Filling Space with Random Fractal Non-Overlapping Simple Shapes

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Abstract

We present an algorithm that randomly places simple shapes (circles, squares, triangles, and others) without overlap in two dimensions. We describe the mathematics of the process in detail with some conjectures about its properties. The distribution of the areas of the shapes is a power law with varying exponents (typically around -1.3 for visual art). When the algorithm continues "to infinity" it fills all space, while the shapes have an infinite total perimeter. We show several uses of this algorithm to produce visual art.

An Illustration

A picture is worth ten thousand words
-- Confucius (?)

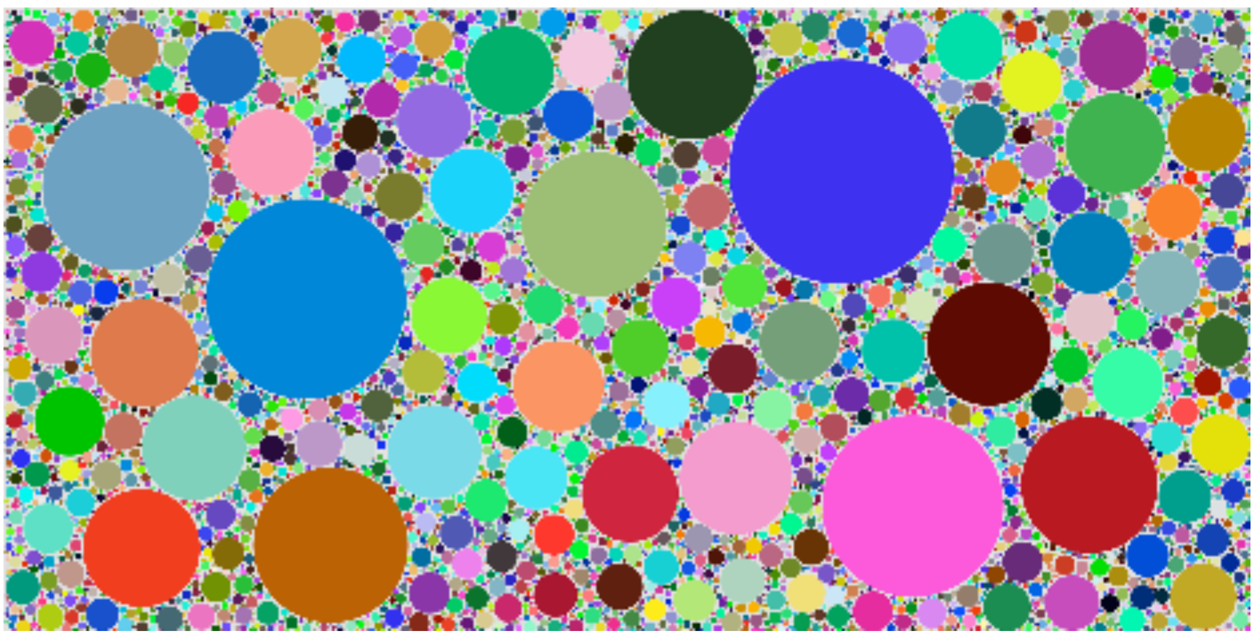

Figure 1: *5000 nonoverlapping fractal circles. The random colors provide the high contrast needed to see the full detail of the image. The successive circle areas decrease by a power law, while their placement is by random search. Such processes have been found to apply to a wide variety of geometric shapes, and in the limit will completely fill all space if properly set up. The image shows the property of "statistical self-similarity", reproducing the same distribution of circle sizes at all length scales.*

Statistical Geometry

Geometry studies the spatial arrangements of shapes (lines, polygons, circles, ...).

"Statistical" and "geometry" are words not usually seen together, so some explanation of this little-explored subject is called for.

Geometry is a huge and ancient subject. Certain branches of geometry have been much used in art and decoration. Tilings of the plane go back a long way, are pleasing to the eye, and have been especially prominent in Islamic art and decoration. Plane tilings pose the question "How do you fill the plane without gaps using a limited number of geometric shapes?" — typically polygons bounded by straight lines. The result is a pattern which covers a bounded region with a finite number of shapes.

