Intro: What's the course about?

# Principles of Mathematical Analysis.

Let us begin with the rational numbers, $\mathbb{Q}$.

Problems of $\mathbb{Q}$

$\mathbb{Q}$ has some problems:

i. algebraic incompleteness: can make simple eq'ns with coeffs in $\mathbb{Q}$ but no sol'n in $\mathbb{Q}$ (e.g. $1x^2 - 2 = 0$).

ii. (analytic) incompleteness: can find a sequence of rational numbers $q_1, q_2, q_3 \ldots$ that approach a point which is not in $\mathbb{Q}$ e.g. $1.4, 1.41, 1.414, 1.4142, \ldots$

(Proof by ※)

A Proof that should be known to any educated person

To back up our claim, let us give the standard proof (by ※)

<u>Claim</u>: $x^2 = 2$ has no sol'n in $\mathbb{Q}$.

<u>Pf:</u> Say it <u>has</u> a sol'n, $x = \frac{p}{q}$; $q \neq 0$.

WLOG we may assume $\frac{p}{q}$ is in lowest terms.

(Q. What do we know?) (Does that imply gcd?) (if not, divide)

So $\frac{p^2}{q^2} = 2$, i.e. $p^2 = 2q^2$.

Observe that $2 \mid p^2$; so $p = 2m$

This implies $4m^2 = 2q^2$

i.e. $2m^2 = q^2$; so $q = 2n$

※

<u>Commentary</u>: Like blind men in the dark stumbling on the light switch

( But this is how Wiles describes discovery!)

<u>Commentary</u>: $\boxed{\text{WYK}}$ . . . . . . → $\boxed{\text{WTS}}$ ?

(in this case WTS ※ !)

<u>Remark</u>: Pythagoras. Reign the Conqueror.

For reasons that will become clear later, we also show:

Opaque claim

Analyst: unsimplifiable

**Claim:** Let $A = \{p \in \mathbb{Q} \mid p > 0,\ p^2 < 2\}$
~~$B = \{p \in \mathbb{Q} \mid p > 0,\ p^2 > 2\}$~~

Then $A$ contains no ~~smallest~~ largest number and $B$ no smallest.

Extremely clever proof: Let $q = p - \frac{p^2-2}{p+2}$ (a rational #) (*)

Easy calculation: $q^2 - 2 = \frac{2(p^2-2)}{(p+2)^2}$ (**)

Now, suppose $p \in A$. Then $q^2 - 2 \overset{(**)}{=} \frac{2(p^2-2)}{(p+2)^2} < 0$

$\Rightarrow q^2 < 2$, i.e. $q \in A$.

And $q \overset{(*)}{=} p - \underbrace{\frac{p^2-2}{p+2}}_{<0}$ so $q > p$.

So for any $p \in A$, $\exists q \in A$ s.t. $q > p$.

End of motivation

Similarly for $p \in B$!

Rk: It is this sort of proof that makes me want to be smarter.

Q. Why is this natural / obvious?

What is $\mathbb{R}$?

We now leave the world of numbers and retreat to the world of sets.

Set notation

Standard set notation: Let $A$ be a set. If $A$ has no elements it is called $\emptyset$ the empty set

- $x \in A$, $x \notin A$
- $A \subset B$ ~~means~~ means "$x \in A \Rightarrow x \in B$"; $A \supset B$ similarly.
- $A = B$ means $A \subset B$ and $A \supset B$; $x \in A \Leftrightarrow x \in B$.

else we write $A \neq B$.

Goal: What's special about $\mathbb{R}$?

order

## Ordered Sets.

Defn: Let $S$ be a set, $<$ a binary truth-relation on $S$ s.t.

(well defined) i. For each pair $x, y \in S \times S$, only one of $x<y$, $x=y$, $y<x$ is true.

(transitivity) ii. If $x<y$ and $y<z \Rightarrow x<z$. $(x,y,z \in S)$

In that case, we call $<$ an <u>order</u> on $S$, and call $S, <$ an <u>ordered set</u>.

questions

Rks: i. You know some ordered sets already. Examples? (What about on $\mathbb{C}$?)

ii. Suppose we define $A<B$ if $A \subset B$. Is this an order?

Bounds

<u>Boundedness:</u> Say $E \subset S$, an ordered set.

Defn: If $\exists \beta \in S$ s.t. $\forall x \in E$, $\beta \geq x$ then we call $\beta$ an <u>upper bound</u> ~~of~~ $E$ and say $E$ is <u>bounded above</u>.

(Similarly for lower bds, bdd below)

<u>Crucial defn:</u> If, in addition (to being an upper bd of $E$) we have that $\forall$ upper bds. $\gamma$ of $E$, $\gamma \geq \beta$

LUB!

then we call $\beta$ a <u>least upper bound</u> of $E$ (aka <u>supremum</u>, $\sup(E)$) (Similarly glb, inf)

E.g. $A = \{p \in \mathbb{Q} \mid p^2 < 2\}$.

$B = \{p \in \mathbb{Q} \mid p^2 > 2\}$.

Clarifying the opacity.

$B$ is the collection of upper bounds of $A$ (in $\mathbb{Q}$); however, there is <u>no</u> l.u.b. $(\sup A \notin \mathbb{Q})$.

LUB prop.

**Defn:** [$S$ an ordered set] Suppose the following is true:

"For any $\emptyset \neq E \subset S$ with an upper bound, $\exists \sup(E) \in S$."

In that case, we say $S$ has the <u>least upper bound property</u>.

Rk: As we saw, $(\mathbb{Q}, <)$ does not have this property.

(lubs and glbs.)

**Thm:** [$S$ an ordered set] $S$ has the l.u.b. prop. $\Rightarrow$ has the glb prop.

Pf: Say $S$ has the l.u.b. property; let $\emptyset \neq B \subset S$ be bdd. below.

WTS $B$ has a glb (inf) in $S$.

Well, let $L$ denote the set of lower bds. of $B$.

* rewritten

Notice! i) $L$ is nonempty (why?)

ii) $L$ is bdd above (every elt. of $B$ is an upper bd.)

By l.u.b. prop., $\exists \sup(L) \in S$.

Claim: $\sup(L) = \inf(B)$

i. $\sup(L)$ is a lower bd. of $B$; i.e. $\sup(L) \leq \gamma \ \forall \gamma \in B$

Pf: i. $\gamma < \sup(L) \Rightarrow \gamma$ is not an upper bd. of $L$

~~Assume~~ Try contraposit'n:

~~If~~ $\gamma < \sup(L) \Rightarrow \gamma \notin B$.

~~Well that's obvious.~~

But every elt. of $B$ is an upper bd of $L$ so

$\gamma < \sup(L) \Rightarrow \gamma \notin B$.

(Contraposition) i.e., $\gamma \in B \Rightarrow \sup(L) \leq \gamma$.

i.e. $\sup(L)$ is a lower bd. of $B$

ii. ~~And~~ By construction $\sup(L)$ is the greatest lower bd. of $B$. ∎

~~Rk: Of course $\Leftrightarrow$~~

Rk: This may require some extra chewing.

LUB prop. **Defn:** [$S$ an ordered set] Suppose the following is true:

"For any $\emptyset \neq E \subset S$ with an upper bound, $\exists \sup(E) \in S$."

In that case, we say $S$ has the least upper bound property.

**Rk:** As we saw, $(\mathbb{Q}, <)$ does not have this property.

(lubs and glbs.) **Thm:** [$S$ an ordered set] $S$ has the l.u.b. prop. $\Rightarrow$ has the glb prop

**Pf:** Say $S$ has the l.u.b. property; let $\emptyset \neq B \subset S$ be bdd. below.

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Notice! i) $L$ is nonempty (why?)

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By l.u.b. prop., $\exists \sup(L) \in S$.

**Claim:** $\sup(L) = \inf(B)$

i. $\sup(L)$ is a lower bd. of $B$; i.e. $\sup(L) \leq b \ \forall\, b \in B$

**Pf:** i. $\gamma < \sup(L) \Rightarrow \gamma$ is not an upper bd. of $L$

But every elt. of $B$ is an upper bd of $L$ so

$\gamma < \sup(L) \Rightarrow \gamma \notin B$.

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ii. And by construction, $\sup(L)$ is the greatest lower bd. of $B$. ∎

**Rk:** Of course $\Leftrightarrow$.

**Rk:** This may require some extra chewing . . .

We now move to a very special class of sets: Fields.
Keep in mind: Example $\mathbb{Q}, \mathbb{R}, \mathbb{C}$. $\mathbb{Q}_{\text{nat}} \ldots$? no.

Field axioms

**Def'n:** Let $F$ be a set possessing 2 binary operations $+, \cdot$, s.t.

A1. (closure) $x, y \in F \Rightarrow x+y \in F$

A2. (commutativity) $\forall x, y \in F$, $x+y = y+x$

A3. (asso.) $\forall x, y, z \in F$, $(x+y)+z = x+(y+z)$

A4. (additive identity) $\exists$ an element $0 \in F$ s.t. $\forall x \in F$, $0 + x = x$.

A5. (additive inverse) For each $x \in F$, $\exists$ "$-x$" $\in F$ s.t. $x + (-x) = 0$.

(Rk: We say nothing about uniqueness of $0$ or $-x$.)

M1. (closure)

M2. (comm.)

M3. (asso.)

M4. (multip. id.)

M5. (multiplicative inverse) $\forall x \in F^*$, $\exists$ "$\frac{1}{x}$" $\in F$ s.t. $x \cdot \left(\frac{1}{x}\right)$:

D. (distributivity) $\forall x, y, z \in F$, $x(y+z) = xy + xz$.

Q) $M_{n\times n}$? a field?

In this case, we call $(F, +, \cdot)$ a field.

minimality of set of axioms

Rk: Of course, there are many other properties which you know and love (?) about fields (e.g. $0x = 0$)

Q. Why don't we include these in the field axioms? Because we need m

They can all be seen to be logical consequences.

Consequences

**Prop'n:**

a. $x+y = x+z \Rightarrow y = z$

b. $x+y = x \Rightarrow y = 0$

c. $x+y = 0 \Rightarrow y = -x$

d. $-(-x) = x$ (Answers for Hands!)

Rk: These may seem obvious to you. Wake up! We're no longer in $\mathbb{R}$

Pf of a: $x+y = x+z$.

$-x + (x+y) = -x + (x+z)$

$(-x+x)+y = (-x+x)+z$

$0 + y = 0 + z$; $y = z$

Other basic properties

**Prop'n:** a. $0x = 0$

b. $x \neq 0, y \neq 0 \Rightarrow xy \neq 0.$

c. $(-x)y = -(xy) = x(-y)$

d. $(-x)(-y) = xy$

**Pf:** a. $0x + 0x = (0+0)x = 0x.$

Cancellation now $\Rightarrow$ $0x = 0.$

b. Proof by contradiction. Say $x \neq 0, y \neq 0, xy = 0.$

Then $\frac{1}{x}\frac{1}{y}xy = \frac{1}{x}\frac{1}{y}0 \Rightarrow 1 = 0$ ※

(Q. Why couldn't $1 = 0$ (A question you probably thought you'd never be asked)

c. $(-x)y + xy = (-x+x)y$

$= 0y$

$= 0$

So $(-x)y = -(xy)$. Similarly ...

d. $(-x)(-y) = -(x(-y))$ (c.)

$= -(-(xy))$ (c)

$= xy$ (previous prop'n) ∎

ordered fields

<u>Ordered fields</u>:

**Defn:** $F$ a field. If $\exists$ an order $<$ on $F$ s.t.

$<$ must jive with $+$: i. $y < z \Rightarrow x+y < x+z$

$<$ must jive w/ $\cdot$: ii. $x > 0, y > 0 \Rightarrow x \cdot y > 0$

then we call $F$ an <u>ordered field</u>.

Example of an ordered field: $\mathbb{Q}$. Can $\mathbb{C}$ be made into an ordered fi

Basic properties

**Prop'n:** [Basic properties] a. $x < 0 \Rightarrow -x > 0$

b. $x > 0, y < z \Rightarrow xy < xz$

c. $x < 0, y < z \Rightarrow xy > xz$

d. $(x \neq 0)$ $x^2 > 0.$

e. $0 < x < y \Rightarrow 0 < \frac{1}{y} < \frac{1}{x}$

[Construction of $\mathbb{R}$]

main thm. **Thm:** $\exists$ an ordered field that i) contains $\mathbb{Q}$ as a subfield

ii) has the l.u.b. property.

Pf: Deferred.

Rk: We call this field $\mathbb{R}$.

warning! Although we are very familiar with $\mathbb{R}$, let's pretend we are not and merely know the above thm. What can we say about $\mathbb{R}$?

prop's of $\mathbb{R}$. Thm: [Properties of $\mathbb{R}$]

i. [Archimedean] Given any $x > 0$, $y \in \mathbb{R}$, $\exists n \in \mathbb{N}$ s.t. $nx > y$.

ii. [$\mathbb{Q}$ "dense in $\mathbb{R}$"] $\forall x, y \in \mathbb{R}$ $(x<y)$ $\exists p \in \mathbb{Q}$ st. $x<p<y$

archimedean Pf: i. By contradiction.

Suppose that $\forall n \in \mathbb{N}$, $nx \le y$.

Then $y$ is an upper bd. of $A := \{nx \mid n \in \mathbb{N}\}$.

$\Rightarrow \exists$ a lub., $M = \sup(A)$.

By def'n, $M - x$ cannot be an upper bd. of $A$

$\Rightarrow M - x < n_0 x$ for some $n_0 \in \mathbb{N}$.

$\Rightarrow M < (n_0+1)x$ ※.

density of $\mathbb{Q}$ in $\mathbb{R}$ ii. Idea: want integers $m, n \neq 0$ s.t. $x \underset{②}{<} \frac{m}{n} \underset{①}{<} y$, i.e. $nx < m < ny$

So we'll b. trap $m-1 \le nx < m$, where

a. choose $n$ s.t. $nx + 1 < ny$

For then ① $nx < m \Rightarrow x < \frac{m}{n}$;

② $m \le nx+1 < ny \Rightarrow \frac{m}{n} < y$ and we're done

a. Want $n$ st. $1 < n(y-x)$. possible by archimedean prop.

b. Again by (i), $\exists m_1$ st. $m_1 > nx$,

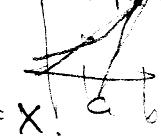
$m_2$ s.t. $-m_2 < nx$. $\Big\} \Rightarrow$ b.

$(m_2 > -nx)$

Lemma: $0<a<b \Rightarrow b^n - a^n \le (b-a)\, n b^{n-1}$

Pf: $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \dots + a^{n-1})$

Another property of the reals: existence of $n^{th}$ roots. Geometric interp.



existence of $\sqrt[n]{x}$ — (1.21) Thm: $\forall$ real $x>0$, $n\in\mathbb{N}$, $\exists!$ real $y>0$ s.t. $y^n = x$!

algebraic lemma — Pf: Uniqueness obvious $(0<y_1<y_2 \Rightarrow y_1^n < y_2^n)$

We have only one trick to play: LUB prop.

Finding the candidate — Obviously we set $E = \{t\in\mathbb{R} \mid t>0,\ t^n < x\}$.

i. $E \neq \emptyset$: Let $t = \dfrac{x}{1+x}$ $\left(\begin{matrix}<x\\<1\end{matrix}\right)$

Since $0<t<1$, $t^n \le t < x$.

ii. $E$ bdd above. Suppose $t > 1+x$. $\begin{matrix}>1\\>x\end{matrix}$

Then $t^n \ge t > x$; i.e. $t \notin E$.

Thus $t\in E \Rightarrow t \le 1+x$; $1+x$ is an upper bd

So $\exists \sup(E) \in \mathbb{R}$; call it "$y$".

Showing the candidate is good. — Claim: $y^n = x$.

Pf: i. Say $y^n < x$. (Our only other trick is pf. by $\divideontimes$)

We will create (a small positive) $h$ s.t. $(y+h)^n < x$ (i.e. $y$ not ~~sup~~ upper bd of $E$.)

i.e. $(y+h)^n - y^n < x - y^n$

The black box — How? Well, lemma $\Rightarrow$ $(y+h)^n - y^n < hn(y+h)^{n-1}$

WMA $h<1$, $< hn(y+1)^{n-1}$!

So we just need $hn(y+1)^{n-1} < x - y^n$.

By density, choose $h\in(0,1)$ s.t. $h < \dfrac{x-y^n}{n(y+1)^{n-1}}$.

Then $(y+h)^n - y^n \overset{\text{(Lemma)}}{<} hn(y+1)^{n-1} < x - y^n$,

i.e. $(y+h)^n < x$

ii. Similarly, $y^n > x \Rightarrow \divideontimes$ (Exerc.) So $y^n = x$ $\divideontimes$ (Why?)

Complex #s

Complex numbers. Let $\mathbb{C} := \{(a,b) \mid a,b \in \mathbb{R}\}$ (ordered pairs)

Define $+$ on $\mathbb{C}$ by $(a,b)+(c,d) = (a+c, b+d)$

$\cdot$ on $\mathbb{C}$ by $(a,b)(c,d) = (ac-bd, ad+bc)$.

Area field!

Thm: $(\mathbb{C}, +, \cdot)$ is a field (with $(0,0)$ additive identity, $(1,0)$ multiplicative identity)

Pf: (Let $x=(a,b)$, $y=(c,d)$, $z=(e,f)$ (notation))

(M2) $xy = (ac-bd, ad+bc)$
$yx = (ca-db, cb+da)$ } which are of course equal.

Similarly, (M3) associativity is true

(M4) $(1,0)(a,b) = (a,b)$

(M5) The only interesting one: Say $(a,b) \neq (0,0)$.

Then $a^2+b^2>0$; $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$ is $(a,b)$

$\mathbb{R} \subset \mathbb{C}$.

Rk: $\mathbb{R}$ is a subfield of $\mathbb{C}$ in the sense that $\mathbb{R} \to \mathbb{C}$ is a field isomorphism: $1 \leftrightarrow (1,0)$, $x \mapsto (x,0)$

$-1 \leftrightarrow (-1,0)$ etc

$i$

Rk: What is $(0,1)\cdot(0,1)$? $(-1,0)$. I.e., $-1$ has a square root (in fact 2)

Thus we could write $(a,b)$ as $(a,0)+(0,b) = a\cdot 1 + b \cdot i$ $(a+bi)$

terminology

Terminology: $z=(a,b) \in \mathbb{C}$; $\mathrm{Re}(z) := a$

$\mathrm{Im}(z) := b$

Complex conjugate: $\bar{z} := (a,-b) = a-bi$.

complex conjugate.

Thm: [Properties of $\bar{z}$]

a. $\overline{z+w} = \bar{z}+\bar{w}$

b. $\overline{zw} = \bar{z}\cdot\bar{w}$

c. $z+\bar{z} = 2\mathrm{Re}(z)$, $z-\bar{z} = 2i\,\mathrm{Im}(z)$

d. $\bar{z}z$ is real & positive (sauf quand $z=$

absolute value

**Defn:** $[z \in \mathbb{C}]$ The <u>absolute value</u> $|z| := (z\bar{z})^{1/2} \geq 0$.

( Rk: Fortunately we now know such numbers exist and are unique.)

Properties of

**Thm:** [Properties of $|\,|$]

a. $|z| > 0$ for all $z \neq 0$.

b. $|\bar{z}| = |z|$

c. $|zw| = |z||w|$

d. $|\mathrm{Re}\, z| \leq |z|$

e. $|z+w| \leq |z| + |w|$.

Pf: [Let $z = a+bi$, $w = c+di$; $a,b,c,d \in \mathbb{R}$.]

c. $|zw|^2 = |z|^2|w|^2$ is an easy calculation.

Thus $|zw|$, $|z||w|$ are both (positive) square roots of the same positive #. By uniqueness, done

d. $|\mathrm{Re}\, z| = |a| = \sqrt{a^2} \leq \sqrt{a^2+b^2} = |z|$.

e.
$$
\begin{aligned}
|z+w|^2 &= (z+w)(\bar{z}+\bar{w}) \\
&= z\bar{z} + w\bar{z} + z\bar{w} + w\bar{w} \\
&= |z|^2 + 2\,\mathrm{Re}(z\bar{w}) + |w|^2 \\
&\leq |z|^2 + 2|z\bar{w}| + |w|^2 = (|z| + |w|)^2.
\end{aligned}
$$

Notation: $\sum$

C-S Ineq.

**Thm:** Cauchy-Schwarz Inequality $\left|\sum_{i=1}^n a_i b_i\right|^2 \leq \sum_{i=1}^n |a_i|^2 \sum_{i=1}^n |b_i|^2$

Pf: Let $A = \sum |a_j|^2$, $B = \sum |b_j|^2$, $C = \sum a_j \bar{b_j}$ (WLOG $B > 0$)

WTS $|C|^2 \leq AB$.

$$
\begin{aligned}
\text{Now } 0 &\leq \sum |Ba_j - Cb_j|^2 \\
&= \sum (Ba_j - Cb_j)(B\bar{a_j} - \bar{C}\bar{b_j}) \\
&= \\
&= B^2A - B|C|^2 = B(AB - |C|^2).
\end{aligned}
$$
■

absolute value

**Defn:** $[z \in \mathbb{C}]$ The absolute value $|z| := (z\bar{z})^{1/2} \geq 0$.
(Rk: Fortunately we now know such numbers exist and are unique.)

