Module 8 Vertex-colorings

Contents

The concept of vertex coloring of a graph can be used to model many scheduling problems, optimal assignment of channels to radio stations and optimal assignment of spectrum frequencies to mobile operations. However, the topic originated with the following "map coloring conjecture".

At most four colors are required to color any map of a country so that adjacent states receive different color.

This problem was first raised by Francis Guthrie in a letter to De Morgan in 1852. For a long time, it was a fascinating open problem among mathematicians. Arthur Cayley (1821 – 1852) and G.D. Birkhoff (1884 – 1944) made it known to a wider community by their presentations and innovative contributions. It was settled in affirmative by K. Appel, W. Haken and J. Kosh in 1977.

8.1 Basic definitions

Definition. *Any function $f : V(G) \to \{1, 2, \ldots, k\}$ is called a* ***k-vertex-coloring*** *of a graph G if $f(u) \neq f(v)$, for any two adjacent vertices $u, v \in G$. The integers $1, 2, \ldots, k$ are called the* ***colors****. Equivalently, a k-vertex-coloring of G is a partition $(V_1, \ldots, V_k)$ of $V(G)$ where each V_i is an independent set of vertices in G.*

Remarks.

- Every graph G admits a n-vertex-coloring. However, given k, G may not admit a k-vertex-coloring.
  Figure 8.1 shows various colorings of $K_4 - e$.
- If G admits a k-vertex-coloring, then it also admits a $(k + 1)$-vertex-coloring.

The above remarks motivate the following important concept.

Definition. *The minimum integer k such that G admits a k-vertex-coloring is called the* ***vertex-chromatic-number*** *of G. It is denoted by $\chi(G)$ or $\chi_0(G)$. In other*

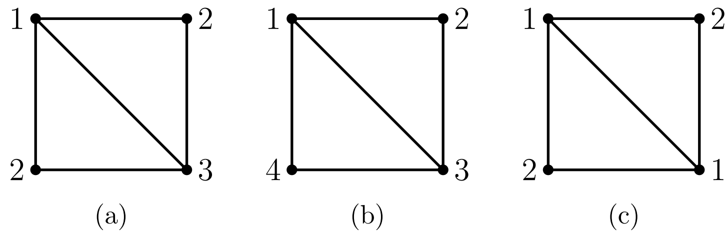


Figure 8.1: (a) shows a 3-vertex-coloring, (b) shows a 4-vertex-coloring and (c) shows a coloring which is not a 2-vertex-coloring.

words, $\chi(G)$ is the minimum number of colors required to color the vertices such that no two adjacent vertices of G receive the same color.

The following table shows the vertex-chromatic number of some standard graphs.

G	K_n	K_n^c	Bipartite graph	Petersen graph
$\chi(G)$	n	1	2	3

Table 8.1: Vertex-chromatic number of some standard graphs.

Remarks.

- If H is the underlying simple graph of G, then $\chi(H) = \chi(G)$.
- If $G_1, G_2, \ldots, G_t$ are the components of G, then
$$\chi(G) = \max\{\chi(G_1), \chi(G_2), \ldots, \chi(G_t)\}.$$
- If $B_1, B_2, \ldots, B_p$ are the blocks of G, then
$$\chi(G) = \max\{\chi(B_1), \chi(B_2), \ldots, \chi(B_p)\}.$$
- In view of the above remarks, the study of vertex-colorings may be restricted to simple connected graphs.

8.2 Cliques and chromatic number

Definition. *A subgraph Q of a graph G is called a* ***clique****, if any two vertices in Q are adjacent. A* ***maximal clique*** *is a clique that is not a subgraph of any other clique. The* ***clique number*** *of G is the integer $\omega(G) = \max\{|V(Q)| : Q \text{ is a clique in } G\}$.*

If H is a subgraph of G, then obviously $\chi(H) \leq \chi(G)$. Hence,

- For any graph G, $\chi(G) \geq \omega(G)$.

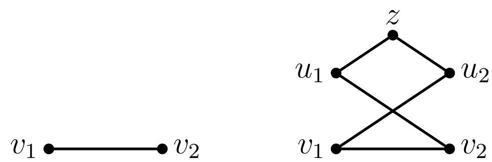
• Mycielski's theorem

There has been intense research to obtain upper bounds for $\chi(G)$ in terms of $\omega(G)$ alone. Questions like for which graphs $\chi(G) = \omega(G)$ or $\chi(G) = \omega(G) + 1$ or $\chi(G) \leq f(\omega(G))$ for a function f, have received wide attention. Mycielski (1955) showed that given any k, there exists a graph G with $\omega(G) = 2$ and $\chi(G) = k$. This result indicates that there is no upper bound for an arbitrary class of all graphs as a function of ω alone.

