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→ Qualitative study
Quantitative study

→ Informally, a random experiment is an experiment —
- All outcomes are known in advance.
- Any performance of experiment results in an outcome that is known in advance.
- Such experiment can be repeated under identical conditions.

→ Example:

(A) RE: A coin is tossed once
Outcome: $\underbrace{\Omega = \{H, T\}}$ (sample space)

(B) RE: A coin is tossed twice
Outcome: $\Omega = \{HH, HT, TH, HH\}$

(C) RE: A coin is tossed n-many times
Outcome: $\Omega = \{(x_1, x_2 \cdots x_n) : x_i \in \{H, T\} \ \forall i \in [n]\}$.

Note: (1) If we are observing how many times the coin flips in the air, then $\Omega = \mathbb{N} \cup \{0\}$

(2) If we observe the velocity of the coin when it touches the ground, then $\Omega = \mathbb{R}^+$

→ Example:
RE: A coin is tossed until head appears.
Outcome: $\Omega = \{H, TH, TTH, \cdots\} = \{(x_1, x_2 \cdots x_n) : x_i = T \ \forall i \in [n-1], \ x_n = H\}$

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• Def^n: (a) A sample space of a random experiment is the pair $(\Omega, \mathcal{F})$ where Ω is the set of all outcomes of the random experiment and $\mathcal{F}$ is a σ-algebra.

(b) The members of $\mathcal{F}$ is called events.

(c) Let Ω be a non-empty set. A collection of subsets of Ω is called a σ-algebra if it satisfies the following conditions :—

(i) $\phi \in \mathcal{F}$

(ii) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

(iii) If $\{E_n\}_{n=0}^{\infty}$ be a sequence of members of $\mathcal{F}$, then $\bigcup_{n=0}^{\infty} E_n \in \mathcal{F}$.

• Example of σ-algebra:

i) $\mathcal{P}(\Omega)$ [for finite sets].

ii) $\{\phi, \Omega\}$

iii) $\{\phi, A, A^c, \Omega\}$.

• Probability: A non-zero func^n $P_r : \mathcal{F} \to [0,1]$ is called probability measure or simply probability if $\{E_n\}_{n=0}^{\infty}$ denotes mutually disjoint sequence of members (events) of $\mathcal{F}$ then $\sum_{n=0}^{\infty} P_r(E_n)$ exists and equals $P_r(\bigsqcup_{n=0}^{\infty} E_n)$ and $P_r(\Omega) = 1$

(the series converges) | $(\Omega, \mathcal{F}, P_r)$ is called probability space.

Kolmogorov model

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- Example:

When Ω is finite, say $\Omega = \{w_1, w_2, \dots, w_n\}$

i.e. $|\Omega| = n$, $P(\Omega)$ is σ-algebra.

$\rightarrow$ $P_r : P(\Omega) \longrightarrow [0,1]$ st

$P_r(\{w_i\}) = \frac{1}{n} \quad \forall\ i \in [n]$.

$\rightarrow$ $A \subseteq \Omega$, $A = \{w_{i_1}, w_{i_2}, \dots, w_{i_m}\}$ i.e. $|A| = m$.

$$\therefore P_r(A) = P_r\left(\bigsqcup_{j=1}^{m} \{w_{i_j}\}\right)$$

$$= \sum_{j=1}^{m} P_r(\{w_{i_j}\})$$

$$= \sum_{j=1}^{m} \frac{1}{n}$$

$$= \frac{m}{n}$$

$$= \frac{|A|}{|\Omega|}$$

Exercises / Properties:

(a) $P_r(\phi) = 0$

Take $A \neq \phi$ st $P_r(A) \neq 0$.

Now, $P_r(\phi \sqcup A) = P_r(\phi) + P_r(A)$

$\Rightarrow P_r(A) = P_r(\phi) + P_r(A)$

$\Rightarrow P_r(\phi) = 0$.

(b) $P_r(\Omega) = 1$

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(c) $Pr(A^c) = 1 - Pr(A)$.

$1 = Pr(\Omega) = Pr(A \sqcup A^c) = Pr(A) + Pr(A^c)$

(d) $Pr(A \cup B) \leq Pr(A) + Pr(B)$

In particular, $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$.

Q If a 3-digit no. (000-999) is chosen at random, find the probability that exactly 1 digit will be > 5.

Ans; RE; Choosing a 3-digit number

Sample space; All possible numbers from 000 to 999 i.e. $|\Omega| = 1000$

Event; $A_1 = \{(x, y, z) ; 9 \geq x \geq 6, 0 \leq y, z \leq 5\}$

$A_2 = \{(x, y, z) ; 9 \geq y \geq 6, 0 \leq x, z \leq 5\}$

$A_3 = \{(x, y, z) ; 9 \geq z \geq 6, 0 \leq y, x \leq 5\}$.

Assuming $Pr(x, y, z) = \frac{1}{10^3}$,

$Pr(A_1) = \frac{4 \times 6 \times 6}{10^3}$

$Pr(A_2) = \frac{4 \times 6 \times 6}{10^3}$

$Pr(A_3) = \frac{4 \times 6 \times 6}{10^3}$

$Pr(A_1 \sqcup A_2 \sqcup A_3)$

$= \sum_{i=1}^{3} Pr(A_i)$

$= \frac{3 \times 144}{10^3}$

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Q) An urn contains 3 red, 8 yellow, 13 green balls, another urn contains 5 red, 7 yellow, 6 green balls. One balls is selected from each urn. Find the probability that both the balls will be of same colour.

Ans; RE; Choosing one ball from each urn.

