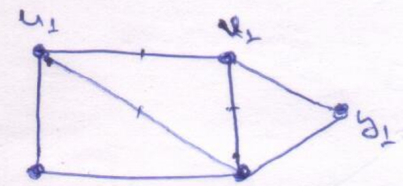
Chordal graph

A chord of a cycle C in a graph G is an edge that joins two non-consecutive vertices of C.

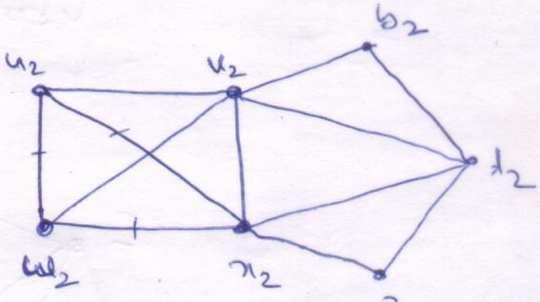
A graph G is chordal if every cycle of length 4 or more in G has a chord.

Ex Every complete graph is a chordal graph.

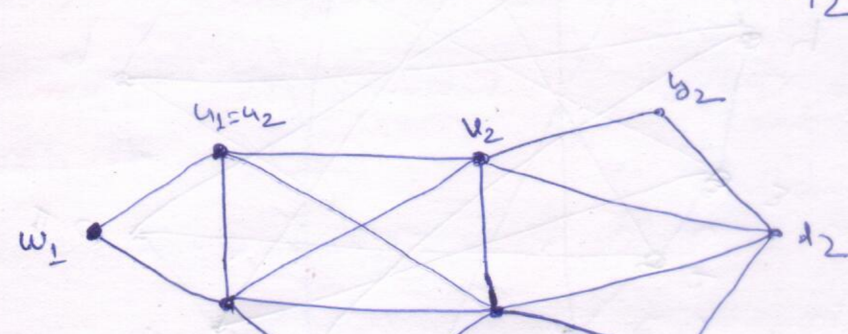
No complete bipartite graph $K_{s,t}$, where $s,t \geq 2$ is chordal.



G_1



G_2



G_3

$\underline{Th^m}$ Let G be a graph obtained by identifying two complete subgraphs of the same order in two graph G_1 and G_2. Then G is chordal if and only if G_1 and G_2 are chordal.

$\underline{Proof}$ Suppose that G_1 and G_2 are two graphs containing complete graph H_1 and H_2, respectively, of the same order and G is the graph obtained by identifying the vertices of H_1 with the vertices of H_2 (in a one-to-one manner). If G contains a cycle C of length 4 or more having no chord then C must belong to G_1 or G_2.

ie. If G_1 and G_2 are chordal then G is chordal.

Suppose G is chordal. If G_1 is not chordal then it would contain a cycle C' of length 4 or more having no chords. Then C' would be a cycle in G having no chord.

Th^m A graph G is chordal if and only if G can be obtained by identifying two complete subgraphs of the same order in two chordal graph.

Proof: If G is complete, say $G = K_n$, then G is chordal and can trivially be obtained by identifying the vertices of $G_1 = K_n$ and the vertices of $G_2 = K_n$ in any one-to-one manner. Hence we may assume that G is a connected chordal graph that is not complete.

Let S be a minimum vertex-cut of G. Now let V_1 be the vertex set of one component of $G-S$ and let $V_2 = V(G) - (V_1 \cup S)$. Consider the two S-branches

$$G_1 = G[V_1 \cup S] \quad \text{and} \quad G_2 = G[V_2 \cup S]$$

of G. Consequently, G is obtained by identifying the vertices of S in G_1 and G_2. We now show that $G[S]$ is complete. Since this is certainly true if $S=1$, we may assume that $|S| \geq 2$.

Each vertex v in S is adjacent to at least one vertex in each component of $G-S$, for otherwise $S - \{v\}$ is a vertex-cut of G, which is impossible.

Let $u, w \in S$. Hence there are u-w paths in G_1, where every vertex except u and w belongs to V_1. Among all such paths, let $P = (u, x_1, x_2, \cdots, x_s, w)$ be one of ~~the~~ minimum length. Similarly, let $P' = (u, y_1, \cdots, y_t, w)$ be a u-w path of minimum length where every vertex except u and w belongs to V_2. Hence

$$C = (u\, x_1 \cdots x_s\, w\, y_t\, y_{t-1} \cdots y_1, u)$$

is a cycle of length 4 or more in G. Since G is chordal, C contains a chord. No vertex x_i $(1 \le i \le s)$ can be adjacent to a vertex y_j $(1 \le j \le t)$ since S is a vertex cut of G. Furthermore no non-consecutive vertices of P or of P' can be adjacent due to the manner in which P and P' are defined. Thus $uw \in E(G)$, implying that $G[S]$ is complete.