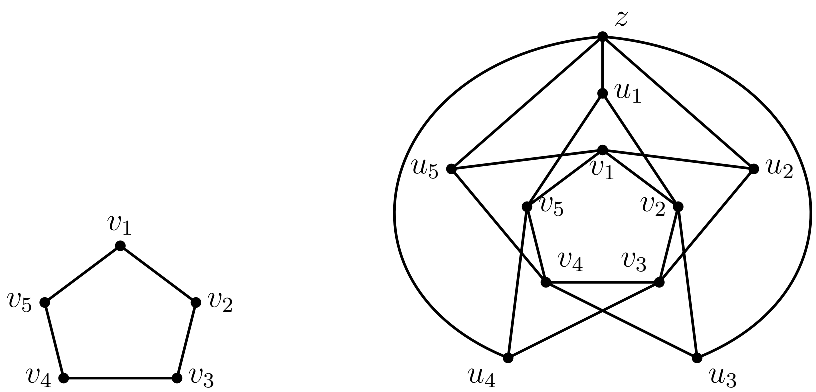
Theorem 8.1. *For any given integer k (≥ 1), there exists a triangle-free graph G_k with vertex-chromatic number k. (Triangle-free $G :=$ G has no K_3 as a subgraph.)*

Proof. G_k is constructed by induction on k. For $k = 1$ and 2, K_1 and K_2 are the required graphs. Next suppose that we have constructed a triangle-free graph G_k on p vertices $v_1, v_2, \ldots, v_p$. Construct G_{k+1} from G_k as follows.

1. Add $p + 1$ new vertices $u_1, u_2, \ldots, u_p$ and z to G_k.
2. Join u_i ($1 \leq i \leq p$) to all those vertices to which v_i is adjacent in G_k.
3. Join z to every u_i.
   Figure 8.2 shows this construction of G_3 from G_2 ($= K_2$) and G_4 from G_3 ($= C_5$).



(a) $G_2 \to G_3$



(b) $G_3 \to G_4$

Figure 8.2: Mycielski's construction.

We shall prove that G_{k+1} is triangle-free and that $\chi(G_{k+1}) = k + 1$.

(a) G_{k+1} is triangle-free.

On the contrary, suppose that G_{k+1} contains a triangle $\mathcal{T}$. Since G_k is triangle-free, one of the vertices of $\mathcal{T}$ is z or u_i (for some i). If $z \in \mathcal{T}$, then the other two vertices of $\mathcal{T}$ are u_g and u_h (for some g and h). This is a contradiction, since no two vertices in $\{u_1, \ldots, u_p\}$ are adjacent. If $u_i \in \mathcal{T}$ and $z \notin \mathcal{T}$, then the other two vertices of $\mathcal{T}$ are v_g and v_h (for some g and h). But then, v_i, v_g and v_h are the vertices of a triangle in G_k, which again is a contradiction. Hence, G_{k+1} is triangle-free.

(b) $\chi(G_{k+1}) = k + 1$.

Since, G_k is a subgraph of G_{k+1}, we have $\chi(G_{k+1}) \geq \chi(G_k) = k$. If $f : V(G_k) \to \{1, 2, \ldots, k\}$ is a k-coloring of G_k, then $g : V(G_{k+1}) \to \{1, 2, \ldots, k+1\}$ defined by

$$\begin{aligned} g(v_i) &= f(v_i), 1 \leq i \leq p, \\ g(u_i) &= f(v_i), 1 \leq i \leq p, \\ g(z) &= k+1, \end{aligned}$$

is a $(k+1)$-coloring of G_{k+1}. Hence, $\chi(G_{k+1}) \leq k+1$.

Next if possible suppose that $\chi(G_{k+1}) = k$, and let $h : V(G_{k+1}) \to \{1, 2, \ldots, k\}$ be a k-coloring of G_{k+1}. Without loss of generality, let $h(z) = k$. So, $h(u_i) \neq k$ for $i = 1, 2, \ldots, k$. Define $g : \{v_1, v_2, \ldots, v_p\} \to \{1, 2, \ldots, k-1\}$ by

$$g(v_i) = \begin{cases} h(v_i), & \text{if } h(v_i) \neq k \\ h(u_i), & \text{if } h(v_i) = k \end{cases}$$

It is easy to verify that g is a $(k-1)$-vertex-coloring of G$_k$. This is a contradiction to $\chi(G_k) = k$.