Sample space; ~~x, y~~

$\Omega = \{(x,y) : x \in U_1, y \in U_2\}$ where

$\therefore |\Omega| = (3+8+13)(5+7+6)$

$= 24 \times 18$

$U_1 = \{3R, 8Y, 13G\}$

$U_2 = \{5R, 7Y, 6G\}$

Event; $A_1 = \{(x,y) : x = R = y\}$

$A_2 = \{(x,y) ; x = Y = y\}$

$A_3 = \{(x,y) ; x = G = y\}$

Assuming $P_r((x,y)) = \frac{1}{24 \times 18}$,

$P_r(A_1) = \frac{3 \times 5}{24 \times 18}$

$P_r(A_2) = \frac{8 \times 7}{24 \times 18}$

$P_r(A_3) = \frac{13 \times 6}{24 \times 18}$

$P_r(A_1 \sqcup A_2 \sqcup A_3)$

$= \sum_{i=1}^{3} P_r(A_i)$

• Why is it necessary to assume that $P_r(\Omega) = 1$

$\Rightarrow$ (MCT) $P_r(E_1) < P_r(E_1 \sqcup E_2) < \dots < P_r(\bigsqcup_{i=1}^{n} E_i) < \dots < P_r(\Omega)$
(Monotonically increasing sequence bounded by 1)

Q) Let $(\Omega, \mathcal{F}, P_r)$ be a probability space: ~~and A^c~~
~~denote~~ Show that —

1) $P_r(A^c) = 1 - P_r(A)$

11) $P_r(A \cup B) \leq P_r(A) + P_r(B)$

111) If $A \subset B$, where $A, B \in \mathcal{F}$, then $P_r(A) \leq P_r(B)$.

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•• Thm: (Boole's Inequality)

Let $(\Omega, \mathcal{F}, P_r)$ be a probability space and $\{A_n\}_{n=1}^{\infty}$ be a sequence of events. Then

$$P_r\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P_r(A_n).$$

Proof: We know, $P_r(A_1 \cup A_2) = P_r(A_1) + P_r(A_2) - P_r(A_1 \cap A_2)$

$\Rightarrow P_r(A_1 \cup A_2) \leq P_r(A_1) + P_r(A_2)$,

Let us assume, $P_r\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P_r(A_i)$ for some $n \in \mathbb{N}$ and $n \geq 2$

$$\begin{aligned}
\text{Now, } P_r\left(\bigcup_{i=1}^{n+1} A_i\right) &= P_r\left(\bigcup_{i=1}^{n} A_i \cup A_{n+1}\right) \\
&= P_r\left(\bigcup_{i=1}^{n} A_i\right) + P_r(A_{n+1}) - P_r\left(\left(\bigcup_{i=1}^{n} A_i\right) \cap A_{n+1}\right) \\
&\leq P_r\left(\bigcup_{i=1}^{n} A_i\right) + P_r(A_{n+1}) \\
&\leq \sum_{i=1}^{n} P_r(A_i) + P_r(A_{n+1}) \\
&\leq \sum_{i=1}^{n+1} P_r(A_i)
\end{aligned}$$

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$\therefore \Pr(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} \Pr(A_i) \quad \forall n \in \mathbb{N}.$

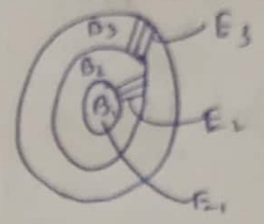
Let $B_n = \bigcup_{i=1}^{n} A_i$

Then $B_1 \subset B_2 \subset \dots \subset B_n$

[why are E_i's events?]

We construct $E_1 = B_1$

$E_2 = B_2 \setminus B_1$



$\vdots$

$E_n = B_n \setminus B_{n-1}$

Then $\{E_i\}_{i=1}^{\infty}$ is a sequence of events and $\forall i, j \in \mathbb{N}$ with $i \neq j$, $E_i \cap E_j = \phi$.

Hence, ~~Pr~~ $\sum_{n=1}^{\infty} \Pr(E_n)$ exists and equals $\Pr(\bigsqcup_{i=1}^{\infty} E_i)$

Note that,

$$\Pr(\bigcup_{n=1}^{\infty} A_n) = \Pr(\bigcup_{n=1}^{\infty} B_n) = \Pr(\bigsqcup_{n=1}^{\infty} E_n)$$

$$= \sum_{n=1}^{\infty} \Pr(E_n)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \Pr(E_i)$$

$$= \lim_{n \to \infty} \Pr(B_n) \quad ?? \text{ [refer to diagram]}$$

$$= \lim_{n \to \infty} \Pr(A_1 \cup A_2 \cup \dots \cup A_n)$$

$$\leq \lim_{n \to \infty} \sum_{i=1}^{n} \Pr(A_i)$$

$$= \sum_{n=1}^{\infty} \Pr(A_n) \qquad \square$$

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HW: If $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ are sequence of events in probability space $(\Omega, \mathcal{F}, P_r)$ st $A_1 \supseteq A_2 \supseteq \cdots$ and $B_1 \subseteq B_2 \subseteq \cdots$, then PT

i) $P_r\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} P_r(A_n)$

ii) $P_r\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n \to \infty} P_r(B_n)$

•• $S = \left\{\frac{1}{n} + \frac{1}{m} : m, n \in \mathbb{N}\right\}$ $\limsup_{m,n \to \infty} S = 1$ (read limsup !!)

$\liminf_{m,n \to \infty} S = 0$

•• Def^n: Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of events in the probability space $(\Omega, \mathcal{F}, P_r)$. The event $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$ denoted as $\limsup_{n \to \infty} A_n$. $\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \liminf_{n \to \infty} A_n\right)$

$A_1 \cup A_2 \cup A_3 \cup \cdots$

$A_2 \cup A_3 \cup \cdots$

$A_3 \cup \cdots$

It means some event A will occur infinitely many times.

•• Example: $A_1 \ A_2 \ \check{A}_3 \ A_4 \ A_5 \ \check{A}_6 \ A_7 \ \cdots$

• Let $A = A_n$ where $n \equiv 0 \bmod 3$.

Then $\limsup_{n \to \infty} A_n = A$

• Let $B = A_n$ where $n \equiv 1 \bmod 3$.

Then $\limsup_{n \to \infty} A_n = A \cup B$.

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Th^m: (Borel Cantelli Theorem)

Let (Ω, F, Pr) be a probability space and $\{A_n\}_{n=1}^{\infty}$ be a seq^n of events. If $\sum_{n=1}^{\infty} Pr(A_n)$ is finite, then $Pr(\limsup_{n\to\infty} A_n) = 0$.