Properties of

**Thm:** [Properties of $|\,|$]
a. $|z| > 0$ for all $z \neq 0$.
b. $|\bar{z}| = |z|$
c. $|zw| = |z||w|$
d. $|\operatorname{Re} z| \leq |z|$
e. $|z+w| \leq |z| + |w|$.

**Pf:** [Let $z = a+bi$, $w = c+di$; $a,b,c,d \in \mathbb{R}$.]

c. $|zw|^2 = |z|^2|w|^2$ is an easy calculation.
Thus $|zw|$, $|z||w|$ are both (positive) square roots of the same positive #. By uniqueness, done

d. $|\operatorname{Re} z| = |a| = \sqrt{a^2} \leq \sqrt{a^2+b^2} = |z|$.

e.
$$\begin{aligned} |z+w|^2 &= (z+w)(\bar{z}+\bar{w}) \\ &= z\bar{z} + w\bar{z} + z\bar{w} + w\bar{w} \\ &= |z|^2 + 2\operatorname{Re}(z\bar{w}) + |w|^2 \\ &\leq |z|^2 + 2|z\bar{w}| + |w|^2 = (|z|+|w|)^2. \end{aligned}$$

Notation: $\sum$

C-S Ineq.

**Thm:** Cauchy-Schwarz Inequality $\left|\sum_{i=1}^{n} a_i b_i\right|^2 \leq \sum_{i=1}^{n}|a_i|^2 \sum_{i=1}^{n}|b_i|^2$

**Pf:** Let $A = \sum |a_j|^2$, $B = \sum |b_j|^2$, $C = \sum a_j \bar{b}_j$ (WLOG $B > 0$)
WTS $|C|^2 \leq AB$.

$$\begin{aligned} \text{Now } 0 &\leq \sum |Ba_j - Cb_j|^2 \\ &= \sum (Ba_j - Cb_j)(B\bar{a}_j - \bar{C}\bar{b}_j) \\ &= \\ &= B^2A - B|C|^2 = B(AB - |C|^2). \end{aligned}$$

∎

## Euclidean space:

$\mathbb{R}^n$ vector space

Defn: i) $n \in \mathbb{N}$. $\mathbb{R}^n := \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\}$.

Rk. Elements of $\mathbb{R}^n$ denoted $\vec{x} = (x_1, \ldots, x_n)$, called <u>vectors</u>.

(vector addn) ii) Given $\vec{x}, \vec{y} \in \mathbb{R}^n$,

we define $\vec{x} + \vec{y} = (x_1 + y_1, \cdots, x_n + y_n) \in \mathbb{R}^n$.

(scalar mult.) iii) Given $\alpha \in \mathbb{R}$, $\vec{x} \in \mathbb{R}^n$, $\alpha\vec{x} := (\alpha x_1, \ldots, \alpha x_n)$.

<u>Rk:</u> In this way we obtain a real vector space.

inner product w/ norm

(inner product) iv) $\vec{x}, \vec{y} \in \mathbb{R}^n \Rightarrow \vec{x} \cdot \vec{y} := \sum x_i y_i \in \mathbb{R}$.

(norm) v) $\vec{x} \in \mathbb{R}^n \Rightarrow$ we define $|\vec{x}| = (\vec{x} \cdot \vec{x})^{\frac{1}{2}}$.

Rk: Thus C-S ineq. can be written $|\vec{x} \cdot \vec{y}| \leq |\vec{x}||\vec{y}|$.

Q. We have multiplication. Is this a field?

properties of norm

Thm: [Properties of the norm] $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$.

a. $|\vec{x}| \geq 0$, b. $|\vec{x}| = 0 \Leftrightarrow \vec{x} = \vec{0}$

c. $|\alpha\vec{x}| = |\alpha||\vec{x}|$, d. $|\vec{x} \cdot \vec{y}| \leq |\vec{x}||\vec{y}|$

e. $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$, f. $|\vec{x} - \vec{z}| \leq |\vec{x} - \vec{y}| + |\vec{y} - \vec{z}|$.

Pf: e)
$$
\begin{aligned}
|\vec{x} + \vec{y}|^2 &= (x+y) \cdot (x+y) \\
&= x \cdot x + 2x \cdot y + y \cdot y \\
(d \Rightarrow) &\leq |x|^2 + 2|x||y| + |y|^2 \\
&= (|x| + |y|)^2
\end{aligned}
$$

# Chapter 2: Basic Topology

1-1 correspondence between sets

Defns: [$A, B$ sets. $f: A \to B$ a mapping of $A$ into $B$.]

i. For $E \subset A$, $f(E) := \{f(x) \in B \mid x \in E\}$. the image of $E$.

ii. If $f(A) = B$ we say $f$ is onto ($f: A \twoheadrightarrow B$)

iii. For $E \subset B$, $f^{-1}(E) := \{x \in A \mid f(x) \in E\}$

the inverse image of $E$.

iv. For $y \in B$, $f^{-1}(y) := \{x \in A \mid f(x) = y\}$.

v. If $\forall y \in B$, $\#(f^{-1}(y)) \le 1$ we say $f$ is 1-1 $A \hookrightarrow B$

($\Leftrightarrow$ If $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$).

vi. If $\exists f: A \hookrightarrow\!\!\!\!\twoheadrightarrow B$ we say $A$ and $B$ have the same cardinality

Rk: Note that this is an equivalence relation: and write $A \sim B$

$A \sim A$, $A \sim B \Rightarrow B \sim A$, $A \sim B, B \sim C \Rightarrow A \sim C$

finite, ctble, unctble

Defns: Let $J_n = \{1, 2, \dots, n\}$, $J = \{1, 2, 3, \dots\}$. $A$ a set.

i. If $A \sim J_n$ for some $n$, we call $A$ finite; (o/w infinite)

ii. If $A \sim J$ we call $A$ countable (countably infinite)

iii. If $A$ is finite or countable we call it at most ctble.

otherwise, we call $A$ uncountable (-bly infinite)

seq.

Defn: Any function $a: J \to A$ is called a sequence (in $A$)

Rk: Usually one writes $a_i$ instead of $a(i)$.

subset of ctble $\Rightarrow$ most ctble

Thm: Every infinite subset of a countable set $A$ is countable.

Pf: Say $E \subset A$ is infinite.

Since $A$ is ctble, $A = \{x_1, x_2, x_3, \dots\}$ can be arranged in a sequence.

Let $n_1$ denote the first integer s.t. $x_{n_1} \in E$, $n_2$ the second, etc

Then $\{e_i := x_{n_i}\}$ is a sequence describing all elts. of $E$.

union
intersection
basic rules
sets and countability

<u>Set notation</u>: $A, \Omega$ sets. i.e. we have a colln of sets indexed by
Say $\forall \alpha \in A \; \exists \; E_\alpha \subset \Omega$, notated $\{E_\alpha\}_{\alpha \in A}$.
Std. define: $\bigcup_{\alpha \in A} E_\alpha := \{x \in \Omega \mid x \in E_\alpha \text{ for some } \alpha\}$.

$\bigcap_{\alpha \in A} E_\alpha := \{x \in \Omega \mid x \in E_\alpha \; \forall \alpha \in A\}$.

Rk: Finite intersection, infinite intersection, etc.
<u>Rk:</u> We say A and B are <u>disjoint</u> if $A \cap B = \emptyset$, else <u>inters</u>

Rules of set theory:
i. $A \subset A \cup B$
ii. $A \cap B \subset A$
iii. $A \cup \emptyset = A$, $A \cap \emptyset = \emptyset$
iv. $A \subset B \Rightarrow A \cup B = B$, $A \cap B = A$.
v. $A \cup B = B \cup A$, $A \cap B = B \cap A$
vi. $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap C = \ldots$
vii. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(The rule which screws the analogy) ix. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. . .

+ <u>countability</u>:
Thm: Let $\{E_n\}_{n=1}^\infty$ be a sequence of ctble sets. Then $\bigcup_{n=1}^\infty E_n =: S$ is ctble

<u>Pf:</u> Arrange $E_n$ in a sequence $\{x_{n1}, x_{n2}, x_{n3}, \ldots\}$ (since ctble)
Consider
$$\begin{matrix} x_{11} & x_{12} & x_{13} & \cdots \\ x_{21} & x_{22} & x_{23} & \cdots \\ x_{31} & x_{32} & x_{33} & \cdots \end{matrix}$$
Thus $S$ can be arranged in a sequence $\{x_{11}, x_{21}, x_{12}, \ldots\}$
(w/ possible duplicates); so $S \sim T \subset \mathbb{N}$.
Since $E_1 \subset S$, and $E_1$ is infinite, so is $S$.

n-tuples ctble

Thm: A ctble, $B_n := \{(a_1, a_2, \ldots, a_n) \mid a_i \in A\}$, the set of n-tuples
Then $B_n$ is ctble.

Pf: A ctble $\Rightarrow A = \{x_1, x_2, x_3, \ldots\}$.
Note: The set $\{x_i\} \times A = \{(x_i, a) \mid a \in A\} \sim A$.
Since $B_2 = \bigcup_{i=1} \{x_i\} \times A$, it is a ctble union of ctble set
$\Rightarrow B_2$ is ctble.

Corollary: $\mathbb{Q}$ is ctble.

$\mathbb{R}$ unctble

Thm: The set A of all sequences whose elts are either 0 or 1 is unctb

Pf: Suppose A is ctble; $A = \{s_1, s_2, s_3, \ldots\}$, each a sequence.
Create a new sequence $s^*$ by requiring its $n^{th}$ element differ from that of $s_n$. $s^* \in A$ ※.
(Cantor's diagonal process.)

Metric Spaces:

metric

Defn: X a set $d: X \times X \to \mathbb{R}$ a fn. If $\forall p, q \in X$,

a. $d(p,q) > 0$ whenever $p \neq q$; $d(p,p) = 0$
b. $d(p,q) = d(q,p)$
c. $d(p,q) \leq d(p,r) + d(r,q)$

then we call d a metric, and X,d a metric s

Rk: A metric is a generalization of the notion of distance.

basic defns

Basic defns: i. $(a,b)$, $[a,b]$, $[a,b)$, $(a,b]$; intervals
ii. $a_i < b_i$, $i = 1, \ldots, k$. $\{\vec{x} = (x_1, \ldots, x_k) \mid a_i < x_i < b_i\}$ is called a k-cell.
iii. Fix $\vec{x} \in \mathbb{R}^k$, $r > 0$. open ball, closed ball.
iv. $E \subset \mathbb{R}^k$. If $\forall \vec{x}, \vec{y} \in E$, $\lambda \in (0,1)$, $\lambda\vec{x} + (1-\lambda)\vec{y} \in E \Rightarrow$ con

Ex: Any (open) ball is convex. Pf?

n-tuples ctble

Thm: A ctble, $B_n := \{(a_1, a_2, \dots, a_n) \mid a_i \in A\}$, the set of n-tuples. Then $B_n$ is ctble.

Pf. A ctble $\Rightarrow$ $A = \{x_1, x_2, x_3, \dots\}$.

Note: The set $\{x_i\} \times A = \{(x_i, a) \mid a \in A\} \sim A$.

Since $B_2 = \bigcup_{i=1} \{x_i\} \times A$, it is a ctble union of ctble set

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Ex: Any (open) ball is convex. Pf?

topological terms

Basic defns of topology: $X$ a metric space

i. $[p \in X, r > 0]$ Let $N_r(p) := \{q \in X \mid d(p,q) < r\}$;
we call such $N_r(p)$ a nbhd. of $p$ of radius $r$.

ii. $[E \subset X]$ $p \in X$. If every nbhd. of $p$ contains a $q \neq p$ s.t. $q \in E$, we say $p$ is a limit pt. of $E$

iii. If $p \in E$ but $p$ is not a limit pt of $E$ we say $p$ is an isolated pt.

iv. If every limit pt. of $E$ is in $E$ we say $E$ is closed.

v. If $\exists N_r(p)$ s.t. $N_r(p) \subset E$ we call $p$ an interior pt. of $E$.

vi. If every pt. in $E$ is an interior pt., we say $E$ is open.

vii. $E^c := \{x \in X \mid x \notin E\}$ is called the complement of $E$.

viii. If $E$ is closed and every pt in $E$ is a limit pt. we say $E$ is perfect

ix. If $\exists q \in X$, $M > 0$ s.t. $d(p,q) < M$ $\forall p \in E$ we say $E$ is bdd

x. If every pt. of $X$ is a limit pt. of $E$ or a pt. of $E$ $\Rightarrow$ we say $E$ is dense in $X$.

(Examples.)

nbhd $\Rightarrow$ open

Thm: Every nbhd. is an open set.

Pf: Let $E = N_r(p)$. Pick any $q \in E$. (thus $d(p,q) = r - h < r$)
Then $d_h(q) \subset E$ (why?), so $q$ is an interior pt.

limit pt. property

Thm: $p$ a limit pt. of $E$ $\Rightarrow$ every nbhd. contains infinitely many pts. of

Pf: Let $N_r(p)$ be a nbhd. of $p$.
$p$ a limit pt. $\Rightarrow$ $\exists q_1 \in N_r(p)$ s.t. $q_1 \neq p$ and $q \in E$
Let $r_2 = d(p,q_1)$; then $N_{r_2}(p) \subset N_r(p)$. (
and $q_1 \notin N_{r_2}(p)$.
$p$ a limit pt $\Rightarrow$ $\exists q_2 \in N_{r_2}(p)$ s.t. $q_2 \neq p, q_1$
$q_2 \in$
Iterate; get an infinite sequence ■

Rules of Sets & Topological Notions:

Basic sets

**Thm:** $\{E_\alpha\}$ a collxn of sets. Then $\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha^c)$.

Pf: Trivial. (E.g. $x \in (\bigcup E_\alpha)^c \Rightarrow x \notin E_\alpha \ \forall \alpha \Rightarrow x \in E_\alpha^c \ \forall \alpha$.)

open $\rightleftarrows^c$ closed

**Thm:** A set $E$ is open $\Leftrightarrow$ $E^c$ is closed.

Pf: $\Rightarrow$) Say $E$ is open, and let $x$ be a limit pt. of $E^c$.

Then every nbhd. of $x$ contains some $p \neq x$ s.t. $p \in E^c$.

$\Rightarrow x$ is not an interior pt. of $E$.

$\Rightarrow x \notin E$, i.e. $x \in E^c$.

$\Leftarrow$) Say $E^c$ is closed, and pick any $x \in E$.

Then $x$ is not a limit pt. of $E^c$

$\Rightarrow \exists$ some nbhd. $N_r(x)$ containing no pts. of $E^c$

i.e. $N_r(x) \subset E$. So $x$ is an interior pt.

Rk: Just definition-chasing

openness + infinite unions, etc.

**Thm:** [$\cup, \cap$, open, closed]

a. $\{G_\alpha\}$ a collxn. of open sets $\Rightarrow \bigcup_\alpha G_\alpha$ is open

b. $\{F_\alpha\}$ collxn. of closed sets $\Rightarrow \bigcap_\alpha F_\alpha$ is closed.

c. $G_1, \dots, G_n$ a finite collxn of open sets $\Rightarrow \bigcap^n G_i$ is open.

d. $F_1, \dots, F_n$ a finite collxn of closed sets $\Rightarrow \bigcup^n F_i$ is closed.

Pf. a. Let $x \in \bigcup G_\alpha \Rightarrow x \in$ some $G_{\alpha_0}$, which is open. ∎

b. STS $\left(\bigcap_\alpha F_\alpha\right)^c$ is open.

c. $x \in \bigcap^n G_i \Rightarrow$ for each $i$, $\exists r_i > 0$ s.t. $x \in N_{r_i}(x) \subset G_i$

Let $r = \min\{r_1, \dots, r_n\}$. $N_r(x) \subset \bigcap G_i$.

d. STS $\left(\bigcup^n F_i\right)^c$ is open.

Rk: Finiteness essential

Rules of Sets & Topological Notions:

Basic sets

Thm: $\{E_\alpha\}$ a collxn. of sets. Then $\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha^c)$.

Pf: Trivial. (E.g. $x \in (\bigcup E_\alpha)^c \Rightarrow x \notin E_\alpha \ \forall \alpha \Rightarrow x \in E_\alpha^c \ \forall \alpha$.)

open $\overset{c}{=}$ closed

Thm: A set $E$ is open $\Leftrightarrow$ $E^c$ is closed.

Pf: $\Rightarrow$) Say $E$ is open, and let $x$ be a limit pt. of $E^c$.

Then every nbhd. of $x$ contains some $p \neq x$ s.t. $p \in E^c$.

$\Rightarrow$ $x$ is not an interior pt. of $E$.

$\Rightarrow x \notin E$, i.e. $x \in E^c$.

$\Leftarrow$) Say $E^c$ is closed, and pick any $x \in E$.

Then $x$ is not a limit pt. of $E^c$

$\Rightarrow \exists$ some nbhd. $N_r(x)$ containing no pts. of $E^c$

i.e. $N_r(x) \subset E$. So $x$ is an interior pt.

Rk: Just definition-chasing

openness + infinite unions, etc.

Thm: [$\cup$, $\cap$, open, closed]

a. $\{G_\alpha\}$ a collxn. of open sets $\Rightarrow \bigcup_\alpha G_\alpha$ is open

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Pf. a. Let $x \in \bigcup G_\alpha \Rightarrow x \in$ some $G_{\alpha_0}$, which is open. ∎

b. STS $\left(\bigcap_\alpha F_\alpha\right)^c$ is open.

c. $x \in \bigcap^n G_i \Rightarrow$ for each $i$, $\exists r_i > 0$ s.t. $x \in N_{r_i}(x) \subset G_i$

Let $r = \min\{r_1, \ldots, r_n\}$. $N_r(x) \subset \bigcap G_i$.

d. STS $\left(\bigcup^n F_i\right)^c$ is open.

Rk: Finiteness essential

$\overline{E}$

**Closure:** Notation: $X$ a metric space, $E \subset X$, $E' :=$ limit pts. of $E$ in $X$

**Defn:** The closure of $E$, $\overline{E} := E \cup E'$.

What is $\overline{E}$?

**Thm:** [Closedness & Closure]

a. $\overline{E}$ is closed

b. $E$ is closed $\Leftrightarrow$ $E = \overline{E}$

c. $[F \subset X]$ Say $\left.\begin{array}{l} E \subset F \\ F \text{ is closed} \end{array}\right\}$ then $\overline{E} \subset F$.

**Pf:** a. Pick any $p \in X \setminus \overline{E}$.

Since $p \notin E'$, $\exists N_{r_1}(p)$ s.t. $N_{r_1}(p) \cap E = \phi$.

Further, $\exists N_{r_2}(p)$ s.t. $N_{r_2}(p) \cap E' = \phi$.

Thus $(\overline{E})^c$ is open, i.e. $\overline{E}$ is closed.

b. $\Rightarrow$) $E$ closed $\overset{\Delta}{\Rightarrow}$ $E' \subset E$ $\Rightarrow$ $E \cup E' \subset E$, so $\overline{E} = E$.

$\Leftarrow$) $E = \overline{E}$ $\Rightarrow$ (a) implies $E$ is closed

c. STS $E' \subset F$. Well, $E \subset F \Rightarrow E' \subset F' \subset F = F$

Example in $\mathbb{R}$

**Thm:** $[E \subset \mathbb{R}]$ $\sup E \in E \cup E'$ (if it exists)

**Pf:** Assume WLOG that $\sup E \notin E$. Take any nbhd $N_r(\sup E) = (\sup E - r, \sup E + r)$.

If $N_r(\sup E) \cap E = \emptyset$, then $\sup E - r$ is an upper bd. for $E$

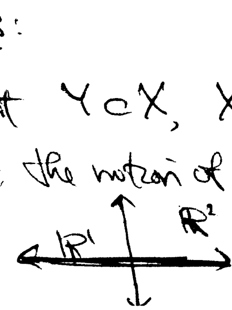
Thus $N_r(\sup E) \cap E \neq \emptyset$ $\forall r > 0$. ∎

Relative openness

**Relative Openness:**

**Rk:** Note that $Y \subset X$, $X$ a metric space $\Rightarrow$ $Y$ is a metric space.

However, the notion of nbhd (and thus the topology) may differ.

E.g. 


