A directed Hamiltonian path of a digraph D is a directed path in D that includes every vertex of D (once and only once)

Th^m Every tournament T has a directed Hamiltonian path.

Proof Assume that T has n-vertices. If $n=1, 2,$ or 3 we can easily check that T has a directed Hamiltonian path. Thus we may assume $n \geq 4$.

Assume that the result is true for all tournaments on $n-1$ vertices.

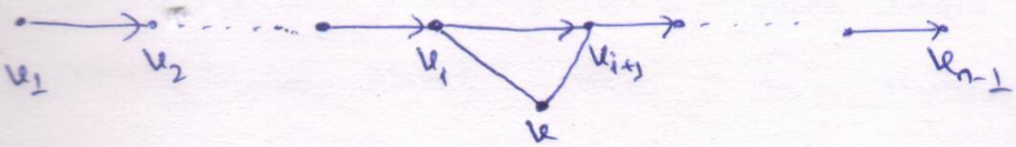
Let v be a vertex of T. Then $T-v$ has $n-1$ vertices and so by hypothesis there is a directed Hamiltonian path in $T-v$. Let $P = v_1 v_2 \dots v_{n-1}$ be such a path.

If there is an arc from v to v_1, then $P' = v v_1 \dots v_{n-1}$ is a directed Hamiltonian path. Similarly, if there is an arc from v_{n-1} to v then $P'' = v_1 \dots v_{n-1} v$ is a directed Hamiltonian path in T.

Hence we may now suppose that there is no arc from v to v_1 and no arc from v_{n-1} to v.

Then there is atleast one vertex v on P with the property that there is an arc from v to v and v is not v_{n-1} (since v_1 has this property).

Let v_i be the last vertex on P having this property, so that the next vertex v_{i+1} does not have this property.



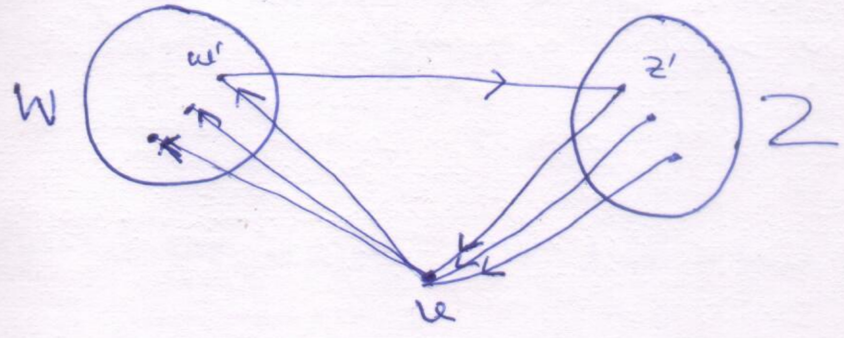
Then in particular, there is an arc from v_i to v and an arc from v to v_{i+1}. But then $Q = v_1 \cdots v_i v v_{i+1} \cdots v_{n-1}$ gives us a directed Hamiltonian path in D. $\blacksquare$

<u>Def</u>^{n} A directed Hamiltonian cycle in a digraph D is a directed cycle which include every vertex of D. If D contains such a cycle then D is called Hamiltonian.

Thm A strongly connected tournament T on n-vertices contains directed cycles of length $3, 4, \ldots n$.

Proof We first show that T contains a directed cycle of length 3. Let v be any vertex of T. Let W denote the set of all vertices w of T for which there is an arc from v to w. Let Z denote the set of all vertices z of T for which there is an arc from z to v. (Note that since T is a tournament, $W \cap Z = \phi$)