Proof: We consider the sequence of events $\{B_n\}_{n=1}^{\infty}$ where $\forall n \in \mathbb{N}$, $B_n = \bigcup_{i \geq n}^{\infty} A_i$.

$$Pr(\limsup_{n\to\infty} A_n) \underbrace{=}_{\text{(def of limsup)}} Pr\left(\bigcap_{n=1}^{\infty} B_n\right) \underbrace{=}_{\text{(HW1)}} \lim_{n\to\infty} Pr(B_n).$$

$$= \lim_{n\to\infty} Pr\left(\bigcup_{i\geq n}^{\infty} A_i\right)$$

$$\leq \lim_{n\to\infty} \sum_{i\geq n}^{\infty} Pr(A_i) = 0. \quad \square$$

$$\left[\because \sum_{n=1}^{\infty} Pr(A_n) < \infty\right]$$

$$\boxed{\text{If } \sum_{n=1}^{\infty} Pr(A_n) = \infty \text{, then } Pr(\limsup_{n\to\infty} A_n) = 1}$$

Th^m: Let $A_1, A_2, \ldots, A_n$ be events in the probability space (Ω, F, Pr). Then

$$Pr(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} Pr(A_i) + (-1)^{2-1}\left(\sum_{1\leq i < j \leq n} Pr(A_i \cap A_j)\right)$$
$$+ \cdots + (-1)^{k-1}\left(\sum_{1\leq i_1 < i_2 < \cdots < i_k \leq n} Pr(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k})\right)$$
$$+ \cdots + (-1)^{n-1} Pr(A_1 \cap A_2 \cap \cdots \cap A_n))$$

Proof: Base case: For $n = 2$

$$Pr(A_1 \cup A_2) = Pr(A_1) + Pr(A_2) - Pr(A_1 \cap A_2)$$

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Induction Hypothesis:

Let $Pr[A_1 \cup A_2 \cup \cdots \cup A_k] = \sum_{i=1}^{k} Pr[A_i] + (-1)^{2-1} \sum_{1 \le i < j \le k} Pr[A_i \cap A_j]$

$$+ \cdots + (-1)^{l-1} \sum_{1 \le i_1 < \cdots < i_l \le k} Pr(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_l})$$

$$+ \cdots + (-1)^{k-1} Pr[A_1 \cap A_2 \cap \cdots \cap A_k]$$

for some $k \in \mathbb{N}$, $k \geq 2$.

Now, $Pr(\bigcup_{i=1}^{k+1} A_i) = Pr[(\bigcup_{i=1}^{k} A_i) \cup A_{k+1}]$

$$= Pr(\bigcup_{i=1}^{k} A_i) + Pr[A_{k+1}] - Pr((\bigcup_{i=1}^{k} A_i) \cap A_{k+1})$$

$$= Pr(\bigcup_{i=1}^{k} A_i) + Pr(A_{k+1}) - Pr(\bigcup_{i=1}^{k} (A_i \cap A_{k+1}))$$

$$= Pr[\bigcup_{i=1}^{k} A_i] + Pr[A_{k+1}] - \Big(\sum_{i=1}^{k} Pr[A_i \cap A_{k+1}]$$

$$+ (-1)^{2-1} \sum_{1 \le i_1 < i_2 \le k} Pr(A_{i_1} \cap A_{i_2} \cap A_{k+1}) + \cdots +$$

$$(-1)^{l-1} \sum_{1 \le i_1 < \cdots < i_l \le k} Pr(\bigcap_{j=1}^{l} A_{i_j} \cap A_{k+1}) + \cdots +$$

$$(-1)^{k-1} Pr(\bigcap_{i=1}^{k+1} A_i)\Big)$$

$$= \sum_{i=1}^{k+1} Pr(A_i) + (-1)^{2-1} \sum_{1 \le i < j \le k+1} Pr[A_i \cap A_j] + \cdots +$$

$$(-1)^{l-1} \sum_{1 \le i_1 < \cdots < i_l \le k+1} Pr(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_l}) + \cdots + (-1)^{k} Pr(\bigcap_{i=1}^{k+1} A_i)$$

$\square$

Q) The Birthday Problem:

Assume that a person's birthday is equally likely to fall on any of the 365 days in a year (neglect leap year). If k many persons are selected, then what is the probability that all the k birthdays are different?

Ans: RE: Selecting k people / k birthdays.

Sample space: 365^k ~~days~~ $= |\Omega|$

$\Omega = \{(x_1, x_2, \ldots, x_k) : x_i \in [365]\ \forall i \in [k]\}$.

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Events:

$A_1 = \{(x_1, x_2, \dots, x_k) : x_1 = x_2 = \dots = x_k = 1^{st}\text{ Jan}\}$

$A_2 = \{(x_1, x_2, \dots, x_k) : x_1 = x_2 = \dots = x_k = 2^{nd}\text{ Jan}\}$

$\vdots$

$A_{365} = \{(x_1, x_2, \dots, x_k) : x_1 = x_2 = \dots = x_k = 31^{st}\text{ Dec}\}$

$E = \{(x_1, x_2, \dots, x_k) \in \Omega : x_1, x_2, \dots, x_k \text{ are different}\}$

Assume, $Pr(\{\omega\}) = \dfrac{1}{(365)^k}$

Now, $Pr(E) = \dfrac{|E|}{|\Omega|}$.

$|E| = 365 \times 364 \times \dots \times (365 - k + 1)$

$\therefore Pr(E) = \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right) \dots \left(1 - \frac{k-1}{365}\right) < \frac{1}{2}$

if $k \geq 23$.

i.e. for $k \geq 23$ $Pr(E^c) > \frac{1}{2}$

i.e. for $k \geq 23$, at least two people have same birthday.

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- (Ω, F, Pr) — probability space
  $Pr : F \longrightarrow [0,1]$ st $Pr(\Omega) = 1$ and
  $Pr(\bigsqcup_{i\geq 1}^{\infty} E_i) = \sum_{i\geq 1}^{\infty} Pr(E_i)$ where $\{E_n\}_{n\geq 1}^{\infty}$ is a sequence of events.

- (Ω, F) — measurable space
  $\mu : F \longrightarrow [0, m]$ st $\mu(\bigsqcup_{i\geq 1}^{\infty} E_i) = \sum_{i\geq 1}^{\infty} Pr(E_i)$ where $\{E_n\}_{n\geq 1}^{\infty}$ is a sequence of events and $\mu(\Omega) = m$.

- Scaling: $Pr : F \longrightarrow [0,1] \ni Pr(E) = \frac{\mu(E)}{m}$.
  Then (Ω, F, Pr) is a probability space.