$\mathbb{R}^1$ inherits a metric, but open in $\mathbb{R}^1$ $\not\Rightarrow$ open in $\mathbb{R}^2$.

definition — **Defn:** $E$ is <u>open relative to $Y$</u> if $\forall p \in E$,
$\exists r > 0$ s.t. $\{q \in \underline{\underline{Y}} \mid d(p,q) < r\} \subset E$.
(i.e. $\exists$ a nbhd $N$ in $Y$ s.t. $p \in N \subset E$.)

alternate interp. of relative openness — **Thm:** [$E \subset Y \subset X$, $X$ a metric space]
$E$ is open relative to $Y \Leftrightarrow E = Y \cap G$ for some open $G \subset X$

Pf: $\Rightarrow$) We know $\forall p \in E\ \exists r_p > 0$ s.t. $\{q \in Y \mid d(p,q) < r_p\} \subset E$.
Well, let $G = \bigcup_{p \in E} \{q \in \underline{\underline{X}} \mid d(p,q) < r_p\}$.
i. $G$ is open.
ii. $E = Y \cap G$.

$\Leftarrow$) $E = Y \cap G \Rightarrow \forall p \in E$, $\exists$ a nbhd $p \in V_p \subset G$.
$\Rightarrow p \in V_p \cap Y \subset G \cap Y = E$. ∎

## Compact Sets:

open cover — **Defn:** [$E \subset X$, $X$ a metric space] Any collxn $\{G_\alpha\}$ of open $G_\alpha \subset X$
s.t. $E \subset \bigcup_\alpha G_\alpha$ is called an open <u>cover</u> of $E$ in $X$

compactness — **Defn:** $K \subset X$. If $K$ has the property that every open cover contains a finite subcover, we call $K$ <u>compact</u>
i.e. Given <u>any</u> open cover $\bigcup_\alpha G_\alpha \supset E$,
$\exists$ a finite set of indices $\alpha_1, \ldots, \alpha_n$ s.t. $\bigcup_{i=1}^{n} G_{\alpha_i} \supset K$

non-compact sets — E.g. i. $\mathbb{R}$ is not compact.
ii. $(0,1)$ is not compact
iii. $(0,1]$ is not compact

can find open covers that have no finite subcovers.

Rk: This is a <u>crucial</u> if somewhat opaque definition.

## Properties of Compactness:

independence of embedding space

Thm: [Compactness indpt. of embedding space.] $K \subset Y \subset X$.

Claim: $K$ is compact rel. to $X \iff K$ is cpt. rel. to $Y$.

Pf: $\Rightarrow$) Say $K$ is compact relative to $X$.

Let $\{G_\alpha\}$ be an open covering of $K$ in $Y$.

(ie. $G_\alpha$ are open relative to $Y$)

$\Rightarrow$ each $G_\alpha = Y \cap H_\alpha$ for some $H_\alpha$ open rel. to $X$

Since $K \subset \bigcup_\alpha H_\alpha$, $K \subset H_{\alpha_1} \cup \ldots \cup H_{\alpha_n}$,

and $K \subset Y \Rightarrow K \subset (H_{\alpha_1} \cup \ldots \cup H_{\alpha_n}) \cap Y$

PE: We intersect the cover in $X$ down to one in $Y$. $= \bigcup_{i=1}^{n} G_{\alpha_i}$.

$\Leftarrow$) Say $K$ is compact relative to $Y$.

Given an open covering $K \subset \bigcup_\alpha H_\alpha$ in $X$,

we know $K \subset \bigcup_\alpha (H_\alpha \cap Y)$ is an open covering in $Y$

So $\exists$ a finite subcover: $K \subset \bigcup_{i=1}^{n} (H_{\alpha_i} \cap Y)$.

PE: We expand the cover in $Y$ to one in $X$. and thus one in $X$: $\subset \bigcup_{i=1}^{n} H_{\alpha_i}$.

compact $\Rightarrow$ closed

Thm: [Compact $\Rightarrow$ closed] Say $K \subset X$ is compact, $X$ a metric space.

Claim: $K$ is closed.

Pf: We shall show $K^c$ is open. Pick any $p \in K^c$.

For each $q \in K$, cover it with a nbhd $N_{r_q}(q)$

where $r_q < \frac{1}{2} \operatorname{dist}(p, q)$.

$K$ compact $\Rightarrow \exists$ a finite subcover $N_{r_{q_1}}(q_1) \cup \ldots \cup N_{r_{q_n}}(q_n)$

Then let $r = \min\{r_{q_1}, \ldots, r_{q_n}\}$.

Q. Why did we need finiteness? $N_r(p)$ doesn't intersect the subcover (or $K$). ∎

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Then let $r = \min\{r_{q_1}, \ldots, r_{q_n}\}$.

Q. Why did we need finiteness? $N_r(p)$ doesn't intersect the subcover (or $K$).

closed subset of compact set is compact

**Thm:** $[F \subset K \subset X,\ K \text{ compact}]$ If $F$ is closed $\Rightarrow$ $F$ is compact.

Pf: Take an open cover $F \subset \bigcup_\alpha G_\alpha$ in $X$.

$F$ closed $\Rightarrow$ $F^c \subset X$ is open; $F^c \cup \bigcup_\alpha G_\alpha$ is an open cover of

$K$ compact $\Rightarrow$ $\exists$ a finite subcover of $K$;

by removing $F^c$ we obtain a subcover of $\bigcup G_\alpha$

(Defn: finite $\cap$ and $\cup$.) that covers $F$. ∎

finite intersection condition $\Rightarrow \bigcap_\alpha K_\alpha \neq \phi$

**Thm:** $X$ a metric space, $\{K_\alpha\}$ compact sets in $X$.

If the intersection of any finite subcollxn is nonempty $\Rightarrow$ $\bigcap_\alpha K_\alpha$ is nonempty

Pf: Suppose it's empty, ie $\bigcap_\alpha K_\alpha = \phi$

ie. $X = \left(\bigcap_\alpha K_\alpha\right)^c = \bigcup_\alpha K_\alpha^c$.

Then for any $K_{\alpha_0}$ we have $K_{\alpha_0} \subset \bigcup_\alpha K_\alpha^c$

$$\Rightarrow K_{\alpha_0} \subset \left(K_{\alpha_1}^c \cup \dots \cup K_{\alpha_n}^c\right) \quad \text{some finite subcollxn}$$

$$\Rightarrow K_{\alpha_0} \cap \left(\quad\quad\right)^c = \phi$$

$$\Leftrightarrow K_{\alpha_0} \cap K_{\alpha_1} \cap \dots \cap K_{\alpha_n} = \phi \quad \times$$

compactness $\Rightarrow$ limit pt. compactness. (aka Bolzano-Weierstraß prop.)

**Thm:** $[E \subset K$ an infinite subset$]$ If $K$ compact $\Rightarrow$ $E$ has a limit pt in (E infinite)

Pf: Suppose no point in $K$ is a limit pt. of $E$,

ie, for each $q \in K$, $\exists$ a nbhd $V_q$ containing no elt. of $E$

No finite subcollxn can cover $E$, let alone $K$. (except at most $q$ itself)

**Thm:** $\{I_n\}$ a nested sequence of closed intervals in $\mathbb{R}^1$. Then $\bigcap_n I_n \neq \phi$

Pf: Notate $I_n = [a_n, b_n]$ (note if we knew $I_n$ cpt. $\Rightarrow$ done

Let $E = \{a_i\}_{i=1}^{\infty}$. Since $a_m \leq a_{m+n} \leq b_{m+n} \leq b_m$,

we know any $b_i$ is an upper bound of $E$ $\Rightarrow \exists \sup(E)$.

Since all $b_i$ are u.b, $x \leq b_m\ \forall m$. Since $x$ is ub, $a_m \leq x$

Rk: Similarly for k-cells.

k-cells compact. Thm: [Bu facts] All k-cells are compact.

Pf: Let $I = \prod_{i=1}^{k} [a_i, b_i]$;

$$\delta = \left[\sum_{i=1}^{k} (b_i - a_i)^2\right]^{1/2} \quad (\text{length of diagonal})$$

Then if $x, y \in I$, $\|x-y\| < \delta$. (why?)

(Pf. by contradiction:) Suppose $\exists$ an open cover $\{G_\alpha\}$ of $I$ for which $\nexists$ a decent finite subcover.

dyadic subcubes — In that case, dyadically divide $I$ into $2^k$ k-cells.

At least one, call it $I_1$, cannot be covered by a finite subcover of $\{G_\alpha$

Iterating, we obtain a nested sequence of k-cells $\{I_n\}$:

i. each $I_n$ cannot be covered by a finite subcover

ii. $\text{diam}(I_n) < 2^{-n}\delta$.

By ... we know $\bigcap I_n \ni x^*$, some pt.

Further, $x^* \in G_\alpha$ for some $\alpha$

$G_\alpha$ open $\Rightarrow \exists N_r(x^*) \subset G_\alpha$.

Now, choose some $n$ so large that $2^{-n}\delta <$

Then $I_n \subset N_r(x^*) \subset G_\alpha$ ※.

($I_n$ couldn't be covered by a finite subcover)

(i.e. s.t. $2^n >$ .)

Heine-Borel — Thm: [Heine-Borel] In $\mathbb{R}^n$, TFAE:

a. $E \subset \mathbb{R}^n$ is closed and bdd

b. $E$ is compact

c. $E$ is limit pt. compact.

Pf: $a \Rightarrow b$ basically done

$b \Rightarrow c$ already done.

So it suffices to show that c ⇒ a)

Pf: Well, suppose $E$ limit pt. compact.

(easy proof)

$E$ not bdd ⇒ can create an infinite set w/o limit pt. ※.

$E$ not closed ⇒ $\exists x_0 \in E^c \subset \mathbb{R}^k$ s.t. $x_0$ is a limit pt. of $E$

Then $\forall n \in \mathbb{N}$, $\exists e_n \in E \cap N_{\frac{1}{n}}(x_0) \setminus \{x_0\}$

$S = \{e_n\}_{n=1}^{\infty}$ is an infinite set w/ $x_0$ as limit.

Rk: Why can no other pt. be the limit pt? (Obvious.)

$$\|e_n - y\| \geq \|x_0 - y\| - \|x_0 - e_n\|$$
$$\geq \|x_0 - y\| - \frac{1}{n} \geq \frac{1}{2}\|x_0 - y\|$$

for $n$ sufficiently large

Connected Sets:

Defn: [$X$ a metric space, $A, B \subset X$]

separated sets

i. If $A \cap \overline{B} = \overline{A} \cap B = \emptyset$ we say $A, B$ are separated.

connected set

ii. If $E$ can be expressed as $A \amalg B$ where $A, B$ are separated then we say $E$ is disconnected. Else connected

Rk: Connxd ⇒ $E = A \cup B$ ⇒ $A$ and $B$ are not separated.

Rk: Separated ⇒ disjoint but disjoint ⇏ separated.

Thm: $E \subset \mathbb{R}^1$ is connxd ⇔ $\forall x, y \in E$, $(x, y) \subset E$

# Chapter 4: Continuity.

limit $\lim_{x\to p} f(x)$

**Defn** [limits]: $[(X,d_X), (Y,d_Y)$ metric spaces. $E\subset X$, $p\in E']$

$f: E\to Y$. If $\exists q\in Y$ s.t.

for each $N_\varepsilon(q)$ $(\varepsilon>0)$

$\exists$ a $N^*_\delta(p)$ s.t. $f[N^*_\delta(p)\cap E]\subset N_\varepsilon(q)$.

Then we say $\lim_{x\to p} f(x) = q$.

and we say <u>$f$ has a limit at $p$</u>.

rephrased in terms of the metric

Rephrased: If $\exists q\in Y$ s.t.

$\forall \varepsilon>0, \exists \delta>0$ s.t.

if $x\in E$ satisfies $d_X(x,p)<\delta$

then $d_Y(f(x),q)<\varepsilon$

then we say $\lim_{x\to p} f(x)=q$.

comment

Rk: $p$ need not be in $E$; further, we do not require $f(p)=\lim_{x\to p} f(x)$ (even if $p\in E$)

uniqueness of limit

**Thm**: If $f$ has a limit at $p$, the limit is unique.

Pf: Suppose there are 2 limits $q_1, q_2$ and $q_1\neq q_2$.

Then take $\varepsilon<\frac{1}{2}d(q_1,q_2)$.

We know $\exists N^*_{\delta_i}(p)$ s.t. $f(N^*_{\delta_i}(p)\cap E)\subset N_\varepsilon(q_i)$; $i=1$.

But then $f(N^*_{\min(\delta_1,\delta_2)}(p)\cap E)\subset N_\varepsilon(q_1)\cap N_\varepsilon(q_2) = \emptyset$.

$p$ was a limit pt. of $E$, so $\times$.

## Continuity:

continuity

**Defn**: $[X, Y$ metric spaces, $E\subset X$, $p\in E$, $f: E\to Y.]$

If $\forall N_\varepsilon(f(p))$ $(\varepsilon>0)$

$\exists N_\delta(p)$ s.t. $f[N_\delta(p)\cap E]\subset N_\varepsilon(f(p))$

at a pt.

on a set.

then we say $f$ is <u>cts. at $p$</u>. (also define <u>cts. on $E$</u>.)

comment | (Q.) What if $p$ is an isolated pt. of $E$? (degenerate case)
Then every $f$ (with $E$ as domain) is cts at $p$.

rel'n w/ limits | Thm: If, in add'n, $p \in E'$ then $f$ is cts at $p \iff \lim_{x \to p} f(x) = f(p)$.

composition of cts. fns | Thm: $X$, $Y$, $Z$ metric spaces.

$$\begin{array}{ccccc} X & & Y & & Z \\ \cup & & \cup & & \cup \\ E & \xrightarrow{f} & f(E) & \xrightarrow{g} & g(f(E)) \end{array}$$

Let $h = g \circ f$.

If $f$ is cts. at $p \in E$, $g$ cts at $f(p) \in Y$ then $h$ is cts at $p$.

Pf: Fix a nbhd of $h(p) = g(f(p))$, $N_\varepsilon(g(f(p)))$, $\varepsilon > 0$.

$g$ cts. at $f(p)$, so $\exists N_\eta(f(p))$, $\eta > 0$ s.t.

$$g(N_\eta(f(p)) \cap f(E)) \subset N_\varepsilon(g(f(p)))$$

$f$ cts. at $p$, so $\exists N_\delta(p)$, $\delta > 0$ s.t.

$$f(N_\delta(p) \cap E) \subset N_\eta(f(p)).$$

Then $g \circ f(N_\delta(p) \cap E) \subset g(N_\eta(f(p)) \cap f(E))$
$\subset N_\varepsilon(g(f(p)))$.

ie $h(N_\delta(p) \cap E) \subset N_\varepsilon(h(p))$. ∎

useful char'n of continuity. | Thm: [Alternate characterization of continuity] [$f: X \to Y$ metric spaces]
$f$ is cts. on $X \iff f^{-1}(V)$ is open in $X$ $\forall V$ open in $Y$.

Pf: $\Rightarrow$) Let $V \subset Y$ be open. WTS $f^{-1}(V)$ is open, ie. every pt. interior.
Well, say $p \in f^{-1}(V)$ ie. $f(p) \in V$.
We know $\exists N_\varepsilon(f(p)) \subset V$ since $V$ is open.
And by continuity, $\exists N_\delta(p)$ s.t. $f(N_\delta(p)) \subset N_\varepsilon(f(p))$ $\subset V$.
ie, $N_\delta(p) \subset f^{-1}(V)$.

$\Leftarrow$) Suppose $f^{-1}(V)$ is open in $X$ $\forall$ open sets $V \subset Y$.
Pick some $p \in X$ and some nbhd. $N_\varepsilon(f(p)) \subset Y$.

Well, $f^{-1}(N_\varepsilon(f(p)))$ is open and contains $p$.

$$\Rightarrow \exists\, N_\delta(p) \subset f^{-1}(N_\varepsilon(f(p))).$$

$$\Rightarrow f(N_\delta(p)) \subset N_\varepsilon(f(p)). \quad \blacksquare$$

dumb corollary

<u>Corollary</u>: [$X, Y$ metric spaces] $f: X \to Y$ is cts.
$\Leftrightarrow$ $f^{-1}(C)$ is closed in $X$ $\forall$ $C$ closed in $Y$.

<u>Pf</u>: $f^{-1}(E^c) = [f^{-1}(E)]^c$.

<u>Thm</u>: [Continuity of vector-valued fns.] ($X$ a metric space)

a) $f_1, \dots, f_k : X \to \mathbb{R}$. Let $\vec{f} : X \to \mathbb{R}^k$
$x \longmapsto (f_1(x), f_2(x), \dots, f_k(x))$

<u>Claim</u>: $\vec{f}$ is cts. $\Leftrightarrow$ all $f_i$ are cts. (ie. $\vec{f}$ has components $f_1, \dots, f_k$)

b) $\vec{f}, \vec{g}$ cts. on $X \to \mathbb{R}^k$ $\Rightarrow$ $\vec{f} + \vec{g}$, $\vec{f} \cdot \vec{g}$ are also.

<u>Pf a)</u>:

commentary

<u>Rk</u>: When speaking of continuity, there's little reason to speak of <u>subsets</u> $E \subset X$.
So we will only talk about cts. mappings of <u>metric spaces</u>.

## Continuity and Compactness

bdd fn.

**Defn**: $f: E \to \mathbb{R}^k$. If $\exists M$ s.t. $|f(x)| < M \ \forall x \in E$, we say $f$ is **bounded**.

cts on cpt set $\Rightarrow$ cpt image

**Thm**: [$f: X \to Y$ metric spaces] $\left.\begin{array}{l} f \text{ cts} \\ X \text{ cpt} \end{array}\right\} \Rightarrow f(X)$ cpt.

**Pf**: Obvious. Given an open cover of $f(X)$, call it $\{G_\alpha\}$ then $\{f^{-1}(G_\alpha)\}_\alpha$ is an open cover of $X$.

$\Rightarrow X \subset f^{-1}(G_{\alpha_1}) \cup \dots \cup f^{-1}(G_{\alpha_n})$ some finite subcover.

$\Rightarrow f(X) \subset \bigcup_{i=1}^{n} f(f^{-1}(G_{\alpha_i})) = \bigcup_{i=1}^{n} G_{\alpha_i}$.

Consequences:

corollaries

**Thm**: $f: X \to \mathbb{R}^k$ $f$ cts, $X$ cpt $\Rightarrow f$ is bdd

(What happens in $\mathbb{R}^1$?) $f$ bdd $\Rightarrow f(X)$ has sup and inf.

extreme value thm.

**Thm**: [Extreme Value Thm.] $f: X \to \mathbb{R}$.

$f$ cts, $X$ cpt $\Rightarrow \exists p, q \in X$ s.t. $f(p) = \sup f(X)$
$f(q) = \inf f(X)$.

**Pf**: $M = \sup_{x \in X} f(x)$ and $m = \inf_{x \in X} f(x)$ are both in $(f(X))'$.

Since $f(X)$ is cpt, it is closed, so $M, m \in f(X)$ ■

**Rk**: This theorem is the missing keystone in (most) introductory calculus texts.

$f$ 1-1 cts. $X$ cpt. $\Rightarrow$ inverse cts

**Thm**: [$f: X \to Y$ metric spaces, $f$ 1-1 and onto (so $f^{-1}: Y \to X$ is defined)]

If $f$ is cts and $X$ cpt $\Rightarrow f^{-1}$ is cts. ($f^{-1}(f(x)) = x$.)

**Pf**: STS that $V \subset X$ open $\Rightarrow f(V)$ is open in $Y$.

Well, $V^c$ is closed $\Rightarrow$ cpt. $\Rightarrow f(V^c)$ is cpt. $\Rightarrow$ closed and $f(V^c) = (f(V))^c$ ■ since $f$ is bijective.

## Continuity and Compactness

bdd fn. **Defn:** $f: E \to \mathbb{R}^k$. If $\exists M$ s.t. $|f(x)| < M \ \forall x \in E$, we say $f$ is **bounded**.

cts. on cpt. set $\Rightarrow$ cpt. image

**Thm:** [$f: X \to Y$ metric spaces] $\left.\begin{array}{l} f \text{ cts.} \\ X \text{ cpt.} \end{array}\right\} \Rightarrow f(X)$ cpt.

**Pf:** Obvious. Given an open cover of $f(X)$, call it $\{G_\alpha\}$ then $\{f^{-1}(G_\alpha)\}_\alpha$ is an open cover of $X$.

$\Rightarrow X \subset f^{-1}(G_{\alpha_1}) \cup \dots \cup f^{-1}(G_{\alpha_n})$ some finite subcover.

$\Rightarrow f(X) \subset \bigcup_{i=1}^{n} f(f^{-1}(G_{\alpha_i})) = \bigcup_{i=1}^{n} G_{\alpha_i}$.

Consequences:

corollaries

**Thm:** $f: X \to \mathbb{R}^k$ $f$ cts., $X$ cpt $\Rightarrow$ $f$ is bdd

(What happens in $\mathbb{R}^1$?) $f$ bdd $\Rightarrow$ $f(X)$ has sup and inf.

extreme value thm.

