Lecture-1 (6 April, 2021)

Let X be a non-empty set.

Definition 1:- A <u>metric</u> on X is a function $d : X \times X \to \mathbb{R}$ which satisfies the following three conditions:

1) $d(x,y) \geq 0$, and $d(x,y) = 0$ iff $x = y$;
2) $d(x,y) = d(y,x)$;
3) $d(x,y) \leq d(x,z) + d(z,y)$.

Example 2 (Discrete metric):-

Let X be a non-empty set. Define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases} \qquad \text{———} (\mathrm{I})$$

Example 3 (Euclidean metric):-

$X = \mathbb{R}/\mathbb{C}$ and define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = |x - y|, \qquad \text{———} (\mathrm{II})$$

for all $x, y \in X$.

Fix $1 \leq p < \infty$. Consider the following set

$$\ell^p = \left\{ \{x_n\} \;\middle|\; x_n \in \mathbb{R}, \sum_{i=1}^{\infty} |x_i|^p < \infty \right\} \qquad \text{———} (\mathrm{III})$$

On ℓ^p, we define $\|\cdot\|_p : \ell^p \to \mathbb{R}$ by

$$\|\{x_n\}\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} \text{. ——— (IV)}$$

Theorem 4 :- For any two sequences $\{x_n\}$ & $\{y_n\}$ the function d on $\ell^p \times \ell^p$, defined by

$$d_p(\{x_n\}, \{y_n\}) = \|\{x_n - y_n\}\|_p \text{ —— (V)}$$

is a metric on ℓ^p.

Theorem 5 (Hölder's inequality):- Let $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$. Let $\{x_n\} \in \ell^p$ and $\{y_n\} \in \ell^q$. Then

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \|\{x_n\}\|_p \|\{y_n\}\|_q \text{. —— (III)}$$

Lemma 6 (Young's inequality) :- Let $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$. Then for any $a, b \geq 0$

$$a^{1/p} b^{1/q} \leq \frac{a}{p} + \frac{b}{q} \text{ ——— (IV)}$$

Proof :- For $t \in [1, \infty)$, define $f(t) = k(t-1) - t^k + 1$, where $0 < k < 1$.

Then, $f'(t) = k - kt^{k-1} = k(1-t^{k-1}) \geqslant 0$.
So, f is an increasing function; hence $f(t) \geqslant f(1)$.

This implies, $k(t-1) - t^k + 1 \geqslant 0 = f(1)$
$$\Rightarrow k(t-1) + 1 \geqslant t^k \quad \text{——— (V)}$$
The statement is clearly true if either $a=0$ or $b=0$.
So, without loss of generality, let $a \geqslant b > 0$.
Then we put $t = a/b$ and $k = 1/p$ in (V):

$$\frac{1}{p}\left(\frac{a}{b} - 1\right) + 1 \geqslant \left(\frac{a}{b}\right)^{1/p}$$
$$\Rightarrow \frac{a}{p} \cdot \frac{1}{b} + \frac{1}{q} \geqslant \left(\frac{a}{b}\right)^{1/p}$$
$$\Rightarrow \frac{a}{p} \cdot b^{1/p - 1} + \frac{1}{q} b^{1/p} \geqslant a^{1/p}$$
$$\Rightarrow \frac{a}{p} + \frac{b}{q} \geqslant a^{1/p} \cdot b^{1/q} \qquad \boxtimes$$

Proof of Theorem 5 :- The statement is trivially true of one of the sequences $\{x_n\}_{n=1}^{\infty}$ or $\{y_n\}_{n=1}^{\infty}$ is a zero sequence.
Therefore, without loss of generality, we assume both the sequences $\{x_n\}$ and $\{y_n\}$ are nonzero.

Now, for a fixed $i \in \mathbb{N}$, set

$$a = \left(|x_i| / \|\{x_n\}\|_p\right)^p, \quad b = \left(|y_i| / \|\{y_n\}\|_q\right)^q.$$

By Lemma 6, we have

$$\frac{|x_i|\,|y_i|}{\|\{x_n\}\|_p \,\|\{y_n\}\|_q} \leq \frac{1}{p}\left(\frac{|x_i|}{\|\{x_n\}\|_p}\right)^p + \frac{1}{q}\left(\frac{|y_i|}{\|\{y_n\}\|_q}\right)^q$$

Now, summing over $i = 1$ to n

$$\frac{\sum_{i=1}^{n} |x_i|\,|y_i|}{\|\{x_n\}\|_p \,\|\{y_n\}\|_q} \leq \frac{1}{p\,\|\{x_n\}\|_p^p} \sum_{i=1}^{n} |x_i|^p + \frac{1}{q\,\|\{y_n\}\|_q^q} \sum_{i=1}^{n} |y_i|^q$$

$$\leq \frac{1}{p\,\|\{x_n\}\|_p^p} \|\{x_n\}\|_p^p + \frac{1}{q\,\|\{y_n\}\|_q^q} \|\{y_n\}\|_q^q$$

$$= \frac{1}{p} + \frac{1}{q} = 1.$$

Since, the last inequality holds for every $n \in \mathbb{N}$ we obtain (III) by letting $n \to \infty$ in the LHS. ☐

<u>Remark 7 (Cauchy-Schwarz inequality):-</u> In particular, if $p = q = 2$, the Hölder's inequality is known as Cauchy-Schwarz inequality

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \|\{x_n\}\|_2 \,\|\{y_n\}\|_2$$

for all $\{x_n\}, \{y_n\} \in \ell^2$. ——— (V)

Lecture-2 (7 April, 2021)

Theorem 8 (Minkowski's inequality):- Let $1 \leq p < \infty$. Let $\{x_n\}, \{y_n\} \in \ell^p$. Then

$$\|\{x_n + y_n\}\|_p \leq \|\{x_n\}\|_p + \|\{y_n\}\|_p \quad \text{——— (VI)}$$

Proof:- For $p = 1$ we prove the statement using triangle inequality for $\mathbb{R}$ with respect to the Euclidean metric.

For any fixed $i \in \mathbb{N}$ we have,

$$|x_i + y_i| \leq |x_i| + |y_i| \leq \sum_{i=1}^{\infty} |x_i| + \sum_{i=1}^{\infty} |y_i|$$
$$= \|\{x_n\}\|_1 + \|\{y_n\}\|_1 .$$

Since, i was chosen arbitrarily we have (VI) for $p = 1$.

For, $1 < p < \infty$ we have the proof using

$$\sum_{i=1}^{m} |x_i + y_i|^p \leq \sum_{i=1}^{m} |x_i + y_i|^{p-1} |x_i + y_i|$$
$$\leq \sum_{i=1}^{m} |x_i + y_i|^{p-1} |x_i| + \sum_{i=1}^{m} |x_i + y_i|^{p-1} |y_i| \quad \text{——— (VII)}$$

Where m is a fixed natural number.

Define $z_n = x_n + y_n$, if $1 \leqslant n \leqslant m$,

$ = 0$, otherwise.

Clearly, $\{z_n\} \in \ell^p$ and by Hölder's inequality we get

$$\sum_{i=1}^{m} |z_i|^{p-1} |x_i|$$

$$\leqslant \sum_{i=1}^{\infty} |z_i|^{p-1} |x_i|$$

$$\leqslant \|\{x_n\}\|_p \times \|\{|z_i|^{p-1}\}\|_q$$

$$= \|\{x_n\}\|_p \times \left(\sum_{i=1}^{m} |x_i + y_i|^{q(p-1)} \right)^{1/q}, \text{ where}$$

$$\frac{1}{p} + \frac{1}{q} = 1 \Longleftrightarrow pq - q = p.$$

Therefore,

$$\sum_{i=1}^{m} |x_i + y_i|^{p-1} |x_i| \leqslant \|\{x_n\}\|_p \times \left(\sum_{i=1}^{m} |x_i + y_i|^{p} \right)^{1/q}.$$

Similarly, we have

$$\sum_{i=1}^{m} |x_i + y_i|^{p-1} |y_i| \leqslant \|\{y_n\}\|_p \times \left(\sum_{i=1}^{m} |x_i + y_i|^{p} \right)^{1/q}.$$

Using the last two inequalities we simplify (VII) as follows

$$\sum_{i=1}^{m} |x_i+y_i|^p \leq \left(\|\{x_n\}\|_p + \|\{y_n\}\|_p\right) \times \left(\sum_{i=1}^{m} |x_i+y_i|^p\right)^{1/q}$$

$$\Rightarrow \left(\sum_{i=1}^{m} |x_i+y_i|^p\right)^{1/p} \leq \|\{x_n\}\|_p + \|\{y_n\}\|_p .$$

Now, letting $m \to \infty$ on the L.H.S. we obtain (VI) for $1 < p < \infty$. $\square$

Proof of Theorem 4:- Let $\{x_n\}, \{y_n\} \in \ell^p$. Then $\{-y_n\} \in \ell^p$.

In fact $\|\{y_n\}\|_p = \|\{-y_n\}\|_p$.

By Minkowski's inequality we have

$$d_p(\{x_n\}, \{y_n\}) = \|\{x_n - y_n\}\|_p \leq \|\{x_n\}\|_p + \|\{-y_n\}\|_p < \infty .$$

Hence, $d_p : \ell^p \times \ell^p \longrightarrow \mathbb{R}$ and by definition it is non-negative.

Moreover, if $d_p(\{x_n\}, \{y_n\}) = 0$ then

$$\sum_{n=1}^{\infty} |x_n - y_n|^p = 0 \iff |x_n - y_n| = 0 \quad \forall n \in \mathbb{N}$$
$$\iff x_n = y_n \quad \forall n \in \mathbb{N}.$$

$d_p(\{x_n\}, \{y_n\}) = d_p(\{y_n\}, \{x_n\})$ is obvious to check.

Finally, once again by Minkowski's inequality, we obtain the triangle inequality:

$\|\{x_n - y_n\}\|_p$
$= \|\{x_n - z_n - y_n + z_n\}\|_p$
$\leq \|\{x_n - z_n\}\|_p + \|z_n - y_n\|_p$, for any $\{z_n\} \in \ell^p$. ◻

Example 9:- $\ell^\infty = \{\{x_n\} \mid x_n \in \mathbb{R},\ \sup_n |x_n| < \infty\}$.

Define $\|\cdot\|_\infty : \ell^\infty \longrightarrow \mathbb{R}$ by

$$\|\{x_n\}\|_\infty = \sup_n |x_n| \quad \text{———} \quad (\text{VIII}).$$

For any $\{x_n\}, \{y_n\} \in \ell^\infty$, observe that

$$|x_n - y_n| \leq |x_n| + |y_n| \leq \|\{x_n\}\|_\infty + \|\{y_n\}\|_\infty .$$

So, $\|\{x_n - y_n\}\|_\infty \leq \|\{x_n\}\|_\infty + \|\{y_n\}\|_\infty$.

Define $d_\infty : \ell^\infty \times \ell^\infty \longrightarrow \mathbb{R}$ by

$$d_\infty(\{x_n\}, \{y_n\}) = \|\{x_n - y_n\}\|_\infty .$$

Then using the property of supremum, it is easy to check that d_∞ is a metric.

Example 10 (p-adic metric):- Let p be a prime number. Every non-zero $x \in \mathbb{Q}$ can be expressed as follows:

$$x = p^k \frac{r}{s}, \quad k \in \mathbb{Z} \text{ unique}, \ r \in \mathbb{Z}, \ s \in \mathbb{N}, \ p \nmid r, \ p \nmid s.$$

Define $|\cdot|_p : \mathbb{Q} \longrightarrow \mathbb{R}$ by

$$|x|_p = \begin{cases} p^{-k} & \text{if } x = p^k \frac{r}{s}, \\ 0 & \text{if } x = 0. \end{cases}$$

In many literatures, k is denoted by $Ord_p(x)$. Ord_p behaves like logarithm:

$$Ord_p(xy) = Ord_p(x) + Ord_p(y),$$
$$Ord_p\left(\frac{x}{y}\right) = Ord_p(x) - Ord_p(y).$$

Let $x = \frac{a}{b}$, $y = \frac{c}{d} \in \mathbb{Q}$. Then $x+y = \frac{ad+bc}{bd}$ and

$$Ord_p(x+y) = Ord_p(ad+bc) - Ord_p(bd).$$

Now the highest power of p dividing $x+y$ is greater than or equal to the minimum of the highest power of p dividing x and the highest power of p dividing y:

$$\begin{aligned}
&Ord_p(x+y) \\
&\geq \min\left(Ord_p(ad), Ord_p(bc)\right) - Ord_p(bd) \\
&= \min\left(Ord_p(a) + Ord_p(d), Ord_p(b) + Ord_p(c)\right) \\
&\quad - Ord_p(b) - Ord_p(d) \\
&= \min\left(Ord_p(a) - Ord_p(b), Ord_p(c) - Ord_p(d)\right) \\
&= \min\left(Ord_p\left(\tfrac{a}{b}\right), Ord_p\left(\tfrac{c}{d}\right)\right) \\
&= \min\left(Ord_p(x), Ord_p(y)\right).
\end{aligned}$$

So, $|x+y|_p = p^{-\mathrm{ord}_p(x+y)}$
$\leq \max\{p^{-\mathrm{ord}_p(x)}, p^{-\mathrm{ord}_p(y)}\}$
$= \max\{|x|_p, |y|_p\}$.

