Vertex and edge connectivity

A vertex cut of a graph G is a set of vertices of G such that $G-S$ is disconnected.

A vertex-cut of minimum cardinality in G is called a minimum vertex-cut of G and this cardinality is called the vertex-connectivity (or connectivity) of G and is denoted by $K(G)$.

The removal of any proper subset of vertices from a complete graph results in a smaller complete graph. The connectivity of the complete graph of order n is defined as $n-1$, ie. $K(K_n) = n-1$.

The connectivity $K(G)$ of a graph G is the smallest number of vertices whose removal from G results in either a disconnected graph or a trivial graph.

For any graph G of order n, $0 \leq K(G) \leq n-1$

Thus a graph G has connectivity 0 if and only if either $G \simeq K_1$ or G is disconnected; a graph G has connectivity 1 if and only if $G \simeq K_2$ or G is a connected graph with cut vertices and a graph G has connectivity 2 or more if and only if G is a nonseparable graph of order 3 or more.

A graph G is k-connected, $k \geq 1$, if $\kappa(G) \geq k$. i.e.

G is k-connected if the removal of fewer than k vertices from G results in neither a disconnected graph nor a trivial graph.

An edge-cut of a graph G is a subset X of $E(G)$ such that $G-X$ is connected. An edge-cut of minimum cardinality is the edge connectivity of G, which is denoted by $\lambda(G)$.

For the trivial graph K_1, we define $\lambda(K_1)=0$.

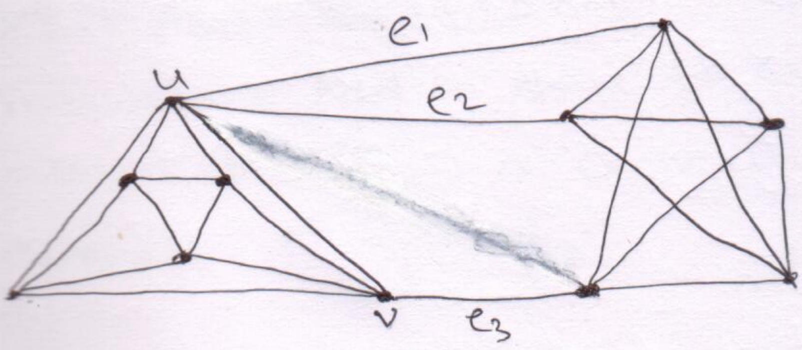
So $\lambda(G)$ is the minimum number of edges whose removal from G results in a disconnected or trivial graph.

Thus $0 \leq \lambda(G) \leq n-1$, for every graph G of order n.

A graph G is k-edge-connected, $k \geq 1$ if $\lambda(G) \geq k$.

Thus a 1-edge connected graph is a nontrivial connected graph and a 2-edge connected graph is a non trivial connected bridgeless graph.

Ex



G

$\kappa(G)=2$

$\lambda(G)=3$

$\delta(G)=4$

$\underline{\underline{Th^m}}$ For every positive integer n, $\lambda(K_n) = n-1$.

$\underline{\underline{Proof}}$ Since the edge connectivity of K_1 is defined to be 0, we may assume that $n \geq 2$. If the $n-1$ edges incident with any vertex of K_n are removed from K_n, then a disconnected graph results. Thus $\lambda(K_n) \leq n-1$.

Now let X be a minimum edge cut of K_n. Then $|X| = \lambda(K_n)$ and $G-X$ consists of two components, say G_1 and G_2. Suppose that G_1 has order k. Then G_2 has order $n-k$. Thus $|X| = k(n-k)$. Since $k \geq 1$ and $n-k \geq 1$, it follows that $(k-1)(n-k-1) \geq 0$ and so $(k-1)(n-k-1) = k(n-k) - (n-1) \geq 0$

Thus $\lambda(K_n) = |X| = k(n-k) \geq n-1$, therefore $\lambda(K_n) = n-1$.

$\underline{\underline{Th^m}}$ For every graph G,

$$K(G) \leq \lambda(G) \leq \delta(G).$$

$\underline{\underline{Proof}}$ Let G be a graph of order n. If G is disconnected then $K(G) = \lambda(G) = 0$ while if G is complete, then $K(G) = \lambda(G) = \delta(G) = n-1$. Thus the desired inequalities hold in these two case. Hence we may assume that G is a connected graph that is not complete.

Since G is not complete, $\delta(G) \leq n-2$. Let v be a vertex of G such that $\deg(v) = \delta(G)$. If the edges incident with v are deleted from G, then a disconnected graph is produced. Hence $\lambda(G) \leq \delta(G) \leq n-2$.

It remains to show that $\kappa(G) \leq \lambda(G)$. Let $X(G)$ be a minimum edgecut of G. Then $|X| = \lambda(G) \leq n-2$. Then $G-X$ ~~has~~ consists of two components, say G_1 and G_2. Suppose the order of G_1 is k. ~~and the~~ Then the order of G_2 is $n-k$, where $k \geq 1$ and $n-k \geq 1$. Also every edge ~~of~~ in X joins a vertex of G_1 and a vertex of G_2. We consider two cases.

<u>Case I</u> Every vertex of G_1 adjacent to every vertex of G_2

Then $|X| = k(n-k) \geq n-1$ as $k-1 \geq 0$ and $n-k-1 \geq 0$

So $\lambda(G) = |X| = k(n-k) \geq n-1$ which contradicts $\lambda(G) \leq n-2$, so Case I cannot occur.

<u>Case 2</u> There exist a vertex u in G_1 and a vertex v in G_2 s.t. $uv \notin E(G)$. We now define a set U of vertice of G. Let $e \in X$. If e is incident with u say $e = uv'$ then the vertex v' is place in U. If e is not incident with u, say $e' = u'v'$ where u' is in G_1, then place the vertex u' in U. Hence for every edge $e \in X$, one of its two incident vertices belongs to U but $u, v \notin U$. Thus $|U| \leq X$ and U is a vertex cut. Therefore

$$\kappa(G) \leq |U| \leq |X| = \lambda(G).$$