Hence, we conclude that $\chi(G_{k+1}) = k+1$. □

8.3 Greedy coloring algorithm

Finding $\chi(G)$ of an arbitrary graph G is a hard problem. So there have been attempts to use greedy and approximate algorithms and obtain tight bounds. However, it has been shown that there can be no approximate algorithm for finding $\chi(G)$ within a factor of $n^{1-\epsilon}$, for any $\epsilon > 0$, where n is the number of vertices in G. This deep result shows that finding the vertex-chromatic number of an arbitrary class of graphs is impossibly hard. Despite this negative result, greedy algorithm does use nearly χ colors if the input graph belongs to several interesting classes of graphs. This sections explores these aspects.

Greedy algorithm (Sequential algorithm/An example of an on-line algorithm.)

Input: A graph G and an ordering $(v_1, v_2, \ldots, v_n)$ of its vertices.

Output: A k-vertex-coloring of G, for some k.

Step 1: Color v_1 with 1.

Step 2: Having colored $v_1, v_2, \ldots, v_i$ with t colors, say $1, \ldots, t$, color v_{i+1} with the following policy. Let T be the set of colors used to color the vertices adjacent with v_i, then

$$color(v_{i+1}) = \begin{cases} \min(\{1, 2, \ldots, t\} - T), & \text{if } T \neq \{1, 2, \ldots, t\} \\ t + 1, & \text{if } T = \{1, 2, \ldots, t\}. \end{cases}$$

Step 3: Stop when v_n is colored.

An illustration:

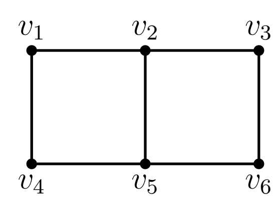
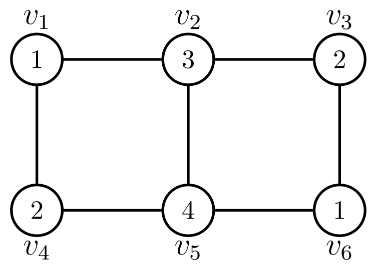
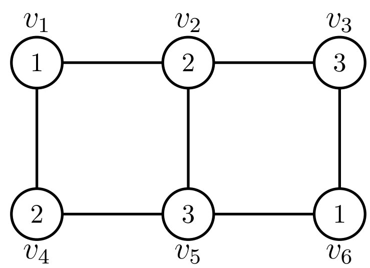


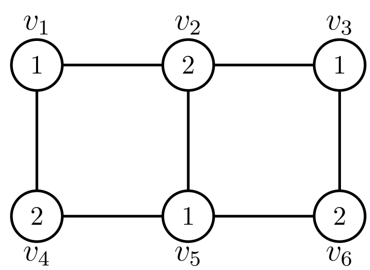
Figure 8.3: A graph G.



(a) An output of the greedy algorithm with input $(G; (v_1, v_6, v_3, v_4, v_2, v_5))$. The number inside the circle indicates its colors.



(b) An output of the greedy algorithm with input $(G; (v_1, v_2, v_6, v_3, v_5, v_4))$.



(c) An output of the greedy algorithm with input $(G; (v_1, v_3, v_5, v_2, v_4, v_6))$. Since $\chi(G) = 2$, with this choice of vertex ordering the algorithm uses the minimum number of colors.

Figure 8.4: Outputs of the greedy algorithm when three different vertex orderings are given as input.

Remarks.

- Algorithm is called a greedy algorithm since we use the first available color at every step.
- It is called a sequential algorithm since the input is a sequence of the vertices and the vertices are colored sequentially.

- It is an example of an on-line algorithm since at every step we use the available data to color the vertex and do not change this color subsequently.
- The illustration indicates that the number of colors used by the algorithm depends on the ordering of the vertices.

Theorem 8.2. *The greedy algorithm uses at most $\Delta(G)+1$ colors.*

Proof. While coloring v_{i+1}, under Step 3, there are at most $\Delta(G)$ neighbours of v_{i+1} in $\{v_1, v_2, \ldots, v_i\}$, that is $|T| \leq \Delta(G)$. Hence, $color(v_{i+1}) \leq 1+\Delta(G)$. This holds for every i. Hence, the result. □

Corollary. *For any graph G, $\chi(G) \leq 1+\Delta(G)$.* □

Remark. If the vertices of G can be ordered $(v_1, v_2, \ldots, v_n)$ so that every v_i has at most $k-1$ neighbours in $\{v_1, v_2, \ldots, v_{i-1}\}$, then the greedy algorithm uses at most k colors.

