# Menger's Theorem

For two nonadjacent vertices $u$ and $v$ of $G$, a $u-v$ separating set is a set $S \subseteq G - \{u, v\}$ such that $u$ and $v$ lie in different components of $G - S$.

A $u-v$ separating set of minimum cardinality is called a minimum $u-v$ separating set.

For two distinct vertices $u$ and $v$ of ~~a~~ $G$, a collection of $u-v$ paths is internally disjoint if every two paths in the collection have only $u$ and $v$ in common

Theorem (Menger)
Let $u$ and $v$ be nonadjacent vertices in $G$. The minimum number of vertices in a $u-v$ separating set equals the maximum number of internally disjoint $u-v$ paths in $G$.

Proof We proceed by induction on the size of graphs. The theorem is certainly true for every empty graph.

Assume that the theorem holds for all graphs of size less than $m$, $m \geq 1$, and let $G$ be a graph of size $m$.

Let $u$ and $v$ be two nonadjacent vertices of $G$. If $u$ and $v$ belong to different component of $G$, then the result follows.

So we may assume that $u$ and $v$ belong to same component of $G$. Suppose a minimum $u-v$ separating set consists of $k \geq 1$ veretices. Then $G$ contains at most $k$ internally disjoint $u-v$ paths.

We show that $G$ contains ~~to~~ $k$ internally disjoint paths. Since this is obviously true of $k=1$, we assume that $k \geq 2$. We now consider three cases:

<u>Case I</u>: Some minimum $u-v$ separating set $X$ in $G$ contains a vertex $x$ that is adjacent to both $u$ and $v$.

Then $X - \{x\}$ is a minimum $u-v$ separating set in $G - x$ consisting of $k-1$ vertices. Since the size of $G - x$ is less than $m$, it follows by induction hypothesis that $G - x$ contains $k-1$ internally disjoint $u-v$ paths. These paths together with the path $P = (u, x, v)$ produce $k$ internally disjoint $u-v$ paths in $G$.

<u>Case II</u>: For every minimum $u-v$ separating set $S$ in $G$, either every vertex in $S$ is adjacent to $u$ and $v$ or every vertex.

Necessarily then $d(u,v) \geq 3$. Let $P = (u, x, y, \ldots, v)$ be a ~~u-v~~ shortest $u-v$ path in $G$ where $e = xy$. Every minimum $u-v$ separating set in $G-e$ contains at least $k-1$ vertices. We show that, ~~in~~ in fact, that every minimum $u-v$ separating set in $G-e$ contains $k$-vertices.

Suppose that there is some minimum $u$-$v$ separating set in $G-e$ with $k-1$ vertices, say $Z = \{z_1, \dots, z_{k-1}\}$

Then ~~$Z = z_1, \dots, z_{k-1}$~~ $Z \cup \{x\}$ is a $u$-$v$ separating set in $G$ and therefore a minimum $u$-$v$ separating set in $G$. Since $x$ is adjacent to $u$ not to $v$, it follows that every vertex $z_i$ ~~i~~, $1 \le i \le k-1$) is also adjacent to $u$ and not adjacent to $v$.

Since $Z \cup \{y\}$ is also a minimum $u$-$v$ separating set in $G$ and each vertex $z_i$ ($1 \le i \le k-1$) is adjacent to $u$ but not $v$, it follows that $y$ is adjacent to $u$. This however contradicts the assumption that $P$ is a shortest $u$-$v$ path.

Thus $k$ is the minimum number of vertices in a $u$-$v$ separating set in $G-e$. Since the size of $G-e$ is less than $m$, it follows by induction hypothesis that there are $k$ internally disjoint $u$-$v$ paths in $G-e$ and in $G$ as well.

Case-III There exists a minimum $u$-$v$ separating set $W$ in $G$ in which no vertex is adjacent to both $u$ and $v$ and containing atleast one vertex not adjacent to $u$ and atleast one vertex not adjacent to $v$.

Let $W = \{w_1, \dots, w_k\}$. Let $G_u$ be the subgraph of $G$ consisting of, for each $i$ with $1 \le i \le k$, all $u$-$w_i$ paths in $G$ in which $w_i$ is the only vertex of the path belonging to $W$.