A related area of geometry is that of "packings" -- incomplete or maximally-dense filling of a region by circles and other simple shapes. Circle packings alone have a large mathematical literature. The usual rule in circle packings is that one finds a set of circles which all touch (are tangent to) each other. Such tangent packings are called "Appolonian" after the ancient Greek mathematician Appolonius of Perga who first described such a pattern. Such packings don't fill the whole region. These packings have seen relatively little use in art. The packings of interest here are non-Appolonian and violate the rules of formal mathematical circle packing.

Traditional decorative geometric patterns are models of order and regularity, with every shape having an exact location and no elements of randomness.

One might ask: "Can you cover a bounded region with an *infinite* number of regular shapes?" Several examples of this are known, such as the Sierpinski carpet [1], but they have found little use in art, perhaps because their appearance is not particularly attractive to the average eye. Such constructions are largely recursive.

The geometric construction described here poses a different question: "How do you cover a bounded region *non-recursively* with an infinite number of ever-smaller randomly-placed simple shapes (triangles, squares, circles) such that in the limit they completely fill it?" Despite much searching, I have not found any prior account of such an algorithm.

Geometry is a subject of great exactitude. There are precise rules for edges, angles, and vertices. There is no place for randomness or uncertainty. But if you look at the pictures hanging on the wall of an art museum what you see combines elements of both randomness and order. A street scene, for example, has the regular structures of streets and buildings, and the turbulent swirl of vehicles and pedestrians. There is an attractiveness to an image which combines elements of both order and randomness. Nature itself combines randomness and order. All oak trees have a regular branching structure which the eye easily recognizes. But the details differ from one tree to another in a random way.

The geometry described here would startle Euclid.

Conventional tilings have exact symmetry -- it is one of their charms. The shapes making up the pattern have rotation, translation, mirror, and other symmetries. The statistical geometry patterns of interest here have individual shapes with symmetry (square, circle, etc.) but there is no symmetry at all in their placement. What they do have is what might be called fractal symmetry (or "statistical self-similarity") —

a regular progression in the sizes of the shapes. The eye recognizes this kind of symmetry. Apparently even untrained observers see this, although they don't know what to call it.

Rules of Construction

Suppose that we have a bounded region of area A. We intend to fill it with similar geometric shapes having a sequence of areas A_1, A_2, ... (to infinity). The areas A_i are to be computed using a mathematical rule with no randomness.

The algorithm begins by placing shape A_1 somewhere within the region. It then proceeds to generate random positions x,y within the region for the following shapes in the sequence, and for each one tests whether the given shape A_n overlaps any previous A_m. If it does not overlap, this is a "successful placement" and x,y and the size and shape of A_n are placed in a file and the process repeated for the next shape A_{n+1}, or else a new trial position is generated.

If the shapes are to completely fill the area A in the limit, it is evident that one must have

$$A = \sum_{n=1}^{\infty} A_n \qquad (1)$$

The area A_n of the n-th shape is to be chosen according to a mathematical rule. It is evident that the rule must be such that the sum above is convergent. The sequence of areas A_n should follow some ever-smaller rule: $A_n = g(n)$ for the n-th shape.

Many functions obey the obvious requirements: $exp(-an)$, $exp(-an^2)$, and power laws $1/n^c$. Here a and c are parameters which need to be chosen such that Eq. (1) is satisfied. (The sum in Eq. (1) does not converge for all values of c when a power law is used. See [2] for details.)

If the sum in Eq. (1) is less than A, the region will never be completely filled. If the sum is greater than A, the process of seeking random unoccupied positions for ever-smaller shapes will come to a halt at some point for lack of space.

Power-law functions $A_n = A_1 n^{-c}$ (exponent c) are the only ones which have been found to work in computer trials. Useful c values for art lie between 1 and 2. The "tailing off" of $g(n)$ must be slow enough that there is always room in the lacy "gasket" of unoccupied space for another placement. The gasket must get narrower at just such a rate that allows this.

For a power law Eq. (1) becomes

$$A = A_1 \sum_{n=1}^{\infty} \frac{1}{n^c} = A_1 \zeta(c) \qquad (2)$$

The sum can be recognized as the series which defines the Riemann zeta function [2] so that

$$c = \zeta^{-1}(A / A_1) \qquad (3)$$

where $\zeta^{-1}(\ )$ is the inverse zeta function. Thus this process does not have a unique power law exponent, but rather an exponent which varies depending on the choice made for the ratio of A_1 to total area A. It may be that this is the first-ever practical application of the Riemann zeta function.