• Coloring of chordal graphs (Optional)

Theorem 8.3. *We can color any chordal graph G with $\omega(G)$ colors using the greedy algorithm.*

Proof. Let S be a PEO of the vertices of G; see Theorem 4.7. Under Step 1 of the greedy algorithm, choose the reverse ordering of S, say $(v_1, v_2, \ldots, v_n)$. By the PEO property, v_{i+1} and its neighbors in $\{v_1, v_2, \ldots, v_i\}$ form a complete subgraph in G. Hence at the $(i+1)$-th iteration, there are at most $\omega(G)-1$ colors appearing on the neighbors of v_{i+1}. Hence the greedy algorithm uses at most $\omega(G)$ colors. □

An illustration:

Corollary. *For any chordal graph G, $\chi(G)=\omega(G)$.* □

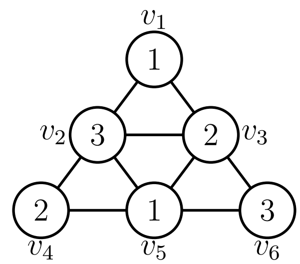


Figure 8.5: A graph G with PEO $(v_1, v_4, v_6, v_2, v_3, v_5)$ and its reverse order $(v_5, v_3, v_2, v_6, v_4, v_1)$ as the input.

We observed in Theorem 8.2 that $\chi(G) \leq 1 + \Delta(G)$, for any graph G. Clearly, $\chi(C_{2n+1}) = 3 = 1 + \Delta(C_{2n+1})$ and $\chi(K_n) = n = 1 + \Delta(K_n)$. Surprisingly, Brooks (1941) showed that these are the only classes of graphs for which the upper bound is attained.

- ## Brooks theorem (Optional)

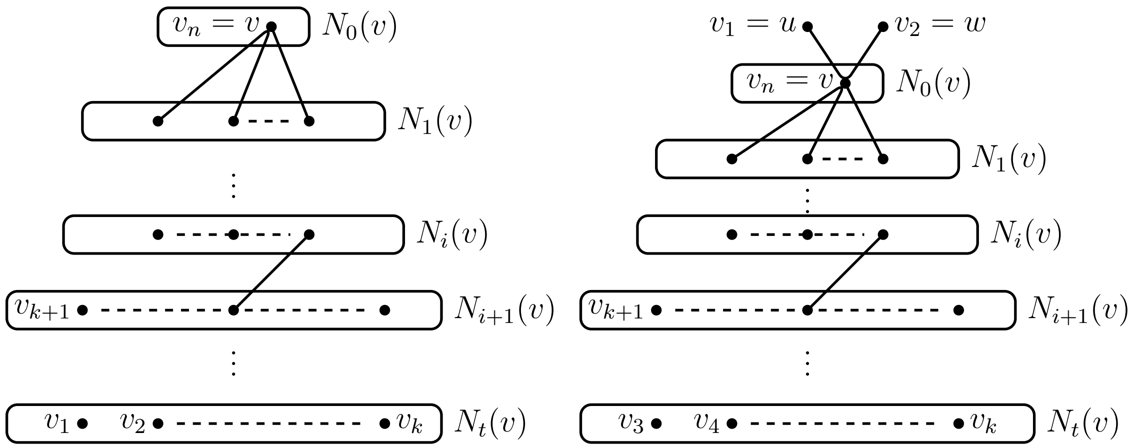
Theorem 8.4 (Brooks, 1941)**.** *If $G \notin \{C_{2p+1}, K_n\}$, then $\chi(G) \leq \Delta(G)$.*

***Proof*.** (Lovasz, 1975) Without loss of generality, we assume that G is 2-connected. As remarked earlier, the idea of the proof is to get an ordering $(v_1, v_2, \ldots, v_n)$ of the vertices of G such that every v_i has at most $\Delta - 1$ neighbors in $(v_1, v_2, \ldots, v_{i-1})$.

Case 1: G is not regular, that is $\Delta(G) \neq \delta(G)$.