**Thm:** [Extreme Value Thm.] $f: X \to \mathbb{R}$.

$f$ cts, $X$ cpt. $\Rightarrow \exists p, q \in X$ s.t. $f(p) = \sup f(X)$
$f(q) = \inf f(X)$.

**Pf:** $M = \sup_{x \in X} f(x)$ and $m = \inf_{x \in X} f(x)$ are both in $(f(X))'$.

Since $f(X)$ is cpt, it is closed, so $M, m \in f(X)$ ∎

**Rk:** This theorem is the missing keystone in (most) introductory calculus texts.

$f$ <-> cts. $X$ cpt. $\Rightarrow$ inverse cts

**Thm:** [$f: X \to Y$ metric spaces, $f$ 1-1 and onto. (so $f^{-1}: Y \to X$ is defined)]

If $f$ is cts. and $X$ cpt. $\Rightarrow f^{-1}$ is cts. $(f^{-1}(f(x)) = x$.

**Pf:** STS that $V \subset X$ open $\Rightarrow f(V)$ is open in $Y$.

Well, $V^c$ is closed $\Rightarrow$ cpt. $\Rightarrow f(V^c)$ is cpt. $\Rightarrow$ closed

and $f(V^c) = (f(V))^c$ ∎ since $f$ is bijective.

## Uniform continuity:

uniform continuity — **Defn:** [$f: X \to Y$ metric spaces]

If, for every $\varepsilon > 0$, $\exists \delta > 0$ s.t.

whenever $d_X(p,q) < \delta$, $d_Y(f(p), f(q)) < \varepsilon$

Then we say $f$ is <u>uniformly continuous</u>.

difference between cty and unif. cty? — **Rk:** What's the difference between cts. and unifly cts.?

The point is that the $\delta$ is independent of the point $p$.

i.e. <u>uniform</u> over the whole space $X$.

cts. on cpt. set $\Rightarrow$ unif. cts. — **Thm:** [$f: X \to Y$ metric spaces]

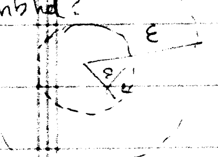
$f$ cts. and $X$ cpt. $\Rightarrow$ $f$ is uniformly cts.

**Pf:** Fix $\varepsilon > 0$.

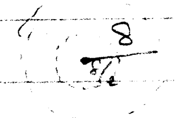
$f$ cts. $\Rightarrow$ $\forall p \in X$, $\exists\, \delta(p) > 0$ s.t.

$f[N_{\delta(p)}(p)] \subset N_{\frac{\varepsilon}{2}}(f(p))$.

Idea: What if we only had one nbhd? (i.e. could cover all of X w/ one nbhd)



Cover each point $p$ with $N_{\frac{\delta(p)}{2}}(p) =: J(p)$.



Since cpt., we have only finitely many

$X$ cpt. $\Rightarrow$ $\exists$ a finite subcover:

$X \subset J(p_1) \cup \dots \cup J(p_n)$.

Let $\delta = \frac{1}{2} \min\{\delta(p_1), \dots, \delta(p_n)\}$.

Let's see this $\delta$ behaves as we wanted:

Pick any $p, q \in X$ s.t. $d(p,q) < \delta$.

$p$ must lie in some $J(p_m) = N_{\frac{\delta(p_m)}{2}}(p_m)$.

Then $d(p, p_m) < \frac{1}{2}\delta(p_m)$

$d(q, p_m) < d(p,q) + d(p, p_m) < \delta + \frac{1}{2}\delta(p_m) \leq \delta(p_m)$.

$\Rightarrow$ $d(f(p), f(q)) \leq d(f(p), f(p_m)) + d(f(q), f(p_m)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

§ Continuity & Connectedness

recall defn connected set

Recall: $E$ is connected if it cannot be expressed as $A \amalg B$ s.t. $A \cap \bar{B} = \bar{A} \cap B = \phi$. i.e. cannot be expressed as a union of (nonempty) separated sets

$f$ cts, $E$ connxd $\Rightarrow f(E)$ connxd

Thm: [$X, Y$ metric spaces, $f: X \to Y$]

If $f$ is cts, $E \subset X$ is connected $\Rightarrow$ $f(E)$ is connected.

Pf: Say $f(E)$ can be expressed as $A \amalg B$, separated sets.

$\Rightarrow A \cap \bar{B} = \phi, \ \bar{A} \cap B = \phi, \ f(E) = A \cup B.$

$\Rightarrow f^{-1}(A \cap \bar{B}) = \phi, \ f^{-1}(\bar{A} \cap B) = \phi, \ E = f^{-1}(A) \cup f^{-1}(B)$

$\Rightarrow f^{-1}(A) \cap f^{-1}(\bar{B}) = \phi, \ f^{-1}(\bar{A}) \cap f^{-1}(B) = \phi.$

$f$ cts, so $f^{-1}(\bar{B})$ is closed and contains $f^{-1}(B)$ (and thus $\overline{f^{-1}(B)}$)

$\Rightarrow f^{-1}(A) \cap \overline{f^{-1}(B)} = \phi, \ \overline{f^{-1}(A)} \cap f^{-1}(B) = \phi$ (similarly for $A$)

$\Rightarrow$ $E$ can be expressed as separated sets. ※ (contrapositive

Intermediate Value Thm.

Corollary: [$f: [a,b] \to \mathbb{R}$ cts.] Say $f(a) < f(b)$.

For any $c \in (f(a), f(b))$

$\exists x \in (a,b)$ s.t. $f(x) = c$

A cts. fn. assumes all intermediate valu

Pf: $[a,b]$ is connected, and so $f([a,b])$ is as well.

discontinuities

Discontinuities on $\mathbb{R}^1$: (Some are worse than ~~others~~)

right limits left limits

Defn: [$f: (a,b) \to \mathbb{R}^1$ $x \in [a,b)$.] If $\exists q \in \mathbb{R}^1$ s.t. $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if $p \in (x, x+\delta) \Rightarrow f(p) \in N_\varepsilon(q)$

then we say $f(x^+) = q$. Sim. $f(x$

(Rk: $f(x^+) = f(x^-) \Rightarrow \lim_{t \to x} f(t)$ exists.)

simple discontinuities

Defn: If $f$ is discts. at $x$, but $f(x^+), f(x^-)$ exist, we say $f$ has a simple discontinuity at $x$. (1st kind, 2nd kind)

## § Continuity & Connectedness

recall def'n connected set

Recall: $E$ is connected if it cannot be expressed as $A \amalg B$ s.t. $A \cap \bar{B} = \bar{A} \cap B = \phi$. i.e. cannot be expressed as a union of (nonempty) separated sets

$f$ cts, $E$ connxd $\Rightarrow f(E)$ connxd

Thm: [$X, Y$ metric spaces, $f: X \to Y$]

$$\left.\begin{array}{l}\text{If } f \text{ is cts} \\ E \subset X \text{ is connected}\end{array}\right\} \Rightarrow f(E) \text{ is connected.}$$

Pf: Say $f(E)$ can be expressed as $A \amalg B$, separated sets.

$\Rightarrow A \cap \bar{B} = \phi$, $\bar{A} \cap B = \phi$, $f(E) = A \cup B$.

$\Rightarrow f^{-1}(A \cap \bar{B}) = \phi$, $f^{-1}(\bar{A} \cap B) = \phi$, $E = f^{-1}(A) \cup f^{-1}(B)$

$\Rightarrow f^{-1}(A) \cap f^{-1}(\bar{B}) = \phi$, $f^{-1}(\bar{A}) \cap f^{-1}(B) = \phi$.

$f$ cts, so $f^{-1}(\bar{B})$ is closed and contains $f^{-1}(B)$ (and thus $\overline{f^{-1}(B)}$)

$\Rightarrow f^{-1}(A) \cap \overline{f^{-1}(B)} = \phi$, $\overline{f^{-1}(A)} \cap f^{-1}(B) = \phi$ (similarly for $A$)

$\Rightarrow$ $E$ can be expressed as separated sets. ※ (contrapositive)

Intermediate Value Thm.

Corollary: [$f: [a,b] \to \mathbb{R}$ cts.] Say $f(a) < f(b)$.

For any $c \in (f(a), f(b))$

$\exists x \in (a,b)$ s.t. $f(x) = c$

"A cts. fn. assumes all intermediate values"

Pf: $[a,b]$ is connected, and so $f([a,b])$ is as well.

discontinuities

## Discontinuities on $\mathbb{R}^1$: (Some are worse than others)

right limits / left limits

Def'n: [$f: (a,b) \to \mathbb{R}^1$, $x \in [a,b)$.] If $\exists q \in \mathbb{R}^1$ s.t. $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if $p \in (x, x+\delta) \Rightarrow f(p) \in N_\varepsilon(q)$

(Pf: $f(x^+) = f(x^-) \Rightarrow \lim_{t \to x} f(t)$ exists.) Then we say $f(x^+) = q$. Sim.: $f(x^-)$

simple discontinuities

Def'n: If $f$ is discts. at $x$, but $f(x^+)$, $f(x^-)$ exist, we say $f$ has a simple discontinuity at $x$. (1st kind, 2nd kind)

## Monotonic fns and discontinuities:

monotonic fnc

**Defn:** $[f : (a,b) \to \mathbb{R}.]$ If $x<y \Rightarrow f(x) \le f(y)$ $(x,y \in (a,b))$ then we call $f$ monotonically increasing (similarly decreasing)

monotonic $\Rightarrow f(x^-)$ $f(x^+)$ exist

**Thm:** $f$ monotonically increasing on $(a,b) \Rightarrow$

i. $\forall x \in (a,b)$, $f(x^+)$ and $f(x^-)$ exist

ii. $a<x<y<b \Rightarrow f(x^+) \le f(y^-)$

Rk: (i) $\Rightarrow$ thus all discontinuities are simple.

**Pf:** i. Fix $x \in (a,b)$. Claim: $f(x^-) = \sup_{a<t<x} f(t) =: A$.

Pf: Fix any $\varepsilon > 0$.

We know $f(t) > A - \varepsilon$ for some $t = x - \delta$. i.e. $t \in (a,x)$,

Since $f$ is increasing $A - \varepsilon < f(t) < A$

for all $x - \delta < t < x$. ∎

ii. $f(x^+) = \inf_{x<t<b} f(t) = \inf_{x<t<y} f(t)$ $\wedge$. ∎

$f(y^-) = \sup_{a<t<y} f(t) = \sup_{x<t<y} f(t)$

monotonic $\Rightarrow$ discontinuities at most ctble.

**Thm:** $f$ monotonic on $(a,b)$. The set of discontinuities is at most countable

**Pf:** Say $f$ is dcts. at $x$. Discontinuity is simple, $\Rightarrow f(x^-) < f(x^+)$

Archimedean principle $\Rightarrow \exists\, r(x) \in \mathbb{Q}$ s.t. $r(x) \in (f(x^-), f(x^+))$

Further, by (ii) above $x_1 < x_2 \Rightarrow f(x_1^+) \le f(x_2^-)$

$\Rightarrow r(x_1) \ne r(x_2)$.

So the discontinuities $\hookrightarrow \mathbb{Q}$ and are thus at most ctble ∎

Limits and infinity:

Recall: $\lim_{x\to p} f(x) = q$ meant $\forall \varepsilon>0, \exists \delta>0$ s.t.

if $0<d(x,p)<\delta$

then $d(f(x),q)<\varepsilon$

i.e., $\forall N_\varepsilon(q)$, $\varepsilon>0$

$\exists N_\delta(p)$ s.t. $f(N^*_\delta(p)) \subset N_\varepsilon(q)$
$\delta>0$

Q. What about infinity?

Defn: For $c\in\mathbb{R}$, we call $\{x\in\mathbb{R} \mid x>c\}$ a nbhd. of infinity $(c,\infty)$.

(Similarly for $-\infty$)

Defn: $[f:\mathbb{R}\to Y]$ $\lim_{x\to\infty} f(x)=q$ means $\forall N_\varepsilon(q)$, $\varepsilon>0$

$\exists (c,\infty)$ s.t. $f((c,\infty))\subset N_\varepsilon(q)$

i.e. $\forall \varepsilon>0$, $\exists c\in\mathbb{R}$ s.t.

if $x>c$

then $d(f(x),q)<\varepsilon$

Concisely: $[f: E\to\mathbb{R}$. For $A,x \in \mathbb{R}\cup\{\infty\}]$

$\lim_{t\to x} f(t) = A$ means $\forall$ nbhd $\mathcal{U}$ of $A$

$\exists$ nbhd $V$ of $x$ s.t. $V\cap E\neq\emptyset$

and $f(V\cap E)\subset \mathcal{U}$.

# Chapter 5: Differentiation

differentiability of a pt.

**Defn:** $[f: [a,b] \to \mathbb{R}]$ If $\lim_{t \to x} \frac{f(t)-f(x)}{t-x}$ exists, we call it $f'(x)$ and say $f$ is <u>differentiable at $x$</u>.

Properties of $'$:

diff'ble at $x$ $\Rightarrow$ cts at $x$

**Thm:** $[f: [a,b] \to \mathbb{R}]$ $f$ diff'ble at $x \in [a,b] \Rightarrow f$ cts at $x$.

**Pf:** STS $\lim_{t\to x} f(t) = f(x)$. Well, $\lim_{t \to x} f(t) - f(x) =$

$$\lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}\right)(t-x) = f'(x)\cdot 0 = 0 \blacksquare$$

differentiation and arithmetic

**Thm:** [Differentiation and arithmetic] $f, g$ diff'ble at $x \in [a,b]$. Then

a) $(f+g)'(x) = f'(x) + g'(x)$

b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

c) $(f/g)'(x) = \dfrac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$

**Pf;** b) $$\frac{(fg)(t)-(fg)(x)}{t-x} = \frac{f(t)g(t) - f(x)g(x)}{t-x}$$

$$= \frac{f(t)[g(t)-g(x)] + [f(t)-f(x)]g(x)}{t-x}$$

Take limits; use continuity of $f$ at $x$ from above.

c) Similarly. $$\frac{(f/g)(t) - (f/g)(x)}{t-x} = \frac{1}{g(t)g(x)}\left[g(x)\frac{f(t)-f(x)}{t-x} - f(x)\frac{g(t)-g(x)}{t-x}\right.$$

chain rule

**Thm:** [the chain rule] $\left[\begin{array}{l} f: [a,b] \to \mathbb{R} \text{ cts.} \\ g: I \to \mathbb{R} \text{ where } I \supset \mathrm{Im}(f) \end{array}\right]$

$\left.\begin{array}{l}\text{If } f \text{ is diff'ble at some } x \in [a,b] \\ g \text{ diff'ble at } f(x)\end{array}\right\} \Rightarrow g \circ f$ is diff'ble at $x$

and $(g\circ f)'(x) = g'(f(x))f'(x)$.

Alternate view of $f'$

**Remark:** It will be profitable to think of $f'$ as follows:

Say $\lim_{t \to x} \frac{f(t)-f(x)}{t-x} = f'(x)$, i.e. $\lim_{t \to x} \underbrace{\frac{f(t)-f(x)}{t-x} - f'(x)}_{\text{call this } u(t)} = 0$

i.e. "$\exists$ a number $f'(x)$ and a fn. $u(t)$ s.t.

$$\frac{f(t)-f(x)}{t-x} = f'(x) + u(t) \quad \text{and} \quad \lim_{t \to x} u(t) = 0."$$

proof of Chain Rule

Pf: WTS $\lim_{t \to x} \frac{g(f(t))-g(f(x))}{t-x} = g'(f(x))f'(x)$.

Well, $g(f(t))-g(f(x)) = [f(t)-f(x)]\,[g'(f(x)) + v(f(t))]$

$$= [t-x][f'(x)+u(t)][g'(f(x)) + v(f(t))]$$

$$\Rightarrow \quad \frac{g(f(t))-g(f(x))}{t-x} = [f'(x)+u(t)][g'(f(x)) + v(f(t))]$$

$$\xrightarrow{t \to x} f'(x)g'(f(x)). \quad \blacksquare$$

($\because f$ is cts. at $x$)

Ex: $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

$\Rightarrow$ $(x \neq 0)$ $\quad f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$.

$x = 0$ $\quad \frac{f(t)-f(0)}{t-0} = \frac{t^2\sin\frac{1}{t} - 0}{t-0}$

$$= t\sin\tfrac{1}{t}. \to 0$$

by "squeeze thm."

(MVT)

MVT:

local max.

**Defn:** [$f: X \to \mathbb{R}$, $X$ a metric space] [$p \in X$]
If $\exists N_\delta(p)$ s.t. $f(q) \leq f(p)$ $\forall q \in N_\delta(p)$
then we say $f$ has a <u>local maximum</u> at $p$.

critical pt. thm.

Thm: [$f: [a,b] \to \mathbb{R}$] If i) $f$ has a local max at $x \in (a,b)$ ii) $f'(x)$ exists $\Rightarrow f'(x) =$

[Rk. Ie. local max $\Rightarrow$ either $f'(x) = 0$ or $f'(x)$ doesn't exist.

Pf: Read yourself.

Rolle's thm.

Thm: [Rolle's Thm] $f$ cts on $[a,b]$, diff$^{ble}$ on $(a,b)$

If $f(a) = f(b) \Rightarrow \exists c \in (a,b)$ s.t. $f'(c) = 0$.

Pf: Extreme val. thm $\Rightarrow$ $f$ attains max. at some $p \in [a,b]$
min. at some $q \in [a,b]$

If $p, q$ are at endpts. $\Rightarrow$ fn. is constant (done)

If one is not an endpt, call it $c$ ∎

Corollary: Rolle's II.

Corollary: Say $f, g$ are cts. on $[a,b]$, diff$^{ble}$ on $(a,b)$
If $f(a) - f(b) = g(a) - g(b) \Rightarrow \exists c \in (a,b)$ s.t. $f'(c) = g'(c$

Corollary: GMVT

Thm: [GMVT] $f, g$ cts. on $[a,b]$, diff$^{ble}$ on $(a,b)$.
Then $\exists c \in (a,b)$ s.t. $[f(b) - f(a)]\, g'(c) = [g(b) - g(a)]\, f'(c)$.

"Pf": Multiply $g$ by $\alpha = \dfrac{f(b) - f(a)}{g(b) - g(a)}$ (if $g(b) - g(a) = 0$, done by Rolle's Thm.)

Then $f$, $\alpha g$ satisfy the hypotheses of the above corollary. ∎

Rk: Physical interpretation: ratio of average velocities = ratio of instantaneous velocities.

Corollary: MVT

Corollary: [MVT]

Corollary: $f'$ and behavior.

Corollary: [$f$ diff$^{ble}$ in $(a,b)$]

a. $f'(x) \geq 0\ \forall x \Rightarrow$ monotonically increasing

b. $f'(x) = 0\ \forall x \Rightarrow$ constant

c. $f'(x) \leq 0\ \forall x \Rightarrow$ monotonically decreasing

Taylor's Thm: Another statement about the meaning of differentiability.

Segue

**Recall:** $f$ differentiable at $x$ means $\exists f'(x) \in \mathbb{R}$, and a function $r(t)$ s.t.

$$f(t) = \underbrace{f(x) + f'(x)(t-x)}_{\text{tangent line}} + \underbrace{r(t)(t-x)}_{\text{error s.t. } \lim_{t\to x} r(t) = 0}.$$

(Q) What does "n-times differentiable" mean?

Statement

Taylor's Thm: $f : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$.
$f^{(n-1)}$ cts. on $[a,b]$, $f^{(n-1)}$ diff'ble on $(a,b)$.

Let $\alpha \in [a,b]$; define $P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$.

Then for any $\beta \in [a,b]$, $\exists x$ between $\alpha$ and $\beta$ s.t.

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n.$$

motivation

E.g., if $n=1$, we get $f(\beta) = f(\alpha) + \frac{f'(x)}{1!}(\beta-\alpha)^1$.

$n=2$, we get $f(\beta) = f(\alpha) + f'(\alpha)(t-\alpha) + \frac{f''(x)}{2!}(\beta-\alpha)^2$

etc

**Rk:** Notice the case $n=1$ is the MVT;
Taylor's Thm is sort of a generalization of the MVT.

Pf: [Case $n=2$] Define $M$ by $f(\beta) - P(\beta) = \underline{M(\beta-\alpha)^2}$.

Let $g(t) := f(t) - [P(t) + M(t-\alpha)^2]$. (degree n poly'l approx of error at $\beta$.)

Notice: $g^{(2)}(t) = f^{(2)}(t) - [P^{(2)}(t) + 2!\,M]$
$= f^{(2)}(t) - 2!\,M$; so STS $g^{(2)}(t) = 0$ somewhere.

