

Lecture-1 (6 April, 2021)

Let X be a non-empty set.

Definition 1 :- A metric on X is a function $d: X \times X \rightarrow \mathbb{R}$ which satisfies the following three conditions:

- 1) $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$;
- 2) $d(x, y) = d(y, x)$;
- 3) $d(x, y) \leq d(x, z) + d(z, y)$.

Example 2 (Discrete metric) :-

Let X be a non-empty set. Define $d: X \times X \rightarrow \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases} \quad (\text{I})$$

Example 3 (Euclidean metric) :-

$X = \mathbb{R}/\mathbb{C}$ and define $d: X \times X \rightarrow \mathbb{R}$ by

$$d(x, y) = |x - y|, \quad (\text{II})$$

for all $x, y \in X$.

Fix $1 \leq p < \infty$. Consider the following set

$$l^p = \left\{ \{x_n\} \mid x_n \in \mathbb{R}, \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}. \quad (\text{III})$$

On ℓ^p , we define $\|\cdot\|_p: \ell^p \rightarrow \mathbb{R}$ by

$$\|\{x_n\}\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}. \quad (\text{I})$$

Theorem 4 :- For any two sequences $\{x_n\}$ & $\{y_n\}$ the function d on $\ell^p \times \ell^p$,

defined by

$$d_p(\{x_n\}, \{y_n\}) = \|\{x_n - y_n\}\|_p \quad (\text{II})$$

is a metric on ℓ^p .

Theorem 5 (Höldel's inequality) :- Let $p, q \in (1, \infty)$ such that

$\frac{1}{p} + \frac{1}{q} = 1$. Let $\{x_n\} \in \ell^p$ and $\{y_n\} \in \ell^q$.

Then

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \|\{x_n\}\|_p \|\{y_n\}\|_q. \quad (\text{III})$$

Lemma 6 (Young's inequality) :- Let $p, q \in (1, \infty)$ such that

$\frac{1}{p} + \frac{1}{q} = 1$. Then for any $a, b \geq 0$

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q} \quad (\text{IV})$$

Proof :- For $t \in [1, \infty)$, define

$$f(t) = kt - t^k + 1,$$

where $0 < k < 1$.

Then, $f'(t) = k - kt^{k-1} = k(1 - t^{k-1}) \geq 0$.

So, f is an increasing function; hence $f(t) \geq f(1)$.

This implies, $k(t-1) - t^k + 1 \geq 0 = f(1)$
 $\Rightarrow k(t-1) + 1 \geq t^k \quad \text{--- (II)}$

The statement is clearly true if either $a=0$ or $b=0$.

So, without loss of generality, let $a \geq b > 0$.

Then we put $t = \frac{a}{b}$ and $k = \frac{1}{p}$ in (II):

$$\begin{aligned} \frac{1}{p} \left(\frac{a}{b} - 1 \right) + 1 &\geq \left(\frac{a}{b} \right)^{\frac{1}{p}} \\ \Rightarrow \frac{a}{p} \cdot \frac{1}{b} + \frac{1}{q} &\geq \left(\frac{a}{b} \right)^{\frac{1}{p}} \\ \Rightarrow \frac{a}{p} \cdot b^{\frac{1}{p}-1} + \frac{1}{q} b^{\frac{1}{p}} &\geq a^{\frac{1}{p}} \\ \Rightarrow \frac{a}{p} + \frac{b}{q} &\geq a^{\frac{1}{p}} \cdot b^{\frac{1}{q}} \quad \square \end{aligned}$$

Proof of Theorem 5 :- The statement is trivially true if one of the sequences $\{x_n\}_{n=1}^{\infty}$ or $\{y_n\}_{n=1}^{\infty}$ is a zero sequence.

Therefore, without loss of generality, we assume both the sequences $\{x_n\}$ and $\{y_n\}$ are non-zero.

Now, for a fixed $i \in \mathbb{N}$, set

$$a = \left(|x_i| / \| \{x_n\} \|_p \right)^p, \quad b = \left(|y_i| / \| \{y_n\} \|_q \right)^q.$$

By Lemma 6, we have

$$\frac{|x_i||y_i|}{\|\{x_n\}\|_p \|\{y_n\}\|_q} \leq \frac{1}{p} \left(\frac{|x_i|}{\|\{x_n\}\|_p} \right)^p + \frac{1}{q} \left(\frac{|y_i|}{\|\{y_n\}\|_q} \right)^q$$

Now, summing over $i = 1 \dots n$

$$\begin{aligned} \frac{\sum_{i=1}^n |x_i||y_i|}{\|\{x_n\}\|_p \|\{y_n\}\|_q} &\leq \frac{1}{p} \frac{\sum_{i=1}^n |x_i|^p}{\|\{x_n\}\|_p^p} + \frac{1}{q} \frac{\sum_{i=1}^n |y_i|^q}{\|\{y_n\}\|_q^q} \\ &\leq \frac{1}{p \|\{x_n\}\|_p^p} \|\{x_n\}\|_p^p + \frac{1}{q \|\{y_n\}\|_q^q} \|\{y_n\}\|_q^q \\ &= \frac{1}{p} + \frac{1}{q} = 1. \end{aligned}$$

Since, the last inequality holds for every $n \in \mathbb{N}$
we obtain (III) by letting $n \rightarrow \infty$ in the LHS. \square

Remark 7 (Cauchy - Schwarz inequality):-

In particular,
if $p = q = 2$,

the Hölder's inequality is known as Cauchy - Schwarz inequality

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \|\{x_n\}\|_2 \|\{y_n\}\|_2$$

for all $\{x_n\}, \{y_n\} \subset \ell^2$. — (IV)

Lecture-2 (7 April, 2021)

Theorem 8 (Minkowski's inequality):- Let $1 \leq p < \infty$.
 Let $\{x_n\}, \{y_n\} \in l^p$. Then
 $\|\{x_n + y_n\}\|_p \leq \|\{x_n\}\|_p + \|\{y_n\}\|_p$ ————— (VI)

Proof :- For $p=1$ we prove the statement using triangle inequality for \mathbb{R} with respect to the Euclidean metric.

For any fixed $i \in \mathbb{N}$ we have,

$$\begin{aligned} |x_i + y_i| &\leq |x_i| + |y_i| \leq \sum_{i=1}^{\infty} |x_i| + \sum_{i=1}^{\infty} |y_i| \\ &= \|\{x_n\}\|_1 + \|\{y_n\}\|_1. \end{aligned}$$

Since, i was chosen arbitrarily we have (VI) for $p=1$.

For, $1 < p < \infty$ we have the proof using

$$\begin{aligned} \sum_{i=1}^m |x_i + y_i|^p &\leq \sum_{i=1}^m |x_i + y_i|^{p-1} |x_i + y_i| \\ &\leq \sum_{i=1}^m |x_i + y_i|^{p-1} (|x_i| + \sum_{i=1}^m |x_i + y_i|^{p-1} |y_i|) \end{aligned} \quad \text{————— (VII)}$$

where m is a fixed natural number.

Define $z_n = x_n + y_n$, if $1 \leq n \leq m$,
 $= 0$, otherwise.

Clearly, $\{z_n\} \in \ell^p$ and by Hölder's inequality we get

$$\sum_{i=1}^m |z_i|^{p-1} |x_i|$$

$$\leq \sum_{i=1}^{\infty} |z_i|^{p-1} |x_i|$$

$$\leq \|\{x_n\}\|_p \times \|\{|z_i|^{p-1}\}\|_q$$

$$= \|\{x_n\}\|_p \times \left(\sum_{i=1}^m |x_i + y_i|^{q(p-1)} \right)^{1/q}, \text{ where}$$

$$\frac{1}{p} + \frac{1}{q} = 1 \iff p+q = pq.$$

Therefore,

$$\sum_{i=1}^m (x_i + y_i)^{p-1} |x_i| \leq \|\{x_n\}\|_p \times \left(\sum_{i=1}^m |x_i + y_i|^p \right)^{1/q}.$$

Similarly, we have

$$\sum_{i=1}^m |x_i + y_i|^{p-1} |y_i| \leq \|\{y_n\}\|_p \times \left(\sum_{i=1}^m |x_i + y_i|^p \right)^{1/q}.$$

Using the last two inequalities we simplify (VII) as follows

$$\sum_{i=1}^m |x_i + y_i|^p \leq (\|\{x_n\}\|_p + \|\{y_n\}\|_p) \times \left(\sum_{i=1}^m |x_i + y_i|^p\right)^{\frac{1}{p}}$$

$$\Rightarrow \left(\sum_{i=1}^m |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \|\{x_n\}\|_p + \|\{y_n\}\|_p.$$

Now, letting $m \rightarrow \infty$ on the L.H.S. we obtain
 (VI) for $1 < p < \infty$. □

Proof of Theorem 4 :- Let $\{x_n\}, \{y_n\} \in l^p$.
 Then $\{x_n - y_n\} \in l^p$.

In fact $\|\{y_n\}\|_p = \|\{x_n - y_n\}\|_p$.

By Minkowski's inequality we have

$$d_p(\{x_n\}, \{y_n\}) = \|\{x_n - y_n\}\|_p \leq \|\{x_n\}\|_p + \|\{x_n - y_n\}\|_p < \infty.$$

Hence, $d_p : l^p \times l^p \rightarrow \mathbb{R}$ and by definition it is non-negative.