In the above calculation it is assumed that all shapes will be placed completely inside the bounded area A. This is easy to do computationally. Other choices such as periodic or cyclical arrangements are possible but so far unexplored.

It has been found that the process also works if the sequence in Eq. (2) does not begin with $n = 1$, but starts with some higher value of n. Here the Hurwitz zeta function [2] replaces the Riemann zeta function. Or one can have various laws for A_i versus i for the first N terms and then go over to a power law for $n > N$, as long as Eq. (1) is satisfied.

The process has been used with circles, squares, nonsquare rectangles, and equilateral triangles. The process has been found to run smoothly when set up as described.

By construction the shapes are non-touching (non-Appolonian). With finite-accuracy computing they sometimes touch and may even seem to be slightly overlapping in images. This results from finite precision and roundoff error.

Observed Properties

The remarks here apply to the case where one starts with $n = 1$ as in Eq. (2).

This process operates within a very narrow window. For a given choice of A and A_1 there is only one value of c which works.

It isn't obvious to me why a power law is the unique choice here. Perhaps a rigorous proof of this is possible for this simple "model" system.

While the total area of the shapes has been set up to go to a particular limit, the perimeter grows without limit as n increases. This is characteristic of fractal sets (e.g., Sierpinski [1]).

It has been found in computational experiments that the process does seem to run on "forever" if a power law is used as described above. Sequences of up to 500000 shapes have been computed in this way with no sign that the process will quit (but it does slow down a lot). If the process described here is viewed as a way of measuring area, it reveals a rather surprising property of space.

The process uses random iterations of x,y to find a successful placement. The total (cumulative) number of iterations n_{it} needed follows an increasing power law in n, $n_{it} = n_0 n^f$, with an exponent f. Study of computed data shows that $f \cong c$, i.e., the (negative) value of c is the same (within statistical error) as the (positive) value of f. (It is not at all obvious to me why this should be so.) Thus there is a smooth and regular increase in the average amount of computation for each new shape. This says that the useful (big enough) space for placement is going down by a power law since the probability of placement is a measure of the available area. This supports the idea that the process will always find a place for a new shape "to infinity" in a finite number of iterations.

The following data was found using estimates from computation runs with the stated c values. The mean-square estimates of f and n_0 are thus subject to some uncertainty since we deal with a random process.

$c = 1.15$ $f = 1.1513$ $n_0=2.70$
$c = 1.24$ $f = 1.2429$ $n_0=8.09$
$c = 1.31$ $f = 1.3038$ $n_0=34.3$

The power law for n_{it} does not apply to the first few placements since they are exceptional. Usually enough "slack" is left after the initial placement that the algorithm has an artificially easy time for the first few placements. As n increases the process goes over more and more to a "steady state".

For a given n, the number of iterations needed can be 1, 2, 3, ... Study of histograms of these numbers shows that for large n the distribution is accurately represented by a decaying exponential function. This agrees with the fact that the Poisson distribution goes over to an exponential form when the probability of an individual event (here a successful placement) is << 1.

With its lengthy searches over the "back list" of shapes and their positions, this is a very slow and inefficient algorithm, although simple and easy to code (less than 50 lines of C code for the central loop). Of simple shapes, the square runs fastest. Improved searches should be possible.

One can define a crude measure of the "effective width" of the lacy "gasket" by taking the ratio A_{gask} (the original area A with holes cut out for every shape) divided by the perimeter P_{gask} of all shapes (both functions of n, where n is the number of shapes placed).

$$eff\_width(n) = A_{gask} / P_{gask} \qquad (4)$$

How does this compare with the size of the i-th shape? In the circle case we can define a dimensionless ratio b by

$$b(n) = A_{gask} / diam \cdot P_{gask} \qquad (5)$$

where *diam* is the diameter of the n-th circle. This has been computed using data from a run of the algorithm, and also from formulas. Data from a computer run with c = 1.24 is as follows:

$n = 1000$ $b = .4197$
$n = 2000$ $b = .4140$
$n = 3000$ $b = .4114$
$n = 4000$ $b = .4096$
$n = 5000$ $b = .4086$

One can see that while b is not quite a constant versus n, it has a very slow variation. A check of this versus computation using formulas gave agreement to nearly 4 decimal places (satisfactory in view of numerical and statistical accuracy). What this means is that as n increases the "effective width" of the gasket falls in step with the size of the shape, which explains why random placements continue to be possible all the way "to infinity".