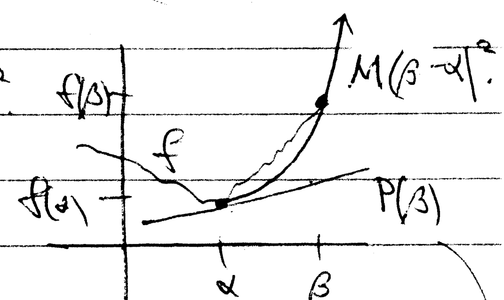
Well, $g(\alpha) = g'(\alpha) = 0$ is obvious.

Since $g(\beta) = 0$, $\exists\ x_1 \in (\alpha, \beta)$ s.t. $g'(x_1) = 0$.

Since $g'(\alpha) = 0$, $g'(x_1) = 0$, $\exists\ x_2 \in (\alpha, x_1)$ s.t. $g^{(2)}(x_2) = 0$.

The general case follows similarly. ∎

<u>Picture</u>: $f(\beta) - P(\beta) = M(\beta)(\beta-\alpha)^2$.




(Hoping that the <u>2nd derivative</u> of $P(t) + M(t-\alpha)^2$ equals that of $f(t)$ at same pt.)

i.e., that $P(t) + M(t-\alpha)^2$ approximates $f(t)$ in some sense.

## The continuity of the derivative:

Rk: Derivatives need not be cts. However, they always possess an intermediate value property (and thus no simple discontinuities.)

(E.g. $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ is diff'ble everywhere, but $f'$ discts. at 0)

Thm: $f : [a,b] \to \mathbb{R}$, $f$ diff'ble on $[a,b]$ (assume $f'(a) < f'(b)$)

For any $\lambda \in (f'(a), f'(b))$, $\exists x \in (a,b)$ s.t. $f'(x) = \lambda$.

Pf: Consider $g(t) := f(t) - \lambda t$. STS $g'(t) = 0$ somewhere $\in (a,b)$

$\Rightarrow$ STS $g$ attains its minimum at some $x \in (a,b)$.

Well, $g'(a) = f'(a) - \lambda < 0$, $g'(b) = f'(b) - \lambda > 0$ $\Rightarrow$ minimum is not attained at endpts.

Clearing up the hand-waving

[Precisely: $g'(a) < 0 \Rightarrow \lim_{t \to a^+} \frac{g(t) - g(a)}{t - a} = g'(a) < 0$.

$\Rightarrow \exists$ nbhd $(a, a+h)$ s.t. $t \in (a, a+h)$ implies $\left| \frac{g(t) - g(a)}{t - a} - g'(a) \right| < \frac{g'(a)}{2}$.

$\Rightarrow$ $g(t) - g(a) < 0$ in that nbhd.]

Since $g$ is cts., it must attain its minimum somewhere, (EVT)

$\Rightarrow$ it must attain it in $(a,b)$ ∎

Comment: Lots of defns & elementary arguments (maybe limit do it order)

# Chapter 3: Numerical Sequences and Series.

convergent sequences.

**Defn:** [ $p: \mathbb{N} \to X$, (also written as $\{p_n\}$) a sequence in a metric space $X$, $p_n := p(n)$ ]

If $\exists$ a point $p \in X$ with the property that

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N} \text{ s.t. } d(p_n, p) < \varepsilon \ \ \forall n \geq N,$$

then we say $\{p_n\}$ <u>converges to $p$</u> in $X$.

<u>Rk</u>: We also say $p_n \to p$ or $\lim_{n\to\infty} p_n = p$.

If such a $p$ doesn't exist we say $\{p_n\}$ <u>diverges</u>.

<u>Rk</u>. Note that convergence depends on $X$

i) in that the limit must be in $X$

ii) in that the convergence depends on $X$'s metric.

E.g.. $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to $0$ in $\mathbb{R}$, but not in $\mathbb{R}^+$ (w/ std. metrics) and doesn't converge at all in the p-adic metric.

implications of convergence

<u>Thm</u> [Implications of convergence] Let $\{p_n\}$ be a sequence in $X$, a metric space

i) $p_n \to p \in X \iff \forall r > 0$, $N_r(p)$ contains almost all of the $\{p_n\}$

ii) If $p_n \to p$, $p_n \to p'$, then $p = p'$ (uniqueness of limit).

iii) If $p_n$ converges, then $\{p_n\}$ is bounded. (<u>Rk</u>: converse false)

iv) If $E \subset X$ and $p \in E' \Rightarrow \exists$ a sequence $\{p_n\}$ in $E$ s.t. $p_n \to p$.

<u>Pf</u>: i) $\Rightarrow$) Say $p_n \to p$. Fix some $N_r(p)$.

By convergence, $\exists N \in \mathbb{N}$ s.t. $d(p_n, p) < r \ \forall n \geq N$ ∎

$\Leftarrow$) Given $\varepsilon > 0$, consider $N_\varepsilon(p)$. By hypothesis, $N_\varepsilon(p)$ contains all but finitely many of the $\{p_n\}$.

Let $N = \max\{p_{n_1}, p_{n_2}, \ldots, p_{n_k}\} + 1$.

Then $n \geq N \Rightarrow p_n \in N_\varepsilon(p)$. ∎

ii) - iv) Exercise in class. (continuous analogues already done.)

## Sequences in Euclidean Spaces:

Complex series — Thm: [Convergence & arithmetic] Let $s_n \to s$, $t_n \to t$ in $\mathbb{C}$. Then

i) $\lim_{n\to\infty} (s_n + t_n) = s + t$.

ii) $\lim_{n\to\infty} cs_n = cs$ $(\forall c \in \mathbb{C})$

iii) $\lim_{n\to\infty} s_n t_n = st$

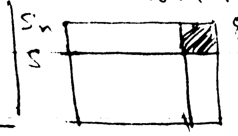
iv) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$ (as long as $s_n \neq 0$ $\forall n$, $s \neq 0$.)

(FACTOR OUT THE KNOWN) Recognize how the known factors into the problem

products — Pf: iii) Think: $s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)$ what's really going on?

(since $s(t_n - t), t(s_n - s) \to 0$ are obvious)

STS $\lim_{n\to\infty} s_n t_n - st = 0$; by (ii), STS $\lim_{n\to\infty} (s_n - s)(t_n - t) = 0$.





Given $\varepsilon > 0$, $\exists N_1, N_2 \in \mathbb{N}$ s.t. $|s_n - s| < \sqrt{\varepsilon}$; $n > N_1$

$|t_n - t| < \sqrt{\varepsilon}$; $n > N_2$.

Where is the known?

Then $|(s_n - s)(t_n - t)| < \varepsilon$ $\forall n > \max\{N_1, N_2\}$

quotients — iv) Given $\varepsilon > 0$, want an $N$ s.t. $n \geq N$ implies

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| < \varepsilon. \quad \text{Notice } \left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s_n - s}{s_n s}\right|$$

the known

a) We can choose $N_1$ s.t. $n \geq N_1 \Rightarrow |s_n - s| < \frac{1}{2}|s|^2 \varepsilon$.

b) Choose $N_2$ s.t. $n \geq N_2 \Rightarrow |s_n - s| < \frac{1}{2}|s|$.

$$\Rightarrow |s_n| > \tfrac{1}{2}|s|.$$

Then for $n \geq \max\{N_1, N_2\}$,

$$\left|\frac{s_n - s}{s_n s}\right| < \frac{\frac{1}{2}|s|^2 \varepsilon}{\frac{1}{2}|s|^2} = \varepsilon.$$

vector convergence $\Leftrightarrow$ component-wise convergence.

Thm: [Sequences in $\mathbb{R}^k$]

a. Let $\vec{x}_n \in \mathbb{R}^k$, $n = 1, 2, 3, \ldots$ be a sequence; notate $\vec{x}_n = (\alpha_{1,n}, \alpha_{2,n}, \ldots, \alpha_{k,n})$.

Then $\lim_{n\to\infty} \vec{x}_n = \vec{x}$ $\left(= (\alpha_1, \alpha_2, \ldots, \alpha_k)\right)$

$$\Longleftrightarrow \lim_{n\to\infty} \alpha_{i,n} = \alpha_i \text{ for } i = 1, \ldots, k.$$

b. $\lim_{n\to\infty} \vec{x}_n + \vec{y}_n = \vec{x} + \vec{y}$, $\lim_{n\to\infty} \vec{x}_n \cdot \vec{y}_n = \vec{x}\cdot\vec{y}$.

(recognize rel'n between known & unknown.)

Pf of a: $\Rightarrow$) $\forall \varepsilon > 0, \exists N$ s.t. $n \geq N \Rightarrow \|\vec{x}_n - \vec{x}\| < \varepsilon$.

Since $|\alpha_{i,n} - \alpha_i| \leq \|\vec{x}_n - \vec{x}\| < \varepsilon$, we're done.

$\Leftarrow$) Choose $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |\alpha_{i,n} - \alpha_i| < \frac{\varepsilon}{\sqrt{k}}$; $i = 1, \ldots, k$.

Then $\|\vec{x}_n - \vec{x}\| = \left\{\sum_{i=1}^{k} |\alpha_{i,n} - \alpha_i|^2\right\}^{1/2} < \varepsilon$.

## Subsequences:

subsequences

Def'n: [$\{p_n\}$ a sequence.] If $\{n_k\}_{k=1}^{\infty}$ is a strictly increasing sequence in $\mathbb{N}$, then we call $\{p_{n_k}\}_{k=1}^{\infty}$ a <u>subsequence</u> of $\{p_n\}$.

Rk: $\{p_n\}$ converges to $p \Leftrightarrow$ every subsequence $p_{n_k} \to p$.

sequence in compact space $\Rightarrow \exists$ cvgnt. subseq.

Thm: [$\{p_n\}$ a sequence in $X$ a metric space]

If $X$ is compact $\Rightarrow$ some subsequence of $\{p_n\}$ converges (in $X$).

Pf: Let $E$ denote the range of $\{p_n\}$. $E$ finite $\Rightarrow$ done.

Exercise

$E$ infinite $\Rightarrow$ $E$ has a limit pt. $p \in X$.

Choose $n_1$ s.t. $d(p, p_{n_1}) < 1$; $n_2 > n_1$ s.t. $d(p, p_{n_2}) < \frac{1}{2}$, etc.

Corollary: Every <u>bdd</u> sequence in $\mathbb{R}^k$ has a convergent subsequence.

## Cauchy sequences: (Checking convergence w/o prior knowledge of the limit.)

Cauchy sequence — **Defn:** [$\{p_n\}$ a sequence in $X$, a metric space]

If $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n, m > N \Rightarrow d(p_n, p_m) < \varepsilon$.

then we call $\{p_n\}$ a Cauchy sequence.

Rk: We shall see that in $\mathbb{R}^k$ $\{p_n\}$ converges $\Leftrightarrow$ $\{p_n\}$ is Cauchy.

diameter — **Defn:** $E \subset X$, metric space. Let $S = \{ d(p,q) \mid p, q \in E \}$.

We define diam $(E) = \sup S$.

Geometric interp$^n$ of Cauchy seq. — Rk: Thus $\{p_n\}$ is Cauchy $\Leftrightarrow \lim_{N \to \infty} \operatorname{diam} E_N = 0$,

where $E_N := \{p_N, p_{N+1}, p_{N+2}, \dots\}$ (the tail)

Facts about diam — **Thm:** [Facts about diam]

i) [$E \subset X$] $\operatorname{diam} \bar{E} = \operatorname{diam} E$.

ii) $\{K_n\}$ a sequence of nested sets in $X$.

If a) the $K_n$ are compact, b) $\lim_{n \to \infty} \operatorname{diam}(K_n) = 0$ $\Rightarrow \bigcap_{n=1}^{\infty} K_n$ consists of exactly one point.

Q. Why is it intuitively obvious?

Pf: i) STS $\operatorname{diam} \bar{E} \leq \operatorname{diam} E$. Pick any $p, q \in \bar{E}$.

Given any $\varepsilon > 0$, $\exists p', q' \in E$ s.t. $d(p, p') < \frac{\varepsilon}{2}$, $d(q, q') < \frac{\varepsilon}{2}$.

$$\Rightarrow d(p,q) \leq d(p,p') + d(p',q') + d(q',q)$$
$$< d(p',q') + \varepsilon \leq \operatorname{diam}(E) + \varepsilon.$$

$\Rightarrow \operatorname{diam} \bar{E} \leq \operatorname{diam}(E) + \varepsilon \quad \forall \varepsilon > 0$.

ii) Since $\{K_n\}$ satisfy the f.i.p., $K = \bigcap K_n$ is nonempty.

If $K = \bigcap K_n$ contained two distinct points then $\operatorname{diam}(K) > 0$.

But $\operatorname{diam} K_n \geq \operatorname{diam} K$ and $\operatorname{diam} K_n \to 0$ ※.

convergent $\Rightarrow$ Cauchy

**Thm:** [$\{p_n\}$ a sequence in $X$, metric sp.] $\{p_n\}$ cvges $\Rightarrow$ $\{p_n\}$ is Cauchy

Pf: Given $\varepsilon > 0$, choose $N$ s.t. $n \geq N$ implies $d(p_n, p) < \frac{\varepsilon}{2}$.

Then $m, n \geq N \Rightarrow d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < \varepsilon$.

Cauchy $\not\Rightarrow$ convergent

Rk: It is not true that $\{p_n\}$ Cauchy $\Rightarrow$ $\{p_n\}$ converges.
(e.g. $p_n = \frac{1}{n}$ in $\mathbb{R}^+$.) (e.g., $1, 1.4, 1.41, \ldots$ in $\mathbb{Q}$) or $\sum_{n=1}^{\infty} \frac{1}{n^2} \to \frac{\pi^2}{6}$.

complete

**Defn:** [$X$ a metric space] If every Cauchy sequence converges in $X$, we say $X$ is <u>complete</u>.

E.g. $\mathbb{Q}$ with $d(x,y) = |x-y|$ is not complete

compact metric sp. $\Rightarrow$ complete

**Thm:** a) $X$ a compact metric space $\Rightarrow$ $X$ is complete
b) $\mathbb{R}^k$ is complete.

Pf: a) Let $\{p_n\}$ be a Cauchy sequence. Why must there be a limit in $X$?
(i.e., if $E_N = \{p_N, p_{N+1}, p_{N+2}, \ldots\}$, then $\lim_{N\to\infty} \operatorname{diam}(E_N) = 0$)

Consider $\overline{E}_N$; $N = 1, 2, 3, \ldots$.

Each $\overline{E}_N$ is compact (why?), $\overline{E}_N \supset \overline{E}_{N+1}$, and $\lim_{N\to\infty} \operatorname{diam}(\overline{E}_N) = 0$

$\Rightarrow \bigcap_{N=1}^{\infty} \overline{E}_N = p \in X$.

Now, given any $\varepsilon > 0$, choose $N_0$ s.t. $N \geq N_0 \Rightarrow \operatorname{diam}(\overline{E}_N) < \varepsilon$.

$\Rightarrow d(p_n, p) < \varepsilon$ for all $n \geq N_0$; thus $p_n \to p$.

obvious corollary: $\mathbb{R}^k$ is complete.

b) Let $\{p_n\}$ be a Cauchy sequence in $\mathbb{R}^k$.
It suffices to show $\{p_n\}$ is bdd; for then $\{p_n\} \subset \overline{B_R(0)}$ for some $R > 0$. ← cpt.
Why is $\{p_n\}$ bdd?

$\{p_n\}$ Cauchy $\Rightarrow$ for some $N$, $\operatorname{diam}(E_N) < 1$.

Since $\{p_n\} = \{p_1, p_2, \ldots, p_{N-1}\} \cup E_N$,
$\{p_n\}$ is bdd.

For p prime, $a \in \mathbb{Z}$, $\operatorname{ord}_p(a)$ = highest power

Rk: Another way of constructing $\mathbb{R}$ is as the completion of $\mathbb{Q}$ w.r.t. $|x-y|$.
(If we take a different metric, and complete $\mathbb{Q}$, what happens? (p-adics)

Question — Rk: Convergent sequences are bdd. Are bdd sequences convergent?

monotonic sequences — Defn: $\{s_n\}$ a sequence in $\mathbb{R}$. If $s_n \leq s_{n+1}$; $n=1,2,3,\ldots$

then we say $\{s_n\}$ is <u>monotonically increasing</u>. (similarly for dec.)

monotone sequence theorem — Thm [Monotonic sequence thm.] $\{s_n\}$ a monotonic sequence.

$\{s_n\}$ converges $\iff$ $\{s_n\}$ is bdd.

Pf: $\Leftarrow$) (WLOG $\{s_n\}$ increasing) $\{s_n\}$ bdd (above), so $E_1 = \{s_1, s_2, \ldots\}$ has lub (why?)

Claim: $s_n \to \sup(E_1)$.

Pf: Given $\varepsilon > 0$, we know $\exists N$ s.t. $\sup(E_1) - \varepsilon < s_N < \sup(E_1)$.

Since $\{s_n\}$ is increasing and $\sup(E_1)$ an upper bd, (why?)

$\sup(E_1) - \varepsilon < s_N \leq s_n < \sup(E_1)$ for all $n > N$;

thus $s_n \in N_\varepsilon(\sup(E_1))$ $\forall n > N$ ∎

limit superior limit inferior — <u>lim inf, lim sup</u>:

"convergence" to infinity — <u>Notation</u>: $\{s_n\}$ a sequence of real numbers

If $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ s.t. $s_n \geq M$ for all $n \geq N$,

then we write $s_n \to +\infty$. (similarly for $-\infty$)

define limsup/inf. — Defn: $\{s_n\}$ a sequence of real numbers.

Let $E \subset \mathbb{R} \cup \{\pm\infty\}$ denote the set of all $x$ s.t. $s_{n_k} \to x$ for some subsequence $\{s_{n_k}\}$ (all subsequential limits)

We define the <u>limit superior</u>, $\limsup_{n\to\infty} s_n = \sup(E)$

(also notated $\overline{\lim}\, s_n$.)

Similarly $\liminf_{n\to\infty} s_n = \inf(E)$.

Example — E.g. $\{s_n = (-1)^n\}$ then $\overline{\lim}\, s_n = 1$, $\underline{\lim}\, s_n = -1$

Commentary — Rk: Unlike the limit, for any sequence $\{s_n\}$, the $\overline{\lim}$ and $\underline{\lim}$ always exist (proof?)

properties of lim sup.

**Thm:** [Character of the limit superior] $\{s_n\}$ a sequence in $\mathbb{R}$, $E$ as above.

i) $\limsup_{n\to\infty} s_n \in E$. (i.e., some subsequence converges to it)

ii) For any $x > \limsup_{n\to\infty} s_n$, $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow s_n < x$ (we can bound above)

(Further, any number satisfying i & ii must be $s^*$.)

lim sup is a subsequential limit (or infinite)

Pf: i) Cases: $s^* = +\infty, -\infty$, or is finite.

$s^* = +\infty$: In this case, $E$ is not bdd above

$\Rightarrow \{s_n\}$ not bdd above $\Rightarrow \exists$ a subseq. $s_{n_k} \to \infty$.

$s^* = -\infty$: Then $E = \{-\infty\}$. $\to \exists$ subseq. $s_{n_k} \to -\infty$.

In case $s^* \in \mathbb{R}$ we need the following lemma:

<u>Lemma:</u> (3.7) The subsequential limits of $\{p_n\}$ in $X$, metric space, form a closed subset.

Then, since $\sup E$ is a limit pt. of the subsequential limits, it itself is a subsequential limit ($\in E$)

Although we can't get arbitrarily close, we can find an upper bound arbitrarily close to lim

ii) By contradiction: Suppose that for each $N \in \mathbb{N}$, $\exists n > N$ s.t. $s_n \geq x > s^*$,

$\Rightarrow s_n \geq x$ for infinitely many indices $n$.

$\Rightarrow E$ contains a number $y$ s.t. $y \geq x > s^*$ $\times$. ■

uniqueness

<u>Uniqueness</u> is obvious: If $p, q$ possess properties i and ii, say $p < q$, then for any $x$ s.t. $p < x < q$,

$\exists N$ s.t. $s_n < x \quad \forall n \geq N$.

But then $q$ doesn't possess property (i).

useful thm.

**Thm:** $\{s_n\}, \{t_n\}$ sequences. If $\exists N$ s.t. $s_n \leq t_n \ \forall n \geq N$, then

$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n$; $\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n$.

(rel'n to series)

Rk: Why use $\overline{\lim}$, $\underline{\lim}$? Many facts you know about series which depended on lim actually still hold for the more general $\overline{\lim}$.

Common Sequences: (Exploiting the binomial theorem)

Rk: [Squeeze Thm.] If a) $0 \le x_n \le s_n \ \forall n > N$ b) $s_n \to 0$ $\Rightarrow x_n \to 0$. (Pf?)

1. $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for all $p > 0$. Pf: Binomial thm.

(trivial one.)

Pf: Given $\varepsilon > 0$, take any $N > \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}$.

Then $n > N \Rightarrow \frac{1}{n} < \frac{1}{N} < \frac{1}{\left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}} \Rightarrow \frac{1}{n^p} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$..