Moreover, if $d_p(\{x_n\}, \{y_n\}) = 0$ then

$$\sum_{n=1}^{\infty} |x_n - y_n|^p = 0 \Leftrightarrow |x_n - y_n| = 0 \quad \forall n \in \mathbb{N} \\ \Leftrightarrow x_n = y_n \quad \forall n \in \mathbb{N}.$$

$d_p(\{x_n\}, \{y_n\}) = d_p(\{y_n\}, \{x_n\})$ is obvious to check.

Finally, once again by Minkowski's inequality, we obtain the triangle inequality:

$$\begin{aligned}
 & \| \{x_n - y_n\} \|_p \\
 &= \| \{x_n - z_n - y_n + z_n\} \|_p \\
 &\leq \| \{x_n - z_n\} \|_p + \| z_n - y_n \|_p, \text{ for any } \{z_n\} \in l^p.
 \end{aligned}$$

□

Example 9 :- $l^\infty = \{ \{x_n\} \mid x_n \in \mathbb{R}, \sup_n |x_n| < \infty \}$.

Define $\|\cdot\|_\infty : l^\infty \rightarrow \mathbb{R}$ by

$$\| \{x_n\} \|_\infty = \sup_n |x_n| \quad (\text{VIII}).$$

For any $\{x_n\}, \{y_n\} \in l^\infty$, observe that

$$|x_n - y_n| \leq |x_n| + |y_n| \leq \| \{x_n\} \|_\infty + \| \{y_n\} \|_\infty.$$

$$\text{So, } \| \{x_n - y_n\} \|_\infty \leq \| \{x_n\} \|_\infty + \| \{y_n\} \|_\infty.$$

Define $d_\infty : l^\infty \times l^\infty \rightarrow \mathbb{R}$ by

$$d_\infty (\{x_n\}, \{y_n\}) = \| \{x_n - y_n\} \|_\infty.$$

Then using the property of supremum, it is easy to check that d_∞ is a metric.

Example 10 (p -adic metric) :-

Let p be a prime number.

Every non-zero $x \in \mathbb{Q}$

can be expressed as follows :

$$x = p^k \frac{r}{s}, \quad k \in \mathbb{Z} \text{ unique, } r \in \mathbb{Z}, s \in \mathbb{N}, p \nmid r, p \nmid s.$$

Define $| \cdot |_p : \mathbb{Q} \rightarrow \mathbb{R}$ by

$$|x|_p = \begin{cases} p^{-k} & \text{if } x = p^k \frac{a}{b}, \\ 0 & \text{if } x = 0. \end{cases}$$

In many literatures, k is denoted by $\text{Ord}_p(x)$. Ord_p behaves like logarithm:

$$\begin{aligned}\text{Ord}_p(xy) &= \text{Ord}_p(x) + \text{Ord}_p(y), \\ \text{Ord}_p\left(\frac{x}{y}\right) &= \text{Ord}_p(x) - \text{Ord}_p(y).\end{aligned}$$

Let $x = \frac{a}{b}$, $y = \frac{c}{d} \in \mathbb{Q}$. Then $x+y = \frac{ad+bc}{bd}$
and

$$\text{Ord}_p(x+y) = \text{Ord}_p(ad+bc) - \text{Ord}_p(bd).$$

Now the highest power of p dividing $x+y$ is greater than or equal to the minimum of the highest power of p dividing x and the highest power of p dividing y :

$$\begin{aligned}&\text{Ord}_p(x+y) \\ &\geq \min(\text{Ord}_p(ad), \text{Ord}_p(bc)) - \text{Ord}_p(bd) \\ &= \min(\text{Ord}_p(a) + \text{Ord}_p(d), \text{Ord}_p(b) + \text{Ord}_p(c)) \\ &\quad - \text{Ord}_p(b) - \text{Ord}_p(d) \\ &= \min(\text{Ord}_p(a) - \text{Ord}_p(b), \text{Ord}_p(c) - \text{Ord}_p(d)) \\ &= \min(\text{Ord}_p\left(\frac{a}{b}\right), \text{Ord}_p\left(\frac{c}{d}\right)) \\ &= \min(\text{Ord}_p(x), \text{Ord}_p(y)).\end{aligned}$$

$$\begin{aligned} \text{So, } |x+y|_p &= p^{-\text{ord}_p(x+y)} \\ &\leq \max\{p^{-\text{ord}_p(x)}, p^{-\text{ord}_p(y)}\} \\ &= \max\{|x|_p, |y|_p\}. \end{aligned}$$

This implies, $|x+y|_p \leq |x|_p + |y|_p$.

Define $d : \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{R}$ by

$$d(x, y) = |x-y|_p.$$

Then d satisfy all the properties of the metric in Definition 1. d is called the p -adic metric.

In fact, $d(x, y) \leq \max\{d(x, z), d(z, y)\}$, which is stronger than triangle inequality. That's why p -adic metric is called non-Archimedean metric.

Example 11 :- Let $\mathcal{F}(S)$ be the set of all finite subsets of a set S .

Define $d : \mathcal{F}(S) \times \mathcal{F}(S) \rightarrow \mathbb{R}$ by

$$d(A, B) = \text{Card}(A \Delta B).$$

Clearly, $d(A, B) \geq 0$ and $d(A, A) = 0$.

Now, $d(A, B) = 0 \Rightarrow A \Delta B = \emptyset$
 $\Rightarrow (A \cup B) \cap (A \cap B)^c = \emptyset$

$$\Rightarrow A \cup B = A \cap B$$

$$\Rightarrow A = B.$$

Finally, observe that

$$(A \Delta B) \Delta (B \Delta C) = A \Delta C$$

implies that

$$A \Delta C \subseteq (A \Delta B) \cup (B \Delta C)$$

$$\Rightarrow d(A, C) \leq d(A, B) + d(B, C).$$

Example 12 :- $C[a, b] = \{f : [a, b] \rightarrow \mathbb{R} \mid f \text{ is continuous}\}.$

Define $\|\cdot\|_\infty : C[a, b] \rightarrow \mathbb{R}$ by

$$\|f\|_\infty = \sup \{|f(x)| \mid x \in [a, b]\}. \quad (\text{IX})$$

Define $d_\infty : C[a, b] \times C[a, b] \rightarrow \mathbb{R}$ by

$$d_\infty(f_1, f_2) = \|f_1 - f_2\|_\infty. \quad (\text{X})$$

d is a metric on $C[a, b]$.

Example 13 :- Once again, on $C[a, b]$ we define another metric d by

$$d(f_1, f_2) = \int_a^b |f_1(x) - f_2(x)| dx.$$

Lecture-3 (8 April, 2021)

Exercise 1 :-

Suppose $n \in \mathbb{N}$, and for each $1 \leq i \leq n$, (X_i, d_i) is a metric space.

Then the following three functions are metrics on $\prod_{i=1}^n X_i$:

- 1) $\mu_1(a, b) = \sum_{i=1}^n d_i(a_i, b_i),$
- 2) $\mu_p(a, b) = \left(\sum_{i=1}^n d_i(a_i, b_i)^p \right)^{\frac{1}{p}}, \quad 1 < p < \infty$
- 3) $\mu_\infty(a, b) = \max \{d_i(a_i, b_i) \mid 1 \leq i \leq n\}.$

In particular, for $X_i = \mathbb{R}$ and $d_i = |\cdot|$, space $\prod_{i=1}^n X_i = \mathbb{R}^n$ and, for each $1 \leq p \leq \infty$, μ_p is the metric induced by the p -norm on \mathbb{R}^n .

Definition 14 :-

Let (X, d) be a metric space.

An open sphere around $x_0 \in X$

with radius $\epsilon > 0$ is the subset of X defined by

$$S_\epsilon(x_0) = \{x \in X \mid d(x, x_0) < \epsilon\}.$$

Definition 15 :-

A subset G of a metric space

(X, d) is an open set if for

any point $x \in G$ there exists a $\epsilon > 0$ such that

$$S_\epsilon(x) \subseteq G.$$

Theorem 16 :- In a metric space (X, d) , the empty set and the full space X are open sets.

Proof :- Trivial.

Theorem 17 :- In a metric space, each open sphere is an open set

Proof :- Let $S_\epsilon(x_0)$ be an open sphere in X and $x \in S_\epsilon(x_0)$.

Since, $d(x_0, x) < \epsilon \Rightarrow r = \epsilon - d(x_0, x) > 0$.

Consider $S_r(x)$ and for any $y \in S_r(x)$ we have

$$d(x, y) < r \Rightarrow d(x, y) < \epsilon - d(x_0, x)$$

$$\Rightarrow d(x_0, x) + d(x, y) < \epsilon$$

$$\Rightarrow d(x_0, y) < \epsilon \Rightarrow y \in S_\epsilon(x_0).$$

So, $S_r(x) \subseteq S_\epsilon(x_0)$; hence $S_\epsilon(x_0)$ is open. \square

Theorem 18 :- Let (X, d) be a metric space. A subset G of X is open iff G is a union of open spheres.

Proof :- Let G is open. For every $x \in G \exists \epsilon_x > 0$ such that $S_{\epsilon_x}(x) \subseteq G$.

Thus, $\bigcup_{x \in G} S_{\epsilon_x}(x) \subseteq G$ and, clearly, $G \subseteq \bigcup_{x \in G} S_{\epsilon_x}(x)$.

So, $G = \bigcup_{x \in G} S_{\epsilon_x}(x)$.