(Read about σ-algebra)

- ~~Restricting~~ $\boxed{\text{Conditional Probability}}$:—
  Let (Ω, F, Pr) be a probability space.
  Let $\Omega' \subset \Omega$, $\Omega' \neq \phi$. "$\Omega' \in F$" (why?)

  To find (Ω', F', Pr') — (unique)
  - $F' = \sigma(S)$ — [minimum sigma algebra generated by S.]
    where $S = \{E \cap \Omega' : E \in F\}$.
    Take,
  - ~~Pr~~ $Pr/_{\sigma(S)}(\Omega') = Pr(\Omega')$ (restricting the func^n)

$$Pr' = \frac{1}{Pr(\Omega')} \times Pr/_{\sigma(S)}$$

$$Pr^{①}(A/\Omega') = \frac{Pr/_{\sigma(S)}(A)}{Pr(\Omega')} = \frac{Pr(A \cap \Omega')}{Pr(\Omega')} \quad \forall A \cap \Omega' \in S$$

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· **Defn**: Let (Ω, F, Pr) be a probability space and A, Ω' be two events with $Pr(\Omega') > 0$. The conditional probability of A given Ω' has occurred, in notion such an event ~~is~~ denoted as $A_{|\Omega'}$, is defined by

$$\boxed{Pr(A_{|\Omega'}) = \frac{Pr(A \cap \Omega')}{Pr(\Omega')}}$$

* **Minimal σ-algebra**;

F – σ algebra, $S \subset F$

$\sigma(S) = \bigcap \{F : F \text{ is a } \sigma \text{ algebra containing } S\}$ = minimal σ-algebra generated by S.

Existence: → $\phi \in \sigma(S) \Rightarrow \sigma(S) \neq \phi$.

→ $A \in S \Rightarrow A^c \in \sigma(S)$

→ $\{E_n\}_{n=1}^{\infty}$ ~~$E_n \in S$~~ ~~$E_i \cap E_j = \phi\ \forall i \neq j$~~, then $\bigcup_{i=1}^{\infty} E_i \in \sigma(S)$ if $E_i \in \sigma(S)\ \forall i \in \mathbb{N}$.

Uniqueness;

Q) **Example**; A number is chosen from the first 100 positive integers. Given that the digit in ten's place is 2, what is the conditional probability that the number is prime?

RE: Choosing a number from 1, 2, ..., 100

Sample space: $\Omega = [100]$

~~[illegible]~~ $\Omega' = \{\omega \in \Omega : \text{ten's place of } \omega \text{ is } 2\}$
$= \{20, 21, \dots, 29\}$

Event: $E = \{\omega \in \Omega : \omega \text{ is prime}\}$
$E|_{\Omega'} = \{\omega \in \Omega' : \omega \text{ is prime}\}$

$$\therefore \Pr(E|_{\Omega'}) = \frac{\Pr(E \cap \Omega')}{\Pr(\Omega')}$$

(Borel's σ-algebra)

$$= \frac{2/|\Omega|}{\frac{|\Omega'|}{|\Omega|}}$$

$$= \frac{2}{|\Omega'|}$$

$$= \frac{2}{10}$$

$$= 1/5.$$

Q) An urn contains 10 ~~white~~ red balls and 10 black balls. Two balls are chosen at random without replacement.

a) Given that first ball drawn is red, what is the conditional probability that the second ball is red?

b) What is unconditional probability of the second ball drawn to be red?

Ans: $U = \{10R, 10B\}$

RE: Choosing two balls from U without replacement

Sample space: ~~$\Omega = U = \{10R, 10B\}$~~

~~$\Omega' = \{9R, 10B\}$~~

$\Omega = \{(b_1, b_2) : b_1, b_2 \in U\}$

$\Omega' = \{(R, b_2) : b_2 \in U \setminus \{R\}\}$

Exercise: a) $E = \cancel{\{b \in \Omega : b = R\}} = \{(b_1, b_2) : b_2 = R\}$

$E|_{\Omega'} = \cancel{\{b \in \Omega' : b = R\}} = \{(R, b_2) : b_2 = R\}$

$$P_r(E|_{\Omega'}) = \frac{P_r(E \cap \Omega')}{P_r(\Omega')} = \frac{\quad}{10/20}$$

b)

Example: (Sampling with replacement from a population)

Define $\Omega = \{(x_1, x_2, \ldots, x_k) : x_i \in [n] \ \forall i \in [k]\}$

Here $[n]$ is the ~~size of the~~ population (so size of the population is n) and size of the sample is k.

The probability assigned on all samples are equally likely i.e. $\boxed{P_r(\{\omega\}) = \frac{1}{n^k}}$ $\forall \omega \in \Omega$.

(Often the language used is an urn with n balls from which k balls are drawn with replacement)

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• (Sampling without replacement from a population)

Here we define,

$\Omega = \{(x_1, x_2, \ldots, x_k) : x_i \in [n] \text{ and } \forall\ 1 \le i < j \le n,\ x_i \neq x_j\}$

$[n]$ is the population and size of the sample is ~~a~~ k.

The probability assigned on all samples are equally likely i.e. $\boxed{P_n(\{\omega\}) = \frac{1}{n(n-1)\cdots(n-k+1)}}$

$\forall\, \omega \in \Omega$

• Examples:

(a) Place k distinguishable balls in n distinguishable urns at random. n^k ~~[illegible]~~

(b) Place k identical balls in n distinguishable urns at random. ${}^nC_k$

(c) Place k labelled balls in n distinguishable urns and erase the label.

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Theorem: Let $\Omega_1, \Omega_2, \cdots, \Omega_n$ be mutually disjoint events st. $\Omega = \bigsqcup_{i=1}^{n} \Omega_i$ and $Pr(\Omega_i) > 0$ $\forall i \in [n]$ in the probability space (Ω, F, Pr). Let A be an event.