2. $\lim_{n\to\infty} p^{\frac{1}{n}} = 1$ for all $p > 0$.

using the ~~first~~ part of the binomial thm. (0th & 1st term)

Pf: [Case $p > 1$.] WTS $x_n := p^{\frac{1}{n}} - 1$ converges to 0.

Notice: $(1+x_n)^n = p$ ;

Binomial thm. $\Rightarrow$ $1 + nx_n \le (1+x_n)^n = p$; $0 \le x_n \le \frac{p-1}{n}$. ∎

3. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

the 2nd term of the binomial thm.

Pf: Let $x_n = n^{\frac{1}{n}} - 1$, WTS $x_n \to 0$. [Can we do the same thing?]

$(1+x_n)^n = n \Rightarrow 1 + nx_n \le n \Rightarrow x_n \le \frac{n-1}{n}$ ... no good.

But we do know $\Rightarrow$ $\frac{n(n-1)}{2} x_n^2 < n \Rightarrow \quad \Rightarrow x_n < \sqrt{\frac{2}{n-1}}$.

4. $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$ for $p > 0$, $\alpha \in \mathbb{R}$.

the general term

Pf: Let $k = [\alpha] + 1$. For $n > 2k$,

$$\frac{n^\alpha}{(1+p)^n} < \frac{n^\alpha}{\binom{n}{k} p^k} = \frac{n^\alpha k!}{\underbrace{n(n-1)\cdots(n-k+1)}_{k \text{ terms each} > \frac{n}{2}} p^k} < \frac{2^k k!}{p^k} n^{\alpha - k}$$

$\alpha - k < 0$, so as $n \to \infty$, get 0, by (1).

5. $\lim_{n\to\infty} x^n = 0$ for $|x| < 1$.

Teach children [illegible]
themselves

Recognize the individual.
Artistry must be approached
through craftsmanship
(2 senses: the only way / without good approach, craftsmanship is empty)

## Series.

$\sum_{n=p}^{q} a_n$

**Notation.** Given a sequence $\{a_n\}$ of complex numbers, let $\sum_{n=p}^{q} a_n = a_p + a_{p+1} + \dots + a_q$. (finite sum)

seq. of partial sums

**Defn:** [$\{a_n\}$ a sequence] The sequence $\{s_n\}$ of partial sums of $\{a_n\}$, is defined $s_n := \sum_{i=1}^{n} a_i$.

series convergence

If $\{s_n\}$ converges to some $s$, we write $\sum_{n=1}^{\infty} a_n = s$ and say "the infinite series sums to $s$."

**Point:** Convergence of $\sum a_i \Leftrightarrow$ convergence of the sequence $\{s_n\}$.

translating results

Translating results on sequences to results on series:

Cauchy Criterion

**Thm:** [Cauchy Criterion] $\sum a_n$ converges $\Leftrightarrow \forall \varepsilon > 0 \; \exists N$ s.t. $\left|\sum_{k=n}^{m} a_k\right| < \varepsilon$ for all $m \geq n \geq N$.

**Pf:** Obvious: $\sum a_n$ converges $\Leftrightarrow \{s_n\}$ converges $\Leftrightarrow \{s_n\}$ is Cauchy

$\Leftrightarrow \forall \varepsilon > 0, \exists N$ s.t. $m, n \geq N \Rightarrow s_n - s_m < \varepsilon$

$\Leftrightarrow \forall \varepsilon > 0, \exists N$ s.t. $m \geq n \geq N \Rightarrow \left|\sum_{k=n}^{m} a_k\right| < \varepsilon$

Corollary: $n^{th}$ term test

**Corollary:** [$n^{th}$ term test] $\sum a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$

main thm for nonnegative series

**Thm:** [Nonnegative series] $\{a_n\}$ a nonnegative series

$\sum a_n$ converges $\Leftrightarrow \{s_n\}$ is bdd.

**Pf:** $\{s_n\}$ is monotonic, so $\{s_n\}$ cvges $\Leftrightarrow \{s_n\}$ bdd.

obvious but quite
Some useful theorems:

comparison test

Thm: [Comparison Test] $\{a_n\}, \{c_n\}, \{d_n\}$ sequences.

i. Say $|a_n| \leq c_n \; \forall n \geq$ some $N_0$. Then $\sum c_n$ converges $\Rightarrow \sum a_n$ cvges.

ii. Say $a_n \geq d_n \geq 0 \; \forall n \geq$ some $N_0$. Then $\sum d_n$ diverges $\Rightarrow \sum a_n$ diverges

Pf: i. $\sum c_n$ converges $\Rightarrow \forall \varepsilon > 0, \exists N$ s.t. $m, n > N \Rightarrow \sum_{k=m}^{n} c_k < \varepsilon$.

But then $\left|\sum_{k=m}^{n} a_k\right| \leq \sum_{k=m}^{n} |a_k| \leq \sum c_k < \varepsilon$,

so $\sum a_k$ is Cauchy too.

Rk: $\{a_n\}$ could be complex.

ii. Follows from i. (Prove ii. by contraposition.)

lacunary test (Cauchy)

Thm: [Lacunary thm.] $\{a_n\}$ a nonnegative, nonincreasing sequence.

$$\sum_{n=1}^{\infty} a_n \text{ converges} \Leftrightarrow \sum_{k=0}^{\infty} 2^k a_{2^k} \text{ converges.}$$

What's the intuition

Why is this obvious?

Pf. Let $s_n = a_1 + a_2 + \dots + a_n$;

$t_k = a_1 + 2a_2 + \dots + 2^k a_{2^k}$.

Now, notice: for any pair $n, k$ s.t. $n < 2^k$, $s_n \leq t_k$.

Because
$$\begin{aligned} s_n &= a_1 + \dots + a_n \\ &\leq a_1 + \dots + a_n + \dots + (a_{2^k} + \dots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + \dots + 2^k a_{2^k} = t_k \end{aligned}$$

So $\{s_n\}$ unbdd $\Rightarrow \{t_k\}$ unbdd.

Similarly, $n > 2^k \Rightarrow s_n \geq \frac{1}{2} t_k$,

since
$$\begin{aligned} s_n &\geq a_1 + a_2 + (a_3 + a_4) + \dots + (a_{2^{k-1}+1} + \dots + a_{2^k}) \\ &\geq \tfrac{1}{2} a_1 + a_2 + 2a_4 + \dots + 2^{k-1} a_{2^k} \\ &= \tfrac{1}{2} t_k. \end{aligned}$$

Rk: An obvious thing, but quite useful.

So $\{t_k\}$ unbdd $\Rightarrow \{s_n\}$ unbdd.

nonneg.-series  Examples of series of nonnegative terms:

geometric  **Thm:** [Geometric series] $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $x \in [0,1)$. ($x \geq 1 \Rightarrow$ diverges)

**Pf:** $S_n = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \dots + x^n = \frac{1-x^{n+1}}{1-x}$. Let $n \to \infty$.

If $x=1 \Rightarrow$ divergence is clear. $x>1$?

p-series (relies on geom.)  **Thm:** [p-series] $\sum \frac{1}{n^p}$ converges $\Leftrightarrow p>1$.

**Pf:** $p \leq 0 \Rightarrow$ diverges (obvious). Assume $p>0$.

The "lacunary" test $\Rightarrow$ suffices to check convergence of $\sum_{k=0}^{\infty} 2^k \frac{1}{2^{kp}}$

$= \sum_{k=0}^{\infty} 2^{(1-p)k}$, which is a geometric series.

$2^{1-p} < 1 \Leftrightarrow 1-p<0$ ■

(relies on p-series)  **Thm:** $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges $\Leftrightarrow p>1$.

**Pf:** We have decreasing terms, so can again apply the lacunary thm: since

$$\sum_{k=1}^{\infty} 2^k \frac{1}{2^k(\log 2^k)^p} = \sum_{k=1}^{\infty} \frac{1}{(k\log 2)^p} = \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}$$

we are done ■

**Rk:** Read section on "e" yourself.

## Root and Ratio Tests:

**Thm:** [Root Test] Given $\sum a_n$, let $\alpha = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}$.

i. $\alpha < 1 \Rightarrow \sum a_n$ converges

ii. $\alpha > 1 \Rightarrow \sum a_n$ diverges

iii. $\alpha = 1 \Rightarrow$ no information (Pf. by example: $\sum \frac{1}{n}$, $\sum \frac{1}{n^2}$)

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[Motivation]

pf. of root test

Pf: Suppose we had that $\sqrt[n]{|a_n|} = \alpha$, i.e. $|a_n| = \alpha^n$.

In that case, (i) and (ii) are clear.

We don't have equality, but the lim sup will suffice:

i. $\alpha < 1 \Rightarrow \exists \beta$ s.t. $\alpha < \beta < 1$.

$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \alpha$, so $\exists N$ s.t. $\sqrt[n]{|a_n|} < \beta \ \forall n \geq N$.

I.e., $|a_n| < \beta^n \ \forall n \geq N$.

By the comparison test since $\sum \beta^n$ converges, done. ☒

ii. $\alpha > 1$. We know $\exists$ a subsequence $\sqrt[n_k]{|a_{n_k}|} \to \alpha$ (why?)

$\Rightarrow |a_{n_k}| > 1$ for all $k > K$ (some $K$)

$\Rightarrow |a_n| > 1$ for infinitely many $n \Rightarrow$ diverges ☒

ratio test

Thm: [Ratio Test] The series $\sum a_n$

i. converges if $\overline{\lim_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| < 1$

ii. diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for $n \geq n_0$ (some fixed $n_0$)

Rk: The weaker test, and inconclusive if $\rho > 1$.

Rk: Note does not say diverges if...

[Motivation]

pf. of ratio test

Pf: Again, suppose we had that $\left|\frac{a_{n+1}}{a_n}\right| = \alpha < 1$; i.e. $|a_{n+1}| = \alpha |a_n|$ for all $n$

I.e., $|a_{n+2}| = \alpha^2 |a_n|$, etc.

Then we'd have a geometric series and the result would be clear.

Again, the lim sup will suffice.

i. $\overline{\lim_{n\to\infty}} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \exists \beta < 1$ s.t. $\left|\frac{a_{n+1}}{a_n}\right| < \beta \ \forall n >$ some $N$.

$\Rightarrow |a_{N+1}| < \beta |a_N|$, $|a_{N+2}| < \beta^2 |a_N|$, etc.

Since $\sum_{k=1}^{\infty} \beta^k |a_N|$ converges, $\sum_{k=1}^{\infty} |a_{N+k}|$ does. ☒

ii. $\left|\frac{a_{n+1}}{a_n}\right| \geq 1 \Rightarrow a_n \not\to 0$. ☒

Rk: Though both seem to be based on geom. series, the tests are not equivalent.
Though ratio test is easier to use, In fact root test is the more powerful.

(Very illustrative example!) **Example:** $\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{2^4}+\frac{1}{3^4}+\dots$

$$\limsup_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{2^n}} = \frac{1}{\sqrt{2}} \implies \text{converges.}$$

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{3}{2}\right)^n = \infty \implies \text{does not apply(!)}$$

<u>Rk</u>: $\underline{\lim} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{3^n}} = \frac{1}{\sqrt{3}}$; $\underline{\lim} \frac{a_{n+1}}{a_n} = {}^{2}\lim_{n\to\infty}\left(\frac{2}{3}\right)^n = 0$

Rk: What if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$?

application of root test: <u>Another application:</u> Power series.

<u>Def'n:</u> A series of the form $\sum_{n=0}^{\infty} c_n z^n$ ($\{c_n\}$ complex numbers (called <u>coefficients</u>), $z$ a complex variable) is called a <u>power series</u>.

Radius of convergence. Rk: We'll leave detailed discussion for $\mathbb{C}$ &, but ...

disk of convergence <u>Thm:</u> [Circle of convergence] $\sum c_n z^n$ a power series;

let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$, $R = \frac{1}{\alpha}$ (If $\alpha = 0$, $R := \infty$, $\alpha = \infty$ $R := 0$)

<u>Claim:</u> $\sum c_n z^n$ converges in $|z| < R$, diverges if $|z| > R$.

(conseq. of root test) <u>Pf</u>: By the root test, we simply need consider

$$\limsup_{n\to\infty} \sqrt[n]{|c_n z^n|} = \frac{|z|}{R}. \quad \blacksquare$$

radius of convergence Rk: R is called the <u>radius of convergence</u> of $\sum c_n z^n$.

illustrative examples <u>Examples</u>: i. $\sum z^n$ diverges for $|z| = 1$.

(All have $R=1$.) ii. $\sum \frac{z^n}{n}$ diverges for $z = 1$, converges for all other $|z| = 1$!

iii. $\sum \frac{z^n}{n^2}$ converges for $|z| = 1$.

## Series of the form $\sum a_n b_n$. (Term-by-term products)

(describing a section of the infinite tail)

**Lemma:** [Partial summation formula] $\{a_n\}, \{b_n\}$ sequences.

Notate $A_{-1}=0$, $A_n = \sum_{k=0}^{n} a_k$; $n \geq 0$.

**Claim:** $$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

**Pf:** $$\text{RHS} = \sum_{n=p}^{q-1} A_n b_n + A_q b_q - \sum_{n=p}^{q-1} A_n b_{n+1} - A_{p-1} b_p \quad \text{(trivial) rearrangement}$$

$$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} \quad \text{couple}$$

$$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1} b_n \quad \text{reindex}$$

$$= \sum_{n=p}^{q} \underbrace{[A_n - A_{n-1}]}_{\text{combine}} b_n = \sum_{n=p}^{q} a_n b_n. \quad \text{done}$$

Rk: This trivial lemma is just a matter of notation ... but it's quite useful:

(convergence of a "product" (term-by-term))

**Thm:** If
a. $\{A_n := \sum_{k=0}^{n} a_k\}$ is bdd.
b. $b_0 \geq b_1 \geq \dots$
c. $\lim_{n\to\infty} b_n = 0$

$\Rightarrow \sum a_n b_n$ converges.

**Pf:** Let $M$ bound $|A_n|$ $\forall n$. Fix $\varepsilon > 0$.

$\lim_{n\to\infty} b_n = 0 \Rightarrow \exists N$ st. $b_N < \frac{\varepsilon}{2M}$.

(a)

Then $q \geq p \geq N \Rightarrow \left|\sum_{n=p}^{q} a_n b_n\right| \underset{\text{(lemma)}}{=} \left|\sum_{p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right|$

$$\leq \sum_{n=p}^{q-1} |A_n(b_n - b_{n+1})| + |A_q b_q| + |A_{p-1} b_p|$$

telescoping series!

$$\leq M \left[ \sum_{n=p}^{q-1} b_n - b_{n+1} + b_q + b_p \right] = M \cdot 2b_p.$$

$$\leq 2Mb_N < \varepsilon.$$

Rk: Pretty slick! By Cauchy Criterion, done. ∎

applications:

Applications of the above:

alternating series test

Thm: [Leibniz's Thm.] $\sum c_n$ a series.

If i. $|c_1| \geq |c_2| \geq |c_3| \geq \cdots$

ii. $c_{2m-1} \geq 0,\ c_{2m} \leq 0$ (ie. an alternating series)

iii. $\lim_{n\to\infty} c_n = 0$

Then $\sum a_n$ converges.

Pf: Let $a_n = (-1)^{n+1}$, $b_n = |c_n|$.

Then $a_n b_n = c_n$, and $\{a_n\}, \{b_n\}$ satisfy the above.

vanishing decreasing coefficient power series.

Thm: $\sum c_n z^n$ a power series with radius of convergence 1.

If i. $c_0 \geq c_1 \geq c_2 \geq \cdots$

ii. $\lim_{n\to\infty} c_n = 0$

$\Rightarrow \sum c_n z^n$ converges on $\{|z|=1\} \setminus \{1\}$.

Pf: In the above thm, take $a_n = z^n$, $b_n = c_n$.

Then $|A_n| = \left| \sum_{k=0}^{n} z^k \right| = \left| \frac{1-z^{n+1}}{1-z} \right| \leq \frac{2}{|1-z|}$,

ie. for $z \neq 1$, $|A_n|$ is bdd $\forall n$.

Rk: We used $|z|=1$ for the last inequality. ($\Delta \neq$)

Q What could we say of the radius $<1$?

example

Ex: $\sum \frac{z^n}{n}$ converges $\forall |z|=1$ except $z=1$.

## Absolutely convergent series:

abs. convergence  **Defn:** If $\sum |a_n|$ converges, we say $\sum a_n$ converges absolutely.

stronger than convergence  **Thm:** $\sum a_n$ converges absolutely $\Rightarrow$ $\sum a_n$ converges.

**Pf:** Cauchy Criterion.

**Rk:** "We shall see that we may operate [arithmetically] with absolutely convergent series very much as with finite sums." (multiplication, rearrangements)

Series and Arithmetic:

addition, scalar mult.  **Thm:** $\sum a_n = A, \sum b_n = B \Rightarrow \sum (a_n + b_n) = A+B, \sum c a_n = cA$.

Cauchy product  **Defn:** $\sum a_n, \sum b_n$ series. Let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

We call $\sum c_n$ the (Cauchy) product of $\sum a_n$ and $\sum b_n$.

Q. If $\sum a_n, \sum b_n$ converge, does the product? Not necessarily (see p. 73, bottom)

abs. convergent $\Rightarrow$ Cauchy product converges  **Thm [Mertens]:** $\left(\sum a_n = A, \sum b_n = B, \; c_n := \sum_{k=0}^{n} a_k b_{n-k}\right)$

If $\sum a_n$ converges absolutely then $\sum c_n = AB$.

**Pf:** Let us first introduce some notation: $A_n = \sum_{k=0}^{n} a_k$; $B_n$, $C_n$; $\beta_n = B_n - B$.

$\gamma_n := a_0\beta_n + a_1\beta_{n-1} + \dots + a_n\beta_0$.

reduction  Step I: Reduction of problem: STS $\lim_{n\to\infty} \gamma_n = 0$

Pf: $C_n := a_0b_0 + (a_0b_1 + a_1b_0) + \dots + (a_0b_n + a_1b_{n-1} + \dots + a_nb_0)$

$= a_0B_n + a_1B_{n-1} + \dots + a_nB_0$

$= a_0(B+\beta_n) + a_1(B+\beta_{n-1}) + \dots + a_n(B+\beta_0)$

$= A_nB + \gamma_n$. Thus STS $\gamma_n \to 0$.

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$= a_0B_n + a_1B_{n-1} + \dots + a_nB_0$

$= a_0(B+\beta_n) + a_1(B+\beta_{n-1}) + \dots + a_n(B+\beta_0)$

$= A_nB + \gamma_n$. Thus STS $\gamma_n \to 0$.

<u>Step II.</u> <u>Claim:</u> $\lim_{n\to\infty} \gamma_n = 0$

<u>Pf:</u> Let $\alpha = \sum_{n=0}^{\infty} |a_n|$. Fix $\varepsilon > 0$.

Since $\beta_n \to 0$, $\exists N$ s.t. $|\beta_n| \le \varepsilon \ \forall n \ge N$.

$$\Rightarrow |\gamma_n| \le \left|a_0\beta_n + a_1\beta_{n-1} + \dots + a_{n-N-1}\beta_{N+1}\right| + \left|a_{n-N}\beta_N + \dots + a_n\beta_0\right|$$

$$\le \varepsilon\alpha + \left|a_{n-N}\beta_N + \dots + a_n\beta_0\right|.$$

Take $n\to\infty$; get $\limsup_{n\to\infty} |\gamma_n| \le \varepsilon\alpha$ (since $a_k \to 0$)

$$\Rightarrow \lim_{n\to\infty} \gamma_n = 0.$$

result of Abel

<u>Thm:</u> [Abel] $\sum a_n$, $\sum b_n$ convergent series; $\sum c_n$ the Cauchy product. If $\sum c_n$ converges, it converges to $(\sum a_n)(\sum b_n)$.

<u>Rearrangements:</u>

# Chapter 7: Sequences and Series of Functions.

ptwise convergence

**Defn**: $E \subset X$ a metric space, $\{f_n : E \to \mathbb{C}\}$ a sequence of functions. If $\lim_{n\to\infty} f_n(x)$ exists for each $x \in E$, (call it $f(x)$) then we say $\{f_n\}$ <u>converges pointwise</u> to $f$.
("$f$ is the pointwise limit of $\{f_n\}$")

(Similarly, if $\sum_{n=1}^{\infty} f_n(x)$ converges $\forall x \in E$, we can define the <u>sum</u> of the series, $f(x) := \sum_{n=1}^{\infty} f_n(x)$.)

Things Fall Apart:

continuity not preserved

Ex: $f_n : [0,1] \to \mathbb{R}$ defined as $f_n(x) = x^n$.

Q What is the pointwise limit of $\{f_n\}$?

Rk: Limit of cts. fns. need not be continuous.

Ex: nonconvergent sequence: "blimps". (Convergence in <u>measure</u>.)

Sum of cts. fns. need not be cts.