Conversely, let $G = \bigcup_{\lambda \in \Lambda} S_{\varepsilon_\lambda}(x_\lambda)$. Let $y \in G$.

$\exists \lambda' \in \Lambda$ such that $y \in S_{\varepsilon_{\lambda'}}(x_{\lambda'})$. So, $\exists \varepsilon > 0$

such that $S_\varepsilon(y) \subseteq S_{\varepsilon_{\lambda'}}(x_{\lambda'}) \subseteq \bigcup_{\lambda \in \Lambda} S_{\varepsilon_\lambda}(x_\lambda) = G$.

So, G is open □

Theorem 19:- Let (X, d) be a metric space. Then

- 1) Any union of open sets in X is open
- 2) Any finite intersection of open sets in X is open.

Proof:- The first part follows from Theorem 18. For the second statement, suppose G_1, G_2, \dots, G_n be open sets in X . Then for any $x \in \bigcap_{i=1}^n G_i$, we have for all $1 \leq i \leq n$,

$$x \in G_i \Rightarrow \exists \varepsilon_i > 0 \text{ s.t. } S_{\varepsilon_i}(x) \subseteq G_i.$$

Set $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$. Then $S_\varepsilon(x) \subseteq S_{\varepsilon_i}(x)$ for all $1 \leq i \leq n$ and this implies $S_\varepsilon(x) \subseteq \bigcap_{i=1}^n S_{\varepsilon_i}(x) \subseteq \bigcap_{i=1}^n G_i$ □

Lecture-4 (9 April, 2021)

Definition 20:- Let (X, d) be a metric space and S be a subset of X . An element $x \in X$ is said to be a limit point of S if $\forall \varepsilon > 0$ $S_\varepsilon(x) \setminus \{x\} \cap S \neq \emptyset$.

Definition 21 :- A subset S of a metric space is closed if it contains each of its limit points.

Definition 22 :- Let $x_0 \in X$ and $\epsilon > 0$. Then the closed sphere $S_\epsilon[x_0]$ with center x_0 and radius ϵ is the subset of X defined by

$$S_\epsilon[x_0] = \{x \in X \mid d(x, x_0) \leq \epsilon\}.$$

Theorem 23 :- In any metric space X , the empty set \emptyset and the full space X are closed sets.

Proof :- Trivial.

Theorem 24 :- Let (X, d) be a metric space. A subset F of X is closed iff $X \setminus F$ is open.

Proof :- Let F is closed and, WLOG, $X \setminus F \neq \emptyset$.

Let $x \in X \setminus F \Rightarrow x$ is not a limit

point of F .

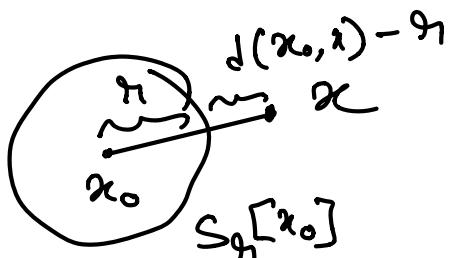
Hence, $\exists r > 0$ such that $S_r(x) \cap F = \emptyset$
 $\Rightarrow S_r(x) \subseteq X \setminus F$. So, $X \setminus F$ is open.

Conversely, let $X \setminus F$ is open. let $x \in X \setminus F$.
There exists $r > 0$ such that $S_r(x) \subseteq X \setminus F$
 $\Rightarrow S_r(x) \cap F = \emptyset$. So, x is not a limit point of F . Negating this statement we get every limit point belongs to F \square

Theorem 25 :- Every closed sphere in a metric space is a closed set.

Proof :- Let (X, d) is a metric space, $x_0 \in X$ and $r > 0$. Consider the closed sphere $S_r[x_0]$ in X .

Let $x \in X \setminus S_r[x_0]$. This means,
 $d(x_0, x) > r$.
 $\Rightarrow d(x_0, x) - r > 0$.



Let $r_1 = d(x_0, x) - r_1$ and consider $S_{r_1}(x)$.

If $y \in S_{r_1}(x)$, then $d(x, y) < r_1 = d(x_0, x) - r_1$.

$$\begin{aligned} \text{Now, } d(x_0, x) &\leq d(x_0, y) + d(y, x) \\ \Rightarrow d(x_0, y) &\geq d(x_0, x) - d(y, x) \\ \Rightarrow d(x_0, y) &> d(x_0, x) - r_1, \\ \Rightarrow d(x_0, y) &> r_1 \\ \Rightarrow y &\in X \setminus S_{r_1}[x_0]. \end{aligned}$$

Therefore, $S_{r_1}(x) \subseteq X \setminus S_{r_1}[x_0]$, hence $X \setminus S_{r_1}[x_0]$ is open. \square

Theorem 26 :- Let (X, d) be a metric space. Then

- 1) Any intersection of closed sets is closed.
 - 2) Any finite union of closed sets is closed.
- Proof :- Directly follows from Theorem 19 and Theorem 24. \square

Let (X, d) be a metric space and $Y \subseteq X$. On Y define a metric $d_Y(y_1, y_2) = d(y_1, y_2) \forall y_1, y_2 \in Y$. Then (Y, d_Y) is a metric subspace of (X, d) .

Example 27:- Let $X = [0, 1]$ with Euclidean metric. Then $[0, 1)$ is both open and closed in $([0, 1), |\cdot|)$. But $[0, 1)$ is neither open nor closed in $(\mathbb{R}, |\cdot|)$.

Additional Reference :- Metric Spaces by Michael Ó Searcoid. Springer Undergrad Math. Series.

Definition 28:- Let (X, d) be a metric space. A sequence of points $\{x_n\}$ in X is convergent if there exists a point $x \in X$ such that for each open sphere $S_\epsilon(x)$ $\exists n_0 \in \mathbb{N}$ such that $x_n \in S_\epsilon(x) \quad \forall n \geq n_0$.

Definition 29:- A sequence $\{x_n\}$ in a metric space (X, d) is Cauchy if $\forall \epsilon > 0 \exists n_0 \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon \quad \forall m, n \geq n_0$.

Proposition 30:- Every convergent sequence in (X, d) is Cauchy.

Proof:- Let $\{x_n\}$ in X is convergent and $\lim_{n \rightarrow \infty} x_n = x$. Let $\epsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \epsilon/2$ for all $n \geq n_0$.

Hence, for any $m \geq n_0$, $d(x_m, x) < \epsilon/2$. So,

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \epsilon/2 + \epsilon/2 = \epsilon$$

$\forall m, n \geq n_0$.



The converse of Proposition 30 is not true in general.
Take $X = [0, 1]$, $d = | \cdot |$ and $x_n = 1 - \frac{1}{n}$, $\forall n \in \mathbb{N}$.
Clearly, $\{x_n\}$ is Cauchy but $\{x_n\}$ does not converge in X .

Lecture-5 (13 April)

Definition 31:- A metric space (X, d) is complete if every Cauchy sequence is convergent in (X, d) .

Example 32 :- $(\mathbb{R}, | \cdot |)$ is complete.

Proposition 33 :- Any non-empty set X equipped with discrete metric is complete.

Proof :- Suppose $\{x_n\}$ be a Cauchy sequence in (X, dis) . Then for all $\epsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that

$$\text{dis}(x_m, x_n) < \epsilon \quad \forall m, n \geq n_0.$$

However, if $x_m \neq x_n$, $\text{dis}(x_m, x_n) = 1$.

In particular, for a fixed $\epsilon < 1$, $\exists n'_0 \in \mathbb{N}$ such that $\text{dis}(x_m, x_n) < \epsilon$, $\forall m, n \geq n'_0$.

$$\Rightarrow x_m = x_n = x \text{ (say)} \quad \forall m, n \geq n_0.$$

Hence, the sequence $\{x_n\}$ is a constant sequence for all $n \geq n_0$. This implies that $x_n \rightarrow x$. \square

Definition 34 :- Suppose d_1 & d_2 be metrics on a non-empty set X . Then d_1 is topologically equivalent to d_2 , denoted by $d_1 \sim d_2$, if the collection of all open sets of (X, d_1) is same as the collection of open sets of (X, d_2) .

Theorem 35 :- Let d_1 & d_2 be metrics on a non-empty set X . Then the following

Conditions are equivalent:

- d_1 and d_2 are topologically equivalent;
- For each $x \in X$ and open sphere $S_{r_1}^{d_1}(x)$, for some $r_1 > 0$, in (X, d_1) there exists $s > 0$ such that $S_s^{d_2}(x) \subseteq S_{r_1}^{d_1}(x)$ for $i, j = 1, 2$ and $i \neq j$.

Proof :- $a) \Rightarrow b)$: Suppose $x \in X$ and $S_{r_1}^{d_1}(x)$ is an open sphere in (X, d_1) , for some $r_1 > 0$.

Since, $d_1 \sim d_2$ and $S_{r_1}^{d_1}(x)$ is open in (X, d_1) , $S_{r_1}^{d_1}(x)$ is also open in (X, d_2) . There exists $s > 0$ such that $S_s^{d_2}(x) \subseteq S_{r_1}^{d_1}(x)$. Similarly, for any $x \in X$ and open sphere $S_{r_2}^{d_2}(x)$ in (X, d_2) for some $r_2 > 0$, $\exists s > 0$ such that $S_s^{d_1}(x) \subseteq S_{r_2}^{d_2}(x)$.

b) \Rightarrow a) Let U be an open set in (X, d_1) . Let $x \in U$. We need to produce an open sphere $S_{\eta}^{d_2}(x) \subseteq U$ to conclude that U is also open in (X, d_2) .