It could be that the weaker variation for large n reflects the approach of the process to "steady state". To date it is unclear whether b really passes to a finite limit for large n.

If one just looks at the formulas it is not at all obvious that b should be nearly flat versus n, since it contains the divergent perimeter P_{gask}. (1/diam also grows without limit.)

The great majority of known mathematical fractal patterns are recursive in nature. This one joins the small set of nonrecursive fractals. In its randomness it resembles natural fractals such as "the coastline of Britain" or "all the islands of the world" discussed by Mandelbrot [1].

As the algorithm proceeds, one can think of the placement process as being in a "critical state". If the exponent c varies even slightly from its precise value for a given A_1, the process will not fill all of the space available, or it will come to an end when it cannot place another shape.

These patterns can be viewed as tessellations if the reader is willing to extend this idea to an infinite number of tiles which cover a given space. The author knows of no natural objects for which this construction could serve as a model, but if the algorithm comes to be known by many people I have little doubt that some will be found.

One might think of an empty world in which the first person to arrive stakes out a territory A_1. As more people arrive they stake out territories A_2, A_3, ... in the unoccupied part. Eventually the entire area is filled by ever-more people occupying ever-smaller territories -- but *they never run out of room for another territory* so peace is preserved.

Conjectures

It would be interesting if it could be shown that the power laws used here are the only laws which work.

It is noted that available data says that the exponents f and c are the same (within statistical error) for sequences beginning with 1. It would be interesting if it could be shown that the most probable value or the expectation value of f is c in this case.

It would be interesting to clarify the asymptotic behavior of the ratio b defined above as n goes to infinity. This problem does not involve randomness since it depends only on nonrandom calculations of the gasket area, perimeter, and size versus n. This problem intimately involves (various sets of terms in) the infinite series for the zeta function.

The quantity b can be defined for any functional rule $A_i = g(i)$. It can be speculated that near-constancy of b as n goes to infinity is a requirement for any successful algorithm of this kind. In fact, by calculating the b parameter on-the-fly as the algorithm progresses, it might be possible to develop an "adaptive" choice of the next circle size.

The author does not know of any formal scheme for describing the statistical properties and ordering of an object of this kind. Statistical *physics* has a vast body of theory developed by several generations of physicists since Boltzmann and Gibbs, but that is lacking here. The physics case is greatly aided by the fact that every atom of a given kind is identical to every other. Here the individual elements (shapes) are all different.

It would be interesting to determine what classes of shapes can be "fractalized" using this algorithm, and what can't. The algorithm works well for a circle or square (low perimeter-to-area ratio). It also works for nonsquare rectangles of mixed orientation. It fails to work for the equilateral triangle without additional requirements such as opposite orientations at each step (Figure 4).

Examples

One of the problems with images of these patterns is that the placed shapes may so nearly fill the area that the eye blends them all into one big blur. For this reason I have limited the filling factor to 90% or less. The background is white. The author has computed patterns with up to 97% filling factor containing 500000 shapes. Further examples can be found at the author's web site [3].

