Real Analysis Chapter 1 Study Guide (for "Real Analysis, A First Course", 2nd Edition, Russell A. Gordon)

Number of Starred Exercises: **10**; Number of Notes: 10; Number of Other (non-starred) Exercises: 34; Minimum Number of Other (non-starred) Exercises to Do (to do at least 25% of them): **9**;

THE most important thing to get out of this chapter: The fundamental nature of the ***completeness axiom***; both in distinguishing the real numbers from the rational numbers, and in proving fundamental facts in real analysis.

Other matters of importance:

- The ordered field properties of real numbers
- The triangle inequality and other properties of absolute value
- The definition of bounded sets and the concepts of supremums and infimums.
- The Archimedean Property of real numbers
- The density of the rationals and irrationals in the real number system
- The notion of a one-to-one correspondence (bijection) and its relationship to the idea of the cardinality of a set.

Reading Guide:

1. *(Answer these before you begin your reading) If a person off the street asked you what the difference is between a rational number and an irrational number, what would you say? How would you define an irrational number? How do you know they exist? Is the previous question a nonsense question or not? Why or why not?
2. **Note:** the reference on page 2 to an article by B. Pourciau [20] was made possible by none other than yours truly…Dr. Kinney…well, that's maybe a bit of an exaggeration…Bruce Pourciau is a math professor at Lawrence University in Appleton, Wisconsin and he wrote that article while on sabbatical during the 1997-1998 school year…and, well,…I was his sabbatical replacement then, which was the year before I came to Bethel…though I suppose he may have written it without me replacing him anyway ☺
3. *Does your answer to the first question in #1 above agree with any of the descriptions given in the text at the bottom of page 3?
4. In the proof that $\sqrt{2}$ is irrational, why may we assume that p and q have no common divisors greater than 1?
5. For those of you who are scientifically inclined, ponder whether there really is a concept of "exact measurement" of continuous quantities (see the bottom of page 3). Does such an idea make sense? How does your answer to this inform your view of rational versus irrational numbers?
6. Come up with a few of your own fraction-to-decimal and decimal-to-fraction examples and work through the calculations similar to the ones in the book on pages 4 and 5. (don't make them trivial).

7. Look up Zeno's paradoxes…which one intrigues you the most? Why?
8. What does it mean for an operation to be defined on a set, as in the definition of a field on page 6? In other words, what is meant by the word "operation" (or, more precisely, "binary operation")? Can you define it more precisely? Hint: it's a certain type of function…but alas, then you may also want to more precisely define what a function is…
9. Try to recall how to define what a field in more brevity by using words from Algebraic Structures such as "ring", "commutative", "unity", and "unit". Of course, you might also want to try to recall how to define each of these words. Write down your thoughts.
10. Do you have any initial thoughts about the axiomatic definition of the real number system? Does this seem like a good way to go about "defining" a real number?
11. **Note:** Property 1 in Definition 1.2 is called the principle of trichotomy and Property 2 of Definition 1.2 is called transitivity (of a binary relation).
12. Use the ordered field properties to prove the property in the middle of page 8.
13. What would the conclusion of this property (in the middle of page 8) be if $z < 0$ instead? Can you prove the resulting statement…perhaps by using the original statement?
14. Does the piecewise formula for the absolute value function on page 10 make sense to you? Explain why it works. You might want to imagine that you are a teacher trying to explain it to a 6th grader.
15. Rewrite property (a) of Theorem 1.4 that would be more informative about the true nature of the relationship between an arbitrary real number *a* and the numbers $|a|$ and $-|a|$. If you're not sure what I mean here at first glance, think harder!
16. It's somewhat surprising how often property (d) of Theorem 1.4 comes up in proofs. Take note of it and think about it. Draw a picture to help you understand it. Use it to help you solve the inequality $|2x+7|<4$. Draw graphs to help you understand the solution set of the inequality visually.
17. *Try to prove the Triangle Inequality (Theorem 1.5) ***without looking at the proof in the book***. If you've already read it, try it anyway. Hint: Use Theorem 1.4. Take note of the fact that the proof of the Triangle Inequality in the book is actually quite simple. Though if you didn't think about using Theorem 1.4, you'd probably have a hard time coming up with the proof (maybe you still had a hard time anyway).
18. Under what conditions on *a* and *b* is the triangle inequality a *strict* inequality? Under what conditions is it an *equality*? Can you generalize these to conditions on vectors **a** and **b** in $\mathbb{R}^2$? (you'll want to look up the more general triangle inequality for vectors first).
19. Is there a picture of vectors (like Figure 1.3) that would help you understand the Reverse Triangle Inequality? Try drawing one and see if you can figure out how to interpret it (Hint: you'll need to recall how to geometrically interpret the difference of two vectors).
20. **Note:** The discussion at the top of page 12 is pretty important because it will often come up in proofs. Read it carefully and think about it.
21. The proof of Theorem 1.7 is "tricky" yet somewhat typical of the style of many kinds of logical arguments in real analysis. Make sure you think about whether

you believe that it *really proves what you want to prove*. Write down any questions you may have about it. Also think carefully about the paragraph above Theorem 1.7 and write down any questions you have about it.

