Vertex and Edge Cover

Def^n

A vertex cover of a graph G is a set $Q \subseteq V(G)$ that contains at least one endvertex of every edge.

Th^m If G is a bipartite graph, then the maximum size of a matching in G equals the minimum size of a vertex cover of G.

Proof: Let (X, Y) be a bipartition of G. Since distinct vertices must be used to cover the edges of a matching, so $|Q| \geq M$ whenever Q is a vertex cover and M is a matching in G. Given a smallest vertex cover Q of G, we construct a matching of size $|Q|$ to prove that equality can always be achieved.

Partition Q by letting $R = Q \cap X$ and $T = Q \cap Y$. Let H and H' be the subgraphs of G induced by $R \cup (Y-T)$ and $T \cup (X-R)$, respectively. We use Hall's theorem to show that H has a matching that saturates R into $Y-T$ and H' has a matching that saturates T. Since H and H' are disjoint, the two matching together form a matching of size $|Q|$ in G.

Since $R \cup T$ is a vertex cover, so G has no edges from $Y-T$ to $X-R$. For each $S \subseteq R$, we consider $N_H(S)$, which is contained in $Y-T$. If $|N_H(S)| < |S|$, then we can substitute $N_H(S)$ for S in Q to obtain a smaller vertex cover, since $N_H(S)$ covers all edges incident

incident to S that are not covered by T.

The minimality of Q thus yields Hall's condⁿ in H and hence H has a matching that saturates R. Applying the same argument to H′ yields the matching that saturates T.

Defⁿ An edge cover of G is a set L of edges such that every vertex of G is incident to some edges of L.

Only graphs without isolated vertices have edge covers. A perfect matching of G form an edge cover with $\frac{|V(G)|}{2}$ edges. In general we can obtain an edge cover by adding edges to a maximum matching.

Defⁿ An independent set in a graph is a set of pairwise non-adjacent vertices. The independence number of a graph is the maximum size of an independent set of vertices. We denote it by $\alpha(G)$.

Defⁿ A clique in a graph is a set of pairwise adjacent ~~vertices~~ vertices. The maximum order of a clique in G is called the clique number of G.

~~Notation~~

Maximum size of independent set - $\alpha(G)$ - Independence number

maximum size of matching - $\alpha'(G)$ - matching number

minimum size of vertex cover - $\beta(G)$ - vertex covering number

minimum size of edge cover - $\beta'(G)$ - edge covering number.

For a graph G, $\alpha'(G) \leq \frac{|V(G)|}{2}$ and equality happens iff it has a perfect matching

For a bipartite graph G, $\alpha'(G) = \beta(G)$.

Since no edge can cover two vertices of an independent set, so $\beta'(G) \geq \alpha(G)$.

<u>Lemma</u> In a graph G, $S \subseteq V(G)$ is an ~~vertex cover~~ independent set if and only if $\bar{S} = V(G) \setminus S$ is a vertex cover, and hence $\alpha(G) + \beta(G) = |V(G)|$.

<u>Proof</u> If S is an independent set, then every edge is incident to at least one vertex of $\bar{S}$. Conversely, if $\bar{S}$ covers all the edges, then there are no edges joining vertices of S. Hence every maximum independent set is the complement of a minimum vertex cover, and $\alpha(G) + \beta(G) = |V(G)|$.

$\underline{\underline{Th^m}}$ If G is a graph without isolated vertices, then $\alpha'(G) + \beta'(G) = |V(G)|$.

Proof: From a maximum matching M, we will construct an edge cover of size $|V(G)| - M$. Since a smallest edge cover is no bigger than this cover, this will imply that $\beta'(G) \leq |V(G)| - \alpha'(G)$. Also, from a minimum edge cover L, we will construct a matching of size $|V(G)| - |L|$. Since a largest matching is no smaller than this matching, this will imply that $\alpha'(G) \geq |V(G)| - \beta'(G)$. These two inequalities complete the proof.

Let M be a maximum matching in G. We construct an edge cover of G by adding to M one edge incident to each unsaturated vertex. We have used one edge for each vertex, except that each edge of M takes care of two vertices, so the total size of edge cover is $|V(G)| - M$ as desired.

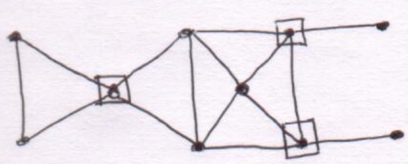
Now let L be a minimum edge cover. If both end-vertices of an edge e belong to edges in L other than e, then $e \notin L$, since $L - e$ is also an edge cover. Hence each component formed by edges of L has at most one vertex of degree exceeding 1 and is a ~~star~~ star. Let k be the number of these components. Then we have $|L| = |V(G)| - k$.

we form a matching in of size $k = |V(G)| - |L|$ by choosing one edge from each star in L.

<u>Cor</u> If G is a bipartite graph with no isolated vertices, then $\alpha(G) = \beta'(G)$.

<u>Defn</u> In a graph G, a set $S \subseteq V(G)$ is a dominating set if every vertex not in S has a neighbour in S. The domination number $\gamma(G)$ is the minimum size of a dominating set in G.

<u>Ex</u>



□ - minimum dominating set

○ - minimal dominating set

G $\qquad \gamma(G) = 3$

When a graph G has no isolated vertices, every vertex cover is a dominating set, so $\gamma(G) \leq \beta(G)$. The difference can be large; $\gamma(K_n) = 1$, $\beta(K_n) = n-1$.