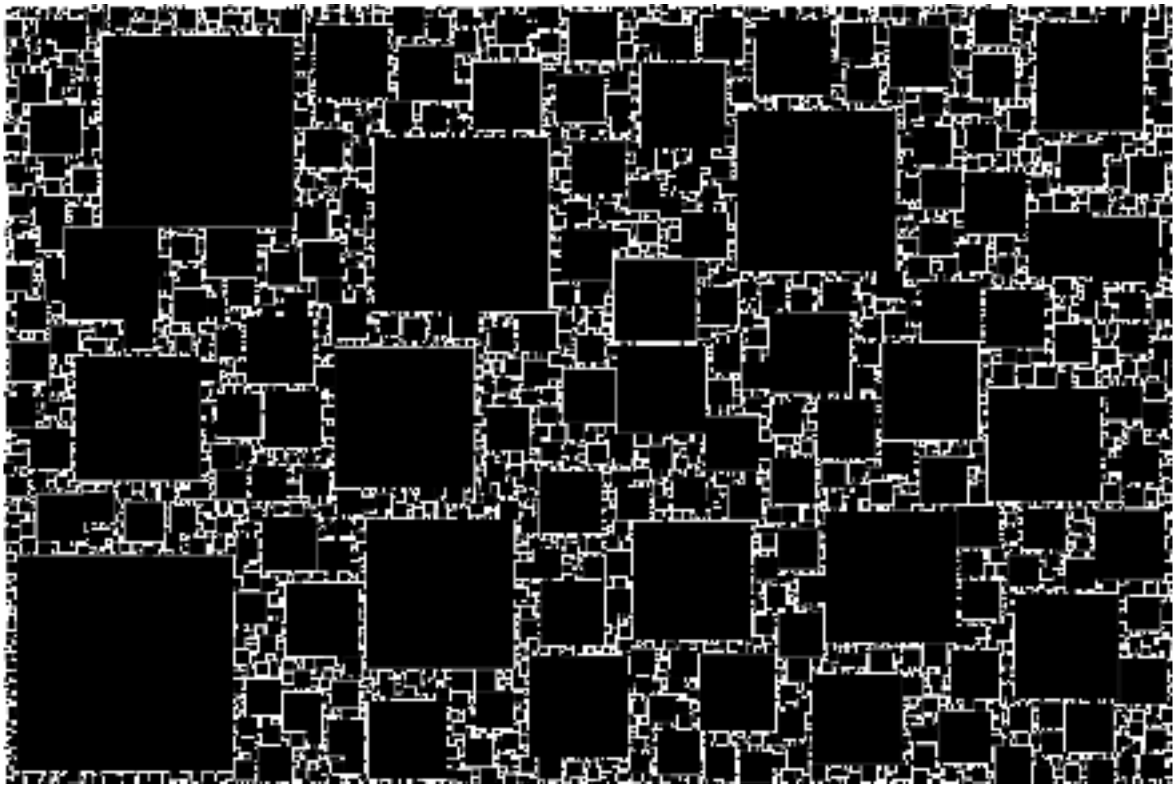

Figure 2: *5000 fractal squares. 83% space filling.*

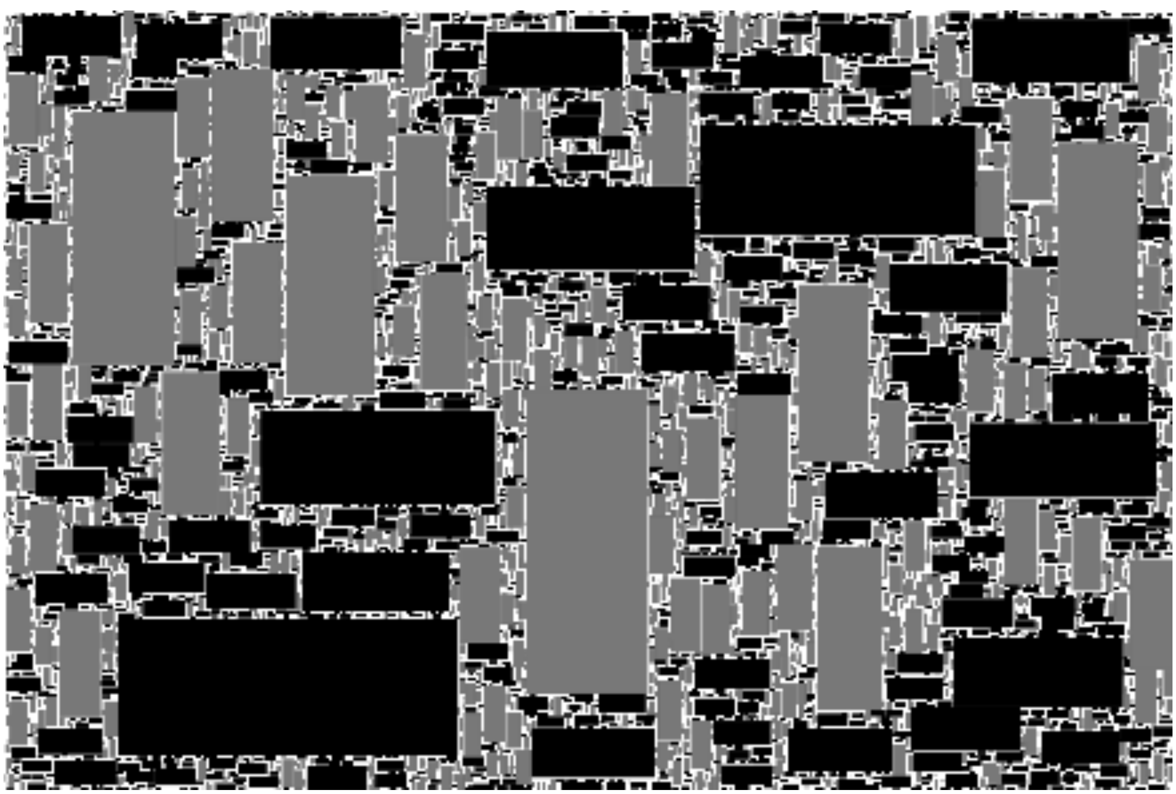

Figure 3: *5000 mixed-orientation fractal rectangles. 2.5 to 1 aspect ratio. 83% space filling.*