Then —

$$Pr(A) = \sum_{i=1}^{n} Pr(A|\Omega_i) Pr(\Omega_i) \quad \text{[total probability rule]}$$

Proof:
$$\begin{aligned} Pr(A) &= Pr(A \cap \Omega) \\ &= Pr\left(A \cap \left(\bigsqcup_{i=1}^{n} \Omega_i\right)\right) \\ &= Pr\left(\bigsqcup_{i=1}^{n} (A \cap \Omega_i)\right) \\ &= \sum_{i=1}^{n} Pr(A \cap \Omega_i) \\ &= \sum_{i=1}^{n} Pr(A|\Omega_i) Pr(\Omega_i) \end{aligned}$$

Remark: $\{\Omega_i\}_{i=1}^{\infty}$ be a sequence of mutually disjoint events st $\Omega = \bigsqcup_{i=1}^{\infty} \Omega_i$ with $Pr(\Omega_i) > 0$ $\forall i \in \mathbb{N}$.

Then $Pr(A) = \sum_{i=1}^{\infty} Pr(A|\Omega_i) Pr(\Omega_i)$

Theorem: - - - -

$$\forall i \in [n],\ Pr(\Omega_i|A) = \frac{Pr(\Omega_i \cap A)}{Pr(A)} = \frac{Pr(A|\Omega_i) Pr(\Omega_i)}{\sum_{i=1}^{n} Pr(A|\Omega_i) Pr(\Omega_i)}$$

[Bay's rule]

(similar rule for infinitely many event)

[interchanging the Ω_i's and A]

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Dependent and Independent Events:

- **Def^n**: Let A and B be two events in the probability space (Ω, F, Pr) with $Pr(A) > 0$ and $Pr(B) > 0$. We call A and B to be <u>independent</u> if $Pr(A|B) = Pr(A)$ (or $Pr(B|A) = Pr(B)$ or $Pr(A \cap B) = Pr(A)$

- <u>Example</u>:

(a) <u>RE:</u> A fair dice is rolled twice.

<u>Sample space:</u> $\Omega = \{(i,j) : i, j \in [6]\}$

<u>Event 1:</u> First roll is an even number.

$E_1 = \{(i,j) : i, j \in [6] \text{ and } i \text{ is even}\}$

$Pr(E_1) = \dfrac{|E_1|}{|\Omega|} = \dfrac{18}{36} = \cancel{\tfrac{1}{6}}\ \dfrac{1}{2}$

<u>Event 2:</u> Sum of the two rolls is an even no.

$E_2 = \{(i,j) : i, j \in [6] \text{ and } i+j \text{ is even}\}$

$Pr(E_2) = \dfrac{|E_2|}{|\Omega|} = \dfrac{18}{36} = \dfrac{1}{2}$.

$E_2 = \{(i, 2-i) : i = 1\} \sqcup \{(i, 4-i) : i = 1, 2, 3\} \sqcup$
$\{(i, 6-i) : i = 1, 2, 3, 4, 5\} \sqcup \{(i, 8-i) : i = 2, 3, 4, 5, 6\} \sqcup$
$\{(i, 10-i) : i = 4, 5, 6\} \sqcup \{(i, 12-i) : i = 6\}$

~~$\therefore Pr(E_2) = 1+3+5+5+3+1 = 18$~~

~~$Pr(A \cap B)$~~ $Pr(E_1 \cap E_2) = ?$

$E_1 \cap E_2 = \{(i,j) : i = 2, 4, 6 \text{ and } i+j = 4, 6, 8, 10, 12\}$
$= \{(i, 4-i) : i = 2\} \sqcup \{(i, 6-i) : i = 2, 4\} \sqcup$
$\{(i, 8-i) : i = 2, 4, 6\} \sqcup \{(i, 10-i) : i = 4, 6\} \sqcup \{(i, 12-i) : i$

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$$Pr(E_1 \cap E_2) = \frac{|E_1 \cap E_2|}{|\Omega|} = \frac{9}{36} = \frac{1}{4}$$

$\therefore Pr(E_1 \cap E_2) = Pr(E_1) \cdot Pr(E_2)$.

$\therefore E_1$ and E_2 are independent events.

(b) RE: A fair coin is tossed twice.

Sample space: $\Omega = \{(i,j) ;\ i,j \in \{H,T\}\}$

Event 1: First toss is Head

$E_1 = \{(i,j) ;\ i,j \in \{H,T\}$ and $i = H\}$

$\therefore E_1 = \{(H,H), (H,T)\}$

$$Pr(E_1) = \frac{|E_1|}{|\Omega|} = \frac{2}{4} = \frac{1}{2}$$

Event 2: Exactly one of the two tosses is Head.

$E_2 = \{(i,j) :\ \cancel{et}\ i,j \in \{H,T\}$ and either $i = H$ or $j = H\}$

$\therefore E_2 = \{(H,T), (T,H)\}$

$$Pr(E_2) = \frac{|E_2|}{|\Omega|} = \frac{2}{4} = \frac{1}{2}.$$

$E_1 \cap E_2 = \{(H,T)\}$

$\therefore Pr(E_1 \cap E_2) = \frac{1}{4} = Pr(E_1)\, Pr(E_2)$

$\therefore E_1$ and E_2 are independent events.

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• ⓒ <u>RE:</u> A number is chosen at random from
1 to 100.

<u>Sample space</u>; $\Omega = \{ i : i \in [100] \}$

<u>Event 1</u>; The digit in 10^{th} place is odd.

$E_1 = \{ 10x + y : x, y \in [9] \cup \{0\}$ and x is odd $\}$

<u>Event 2</u>; The no. is divisible by 5.

<u>Exercise</u>; Let A and B be two independent events in the prob space $(\Omega, \mathcal{F}, Pr)$. Then —

$\left.\begin{array}{l} \cdot\ A, B^c \\ \cdot\ A^c, B^c \\ \cdot\ A^c, B \end{array}\right\}$ are independent.

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• **Theorem**: (Borel- Cantelli Lemma)

Let $\{A_n\}_{n\geq 1}^{\infty}$ be a sequence of events in the probability space (Ω, F, Pr). If for each integer m, $A_1, A_2, \ldots, A_m$ are independent events and $\sum_{n\geq 1}^{\infty} Pr(A_n) = \infty$, then $Pr(\limsup_{n\to\infty} A_n) = 1$.

