

Calculating the area and centroid of a polygon in 2d

Let $\{(x_i, y_i)\}_{i=0}^{N-1} \subset \mathbb{R}^2$ be a closed polygon in the plane, and let the vertices be ordered counter clockwise. Then it is well-known that the polygon encloses the area

$$A = \frac{1}{2} \sum_{i=0}^{N-1} (x_i y_{i+1} - x_{i+1} y_i),$$

and its centroid is given by

$$\frac{1}{6A} \left(\sum_{i=0}^{N-1} (x_i + x_{i+1}) (x_i y_{i+1} - x_{i+1} y_i), \sum_{i=0}^{N-1} (y_i + y_{i+1}) (x_i y_{i+1} - x_{i+1} y_i) \right)^T \in \mathbb{R}^2;$$

see e.g. paulbourke.net/geometry/polygonmesh.

Calculating the volume and centroid of a polyhedron in 3d

Similar formulas exist for the enclosed volume and centroid of a polyhedron P in \mathbb{R}^3 , but these appear to be less well-known. In the following we assume without loss of generality that the boundary of the polyhedron is given by a union of triangles. (More general facets can easily be subdivided into triangles.) We stress that P need not be convex.

Let A_i , $i = 0, \dots, N-1$, be the N triangular faces of the polyhedron, with vertices (a_i, b_i, c_i) , which are assumed to be ordered counter clockwise on A_i . This means that we can define the outer unit normal n to P on each A_i as $n_i = \hat{n}_i / |\hat{n}_i|$, where $\hat{n}_i = (b_i - a_i) \otimes (c_i - a_i)$. Then the volume of P is given by

$$V = \int_P 1 = \frac{1}{3} \int_{\partial P} x \cdot n = \frac{1}{3} \sum_{i=0}^{N-1} \int_{A_i} a_i \cdot n_i = \frac{1}{6} \sum_{i=0}^{N-1} a_i \cdot \hat{n}_i,$$

where we have used the divergence theorem, the fact that $x \cdot n_i$ is constant on each A_i , and the fact that the area of A_i is given by $\frac{1}{2} |\hat{n}_i|$.

Let $c \in \mathbb{R}^3$ denote the centroid of P , i.e. $c = \frac{1}{V} \int_P x$. Applying the divergence theorem once again, and on denoting the standard basis in \mathbb{R}^3 by $\{e_1, e_2, e_3\}$, we obtain for the three coordinates of the centroid that

$$c \cdot e_d = \frac{1}{V} \int_{\partial P} \frac{1}{2} (x \cdot e_d)^2 (n \cdot e_d) = \frac{1}{2V} \sum_{i=0}^{N-1} \int_{A_i} (x \cdot e_d)^2 (n_i \cdot e_d), \quad d = 1, 2, 3.$$

It remains to compute that

$$\begin{aligned} \int_{A_i} (x \cdot e_d)^2 (n_i \cdot e_d) &= \frac{1}{6} \hat{n}_i \cdot e_d \left(\left[\frac{1}{2} (a_i + b_i) \cdot e_d \right]^2 + \left[\frac{1}{2} (b_i + c_i) \cdot e_d \right]^2 + \left[\frac{1}{2} (c_i + a_i) \cdot e_d \right]^2 \right) \\ &= \frac{1}{24} \hat{n}_i \cdot e_d \left([(a_i + b_i) \cdot e_d]^2 + [(b_i + c_i) \cdot e_d]^2 + [(c_i + a_i) \cdot e_d]^2 \right), \end{aligned}$$

where we have observed that the integrand is a quadratic function on A_i , so that the standard midpoint sampling quadrature formula for triangles yields the integral exactly, see e.g. [1].

References

- [1] A. H. STROUD, *Approximate calculation of multiple integrals*, Prentice-Hall Inc., Englewood Cliffs, N. J., 1971.