Wiener index

Let G be a connected graph and $V(G)=\{v_1,\dots,v_n\}$. The Weiner index $W(G)$ of G is defined as $W(G)=\sum_{i<j} d(v_i,v_j)$

Thm Among trees with n-vertices, the Wiener index is minimized by the star and maximized by the path, both uniquely.

Proof Since a tree has $n-1$ edges, it has $n-1$ pairs of vertices at distance 1, and all other pairs have distance at least 2. The star achieves this and hence minimizes ~~W(T)~~ the Wiener index among all trees on n-vertices.

To show that no other tree achieves this, consider a pendant vertex x of T, and let v be its neighbour. If all other vertices have distance 2 from x, then they must be neighbours of v, and T is a star. The value is $W(K_{1,n-1})=(n-1)+2\binom{n-1}{2}=(n-1)^2$

For the maximization, consider first $W(P_n)$. This equals the sum of the distances from an endvertex u to the other vertices, plus $W(P_{n-1})$. We have $\sum_{v\in V(P_n)} d(u,v)=\sum_{i=1}^{n-1} i=\binom{n}{2}$

Thus $W(P_n)=W(P_{n-1})+\binom{n}{2}=\binom{n+1}{3}$.

We now prove by induction on n that P_n is the only tree that maximizes $W(T)$

For $n=1$, The only tree with one vertex is P_1.

Assume that the result is true of trees on $n-1$ $(n>1)$ vertices.

Let ~~the~~ T be a tree on n-vertices $(n>1)$ and let u be a pendant vertex of T. Now $W(T) = W(T-u) + \sum_{v \in V(T)} d(u,v)$

By induction hypothesis $W(T-u) \leq W(P_{n-1})$ with equality of and only of $T-u$ is a path. Thus it is sufficient to show that $\sum_{v \in V(T)} d(u,v)$ is maximized only when T is a path and u is an end vertex of T.

Consider the list of distances from u. In P_n, this list is $1, 2, \ldots, n-1$, all distinct. A shortest path from u to a vertex farthest from u contains vertices at all distances from u, so in any tree, ~~the~~ the set of distance from u to the other vertices has no gaps. Thus any repetition makes $\sum_{v \in V(T)} d(u,v)$ smaller than when u is a pendant vertex of a path. When T is not a path, such repetition occurs.

Eulerian graph

A trail in a graph G is a walk in G in which the edges are distinct.

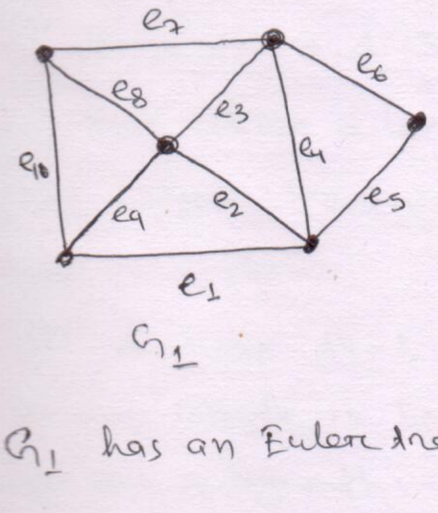
A trail in G is called an Euler trail if it includes every edge of G.

A tour of G is a closed walk of G which includes every edge of G at least once.

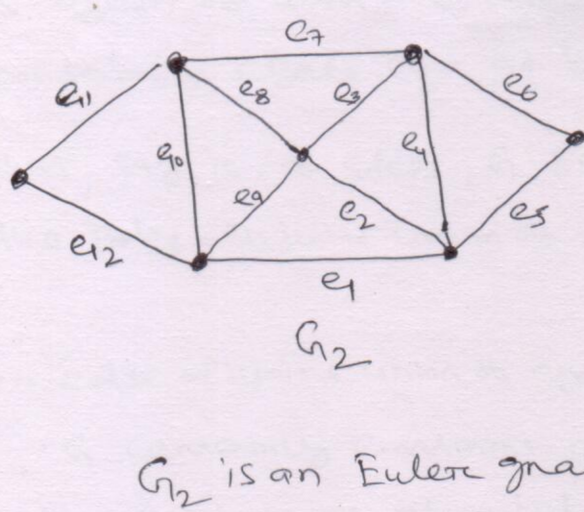
An Euler tour of G is a tour which includes each edge of G exactly once.

A graph G is called Eulerian or Euler if it has an Euler tour.

Example:



G_1



G_2

G_1 has an Euler trail $\qquad$ G_2 is an Euler graph

$\underline{\underline{Th^m}}$ For a connected graph G, the following statements are equivalent:

(1) G is Eulerian

(2) The degree of each vertex of G is an even integer

(3) G is an edge disjoint union of cycles.

Proof (1) ⇒ (2): Let T be an Euler tour of G starting from some vertex $v_0 \in V(G)$. If $v \in V(G)$, and $v \neq v_0$, then every time T enters v, it must move out of v to get back to v_0. Hence two edges incident with v are used during visit to v, and hence, $d(v)$ is even. At v_0, every time T moves out of v_0 it must get back to v_0; Hence $d(v_0)$ is also even. Thus degree of each vertex of G is even.

(2) ⇒ (3): As $\delta(G) \geq 2$, G contains a cycle, say C_1. In $G \setminus E(C_1)$, remove the isolated vertices, if there are any. Let the resulting subgraph of G be G_1. If G_1 is non-empty, each vertex of G_1 is again of even the ~~integer~~ degree. Hence $\delta(G_1) \geq 2$ and so G_1 contains a cycle C_2. It follows that ~~say~~ after a finite number, say r, of steps, $G \setminus E(C_1 \cup \cdots \cup C_r)$ has no edges. Then G is the ~~edge~~ edge disjoint union of the cycles $C_1, \ldots, C_r$.

(3) ⇒ (1) Assume that G is an edge disjoint union of cycles. Since any cycle is Eulerian, G certainly contains an Eulerian subgraph. Let G_1 be the longest closed trail in G. Then G_1 must be G. ~~If~~

If not, let $G_2 = G \setminus E(G_1)$. Since G is an edge disjoint union of cycles, every vertex of G is of even degree ≥ 2. Further since G_1 is Eulerian, each vertex of G_1 is of even degree ≥ 2. Hence each vertex of G_2 is of even degree. Since G_2 is not totally disconnected and G is connected, G_2 contains a cycle C having a vertex v in common with G_1. ~~Describe the Des~~ Describe the Euler tour of G_1 starting and ending at v and follow it by C. Then $G_1 \cup C$ is a closed trail in G longer than G_1. This contradicts the choice of G_1 and so G_1 must be G. Hence G is Eulerian.

If $G_1, \ldots, G_r$ are subgraphs of a graph G that are pairwise edge disjoint and their union is G, the this is denoted by $G = G_1 \oplus \cdots \oplus G_r$.

If $G_i = C_i$, a cycle of G for each i, then $G = C_1 \oplus \cdots \oplus C_r$. The set of cycles $S = \{C_1, \ldots, C_r\}$ is called a cycle decomposition of G.

Thus a connected graph is Eulerian if and only if it admits a cycle decomposition.

Thm A graph is Eulerian if and only if each edge e of G belongs to an odd number of cycles of G.