This implies, $|x+y|_p \leq |x|_p + |y|_p$.

Define $d : \mathbb{Q} \times \mathbb{Q} \longrightarrow \mathbb{R}$ by

$$d(x,y) = |x-y|_p.$$

Then d satisfy all the properties of the metric in Definition 1. d is called the <u>p-adic</u> metric.

In fact, $d(x,y) \leq \max\{d(x,z), d(z,y)\}$, which is stronger than triangle inequality. That's why p-adic metric is called <u>non-Archimedean metric.</u>

Example 11 :- Let $\mathcal{F}(S)$ be the set of all finite subsets of a set S.

Define $d : \mathcal{F}(S) \times \mathcal{F}(S) \longrightarrow \mathbb{R}$ by

$$d(A,B) = \mathrm{Card}(A \Delta B).$$

Clearly, $d(A,B) \geq 0$ and $d(A,A) = 0$.

Now, $d(A,B) = 0 \Rightarrow A \Delta B = \phi$
$\Rightarrow (A \cup B) \cap (A \cap B)^c = \phi$

$$\Rightarrow A \cup B = A \cap B$$
$$\Rightarrow A = B.$$

Finally, observe that

$$(A \Delta B) \Delta (B \Delta C) = A \Delta C$$

implies that

$$A \Delta C \subseteq (A \Delta B) \cup (B \Delta C)$$
$$\Rightarrow d(A, C) \leq d(A, B) + d(B, C).$$

Example 12 :- $C[a,b] = \{ f : [a,b] \to \mathbb{R} \mid f \text{ is Continuous} \}$.

Define $\|\cdot\|_\infty : C[a,b] \to \mathbb{R}$ by

$$\|f\|_\infty = \sup \{ |f(x)| \mid x \in [a,b] \}. \quad \text{———} \quad (\overline{\underline{IX}})$$

Define $d_\infty : C[a,b] \times C[a,b] \to \mathbb{R}$ by

$$d_\infty(f_1, f_2) = \|f_1 - f_2\|_\infty. \quad \text{———} \quad (\overline{\underline{X}})$$

d is a metric on $C[a,b]$.

Example 13 :- Once again, on $C[a,b]$ we define another metric d by

$$d(f_1, f_2) = \int_a^b |f_1(x) - f_2(x)| \, dx.$$

Lecture-3 (8 April, 2021)

Exercise 1 :- Suppose $n \in \mathbb{N}$, and for each $1 \leq i \leq n$, (X_i, d_i) is a metric space. Then, the following three functions are metrics on $\prod_{i=1}^{n} X_i$:

1) $\mu_1(a,b) = \sum_{i=1}^{n} d_i(a_i, b_i)$,
2) $\mu_p(a,b) = \left(\sum_{i=1}^{n} d_i(a_i,b_i)^p\right)^{1/p}$, $\quad 1 < p < \infty$
3) $\mu_\infty(a,b) = \max\{d_i(a_i,b_i) \mid 1 \leq i \leq n\}$.

In particular, for $X_i = \mathbb{R}$ and $d_i = |\cdot|$, space $\prod_{i=1}^{n} X_i = \mathbb{R}^n$ and, for each $1 \leq p \leq \infty$, μ_p is the metric induced by the p-norm on $\mathbb{R}^n$.

Definition 14 :- Let (X,d) be a metric space. An open sphere around $x_0 \in X$ with radius $\varepsilon > 0$ is the subset of X defined by
$$S_\varepsilon(x_0) = \{x \in X \mid d(x, x_0) < \varepsilon\}.$$

Definition 15 :- A subset G of a metric space (X,d) is an open set if for any point $x \in G$ there exists a $\varepsilon > 0$ such that
$$S_\varepsilon(x) \subseteq G.$$

Theorem 16 :- In a metric space (X, d), the empty set and the full space X are open sets.

Proof :- Trivial.

Theorem 17 :- In a metric space, each open sphere is an open set

Proof :- Let $S_\varepsilon(x_0)$ be an open sphere in X and $x \in S_\varepsilon(x_0)$.

Since, $d(x_0, x) < \varepsilon \Rightarrow r = \varepsilon - d(x_0, x) > 0$.

Consider $S_r(x)$ and for any $y \in S_r(x)$ we have

$$\begin{aligned} d(x,y) < r \Rightarrow\ & d(x,y) < \varepsilon - d(x_0,x) \\ \Rightarrow\ & d(x_0,x) + d(x,y) < \varepsilon \\ \Rightarrow\ & d(x_0,y) < \varepsilon \Rightarrow y \in S_\varepsilon(x_0). \end{aligned}$$

So, $S_r(x) \subseteq S_\varepsilon(x_0)$; hence $S_\varepsilon(x_0)$ is open. ☒

Theorem 18 :- Let (X, d) be a metric space. A subset G of X is open iff G is a union of open spheres.

Proof :- Let G is open. For every $x \in G$ $\exists$ $\varepsilon_x > 0$ such that $S_{\varepsilon_x}(x) \subseteq G$.

Thus, $\bigcup_{x \in G} S_{\varepsilon_x}(x) \subseteq G$ and, clearly, $G \subseteq \bigcup_{x \in G} S_{\varepsilon_x}(x)$.

So, $G = \bigcup_{x \in G} S_{\varepsilon_x}(x)$.

Conversely, let $G = \bigcup_{\lambda \in \Lambda} S_{\varepsilon_\lambda}(x_\lambda)$. Let $y \in G$.
$\exists\ \lambda' \in \Lambda$ such that $y \in S_{\varepsilon_{\lambda'}}(x_{\lambda'})$. So, $\exists \varepsilon > 0$
such that $S_\varepsilon(y) \subseteq S_{\varepsilon_{\lambda'}}(x_{\lambda'}) \subseteq \bigcup_{\lambda \in \Lambda} S_{\varepsilon_\lambda}(x_\lambda) = G$.
So, G is open $\boxtimes$

Theorem 19:- Let (X, d) be a metric space. Then
1) Any union of open sets in X is open
2) Any finite intersection of open sets in X is open.

Proof:- The first part follows from Theorem 18. For the second statement, suppose $G_1, G_2, \dots G_n$ be open sets in X. Then for any $x \in \bigcap_{i=1}^{n} G_i$ we have for all $1 \le i \le n$,

$$x \in G_i \Rightarrow \exists\ \varepsilon_i > 0 \text{ s.t. } S_{\varepsilon_i}(x) \subseteq G_i.$$

Set $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \dots \varepsilon_n\}$. Then $S_\varepsilon(x) \subseteq S_{\varepsilon_i}(x)$ for all $1 \le i \le n$ and this implies $S_\varepsilon(x) \subseteq \bigcap_{i=1}^{n} S_{\varepsilon_i}(x) \subseteq \bigcap_{i=1}^{n} G_i$ $\boxtimes$

Lecture-4 (9 April, 2021)

Definition 20:- Let (X, d) be a metric space and S be a subset of X. An element $x \in X$ is said to be a <u>limit point</u> of S if $\forall \varepsilon > 0$ $S_\varepsilon(x) \setminus \{x\} \cap S \neq \phi$.

Definition 21 :- A subset S of a metric space is closed if it contains each of its limit points.

Definition 22 :- Let $x_0 \in X$ and $\varepsilon > 0$. Then the closed sphere $S_\varepsilon[x_0]$ with center x_0 and radius ε is the subset of defined by

$$S_\varepsilon[x_0] = \{x \in X \mid d(x, x_0) \leq \varepsilon\}.$$

Theorem 23 :- In any metric space X, the empty set ϕ and the full space X are closed sets.

Proof :- Trivial.

Theorem 24 :- Let (X, d) be a metric space. A subset F of X is closed iff $X \setminus F$ is open.

Proof :- Let F is closed and, WLOG, $X \setminus F \neq \phi$.

Let $x \in X \setminus F \Rightarrow x$ is not a limit point of F.

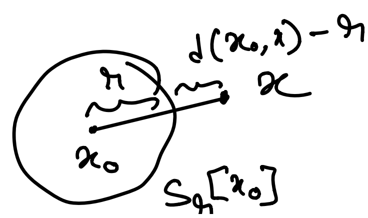
Hence, $\exists\, r>0$ such that $S_r(x) \cap F = \phi$
$\Rightarrow$ $S_r(x) \subseteq X \setminus F$. So, $X\setminus F$ is open.

Conversely, let $X\setminus F$ is open. Let $x \in X\setminus F$. There exists $r>0$ such that $S_r(x) \subseteq X\setminus F$
$\Rightarrow$ $S_r(x) \cap F = \phi$. So, x is not a limit point of F. Negating this statement we get every limit point belongs to F ⊠

Theorem 25:- Every closed sphere in a metric space is a closed set.

Proof:- Let (X,d) is a metric space, $x_0 \in X$ and $r>0$. Consider the closed sphere $S_r[x_0]$ in X.

Let $x \in X \setminus S_r[x_0]$. This means,
$d(x_0, x) > r$.
$\Rightarrow d(x_0,x) - r > 0$.



Let $r_1 = d(x_0, x) - r$ and consider $S_{r_1}(x)$.
If $y \in S_{r_1}(x)$, then $d(x, y) < r_1 = d(x_0, x) - r$.

Now, $d(x_0, x) \leq d(x_0, y) + d(y, x)$

$$\Rightarrow d(x_0, y) \geq d(x_0, x) - d(y, x)$$
$$\Rightarrow d(x_0, y) > d(x_0, x) - r_1$$
$$\Rightarrow d(x_0, y) > r$$
$$\Rightarrow y \in X \setminus S_r[x_0].$$

Therefore, $S_{r_1}(x) \subseteq X \setminus S_r[x_0]$, hence $X \setminus S_r[x_0]$ is open. $\boxtimes$

Theorem 26:- Let (X, d) be a metric space. Then

1) Any intersection of closed sets is closed.
2) Any finite union of closed sets is closed.

Proof:- Directly follows from Theorem 19 and Theorem 24. $\boxtimes$

Let (X, d) be a metric space and $Y \subseteq X$. On Y define a metric $d_Y(y_1, y_2) = d(y_1, y_2)$ $\forall$ $y_1, y_2 \in Y$. Then (Y, d_Y) is a metric subspace of (X, d).

Example 27:- Let $X = [0,1)$ with Euclidean metric. Then $[0,1)$ is both open and closed in $([0,1), |\cdot|)$. But $[0,1)$ neither open nor closed in $(\mathbb{R}, |\cdot|)$.

Additional reference:- Metric Spaces by Mícheál Ó Searcóid. Springer Undergrad Math. Series.

Definition 28:- Let (X,d) be a metric space. A sequence of points $\{x_n\}$ in X is <u>Convergent</u> if there exists a point $x \in X$ such that for each open sphere $S_\varepsilon(x)$ $\exists\, n_0 \in \mathbb{N}$ such that $x_n \in S_\varepsilon(x)$ $\forall\, n \geq n_0$.

Definition 29:- A sequence $\{x_n\}$ in a metric space (X,d) is <u>Cauchy</u> if $\forall \varepsilon > 0$ $\exists\, n_0 \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ $\forall\, m,n \geq n_0$.

Proposition 30:- Every Convergent sequence in (X,d) is Cauchy.

Proof:- Let $\{x_n\}$ in X is convergent and $\lim\limits_{n\to\infty} x_n = x$. Let $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon/2$ for all $n \geq n_0$.

Hence, for any $m \geq n_0$, $d(x_m, x) < \varepsilon/2$. So,

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\forall\ m,n \geq n_0.$ $\boxed{h}$

The converse of Proposition 30 is not true in general. Take $X = [0,1)$, $d = |\cdot|$ and $x_n = 1 - \frac{1}{n}$, $\forall\ n \in \mathbb{N}$. Clearly, $\{x_n\}$ is Cauchy but $\{x_n\}$ does not converge in X.

Lecture-5 (13 April)

Definition 31:- A metric space (X,d) is <u>Complete</u> if every Cauchy sequence is convergent in (X,d).

Example 32:- $(\mathbb{R}, |\cdot|)$ is Complete.

Proposition 33:- Any non-empty set X equipped with discrete metric is complete.

Proof:- Suppose $\{x_n\}$ be a Cauchy sequence in (X, dis). Then for all $\varepsilon > 0$ $\exists\ n_0 \in \mathbb{N}$ such that

$$dis(x_m, x_n) < \varepsilon \quad \forall\ m,n \geq n_0.$$

However, if $x_m \neq x_n$, $dis(x_m, x_n) = 1$. In particular, for a fixed $\varepsilon < 1$, $\exists\ n_0' \in \mathbb{N}$ such that $dis(x_m, x_n) < 1/2$ $\ \forall\ m,n \geq n_0'$.