Since, U is open in (X, d_1) , there exists $r > 0$ such that $S_r^{d_1}(x) \subseteq U$. Then $\exists s > 0$ such that

$$S_s^{d_2}(x) \subseteq S_r^{d_1}(x) \subseteq U.$$

Hence, U is open in (X, d_2) .

Similarly, every open set in (X, d_2) is also open in (X, d_1) \square

Corollary 36:- Two metrics d_1 & d_2 on a non-empty set X are topologically equivalent if there exists real numbers $r, s > 0$ such that

$$\left(\begin{array}{l} d_1(x, y) \leq r d_2(x, y) \\ \text{(and } d_2(x, y) \leq s d_1(x, y)) \\ \forall x, y \in X \end{array} \right) \Leftrightarrow \left(\begin{array}{l} r d_1(x, y) \leq d_2(x, y) \leq s d_1(x, y) \\ \forall x, y \in X. \end{array} \right)$$

This condition is known as strong equivalence of d_1 & d_2 .

Proof :- Let $x \in X$ and $S_t^{d_1}(x)$ be an open sphere in (X, d_1) for some $t > 0$. Now $\exists r > 0$ such that $d_1(x, y) \leq r d_2(x, y) \quad \forall x, y \in X$.

Choose, $\phi = \frac{t}{r}$. Then $S_\phi^{d_2}(x) \subseteq S_t^{d_1}(x)$, because for all $y \in S_\phi^{d_2}(x) \Rightarrow d_2(x, y) < \phi \Rightarrow d_1(x, y) < t \Rightarrow y \in S_t^{d_1}(x)$.

Similarly, we can show that for any open sphere $S_t^{d_2}(x)$

$\exists \delta > 0$ such that $S_{\delta}^{d_1}(x) \subseteq S_{\delta}^{d_2}(x)$; hence $d_1 \sim d_2$ \square

Exercise - 2 :- Converse of Corollary 36 is false.

For instance, take $X = (0, 1]$,

$$d_1(x, y) = |x - y| \text{ and } d_2(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right|.$$

Show that d_1 & d_2 are topologically equivalent but not strongly equivalent.

Notation :- If two metrics d_1 & d_2 are strongly equivalent, then we denote $d_1 \sim d_2$ and say d_1 is equivalent to d_2 .

Observe that \sim defines an equivalence relation on the set of all metrics on a non-empty set X .

Example 37 :- For $n \in \mathbb{N}$, let (X_i, d_i) is a metric space for $1 \leq i \leq n$. Recall the metrics μ_p for $1 \leq p \leq \infty$ on $\prod_{i=1}^n X_i$. Then $\mu_{p_1} \sim \mu_{p_2}$ for all $1 \leq p_1, p_2 \leq \infty$.

In fact, it is sufficient to show that $\mu_p \sim \mu_\infty$ for all $1 \leq p < \infty$. By definition,

$$\mu_p(a, b) = \left[\sum_{i=1}^n |a_i - b_i|^p \right]^{1/p}$$

$$\text{and } \mu_\infty(a, b) = \max \{ |a_i - b_i| \mid 1 \leq i \leq n \}.$$

Then $d_p(a, b) \leq n^{\frac{1}{p}} d_\infty(a, b)$ and $d_p(a, b) \geq d_\infty(a, b)$ for all $1 \leq p < \infty$.

In particular, if $X_i = \mathbb{R}$ and $d_i = | \cdot |$, then the metrics μ_p , for $1 \leq p \leq \infty$, are equivalent on \mathbb{R}^n .

Lecture-6 (15 April, 2021)

Theorem 38 :- Let d_1, d_2 be equivalent metrics on X . Suppose (X, d_1) is complete.

Then (X, d_2) is also complete.

Proof :- Since d_1 and d_2 are equivalent $\exists \alpha, \beta > 0$ such that

$$\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_1(x, y) \quad \forall x, y \in X.$$

Let $\{x_n\}$ be a Cauchy sequence in (X, d_2) . Then for every $\epsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that

$$d_2(x_m, x_n) < \epsilon \quad \forall m, n \geq n_0.$$

This implies $\{x_n\}$ is also a Cauchy sequence in (X, d_1) :

$$d_1(x_m, x_n) \leq \frac{1}{\beta} d_2(x_m, x_n) < \epsilon \quad \forall m, n \geq n_0.$$

Let $\lim_{n \rightarrow \infty} x_n = x$ in (X, d_1) . Hence for $\forall \epsilon' > 0 \exists n'_0 \in \mathbb{N}$ such that $d_1(x_n, x) < \epsilon'/\beta \quad \forall n \geq n'_0$.

Then $d_2(x_n, x) \leq \beta d_1(x_n, x) < \epsilon' \quad \forall n \geq n'_0$.

So, $\lim_{n \rightarrow \infty} x_n = x$ in (X, d_2) as well. \square

Example 39:- \mathbb{R}^n is complete with respect to μ_p for all $1 \leq p \leq \infty$.

Let $\bar{x}_m = (x_m^1, x_m^2, \dots, x_m^n) \in \mathbb{R}^n$ be a Cauchy sequence in $(\mathbb{R}^n, \mu_\infty)$. Then for $\epsilon > 0 \exists n_0 \in \mathbb{N}$ $m, m' \geq n_0$ such that $\mu_\infty(\bar{x}_m, \bar{x}_{m'}) < \epsilon \forall m, m' \geq n_0$.

This implies that for all $1 \leq i \leq n$,

$$|x_m^i - x_{m'}^i| < \epsilon \quad \forall m, m' \geq n_0.$$

Hence, $\{x_m^i\}$ is a Cauchy sequence in \mathbb{R} for all $1 \leq i \leq n$. This means $x_m^i \rightarrow x^i$ as $m \rightarrow \infty$ in \mathbb{R} for all $1 \leq i \leq n$.

We claim that $\bar{x}_m \rightarrow \bar{x} = (x^1, x^2, \dots, x^n)$ as $m \rightarrow \infty$ in $(\mathbb{R}^n, \mu_\infty)$.

For $\epsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $|x_m^i - x^i| < \epsilon \forall m \geq n_0$.

Let $n_0 = \max\{n_0^i \mid 1 \leq i \leq n\}$. Then $|x_m^i - x^i| < \epsilon \forall m \geq n_0, 1 \leq i \leq n$.

Hence, $\mu_\infty(\bar{x}_m, \bar{x}) < \epsilon \forall m \geq n_0$.

Consequently, $(\mathbb{R}^n, \mu_\infty)$ is complete. Finally, $\mu_\infty \sim \mu_p$ for all $1 \leq p < \infty$. Therefore by Theorem 38, (\mathbb{R}^n, μ_p) is also complete for all $1 \leq p \leq \infty$. \square

Example 40:- For any $1 \leq p < \infty$, (l^p, d_p) is complete. Let $\{x_k\}$ be a Cauchy sequence in l^p . Then for any fixed $K \in \mathbb{N}$,

$$x_k = \{x_k^1, x_k^2, x_k^3, \dots\} \in l^p.$$

Suppose $\epsilon > 0$. Then $\exists n_0 \in \mathbb{N}$ such that

$$d_p(\{x_m\}, \{x_n\}) < \epsilon \quad \forall m, n \geq n_0.$$

$$\Rightarrow |x_m^i - x_n^i| \leq d_p(\{x_m\}, \{x_n\}) < \epsilon \quad \forall m, n \geq n_0.$$

This means for every $i \in \mathbb{N}$, $\{x_n^i\}$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$. Suppose $x_n^i \rightarrow x^i$ as $n \rightarrow \infty$ in $(\mathbb{R}, |\cdot|)$ for every $i \in \mathbb{N}$.

Let $\bar{x} = \{x^i\}_{i=1}^{\infty}$. Fix $t \in \mathbb{N}$. Then

$$\left[\sum_{l=1}^t |x_m^l - x_n^l|^p \right]^{\frac{1}{p}} \leq d_p(\{x_m\}, \{x_n\}) < \epsilon, \quad \forall m, n \geq n$$

Letting $m \rightarrow \infty$ and keeping m fixed, we obtain

$$\left[\sum_{l=1}^t |x^l - x_n^l|^p \right]^{\frac{1}{p}} < \epsilon \quad \forall n \geq n_0.$$

Since, t was chosen arbitrarily, we have

$$\left[\sum_{l=1}^{\infty} |x^l - x_n^l|^p \right]^{\frac{1}{p}} < \epsilon \quad \forall n \geq n_0.$$

This shows that $\{x_n^i - x^i\} \in l^p \Rightarrow \bar{x} = \{x^i\} \in l^p$ and $d_p(\{x_n\}, \bar{x}) < \epsilon \quad \forall n \geq n_0$. \square

Exercise 3:- (l^∞, d_∞) is complete.

Lecture-7 (16 April, 2021)

Example 41 :- $(C[a,b], \|\cdot\|_\infty)$ is complete.

