Real Analysis Chapter 3 Study Guide (for "Real Analysis, A First Course", 2nd Edition, Russell A. Gordon)

Number of Starred Exercises: **4**; Number of Notes: 16; Number of Other (non-starred) Exercises: 43; Minimum Number of Other (non-starred) Exercises to Do (to do at least 25% of them): **11**

The most important things to get out of this chapter: (1) A full understanding of the definition of the limit of a function and the use of this definition in proofs. (2) A full understanding of the definition of continuity and how it is related to the limit concept. (3) Fundamental consequences of continuity (the Intermediate Value Theorem (IVT) and the Extreme Value Theorem (EVT)). (4) The definition of uniform continuity and how it differs from the definition of continuity. (5) Functions which are continuous on a closed and bounded interval are also uniformly continuous on that closed and bounded interval.

Other matters of importance:

- The relationship between limits of functions and limits of sequences and how this is used in proofs
- How to prove that a limit does not exist by using sequences
- The algebraic properties of limits of functions and the squeeze theorem for functions
- The relationship between one-sided limits and "ordinary" two-sided limits
- The definitions of "infinite" limits (a special kind of divergence)
- Continuous functions are "nice"
- Most functions encountered in Calculus are continuous where they are defined
- Monotone functions are "nice"
- Gaining an ability to plow through many difficult proofs

Reading Guide:

1. *In one paragraph and with pictures, summarize your understanding of the limit of a function $f(x)$ at this point (before you read Chapter 3).
2. Does the function $f(x) = \varepsilon^{-\frac{1}{\xi^2}}$ have a limit as $x \to 0$? If so, what is its value? Can you explain your answer?
3. Rewrite Definition 3.1 in a less precise way, perhaps using phrases like "arbitrarily close" and "sufficiently close".
4. Rewrite Definition 3.1 with mostly symbols (like $\forall$, $\exists$, etc...).
5. Why is it not necessary for *c* to be in *I* in Definition 3.1? Is there an example that justifies this?
6. *Write the negation of Definition 3.1.
7. Create an example of a function and a value of *L* which is not the correct value of the limit of the function as $x \to \chi$. Then find an $\varepsilon > 0$ such that there are no values of δ that will result in Definition 3.1 being satisfied (keep your example simple).

8. Explain why the fact that $\varepsilon > 0$ is arbitrary is essential in Definition 3.1.

9. *Use a technique similar to the book's example at the bottom for page 84 to show that $\lim\limits_{x \to 2} \xi^2 = 4$. Draw a picture to illustrate the idea of your proof.

10. Note: take note of how a sequence $\{x_n\}$ converging to *c* is constructed in the proof of Theorem 3.2. The method used to construct this sequence is quite common in the book's proofs from here on out. Study it carefully like the author emphasizes.

11. Prove that the function *h* defined near the bottom of pate 85 does have a limit at 0.

12. Prove Theorem 3.3(d) using two methods. First, use Definition 3.1. Second, use Theorem 3.2. Looking at the proof of Theorem 2.7(d) may be helpful.

13. Prove that $L \leq M$ in Theorem 3.4 by showing that the assumption $L > M$ leads to a contradiction.

14. Draw a picture illustrating the Squeeze Theorem for Functions.

15. Use one-sided limits to help you determine the value, if any, of $\lim\limits_{x \to 0} \frac{\xi}{|\xi|}$. Explain.

16. Note: make a note of the strange notation sometimes used for one-sided limits.

17. Write a definition for what you think the notation $\lim\limits_{x \to \infty} \phi(\xi) = \infty$ should mean.

18. Before reading the paragraph after Definition 3.9, compare Definition 3.9 with Definition 3.1. How do they seem to be related? What kinds of differences are there?

19. Note: carefully read the last paragraph of page 93 and think about the examples that go onto the top of page 94 to understand some of the subtleties involved with the function and its domain when you define continuity.

20. Based on thinking about the formulation of continuity at a point given in statement 2 of Theorem 3.10, draw pictures that illustrate different ways a function can fail to be continuous at a point *c*.

21. Note: in the proof of Theorem 3.11 for quotients, it's not necessary to use ε's and δ's. The truly hard work has already been done in the proof of Theorem 3.3.

