Given any walk $W = v_0 v_1 \cdots v_k$ in the underlying graph G of a network N, then the associated arcs in N are ~~neither~~ either $v_{i-1}v_i$ or of the form $v_i v_{i-1}$. A arc of the first form is called a forward arc of W while one of the second form is called a reverse arc of W.

~~of a flow~~

If f is a flow in N we associate to the walk W, in the underlying graph G, a non negative integer $i(W)$, called increment of W, defined by

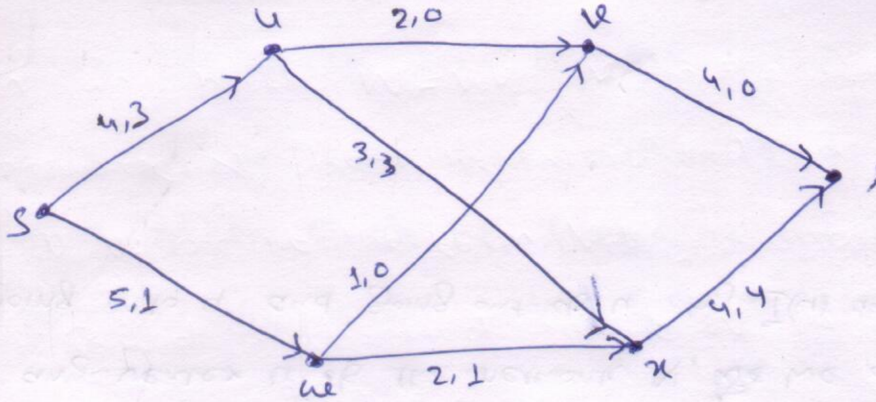
$$i(W) = \min\{i(a) : a \text{ is an arc associated with } W\}$$

where

$$i(a) = \begin{cases} c(a) - f(a) & \text{if } a \text{ is a forward arc of } W \\ f(a) & \text{if } a \text{ is a reverse arc of } W \end{cases}$$

The walk W is ~~called~~ said to be f-saturated if $i(W) = 0$ and f-unsaturated if $i(W) > 0$.

An f-increment walk is an f-unsaturated walk from the source s to the sink t.



Consider the walk $W = swxuv$

The forward arcs of W are sw, wx, uv and xu is the only ~~The reverse arcs of W~~ reverse arc of W. Thus

$i(sw) = c(sw) - f(sw) = 5-1=4$, $i(wx) = c(wx) - f(wx) = 2-1 = 1$

$i(ux) = f(ux) = 3$, $i(uv) = c(uv) - f(uv) = 2-0 = 2$

So $i(W) = \min\{4,3,2,1\} = 1$.

$\underline{\text{Th}}^{m}$ (The Max-Flow, Min-Cut Theorem)

Let N be a network with capacity function c. Then there exists a maximum flow in N, ie. there exists a flow f in N with value $\min\{c(X,\bar{X}) : A(X,\bar{X}) \text{ is a cut}\}$

<u>Proof</u> We know that for any flow f in N with value d, we have $d \leq \min\{c(X,\bar{X}) : A(X,\bar{X}) \text{ is a cut}\}$.

Given an arebitary flow f, we let X be the set of vertices z in N such that either $z = s$ or in the underlying graph G of N there is an f-unsaturated walk $W = v_0 \cdots v_n$ from s to z (so that $s = v_0$, $z = v_n$)

Eithere the sink t is in X or it is in $\overline{X}$. Let us suppose firest that t is in X. Then there must be an f-unsaturated walk W from s to t, i.e. an f-incrementing walk

Choose such a walk W and let $i(W) = \varepsilon$, ~~stet~~ so that $\varepsilon > 0$

We now define a function f_1 on the arcs of N by

$$f_1(a) = \begin{cases} f(a) + \varepsilon & \text{if } a \text{ is a forward arc in } W \\ f(a) - \varepsilon & \text{if } a \text{ is a } \cancel{\text{backward}}\ \text{reverse arc in } W \\ f(a) & \text{if } a \text{ is any other arc in } W \end{cases}$$

Then f_1 is a flow with value $d + \varepsilon$. Since ε is a +ve integer, we have increased the flow f, with value d, to a new flow f_1 with value $d + \varepsilon$.

This new flow f_1 is called the revised flow based on (the f-incrementing walk) W.

This Procedure of increasing the flow is always possible provided the sink t is in the set X.

Thus we may repeat the process, progressively revising the flow based on incrementing walks until we reach a stage, there is no longer an increment walk available to us. Also, since $t \notin X$, $A(X, \bar{X})$ is a cut.

We have reached a flow, call it f', which has associated set X with $t \notin X$.

Then $A(X, \bar{X})$ is a cut. Now if the vertex x is in X then by definition of X there is an f'-unsaturated walk $W = v_0 \cdots v_k$ from the source s to x (so that $v_0 = s$ and $v_k = x$). Suppose that y is a vertex not in X, i.e. $y \in \bar{X}$. If there is an arc from x to y satisfying $f'(xy) < c(xy)$, then the walk $W_1 = v_0 \cdots v_k y$ from s to y would also be f-unsaturated, implying that y is in X, not in $\bar{X}$ a contradiction.

Similarly if there is an arc yx from y to x satisfying $f'(yx) > 0$, then the walk $W_2 = v_0 \cdots v_k y$ from s to y would be f-unsaturated, again a contradiction.

Thus any arc of the form xy where $x \in X$ and $y \in \bar{X}$ must have $f'(xy) = c(x,y)$, while any arc of the form yx where $x \in X$ and $y \in \bar{X}$ must have $f'(yx) = 0$.

This shows that

$$f'(X,\bar{X}) = c(X,\bar{X}) \quad \text{while} \quad f(\bar{X},X) = 0$$

Now if f has value d then, since $A(X,\bar{X})$ is a cut, we have by above theorem

$$d = f'(X,\bar{X}) - f'(\bar{X},X) .$$

Thus $d = c(X,\bar{X}) - 0 = c(X,\bar{X})$.

So the value of the flow f' equals the capacity of the cut $A(X,\bar{X})$. ~~This~~ Hence f' is a maximal flow.