Let $G_u'$ be the graph constructed from $G_u$ by adding a new vertex $v'$ and joining $v'$ to each vertex $w_i$, for $1 \leq i \leq k$. The graph $G_v$ and $G_v'$ are defined similarly.

Since $W$ contains a vertex that is not adjacent to $u$ and a vertex that is not adjacent, the size of both $G_u'$ and $G_v'$ are less than $m$. So $G_u'$ contains $k$ internally disjoint $u$-$v'$ paths $A_i$ $(1 \leq i \leq k)$ where $A_i$ contains $w_i$. Also, $G_v'$ contains $k$ internally disjoint ~~paths~~ $u'$-$v$ paths $B_i$ $(1 \leq i \leq k)$, where $B_i$ contains $w_i$.

Let $A_i'$ be the $u$-$w_i$ ~~path~~ subpath of $A_i$ and $B_i'$ be the $w_i$-$v$ subpath of $B_i$ for $1 \leq i \leq k$. The $k$ paths constructed for $A_i'$ and $B_i'$ for each $i$ $(1 \leq i \leq k)$ are internally disjoint $u$-$v$ paths in $G$.

~~Problem~~ __Exercise__ Let $G$ be a $k$-connected graph $(k \geq 1)$. Prove that (i) if $v \in V(G)$ then $G-v$ is $(k-1)$-connected (ii) if $e \in E(G)$ then $G-e$ is also $(k-1)$-connected.

$\underline{\underline{Th^m}}$ (Whitney)

A nontrivial graph $G$ is $k$-connected for some integer $k \geq 2$ if and only if for each pair $u, v$ of distinct vertices of $G$ there are atleast $k$ internally disjoint $u-v$ paths in $G$.

Proof: First suppose that $G$ is a $k$-connected graph, where $k \geq 2$, and let $u$ and $v$ be two distinct vertices of $G$.

Assume first that $u$ and $v$ are not adjacent. Let $U$ be a minimum $u-v$ separating set. Then

$$k \leq \kappa(G) \leq |U|$$

By Menger's theorem, $G$ contains at least $k$ internally disjoint $u-v$ paths.

Next assume that $u$ and $v$ are adjacent, where $e = uv$. Then $G-e$ is $(k-1)$-connected. Let $W$ be a minimum $u-v$ separating set in $G-e$, so

$$k-1 \leq \kappa(G-e) \leq |W|$$

By Menger's theorem, $G-e$ contains at least $k-1$ internally disjoint $u-v$ paths, implying that $G$ contains atleast $k$ internally disjoint $u-v$ paths.

For the converse, assume that $G$ contains at least $k$ internally disjoint $u-v$ paths for every pair of $u, v$ of distinct vertices of $G$.

If $G$ is complete, then $G = K_n$, where $n \geq k+1$ and so $\kappa(G) = n-1 \geq k$. Hence $G$ is $k$-connected. Thus we may assume that $G$ is not complete.

Let $U$ be a minimum vertex-cut of $G$. Then $|U| = \kappa(G)$. Let $x$ and $y$ be vertices in distinct components of $G-U$. Thus $U$ is an $x$-$y$ separating set of $G$. Since there are ~~k is~~ at least $k$ internally disjoint $x$-$y$ paths in $G$, it follows by Menger's theorem that $k \leq |U| = \kappa(G)$ and so $G$ is $k$-connected.

**Cor** Let $G$ be a $k$-connected graph, $k \geq 1$ and let $S$ be any set of $k$ vertices of $G$. If a graph $H$ is obtained from $G$ by adding a new vertex and joining this vertex to the vertices of $S$, then $H$ is also $k$-connected.

**Cor** If $G$ is a $k$-connected graph, $k \geq 2$ and $u, v_1, v_2, \ldots, v_t$ are $t+1$ distinct vertices of $G$, where $2 \leq t \leq k$, then $G$ contains a $u$-$v_i$ path for each $i$ $(1 \leq i \leq t)$, every two paths of which have only $u$ in common.