Let v be a vertex of minimum degree and let $t = \max\{d(v, x) : x \in V(G)\}$. Let $N_i(v) = \{x \in V(G) : d(v, x) = i\}$, for $i = 0, 1, \ldots, t$. Clearly, $V(G) = \cup_{i=0}^{t} N_i(v)$, and every $x \in N_i(v)$ is adjacent to some vertex in $N_{i-1}(v)$, for every i, $i = 1, \ldots, t$. Let $S = (N_t^+(v), N_{t-1}^+(v), \ldots, N_0(v))$ be an ordering of the vertices of G where $N_i^+(v)$ denotes a sequence of the vertices of $N_i(v)$ ordered arbitrarily; see Figure 8.6a.

We apply greedy algorithm with S as the input. Every vertex $v_p(\neq v)$ in $N_{i+1}(v)$ is adjacent with a vertex in $N_i(v)$, $0 \leq i \leq t - 1$. Hence when v_p is getting



(a) Vertex ordering for Case 1 (b) Vertex ordering for Case 2

Figure 8.6: An ordering of the vertices of G.

colored there are at most at most $\Delta - 1$ neighbors in $\{v_1, v_2, \ldots, v_{p-1}\}$. Therefore, v_p is colored with one of the colors in $\{1, 2, \ldots, \Delta\}$. Since $v_n (= v)$ has degree $\delta(G) < \Delta(G)$, v_n too has at most $\Delta - 1$ neighbors in $\{v_1, v_2, \ldots, v_{n-1}\}$. Therefore v_n is colored with one of the colors in $\{1, 2, \ldots, \Delta\}$.

We illustrate the above case with an example.

Illustration: Consider the graph shown in Figure 8.7. The graph G is a non-regular graph with $\Delta(G) = 4$. By proceeding as in Case 1 with e as the initial vertex, the neighborhood decomposition of the graph yields an ordering of the vertices (chosen arbitrarily within each level) given by $(h, i, a, d, g, j, b, c, f, e)$. Application of the greedy algorithm to this vertex ordering, gives a 4-coloring $(1, 2, 1, 2, 3, 3, 3, 4, 1, 2)$ to the graph as desired.

Case 2: G is d-regular for some d; $2 \leq d \leq n - 2$, since $G \neq K_n$.

If $d = 2$, then G is an even cycle, and so $\chi(G) = 2$. Next assume that $3 \leq d \leq n - 2$. Note that the greedy algorithm uses at most Δ colors in this case

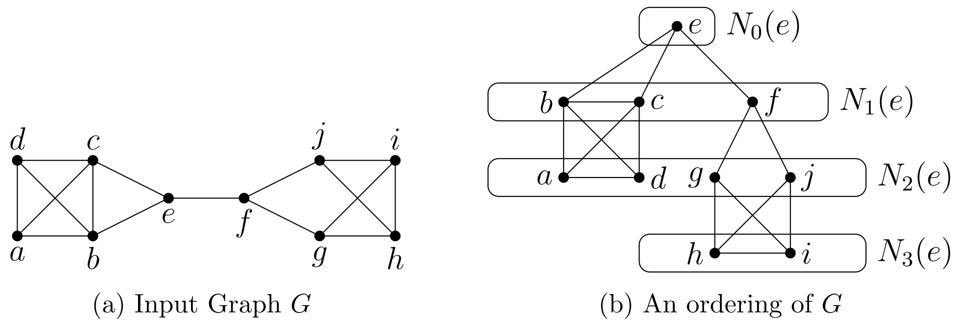


Figure 8.7: A graph G with $\Delta = 4$ with an ordering $(h, i, a, d, g, j, b, c, f, e)$ of the vertices.

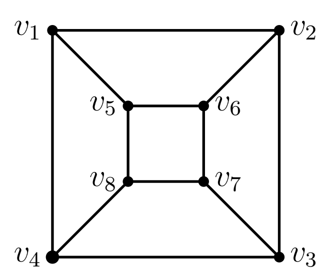
too, if we can get an order $(v_1, v_2, \ldots, v_n)$ of the vertices such that every v_i $(\neq v_n)$ is adjacent with some vertex succeeding v_i and that v_n has at least two neighbors of the **same** color in $(v_1, v_2, \ldots, v_{n-1})$. Such an ordering is achieved by appealing to the following claim (whose proof is left as an exercise).

Claim: In any 2-connected k-regular graph $(3 \leq k \leq n-2)$, there exist three vertices u, v and w such that $(u, v), (v, w) \in E(G)$, $(u, w) \notin E(G)$ and $G - \{u, w\}$ is connected.