$\Rightarrow x_m = x_n = x \text{ (say)} \ \forall \ m, n \geq n_0'$.

Hence, the sequence $\{x_n\}$ is a constant sequence for all $n \geq n_0'$. This implies that $x_n \to x$. $\blacksquare$

Definition 34:- Suppose d_1 & d_2 be metrics on a non-empty set X. Then d_1 is <u>topologically equivalent</u> to d_2, denoted by $d_1 \sim d_2$, if the collection of all open sets of (X, d_1) is same as the collection of open sets of (X, d_2).

Theorem 35:- Let d_1 & d_2 be metrics on a non-empty set X. Then the following conditions are equivalent:

a) d_1 and d_2 are topologically equivalent;

b) For each $x \in X$ and open sphere $S_r^{d_i}(x)$, for some $r > 0$, in (X, d_i) there exists $s > 0$ such that $S_s^{d_i}(x) \subseteq S_r^{d_j}(x)$ for $i, j = 1, 2$ and $i \neq j$.

Proof:- a) $\Rightarrow$ b): Suppose $x \in X$ and $S_r^{d_1}(x)$ is an open sphere in (X, d_1), for some $r > 0$. Since, $d_1 \sim d_2$ and $S_r^{d_1}(x)$ is open in (X, d_1), $S_r^{d_1}(x)$ is also open in (X, d_2). There exists $s > 0$ such that $S_s^{d_2}(x) \subseteq S_r^{d_1}(x)$. Similarly, for any $x \in X$ and open sphere $S_r^{d_2}(x)$ in (X, d_2) for some $r > 0$, $\exists$ $s > 0$ such that $S_s^{d_1}(x) \subseteq S_r^{d_2}(x)$.

b) $\Rightarrow$ a) Let U be an open set in (X, d_1). Let $x \in U$. We need to produce an open sphere $S_r^{d_2}(x) \subseteq U$ to conclude that U is also open in (X, d_2).
Since, U is open in (X, d_1), there exists $r > 0$ such that $S_r^{d_1}(x) \subseteq U$. Then $\exists\, s > 0$ such that

$$S_s^{d_2}(x) \subseteq S_r^{d_1}(x) \subseteq U.$$

Hence, U is open in (X, d_2).
Similarly, every open set in (X, d_2) is also open in (X, d_1) $\boxtimes$

Corollary 36:- Two metrics d_1 & d_2 on a non-empty set X are topologically equivalent if there exists real numbers $r, s > 0$ such that

$$\left(\begin{array}{c} d_1(x,y) \le r\, d_2(x,y) \\ \text{and } d_2(x,y) \le s\, d_1(x,y) \\ \forall\ x, y \in X \end{array}\right) \Longleftrightarrow \left(\begin{array}{c} r d_1(x,y) \le d_2(x,y) \le s d_1(x,y) \\ \forall\ x, y \in X. \end{array}\right)$$

This condition is known as <u>strong equivalence</u> of d_1 & d_2.

Proof:- Let $x \in X$ and $S_t^{d_1}(x)$ be an open sphere in (X, d_1) for some $t > 0$. Now $\exists\, r > 0$ such that $\quad d_1(x,y) \le r\, d_2(x,y) \quad \forall\, x, y \in X$.
Choose, $p = \frac{t}{r}$. Then $S_p^{d_2}(x) \subseteq S_t^{d_1}(x)$, because for all $y \in S_p^{d_2}(x) \Rightarrow d_2(x,y) < p \Rightarrow d_1(x,y) < t$

$$\Rightarrow y \in S_t^{d_1}(x).$$

Similarly, we can show that for any open sphere $S_t^{d_2}(x)$

$\exists\ s>0$ such that $S_s^{d_1}(x) \subseteq S_t^{d_2}(x)$; hence $d_1 \sim d_2$ $\boxtimes$

Exercise - 2 :- Converse of Corollary 36 is false. For instance, take $X = (0,1]$, $d_1(x,y) = |x-y|$ and $d_2(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$. Show that d_1 & d_2 are topologically equivalent but not strongly equivalent.

Notation :- If two metrics d_1 & d_2 are strongly equivalent, the we denote $d_1 \sim d_2$ and say <u>d_1 is equivalent to d_2</u>.

Observe that $\sim$ defines an equivalence relation on the set of all metrics on a non-empty set X.

Example 37 :- For $n \in \mathbb{N}$, let (X_i, d_i) is a metric space for $1 \le i \le n$. Recall the metrics μ_p for $1 \le p \le \infty$ on $\prod_{i=1}^{n} X_i$. Then $\mu_{p_1} \sim \mu_{p_2}$ for all $1 \le p_1, p_2 \le \infty$.

In fact, it is sufficient to show that $\mu_p \sim \mu_\infty$ for all $1 \le p < \infty$. By definition,

$$d_p(a,b) = \left[\sum_{i=1}^{n} |a_i - b_i|^p\right]^{1/p}$$

and $d_\infty(a,b) = \max\{|a_i - b_i| \mid 1 \le i \le n\}$.

Then $d_p(a,b) \leq n^{1/p} d_\infty(a,b)$ and $d_p(a,b) \geq d_\infty(a,b)$ for all $1 \leq p < \infty$.

In particular, if $X_i = \mathbb{R}$ and $d_i = |\cdot|$, then the metrics μ_p, for $1 \leq p \leq \infty$, are equivalent on $\mathbb{R}^n$.

Lecture-6 (15 April, 2021)

Theorem 38:- Let d_1 & d_2 be equivalent metrics on X. Suppose (X, d_1) is complete. Then (X, d_2) is also complete.

Proof:- Since d_1 and d_2 are equivalent $\exists\ r, s > 0$ such that

$$r\, d_1(x,y) \leq d_2(x,y) \leq s\, d_1(x,y) \quad \forall\ x, y \in X.$$

Let $\{x_n\}$ be a Cauchy sequence in (X, d_2). Then for every $\varepsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ such that

$$d_2(x_m, x_n) < r\varepsilon \quad \forall\ m, n \geq n_0.$$

This implies $\{x_n\}$ is also a Cauchy sequence in (X, d_1):

$$d_1(x_m, x_n) \leq \frac{1}{r} d_2(x_m, x_n) < \varepsilon \quad \forall\ m, n \geq n_0.$$

Let $\lim_{n\to\infty} x_n = x$ in (X, d_1). Hence for $\forall\, \varepsilon' > 0\ \exists\ n_0' \in \mathbb{N}$ such that $d_1(x_n, x) < \varepsilon'/s\ \forall\ n \geq n_0'$.

Then $d_2(x_n, x) \leq s\, d_1(x_n, x) < \varepsilon' \quad \forall\ n \geq n_0'$.

So, $\lim_{n\to\infty} x_n = x$ in (X, d_2) as well. $\square$

Example 39:- $\mathbb{R}^n$ is Complete with respect to μ_p for all $1 \le p \le \infty$.

Let $\bar{x}_m = (x_m^1, x_m^2, \ldots, x_m^n) \in \mathbb{R}^n$ be a Cauchy sequence in $(\mathbb{R}^n, \mu_\infty)$. Then for $\varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ m, m' such that $\mu_\infty(\bar{x}_m, \bar{x}_{m'}) < \varepsilon$ $\forall$ $m, m' \ge n_0$.

This implies that for all $1 \le i \le n$,

$$|x_m^i - x_{m'}^i| < \varepsilon \quad \forall \quad m, m' \ge n_0.$$

Hence, $\{x_m^i\}$ is a Cauchy sequence in $\mathbb{R}$ for all $1 \le i \le n$. This means $x_m^i \longrightarrow x^i$ as $m \to \infty$ in $\mathbb{R}$ for all $1 \le i \le n$.

We claim that $\bar{x}_m \to \bar{x} = (x^1, x^2, \ldots, x^n)$ as $m \to \infty$ in $(\mathbb{R}^n, \mu_\infty)$.

For $\varepsilon > 0$, $\exists$ $n_0^i \in \mathbb{N}$ such that $|x_m^i - x^i| < \varepsilon$ $\forall m \ge n_0^i$.

Let $n_0 = \max\{n_0^i \mid 1 \le i \le n\}$. Then $|x_m^i - x^i| < \varepsilon$ $\forall$ $m \ge n_0$, $1 \le i \le n$.

Hence, $\mu_\infty(\bar{x}_m, \bar{x}) < \varepsilon$ $\forall$ $m \ge n_0$.

Consequently, $(\mathbb{R}^n, \mu_\infty)$ is Complete. Finally, $\mu_\infty \sim \mu_p$ for all $1 \le p < \infty$. Therefore by Theorem 38, $(\mathbb{R}^n, \mu_p)$ is also complete for all $1 \le p \le \infty$. $\boxtimes$

Example 40:- For any $1 \le p < \infty$, (ℓ^p, d_p) is Complete. Let $\{x_k\}$ be a Cauchy sequence in ℓ^p. Then for any fixed $k \in \mathbb{N}$,

$$x_k = \{x_k^1, x_k^2, x_k^3, \ldots\ldots\} \in \ell^p.$$

Suppose $\varepsilon > 0$. Then $\exists$ $n_0 \in \mathbb{N}$ such that

$$d_p(\{x_m\}, \{x_n\}) < \varepsilon \quad \forall \quad m, n \ge n_0.$$

$$\Rightarrow |x_m^i - x_n^i| \leq d_p(\{x_m\},\{x_n\}) < \varepsilon \quad \forall\ m,n \geq n_0.$$

This means for every $i \in \mathbb{N}$, $\{x_n^i\}$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$. Suppose $x_n^i \to x^i$ as $n \to \infty$ in $(\mathbb{R}, |\cdot|)$ for every $i \in \mathbb{N}$.

Let $\overline{x} = \{x^i\}_{i=1}^{\infty}$. Fix $t \in \mathbb{N}$. Then

$$\left[\sum_{l=1}^{t} |x_m^l - x_n^l|^p\right]^{1/p} \leq d_p(\{x_n\},\{x_n\}) < \varepsilon, \forall\ m,n \geq n$$

Letting $m \to \infty$ and keeping n fixed, we obtain

$$\left[\sum_{l=1}^{t} |x^l - x_n^l|^p\right]^{1/p} < \varepsilon \quad \forall\ n \geq n_0.$$

Since, t was chosen arbitrarily, we have

$$\left[\sum_{l=1}^{\infty} |x^l - x_n^l|^p\right]^{1/p} < \varepsilon \quad \forall\ n \geq n_0.$$

This shows that $\{x_n^i - x^i\} \in \ell^p \Rightarrow \overline{x} = \{x^i\} \in \ell^p$ and $d_p(\{x_n\}, \overline{x}) < \varepsilon \quad \forall\ n \geq n_0.$ ∎

Exercise 3:- (ℓ^∞, d_∞) is complete.

Lecture-7 (16 April, 2021)

Example 41 :- $(C[a,b], \|\cdot\|_\infty)$ is complete.

Let $\{f_n\}$ be a Cauchy sequence in $C[a,b]$. For $\varepsilon>0$ $\exists$ $n_0 \in \mathbb{N}$ such that

$$\|f_m - f_n\|_\infty < \varepsilon \quad \forall \; m,n \geq n_0.$$

Let $t \in [a,b]$. Then

$$|f_m(t) - f_n(t)| \leq \|f_m - f_n\|_\infty < \varepsilon \quad \forall \; m,n \geq n_0.$$

So, $\{f_n(t)\}$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$. Then we get a function $f:[a,b] \longrightarrow \mathbb{R}$ defined by

$$\lim_{n\to\infty} f_n(t) = f(t) \quad \text{for all } t \in [a,b].$$

To prove that $f_n \longrightarrow f$ in $\|\cdot\|_\infty$, recall that $\exists$ $n_0' \in \mathbb{N}$ such that $|f_m(t) - f_n(t)| < \varepsilon/2$ for all $m,n \geq n_0'$, $t \in [a,b]$.

Now fix $t_0 \in [a,b]$. Since, $\lim_{n\to\infty} f_n(t_0) = f(t_0)$, $\exists$ $m \in \mathbb{N}$ (depending on t_0 & ε) such that $m \geq n_0'$

$$|f_m(t_0) - f(t_0)| < \varepsilon/2 .$$

Then for all $n \geq n_0'$

$$|f_n(t_0) - f(t_0)| \leq |f_n(t_0) - f_m(t_0)| + |f_m(t_0) - f(t_0)|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Observe that n_0' is independent of t_0. So, $\forall \; n \geq n_0'$

$$|f_n(t) - f(t)| < \varepsilon \quad \text{for all } a \leq t \leq b.$$
$$\Rightarrow \|f_n - f\|_\infty < \varepsilon \quad \forall \; n \geq n_0'.$$

So, $f_n \xrightarrow{\|\cdot\|_\infty} f$.