22. Before reading the proof of Theorem 3.13, try to prove it on your own (if you've already read it, try to prove it an hour later without looking at the book's proof).

23. Compare the paragraph at the bottom of page 95 to your work on #20 above. Did you omit any possible ways a function can fail to be continuous at *c*?

24. Draw a graph of $f(x) = \lfloor \xi \rfloor + \lfloor -\xi \rfloor$ to illustrate the idea in the first paragraph on page 96.

25. Draw some graphs of discontinuous functions that either have no extreme values or do not attain every intermediate value over an interval [a, b] (see the first paragraph on page 99).

26. Prove the fact at the bottom of page 99 (hint: use something similar to the "common technique" for generating a sequence $\{x_n\}$ in the proof of Theorem 3.2 (see #10 above)).

27. In the proof of the IVT (Theorem 3.16), explain why $c_n \to \chi$, $c_n \in \Sigma$ for all *n*, and $f(c_n) < \varpi\phi o\rho\,\alpha\lambda\lambda\,\nu$ imply that $f(c) \leq \varpi$.

28. In the proof of the IVT, explain why $\{d_n\}$ is a sequence in $[a,b] \setminus S$ and why $f(d_n) \geq \varpi\phi o\rho\,\alpha\lambda\lambda\,\nu$.

29. From a topological viewpoint, the IVT can be interpreted as saying that continuous functions map intervals onto intervals (though we need to think of one-point sets as being “intervals” for this to be a completely valid observation). Since intervals are “connected” sets, this can be generalized to saying that continuous functions map connected sets onto connected sets. They don’t “break” the set into pieces.

30. Note: take note of the technique in the first paragraph on page 101 that allows us to apply the IVT to prove that a certain equation has a solution. This trick is common.

31. Note: both paragraphs of the proof of the EVT (Theorem 3.17) require the use of the Bolzano-Weierstrass Theorem. Note this means we *do not care* whether the *originally constructed* sequence $\{x_n\}$ or $\{d_n\}$ converges or not. All we care about is the existence of *some* sequence that converges (in this case, as in many cases, the convergent sequence is a subsequence of $\{x_n\}$ or $\{d_n\}$, in the frist and second paragraphs, respectively).

32. Note: notice once more the slick ways of constructing sequences in the proof of the EVT (see #10 above).

33. A topological interpretation of the EVT would say that continuous functions map closed and bounded intervals onto closed and bounded intervals. There’s no way a continuous function could map a closed and bounded interval onto an open and bounded interval or onto an unbounded interval (draw some pictures of this to convince yourself). However, continuous functions *can* map open and bounded intervals onto unbounded intervals and can also map open intervals onto closed intervals (draw pictures to convince yourself). The distinctions in properties between closed and open intervals are sometimes very subtle.

34. Note: Theorem 3.18 codifies the topological interpretations in #29 and #33 above).

35. Prove the statement at the bottom of page 102.

36. Think back to Algebraic Structures. Was the converse of Lagrange’s Theorem true? If not, can you think of any “partial” converses of Lagrange’s Theorem that were true? What did these statements say?

37. Note: “proof by contraposition” (in the proof of Theorem 3.20) means “a proof of the contrapositive”. Also note that the negation of the definition of an interval (look up Definition 1.8) is used in the proof of this theorem.

38. In the proof of Theorem 3.21, why does the fact that *f* is strictly increasing on *I* imply that $f_{inv}(u) < f_{inv}(v)$ when *u* < *v*? Also take note of the fact that Theorem 3.20 is used to prove the inverse function is continuous. In other words, take note of the fact that Theorem 3.20 might be useful for you in proofs that certain monotone functions are continuous.

39. Note: make sure you are clear about the fact that continuous functions satisfy the intermediate value property, but there are some discontinuous functions which satisfy it too (see the example at the bottom of page 104 which goes to the top of page 105).

40. What are the other three cases in the proof of Theorem 3.24?

41. Note: note the fact that derivatives always satisfy the intermediate value property (when they exist), even though they might not always be continuous (think about why the function $f(x) = \xi^2\ \alpha\nu(1/\xi)\ \omega\eta\varepsilon\nu\ \xi \neq 0$ and $f(0) = 0$ provides one such example).