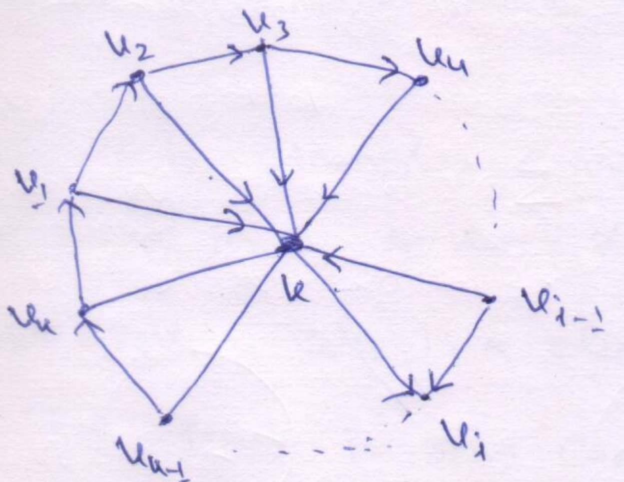
Since T is strongly connected, so W and Z must both be nonempty. Moreover again because T is strongly connected, there must be an arc in T joining from $w' \in W$ to some $z' \in Z$. This gives the directed cycle $vw'z'v$ of length 3.



We now use induction to complete the proof.

We suppose that T has a directed cycle C of length k where $k<n$ (and $k \geq 3$), and using this, we prove that T has a directed cycle of length $k+1$.

Let C be given by $v_1 v_2 \ldots v_k v_1$. Suppose that there is a vertex v, not on the cycle C, with the property that there is an arc from v to v_i and an arc to v_j to v for some v_i, v_j on C. Then there must be a vertex v_i on C with an arc from v_{i-1} to v and an arc from v to v_i. Then $C' = v_1 v_2 \ldots v_{i-1} v v_i v_{i+1} \ldots v_k$ is a directed cycle of length $k+1$.



If no vertex exists with the above property, then the set of vertices not contained in the cycle can be divided into two distinct sets W and Z, where W is the set of vertices w such that for each i, $1 \le i \le k$, there is an arc from v_i to w and ~~Z~~ Z is the set of vertices z such that for each i $1 \le i \le k$, there is an arc from z to v_i.

If W is empty, then the vertices of C and the vertices of Z together make up all the vertices in T. However, by the definition of Z there is no arc from a vertex C to a vertex in Z, a contradiction since T is strongly connected. Thus W must be nonempty. A similar argument shows that Z is nonempty.

Since T is strongly connected, there must be an arc from w' in W to some z' in Z. Then $C' = v_1 w' z' v_3 v_4 \dots v_k v_1$ is a directed cycle of length $k+1$. The proof is now complete by induction.

Cor A tournament T is Hamiltonian if and only if it is strongly connected.

Proof Suppose T has n-vertices. If T is strongly

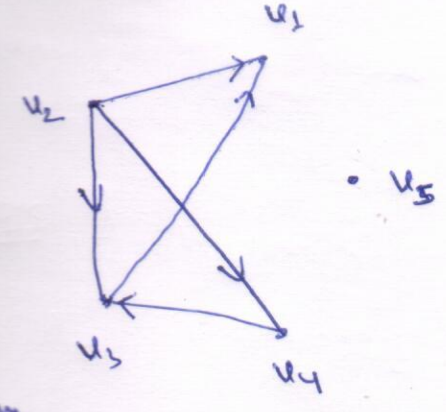
Connected then by the above theorem T is Hamiltonian.

Conversely if T is Hamiltonian with directed Hamiltonian cycle $C = v_1 \cdots v_n v_1$ then given any v_i, v_j in the vertex set of T there is a directed path from v_i to v_j. Thus each vertex is reachable from any other vertex, so T is strongly connected.

Adjacency matrix of a digraph

Let $D=(V, A)$ be a digraph with $V=\{v_1, \cdots, v_n\}$. The adjacency matrix $A(D)$ of D is the $n \times n$ matrix $[a_{ij}]$ with $a_{ij}=1$ if v_iv_j is an arc of D, and 0 otherwise.

Ex



$$A = A(D) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Th The (i,j)th entry $a_{ij}^{(n)}$ of A^k is the number of walks of length k from v_i to v_j.