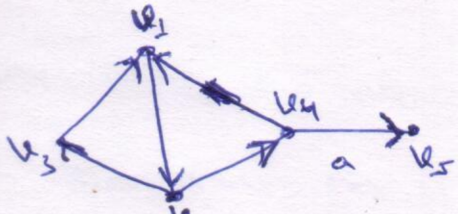
Directed graph

A directed graph $D=(V,A)$ consists of two sets:

V, the vertex set, a nonempty set of elements called the vertices of D and A, the arc set a (possibly empty) set of elements called the arcs of D, such that each arc $a \in A$ is assigned a ordered pair of vertices (u,v).

If a is an arc in D with associated ordered pair of vertices (u,v), then a is said to join u to v, u is called origin, or initial vertex or the tail of a and v is called the terminous, or the terminal vertex or head of a.

Ex



$a=(v_4, v_5)$ head tail

Given a digraph D, we can obtain a graph G from D by removing all the arrows from the arcs. This graph G same vertex set as D and corresponding to each arc a in D with associated ordered pair of vertices (u,v), there is an edge e in G with

associated unordered pair of vertices $\{u, v\}$.
G is called the underlying graph of G.

Let D be a digraph. A directed walk in D is a finite sequence $W = v_0 a_1 v_1 a_2 v_2 \cdots a_k v_k$ whose terms are alternately vertices and arcs such that for $i = 1, \ldots, k$, the arc a_i has origin v_{i-1} and terminus v_i. The number k of arcs in W is called the length of W. ($\underset{\text{origin}}{v_0}$-$\underset{\text{terminus}}{v_k}$ walk)

<u>directed trail</u>, <u>directed path</u>, <u>directed cycle</u>, <u>directed tour</u>

A vertex v of the directed graph D is said to be reachable from a vertex u if there is a directed path in D from u to v.

A digraph D is said to be weakly connected if its underlying graph is connected.

A digraph D is strongly connected if for any pair of vertices u and v in D there is a directed path from u to v.

Given a graph G, we can obtain a digraph from G by specifying for each edge in G an order to its end vertices. Such a digraph D is called an orientation of G.

Two digraph D_1 and D_2 are said to be isomorphic if there is a one-to-one and onto correspondence between $V(D_1)$ and $V(D_2)$ and a one-to-one and onto correspondence between $A(D_1)$ and $A(D_2)$ such that each arc a_1 in D_1 goes from vertex u_1 to v_1 then the corresponding arc a_2 in D_2 goes from u_2 to v_2, where u_2, v_2 are vertices in D_2 corresponding to u_1 and v_1, respectively.

A digraph D is called simple if for any ordered pair of vertices u and v of D, there is at most one arc from u to v and there is no arc from u to itself.

Let v be a vertex in a digraph D. The indegree $id(v)$ of v is the number of arcs in D that have v as its head.

The out degree $od(v)$ of v is the number of arcs of D that have v as its tail.

$\underline{\underline{Th^m}}$ Let D be a digraph with n-vertices and q-arcs if $\{v_1, \ldots, v_n\}$ is the set of vertices of D then

$$\sum_{j=1}^{n} id(v_j) = \sum_{j=1}^{n} od(v_j) = q$$

Let D be a weakly connected digraph. Then a directed Euler trail in D is a directed open trail in D containing all the arcs of D (once and only once).

A directed Euler tour of D is a directed closed trail of D containing all arcs of D (once and only once)

A digraph D containing a directed Euler tour is called an Euler digraph.

$\underline{\underline{Th^m}}$ Let D be a weakly connected digraph with atleast one arc. Then D is Euler iff $od(v) = id(v)$ for every vertex v of D.

Def^n A tournament is an orientation of a complete graph.

Def^n In a diagraph, a king is a vertex from which every vertex is reachable by a ~~path~~ directed path of length at most 2.

Th^m Every tournament has a king.

Proof: Let x be a vertex in a tournament T. If x is not a king, then some vertex y is not reachable from x by a directed path of length at most 2. Hence no successor of x is a predecessor of y. Since T is a orientation of K_n, every successor of x must therefore be a successor of y. Also $y \to x$. Hence $od(y) > od(x)$

If y is not a king, then we repeat the argument to find z with yet larger outdegree. Since T is finite, we cannot ~~forever~~ obtain vertices of successively higher out degree. The procedure must terminate only when we have found a king.