Let $\{f_n\}$ be a Cauchy sequence in $C[a,b]$. For $\epsilon > 0$ $\exists n_0 \in \mathbb{N}$ such that

$$\|f_m - f_n\|_\infty < \epsilon \quad \forall m, n \geq n_0.$$

Let $t \in [a,b]$. Then

$$|f_m(t) - f_n(t)| \leq \|f_m - f_n\|_\infty < \epsilon \quad \forall m, n \geq n_0.$$

So, $\{f_n(t)\}$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$. Then we get a function $f: [a,b] \rightarrow \mathbb{R}$ defined by

$$\lim_{n \rightarrow \infty} f_n(t) = f(t) \quad \text{for all } t \in [a,b].$$

To prove that $f_n \rightarrow f$ in $\|\cdot\|_\infty$, recall that $\exists n'_0 \in \mathbb{N}$ such that $|f_m(t) - f_n(t)| < \epsilon/2$ for all $m, n \geq n'_0$, $t \in [a,b]$.

Now fix $t_0 \in [a,b]$. Since, $\lim_{n \rightarrow \infty} f_n(t_0) = f(t_0)$, $\exists m \in \mathbb{N}$ (depending on t_0 & ϵ) such that $m \geq n'_0$

$$|f_m(t_0) - f(t_0)| < \epsilon/2.$$

Then for all $n \geq n'_0$

$$\begin{aligned} |f_n(t_0) - f(t_0)| &\leq |f_n(t_0) - f_m(t_0)| + |f_m(t_0) - f(t_0)| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

Observe that n'_0 is independent of t_0 . So, $\forall n \geq n'_0$

$$|f_n(t) - f(t)| < \epsilon \quad \text{for all } a \leq t \leq b.$$

$$\Rightarrow \|f_n - f\|_\infty < \epsilon \quad \forall n \geq n'_0.$$

So, $f_n \xrightarrow{\|\cdot\|_\infty} f$.

Finally, we need to show that f is continuous. Let $t' \in [a, b]$. Fix $\epsilon > 0$.

Since, $\lim_{n \rightarrow \infty} f_n = f$, $\exists m \in \mathbb{N}$ such that $d_\infty(f_m, f) < \epsilon/3$.

Now, $f_m \in C[a, b]$, so it is continuous at t_0 . There exists a neighbourhood U of t_0 such that

$$|f_m(y) - f_m(t_0)| < \epsilon/3 \quad \forall y \in U.$$

Then for any $y \in U$, we have

$$\begin{aligned} |f(y) - f(t_0)| &\leq |f(y) - f_m(y)| + |f_m(y) - f_m(t_0)| + |f_m(t_0) - f(t_0)| \\ &\leq d_\infty(f, f_m) + |f_m(y) - f_m(t_0)| + d_\infty(f_m, f) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

Hence $f \in C[a, b]$. \square

Definition 4.2 :- Suppose (X, d) is a metric space and $\emptyset \neq A \subseteq X$. The diameter of A is defined by $\sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$ and it is denoted by $\text{diam}(A)$. Moreover, A is bounded if $\text{diam}(A) < \infty$.

Example 4.3 :- In any metric space (X, d) , $\text{diam}\{x\} = 0 \quad \forall x \in X$.

Proposition 4.4 :- Suppose (X, d) is a metric space and $\emptyset \neq A \subseteq B \subseteq X$. Then $\text{diam}(A) \leq \text{diam}(B)$.

Proof :- Since $A \subseteq B$, we have
 $\{d(a_1, a_2) \mid a_1, a_2 \in A\} \subseteq \{d(b_1, b_2) \mid b_1, b_2 \in B\}$.
 $\Rightarrow \text{diam}(A) \leq \text{diam}(B)$ □

Proposition 45 :- Let (X, d) be a metric space and
 $\phi \neq A \subseteq X$. Then $\text{diam}(A) = \text{diam}(\bar{A})$,
where \bar{A} is the closure of A in (X, d) .

Proof :- Clearly, $\text{diam}(A) \leq \text{diam}(\bar{A})$ as $A \subseteq \bar{A}$.
To prove the other inequality, fix $\varepsilon > 0$ and
let $x, y \in \bar{A}$. Then $S_{\varepsilon/2}(x) \cap A \neq \phi$ and $S_{\varepsilon/2}(y) \cap A \neq \phi$.
Let $x_1 \in S_{\varepsilon/2}(x) \cap A$ and $y_1 \in S_{\varepsilon/2}(y) \cap A$.
Then $x_1, y_1 \in A$ such that $d(x, x_1) < \varepsilon/2$ and
 $d(y, y_1) < \varepsilon/2$.

Thus

$$\begin{aligned} d(x, y) &\leq d(x, x_1) + d(x_1, y_1) + d(y_1, y) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + d(x_1, y_1) \end{aligned}$$

$$\Rightarrow d(x, y) < \varepsilon + d(x_1, y_1) < \varepsilon + \text{diam}(A).$$

Varying $x, y \in \bar{A}$ and taking the supremum on
the left hand side, we have

$$\text{diam}(\bar{A}) \leq \varepsilon + \text{diam}(A).$$

Since, $\varepsilon > 0$ was arbitrarily chosen we have
 $\text{diam}(\bar{A}) \leq \text{diam}(A)$. □

Lecture-8 (20 April, 2021)

Theorem (Cantor's Intersection Theorem) 46 :- A metric

Space (X, d)

is complete. Let $\{F_n\}$ be a sequence of non-empty closed subsets of X such that

a) $\{F_n\}$ is decreasing :- $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$.

b) $\text{diam}(F_n) \rightarrow 0$ as $n \rightarrow \infty$.

Then (X, d) is complete iff $F = \bigcap_{n=1}^{\infty} F_n$ is singleton.

Proof :- Let (X, d) be a complete metric space.

Since each F_n is non-empty, choose $x_n \in F_n$ for all $n \in \mathbb{N}$. We claim that $\{x_n\}$ is Cauchy in (X, d) . In fact, if $m, n \in \mathbb{N}$ such that $m > n$, then $F_m \subseteq F_n$. Consequently, $x_m, x_n \in F_n$ which implies that

$$d(x_m, x_n) \leq \text{diam}(F_n).$$

Fix $\epsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that

$$\text{diam}(F_n) < \epsilon \quad \forall n \geq n_0.$$

This implies, $d(x_m, x_n) < \epsilon \quad \forall m > n \geq n_0$.

This shows that $\{x_n\}$ is a Cauchy sequence in (X, d) and $\exists x \in X$ such that $x_n \xrightarrow{d} x$ as $n \rightarrow \infty$.

Again, fix $\epsilon > 0$. Then $\exists n'_0 \in \mathbb{N}$ such that

$$x_n \in S_\epsilon(x) \quad \forall n \geq n'_0.$$

Also, $x_n \in F_n \subseteq F_{n'_0}$ for all $n \geq n'_0$. Hence,

$S_\epsilon(x) \cap F_n \neq \emptyset$ for all $n \in \mathbb{N}$. Equivalently, x is a limit point of F_n for all $n \in \mathbb{N}$. Now F_n is closed for every $n \in \mathbb{N}$, hence $x \in F_n$ for

all $n \in \mathbb{N}$. So, $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

If possible, let $y \in \bigcap_{n=1}^{\infty} F_n \setminus \{x\}$. Then $\text{diam}(F_n) \geq d(x, y) > 0$ for all $n \in \mathbb{N}$. This contradicts the fact that $\text{diam}(F_n) \xrightarrow{n \rightarrow \infty} 0$. Thus $\bigcap_{n=1}^{\infty} F_n$ is singleton.

To prove the converse, suppose $\{x_n\}$ is a Cauchy sequence in (X, d) . For each $n \in \mathbb{N}$ define

$$F_n = \{x_n, x_{n+1}, \dots\}.$$

Clearly, $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$ and hence $\overline{F}_1 \supseteq \overline{F}_2 \supseteq \overline{F}_3 \dots$

Fix $\varepsilon > 0$. There exists $l \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon/2$ $\forall m, n \geq l$. So, for all $n \geq l$ $F_n \subseteq \overline{F}_l$ and $\text{diam}(F_n) \leq \text{diam}(\overline{F}_l) < \varepsilon/2 < \varepsilon$.

Hence, $\text{diam}(\overline{F}_n) \rightarrow 0$ as $n \rightarrow \infty$. Then by the previous Proposition 46, we have $\text{diam}(\overline{F}_n) \rightarrow 0$ as $n \rightarrow \infty$.

By hypothesis $\bigcap_{n=1}^{\infty} \overline{F}_n = \{x\}$ for some $x \in X$.

Thus $d(x, x_n) \leq \text{diam}(\overline{F}_n)$ and $\text{diam}(\overline{F}_n) \rightarrow 0$ as $n \rightarrow \infty$. Hence $x_n \xrightarrow{d} x$ as $n \rightarrow \infty$ in (X, d) \square

Exercise - 4 :- Verify that none of the conditions that F_n 's are closed sets and $\text{diam}(F_n) \rightarrow 0$ as $n \rightarrow \infty$ can be dropped for getting $\bigcap_{n=1}^{\infty} F_n$ to be singleton.

Proposition 47 :- Suppose (X, d) is a metric space.
Let $A \subseteq X$. Then

- If (X, d) is complete and A is closed, then (A, d) is complete.
- If (A, d) is complete, then A is closed in (X, d) .