In Figure 3 all of the rectangles have the same areas as in Figure 2. They are elongated with a 2.5 to 1 aspect ratio, and the "vertical" shapes are gray while the "horizontal" ones are black. The aspect ratio changes each cycle, so that even-numbered shapes are gray and odd-numbered ones black, etc. The reader may note that there is an ordering property here. If a large gray shape got an early placement in a given area, it is surrounded by mostly gray rectangles, etc. While this is a random process, the randomness is *constrained* by all of the previous placements.

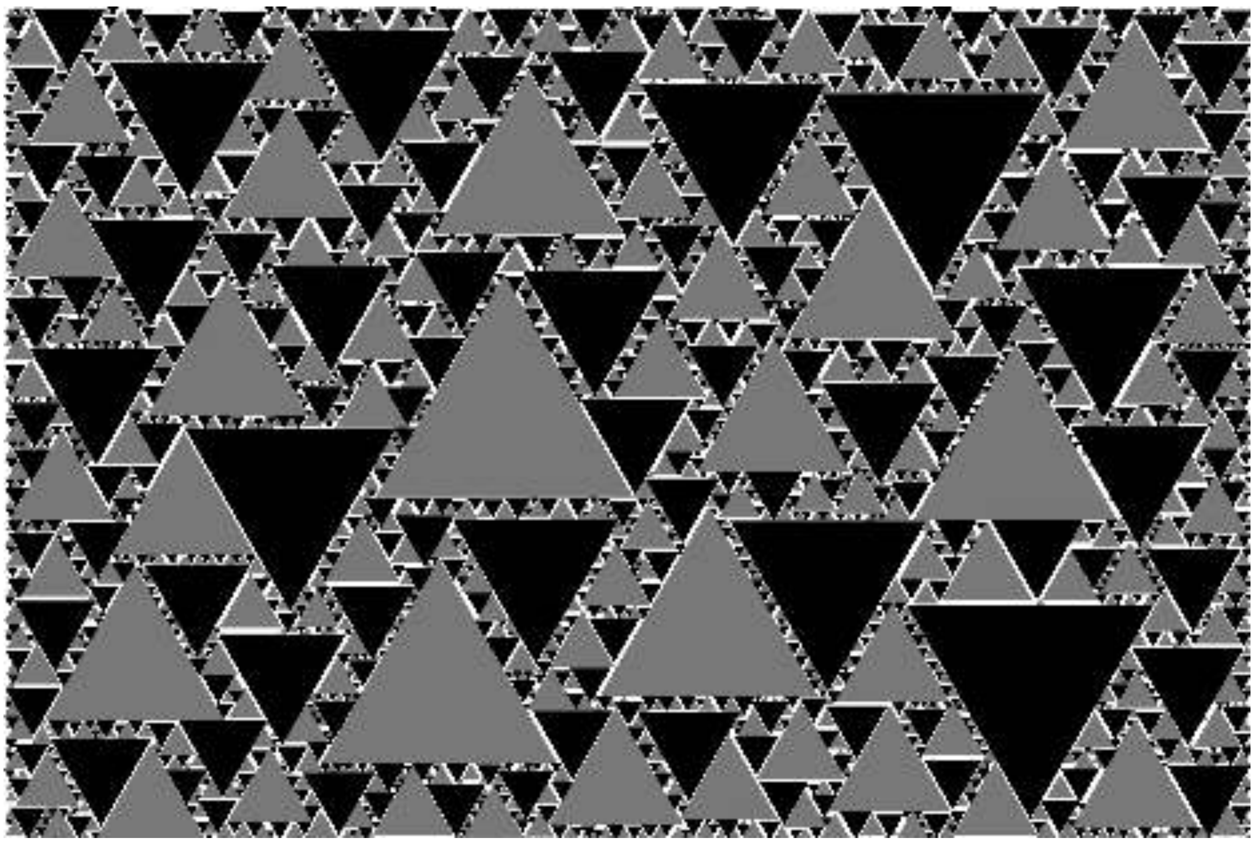

Figure 4: *2500 fractal equilateral triangles. 88% space filling. c = 1.4214. One suggested title for this image is "Sierpinski exploded".*

