(u) + d_G(v) \leq n-1$, a contradiction.

$\underline{\text{Th}^m}$ Let G be a simple graph of order $n \geq 3$ vertices. Then G is Hamiltonian if and only if $G+uv$ is Hamiltonian for every pair of non adjacent vertices u and v with $d(u)+d(v) \geq n$.

Def^n The closure of a graph G, denoted by $cl(G)$ is defined to be the supergraph of G obtained from G by recursively joining pairs of non-adjacent vertices whose degree sum is atleast n until no such pair exists.

Th^m The closure $cl(G)$ of a graph G ~~is well~~ is well defined.

Proof Let G_1 and G_2 be two graphs obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is atleast n until no such pair exists. We have to prove that $G_1 = G_2$.

Let $\{e_1, \dots, e_p\}$ and $\{f_1, \dots, f_q\}$ be the sets of new edges added to G to get G_1 and G_2, respectively. We want to show that each e_i is some f_j (and therefore belongs to G_2) and that each f_k is some e_l (and ~~the~~ therefore belongs to G_1).

Let e_i be the first edge in $\{e_1, \dots, e_p\}$ not belonging to G_2. Then $e_1, \dots, e_{i-1}$ are all in both G_1 and G_2, and $uv = e_i \notin E(G_2)$. Let $H = G + \{e_1, \dots, e_{i-1}\}$. Then H is a subgraph of both G_1 and G_2. The way $cl(G)$ is defined

$$d_H(u) + d_H(v) \geq n,$$

and hence $d_{G_2}(u) + d_{G_2}(v) \geq n$.

But this is a contradiction since u and v are nonadjacent vertices of G_2 and G_2 is a closure of G. Thus $e_i \in E(G_2)$ and similarly each $b_k \in E(G_1)$.

$\underline{\underline{Th^m}}$ G is Hamiltonian iff and only iff ~~closure~~ $cl(G)$ is Hamiltonian.

A graph G with atleast three vertices is Hamiltonian-connected if any two vertices of G ~~are~~ connected by a Hamiltonian path in G.

$\underline{\underline{Ex}}$ For $n \geq 3$, K_n is Hamiltonian connected,

For $n \geq 4$, C_n is not Hamiltonian connected.

$\underline{\underline{Th^m}}$ If G is a simple graph with $n \geq 3$ vertices such that $d(u) + d(v) \geq n+1$ for every pair of nonadjacent vertices of G, then G is Hamiltonian connected.

$\underline{\underline{Proof}}$ Let u and v be any two vertices of G. Our aim is to show that there exists a Hamiltonian path from u to v in G. Choose a new vertex w, and let $G^* = G \cup \{wu, wv\}$.

We claim that $cl(G^*) = K_{n+1}$.

First, the recursive addition of the pairs of nonadjacent vertices u and v of G with $d(u) + d(v) \geq n+1$ gives K_n.

Further each vertex of K_n is of degree $n-1$ in K_n and $d_{G^*}(w) = 2$. Hence $cl(G^*) = K_{n+1}$. So G^* is Hamiltonian. Let C be a Hamiltonian cycle in G^*. Then $C - w$ is a Hamiltonian Path in G from u to v.