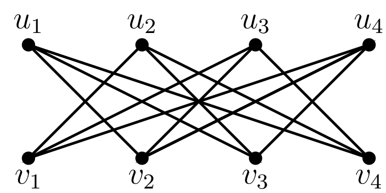
Let u, v, w be as in the claim and let $S = (v_1 = u, v_2 = w, N_t^+(v), \ldots, N_0^+(v))$, where $N_i(v) = \{x \in V(G - \{u, w\}) : d(x, v) = i\}$, $i = 0, 1, 2, \ldots, t$ and $N_i^+(v)$ is the set of vertices in $N_i(v)$ ordered arbitrarily; see Figure 8.6b. With S as the input, the greedy algorithm colors both the vertices v_1 and v_2 with color 1. As in case 1, every vertex v_i $(3 \leq i \leq n-1)$ is adjacent with some vertex succeeding v_i in S. Hence when v_i is getting colored it has at most $\Delta - 1$ colors appearing on its neighbors in $(v_1, v_2, \ldots, v_{i-1})$. Therefore, the greedy algorithm colors v_i with one of the colors in $\{1, 2, \ldots, \Delta\}$. Finally when v_n is getting colored it has at most $\Delta - 1$ colors appearing on its neighbors in $(v_1, v_2, \ldots, v_{n-1})$ since v_1 and v_2 are its neighbors and both are colored 1. Therefore, the greedy algorithm colors v_n too with one of the colors in $\{1, 2, \ldots, \Delta\}$. $\square$

Exercises

1. Find the vertex chromatic number of :
   (a) a tree.
   (b) a graph with exactly one cycle.
   (c) the complement of a cycle.
   (d) $C_{2s} + C_{2t+1}$
   (e) Petersen graph.
   (f) $K_{n_1,n_2,\ldots,n_s}$.
2. Prove: $\chi(G_1 + G_2) = \chi(G_1) + \chi(G_2)$.
3. For every graph G, show that $\frac{n(G)}{\alpha_0(G)} \leq \chi(G) \leq n + 1 - \alpha_0(G)$.
4. Show that every graph with $\chi(G) = k$, contains
   (a) at least $k(k-1)/2$ edges, and
   (b) a cycle of length at least k.
5. Find $n(G_k)$, where G_k is the k-vertex-chromatic, triangle-free graph constructed using Mycielski's method.
6. Draw a graph having maximum number of edges with $\chi(G) = 4$, $n(G) = 10$.
7. Let G be the graph shown below: Order the vertices of G in 3 different ways, say S_1, S_2 and S_3 so that the greedy algorithm
   (a) uses 2 colors, if S_1 is the input,
   (b) uses 3 colors, if S_2 is the input, and
   (c) uses 4 colors, if S_3 is the input.



8. Order the vertices of the graph shown below, in such a way that the greedy algorithm uses 4 colors.



9. Let G be a simple graph such that $\delta(H) \leq k$, for every subgraph H of G (such a graph is called a ***k-degenerate graph***). Order the vertices of G so that if one applies greedy algorithm to color the vertices of G, then it uses at most $k+1$ colors.

10. There are 7 chemicals A, B, C, D, E, F , G. For safety reasons, the following pairs of chemicals cannot be stored in one room: $A\&B$, $A\&C$, $A\&G$, $B\&C$, $B\&D$, $C\&D$, $D\&E$, $D\&F$, $E\&F$, $E\&G$, $F\&G$. If the problem is to find the minimum number of rooms (say k) required to store all the chemicals, how would you model it as a graph theoretical problem and find k.

A graph G is said to be ***k-critical*** if

- $\chi(G) = k$, and
- $\chi(H) < k$, for every proper subgraph H of G.

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11. If G is k-critical, then show that G is connected and that $\delta(G) \geq k-1$.

12. If S is a set of independent vertices in a k-critical graph, then what is the chromatic number of $G - S$.

13. If G_1 and G_2 are simple graphs, show that $G_1 + G_2$ is a critical graph iff G_1 and G_2 are critical.

A partition $(V_1, V_2, \ldots, V_p)$ of $V(G)$ is called a ***clique partition*** of G if every induced subgraph $[V_i]$, $1 \leq i \leq p$, is a clique in G. The minimum integer k such that G admits a clique partition is called the ***clique partition number*** of G; it is denoted by $\theta_0(G)$.

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14. Show that:

    (a) $\theta_0(G) = \chi(G^c)$.

    (b) $\theta_0(G_1 + G_2) = \max\{\theta_0(G_1), \theta_0(G_2)\}$.

    (c) $\theta_0(G) \geq \alpha_0(G)$.