Figure 4 shows equilateral triangles. It is interesting that the algorithm fails (by stopping) if all of the triangles have the same orientation. If the process is modified so that odd numbered triangles are "up arrows" while evens are "down arrows" the process works quite well, and that is the case shown here. The black triangles are "up arrows" as drawn, and the gray ones "down arrows". The viewer will note a strong ordering here; the immediate neighbors of an "up" are mostly "downs", etc.

Another case studied was "L-shaped" polygons (not shown). Such a polygon is non-convex and it was thought this might make a difference. The algorithm ran flawlessly.

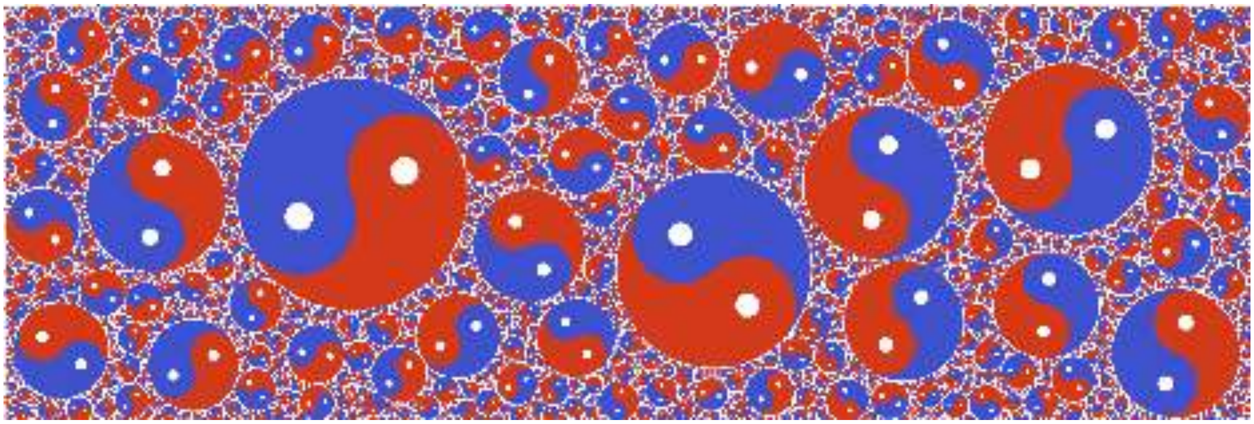

Figure 5: Geometric patterns often lend themselves to decorative uses. This example, with 4000 yin-yang symbols, should please east Asians. Happy Chinese New Year!

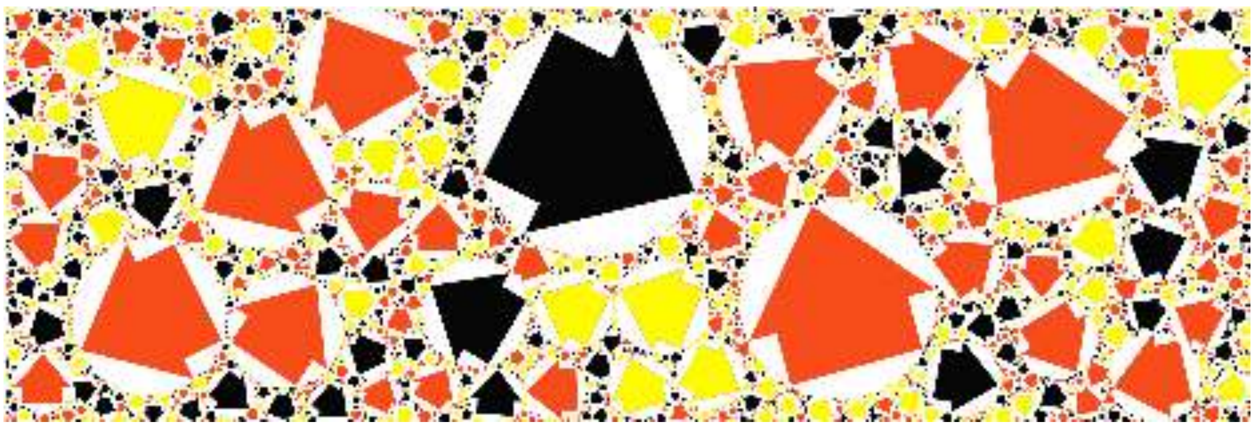

Figure 6: *Modern life offers a confusing chaos of directions as expressed in this image. The arrows are inscribed at locations of circles. With more elaborate programming the entire space could be filled with arrows. Three random colors and random orientations.*