Proof: For each integer n, let $B_n = \bigcup_{k=n}^{\infty} A_k$

$$0 \leq Pr(B_n^c) = Pr\left(\bigcap_{k=n}^{\infty} A_k^c\right) = Pr\left(\lim_{N\to\infty} \bigcap_{k=n}^{N} A_k^c\right)$$

$$= \lim_{N\to\infty} Pr\left(\bigcap_{k=n}^{N} A_k^c\right)$$

$$= \lim_{N\to\infty} \prod_{k=n}^{N} Pr(A_k^c)$$

$$= \lim_{N\to\infty} \prod_{k=n}^{N} (1 - Pr(A_k))$$

$$\leq \lim_{N\to\infty} \prod_{k=n}^{N} e^{-Pr(A_k)}$$

$$= \lim_{N\to\infty} \frac{1}{e^{Pr(A_n)}} \cdot \frac{1}{e^{Pr(A_{n+1})}} \cdots \frac{1}{e^{Pr(A_N)}}$$

$$= \lim_{N\to\infty} \frac{1}{e^{\sum_{k=n}^{N} Pr(A_k)}}$$

$$= 0$$ □

• **Corollary**: $\{x_n\}_{n=1}^{\infty}$ be a sequence with $0 \leq x_n < 1$ st $\sum_{n\geq 1}^{\infty} x_n = \infty$, then $\prod_{n\geq 1}^{\infty} (1 - x_n) = 0$.

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Theorem: (Lovász Local Lemma)

Let $A_1, A_2, \ldots, A_n$ be n-many event in the probability space (Ω, F, P_r) with the property that $\forall\, i \in [n], A_i$ is dependent on at most d-many events (where $d \geqslant 2$) among the n-many events and $0 < P_r(A_i) \leq \frac{1}{4d}$. Then —

$$P_r(A_1^c \cap A_2^c \cap \ldots \cap A_n^c) > 0.$$

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[Remark: Here we prove $P_r(A_1 | A_2^c \cap A_3^c \cap \ldots \cap A_n^c) \leq \frac{1}{2d}$.

The expansion $P_r(A_1 | A_2^c \cap \ldots \cap A_n^c)$ makes sense as (by induction) $P_r(A_2^c \cap \ldots \cap A_n^c) > 0$]

$$P_r(A_1^c \cap A_2^c \cap \ldots \cap A_n^c) = P_r(A_1^c | A_2^c \cap \ldots \cap A_n^c)\, P_r(A_2^c \cap \ldots \cap A_n^c)$$
$$= (1 - P_r(A_1 | A_2^c \cap \ldots \cap A_n^c))\, P_r(A_2^c \cap \ldots \cap A_n^c)$$
$$\geqslant (1 - \tfrac{1}{2d})\, P_r(A_2^c \cap \ldots \cap A_n^c) > 0$$

Proof: We have $P_r(A_n^c) = (1 - P_r(A_n)) \geqslant (1 - \frac{1}{4d}) \geqslant \frac{1}{2}$

Now $P_r(A_{n-1} | A_n) = \dfrac{P_r(A_{n-1} \cap A_n^c)}{P_r(A_n^c)} \leq \dfrac{P_r(A_{n-1})}{P_r(A_n^c)}$ (base case)

$$\leq \frac{1/4d}{1/2} = \frac{1}{2d}$$

Suppose it is true that — (induction hypothesis)

$$P_r(A_{k+1}^c \cap \ldots \cap A_n^c) > 0.$$

We show that $P_r(A_k | A_{k+1}^c \cap \ldots \cap A_n^c) \leq \frac{1}{2d}$ and

$$P_r(A_k^c \cap A_{k+1}^c \cap \ldots \cap A_n^c) > 0$$

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Note that,

$$\Pr(A_k \mid A_{k+1}^c \cap \cdots \cap A_n^c) = \begin{cases} \Pr(A_k) & \text{if } A_k \text{ is independent of } A_{k+1}^c \cap \cdots \cap A_n^c \\ \dfrac{\Pr(A_k \cap A_{k+1}^c \cap \cdots \cap A_l^c \mid A_{l+1}^c \cap \cdots \cap A_n^c)}{\Pr(A_{k+1}^c \cap \cdots \cap A_l^c \mid A_{l+1}^c \cap \cdots \cap A_n^c)} & \end{cases}$$

if A_k is independent of $A_{l+1}^c \cap \cdots \cap A_n^c$ and dependent on $A_{k+1}^c \cap \cdots \cap A_l^c$.

If A_k is independent of $A_{k+1}^c \cap \cdots \cap A_n^c$, then

$$\Pr(A_k \mid A_{k+1}^c \cap \cdots \cap A_n^c) = \Pr(A_k) \leq \frac{1}{4d} < \frac{1}{2d}.$$

Again, if A_k is is dependent on $A_{k+1}^c \cap \cdots \cap A_l^c$ and independent of $A_{l+1}^c \cap \cdots \cap A_n^c$, then

$$\cancel{\Pr(A_k \mid A_{k+1}^c \cap \cdots \cap A_n^c) \geq \Pr(A_k \cap A_{k+1}^c \cap \cdots A_l^c \mid A_{l+1}^c \cap \cdots \cap A_n^c)}$$

$$\Pr(A_k \cap A_{k+1}^c \cap \cdots \cap A_l^c \mid A_{l+1}^c \cap \cdots \cap A_n^c) \leq \Pr(A_k \mid A_{l+1}^c \cap \cdots \cap A_n^c)$$
$$= \Pr(A_k)$$
$$\leq \frac{1}{4d}$$

and $\Pr(A_{k+1}^c \cap \cdots \cap A_l^c \mid A_{l+1}^c \cap \cdots \cap A_n^c)$

$$= 1 - \Pr(A_{k+1} \cup \cdots \cup A_l \mid A_{l+1}^c \cap \cdots \cap A_n^c)$$

$$\geq 1 - \sum_{i=k+1}^{l} \Pr(A_i \mid A_{l+1}^c \cap \cdots \cap A_n^c)$$

$$\geq 1 - \sum_{i=k+1}^{l} \frac{1}{2d} \geq \frac{1}{2}.$$

Hence, $\Pr(A_k \mid A_{k+1}^c \cap \cdots \cap A_n^c) \leq \dfrac{1/4d}{1/2} = \dfrac{1}{2d}.$

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This implies,

$$\Pr(A_k^c \mid A_{k+1}^c \cap \cdots \cap A_n^c) = 1 - \Pr(A_k \mid A_{k+1}^c \cap \cdots \cap A_n^c)$$

$$\geq 1 - \frac{1}{2d} > 0 .$$

Since both $\Pr(A_{k+1}^c \cap \cdots \cap A_n^c) > 0$ and $\Pr(A_k^c \mid A_{k+1}^c \cap \cdots \cap A_n^c) > 0$, we have —

$$\Pr(A_k \mid A_{k+1}^c \cap \cdots \cap A_n^c) = \Pr(A_k^c \mid A_{k+1}^c \cap \cdots \cap A_n^c) \Pr(A_{k+1}^c \cap \cdots \cap A_n^c)$$

$$> 0 .$$

□

Q] In a village, 20% population has covid-19. A test is administrated which has a property that if a person has symptoms of covid-19, the test will be positive 90% time and if the person does not have any covid-19 symptoms, then the test is positive 30% time.