Proof :-

- Let $\{x_n\}$ be a Cauchy sequence in A . Since (X, d) is complete, $\exists x \in X$ such that $x_n \xrightarrow{d} x$ as $n \rightarrow \infty$. Since, A is closed, then $x \in A$; hence (A, d) is complete.
- If $x \in \bar{A}$, we pick a sequence $\{x_n\}$ in A such that $x_n \xrightarrow{d} x$ as $n \rightarrow \infty$. Since $\{x_n\}$ is convergent, it is a Cauchy sequence. But (A, d) is complete, it implies that $x_n \xrightarrow{d} a$ as $n \rightarrow \infty$ for some $a \in A \subseteq X$. Then by uniqueness of the limit of a sequence, $x = a$. Hence, $x \in \bar{A} \Rightarrow x \in A \Rightarrow \bar{A} = A$. □

Lecture-9 (22 April, 2021)

Definition 48 :- Let (X, d_X) and (Y, d_Y) be metric spaces. A bijection $f: (X, d_X) \rightarrow (Y, d_Y)$ is an isometry if for any $x_1, x_2 \in X$, $d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$. Then (X, d_X) is said to be isometric with (Y, d_Y) .

Definition 49:- A complete metric space (Y, d^*) is said to be a completion of a metric space (X, d) , if (X, d) is isometric with a dense subspace of (Y, d^*) . Equivalently, there exists an isometry f from (X, d) onto $(f(X), d^*)$ such that $f(X) = Y$.

Theorem 50:- Every metric space has a completion.

Lemma 51:- Let $\{x_n\}, \{y_n\}$ be Cauchy sequences in a metric space (X, d) . If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences then $\{d(x_n, y_n)\}$ is a convergent sequence in \mathbb{R} .

Proof :- Let $\epsilon > 0$. There exists $m, k \in \mathbb{N}$ such that $d(x_i, x_j) < \epsilon/2$ $\forall i, j \geq m$ and $d(y_i, y_j) < \epsilon/2$ $\forall i, j \geq k$.

Then for all $i, j \geq \max\{m, k\} = n$ we have,

$$\begin{aligned} d(x_i, y_j) &\leq d(x_i, x_j) + d(x_j, y_j) + d(y_j, y_i) \\ &< \epsilon + d(x_j, y_j) \end{aligned}$$

$$\Rightarrow d(x_i, y_j) - d(x_j, y_j) < \epsilon \quad \forall i, j \geq n.$$

Interchanging i & j we obtain

$$d(x_j, y_i) - d(x_i, y_i) < \epsilon \quad \forall i, j \geq n.$$

$$\text{Hence, } |d(x_i, y_i) - d(x_j, y_j)| < \epsilon \quad \forall i, j \geq n.$$

So, $\{d(x_n, y_n)\}$ is a Cauchy sequence in \mathbb{R} , hence convergent.



Proof of Theorem 50:- Let \mathcal{E} be the set of all of Cauchy sequences in a metric space (X, d) .

Define $d_1 : \mathcal{E} \times \mathcal{E} \rightarrow \mathbb{R}$ by

$$d_1(\{x_n\}, \{y_n\}) = \lim_{n \rightarrow \infty} d(x_n, y_n) \text{ for all } \{x_n\}, \{y_n\} \in \mathcal{E}.$$

Since $\{d(x_n, y_n)\}$ is a sequence of non-negative real numbers, $d_1(x, y) \geq 0$ and $d_1(x, y) = 0$ if $x = y$, $\forall x, y \in \mathcal{E}$. Also, $d_1(x, y) = d_1(y, x)$ and $d_1(x, z) \leq d_1(x, y) + d_1(y, z)$ $\forall x, y, z \in \mathcal{E}$.

Observe that $d_1(x, y) = 0$ does not imply $x = y$. So, d_1 is not a metric on \mathcal{E} .

Define a relation \sim on \mathcal{E} as $x \sim y$ iff $d_1(x, y) = 0$ $\forall x, y \in \mathcal{E}$.

\sim is a equivalence relation on \mathcal{E} :

- i) $x \sim x$ as $d_1(x, x) = 0 \quad \forall x \in \mathcal{E}$.
- ii) If $x \sim y$ iff $d_1(x, y) = d_1(y, x) \Rightarrow y \sim x \quad \forall x, y \in \mathcal{E}$.
- iii) If $x \sim y$ & $y \sim z$ iff $d_1(x, y) = 0$ and $d_1(y, z) = 0$.

Then $d_1(x, z) \leq d_1(x, y) + d_1(y, z) = 0$

$$\Rightarrow d_1(x, z) = 0$$

$$\Rightarrow x \sim z, \quad \forall x, y, z \in \mathcal{E}.$$

Thus \sim is an equivalence relation on \mathcal{E} . For any $x \in \mathcal{E}$ denote it's equivalence class by $[x] = \{y \in \mathcal{E} \mid x \sim y\}$. Let $\mathcal{E}/\sim = \{[x] \mid x \in \mathcal{E}\}$, the quotient set of all \sim -equivalence classes.

Define $d^*: \mathcal{E}/\sim \times \mathcal{E}/\sim \rightarrow \mathbb{R}$ by
 $d^*([x], [y]) = d_1(x, y)$.

Suppose $A \in [x]$ and $B \in [y]$. Then

$$\begin{aligned} d_1(A, B) &\leq d_1(A, x) + d_1(x, y) + d_1(y, B) \\ &= d_1(x, y). \end{aligned}$$

Similarly, $d_1(x, y) \leq d_1(x, A) + d_1(A, B) + d_1(B, y)$
 $= d_1(A, B)$.

So, $d_1(A, B) = d_1(x, y)$ for all $A \in [x], B \in [y]$.

Thus, d^* is well-defined.

Suppose $d^*([x] - [y]) = 0$
 $\Rightarrow d_1(x, y) = 0$
 $\Rightarrow x \sim y$
 $\Rightarrow [x] = [y]$.

Then using the properties of d_1 , it is easy to verify
 that d^* is a metric on \mathcal{E}/\sim .

Lecture-10 (23 April, 2021)

For each $x \in X$, let S_x denotes the class in \mathcal{E}/\sim .

Observe that the constant sequence $\{x_n = x\} \in S_x$.
 Let $A = \{S_x \mid x \in X\}$.

Define $f: X \rightarrow A$ by $f(x) = S_x$.

f is clearly surjective. If $f(x_1) = f(x_2)$ then
 $S_{x_1} = S_{x_2} \Rightarrow x_1 = x_2$. (Why??)

So, f is a bijection. Furthermore,

$$d^*(f(x), f(y)) = d^*(s_x, s_y) = d_1(\{x_n = x\}, \{y_n = y\}) \\ = d(x, y).$$

So, f is an isometry.

Now we need to show that $\overline{A} = \mathcal{E}/n$.

Let $[x] \in \mathcal{E}/n$, where $x = (x_1, x_2, \dots) \in \mathcal{E}$.

Let $\epsilon > 0$ and consider the open sphere $S_\epsilon^{d^*}([x])$.

Since x is Cauchy sequence in (X, d) , $\exists n \in \mathbb{N}$
such that $m \geq n \Rightarrow d(x_m, x_n) < \epsilon/2$.

Let $Z = (x_n, x_n, \dots)$.

$$\text{Then } d^*([x], s_{x_n}) = d_1(x, z)$$

$$= \lim_{m \rightarrow \infty} d(x_m, x_n) \leq \epsilon/2 < \epsilon.$$

This shows that $s_{x_n} \in S_\epsilon^{d^*}([x]) \cap A \Rightarrow [x] \in \overline{A}$.

So, $\overline{A} = \mathcal{E}/n$.

The last step is to show that $(\mathcal{E}/n, d^*)$ is complete.

let $([x_1], [x_2], \dots)$ be a Cauchy sequence
in \mathcal{E}/n . Hence, $x_n \in \mathcal{E} \forall n \in \mathbb{N}$.

We denote $x_n = \{x_n^1, x_n^2, \dots\}$ where $x_i^j \in X$.

$$x_1: x_1^1 \ x_1^2 \ \dots \ x_1^m \ \dots \ \dots$$

$$x_2: x_2^1 \ x_2^2 \ \dots \ x_2^m \ \dots \ \dots$$

,

:

$$x_n: x_n^1 \ x_n^2 \ \dots \ x_n^m \ \dots \ \dots$$

:

:

:

For each $n \in \mathbb{N}$, Since x_n is Cauchy in X , $\exists M_n \in \mathbb{N}$ such that

$$d(x_n^i, x_n^{M_n}) < \gamma_n = \epsilon \quad \forall i > M_n \quad (\text{---})$$

let $y_n = (x_n^{M_n}, x_n^{M_n}, \dots)$ be the constant sequence in (X, d) .

By the previous scheme

$$y_1 : x_1^{M_1} \quad x_1^{M_1} \quad \dots \quad x_1^{M_1} \quad \dots$$

$$y_2 : x_2^{M_2} \quad x_2^{M_2} \quad \dots \quad x_2^{M_2} \quad \dots$$

:

:

$$y_n : x_n^{M_n} \quad x_n^{M_n} \quad \dots \quad x_n^{M_n} \quad \dots$$

:

:

Let us consider the element $T_n = [y_n] \quad \forall n \in \mathbb{N}$.

Then $T_n \in \mathcal{E}/n \quad \forall n \in \mathbb{N}$. Consider the sequence $\{T_n\}$ in \mathcal{E}/n .

By (---) we observe that

$$d^*([x_n], [y_n]) = d_1(x_n, y_n)$$

$$= \lim_{i \rightarrow \infty} d(x_n^i, x_n^{M_n}) \leq \frac{1}{n}.$$

$$\Rightarrow \lim_{n \rightarrow \infty} d^*([x_n], [y_n]) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Exercise 5 :-

let $\{x_n\}, \{y_n\}$ be sequences in (X, d) . If $d(x_n, y_n) \rightarrow 0$ in $(\mathbb{R}, 1.1)$

and $\{x_n\}$ is Cauchy, then $\{y_n\}$ is also Cauchy.