Figure 7: *Numbers have a continuing fascination for mathematicians, gamblers, and anybody who works with a computer. Here we see the 9 digits in a rather squarish font placed at the locations of fractal squares. The colors are chosen by random walk in color space, in the order largest to smallest. Each number size has a corresponding color. The winning lottery number is in here somewhere.*

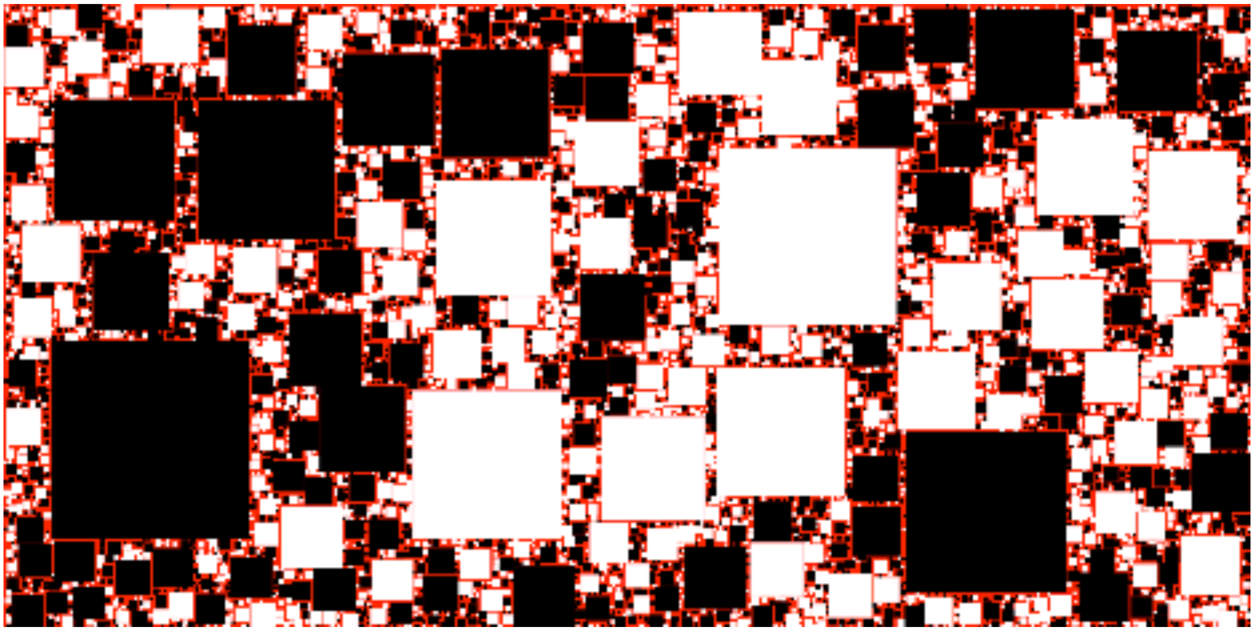

Figure 8: *"The Devil's Checkerboard". In a checkerboard one colors alternating squares of a regular grid black or white (or two other colors). Here the same thing has been done for fractal squares. The largest is black, the second-largest white, in alternation black-white-black- ... . The red color is the part of the original plane which has not been covered with any squares (the "gasket"). This illustrates the random nature of the process, and the regular progression in the areas of the squares. When the filling factor exceeds about 95% the "gasket" becomes difficult to see..*

References

[1] Benoit B. Mandelbrot, *Form, Chance, and Dimension*, W. H. Freeman, San Francisco (1977).

[2] The Riemann zeta function is famous among mathematicians for its link with the theory of prime numbers. Wikipedia has articles on both the Riemann and Hurwitz zeta functions.
http://en.wikipedia.org/wiki/Riemann_zeta_function
http://en.wikipedia.org/wiki/Hurwitz_zeta_function

[3] The author's web site is http://john-art.com