Ex: $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \dfrac{x^2}{(1+x^2)^n}$; $n = 0, 1, 2, \ldots$

Q What is $\sum_{n=0} f_n$? (the ptwise limit) (Does it even exist?)

<u>Case</u>: $x = 0$. Then $\sum f_n(x) = 0$.

<u>Case</u>: $x \neq 0$. $\sum f_n(x) = \sum \dfrac{x^2}{(1+x^2)^n} \overset{?}{=} x^2 \sum \dfrac{1}{(1+x^2)^n}$

(geometric series) $= x^2 \dfrac{1}{1-\frac{1}{1+x^2}} = 1+x^2$.

Rk: Not even a <u>sum</u> of cts. fns. need be continuous.

limit and deriv. need not commute

Q Must a convergent sequence of diff'ble fns converge to a diff'ble fn?

Ex: $f_n(x) = \frac{1}{\sqrt{n}} \sin(nx)$; $n = 1, 2, 3, \ldots$ Q $\lim_{n\to\infty} f_n(x) \equiv 0$.

$f_n'(x) = \sqrt{n}\cos(nx)$ ($f'(x) \equiv 0$.)

Rk: differentiation and limit do not commute.

integrability not preserved

Ex: [Limit of a convergent sequence of integrable fns. need not be integrable.]

Let $f_m(x) := \lim_{n\to\infty} [\cos(m!\pi x)]^{2n}$.

$$= \begin{cases} 1 & \text{if } m!x \in \mathbb{Z} \quad (\text{i.e. } x = \frac{p}{m!}, p \in \mathbb{Z}) \\ 0 & \text{if } m!x \notin \mathbb{Z} \end{cases}$$

Consider $\lim_{m\to\infty} f_m(x)$.

If $x \notin \mathbb{Q}$, then $f_m(x) = 0 \ \forall m$; $\lim_{m\to\infty} f_m(x) = 0$.

If $x \in \mathbb{Q}$, say $x = \frac{p}{q}$,

then for all $m \geq q$ we know $m!x \in \mathbb{Z}$.

$\Rightarrow f_m(x) = 1$. $(\forall m \geq q)$

$\Rightarrow \lim_{m\to\infty} f_m(x) = 1$

So $f_m$ converges to the fn. $f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ 1 & x \in \mathbb{Q} \end{cases}$.

Rk: Although each $f_m$ was Riemann integrable on $[0,1]$, their limit is not.

limit and integration don't commute

Ex: [Limit and integration do not commute]

Simple examples: i) $f_n(x) := \chi_{[n,n+1]}(x)$

ii) $f_n(x) = n\chi_{(0,\frac{1}{n})}(x)$.

More complicated: $f_n(x) := n^2 x(1-x^2)^n$, on $[0,1]$.

Although $\lim_{n\to\infty} f_n(x) \equiv 0$ on $[0,1]$,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \frac{n^2}{2n+2} \to +\infty.$$

ray of hope — However! All hope is not lost! There is a stronger type of convergence that "works."

translation of ptwise cvgnce — Consider: "$f = \lim_{n\to\infty} f_n$ on $E$" means what in $\varepsilon, N$ language?

For each $x \in E$, given $\varepsilon > 0$, $\exists N$ s.t. $n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon$

uniform convergence — **Defn:** [Uniform convergence] $f_n : E \to \mathbb{C}$; $n = 1, 2, 3, \ldots$; $f : E \to \mathbb{C}$.

If, $\forall \varepsilon > 0, \exists N$ s.t. $n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon \ \underline{\forall x \in E}$

then we say $\{f_n\}$ converges to $f$ uniformly.

~~Intuition: The "tube"~~

commentary — Rk: The difference is that in the pointwise case $N = N(x, \varepsilon)$

in the uniform case $N$ is uniform over $E$ $(N = N(\varepsilon))$

Rk: "$\sum_{n=1}^{\infty} f_n$" ~~converges to $f$ uniformly if...~~

Rk: As before, we can express this in terms of a Cauchy criterion:

Cauchy criterion — **Thm:** [Cauchy criterion for uniform convergence] $\{f_n : E \to \mathbb{C}\}$.

$\{f_n\}$ converges uniformly on $E \Leftrightarrow \forall \varepsilon > 0\ \exists N$ s.t. $m, n \geq N$

$\Rightarrow |f_n(x) - f_m(x)| < \varepsilon\ \forall x \in E$.

Pf: $\Rightarrow$) Fix $\varepsilon > 0$. $f_n \to f$ uniformly on $E$, so

$\exists N$ s.t. $n > N \Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{2}\ \forall x \in E$.

Then $n, m > N \Rightarrow |f_n(x) - f_m(x)| < \varepsilon$ by $\Delta \neq$.

$\Leftarrow$) Note that at each $x \in E$, $\{f_n(x)\}$ is a Cauchy sequence, so converges to some $f(x)$ (i.e. we have a ptwise limit).

By hypothesis, given $\varepsilon > 0$,

$\exists N$ s.t. $n > N \Rightarrow |f_n(x) - f_m(x)| < \frac{\varepsilon}{2}\ \forall n, m \geq N$.

Let $m \to \infty$: $|f_n(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon$,

so we in fact have uniform convergence to the ptwise limit ∎

(Q) Could we ever have ptwise cvgnce to one limit & uniform convergence to another?

Reminder: steal my art.
ask: What makes this obvious? natural?

dumb charc'n of uniform convergence

Dumb recharacterization of uniform convergence:

"Thm:" Say $\lim_{n\to\infty} f_n(x) = f(x)$. Let $M_n := \sup_{x\in E} |f_n(x) - f(x)|$

$$f_n \to f \text{ unifly on } E \iff \lim_{n\to\infty} M_n = 0$$ could I do it?

Pf: Exercise

Weierstraß M-Test

Thm: [Weierstraß M-Test] $\{f_n : E \to \mathbb{C}\}$.

If i. $|f_n(x)| \le M_n \ \forall x \in E$ (i.e. $f_n$ bdd)

ii. $\sum M_n$ converges

then $\sum f_n$ converges uniformly

Pf: By Cauchy Criterion, it suffices to show that $\forall \varepsilon > 0, \exists N$ s.t. $m \ge n \ge N \Rightarrow \left|\sum_{k=n}^{m} f_k(x)\right| < \varepsilon \ \forall x \in E$.

Well, $\left|\sum_{k=n}^{m} f_k(x)\right| \le \sum_{k=n}^{m} M_k$ and, by Cauchy Criterion, $\exists N$ s.t. $m \ge n \ge N \Rightarrow \sum_{k=n}^{m} M_k < \varepsilon$.

Properties of uniform convergence

Properties preserved in the limit by uniformly convergent sequences:

continuity preserved

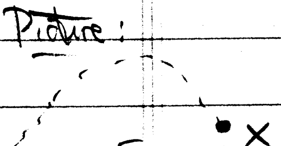
Thm: $\{f_n : E \to \mathbb{C}\}$ a sequence of fns; $x \in E'$.

If $f_n \to f$ uniformly on $E$, then $\lim_{t\to x} f(t) = \lim_{n\to\infty}\left(\lim_{t\to x} f_n(t)\right)$

$\left(\text{i.e., } \lim_{t\to x}\left(\lim_{n\to\infty} f_n(t)\right) = \lim_{n\to\infty}\left(\lim_{t\to x} f_n(t)\right)\right)$. ($= f_n(x)$ if $f_n$ cts. at $x$)

(In particular, $\left\{\lim_{t\to x} f_n(t) = A_n\right\}_{n=1}^{\infty}$ converges.)

Picture:




$A_n$ = values of $f_n$ as $t \to x$.

1. Take $\lim_{n\to\infty}$ then let $t \to x$
2. Take $t \to x$ then let $n \to \infty$.

(doesn't matter)

Reminder: steal my art.

ask: What makes this obvious? natural?

dumb charc'n of uniform convergence

Dumb recharacterization of uniform convergence:

"Thm:" Say $\lim_{n\to\infty} f_n(x) = f(x)$. Let $M_n := \sup_{x\in E} |f_n(x) - f(x)|$

$$f_n \to f \text{ unifly on } E \iff \lim_{n\to\infty} M_n = 0$$ could I do it?

Pf: Exercise

Weierstraß M-Test

Thm: [Weierstraß M-Test] $\{f_n : E \to \mathbb{C}\}$.

If i. $|f_n(x)| \le M_n \ \forall x \in E$ (i.e. $f_n$ bdd)

ii. $\sum M_n$ converges

then $\sum f_n$ converges uniformly

Pf: By Cauchy Criterion, it suffices to show that
$\forall \varepsilon > 0, \exists N$ s.t. $m \ge n \ge N \Rightarrow \left|\sum_{k=n}^{m} f_k(x)\right| < \varepsilon \ \forall x \in E$.

Well, $\left|\sum_{k=n}^{m} f_k(x)\right| \le \sum_{k=n}^{m} M_k$ and, by Cauchy Criterion, $\exists N$ s.t. $m \ge n \ge N \Rightarrow \sum_{k=n}^{m} M_k < \varepsilon$.

Properties of uniform convergence

Properties preserved in the limit by uniformly convergent sequences:

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Thm: $\{f_n : E \to \mathbb{C}\}$ a sequence of fns; $x \in E'$.

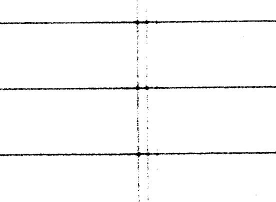
If $f_n \to f$ uniformly on $E$, then $\lim_{t\to x} f(t) = \lim_{n\to\infty}\left(\lim_{t\to x} f_n(t)\right)$

$$\left(\text{i.e., } \lim_{t\to x}\left(\lim_{n\to\infty} f_n(t)\right) = \lim_{n\to\infty}\left(\lim_{t\to x} f_n(t)\right)\right).$$

$\left(\begin{array}{c} \| \\ f_n(x) \\ \text{if } f_n \text{ cts. at } x \end{array}\right)$

(In particular, $\left\{\lim_{t\to x} f_n(t) = A_n\right\}_{n=1}^{\infty}$ converges.)

Picture:




$A_n$ = values of $f_n$ as $t \to x$.

1. Take $\lim_{n\to\infty}$ then let $t \to x$
2. Take $t \to x$ then let $n \to \infty$.

(doesn't matter)

(Notation:)

Pf: i) Let $A_n := \lim_{t\to x} f_n(t)$. Claim: $\{A_n\}$ converges.

STS $A_n$ is Cauchy.

Pf: Obvious. Given $\varepsilon > 0$, uniform convergence implies
$\exists N$ s.t. $m, n > N \Rightarrow |f_n(t) - f_m(t)| < \frac{\varepsilon}{2}$
$\Rightarrow |A_n - A_m| \leq \frac{\varepsilon}{2} < \varepsilon$.

Since $\{A_n\}$ is thus Cauchy, it converges to some $A$.

ii) Claim: $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n (= A)$

Pf: WTS that $\forall \varepsilon > 0$, $\exists$ nbhd $V$ of $x$ s.t. $|f(t) - A| < \varepsilon$ $\forall t \in (V \cap E) \setminus \{x\}$

(Q) What information do we have? $f_n \to f$, $A_n \to A$, $f_n(t) \to A_n$.

Fix $\varepsilon > 0$. Well, $|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$ for each $n = 1, 2, 3, \ldots$

Rk: uninspired proof.

Now, since $f_n \to f$ uniformly, and $A_n \to A$,
$\exists n$ s.t. i) $|f(t) - f_n(t)| < \frac{\varepsilon}{3}$ $\forall t \in E$.
ii) $|A_n - A| < \frac{\varepsilon}{3}$.

Further, since $A_n = \lim_{t\to x} f_n(t)$,

$\exists$ nbhd. $V$ of $x$ s.t. $|f_n(t) - A_n| < \frac{\varepsilon}{3}$ $\forall t \in (V \cap E) \setminus \{x\}$.

Thus $\forall t \in (V \cap E) \setminus \{x\}$, $|f(t) - A| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.

So $\lim_{t\to x} \lim_{n\to\infty} f(t) = \lim_{n\to\infty} \lim_{t\to x} f(t)$

(Rk: We needed it only for one specific n.)

continuity preserved

Corollary: $\{f_n\}$ a sequence of fns.
If i. $\{f_n\}$ are cts.
ii. $f_n \to f$ uniformly $\Big\} \Rightarrow f$ is cts.

Rk: Just because $f_n$ cts $\to f$ cts, there's no reason that the cvgnce be uniform

Rk: It is not even true that pointwise convergence on a cpt. set $\Rightarrow$ uniform cvgnce. However, we do have the following Thm:

partial converse implying uniform convergence

Thm: $\left[f_n \to f^{(\text{ptwise})} \text{ on some set } K\right]$

If i. $\{f_n\}, f$ are all cts.
ii. $f_n(x) \geq f_{n+1}(x)$ $\forall x \in K$, $n = 1, 2, 3, \ldots$
iii. $K$ is compact

} convergence is uniform.

Pf: WTS given $\varepsilon > 0$, $\exists N$ s.t. $n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon$ $\forall x \in K$

Note: Since $f_n$ is decreasing, it suffices to show that for some $N$,
$$f_N(x) - f(x) < \varepsilon \quad \forall x \in K.$$

cool proof

Rephrasal: Let $K_n = \{x \in K \mid f_n(x) - f(x) \geq \varepsilon\}$. STS $K_N = \phi$ for some $N$.

Observe! i. $K_n \supset K_{n+1}$ (nested)
ii. $K_n$ are all compact
iii. $\bigcap K_n = \phi$.
(Pick any $x \in K$. Eventually $x \notin K_n$.)

Thus the $\{K_n\}$ must fail the f.i.p.; some $K_N = \phi$ ∎

Rk: This is a pretty cool strategy.

~~Different Point of View:~~
Backtrack: The sup norm metric on $\mathscr{C}(X)$

$\mathscr{C}(X)$

Defn: i. [X a metric space] $\mathscr{C}(X) := \{f : X \to \mathbb{C} \mid f \text{ cts. \& bdd}\}$.
ii. For $f \in \mathscr{C}(X)$, let $\|f\| := \sup_{x \in X} |f(x)|$ the supremum norm
iii. For $f, g \in \mathscr{C}(X)$, define $d_{\mathscr{C}(X)}(f, g) = \|f - g\|$.

Exerc. $d$ is a metric (i.e., ..., ...)

Rk. So $\{\mathscr{C}(X), d_{\mathscr{C}(X)}\}$ is a metric space convergence in the sup norm metric.

(Q) What does convergence $f_n \to f$ in this metric mean?

Given $\varepsilon > 0$, $\exists N$ s.t. $n \geq N \Rightarrow \sup_{x \in X} |f_n - f| < \varepsilon$.

i.e., $|f_n(x) - f(x)| < \varepsilon \ \forall x \in X$. Look familiar?

$$f_n \to f \text{ uniformly} \iff f_n \to f \text{ in the sup norm metric}$$

$\mathcal{C}(X)$ is complete w.r.t the metric

**Thm:** $\{\mathcal{C}(X), d_{\mathcal{C}(X)}\}$ is a complete metric space.

(Q. WTS?)

**Pf:** Say $\{f_n\}$ is a Cauchy sequence in $\mathcal{C}(X)$, i.e.

$$\forall \varepsilon > 0, \exists N \text{ s.t. } n, m \geq N \Rightarrow \|f_n - f_m\| < \varepsilon.$$

make them do it.

This is the Cauchy criterion for uniform convergence; so

$$\exists f \text{ s.t. } f_n \to f \text{ uniformly.}$$

Further this $f \in \mathcal{C}(X)$, as it is cts (why?) and bounded (why?).

Rk: In fact $\mathcal{C}(X)$ is our first nontrivial example of what is called a Banach space.

## Uniform Convergence, Integration and Differentiation:

(Q) What is the key part of the hypotheses?

Recall Riemann integrable.

Recall: $\overline{\int} f\,dx := \inf_P U(P,f)$; $\underline{\int} f\,dx = \sup_P L(P,f)$.

uniform cgce & integrability.

**Thm:** [$f_n : [a,b] \to \mathbb{C}$ Riemann integrable fns] If $f_n \to f$ unif'ly on $[a,b]$ then $f$ is Riemann integrable and $\int_a^b f\,dx = \lim_{n\to\infty} \int_a^b f_n\,dx$.

**Pf:** Let $\varepsilon_n = \sup_{[a,b]} |f_n(x) - f(x)|$; so $f_n - \varepsilon_n \leq f \leq f_n + \varepsilon_n$.

$$\Rightarrow \int_a^b (f_n - \varepsilon_n)\,dx \leq \underline{\int} f \leq \overline{\int} f \leq \int_a^b (f_n + \varepsilon_n)\,dx$$

$$\Rightarrow 0 \leq \overline{\int} f - \underline{\int} f \leq \int_a^b 2\varepsilon_n\,dx = 2\varepsilon_n(b-a).$$

Since $f_n \to f$ unif'ly, $\varepsilon_n \to 0$ as $n \to \infty$. So $\underline{\int} f = \overline{\int} f$.

Further, $\int_a^b (f_n - \varepsilon_n)\,dx \leq \int_a^b f\,dx \leq \int_a^b (f_n + \varepsilon_n)\,dx$ finishes.

## Uniform convergence and differentiation:

Rk: Regrettably, even uniform convergence of $f_n \to f$ does not imply the desirable $f_n' \to f'$. However...

Thm: $\{f_n : [a,b] \to \mathbb{C}\}$ differentiable functions.

If i. $f_n'$ converges uniformly on $[a,b]$

ii. $\{f_n(x_0)\}_{n=1}^{\infty}$ converges for some $x_0 \in [a,b]$

Then a. $\{f_n\}$ converges uniformly on $[a,b]$ to some (differentiable) fn. $f$

b. $f'(x) = \lim_{n\to\infty} f_n'(x)$ ; $x \in [a,b]$.

(Rk) So can one define a relevant complete metric space?

Rk: I.e., $\frac{d}{dx}\left[\lim_{n\to\infty} f_n\right](x) = \lim_{n\to\infty}\left[\frac{d}{dx} f_n(x)\right]$.

Rk: We see (i & ii) is stronger than unif. cgnce.

Rk: All proofs should be read backwards.

Pf: Step I: $\{f_n\}$ converge uniformly to some fn. $f$.

Pf: WTS $\forall \varepsilon > 0$, $\exists N$ s.t. $n,m \geq N \Rightarrow |f_n(x) - f_m(x)| < \varepsilon \;\; \forall x \in [a,b]$.

Fix $\varepsilon > 0$.

"modding out the easy part" Now, $\{f_n(x_0)\}$ converges $\Rightarrow \exists N_1$ s.t. $n,m \geq N \Rightarrow |f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}$.

Further, $\{f_n'\}$ converges uniformly $\Rightarrow \exists N_2$ s.t. $n,m \geq N \Rightarrow$

$$|(f_n - f_m)'(x)| < \frac{\varepsilon}{2(b-a)}.$$

Thus for $n,m \geq \max\{N_1, N_2\}$,

$$|f_n(x) - f_m(x)| = |(f_n - f_m)(x) - (f_n - f_m)(x_0) + (f_n - f_m)(x_0)|$$

MVT.

$$\leq |(f_n - f_m)(x) - (f_n - f_m)(x_0)| + |(f_n - f_m)(x_0)|$$

$$\leq |x - x_0| \frac{\varepsilon}{2(b-a)} + \frac{\varepsilon}{2} < \varepsilon.$$

(Rk: Perhaps present in stages: mod out easy part, see how (i) controls remaining part.)

**Step II:** Claim: $f'(x) = \lim_{n\to\infty} f_n'(x)$. (Q) Remind you of anything? Reminiscent

Pf: Let $\phi(t) := \dfrac{f(t)-f(x)}{t-x}$, $\phi_n(t) = \dfrac{f_n(t)-f_n(x)}{t-x}$.

Rk: Easy if one recognizes this to be an instance of a previous thm.

Notice $\phi_n \to \phi$ ptwise. STS $\phi_n \to \phi$ uniformly on $[a,b]$, for then $\lim_{t\to x} \phi(t) = \lim_{n\to\infty} \lim_{t\to x} \phi_n(t)$,
we're done by the previous thm:

i.e. $f'(x) = \lim_{n\to\infty} f_n'(x)$.

As before, $\{f_n'\}$ converges uniformly, so $\forall \varepsilon > 0$, $\exists N$ s.t. $n,m \geq N$

$$\Rightarrow \quad |f_n'(c) - f_m'(c)| < \varepsilon \quad \forall c \in [a,b]$$

Then
$$|\phi_n(t) - \phi_m(t)| = \left| \frac{f_n(t) - f_n(x) - f_m(t) + f_m(x)}{t-x} \right|$$
$$\leq |(f_n - f_m)'(c)| \text{ for some } c \text{ between } t \text{ and } x.$$
$$< \varepsilon.$$

So $\{\phi_n\}$ converges uniformly. ∎

**A bit of fun (?)**: a continuous, nowhere-differentiable function

Construction: Let $\varphi(x) = |x|$; $-1 \leq x \leq 1$; extend to a periodic fn. via $\varphi(x+2) = \varphi(x)$.

