Perfect Graphs

$\chi(G)$ - Chromatic number of the graph G

$\omega(G)$ - Clique number of the graph G

For any graph G, $\omega(G) \leq \chi(G)$.

For a given graph H, a graph G is called H-free if no induced subgraph of G is isomorphic to H.

A K_3-free graph is known as triangle free graph.

Thm For every possitive integer k, there exists a triangle free k-chromatic graph.

Using the above theorem it can be proved that for every two integers l and k with $2 \leq l \leq k$, there exists a graph G with $\omega(G) = l$ and $\chi(G) = k$.

A graph G is called Perfect if $\chi(H) = \omega(H)$ for every induced subgraph H of G.

Ex If $G = K_n$, then $\chi(G) = \omega(G) = n$. Furthermore, every induced subgraph H of K_n is also a complete graph and so $\chi(H) = \omega(H)$. Thus every complete graph is perfect.

If $G = \overline{K_n}$ and H is any induced subgraph ~~H of Kn~~ of G, then $\chi(H) = \omega(H) = 1$. So every empty graph is also perfect.

Thm Every bipartite graph is perfect.

Proof: Let G be a bipartite graph and let H be an induced subgraph of G. If H is nonempty then $\chi(H) = \omega(H) = 2$ while if H is empty, then $\chi(H) = \omega(H) = 1$. In either case $\chi(H) = \omega(H)$ and so G is perfect.

Thm Every graph whose complement is bipartite is perfect.

Proof: Let G be a graph of order n such that $\overline{G}$ is bipartite. Since the complement of every (nontrivial) induced subgraph of G is also bipartite, to verify that G is perfect it suffices to show that

$\chi(G) = \omega(G)$.

Suppose that $\chi(G) = k$ and $\omega(G) = l$. Then $k \geq l$. Let there be given a k-coloring of G. Then each color class of G consists either of one or two vertices; for if G contains a color class with three or more vertices, then this would imply that $\bar{G}$ has a triangle, which is impossible.

Of the k color classes, suppose that p of these classes consist of a single vertex and that each of the remaining q classes consists of two vertices. Hence $p+q=k$ and $p+2q=n$. Let W be the set of vertices of G belonging to a singleton color class. Since every two vertices of W are necessarily adjacent, $G[W] = K_p$ and so $\bar{G}(W) = \bar{K}_p$.

Since no k-coloring of G results in more than q color classes having two vertices, it follows that $\bar{G}$ has a maximum matching M with q edges.

We claim that for each edge $uv \in M$, either u is adjacent to no vertex of W or v is adjacent to no vertex of W. Suppose that this is not the case. Then we may assume that u is adjacent to some vertex $w_1 \in W$ and v is adjacent to some vertex $w_2 \in W$. Since $\bar{G}$ is triangle free ~~t~~ $w_1 \neq w_2$. However then, $(M - \{u,v\}) \cup \{uw_1, vw_2\}$ is a matching in $\bar{G}$ containing more than $|M|$ edges. This however is impossible and so as claimed, for each edge $uv \in M$, either u is adjacent to no vertex of W or v is adjacent to no vertex of W.

Therefore $\bar{G}$ contains an independent set of at least $p+q = k$ vertices and so $\omega(G) = l \geq k$. Hence $\chi(G) = \omega(G)$.

The Perfect graph theorem

A graph is perfect if and only if its complement is perfect.

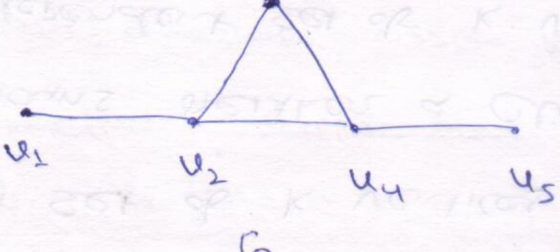
Defn A graph G with $V(G) = \{v_1, v_2, \cdots, v_n\}$ is an interval graph if there exists a collection S of n closed intervals of real numbers, say

$$S = \{ [a_i, b_i] \mid a_i < b_i,\ 1 \le i \le n \}$$

Such that v_i and v_j are adjacent if and only if $[a_i, b_i]$ and $[a_j, b_j]$ have a nonempty intersection.

If G is an interval graph, then every induced subgraph of G is also an interval graph.

Ex $I_1 = [0, 2]$, $I_2 = [1, 5]$, $I_3 = [3, 6]$, $I_4 = [4, 8]$, $I_5 = [7, 9]$



$\chi(G) = \omega(G) = 3$

G

Th^m Every interval graph is Perfect.

Proof

Let G be an interval graph with $V(G) = \{v_1, \dots, v_n\}$. Since every induced subgraph of an interval graph is also an interval graph it is sufficient to show that $\chi(G) = \omega(G)$.

Because G is an interval graph, there exist n closed interval $I_i = [a_i, b_i]$, $1 \le i \le n$, such that v_i is adjacent to v_j $(i \ne j)$ iff $I_i \cap I_j \ne \phi$. We may assume that the Intervals (and Consequently, the vertices of G) have been labeled so that $a_1 \le a_2 \le \dots \le a_n$

We now define a vertex coloring of G. First assign v_1 the color 1. If v_1 and v_2 are not adjacent (that is, if I_1 and I_2 are disjoint), then assign v_2 color 1 as well; otherwise, assign v_2 the color 2. Proceeding inductively, suppose that we have assigned colors to $v_1, \dots, v_r$ where $1 \le r < n$. We now assign v_{r+1} the smallest color (positive integer) that has not been assigned to any neighboure of v_{r+1} in the set $\{v_1, \dots, v_r\}$

This gives a k-coloring of G for some +ve integer k and so $\chi(G) \leq k$. If $k=1$, then $G = \overline{K}_n$ and $\chi(G) = \omega(G) = 1$. Hence we may assume that $k \geq 2$.

Suppose that the vertex v_t has been assigned the color k. Since it was not possible to assign v_t any of the color $1, 2, \ldots, k-1$, this means that the interval $I_t = [a_t, b_t]$ must have a nonempty intersection with $k-1$ intervals $I_{j_1}, \ldots, I_{j_{k-1}}$, where say $1 \leq j_1 < j_2 < \cdots < j_{k-1} < t$. Thus $a_{j_1} \leq a_{j_2} \leq \cdots \leq a_{j_{k-1}} \leq a_t$.

Since $I_{j_i} \cap I_t \neq \phi$ for $1 \leq i \leq k-1$, it follows that

$$a_t \in I_{j_1} \cap I_{j_2} \cdots \cap I_{j_{k-1}} \cap I_t$$

Thus for $U = \{v_{j_1}, \ldots, v_{j_{k-1}}, v_t\}$, $G[U] = K_k$ and so $\chi(G) \leq k \leq \omega(G)$. Since $\chi(G) \geq \omega(G)$, we have $\chi(G) = \omega(G)$, as desired.