By the above theorem G_1 and G_2 are chordal.

Cor: Every chordal graph is perfect.

Proof: Since every induced subgraph of a chordal graph is also a chordal graph, it suffices to show that if G is a connected chordal graph then $\chi(G) = \omega(G)$.

We proceed by induction on the order n of G.

If $n=1$, then $G=K_1$ and $\chi(G)=\omega(G)=1$. Assume therefore that $\chi(H)=\omega(H)$ for every chordal graph H of order less than n, where $n\geq 2$ and let G be a chordal graph of order $n\geq 2$.

If G is complete, then $\chi(G)=\omega(G)=n$. Hence we may assume that G is not complete. By the above theorem G can be obtained from two chordal graphs G_1 and G_2 by identifying two complete subgraphs of same order in G_1 and G_2.

Observe that

$$\chi(G)\leq \max\{\chi(G_1),\chi(G_2)\}=k$$

By induction hypothesis, $\chi(G_1)=\omega(G_1)$ and $\chi(G_2)=\omega(G_2)$.
Thus $\chi(G)\leq \max\{\omega(G_1),\omega(G_2)\}=k$.

On the other hand, let S denote the set of vertices in G that belong to G_1 and G_2. Thus $G[S]$ is complete and no vertex in $V(G_1)-S$ is adjacent to vertex in $V(G_2)-S$. Hence

$$\omega(G)=\max\{\omega(G_1),\omega(G_2)\}=k.$$

Thus $\chi(G)\geq k$. Therefore $\chi(G)=k=\omega(G)$.

Let G be a graph and $v \in V(G)$. The replication graph $R_v(G)$ of G (with respect to v) is that graph obtained from G by adding a new vertex v' to G and joining v' to the vertices in the closed neighbourhood $N[v]$ of v.

Th^m (The Replication lemma)

Let G be a graph with $v \in V(G)$. If G is perfect then $R_v(G)$ is perfect.

Proof Let $G' = R_v(G)$. First we show $\chi(G') = \omega(G')$.

We consider two cases, depending on whether v belongs to a maximum clique of G.

Case-1 v belongs to a maximal clique of G.

Then $\omega(G') = \omega(G)+1$. Since

$$\chi(G') \leq \chi(G)+1 = \omega(G)+1 = \omega(G'),$$

it follows that $\chi(G') = \omega(G')$.

Case-2 v does not belong to any maximal clique of G.

Suppose that $\chi(G) = \omega(G) = k$. Let there be given a k-coloring of G using the colors $1, 2, \dots, k$. We may assume that v is assigned the color 1.

Let V_1 be the color class consisting of the vertices of G that are colored 1. Thus $v \in V_1$. Since $\omega(G) = k$, every maximum clique of G must contain a vertex of each color. Since v does not belong to a maximal clique, it follows that $|V_1| \geq 2$.

Let $U_1 = V_1 - \{v\}$. Because every maximal clique of G contains a vertex of U_1, it follows that $\omega(G - U_1) = \omega(G) - 1 = k - 1$. Since G is perfect $\chi(G - U_1) = k - 1$.

Let a $k-1$ coloring of $G - U_1$ be given, using the colors $1, 2, \ldots, k-1$. Since V_1 is an independent set of vertices, so is $U_1 \cup \{v'\}$. Assigning the vertices of $U_1 \cup \{v'\}$ the color k produces a k-coloring of G'. Therefore, $k = \omega(G) \leq \omega(G') \leq \chi(G') \leq k$, and so $\chi(G') = \omega(G')$.

It remains to show that $\chi(H) = \omega(H)$ for every induced subgraph H of G'. This is certainly the case if H is a subgraph of G. If H contains v' but not v, then $H \cong G[V(H) - \{v'\} \cup \{v\}]$ and so

$\chi(H) = \omega(H)$. If H contains both v and v' but $H \neq G'$, then H is the replication graph of $G[V(H) - \{v'\}]$ and the argument used to show that $\chi(G') = \omega(G')$ can be applied to show that $\chi(H) = \omega(H)$.

The Strong Perfect Graph Theorem

A graph G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length 5 or more.