Finally, we need to show that f is continuous. Let $t' \in [a,b]$. Fix $\varepsilon > 0$.

Since, $\lim_{n\to\infty} f_n = f$, $\exists\ m \in \mathbb{N}$ such that
$$d_\infty(f_m, f) < \varepsilon/3.$$

Now, $f_m \in C[a,b]$, so it is continuous at t_0. There exists a neighbourhood U of t_0 such that
$$|f_m(y) - f_m(t_0)| < \varepsilon/3 \quad \forall\ y \in U.$$

Then for any $y \in U$, we have

$$\begin{aligned}
|f(y) - f(t_0)| &\leq |f(y) - f_m(y)| + |f_m(y) - f_m(t_0)| + |f_m(t_0) - f(t_0)| \\
&\leq d_\infty(f, f_m) + |f_m(y) - f_m(t_0)| + d_\infty(f_m, f) \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}$$

Hence $f \in C[a,b]$. ☒

Definition 42:- Suppose (X,d) is a metric space and $\phi \neq A \subseteq X$. The diameter of A is defined by $\sup\{d(a_1,a_2) \mid a_1, a_2 \in A\}$ and it is denoted by $\operatorname{diam}(A)$. Moreover, A is <u>bounded</u> if $\operatorname{diam}(A) < \infty$.

Example 43:- In any metric space (X,d), $\operatorname{diam}(\{x\}) = 0 \ \forall\ x \in X$.

Proposition 44:- Suppose (X,d) is a metric space and $\phi \neq A \subseteq B \subseteq X$. Then $\operatorname{diam}(A) \leq \operatorname{diam}(B)$.

Proof:- Since $A \subseteq B$, we have

$$\{d(a_1,a_2) \mid a_1,a_2 \in A\} \subseteq \{d(b_1,b_2) \mid b_1,b_2 \in B\}.$$

$$\Rightarrow \operatorname{diam}(A) \leq \operatorname{diam}(B)$$ ☐

Proposition 45:- Let (X,d) be a metric space and $\phi \neq A \subseteq X$. Then $\operatorname{diam}(A) = \operatorname{diam}(\overline{A})$, where $\overline{A}$ is the closure of A in (X,d).

Proof:- Clearly, $\operatorname{diam}(A) \leq \operatorname{diam}(\overline{A})$ as $A \subseteq \overline{A}$.

To prove the other inequality, fix $\varepsilon > 0$ and let $x, y \in \overline{A}$. Then $S_{\varepsilon/2}(x) \cap A \neq \phi$ and $S_{\varepsilon/2}(y) \cap A \neq \phi$.

Let $x_1 \in S_{\varepsilon/2}(x) \cap A$ and $y_1 \in S_{\varepsilon/2}(y) \cap A$.

Then $x_1, y_1 \in A$ such that $d(x,x_1) < \varepsilon/2$ and $d(y,y_1) < \varepsilon/2$.

Thus

$$d(x,y) \leq d(x,x_1) + d(x_1,y_1) + d(y_1,y)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + d(x_1,y_1)$$

$$\Rightarrow d(x,y) < \varepsilon + d(x_1,y_1) < \varepsilon + \operatorname{diam}(A).$$

Varying $x, y \in \overline{A}$ and taking the supremum on the left hand side, we have

$$\operatorname{diam}(\overline{A}) \leq \varepsilon + \operatorname{diam}(A).$$

Since, $\varepsilon > 0$ was arbitrarily chosen we have

$$\operatorname{diam}(\overline{A}) \leq \operatorname{diam}(A).$$ ☐

Lecture-8 (20 April, 2021)

Theorem Cantor's Intersection Theorem) 46 :- A metric

space (X,d)
is complete. Let $\{F_n\}$ be a sequence of non-empty closed subsets of X such that

a) $\{F_n\}$ is decreasing :- $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$.

b) $\text{diam}(F_n) \to 0$ as $n \to \infty$.

Then (X,d) is Complete iff $F = \bigcap_{n=1}^{\infty} F_n$ is singleton.

Proof :- Let (X,d) be a complete metric space.
Since each F_n is non-empty, choose $x_n \in F_n$ for all $n \in \mathbb{N}$. We claim that $\{x_n\}$ is Cauchy in (X,d). In fact, if $m,n \in \mathbb{N}$ such that $m > n$, then $F_m \subseteq F_n$. Consequently, $x_m, x_n \in F_n$ which implies that

$$d(x_m, x_n) \leq \text{diam}(F_n).$$

Fix $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that

$$\text{diam}(F_n) < \varepsilon \quad \forall \ n \geq n_0.$$

This implies, $d(x_m, x_n) < \varepsilon \quad \forall \quad m > n \geq n_0$.
This show that $\{x_n\}$ is a Cauchy sequence in (X,d) and $\exists \ x \in X$ such that $x_n \xrightarrow{d} x$ as $n \to \infty$.
Again, fix $\varepsilon > 0$. Then $\exists \ n_0' \in \mathbb{N}$ such that

$$x_n \in S_\varepsilon(x) \quad \forall \quad n \geq n_0'.$$

Also, $x_n \in F_n \subseteq F_{n_0'}$ for all $n \geq n_0'$. Hence, $S_\varepsilon(x) \cap F_n \neq \phi$ for all $n \in \mathbb{N}$. Equivalently, x is a limit point of F_n for all $n \in \mathbb{N}$. Now F_n is closed for every $n \in \mathbb{N}$, hence $x \in F_n$ for all $n \in \mathbb{N}$. So, $\bigcap_{n=1}^{\infty} F_n \neq \phi$.

If possible, let $y \in \bigcap_{n=1}^{\infty} F_n \setminus \{x\}$. Then $\operatorname{diam}(F_n) \geq d(x,y) > 0$ for all $n \in \mathbb{N}$. This contradicts the fact that $\operatorname{diam}(F_n) \xrightarrow{d} 0$ as $n \to \infty$. Thus $\bigcap_{n=1}^{\infty} F_n$ is singleton.

To prove the converse, suppose $\{x_n\}$ is a Cauchy sequence in (X,d). For each $n \in \mathbb{N}$ define

$$F_n = \{x_n, x_{n+1}, \dots \}.$$

Clearly, $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ and hence $\overline{F_1} \supseteq \overline{F_2} \supseteq \overline{F_3} \cdots$. Fix $\varepsilon > 0$. There exists $l \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon/2$ $\forall\, m, n \geq l$. So, for all $n \geq l$ $F_n \subseteq F_l$ and

$$\operatorname{diam}(F_n) \leq \operatorname{diam}(F_l) < \varepsilon/2 < \varepsilon.$$

Hence, $\operatorname{diam}(F_n) \longrightarrow 0$ as $n \to \infty$. Then by the previous Proposition 46, we have $\operatorname{diam}(\overline{F_n}) \longrightarrow 0$ as $n \to \infty$. By hypothesis $\bigcap_{i=1}^{\infty} \overline{F_n} = \{x\}$ for some $x \in X$.

Thus $d(x, x_n) \leq \operatorname{diam}(\overline{F_n})$ and $\operatorname{diam}(\overline{F_n}) \to 0$ as $n \to \infty$. Hence $x_n \xrightarrow{d} x$ as $n \to \infty$ in (X,d) ▭

Exercise - 4 :- Verify that none of the conditions that F_n's are closed sets and $\operatorname{diam}(F_n) \to 0$ as $n \to \infty$ can be dropped for getting $\bigcap_{i=1}^{\infty} F_n$ to be singleton.

Proposition 47:- Suppose (X,d) is a metric space. Let $A \subseteq X$. Then

a) If (X,d) is complete and A is closed, then (A,d) is complete.

b) If (A,d) is complete, then A is closed in (X,d).

Proof:-

a) Let $\{x_n\}$ be a Cauchy sequence in A. Since (X,d) is complete, $\exists\, x \in X$ such that $x_n \xrightarrow{d} x$ as $n \to \infty$. Since, A is closed, then $x \in A$; hence (A,d) is complete.

b) If $x \in \bar{A}$, we pick a sequence $\{x_n\}$ in such that $x_n \xrightarrow{d} x$ as $n \to \infty$. Since $\{x_n\}$ is convergent, it is a Cauchy sequence. But (A,d) is complete, it implies that $x_n \xrightarrow{d} a$ as $n \to \infty$ for some $a \in A \subseteq X$. Then by uniqueness of the limit of a sequence, $x = a$. Hence, $x \in \bar{A} \Rightarrow x \in A \Rightarrow \bar{A} = A$. $\square$

Lecture-9 (22 April, 2021)

Definition 48:- Let (X,d_X) and (Y,d_Y) be metric spaces. A bijection $f:(X,d_X) \to (Y,d_Y)$ is an <u>isometry</u> if for any $x_1, x_2 \in X$, $d_X(x_1,x_2) = d_Y(f(x_1), f(x_2))$. Then (X,d_X) is said to be isometric with (Y,d_Y).

Definition 49:- A complete metric space (Y, d^*) is said to be a completion of a metric space (X, d), if (X, d) is isometric with a dense subspace of (Y, d^*). Equivalently, there exists an isometry f from (X, d) onto $(f(X), d^*)$ such that $\overline{f(X)} = Y$.

Theorem 50:- Every metric space has a completion.

Lemma 51:- Let $\{x_n\}, \{y_n\}$ be Cauchy sequences in a metric space (X, d). If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences then $\{d(x_n, y_n)\}$ is a convergent sequence in $\mathbb{R}$.

Proof:- Let $\varepsilon > 0$. There exists $m, k \in \mathbb{N}$ such that

$$d(x_i, x_j) < \varepsilon/2 \quad \forall \ i, j \geq m$$

and

$$d(y_i, y_j) < \varepsilon/2 \quad \forall \ i, j \geq k.$$

Then for all $i, j \geq \max\{m, k\} = n$ we have,

$$d(x_i, y_i) \leq d(x_i, x_j) + d(x_j, y_j) + d(y_j, y_i)$$
$$< \varepsilon + d(x_j, y_j)$$

$$\Rightarrow d(x_i, y_i) - d(x_j, y_j) < \varepsilon \quad \forall \ i, j \geq n.$$

Interchanging i & j we obtain

$$d(x_j, y_j) - d(x_i, y_i) < \varepsilon \quad \forall \ i, j \geq n.$$

Hence, $|d(x_i, y_i) - d(x_j, y_j)| < \varepsilon \quad \forall \ i, j \geq n.$

So, $\{d(x_n, y_n)\}$ is a Cauchy sequence in $\mathbb{R}$, hence convergent.

$\boxed{\Delta}$

Proof of Theorem 50:- Let $\mathcal{E}$ be the set of all of Cauchy sequences in a metric space (X,d).

Define $d_1 : \mathcal{E} \times \mathcal{E} \longrightarrow \mathbb{R}$ by

$d_1(\{x_n\}, \{y_n\}) = \lim_{n\to\infty} d(x_n, y_n)$ for all $\{x_n\}, \{y_n\} \in \mathcal{E}$.

Since $\{d(x_n, y_n)\}$ is a sequence of non-negative real numbers, $d_1(X,Y) \geq 0$ and $d_1(X,Y) = 0$ if $X=Y$, $\forall\ X,Y \in \mathcal{E}$.

Also, $d_1(X,Y) = d_1(Y,X)$ and $d_1(X,Z) \leq d_1(X,Y) + d_1(Y,Z)$ $\forall\ X,Y,Z \in \mathcal{E}$.

Observe that $d_1(X,Y) = 0$ does not imply $X=Y$. So, d_1 is not a metric on $\mathcal{E}$.

Define a relation $\sim$ on $\mathcal{E}$ as $X \sim Y$ iff $d_1(X,Y) = 0$ $\forall\ X,Y \in \mathcal{E}$.

$\sim$ is a equivalence relation on $\mathcal{E}$:

i) $X \sim X$ as $d_1(X,X) = 0$ $\forall\ X \in \mathcal{E}$.

ii) If $X \sim Y$ iff $d_1(X,Y) = d_1(Y,X) \Rightarrow Y \sim X$ $\forall\ X,Y \in \mathcal{E}$.

iii) If $X \sim Y$ & $Y \sim Z$ iff $d_1(X,Y) = 0$ and $d_1(Y,Z)$.

Then $d_1(X,Z) \leq d_1(X,Y) + d_1(Y,Z) = 0$

$\Rightarrow d_1(X,Z) = 0$

$\Rightarrow X \sim Z$, $\forall\ X,Y,Z \in \mathcal{E}$.

Thus $\sim$ is an equivalence relation on $\mathcal{E}$. For any $X \in \mathcal{E}$ denote it's equivalence class by $[X] = \{Y \in \mathcal{E} \mid X \sim Y\}$.

Let $\mathcal{E}/\sim\ = \{[X] \mid X \in \mathcal{E}\}$, the quotient set of all $\sim$-equivalence classes.