42. Find a function which is defined everywhere but not locally bounded at one point.

43. Explain why $\delta = 0.001$ and $M = 1000$ work to prove that $g(x) = 1/\xi$ is locally bounded at 0.002.

44. Ponder how a function which is locally bounded at every point in an interval may not be bounded on that interval.

45. Draw pictures which help you understand examples 1 and 2 on pages 110 and 111. Make up your own example and draw pictures for it.

46. *Negate the definition of a uniformly continuous function on an interval. Can you find an example of a function that is continuous on an interval but which seems to be not uniformly continuous on that interval? (hint: infinite intervals are allowed).

47. For a given $\varepsilon > 0$, how should we choose $\delta > 0$ to prove that the sine function is uniformly continuous on ° ?

48. Prove that $h(x) = 1/\xi$ is uniformly continuous on the interval $(0.002, 5)$.

49. Compare the book's negation of the definition of uniform continuity on page 112 with your own from #46 above.

50. Prove that x^3 is not uniformly continuous on ° .

51. Can you find a function which is bounded on ° but which seems to be not uniformly continuous on ° ?

52. **Note:** in the proof of Theorem 3.28, once again take note of the trick of how to generate sequences that have certain properties and take note of the use of the Bolzano-Weierstrass Theorem.

53. Fill in any details in the proof of Theorem 3.29 which are confusing to you.

54. **Note:** Theorem 3.30 gives a nice and simple criterion for determining whether a function *f* which is continuous on an open interval (*a*, *b*) is uniformly continuous on that interval.

55. **Note:** even though the content of Section 3.5 gets quite technical, it's main theme is not so hard to grasp: the main theme is "monotone functions are relatively nice" or, at the very least, "monotone functions are not super-nasty". See the top of page 116 for more explanation of this.

56. **Note:** when you see equations like $f(x-) = \sigma\upsilon\pi\{\phi(\tau) : \tau \in [\alpha, \xi)\}$ and $f(x+) = \iota\nu\phi\{\phi(\tau) : \tau \in (\xi, \beta]\}$ in Theorem 3.31, you should immediately say to yourself: "hey, in this situation the set $\{f(t) : t \in [\alpha, \xi)\}$ must be bounded above for each $x \in [\alpha, \beta]$ and the set $\{f(t) : t \in (\xi, \beta]\}$ must be bounded below for each $x \in [\alpha, \beta]$". Draw pictures of monotone (and possibly discontinuous) functions to help you believe this.

57. Come up with an example of a function defined on an interval [*a*, *b*] where the set of discontinuities is uncountable.

58. Prove the result in part (d) of Theorem 3.35 for sums (the proof for differences would be similar).

59. Can you think of a statement and equation similar to the statement and equation of Theorem 3.36 for definite integrals? Write down that statement.

60. Note: the proofs of Theorems 3.35 and 3.36 are difficult, but taking the time to understand them will pay dividends for helping you understand proofs about definite integrals in Chapter 5.

61. Fill in any details of the proofs of Theorems 3.35 and 3.36 that you find to be confusing. It might also be helpful to make notes to yourself about the overall strategies of the proofs.

62. Draw a picture to help you understand the argument that $f(x) = \sqrt{\xi}\chi o\sigma(\pi / \xi)$ if $x \neq 0$ and $f(0) = 0$ is not of bounded variation on [0, 1].

63. Draw pictures to help you understand the proof of Theorem 3.37. In particular, for graphs of various (relatively simple) functions *f*, see if you can draw the graphs of f_1 and f_2 defined in the proof.

Deep Thoughts to Ponder (but not necessarily answer):

- Why is the “definition” of continuous function as being one whose graph can be drawn without picking up your pencil not a sufficient definition for Real Analysis?
- Do the ideas of limits and continuity seem easier or harder than they used to? Do you feel it has been beneficial to you to study this chapter (either personally or for your future job…perhaps in teaching)? Do you have more of an appreciation for the subtleties that are glossed over in calculus and other pre-calculus mathematics?