22. The equations for the maximum $x \vee \psi$ of two numbers and the minimum $x \wedge \psi$ of two numbers on page 12 are kind of wild. You probably never would have imagined that you could come up with such equations. Briefly spend time thinking about why they are true. See if you can verify them.
23. Before looking at Definition 1.8 on page 13, try to come up with your own definition of an ***interval*** of real numbers. Then compare your definition with the book's definition. How close did you get?
24. Write down the logical negation of the definition of an interval of real numbers. In other words, what does it mean for a subset *S* of ° to **not** be an interval. You might want to review parts of Appendix A.
25. The proof of Theorem 1.10 is a very nice trick. This theorem can also be proved by induction. See if you can give a nice proof by induction.
26. The material on page 15 through the top of page 17 is not essential for our course. However, if you find it interesting, go ahead and check the calculations and proofs and jot some notes about it. The Cauchy-Schwarz inequality actually is a very important inequality for more advanced Real Analysis. You should spend time thinking about it if you are thinking about going to graduate school in math someday.
27. *Write the logical negations of the definitions in Definition 1.14. Draw pictures to illustrate the definitions. Review parts of Appendix A if necessary.
28. What do you think the definition of "bounded" should be for a set of points $S \subseteq$ ° $^{\nu}$? Is there such a thing as an upper and/or lower bound for such a set?
29. *Determine the supremum and the infimum of some examples of sets that you generate. Try to make them nontrivial examples. (Keep generating these examples until you feel you have a reasonable handle on this concept). Is the infimum of a set of positive numbers necessarily a positive number? (Note: be careful about what the definition of a positive number is.)
30. Verify to your own level of satisfaction the fact that the statements near the bottom of page 22 lead to contradictions.
31. **Note:** For clarity, I think the statement of the Completeness Axiom on page 23 should include the phrase "in the real numbers'" at the end of it (this is implied by the author, but not stated). After all, every rational number is also a real number so the following statement is true, but could be misinterpreted: "Each nonempty set of rational numbers that is bounded above has a supremum". This is true, but the supremum might be an irrational number. If someone took this to implicitly imply that the supremum were a rational number, then it would be a false statement.
32. *Write down the statements of the Intermediate Value Theorem and the Mean Value Theorem from your Calculus text (try to write them down without looking them up first, then look them up if you need to). Why are these "existence results"?
33. **Note:** The equivalence of part 4 of Theorem 1.17 to part 1 (the Archimedean property) will be used very frequently.

34. **Note:** Theorem 1.18 is sometimes called the "density" property of the rationals and irrationals in the reals. The conclusion of this theorem can actually be strengthened to say that between any two distinct real numbers there are *infinitely* many rational numbers and *infinitely* many irrational numbers. Do you think you see how you could prove this stronger version? Explain.
35. *Mimic the proof of Theorem 1.19 to show that there exists a real number *x* such that $x^2 = 2$.
36. In the proof of Theorem 1.20, why is it true $0 \le \delta_\nu \le 9$ for each *n*?
37. A Philosophical Question: do you think the definition of what it means for two sets to have the **same size** (the same "cardinality") on page 30 is the "best" definition that could be given, especially in the case where the two sets are infinite?
38. **Note:** a synonym for "countable" that you'll often see in other math textbooks is "denumerable".
39. See whether or not you can find a formula for the one-to-one correspondence between the set of positive integers and the set of all integers pictured as a pairing in the middle of page 31.
40. In the proof of Theorem 1.23, is it clear to you that *f* is a one-to-one function? Briefly explain.
41. In the proof of Theorem 1.23, explain why *a* is the smallest integer in the set $A \setminus \{f(1), f(2), \mathrm{K}\ , f(p)\}$.
42. **Note:** the statement of Theorem 1.25 can be confusing if you misinterpret it (it might seem to be giving a conclusion that is assumed in the hypthesis). The "countable union" means that the *collection consisting of all the sets* in the union is itself a countable collection (set). This allows us to put the *n* = 1 to ∞ below and above the union symbol in the 6th line of the proof.
43. In the proof of Theorem 1.25, why is it important to assume that none of the sets have elements in common with any of the others?
44. Elaborate on why the Fundamental Theorem of Arithmetic implies that the pairing at the bottom of page 32 is the described one-to-one correspondence.
45. *Carefully prove Lemma 1.28 on page 34. Note: the collection of intervals in this lemma is called a **nested** collection.
46. Draw pictures to illustrate the idea of the proof of Theorem 1.29. In what fundamental way(s) does the completeness property of the real numbers enter into this proof?
47. *Are the real numbers as ``trivial" to understand as you may have once thought? Elaborate.
48. Think about the ways of defining the sine function in the middle of page 38. Do you understand these approaches? Try to figure out why they work and the advantages/disadvantages of each. How else can the sine function be defined for all real numbers? (Hint: see the top of page 39). Can you describe this other definition?
49. Give a definition of arctan(*x*) that is similar to the way arsin(*x*) is defined on page 39.

50. *See if you can prove some of the statements in 1 – 4 in the middle of page 40 and the statements 1 – 4 in the middle of page 41. It's an "extra bonus" if you can prove them without resorting to using Calculus.
51. Prove the statement that saying that a nonempty bounded set has a maximum value if and only if it contains its supremum.
52. **Note:** I like to add the word "global" in front of the bold-faced words in parts a, b, and c of Definition 1.33.
53. **Note:** take careful note of the middle paragraph on page 42.
54. Does the collection of real-valued functions defined on an interval form a group? Does it form a ring? (careful…what are the group/ring operations?)

Deep Thoughts to Ponder (but not necessarily answer):

- Are there sets whose cardinality (loosely speaking, "number" of elements) is a "size of infinity" larger than the "size of infinity" of the cardinality of the real numbers? Do some research into the ***power set*** of a given set to find out.
- Since the set of irrational numbers and the set of real numbers are both uncountable, does that mean they can be put in one-to-one correspondence with each other? Can the real numbers be put in one-to-one correspondence with the points in the plane $\mathbb{R}^2$? With $\mathbb{R}^n$ for any $n \in \mathbb{Z}^+$? Do some research into the ***continuum hypothesis*** and the works of ***Georg Cantor*** to find out.