Define $d^*: \mathcal{C}/\sim \times \mathcal{C}/\sim \longrightarrow \mathbb{R}$ by

$$d^*([X],[Y]) = d_1(X,Y).$$

Suppose $A \in [X]$ and $B \in [Y]$. Then

$$d_1(A,B) \leq d_1(A,X) + d_1(X,Y) + d_1(Y,B)$$
$$= d_1(X,Y).$$

Similarly, $d_1(X,Y) \leq d_1(X,A) + d_1(A,B) + d(B,Y)$

$$= d_1(A,B).$$

So, $d_1(A,B) = d_1(X,Y)$ for all $A \in [X]$, $B \in [Y]$.

Thus, d^* is well-defined.

Suppose $d^*([X],[Y]) = 0$

$$\Rightarrow d_1(X,Y) = 0$$
$$\Rightarrow X \sim Y$$
$$\Rightarrow [X] = [Y].$$

Then using the properties of d_1 it is easy to verify that d^* is a metric on $\mathcal{C}/\sim$.

Lecture-10 (23 April, 2021)

For each $x \in X$, let S_x denotes the class in $\mathcal{C}/\sim$.

Observe that the constant sequence $\{x_n = x\} \in S_x$.

Let $A = \{S_x \mid x \in X\}$.

Define $f: X \longrightarrow A$ by $f(x) = S_x$.

f is clearly surjective. If $f(x_1) = f(x_2)$ then

$S_{x_1} = S_{x_2} \Rightarrow x_1 = x_2$. (Why??)

So, f is a bijection. Furthermore,

$$d^*(f(x), f(y)) = d^*(\rho_x, \rho_y) = d_1(\{x_n = x\}, \{y_n = y\})$$
$$= d(x, y).$$

So, f is an isometry.

Now we need to show that $\overline{A} = \mathcal{C}/\sim$.

Let $[X] \in \mathcal{C}/\sim$, where $X = (x_1, x_2, \ldots) \in \mathcal{C}$.

Let $\varepsilon > 0$ and consider the open sphere $S_\varepsilon^{d^*}([X])$.

Since X is Cauchy sequence in (X, d), $\exists\, r \in \mathbb{N}$ such that $m \geq r \Rightarrow d(x_m, x_r) < \varepsilon/2$.

Let $Z = (x_r, x_r, \ldots)$.

Then $d^*([X], \rho_{x_r}) = d_1(X, Z)$
$$= \lim_{m \to \infty} d(x_m, x_r) \leq \varepsilon/2 < \varepsilon.$$

This shows that $\rho_{x_r} \in S_\varepsilon^{d^*}([X]) \cap A \Rightarrow [X] \in \overline{A}$.

So, $\overline{A} = \mathcal{C}/\sim$.

The last step is to show that $(\mathcal{C}/\sim, d^*)$ is complete.

Let $([X_1], [X_2], \ldots)$ be a Cauchy sequence in $\mathcal{C}/\sim$. Hence, $X_n \in \mathcal{C}$ $\forall\, n \in \mathbb{N}$.

We denote $X_n = \{x_n^1, x_n^2, \ldots\}$ where $x_i^j \in X$.

$$\begin{array}{lllll}
X_1 : & x_1^1 & x_1^2 \; \cdots & x_1^m, & \cdots \\
X_2 : & x_2^1 & x_2^2 \; \cdots & x_2^m, & \cdots \\
\vdots & & & & \\
X_n : & x_n^1 & x_n^2 \; \cdots & x_n^m, & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

For each $n \in \mathbb{N}$, Since X_n is Cauchy in X, $\exists\, M_n \in \mathbb{N}$ such that

$$d(x_n^i, x_n^{M_n}) < 1/n = \varepsilon \quad \forall\ i \geq M_n \ \text{—}\ (\cancel{XI})$$

Let $Y_n = (x_n^{M_n}, x_n^{M_n}, \ldots\ldots)$ be the Constant sequence in (X,d).

By the previous scheme

$$\begin{array}{lllllll}
Y_1 : & x_1^{M_1} & x_1^{M_1} & \ldots & x_1^{M_1} & \ldots \\
Y_2 : & x_2^{M_2} & x_2^{M_2} & \ldots & x_2^{M_2} & \ldots \\
\vdots & & & & & \\
Y_n : & x_n^{M_n} & x_n^{M_n} & \ldots & x_n^{M_n} & \ldots \\
\vdots & \vdots & \vdots & & \vdots &
\end{array}$$

Let us consider the element $T_n = [Y_n]$ $\forall\, n \in \mathbb{N}$.
Then $T_n \in \mathcal{E}/\sim$ $\forall\, n \in \mathbb{N}$. Consider the sequence $\{T_n\}$ in $\mathcal{E}/\sim$.
By $(\cancel{XI})$ we observe that

$$d^*([X_n],[Y_n]) = d_1(X_n, Y_n)$$
$$= \lim_{i\to\infty} d(x_n^i, x_n^{M_n}) \leq \frac{1}{n}.$$

$$\Rightarrow \lim_{n\to\infty} d^*([X_n],[Y_n]) \longrightarrow 0 \text{ as } n \to \infty.$$

Exercise 5 :- Let $\{x_n\}, \{y_n\}$ be sequences in (X,d). If $d(x_n, y_n) \to 0$ in $(\mathbb{R}, |\cdot|)$ and $\{x_n\}$ is Cauchy, then $\{y_n\}$ is also Cauchy.

Then by Exercise 5, $\{T_n\}$ is a Cauchy sequence in $(\mathscr{C}/\sim, d^*)$ and $\{T_n\} \sim \{[x_n]\}$ with respect to d^*.

Define $z_n = x_n^{M_n}$ $\forall\, n \in \mathbb{N}$. Then $\{z_n\}$ is a sequence in (X, d).

Let $\varepsilon > 0$. Since $\{T_n\}$ is Cauchy, $\exists\, \eta \in \mathbb{N}$ such that

$$d^*(T_n, T_m) < \varepsilon/2 \quad \forall\, m, n \geq \eta.$$

Now,

$$\begin{aligned} d^*(T_n, T_m) &= d^*([Y_n], [Y_m]) \\ &= d_1(Y_n, Y_m) \\ &= d(x_n^{M_n}, x_m^{M_m}) \\ &= d(z_n, z_m). \end{aligned}$$

So, $\{z_n\}$ is a Cauchy sequence in (X, d). Hence $\{z_n\} \in \mathscr{C}$ and

$$\begin{aligned} d^*(T_n, [\{z_n\}]) &= d_1(Y_n, \{z_n\}) \\ &= \lim_{m \to \infty} d(x_n^{M_n}, x_m^{M_m}) \leq \varepsilon/2 < \varepsilon \\ &\qquad \forall\, n \geq \eta. \end{aligned}$$

So, $T_n \to [\{z_n\}]$ in $\mathscr{C}/\sim$ as $n \to \infty$.

Now, $\{[x_n]\} \sim \{T_n\}$ and $T_n \to [\{z_n\}]$ imply $[x_n] \to [\{z_n\}]$ as $n \to \infty$ in $\mathscr{C}/\sim$.

$\blacksquare$

Lecture-11 (27 April, 2021)

Theorem 52:- Any two completions of a metric space (X,d) are isometric.

Proof:- Let (Y_1,d_1) and (Y_2,d_2) be completions of (X,d).

Then (X,d) is isometric to (A,d_1) for some $A \subseteq Y_1$ and to (B,d_2) for some $B \subseteq Y_2$.

Moreover, $\overline{A}^{d_1} = Y_1$ & $\overline{B}^{d_2} = Y_2$.

Let $y_1 \in Y_1$. Since, $\overline{A}^{d_1} = Y_1$ $\exists \{x_n\}$ in A such that $x_n \xrightarrow{d_1} y_1$ as $n \to \infty$.

Since, (A,d_1) is isometric to (B,d_2) there exists a unique sequence $\{y_n\}$ in B such that $d_1(x_i,x_j) = d_2(y_i,y_j)$ $\forall\, i,j \in \mathbb{N}$.

Hence $\{y_n\}$ is a Cauchy sequence in (B,d_2).

Since, (Y_2,d_2) is complete, $\exists\, y_2 \in Y_2$ such that $y_n \xrightarrow{d_2} y_2$ as $n \to \infty$.

Define $f : (Y_1,d_1) \longrightarrow (Y_2,d_2)$ by

$$f(y_1) = y_2.$$

Exercise-6:- Show that f is well-defined: verify that y_2 does not depend on the choice of sequence $\{x_n\}$.

Then, $d_1(y_1, y_1') = \lim_{n\to\infty} d_1(x_n, x_n') = \lim_{n\to\infty} d_2(y_n, y_n')$
$= d_2(y_2, y_2')$.

So, (Y_1, d_1) and $(Y_2 d_2)$ are isometric. ⊠

Theorem 53:- Suppose (X, d_X) and (Y, d_Y) be metric space. Let $f: X \to Y$. Fix $a \in X$. Then the following statements are equivalent:

(a) For all $\varepsilon > 0$, $\exists\ \delta > 0$ such that $d_X(x,a) < \delta$ implies $d_Y(f(x), f(a)) < \varepsilon$.

(b) For each $\{x_n\}$ in X converging to a in (X, d_X), the sequence $\{f(x_n)\}$ in Y converges to $f(a)$ in (Y, d_Y).

(c) For every open set $V \subseteq Y$ containing $f(a)$, there exists an open set $U \subseteq X$ containing a such that $f(U) \subseteq V$.

Proof:- $(a) \Rightarrow (b)$: Let $\{x_n\}$ be a sequence in X that converges to a in (X, d_X).

There exists $m_\delta \in \mathbb{N}$ such that

$$d_X(x_n, a) < \delta \quad \forall\ n \geq m_\delta.$$

$$\Rightarrow \quad d_Y(f(x_n), f(a)) < \varepsilon \quad \forall\ n \geq m_\delta.$$

$\Rightarrow$ $\{f(x_n)\}$ converges to $f(a)$ in (Y, d_Y).

$(b) \Rightarrow (a)$: If possible, let the statement (a) is not true. Then there exists $\varepsilon > 0$ such that for each $\delta > 0$, there exists $x_\delta \in X$ with $d_X(x_\delta, a) < \delta$ but $d_2(f(x_\delta), f(a)) \geq \varepsilon$.

For each $n \in \mathbb{N}$, $\exists\ x_n \in X$ with

$d_X(x_n, a) < \frac{1}{n}$ such that $d_Y(f(x_n), f(a)) \geq \varepsilon$.

This implies that $x_n \xrightarrow{d_X} a$ in (X, d_X) as $n \to \infty$
but $f(x_n) \xrightarrow{d_Y}\!\!\!\!\!\!/\ \ f(a)$ in (Y, d_Y) as $f(x_n) \notin S_\varepsilon^{d_Y}(f(a))$
$\forall\ n \in \mathbb{N}$, which is a contradiction to the hypothesis (b).
So, (b) $\Rightarrow$ (a).

(a) $\Rightarrow$ (c): Let V be an open set in (Y, d_Y) containing $f(a)$.
There exists $\varepsilon > 0$ such that $S_\varepsilon^{d_Y}(f(a)) \subseteq V$.
By (a), $\exists\ \delta > 0$ such that $d_X(x, a) < \delta$ implies
$d_Y(f(x), f(a)) < \varepsilon$. Choose, $U = S_\delta^{d_X}(a)$. Then
$f(U) \subseteq S_\varepsilon^{d_Y}(f(a)) \subseteq V$.

Lecture-12 (April 28, 2021)

(c) $\Rightarrow$ (a): Let $\varepsilon > 0$. Then $V = S_\varepsilon(f(a))$ is an open subset of (Y, d_Y). There exists an open subset U of (X, d_X) such that $a \in U$ and $f(U) \subseteq V$. Since, U is open, $\exists\ \delta > 0$ such that $S_\delta(a) \subseteq U$ (as $a \in U$). This implies
$f(S_\delta(a)) \subseteq f(U) \subseteq S_\varepsilon(f(a))$ and this is equivalent to (a). $\blacksquare$

Exercise - 7 :- Show that (a) in Theorem 53 is equivalent to $\forall \varepsilon > 0$, $\exists\ \delta > 0$ such that
$f(S_\delta(a)) \subseteq S_\varepsilon(f(a))$.

Definition 54:- A function $f:(X,d_X)\longrightarrow(Y,d_Y)$ is said to be <u>Continuous at a point</u> $a\in X$ if one of the statements in Theorem 53 holds true. Furthermore, f is <u>Continuous</u> if it is continuous at every $x\in X$.

Theorem 55:- A function $f:(X,d_X)\longrightarrow(Y,d_Y)$ is continuous if and only if $f^{-1}(V)$ is open in (X,d_X) for all open subsets V of (Y,d_Y).

Proof:- Exercise - 8.

Theorem 56:- The composition of continuous functions is continuous.

Proof:- Exercise - 9.