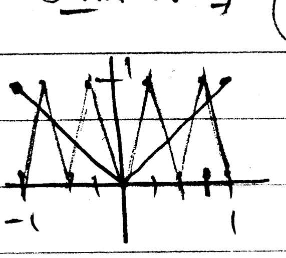
Observe that $|\varphi(s) - \varphi(t)| \leq |s-t|$, (exerc.)

so $\varphi$ is (Lipschitz) continuous.

Define $f(x) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$:

~~Claim: unifly converges.~~ Why?

(Q) The Weierstraß M-Test implies uniform convergence (of cts. fns.) $\Rightarrow$ $f$ is cts.



Claim: $f$ is nowhere differentiable.

Pf: Pick any $x \in \mathbb{R}$.

For each $m \in \mathbb{N}$, let $\delta_m = \pm\left(\frac{1}{2}\right) 4^{-m}$ $=: \delta_m(x)$, where $\pm$ is chosen s.t. no integer lies in between $4^m x$ and $4^m(x+\delta_m)$. (notice $|4^m \delta_m| = \frac{1}{2}$)

Consider $\left|\dfrac{f(x+\delta_m)-f(x)}{\delta_m}\right| = \left|\dfrac{\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \left[\varphi(4^n(x+\delta_m)) - \varphi(4^n x)\right]}{\delta_m}\right|$,

We shall see that this can be made arbitrarily large for large $m$. (i.e., small $\delta_m$)

Now, for $n > m$, $\varphi(4^n x + 4^n \delta_m) - \varphi(4^n x) = 0$;

for $n < m$, $\left|\dfrac{\varphi(4^n x + 4^n\delta_m) - \varphi(4^n x)}{\delta_m}\right| \le 4^n$.

$n = m$ $\left|\dfrac{\varphi(4^n x \pm \frac{1}{2}) - \varphi(4^n x)}{\pm\frac{1}{2}(4^{-m})}\right| = 4^m$. (no integer between)

Thus $= \left|\sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \dfrac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m}\right| = \left|3^m + \sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n \gamma_n\right|$

$\ge 3^m - \sum_{n=0}^{m-1} 3^n$ $\left( \left|3^m + \sum\left(\frac{3}{4}\right)^n \gamma_n\right| \ge \left||3^m| - |\sum\left(\frac{3}{4}\right)^n\gamma_n|\right| \ge \left|3^m - \sum_{n=0}^{m-1} 3^n\right| \right)$,

$= \frac{1}{2}(3^m + 1)$. So $\lim_{h\to 0}\left|\dfrac{f(x+h)-f(x)}{h}\right|$ diverges.

Rk: This is an example of a so-called "fractal".

Commentary: Okay, so uniform convergence is very good.

The obvious question is: when can we obtain such a sequence?

Or, a weaker question: when does a sequence of functions have a convergent (ptwise/unif'ly) subsequence?

Sequential Compactness for spaces of functions

**Boundedness of sequences of functions:**

Def'n: $\{f_n\}$ a sequence of functions on a set $E$.

i. If $\forall x \in E$, $\{f_n(x)\}_{n=1}^{\infty}$ is a bdd sequence, then we say $\{f_n\}$ is ptwise bdd on $E$.

ptwise bdd sequence

(Rk: This is weak!)

Rk: Equivalently: $\exists \phi : E \to \mathbb{R}$ s.t. $|f_n(x)| < \phi(x)$; $n = 1, 2, 3, \dots$

ii. If $\exists M$ s.t. $|f_n(x)| < M$ $\forall x \in E$, $n = 1, 2, 3, \dots$ then we say $\{f_n\}$ is uniformly bdd on $E$.

uniformly bdd sequence

Topic I: Existence of Pointwise convergent subsequences.

(Q) Under what conditions on $\{f_n\}$ do we definitely have a (ptwise) cvgnt subseq?

One failed hypothesis:

(Q): If $\{f_n\}$ are uniformly bdd, cts fns on a compact set, must there exist a ptwise convergent subsequence?

(A) No: Counterexample: $f_n(x) = \sin(nx)$ on $[0, 2\pi]$.

If some subsequence $\{f_{n_k}\}$ converged, then $\lim\limits_{k\to\infty} (\sin(n_k x) - \sin(n_{k+1} x)) = 0$

$$\Rightarrow \lim_{k\to\infty} (\sin(n_k x) - \sin(n_{k+1} x))^2 = 0$$

(Dominated convergence theorem) $\Rightarrow \lim\limits_{k\to\infty} \int_0^{2\pi} (\sin(n_k x) - \sin(n_{k+1} x))^2 \, dx = 0$.

But it $= 2\pi$. ※

Rk: We can prove a weak (but useful) result in this direction using the diagonal process:

weak result: ctble sets easy.

**Thm:** $\{f_n\}$ a sequence of $\mathbb{C}$-valued fns on $E$.

If i. $E$ is ctble
ii. $\{f_n\}$ is ptwise bdd.
$\Rightarrow \exists$ a ptwise cvgnt subsequence $\{f_{n_k}\}$ on $E$.

Hint: Diagonal process.

Pf: Denote $E = \{x_i\}_{i=1}^{\infty}$

Do it point by point.

Since $\{f_n(x_1)\}_{n=1}^{\infty}$ is bdd, $\exists$ a convergent subsequence $\{f_{n_k}(x_1)\}_{k=1}^{\infty}$

Notate $f_{1,k} = f_{n_k}$

~~kth member of subsequence that~~ converges on $x_1$.

Iterate: since $\{f_{1,n}(x_2)\}_{n=1}^{\infty}$ is bdd, $\exists$ cvgnt subseq. $\{f_{1,n_k}(x_2)\}_{k=1}^{\infty}$

Notate $f_{2,k} = f_{1,n_k}$.

In this manner we obtain an array

$$\begin{matrix} f_{1,1} & f_{1,2} & f_{1,3} & f_{1,4} & \cdots \\ f_{2,1} & f_{2,2} & f_{2,3} & f_{2,4} & \cdots \\ f_{3,1} & f_{3,2} & f_{3,3} & f_{3,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

s.t. i. Each row is a subsequence of the prior row.

ii. The $n^{th}$ row converges on $x_n$.

iii. the $k^{th}$ element in the $n^{th}$ row occurs in the $\geq k^{th}$ place in the $(n-1)^{th}$ row.

Then $\{f_{n,n}(x_k)\}_{n=1}^{\infty}$ converges for every $k$; (verify rigorously!)

i.e. the diagonal sequence converges ptwise on $E$.

Topic II: uniformly convergent subsequences.

Conditions which guarantee uniformly convergent subsequences:

Q Does $\{f_n\}$ uniformly bdd and ptwise convergent on a cpt. set $\Rightarrow$ $\exists$ unif'ly cvgnt subseq.?

A. No. Consider $f_n(x) = \dfrac{x^2}{x^2 + (1-nx)^2}$ on $[0,1]$

i. $|f_n(x)| \leq 1$, so unif'y bdd

ii. Converges to 0 at each $x \in [0,1]$

iii. However for $\varepsilon < 1$, no $N$ works $\forall n$, since $f_n\left(\frac{1}{n}\right) = 1$ $\forall n$

The necessary condition: equicontinuity.

equicontinuous families

**Defn:** [$\mathcal{F}$ a colln. of functions $E \to \mathbb{C}$.]

If $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $d(x,y) < \delta$,

then $|f(x) - f(y)| < \varepsilon$ $\forall f \in \mathcal{F}$

then we say $\mathcal{F}$ is an equicontinuous family on $E$.

Rk: How is this related to uniform continuity?

In this case we have a family of uniformly cts. fns., for which $\delta$ is independent of $f$. ($\delta = \delta(\varepsilon)$)

Rk: Any finite family of unifly cts. fns. is equicontinuous. Why?

Conditions implying equicontinuity:

uniform cvgnce of cts fns. on compact set $\Rightarrow$ equicontinuous

**Thm:** [$\{f_n\}_{n=1}^{\infty}$ cts. fns. on metric space $K$]

If i. $K$ is compact

ii. $\{f_n\}_{n=1}^{\infty}$ cvges uniformly on $K$

$\Rightarrow$ $\{f_n\}$ is equicontinuous on $K$

**Proof:** WTS $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f_n(x) - f_n(y)| < \varepsilon$ $\forall n$, $d(x,y) < \delta$.

Fix $\varepsilon > 0$.

Uniform convergence $\Rightarrow$ $\exists N$ s.t. $n > N$ implies $|f_n(x) - f_N(x)| < \frac{\varepsilon}{3}$ $\forall x \in K$.

By previous remark, $\{f_1, \ldots, f_N\}$ is equicontinuous,

so $\exists \delta > 0$ s.t. $|f_i(x) - f_i(y)| < \frac{\varepsilon}{3}$ if $d(x,y) < \delta$, $i = 1, \ldots, N$.

For $n > N$, if $d(x,y) < \delta$, we have

$$|f_n(x) - f_n(y)| \le |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)|$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

So this $\delta$ works for all $n$. ∎

(Rk: Make them do this! Easy!)

## The climax: the theorem of Ascoli-Arzela! (Sequential Compactness)

**Thm:** $\{f_n\}_{n=1}^{\infty}$ a sequence of $\mathbb{C}$-valued fns. on $K$.

If i. $K$ is compact
ii. $\{f_n\}$ is ptwise bdd. on $K$
iii. $\{f_n\}$ is equicontinuous
$\Rightarrow$ a. $\{f_n\}$ is unifly bdd. on $K$.
b. $\{f_n\}$ has a unifly convergent subsequence

equicontinuity + compactness + bds at the centers.

Pf: a) Given $\varepsilon > 0$, $\exists \delta > 0$ s.t. if $d(x,y) < \delta$ then $d(f_n(x), f_n(y)) < \varepsilon$ $\forall n$.

Cover $K$ with $\{N_\delta(x)\}_{x \in K}$; then $\exists$ a finite subcover $\{N_\delta(x_i)\}_{i=1}^{n}$.

Pointwise bddness $\Rightarrow$ $\exists$ bounds $\{M_i\}_{i=1}^{n}$ s.t. $|f_n(x_i)| < M_i$ $\forall n$.

Then for any $x \in K$, $|f_n(x)| < \max\{M_1, \dots, M_n\} + \varepsilon$,

(since $|f_n(x)| = |f_n(x) - f_n(x_i) + f_n(x_i)|$ for some $i = 1, \dots, n$.)

Key fact: separability of $K$

b) FACT: $K$ cpt. metric space $\Rightarrow$ $\exists$ ctble dense subset $E \subset K$. (exerc.)

Then, 7.23 $\Rightarrow$ $\exists$ subsequence $\{f_{n_k}\}$, convergent on $E$.

Claim: $\{f_{n_k}\}$ is uniformly convergent.

(Exercise) Pf: (Easy) Fix $\varepsilon > 0$. By equicontinuity, $\exists \delta > 0$ s.t. ...

* $E$ dense in $K$, $K$ cpt. $\Rightarrow$ $\exists$ a finite subcover $\{N_\delta(e_i)\}_{i=1}^{p}$ of $K$, with centers $\{e_1, \dots, e_p\}$ in $E$.

convergence

$\{f_{n_k}\}$ convergent on $E$ $\Rightarrow$ $\exists N = \max\{N_1, \dots, N_p\}$ s.t.

$k, l > N \Rightarrow |f_{n_k}(e_i) - f_{n_l}(e_i)| < \varepsilon$ for $i = 1, \dots, p$.

Then, for any $x \in K$, if $k, l > N$,

$$|f_{n_k}(x) - f_{n_l}(x)| \le |f_{n_k}(x) - f_{n_k}(e_i)| + |f_{n_k}(e_i) - f_{n_l}(e_i)| + |f_{n_l}(e_i) - f_{n_l}(x)|$$

$\le 3\varepsilon$, since for some $i$, $d(x, e_i) < \delta$. ■

Rk: this is a pretty complex theorem. Rk. This is a "compactness" statement.

Stone-Weierstraß!

**Thm:** [Stone-Weierstraß] Given any $f \in \mathcal{C}([a,b])$, $\exists$ a sequence $\{P_n\}$ of polynomials s.t. $P_n \to f$ uniformly on $[a,b]$.

define $Q_n$

**Pf:** Via method of approximation of the identity.

Let $Q_n(x) = c_n(1-x^2)^n$ on $[-1,1]$, $c_n := \left(\int_{-1}^{1} (1-x^2)^n dx\right)^{-1}$.

properties of $Q_n$

Observe: i. $\int_{-1}^{1} Q_n(x)\,dx = 1$. (defn.)

ii. For any $0 < \delta < 1$, $Q_n \to 0$ uniformly on $\delta \le |x| \le 1$.

Rk: I.e., $\{Q_n\}$ is an approximation of the identity.

proof of properties

Q. Why is (ii) true? Suppose you want to control $Q_n(x) = c_n(1-x^2)^n$.

Observe: a. $c_n < \sqrt{n}$ (check!)

b. On $\{\delta \le |x| \le 1\}$, $1-x^2 \le 1-\delta^2$.

So $Q_n(x) \le \sqrt{n}\,(1-\delta^2)^n$ on $\delta \le |x| \le 1$;

since RHS $\to 0$ as $n \uparrow \infty$, (ii) is true.
(check!)

the approx'n: $P_n = f * Q_n$.

Now, let $P_n(x) := \int_{-1}^{1} f(x+t)\,Q_n(t)\,dt$

Rk: This is what one does with an approximate identity: use it (via convolution) to approx. $f$ with "nice guys".

Claim I: These $\{P_n\}$ are polynomials. (obvious)

Claim II: $P_n \to f$ uniformly on $[0,1]$.

Claim: For each $0<\delta<1$, $Q_n \to 0$ uniformly in $\delta \le |x| \le 1$

Pf: We can show that $c_n < \sqrt{n}$ ; $n=1,2,3,\ldots$ (read yourself!)

$$\text{So } Q_n(x) < \sqrt{n}(1-x^2)^n.$$

$$\Rightarrow Q_n(x) \le \sqrt{n}(1-\delta^2)^n \quad \text{on} \quad \delta \le |x| \le 1.$$

which $\to 0$ as $n \uparrow \infty$ (exercise)

Now, let $P_n(x) = \int_{-1}^{1} f(x+t)\, Q_n(t)\, dt$: these will be our poly'l approx'ns.

(Rk: this is how one uses an approximate identity)

Q Why are these poly'ls?

$$\int_{-1}^{1} f(x+t)\, Q_n(t)\, dt = \int_{-x}^{1-x} f(x+t)\, Q_n(t)\, dt$$

$$= \int_0^1 f(s)\, Q_n(s-x)\, ds = \int_0^1 f(s)\left[1-(s-x)^2\right]^n ds,$$

a poly'l in $x$.

Claim: $P_n \to f$ uniformly on $[a,b]$

Pf: WTS $\forall \varepsilon>0,\ \exists N$ s.t. $n>N \Rightarrow |P_n(x)-f(x)|<\varepsilon\ \forall x\in[a,b]$.

Well, Fix $\varepsilon>0$. consider $|P_n(x)-f(x)| = \left|\int_{-1}^{1} f(x+t)\, Q_n(t)\, dt - f(x)\right|$

$$\overset{@}{=} \left|\int_{-1}^{1}\left[f(x+t)-f(x)\right] Q_n(t)\, dt\right| \le \int_{-1}^{1} |f(x+t)-f(x)|\, Q_n(t)\, dt$$

How can we control this? We use three facts.

i. $f$ unif'ly cts (why?), so $\exists \delta$ s.t. $|f(x)-f(y)|<\frac{\varepsilon}{2}$ for $d(x,y)<\delta$.

ii. $Q_n(x) \le \sqrt{n}(1-\delta^2)^n$ on $\delta \le |x| \le 1$.

iii. $|f| \le M$ on $[0,1]$.

$$\text{So } \int_{-1}^{1} |f(x+t)-f(x)|\, Q_n(t)\, dt \le 2M \int_{\delta\le|x|\le 1} Q_n(t)\, dt + \frac{\varepsilon}{2}\int_{-\delta}^{\delta} Q_n(t)\, dt \le 4M\sqrt{n}(1-\delta^2)^n + \frac{\varepsilon}{2} < \varepsilon.$$

Commentary: The next theorem will be of a slightly different flavor:
It is not a "compactness" theorem, but a "density" theorem.

Thm: [Stone-Weierstraß] Given any $f \in \mathcal{C}([a,b])$,
$\exists$ a sequence of polynomials $\{P_n\}$ s.t. $P_n \to f$ uniformly on $[a,b]$.

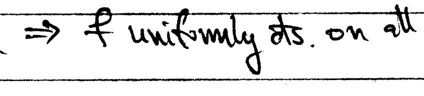
Rk: Recall that convergence w.r.t. the sup norm metric on $\mathcal{C}(K)$ $\Leftrightarrow$ uniform convergence as fns.
So we can rephrase this as saying "$\mathcal{P}$ (poly'ls) are dense in $\mathcal{C}([a,b])$ w.r.t. sup norm metric."

Rk: This is a significant theorem; however, the technique used is even more important than the result.

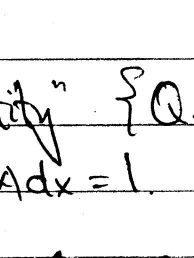
Rk: WLOG we may assume
i. $[a,b]$ is $[0,1]$
ii. $f(0) = f(1) = 0$ (check yourself!)
iii. $f(x) = 0$ outside of $[0,1]$
($\Rightarrow f$ uniformly cts. on all of $\mathbb{R}$)

Okay, now let's get to the proof!

Pf: Idea: we create an appropriate "approximate identity" $\{Q_n\}_{n=1}^{\infty}$,
i.e., a sequence of functions on $[-1,1]$ s.t. i. $\int_{-1}^{1} Q_n(x)\,dx = 1$.
ii. $\forall \delta \in (0,1)$, $Q_n \to 0$ uniformly on $\delta \le |x| \le 1$




Let $Q_n(x) = (1-x^2)^n$; $n = 1, 2, 3, \ldots$



Q. What does it look like?

!! Rk: Of course this is a lie. $\int_{-1}^{1} (1-x^2)^n\,dx \neq 1$.

Let $Q_n(x) = c_n (1-x^2)^n$, where $c_n = \left[\int_{-1}^{1} (1-x^2)^n\,dx\right]^{-1}$.
Then obviously $\int Q_n = 1$.

<u>Proof of II</u>: Fix $\varepsilon>0$. WTS $\exists N$ s.t. $n>N \Rightarrow |P_n(x)-f(x)|<\varepsilon \ \forall x\in[0,1]$.

Well, think: $$|P_n(x)-f(x)| := \left|\int_{-1}^{1} f(x+t)Q_n(t)\,dt - f(x)\right|$$

$$= \left|\int_{-1}^{1}[f(x+t)-f(x)]\,Q_n(t)\,dt\right| \qquad \text{(Rk: The standard trick.)}$$

$$\leq \int_{-1}^{1} |f(x+t)-f(x)|\,Q_n(t)\,dt \qquad \text{(Property of integrals; } Q_n(x)\geq 0\text{)}$$

$$= \int_{|t|\leq\delta} |f(x+t)-f(x)|\,Q_n(t)\,dt + \int_{\delta\leq|t|\leq 1} |f(x+t)-f(x)|\,Q_n(t)\,dt. \qquad \text{(splitting trick)}$$

(How to control this? $f$ uniformly cts $\Rightarrow \exists\delta$ s.t. $|t|<\delta \Rightarrow |f(x+t)-f(x)|<\frac{\varepsilon}{2}$)

$$\leq \frac{\varepsilon}{2}\int_{|t|\leq\delta} Q_n(t)\,dt + \int_{\delta\leq|t|\leq 1}|f(x+t)-f(x)|\,Q_n(t)\,dt$$

We have only crude bounds on $\{\delta\leq|t|\leq 1\}$: $|f(x+t)-f(x)|\leq 2M$ ($f$ bdd.).
But fortunately we have property ii. $Q_n\to 0$ uniformly on $\delta\leq|t|\leq 1$

$$\leq \underbrace{\frac{\varepsilon}{2}\cdot 1}_{\text{(why?)}} + 2M\int_{\delta\leq|t|\leq 1} Q_n(t)\,dt.$$

(We use our theorem about $\int$ and uniform convergence)

Since $$\lim_{n\to\infty}\int_{\delta\leq|t|\leq 1} Q_n(t)\,dt = \int_{\delta\leq|t|\leq 1}\lim_{n\to\infty} Q_n(t)\,dt = 0,$$

$$\exists N \text{ s.t. } n\geq N \Rightarrow \int_{\delta\leq|t|\leq 1} Q_n(t)\,dt < \frac{\varepsilon}{4M}.$$

and thus for which we have

$$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon. \quad \blacksquare$$