Then by Exercise 5, $\{T_n\}$ is a Cauchy sequence in $(\mathcal{E}/n, d^*)$ and $\{T_n\} \sim \{[x_n]\}$ with respect to d^* .

Define $z_n = x_n^{M_n} \quad \forall n \in \mathbb{N}$. Then $\{z_n\}$ is a sequence in (X, d) .

Let $\epsilon > 0$. Since $\{T_n\}$ is Cauchy, $\exists R \in \mathbb{N}$ such that

$$d^*(T_n, T_m) < \epsilon/2 \quad \forall m, n \geq R.$$

$$\begin{aligned} \text{Now, } d^*(T_n, T_m) &= d^*([y_n], [y_m]) \\ &= d_1(y_n, y_m) \\ &= d(x_n^{M_n}, x_m^{M_m}) \\ &= d(z_n, z_m). \end{aligned}$$

So, $\{z_n\}$ is a Cauchy sequence in (X, d) .

Hence $\{z_n\} \in \mathcal{S}$ and

$$\begin{aligned} d^*(T_n, [z_n]) &= d_1(y_n, z_n) \\ &= \lim_{m \rightarrow \infty} d(x_n^{M_n}, x_m^{M_m}) \leq \epsilon/2 < \epsilon \\ &\quad \forall n \geq R. \end{aligned}$$

So, $T_n \rightarrow [z_n]$ in \mathcal{E}/n as $n \rightarrow \infty$.

Now, $\{[x_n]\} \sim \{T_n\}$ and $T_n \rightarrow [z_n]$

imply $[x_n] \rightarrow [z_n]$ as $n \rightarrow \infty$ in \mathcal{E}/n .



Lecture-11 (27 April, 2021)

Theorem 52 :- Any two completions of a metric space (X, d) are isometric.

Proof :- Let (Y_1, d_1) and (Y_2, d_2) be completions of (X, d) .

Then (X, d) is isometric to (A, d_1) for some $A \subseteq Y_1$ and to (B, d_2) for some $B \subseteq Y_2$.

Moreover, $\overline{A}^{d_1} = Y_1$ & $\overline{B}^{d_2} = Y_2$.

Let $y_1 \in Y_1$. Since, $\overline{A}^{d_1} = Y_1$, $\exists \{x_n\}$ in A such that $x_n \xrightarrow{d_1} y_1$ as $n \rightarrow \infty$.

Since, (A, d_1) is isometric to (B, d_2) there exists a unique sequence $\{y_n\}$ in B such that $d_1(x_i, x_j) = d_2(y_i, y_j) \quad \forall i, j \in \mathbb{N}$.

Hence $\{y_n\}$ is a Cauchy sequence in (B, d_2) .

Since, (Y_2, d_2) is complete, $\exists y_2 \in Y_2$ such that $y_n \xrightarrow{d_2} y_2$ as $n \rightarrow \infty$.

Define $f: (Y_1, d_1) \rightarrow (Y_2, d_2)$ by
$$f(y_1) = y_2.$$

Exercise - 6 :- Show that f is well-defined: Verify that y_2 does not depend on the choice of sequence $\{x_n\}$.

$$\text{Then, } d_1(y_1, y'_1) = \lim_{n \rightarrow \infty} d_1(x_n, x'_n) = \lim_{n \rightarrow \infty} d_2(y_n, y'_n) \\ = d_2(y_2, y'_2).$$

S_α , (Y_1, d_1) and (Y_2, d_2) are isometric. \square

Theorem 53:- Suppose (X, d_X) and (Y, d_Y) be metric spaces. Let $f: X \rightarrow Y$.

Fix $a \in X$. Then the following statements are equivalent:

- (a) For all $\epsilon > 0$, $\exists s > 0$ such that $d_X(x, a) < s$ implies $d_Y(f(x), f(a)) < \epsilon$.
- (b) For each $\{x_n\}$ in X converging to a in (X, d_X) , the sequence $\{f(x_n)\}$ in Y converges to $f(a)$ in (Y, d_Y) .
- (c) For every open set $V \subseteq Y$ containing $f(a)$ implies $f^{-1}(V)$ is open in (X, d_X) containing a .

Proof :- (a) \Rightarrow (b): let $\{x_n\}$ be a sequence in X that converges to a in (X, d_X) .

There exists $m_s \in \mathbb{N}$ such that

$$d_X(x_n, a) < s \quad \forall n \geq m_s.$$

$$\Rightarrow d_Y(f(x_n), f(a)) < \epsilon \quad \forall n \geq m_s.$$

$\Rightarrow \{f(x_n)\}$ converges to $f(a)$ in (Y, d_Y) .

(b) \Rightarrow (a): If possible, let the statement (a) is not true. Then there exists $\epsilon > 0$ such

that for each $s > 0$, there exists $x_s \in X$ with $d_X(x_s, a) < s$ but $d_2(f(x_s), f(a)) \geq \epsilon$.

For each $n \in \mathbb{N}$, $\exists x_n \in X$ with
 $d_X(x_n, a) < \frac{1}{n}$ such that $d_Y(f(x_n), f(a)) \geq \varepsilon$.

This implies that $x_n \xrightarrow{d_X} a$ in (X, d_X) as $n \rightarrow \infty$
but $f(x_n) \xrightarrow{d_Y} f(a)$ in (Y, d_Y) as $f(x_n) \notin S_\varepsilon^{d_Y}(f(a))$
 $\forall n \in \mathbb{N}$, which is a contradiction to the hypothesis (b).
So, (b) \Rightarrow (a).

(a) \Rightarrow (c): Let V be an open set in (Y, d_Y) containing a .
let $x \in f^{-1}(V)$. Then $f(x) \in V$. Since,
 V is open in (Y, d_Y) , $\exists \varepsilon > 0$ such that $S_\varepsilon^{d_Y}(f(x)) \subseteq V$.
By (a), $\exists s > 0$ such that $d_X(x, a) < s$ implies
 $d_Y(f(x), f(a)) < \varepsilon$. Hence, $a \in S_s^{d_X}(x) \subseteq f^{-1}(V)$.

Lecture-12 (April 28, 2021)

(c) \Rightarrow (a): let $\varepsilon > 0$. Then $V = S_\varepsilon(f(a))$ is an open subset of (Y, d_Y) . Then $U = f^{-1}(V)$ is an open subset of (X, d_X) . Since, U is open, $s > 0$ such that $S_s(a) \subseteq U$ (as $a \in U$). This implies $f(S_s(a)) \subseteq S_\varepsilon(f(a))$ and this is equivalent to (a).

□

Exercise - 7 :- Show that (a) in Theorem 53 is equivalent to $\forall \varepsilon > 0, \exists s > 0$ such that $f(S_s(a)) \subseteq S_\varepsilon(f(a))$.

Definition 54 :- A function $f: (X, d_X) \rightarrow (Y, d_Y)$ is said to be Continuous at a point $a \in X$ if one of the statements in Theorem 53 holds true. Furthermore, f is Continuous if it is continuous at every $x \in X$.

Theorem 55 :- A function $f: (X, d_X) \rightarrow (Y, d_Y)$ is continuous if and only if $f^{-1}(V)$ is open in (X, d_X) for all open subsets V of (Y, d_Y) .

Proof :- Exercise - 8.

Theorem 56 :- The composition of continuous functions is continuous.

Proof :- Exercise - 9.

A sequence $\{x_n\}$ in \mathbb{R} is nothing but a function $h: \mathbb{N} \rightarrow \mathbb{R}$ defined by $h(n) = x_n$.

Let $\text{Func}(X, Y) = \{f: (X, d_X) \rightarrow (Y, d_Y)\}$.

Definition 57 :- A sequence of functions from (X, d_X) to (Y, d_Y) is a function $h: \mathbb{N} \rightarrow \text{Func}(X, Y)$. Equivalently, for every $n \in \mathbb{N}$, $h(n): X \rightarrow Y$. Denote $h(n) = f_n$. Then

Then h is equivalent to $\{f_n\}$ where $f_n: X \rightarrow Y$ $\forall n \in \mathbb{N}$.

Definition 58:- Let $\{f_n\}$ be a sequence of functions from (X, d_X) to (Y, d_Y) . Then $\{f_n\}$ is said to be

a) pointwise convergent to a function $f: X \rightarrow Y$ if $\forall x \in X, \forall \epsilon > 0 \exists N \in \mathbb{N}$ such that $d_Y(f_n(x), f(x)) < \epsilon$ $\forall n \geq N$.

b) uniformly convergent to a function $f: X \rightarrow Y$ if $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that $d_Y(f_n(x), f(x)) < \epsilon$ $\forall n \geq N, \forall x \in X$.

Example 59:- Suppose $f_n(x) = x^n \quad \forall x \in [0, 1]$.

If $0 \leq x < 1$, then $\lim_{n \rightarrow \infty} f_n(x) = 0$.

On the other hand if $x = 1$, then $x^n \rightarrow 1$ as $n \rightarrow \infty$.
So, $f_n \rightarrow f$ pointwise where

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Suppose, $f_n \rightarrow f$ uniformly. Let $\epsilon = \frac{1}{2}$.