A sequence $\{x_n\}$ in $\mathbb{R}$ is nothing but a function $h:\mathbb{N}\to\mathbb{R}$ defined by $h(n)=x_n$.

Let $Func(X,Y)=\{f:(X,d_X)\longrightarrow(Y,d_Y)\}$.

Definition 57:- A <u>sequence of functions</u> from (X,d_X) to (Y,d_Y) is a function $h:\mathbb{N}\longrightarrow Func(X,Y)$. Equivalently, for every $n\in\mathbb{N}$, $h(n):X\longrightarrow Y$. Denote $h(n)=f_n$. Then

Then h is equivalent to $\{f_n\}$ where $f_n : X \longrightarrow Y$ $\forall\, n \in \mathbb{N}$.

Definition 58:- Let $\{f_n\}$ be a sequence of functions from (X, d_X) to (Y, d_Y). Then $\{f_n\}$ is said to be

a) <u>pointwise Convergent</u> to a function $f : X \longrightarrow Y$ if $\forall\, x \in X$, $\forall\, \varepsilon > 0$ $\exists\, N \in \mathbb{N}$ such that $d_Y(f_n(x), f(x)) < \varepsilon$ $\forall\, n \geqslant N$.

b) <u>Uniformly Convergent</u> to a function $f : X \longrightarrow Y$ if $\forall\, \varepsilon > 0$ $\exists\, N \in \mathbb{N}$ such that $d_Y(f_n(x), f(x)) < \varepsilon$ $\forall\, n \geqslant N$, $\forall\, x \in X$.

Example 59:- Suppose $f_n(x) = x^n$ $\forall\, x \in [0,1]$. If $0 \leq x < 1$, then $\lim_{n\to\infty} f_n(x) = 0$. On the other hand if $x = 1$, then $x^n \longrightarrow 1$ as $n \to \infty$. So, $f_n \longrightarrow f$ pointwise where

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Suppose, $f_n \longrightarrow f$ uniformly. Let $\varepsilon = 1/2$.
Now,

$$|f_n(x) - f(x)| = \begin{cases} x^n & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x = 1. \end{cases}$$

By hypothesis we must have a fix $N \in \mathbb{N}$ such that

$$|x|^n < 1/2 \quad \forall \; n \geq N, \; \forall \; x \in [0,1).$$

$x = 1 - \frac{1}{3N}$, then $x^N = \left(1 - \frac{1}{3N}\right)^N \geq 1 - \frac{1}{3} = \frac{2}{3}$.

This shows that $f_n \not\to f$ uniformly.

Example 60 :- Suppose $f_n : (0,1) \longrightarrow \mathbb{R}$ defined by $f_n(x) = \frac{n}{nx+1}$ $\forall \; n \in \mathbb{N}$.

Since, $x \neq 0$,

$$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{1}{x + 1/n} = \frac{1}{x}.$$

Now, $|f_n(x)| < n$ for all $x \in (0,1)$. So each f_n is bounded on $(0,1)$. But $f(x)$ is not bounded.

Lecture-13 (29 April, 2021)

Definition 61 :- A function $f : X \longrightarrow Y$ is <u>bounded</u> if $f(X) \subset Y$ is bounded.

Proposition 62 :- Let $\{f_n\}$ be a sequence of sequence of functions $f_n : X \longrightarrow Y$. If each f_n is bounded and $f_n \longrightarrow f$ uniformly, then $f : X \longrightarrow Y$ bounded.

Proof :- By uniform convergence, there exists $m \in \mathbb{N}$

such that

$$d_Y(f_m(x), f(x)) \leq 1 \quad \forall\ x \in X.$$

Since, f_m is bounded, there exists $y \in Y$ and $R > 0$ such that $d_Y(f_m(x), y) \leq R$ for all $x \in X$.

Then by triangle inequality,

$$d_Y(f(x), y) \leq d_Y(f(x), f_m(x)) + d_Y(f_m(x), y)$$
$$\leq R + 1 \quad \forall\ x \in X.$$

Hence, $f : X \longrightarrow Y$ is bounded. $\boxtimes$

Theorem 63:- Let $\{f_n\}$ be a sequence of functions $f_n : X \longrightarrow Y$. If each f_n is continuous at $a \in X$ and $f_n \longrightarrow f$ uniformly, then $f : X \longrightarrow Y$ is continuous at $a \in X$.

Proof:- Let $a \in X$. Since $f_n \longrightarrow f$ uniformly, given $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$d_Y(f_m(x), f(x)) < \varepsilon/3 \quad \forall\ x \in X.$$

Also, continuity of f_m at $a \in X$ implies, there exists $\delta > 0$ such that

$$d_Y(f_m(x), f_m(a)) < \varepsilon/3 \quad \text{if } d_X(x, a) < \delta.$$

Then for $x \in X$ such that $d_X(x, a) < \delta$ implies

$$d_Y(f(x), f(a)) \leq d_Y(f(x), f_m(x)) + d_Y(f_m(x), f_m(a))$$
$$+ d_Y(f_m(a), f(a))$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \quad \boxtimes$$

Definition 64:- A sequence of functions $f_n : X \longrightarrow Y$ is uniformly Cauchy if $\forall\, \varepsilon > 0\ \exists\, N \in \mathbb{N}$ such that $m, n \geq N$ implies that $d_Y(f_m(x), f_n(x)) < \varepsilon$ $\forall\, x \in X$.

Consider $X = \{x\}$ and d_X = discrete metric and $(Y, d_Y) = (\mathbb{R}, |\cdot|)$. Let $f_n : X \longrightarrow Y$ be a sequence of functions. This means for every $n \in \mathbb{N}$ we have $f_n(x) \in \mathbb{R}$. So, f_n is nothing but a sequence $\{x_n = f_n(x)\}$ in $(\mathbb{R}, |\cdot|)$.
Suppose $\{f_n\}$ is uniformly Cauchy. Then for all $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that

$$d_Y(f_m(x), f_n(x)) < \varepsilon \quad \forall\ m, n \geq N$$
$$\Rightarrow \quad |f_m(x) - f_n(x)| < \varepsilon \quad \forall\ m, n \geq N$$
$$\Rightarrow \quad |x_m - x_n| < \varepsilon \quad \forall\ m, n \geq N.$$

So, $\{x_n\}$ is a Cauchy sequence in (Y, d_Y). Hence, $x_n \longrightarrow a \in \mathbb{R}$. This means $f_n \longrightarrow f$ where $f(x) = a$.

Theorem 65:- Let (Y, d_Y) be a complete metric space. Then a uniformly Cauchy sequence of functions $f_n : X \longrightarrow Y$ converges uniformly to a function $f : X \longrightarrow Y$.

Proof :- The uniform Cauchy condition implies that the sequence $\{f_n(x)\}$ is a Cauchy sequence in (Y, d_Y) for every $x \in X$. Since, Y is complete $f_n(x) \longrightarrow z_x \in Y$. Define $f : X \longrightarrow Y$ by

$$f(x) = z_x \;\; \forall \; x \in X.$$

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have

$$d_Y(f_m(x), f_n(x)) < \varepsilon \quad \forall \; x \in X.$$

Taking $m \to \infty$ we have

$$d_Y(f(x), f_n(x)) < \varepsilon \quad \forall \; x \in X, \; \forall n \geq N.$$

This shows that $\{f_n\}$ converges to f uniformly. $\square$

Exercise 10 :- Let $B(X,Y) = \{f : X \to Y \mid f \text{ is bounded}\}$. Define $d_\infty : B(X,Y) \times B(X,Y) \longrightarrow \mathbb{R}$ by $d_\infty(f_1, f_2) = \sup\{d_Y(f_1(x), f_2(x)) \mid x \in X\}$. Show that $(B(X,Y), d_\infty)$ is a metric space. Furthermore, $(B(X,Y), d_\infty)$ is complete whenever (Y, d_Y) is complete.

Lecture-14 (30 April, 2021)

Definition 66 :- The spaces

$$C(X,Y) = \{f : X \to Y \mid f \text{ is continuous}\},$$

and $C_b(X,Y) = \{f : X \to Y \mid f \text{ is continuous \& bounded}\}$.

Theorem 67 :- If (Y, d_Y) is a complete metric space then $(C_b(X,Y), d_\infty)$ is also complete.

Proof :- Clearly, $(C_b(X,Y), d_\infty)$ is a subspace of $(B(X,Y), d_\infty)$. Since, (Y, d_Y) is complete, by Exercise-10, $(B(X,Y), d_\infty)$ is also complete. Hence, it is sufficient to show that $C_b(X,Y)$ is a closed subspace of $(B(X,Y), d_\infty)$ and Proposition 47 will ensure its completeness.

Let $\{f_n\}$ be a sequence in $(C_b(X,Y), d_\infty)$ such that $f_n \xrightarrow{d_\infty} f$. This means for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_\infty(f_n, f) < \varepsilon \quad \forall\, n \geq N$.

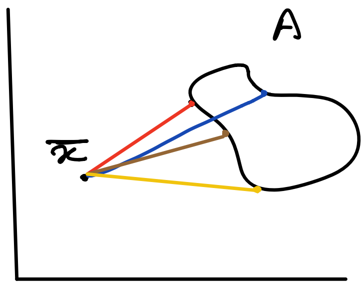
By definition of d_∞, we have

$$d_Y(f_n(x), f(x)) < \varepsilon \quad \forall\, n \geq N,\ \forall\, x \in X.$$

This means $f_n \to f$ uniformly. By Theorem 63, we conclude that $f \in C(X,Y)$. On the other hand, $C_b(X,Y) \subseteq B(X,Y)$ and $(B(X,Y), d_\infty)$ is complete. So, $f \in B(X,Y)$; hence $f \in C_b(X,Y)$. $\boxtimes$

Theorem 68 :- Let A be a subset of a metric space (X,d). Then the function $d(\cdot, A) : X \to \mathbb{R}$ defined by $d(x, A) = \inf\{ d_X(x,a) \mid a \in A\}$ is a continuous function.

In fact, $d(\cdot, A)$ the <u>distance between x and the subset $A \subseteq X$</u>.



Proof :- Let $x_0 \in X$ arbitrarily chosen. Fix $\varepsilon > 0$.
Then it is enough to show that
$x \in S_{\varepsilon/2}(x_0) \Rightarrow |d(x,A) - d(x_0,A)| < \varepsilon.$

Suppose $x \in S_{\varepsilon/2}(x_0)$ and $a \in A$ arbitrarily.
Then $\quad d(x,a) \leq d(x,x_0) + d(x_0,a)$
$$\Rightarrow \inf\{d(x,a) \mid a \in A\} \leq d(x,x_0) + \inf\{d(x_0,a) \mid a \in A\}$$
$$\Rightarrow d(x,A) \leq d(x,x_0) + d(x_0,A)$$
$$\Rightarrow d(x,A) - d(x_0,A) \leq d(x,x_0) < \varepsilon/2$$
Interchanging the roles of x and x_0 we get
$$d(x_0,A) - d(x,A) \leq d(x_0,x) = d(x,x_0) < \varepsilon/2.$$
Thus, $\quad |d(x,A) - d(x_0,A)| < \varepsilon/2. \quad \square$

Theorem 69 (Urysohn's Lemma) :- For any two non-empty disjoint closed subsets A and B of a metric space (X,d), there exists a continuous function $f: X \longrightarrow \mathbb{R}$ such that $f(A) = \{0\}$, $f(B) = \{1\}$ and $0 \leq f(x) \leq 1 \ \forall x \in X.$

Lecture-15 (4 May, 2021)

Suppose A is a closed subset of (X,d). Fix $y \in X \setminus A$. Then $d_A(y) = d(y,A) > 0$ (Check!!). Clearly, $d_A(x) = 0 \;\; \forall\, x \in A$. Define $f : (X,d) \longrightarrow \mathbb{R}$ by $f(x) = \dfrac{d_A(x)}{d_A(y)} \;\; \forall\, x \in X$. Then we observe

that $f(y) = 1$ and $f(A) = \{0\}$.

In this way we can seperate $\{y\}$ and A by a continuous function f.

Proof:- Denote $d_A(x) = d(x,A) \;\; \forall\, x \in A \subseteq X$ and $x \in X$. Define

$$f(x) = \frac{d_A(x)}{d_A(x) + d_B(x)} \quad \forall\, x \in X.$$

If $d_A(x) + d_B(x) = 0$ for some $x \in X$, then $d_A(x) = d_B(x) = 0$. This means $x \in A \cap B = \phi$.
So, f is a well defined continuous function on (X,d). Also it is clear that $0 \le f(x) \le 1$ and $f(A) = \{0\}$ and $f(B) = 1$. ∎

Proposition 70:- Let (X, d_X) and (Y, d_Y) be metric spaces and $f, g : X \longrightarrow Y$ are continuous functions. Then the set $\{x \in X \mid f(x) = g(x)\}$

is closed in X. Equivalently $\{x \in X \mid f(x) \neq g(x)\}$ is an open subset of X.