A drug was given to all those who has tested positive, with side effect of skin rash 25% of time.

Given that a person is picked at random has skin rash, what is the probability that he had covid-19?

Random Variable :—

Discrete Random Variable:

Def^n: Let (Ω, F, Pr) be a probability space. A fun^n $X : \Omega \rightarrow \mathbb{R}$ is said to be discrete random variable if $X(\Omega)$ is an at most countable subset of $\mathbb{R}$ and $\forall x \in X(\Omega)$, $X^{-1}(\{x\}) \in F$.

Example:

(a) RE: Tossing a coin thrice.

Sample space: $\Omega = \{(x,y,z) : x,y,z \in \{H,T\}\}$.

$|\Omega| = 2^3 = 8$ with $Pr(\{x,y,z\}) = \frac{1}{8}$.

$X : \Omega \rightarrow \mathbb{R}$ st $X(w) =$ no. of heads in w.

$\therefore X(\Omega) = \{0,1,2,3\}$.

(distribution of RV)

$Pr(\{w \in \Omega : X(w) = 0\}) = \frac{1}{8} = Pr(X=0)$

$Pr(\{w \in \Omega : X(w) = 1\}) = \frac{3}{8} = Pr(X=1)$

$Pr(\{w \in \Omega : X(w) = 2\}) = \frac{3}{8} = Pr(X=2)$

$Pr(\{w \in \Omega : X(w) = 3\}) = \frac{1}{8} = Pr(X=3)$

$X=i$	$i=0$	$i=1$	$i=2$	$i=3$
$Pr(X=i)$	$1/8$	$3/8$	$3/8$	$1/8$

(b) RE: Throwing a 6-faced dice twice

Sample space: $\Omega = \{(i,j) \mid i,j \in [6]\}$

$|\Omega| = \frac{1}{6^2} = \frac{1}{36}$ with $Pr(\{i,j\}) = \frac{1}{36}$.

$(\Omega, F (= P(\Omega)), Pr)$ is the prob space.

$X : \Omega \rightarrow \mathbb{R}$ st $X(i,j) = i+j$

$\therefore X(\Omega) = \{2, \dots, 12\}$.

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$X=i$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$	$i=10$	$i=11$
$P_r(X)$	$\frac{1}{36}$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$

(Roughly saying, symmetric distribution of random variable is a property of RV).

•• ~~Continuous~~ Bernoulli Random Variable;

• Def^n; Let (Ω, F, P_r) be a probability space and $p \in \mathbb{R}$ with $0 \leq p \leq 1$. A fun^c $X : \Omega \longrightarrow \mathbb{R}$ with $X(\Omega) = \{0, 1\}$ with $P_r(X=1) = p$ (hence $P_r(X=0) = 1-p$) is called Bernoulli random variable with parameter p.

In notation, we write $X \sim Ber(p)$
(read as X follows Bernoulli p).

• Example;

(a) RE; A coin is tossed once.
Sample space; $\Omega = \{H, T\}$.

$$X : \Omega \longrightarrow \mathbb{R} \text{ st } X(\omega) = \begin{cases} 1 & \text{if } \omega = H \\ 0 & \text{if } \omega = T \end{cases}$$

Here $P_r(X=1) = p$
$P_r(X=0) = 1-p$.
| If the coin is unbiased then $p = 1/2$.

$\therefore X \sim Ber(p)$.

(b) <u>RE:</u> A coin is tossed and a dice is rolled.

<u>Sample space:</u> $\Omega = \{H,T\} \times [6]$

$= \{(x,i) ; x \in \{H,T\}, i \in [6]\}$.

$Pr(\{x,i\}) = \frac{1}{12}$.

$Y : \Omega \longrightarrow \{0,1\}$ st $Y(\omega) = \begin{cases} 1 & \text{if } \omega \in \{(T,1),(T,2)\} \\ 0 & \text{otherwise} \end{cases}$.

$\therefore Pr(Y=1) = \frac{1}{6}$ and $Pr(Y=0) = \frac{5}{6}$.

$\therefore X \sim Ber(1/6)$.

.. <u>Binomial Random Variable;</u>

• <u>Def^n:</u> Let (Ω, F, Pr) be a probability space, $p \in \mathbb{R}$ st $0 \le p \le 1$ and $n \in \mathbb{N}$. A func^n $X : \Omega \rightarrow \mathbb{R}$ with $X(\Omega) = \{0,1,\cdots,n\}$ and $Pr(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$ is called binomial random variable with parameters n and p.

In notation, $X \sim binom(n,p)$.

• <u>Example;</u>

(a) <u>RE:</u> A coin is tossed n times / n many identical coins are tossed once.

<u>Sample space:</u> $\Omega = \{H,T\}^n$.

$X : \Omega \longrightarrow \{0,1,\cdots,n\}$ st $X(\omega)$ = no. of heads in ω.

Then $X(\omega) = \{0,1,\cdots,n\}$ and $Pr(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$ $\forall i \in \{0,1,\cdots,n\}$

$\therefore X \sim binom(n,p)$.

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Independence of Random Variable;

(Ω, F, Pr) $\quad X : \Omega \to \mathbb{R}$, $Y : \Omega \to \mathbb{R}$

• **Def^n:** Let $n \geq 2$ be an integer and $\forall i \in [n]$, $X_i : \Omega \to \mathbb{R}$ be a discrete random variable in probability space (Ω, F, Pr).