Now,

$$|f_n(x) - f(x)| = \begin{cases} x^n & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x = 1. \end{cases}$$

By hypothesis we must have a fix $N \in \mathbb{N}$ such that

$$|x|^n < 1/2 \quad \forall n \geq N, \forall x \in [0,1].$$

$$x = 1 - \frac{1}{3N}, \text{ then } x^N = \left(1 - \frac{1}{3N}\right)^N \geq 1 - \frac{1}{3} = \frac{2}{3}.$$

This shows that $f_n \not\rightarrow f$ uniformly.

Example 60 :-

Suppose $f_n : (0,1) \rightarrow \mathbb{R}$ defined by $f_n(x) = \frac{n}{nx+1} \quad \forall n \in \mathbb{N}$.

Since, $x \neq 0$,

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{1}{x + 1/n} = \frac{1}{x}.$$

Now, $|f_n(x)| < n$ for all $x \in (0,1)$. So each f_n is bounded on $(0,1)$. But $f(x)$ is not bounded.

Lecture-13 (29 April, 2021)

Definition 61 :- A function $f : X \rightarrow Y$ is bounded if $f(X) \subset Y$ is bounded.

Proposition 62 :- Let $\{f_n\}$ be a sequence of functions $f_n : X \rightarrow Y$. If each f_n is bounded and $f_n \rightarrow f$ uniformly, then $f : X \rightarrow Y$ bounded.

Proof :- By uniform convergence, there exists $m \in \mathbb{N}$

such that

$$d_Y(f_m(x), f(x)) \leq 1 \quad \forall x \in X.$$

Since, f_m is bounded, there exists $y \in Y$ and $R > 0$ such that $d_Y(f_m(x), y) \leq R$ for all $x \in X$. Then by triangle inequality,

$$\begin{aligned} d_Y(f(x), y) &\leq d_Y(f(x), f_m(x)) + d_Y(f_m(x), y) \\ &\leq R + 1 \quad \forall x \in X. \end{aligned}$$

Hence, $f: X \rightarrow Y$ is bounded. □

Theorem 6.3 :- Let $\{f_n\}$ be a sequence of functions $f_n: X \rightarrow Y$. If each f_n is continuous at $a \in X$ and $f_n \rightarrow f$ uniformly, then $f: X \rightarrow Y$ is continuous at $a \in X$.

Proof :- Let $a \in X$. Since $f_n \rightarrow f$ uniformly, given $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$d_Y(f_m(x), f(x)) < \epsilon/3 \quad \forall x \in X.$$

Also, Continuity of f_m at $a \in X$ implies, there exists $s > 0$ such that

$$d_Y(f_m(x), f_m(a)) < \epsilon/3 \text{ if } d_X(x, a) < s.$$

Then for $x \in X$ such that $d_X(x, a) < s$ implies

$$\begin{aligned} d_Y(f(x), f(a)) &\leq d_Y(f(x), f_m(x)) + d_Y(f_m(x), f_m(a)) \\ &\quad + d_Y(f_m(a), f(a)) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$
□

Definition 64:- A sequence of functions $f_n: X \rightarrow Y$ is uniformly Cauchy if $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that $m, n \geq N$ implies that $d_Y(f_m(x), f_n(x)) < \varepsilon \quad \forall x \in X$.

Consider $X = \{x\}$ and d_X = discrete metric and $(Y, d_Y) = (\mathbb{R}, |\cdot|)$. Let $f_n: X \rightarrow Y$ be a sequence of functions. This means for every $n \in \mathbb{N}$ we have $f_n(x) \in \mathbb{R}$. So, f_n is nothing but a sequence $\{x_n = f_n(x)\}$ in $(\mathbb{R}, |\cdot|)$.

Suppose $\{f_n\}$ is uniformly Cauchy. Then for all $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\begin{aligned} d_Y(f_m(x), f_n(x)) &< \varepsilon \quad \forall m, n \geq N \\ \Rightarrow |f_m(x) - f_n(x)| &< \varepsilon \quad \forall m, n \geq N \\ \Rightarrow |x_m - x_n| &< \varepsilon \quad \forall m, n \geq N. \end{aligned}$$

So, $\{x_n\}$ is a Cauchy sequence in (Y, d_Y) . Hence, $x_n \rightarrow a \in \mathbb{R}$. This means $f_n \rightarrow f$ where $f(x) = a$.

Theorem 65:- Let (Y, d_Y) be a complete metric space.

Then a uniformly Cauchy sequence of functions $f_n: X \rightarrow Y$ converges uniformly to a function $f: X \rightarrow Y$.

Proof :- The uniform Cauchy condition implies that the sequence $\{f_n(x)\}$ is a Cauchy sequence in (Y, d_Y) for every $x \in X$. Since, Y is complete $f_n(x) \rightarrow z_x \in Y$. Define $f: X \rightarrow Y$ by $f(x) = z_x \forall x \in X$.

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have

$$d_Y(f_m(x), f_n(x)) < \varepsilon \quad \forall x \in X.$$

Taking $m \rightarrow \infty$ we have

$$d_Y(f(x), f_n(x)) < \varepsilon \quad \forall x \in X, \forall n \geq N.$$

This shows that $\{f_n\}$ converges to f uniformly. \square

Exercise 10 :- Let $B(X, Y) = \{f: X \rightarrow Y \mid f \text{ is bounded}\}$.

Define $d_\infty: B(X, Y) \times B(X, Y) \rightarrow \mathbb{R}$

by $d_\infty(f_1, f_2) = \sup \{d_Y(f_1(x), f_2(x)) \mid x \in X\}$.

Show that $(B(X, Y), d_\infty)$ is a metric space.

Furthermore, $(B(X, Y), d_\infty)$ is complete whenever (Y, d_Y) is complete.

Lecture-14 (30 April, 2021)

Definition 66 :- The spaces

$$C(X, Y) = \{f: X \rightarrow Y \mid f \text{ is continuous}\},$$

$$\text{and } C_b(X, Y) = \{f: X \rightarrow Y \mid f \text{ is continuous \& bounded}\}.$$

Theorem 67 :- If (Y, d_Y) is a complete metric space then $(C_b(X, Y), d_\infty)$ is also complete.

Proof :- Clearly, $(C_b(X, Y), d_\infty)$ is a subspace of $(B(X, Y), d_\infty)$. Since, (Y, d_Y) is complete, by Exercise-10, $(B(X, Y), d_\infty)$ is also complete. Hence, it is sufficient to show that $C_b(X, Y)$ is a closed subspace of $(B(X, Y), d_\infty)$ and Proposition 47 will ensure its completeness.

Let $\{f_n\}$ be a sequence in $(C_b(X, Y), d_\infty)$ such that $f_n \xrightarrow{d_\infty} f$. This means for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_\infty(f_n, f) < \epsilon \quad \forall n \geq N$.

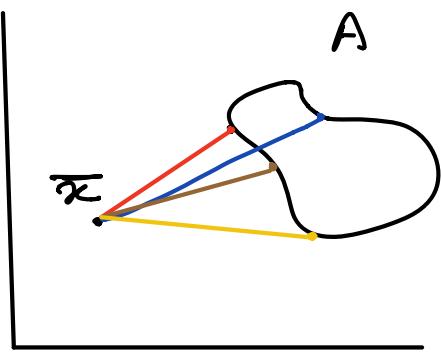
By definition of d_∞ , we have

$$d_Y(f_n(x), f(x)) < \epsilon \quad \forall n \geq N, \forall x \in X.$$

This means $f_n \rightarrow f$ uniformly. By Theorem 63, we conclude that $f \in C(X, Y)$. On the other hand, $C_b(X, Y) \subseteq B(X, Y)$ and $(B(X, Y), d_\infty)$ is complete. So, $f \in B(X, Y)$; hence $f \in C_b(X, Y)$. \square

Theorem 68 :- Let A be a subset of a metric space (X, d) . Then the function $d(\cdot, A): X \rightarrow \mathbb{R}$ defined by $d(x, A) = \inf \{d_X(x, a) | a \in A\}$ is a continuous function.

In fact, $d(x, A)$ is the distance between x and the subset $A \subseteq X$.



Proof :- Let $x_0 \in X$ arbitrarily chosen. Fix $\epsilon > 0$.
 Then it is enough to show that
 $x \in S_{\epsilon/2}(x_0) \Rightarrow |d(x, A) - d(x_0, A)| < \epsilon$.

Suppose $x \in S_{\epsilon/2}(x_0)$ and $a \in A$ arbitrarily.
 Then $d(x, a) \leq d(x, x_0) + d(x_0, a)$
 $\Rightarrow \inf \{d(x, a) \mid a \in A\} \leq d(x, x_0) + \inf \{d(x_0, a) \mid a \in A\}$
 $\Rightarrow d(x, A) \leq d(x, x_0) + d(x_0, A)$
 $\Rightarrow d(x, A) - d(x_0, A) \leq d(x, x_0) < \epsilon/2$

Interchanging the roles of x and x_0 we get

$$d(x_0, A) - d(x, A) \leq d(x_0, x) = d(x, x_0) < \epsilon/2.$$

Thus, $|d(x, A) - d(x_0, A)| < \epsilon/2$. □

Theorem 69 (Urysohn's Lemma) :- For any two non-empty disjoint closed

subsets A and B of a metric space (X, d) , there exists a continuous function $f: X \rightarrow \mathbb{R}$ such that $f(A) = \{0\}$, $f(B) = \{1\}$ and $0 \leq f(x) \leq 1 \quad \forall x \in X$.

Lecture-15 (4 May, 2021)

Proof :-