Proof:- Let $A = \{x \in X \mid f(x) = g(x)\}$ and $a \in X \setminus A$. Then $f(a) \neq g(a)$.

Let $\varepsilon = d_Y(f(a), g(a)) > 0$. Since, f, g are continuous, there exists $\delta_1, \delta_2 > 0$ such that

$$d_X(x,a) < \delta_1 \Rightarrow d_Y(f(x), f(a)) < \varepsilon/3 \text{ and}$$
$$d_Y(x,a) < \delta_2 \Rightarrow d_Y(g(x), g(a)) < \varepsilon/3.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$ and $d_X(x,a) < \delta$

$$\Rightarrow \quad d_Y(f(a), g(a)) \leq d_Y(f(a), f(x)) + d_Y(f(x), g(x)) + d_Y(g(x), g(a))$$
$$< 2\varepsilon/3 + d_Y(f(x), g(x))$$

This implies, $d_Y(f(x), g(x)) > d_Y(f(a), g(a)) - 2\varepsilon/3 = \varepsilon/3 > 0$.

Hence, $x \in X \setminus A$. So, for each $x \in X \setminus A$, $S_\delta^{d_X}(a) \subseteq X \setminus A$. Therefore, $X \setminus A$ is open, so, A is closed. $\square$

Corollary 71:- In particular,

a) If $Y = \mathbb{R}$, $d_Y = |\cdot|$. Fix $c \in \mathbb{R}$ and let $g : X \longrightarrow \mathbb{R}$ defined by $g(x) = c \ \forall x \in X$. Then $\{x \in X \mid f(x) = c\}$ is a closed subset of (X, d).

b) If $Y = X$, $d_Y = d_X$ and $g : X \longrightarrow X$ the identity function on X, i.e. $g(x) = x$ for all $x \in X$. Then $\{x \in X \mid f(x) = x\}$ is a closed subset of (X, d).

Consider $X = \mathbb{R}^{n^2}$ equipped with product metric d_X.
Any point $\overline{x} \in X$ can be written as

$$\overline{x} = (x_{11}, x_{12}, \ldots x_{1n}, x_{21}, x_{22}, \ldots, x_{2n}, \ldots, x_{n1}, x_{n2}, \ldots x_{nn}).$$

$\Updownarrow$

$$A = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}$$

Consider $M_n(\mathbb{R}) = \{A \mid A \text{ is a } n \times n \text{ matrix with real entries}\}$.

Then $f : X \longrightarrow M_n(\mathbb{R})$ defined by
$$f(\overline{x}) = A \text{ is a bijection.}$$
Then we can define a metric d on $M_n(\mathbb{R})$ by
$$d(A, B) = d_X(f(\overline{x}), f(\overline{y})), \text{ where } \overline{x} \,\&\, \overline{y}$$
are pre-images of A & B under f. Thus, $(M_n(\mathbb{R}), d)$ is a metric space.

Then $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is a Continuous function on X (Check!!)

So, $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$ is an open subset of $M_n(\mathbb{R})$.

Similarly $SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) = 1\}$ is a closed subset of $M_n(\mathbb{R})$.

Lecture-16 (May 5, 2021)

Proposition 72:- Let (X, d_X) and (Y, d_Y) be metric spaces and $f, g : X \longrightarrow Y$ continuous functions. If $A \subseteq X$ is dense in X and $f(x) = g(x)$ $\forall x \in A$, then $f(x) = g(x)$ $\forall x \in X$.

Proof:- By Proposition 70, the set $B = \{x \in X \mid f(x) = g(x)\}$ is closed in (X, d_X). Observe that $A \subseteq B$. Then $X = \overline{A}^{d_X} \subseteq \overline{B}^{d_X} = B$. $\boxed{\checkmark}$

Definition 73:- Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \longrightarrow Y$ is said to be uniformly continuous if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $x_1, x_2 \in X$ and $d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon$.

Example 74:- Let $X = (0, 1]$ equipped with the Euclidean metric. Consider the function $f(x) = \frac{1}{x}$ for all $x \in X$. f is continuous but not uniformly continuous. If possible, let f be uniformly continuous. Then for $\varepsilon = 1$, $\exists\, \delta > 0$ such that

$\forall\, x_1, x_2 \in X$ and $|x_1 - x_2| < \delta \Rightarrow \left|\frac{1}{x_1} - \frac{1}{x_2}\right| < 1$.

Choose $n \in \mathbb{N}$ such that $\frac{2}{n(n+2)} < \delta$. For $x_1 = \frac{1}{n}$ and $x_2 = \frac{1}{n+2}$, we get a contradiction.

Consider $x_n = \frac{1}{n}$ $\forall\, n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence in X. But $\{f(x_n)\} = \{n\}$ is not Cauchy in $\mathbb{R}$.

Hence, the continuous functions between metric spaces do not preserve Cauchy sequences.

Proposition 75:- Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \longrightarrow Y$ be uniformly continuous function. If $\{x_n\}$ is a Cauchy sequence in (X, d_X) then $\{f(x_n)\}$ is a Cauchy sequence in (Y, d_Y).

Proof:- Let $\varepsilon > 0$. Since, f is uniformly continuous, $\exists\, \delta > 0$ such that $x, x' \in X$ and $d_X(x, x') < \delta$ implies $d_Y(f(x), f(x')) < \varepsilon$.

Now, $\{x_n\}$ is a Cauchy sequence in (X, d_X). So, $\exists\, N \in \mathbb{N}$ such that $d_X(x_m, x_n) < \delta$ $\forall\, m, n \geq N$. This implies,

$d_Y(f(x_m), f(x_n)) < \varepsilon$ $\forall\, m, n \geq N$. $\boxtimes$

Theorem 76:- Let (X, d_X) be a metric space and $A \subseteq X$ be dense in X. Then any uniformly continuous function $f: A \longrightarrow (Y, d_Y)$, where (Y, d_Y) is a complete metric space, can be extended uniquely to a uniformly continuous function $g: X \longrightarrow Y$.

Proof:- If $A = X$, then the statement is obvious. So, we assume $A \neq X$. Define $g: X \longrightarrow Y$ as follows. For $x \in A$, $g(x) = f(x)$.
If $x \in X \setminus A$, then x is a limit point of A. Hence, $\exists \{a_n\}$ in A such that $a_n \xrightarrow{d_X} x$ as $n \to \infty$.
Since, every convergent sequence is Cauchy, $\{a_n\}$ is Cauchy, and f being uniformly continuous implies $\{f(a_n)\}$ is Cauchy in (Y, d_Y).

Since, (Y, d_Y) is complete, there exists a unique $y \in Y$ such that $f(a_n) \xrightarrow{d_Y} y$ as $n \to \infty$. Then define $g(x) = y$.

Exercise 11 :- Show that g is well defined.

Then we have $g: X \longrightarrow Y$ extending $f: A \longrightarrow Y$.

To show that g is uniformly continuous, fix $\varepsilon > 0$.

Uniform continuity of f on A implies, there exists $\delta > 0$ such that

$$a, a' \in A \text{ and } d_X(a, a') < \delta \Rightarrow d_Y(f(a), f(a')) < \varepsilon/2.$$

Let $x, x' \in X$ such that $d_X(x, x') < \delta/2$.

Since, $\overline{A} = X$, there are sequences $\{a_n\}$ and $\{b_n\}$ in A such that $a_n \xrightarrow{d_X} x$ and $b_n \xrightarrow{d_X} x'$ as $n \to \infty$.

Then $\exists\ N \in \mathbb{N}$ such that

$$d_X(a_n, x) < \delta/4, \quad d_X(b_n, x') < \delta/4$$

$\forall\ n \geq N$.

Now,

$$\begin{aligned} d_X(a_n, b_n) &\leq d_X(a_n, x) + d_X(x, x') + d_X(x', b_n) \\ &< \delta/4 + \delta/2 + \delta/4 = \delta \quad \forall\ n \geq N. \end{aligned}$$

Hence, $d_Y(f(a_n), f(b_n)) < \varepsilon/2 \quad \forall\ n \geq N$.

This shows that $\lim\limits_{n \to \infty} d_Y(f(a_n), f(b_n)) = 0$.

So, $$\lim_{n \to \infty} d_Y(f(a_n), f(b_n)) = d_Y(g(x), g(x')) \leq \varepsilon/2 < \varepsilon.$$

So, g is uniformly continuous. Finally, by Proposition 72, g is unique. $\boxed{L}$

Tutorial-1 (May 11, 2021)

<u>Theorem (Dini):-</u> Let D be a compact subset of $\mathbb{R}$ and $f_n : D \to \mathbb{R}$ is a continuous monotonically increasing sequence, that is $f_n(x) \leq f_{n+1}(x)\ \forall\ n \in \mathbb{N}$,

$\forall x \in D$, and converges to a continuous $f : D \to \mathbb{R}$ pointwise. Then $f_n \to f$ uniformly as $n \to \infty$.

Proof:– Let $\varepsilon > 0$. For each $n \in \mathbb{N}$, define

$$g_n = f - f_n \quad \text{and}$$

$$E_n = \{ x \in D \mid g_n(x) < \varepsilon \}.$$

Clearly, g_n is continuous. Hence E_n is open.

Also, $g_n - g_{n+1} = f - f_n - f + f_{n+1} = f_{n+1} - f_n \geq 0$.

So, $g_{n+1} \leq g_n$ for all $n \in \mathbb{N}$. Equivalently $\{g_n\}$ is a sequence of continuous monotonically decreasing function.

$$\text{If } y \in E_n \Rightarrow g_n(y) < \varepsilon$$

$$\Rightarrow g_{n+1}(y) \leq g_n(x) < \varepsilon$$

$$\Rightarrow y \in E_{n+1}.$$

This shows that $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$.

Moreover, $\{E_n\}$ is an open cover of D. This is because for any $x \in D$, $\exists\, N' \in \mathbb{N}$ such that

$$|f_{N'}(x) - f(x)| < \varepsilon.$$

$$\Rightarrow |g_{N'}(x)| < \varepsilon.$$

$$\Rightarrow -\varepsilon < g_{N'}(x) < \varepsilon$$

$$\Rightarrow x \in E_{N'}$$

Now D is compact. Hence, $\exists\, N \in \mathbb{N}$ such that

$$D \subseteq \bigcup_{n=1}^{N} E_n = E_N.$$

This means, for any $x \in D \subseteq E_N$, we get

$$g_N(x) < \varepsilon$$
$$\Rightarrow f_N(x) - f(x) < \varepsilon$$
$$(22) \Rightarrow |f_N(x) - f(x)| < \varepsilon$$
$$\Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall\ n \geq N.$$

This says that $f_n \to f$ uniformly as $n \to \infty$.

Problem :- For each $n \in \mathbb{N}$, let $f_n(x) = x^{n-1} - x^n$ for all $x \in [0,1]$. Show that $f_n \to f$ uniformly.

Indeed, for a fixed $x \in [0,1]$, we have,
$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^{n-1}(1-x) = 0.$$

So, $f_n \to f \equiv 0$ pointwise as $n \to \infty$.
Observe that f_n and f are Continuous on $[0,1]$.
For each $x \in [0,1]$,
$$\begin{aligned} f_{n+1}(x) - f_n(x) &= x^n - x^{n+1} - x^{n-1} + x^n \\ &= -x^{n-1}(1 - 2x + x^2) \\ &= -x^{n-1}(1-x)^2 \leq 0. \end{aligned}$$

So, $\{f_n\}$ is monotonically decreasing. Hence, by Dini's theorem $\{-f_n\}$ Converges to $-f$ uniformly.
So, $f_n \to f$ uniformly.

Problem :- Define $f_1(x) = \sqrt{x}$ & $\forall n \geq 2$
$$f_n(x) = \sqrt{x f_{n-1}(x)} \quad \forall\ x \in [0,1].$$

Show that $f_n \to f$ uniformly, where f is a Continuous function on $[0,1]$.

$f_1(x) = x^{1/2}$, $f_2(x) = x^{1/2 + 1/2^2}$. More generally,

$$f_n(x) = x^{1/2 + 1/2^2 + 1/2^3 + \cdots + 1/2^n} = x^{1 - 1/2^n}.$$

So, f_n is Continuous on $[0,1]$ for all $n \in \mathbb{N}$.

At $x = 0$, $f_n(x) = 0 \quad \forall\, n \in \mathbb{N}$.

When $x > 0$, $\lim_{n \to \infty} x^{1 - 1/2^n} = x$.

So, $f_n \longrightarrow f$ pointwise, where $f(x) = x \;\forall x \in [0,1]$.

Now, for a fixed $x \in (0,1]$

$$f_{n+1}(x) - f_n(x) = x^{1 - 1/2^{n+1}} - x^{1 - 1/2^n}$$
$$= x^{1 - 1/2^n}\left[x^{1/2^{n+1}} - 1\right]$$
$$\leq 0$$

So, $\{f_n\}$ is monotonically decreasing, hence $f_n \to f$ uniformly.