→ The discrete random variables $X_1, X_2, \ldots, X_n$ are said to be independent if $\forall i \in [n]$ and $\forall x_i \in X_i(\Omega)$, ~~$Pr(X_1 = x_1 \wedge \ldots \wedge X_n = x_n) =$~~

$$Pr(X_1 = x_1, \ldots, X_n = x_n) = Pr(X_1 = x_1) \cdots Pr(X_n = x_n).$$

→ The discrete random variables $X_1, X_2, \ldots, X_n$ are said to be identically distributed if $\exists$ a discrete random variable $X : \Omega \to \mathbb{R}$ st $\forall i \in [n]$ and $\forall x \in X(\Omega)$, $Pr(X_i = x) = Pr(X = x)$.

→ If both properties hold for discrete random variables $X_1, \ldots, X_n$, then they are called independent and identically distributed discrete random variable.

Proposition;

If X and Y are two random variables with parameters m, p and n, p respectively, then X+Y follows binomial random variable with parameters m+n and p.

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Proof: $\forall k \in \mathbb{N}$ with $0 \le k \le m+n$,

$$Pr(X+Y=k) = Pr\left(\bigsqcup_{i=0}^{m} \{\omega \in \Omega : X(\omega)=i,\ Y(\omega)=k-i\}\right)$$

$$= \bigsqcup_{i=0}^{m} Pr(X=i,\ Y=k-i)$$

$$= \bigsqcup_{i=0}^{m} \left(Pr(X=i)\cdot Pr(Y=k-i)\right)$$

(take $\bigsqcup_{i=0}^{m}$ as $\sum_{i=0}^{m}$)

$$= \bigsqcup_{i=0}^{m} \left(\binom{m}{i} p^i (1-p)^{m-i} \binom{n}{k-i} p^{k-i} (1-p)^{n-k+i}\right)$$

$$= \bigsqcup_{i=0}^{m} \left(\binom{m}{i}\binom{n}{k-i} p^k (1-p)^{m+n-k}\right)$$

$$= p^k (1-p)^{m+n-k} \left(\bigsqcup_{i=0}^{m} \binom{m}{i}\binom{n}{k-i}\right)$$

$$= \binom{m+n}{k} p^k (1-p)^{m+n-k}. \qquad \square$$

• Poisson Random Variable;

• Def^n; Let $(\Omega, \mathcal{F}, Pr)$ be a probability space and $\lambda \in \mathbb{R}^+$. The discrete random variable $X: \Omega \to \mathbb{R}$ with $X(\Omega) = \{0, 1, \cdots\}$ with $\boxed{Pr(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}}$ is called Poisson random variable with parameter λ.

•• Proposition: (Ω, F, P_r), $X : \Omega \to \mathbb{R}$, $Y : \Omega \to \mathbb{R}$.

If X and Y are two independent Poisson random variables with parameters λ and μ respectively, then $X+Y$ is a Poisson random variable with parameter $\lambda+\mu$.

Proof: $\forall$ non-negative integer k,

$$P_r(X+Y=k) = \sum_{i=0}^{k} P_r(X=i, Y=k-i)$$

$$= \sum_{i=0}^{k} \left(P_r(X=i) \cdot P_r(Y=k-i)\right)$$

$$= \sum_{i=0}^{k} \left(\left(e^{-\lambda} \frac{\lambda^i}{i!}\right)\left(e^{-\mu} \frac{\mu^{k-i}}{(k-i)!}\right)\right)$$

$$= \frac{e^{-(\lambda+\mu)}}{k!} \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \lambda^i \mu^{k-i}$$

$$= \frac{e^{-(\lambda+\mu)}}{k!} (\lambda+\mu)^k$$

$$= e^{-(\lambda+\mu)} \cdot \frac{(\lambda+\mu)^k}{k!} \qquad \square$$

•• Theorem:

Let $n \geq 2$ be an integer and $X_1, \dots, X_n$ be independent and identically distributed (iid) random variables in a probability space (Ω, F, P_r). $\forall i \in [n]$, X_i follows $Y : \Omega \to \{0,1\}$ with $P_r(Y=1) = p$ ~~and~~ (i.e. X_i follows Bernoulli rv with parameter p). Then the discrete random variable

$X_1 + \cdots + X_n = X$ follows binomial random variable with parameter n and p,

<u>Proof:</u> We note that $X(\Omega) = \{0, 1, \ldots, n\}$.

$$
\begin{aligned}
\Pr(X = i) &= \Pr(X_1 + X_2 + \cdots + X_n = i) \\
&= \Pr(\{\omega \in \Omega ; X_1(\omega) + \cdots + X_n(\omega) = i\}) \\
&= \Pr\Big(\bigsqcup_{\substack{\sum_{j=1}^n x_j = i, \\ x_j \in \{0,1\} \, \forall j \in [n]}} \{\omega \in \Omega ; X_1(\omega) = x_1, \ldots, X_n(\omega) = x_n\}\Big) \\
&= \sum_{\sum_{j=1}^n x_j = i, \; x_j \in \{0,1\} \, \forall j \in [n]} \Pr(\{\omega \in \Omega : X_1(\omega) = x_1, \ldots, X_n(\omega) = x_n\}) \\
&= \sum_{\sum_{j=1}^n x_j = i, \; x_j \in \{0,1\} \, \forall j \in [n]} \Pr(X_1 = x_1, \ldots, X_n = x_n) \\
&= \sum_{\sum_{j=1}^n x_j = i, \; x_j \in \{0,1\} \, \forall j \in [n]} p^i (1-p)^{n-i} \\
&= p^i (1-p)^{n-i} \sum_{\sum_{j=1}^n x_j = i, \; x_j \in \{0,1\} \, \forall j \in [n]} 1 \\
&= \binom{n}{i} p^i (1-p)^{n-i} .
\end{aligned}
$$

□

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• Theorem;

Let $\forall n \in \mathbb{N}$, X_n follows binomial random variable with parameters n and p_n where $\lim_{n \to \infty} n p_n = \lambda > 0$, in probability space $(\Omega, \mathcal{F}, P_n)$

Then $\forall k \geq 0$, $\lim_{n \to \infty} P_n(X_n = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$.

Proof;

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