Tutorial-2 (May 12, 2021)

Theorem :- Let $I = [a,b]$ be a closed and bounded interval and for each $n \in \mathbb{N}$, $f_n : I \to \mathbb{R}$ be Riemann integrable on I. If $\{f_n\}$ converges uniformly to a function $f : I \to \mathbb{R}$, then f is Riemann integrable. Moreover, $\int_a^b f(x)\,dx = \lim \int_a^b f_n(x)\,dx$.

$n \to \infty \qquad a$

<u>Proof :-</u> Let us choose $\varepsilon > 0$. Since $\{f_n\}$ is uniformly Convergent on $[a, b]$ to the function f, there exists a natural number K such that for all $x \in [a, b]$,

$$|f_n(x) - f(x)| < \varepsilon/4(b-a) \quad \forall\, n \geq K.$$

Let $R_n(x) = f_n(x) - f(x)$, $\forall\, x \in [a, b]$.
Then $|R_K(x)| < \varepsilon/4(b-a)$ $\forall\, x \in [a, b]$; hence R_K is bounded on $[a, b]$.

Let us take a partition $P = \{x_0, x_1, \dots x_n\}$ of $[a, b]$ such that $a = x_0 < x_1 < x_2 < \dots < x_n = b$.

Let $M_r = \sup\limits_{x \in [x_{r-1}, x_r]} R_K(x)$, $m_r = \inf\limits_{x \in [x_{r-1}, x_r]} R_K(x)$,

$r = 1, 2, \dots n$.

Then $U(P, R_K) - L(P, R_K)$

$$= (M_1 - m_1)(x_1 - x_0) + (M_2 - m_2)(x_2 - x_1) + \dots + (M_n - m_n)(x_n - x_{n-1})$$

Since, $-\dfrac{\varepsilon}{4(b-a)} < R_K(x) < \dfrac{\varepsilon}{4(b-a)}$ $\forall\, x \in [a, b]$

$M_{r} \leqslant \varepsilon/4(b-a)$ and $m_{r} \geqslant -\varepsilon/4(b-a)$.

This gives $M_r - m_r \leqslant \varepsilon/2(b-a)$ gives

$$U(P, R_K) - L(P, R_K) \leqslant \varepsilon/2(b-a) \times (b-a)$$
$$= \varepsilon/2 < \varepsilon.$$

This proves R_K is Riemann integrable on $[a,b]$.

Now, $f(x) = f_K(x) - R_K(x)$ and as both f_K and R_K are Riemann integrable on $[a,b]$, f is also Riemann integrable on $[a,b]$.

For the second part, since $\{f_n\}$ converges uniformly to f on $[a,b]$, $\lim_{n\to\infty} M_n = 0$,

where $M_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$.

We have $\left|\int_a^b [f_n(x) - f(x)]\,dx\right|$
$$\leqslant \int_a^b |f_n(x) - f(x)|\,dx$$
$$\leqslant M_n(b-a)$$

Let us choose $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $M_n < \varepsilon \;\; \forall\, n \geqslant N$.

Hence, $\left|\int_a^b [f_n(x) - f(x)]\,dx\right| < \varepsilon \;\; \forall\, n \geqslant N$.

So, $\lim_{n\to\infty} \int_a^b f_n(x) = \int_a^b f(x)\,dx$. ☒

Example :- Let $f_n(x) = \frac{nx}{1+n^2x^2}$ for all $x \in [0,1]$.

For a fixed $x \in [0,1]$, we have $\lim_{n\to\infty} f_n(x) = 0$. So, $f_n(x) \longrightarrow f(x) = 0$ for all $x \in [0,1]$.

Each $f_n(x)$ is integrable on $[0,1]$:

$$\int_0^1 f_n(x)\,dx = \frac{1}{2n} \ln(1+n^2x^2)\Big|_0^1 = \frac{1}{2n}\ln(1+n^2).$$

Now, $$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \frac{\ln(1+n^2)}{2n} = 0.$$

So, $\{\int_0^1 f_n(x)\,dx\}$ converges to $\int_0^1 f(x)\,dx$.

However, $\{f_n\}$ does not converge to f uniformly on $[0,1]$.

Let $\varepsilon = 1/4$. If $\{f_n\}$ converges uniformly to f, $\exists\ K \in \mathbb{N}$ such that $\forall\ x \in [0,1]$

$$|f_n(x) - f(x)| < 1/4 \quad \forall\ n \geq K. \quad \text{——} (*)$$

In particular,

$$|f_K(x) - f(x)| < 1/4 \quad \forall\ x \in [0,1].$$

So, we must have

$$|f_K(1/K)| < 1/4.$$

But $$f_K(1/K) = \frac{K\cdot 1/K}{1+K^2\cdot 1/K^2} = \frac{1}{2}.$$

So, there does not exists any such K for which $(*)$ holds true.

<u>Example:-</u> For each $n \in \mathbb{N}$, let $f_n(x) = nxe^{-nx^2}$ for all $x \in [0,1]$.

For all $x \in (0,1]$, $e^{nx^2} > \frac{n^2x^4}{2} > 0$.

We have $0 < nxe^{-nx^2} < 2/nx^3$ for all $x \in (0,1]$

By Sandwich Theorem / Squeeze theorem,

$$f(x) = \lim_{n\to\infty} f_n(x) = 0 \text{ for all } x \in [0,1].$$

Each f_n is integrable on $[0,1]$:

$$\int_0^1 f_n(x)\,dx = \int_0^1 nxe^{-nx^2}\,dx = \left[-\tfrac{1}{2}e^{-nx^2}\right]_0^1 = \frac{1}{2}(1-e^{-n}).$$

f is also integrable on $[0,1]$ and $\int_0^1 f(x)\,dx = 0$.

But $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = 1/2 \neq \int_0^1 f(x)\,dx = 0$.

So, $\{f_n\}$ does not converge to f uniformly.

Tutorial-3 (May 13, 2021)

Let $\{f_n\}$ be a sequence of functions on $[a,b]$ such that for each $n \in \mathbb{N}$, $f_n'(x)$ exists for all $x \in [a,b]$. Let $\{f_n\}$ converge uniformly to a function f on $[a,b]$.

<u>Question:-</u> Does $\{f_n'\}$ converge to f' on $[a,b]$.

Answer :- NO!!

For example, $f_n(x) = \frac{\sin nx}{n}$ $\forall\, x \in [0,1]$.
Then $\{f_n\}$ converge uniformly to the function f, where $f(x) = 0$ $\forall\, x \in [0,1]$.
Clearly,

$$f'(x) = 0 \ \forall\, x \in [0,1] \ \text{and}$$
$$f_n'(x) = \cos nx \ \forall\, x \in [0,1].$$

At $x = 0$, $f_n'(0) = 1$ and $f'(0) = 0$.

Theorem :- Let $\{f_n\}$ be a sequence of functions on $[a,b]$ such that for each $n \in \mathbb{N}$, $f_n'(x)$ exists for all $x \in [a,b]$. If the sequence of derivatives $\{f_n'\}$ converge uniformly on $[a,b]$ to a function g and the sequence $\{f_n\}$ converges at least at one point $x_0 \in [a,b]$ and if the limit function be f, then $f'(x) = g(x)$ $\forall\, x \in [a,b]$.

Proof :- Choose $\varepsilon > 0$. Since $\{f_n'\}$ converges to g on $[a,b]$ uniformly, there exists a natural number k_1 such that
$\forall\, x \in [a,b]$, $|f_{n+p}'(x) - f_n'(x)| < \varepsilon/2(b-a)$ for all $n \geq k_1$ and $p \in \mathbb{N}$.
Also, since $\{f_n(x_0)\}$ is convergent, there exists $k_2 \in \mathbb{N}$ such that

$|f_{n+p}(x_0) - f_n(x_0)| < \varepsilon/2 \quad \forall \ n \geq k_2$ and $p \in \mathbb{N}$.

Let $K = \max\{k_1, k_2\}$. Then for all $x \in [a,b]$,

$|f'_{n+p}(x) - f'_n(x)| < \varepsilon/2(b-a)$ and $|f_{n+p}(x_0) - f_n(x_0)| < \varepsilon/2$

for all $p \in \mathbb{N}$.

Applying Lagrange's mean value theorem to $f_{n+p} - f_n$ on $[x_0, x]$ or $[x, x_0]$ for $x \in [a,b]$

$$|f_{n+p}(x) - f_n(x) - f_{n+p}(x_0) + f_n(x_0)|$$
$$= |x - x_0| \, |f'_{n+p}(\xi) - f'_n(\xi)|, \text{ where } x_0 < \xi < x \text{ or } x < \xi < x_0.$$

Now, $|f'_{n+p}(\xi) - f'_n(\xi)| < \varepsilon/2(b-a)$ for all $n \geq K$, $p \in \mathbb{N}$ and $|x - x_0| < (b-a)$.

It follows that for all $x \in [a,b]$,

$|f_{n+p}(x) - f_n(x) - f_{n+p}(x_0) + f_n(x_0)| < \varepsilon/2$, for all $n \geq K$ and $p \in \mathbb{N}$.

Using triangle inequality, for all $x \in [a,b]$,

$$|f_{n+p}(x) - f_n(x)| \leq |f_{n+p}(x) - f_n(x) - f_{n+p}(x_0) + f_n(x_0)| + |f_{n+p}(x_0) - f_n(x_0)| < \varepsilon$$

for all $n \geq K$ and $p \in \mathbb{N}$.

This proves that $\{f_n\}$ is uniformly convergent on $[a,b]$.

Let f be the uniform limit of $\{f_n\}$ on $[a,b]$.
We now prove that $f'(x) = g(x)$, $\forall\, x \in [a,b]$.
Let $c \in [a,b]$.
Applying Lagrange's mean value theorem to $f_m - f_n$ on $[c,x]$ or $[x,c]$,

$$|f_m(x) - f_n(x) - f_m(c) + f_n(c)| = |x-c|\,|f_m'(\eta) - f_n'(\eta)|$$

where $c < \eta < x$ or $x < \eta < c$.

Therefore,
$$\left| \frac{f_m(x) - f_m(c)}{x-c} - \frac{f_n(x) - f_n(c)}{x-c} \right|$$
$$= |f_m'(\eta) - f_n'(\eta)| \quad \text{for all } x \in [a,b]\setminus\{c\}.$$

Since, $\{f_n'\}$ Converges uniformly to g on $[a,b]$, for a preassigned $\varepsilon > 0$, there exists $k_1 \in \mathbb{N}$ such that for all $x \in [a,b]$,

$$|f_m'(x) - f_n'(x)| < \varepsilon/4 \quad \forall\, m,n \geq k_1.$$

In particular, $|f_m'(\eta) - f_n'(\eta)| < \varepsilon/4 \quad \forall\, m,n \geq k_1.$

Hence, for all $x \in [a,b]\setminus\{c\}$.

$$\left| \frac{f_m(x) - f_m(c)}{x-c} - \frac{f_n(x) - f_n(c)}{x-c} \right| < \frac{\varepsilon}{4}$$

for all $m,n \geq k_1$.

Keeping n fixed and letting $m \to \infty$ and noting that $\{f_m\}$ converges to f uniformly, we have

for all $x \in [a,b] \setminus \{c\}$,

$$\left| \frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c} \right| \leq \frac{\varepsilon}{4} < \frac{\varepsilon}{3}.$$

for all $n \geq k_1$.

Since $\lim_{n\to\infty} f_n'(c) = g(c)$, there exists $k_2 \in \mathbb{N}$

such that

$$|f_n'(c) - g(c)| < \varepsilon/3 \quad \forall\, n \geq k_2.$$

Let $k = \max\{k_1, k_2\}$.

In particular,

$$\left| \frac{f(x)-f(c)}{x-c} - \frac{f_k(x)-f_k(c)}{x-c} \right| < \varepsilon/3 \quad \text{and}$$

$$|f_k'(c) - g(c)| < \varepsilon/3 .$$

Again, since $f_k'(c)$ exists, there is a $\delta > 0$ such that

$$\left| \frac{f_k(x) - f_k(c)}{x - c} - f_k'(c) \right| < \varepsilon/3$$

for all $x \in S_\delta(c) \setminus \{c\} \cap [a, b]$.

By triangle inequality,

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right|$$

$$\leq \left| \frac{f(x) - f(c)}{x - c} - \frac{f_k(x) - f_k(c)}{x - c} \right|$$

$$+ \left| \frac{f_k(x) - f_k(c)}{x - c} - f_k'(c) \right|$$

$$+ \left| f_k'(c) - g(c) \right| < \varepsilon$$

for all $x \in S_\delta(c) \setminus \{c\} \cap [a, b]$.

This implies $\displaystyle\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = g(c)$

$$\Rightarrow f'(c) = g(c).$$

Now, $c \in [a, b]$ was chosen arbitrarily, hence $f'(x) = g(x)$ for all $x \in [a, b]$. $\blacksquare$