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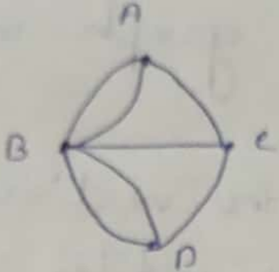
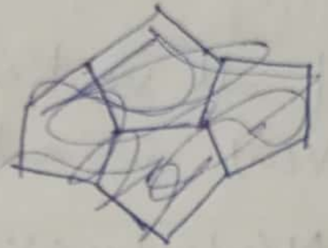
- Konigberg's Problem:



cross each edge exactly once and come back to original point

- Hamilton's Problem:

cross each point exactly once and come back to original point



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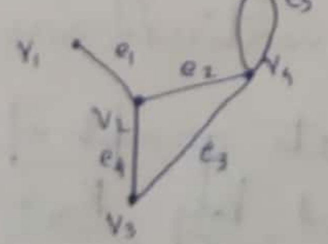
Graph :—

- Def^n: A graph G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$ of edges with each edge of G being an unordered pair of (not necessarily distinct) vertices of G.

- Notation: $G = [V, E]$.

- Example:



$V = \{v_1, v_2, v_3, v_4\}$

$E = \{e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_4\}, \ldots, e_5 = \{v_4, v_4\}\}$

Definitions :

- A loop is an edge whose end vertices are same.
- Parallel / multiple edges are edges having the same end point.

- Simple graph is a graph with no loops / multiple edges.

- Adjacency — b/w two vertices
  Incidence — b/w one vertex and one edge

- The set of all vertices which are adjacent with $v \in V(G)$ is called the neighbourhood of v.

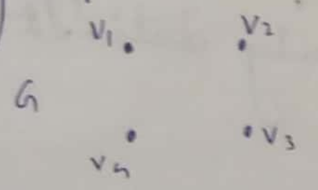
  $N(v) = \{u \in V(G) : u \text{ is adjacent to } v\} \subseteq V(G)$ (open nhbd)

  $N[v] = N(v) \cup \{v\}$ (closed nhbd)

  for simple graphs

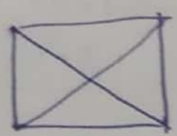
- A graph G is finite if both $E(G)$ and $V(G)$ are finite set.
  Otherwise the graph is infinite.

- A graph can have null $E(G)$,



(Empty graph)/null graph

- A complete graph is a simple graph in which every pair of distinct vertices are adjacent



  * A complete graph of n vertices is denoted by $\boxed{K_n}$
  * maximum no. of edge = $\boxed{{}^nC_2}$.
  * $0 \leq |E(G)| \leq \frac{n(n-1)}{2}$

- No. of vertices = order
  No. of edges = size.

- The degree of vertex v in a graph G ~~is~~ denoted by $d_G(v)$ is the number of edges of G incident with v, each loop counting twice.
  - * If $d_G(v) = 0$, v is called isolated vertex.
  - * If $d_G(v)$ is even, v is called an even vertex.
  - & If $d_G(v)$ is odd, v is called an odd vertex.

- Th^m: Let G be a graph with n vertices and m edges. Then $\boxed{\sum_{i=1}^{n} d_G(v_i) = 2m}$.

  Handshake Lemma

- Corrollary: $\nexists$ a graph with odd no. of odd vertices.

* Minimal degree, $\delta(G) = \min\{d_G(v) : v \in V(G)\}$
Maximal degree, $\Delta(G) = \max\{d_G(v) ; v \in V(G)\}$

* Thus $0 \leq \delta(G) \leq \Delta(G)$.

* If in a graph G, $\delta(G) = \Delta(G) = k$, then G is called a k-regular graph.

* G is a simple graph of n vertices, then $\Delta(G) \leq n-1$

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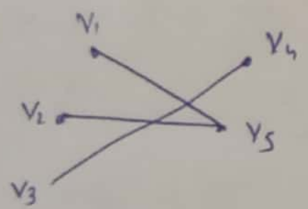
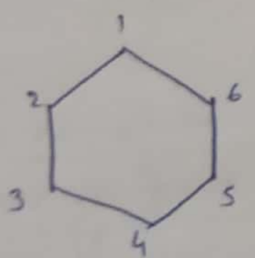
* **Bipartite graph:**

Let G be a graph such that it can be partitioned into two non-empty sets X and Y (i.e. $X \sqcup Y = G$) in such a way that each edge of G has one end in X and the other in Y, then G is called a bipartite graph.

Example:



$X = \{v_1, v_2, v_3\}$
$Y = \{v_4, v_5\}$



$X = \{1, 3, 5\}$
$Y = \{2, 4, 6\}$

* A complete bipartite graph is a simple bipartite graph G with bipartition $G = X \sqcup Y$, in which every vertex in X is joined to every vertex in Y.

→ If $|X| = m$, $|Y| = n$, then denote it by $\boxed{K_{m,n}}$

→ $|E(G)| = mn$??

A walk in a Graph;

* **Def^n:** A walk in a graph is an alternating sequence $\boxed{W: v_0 e_1 v_1 e_2 v_2 \cdots e_n v_n}$ of vertices and edges begining and ending with with vertices in which v_{i-1} and v_i are endpoints of e_i $\forall i \in [n]$.

* If $v_0 = v_n$, then the walk is <u>closed</u>.
* If $e_i \neq e_j$ $\forall i, j \in [n]$, then we call it a <u>trail</u>.
* <u>Length</u> of the walk = no. of edges in W.

Cycle:

A cycle is a closed trail in which all the vertices are distinct except the endpoint (/starting point).

* In cycle, no. of edges = no. of vertices.
* Even cycle $\rightarrow$ no. of edge is even
  Odd cycle $\rightarrow$ no. of edge is odd.
* Cycle of length n $\rightarrow$ $\boxed{C_n}$
* A loop is cycle with length 1.
* In a simple graph, length of a cycle has to be at least 3.

Path:

Subgraph;

A subgraph of a graph G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

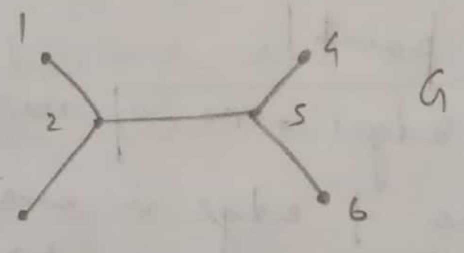
* Spanning Subgraph;

A subgraph H of G is called a spanning subgraph of G if $V(H) = V(G)$.

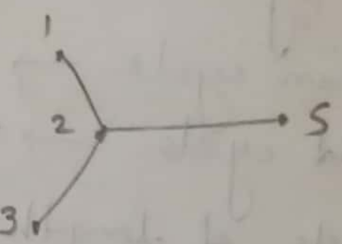
* Induced Subgraph;

Let G be a graph and $S \subseteq V(G)$. Then $G[S]$, the induced subgraph of G with vertex set S (i.e. $V(G[S]) = S$) and two vertices in $G[S]$ is adjacent iff they are adjacent in G.

Example;



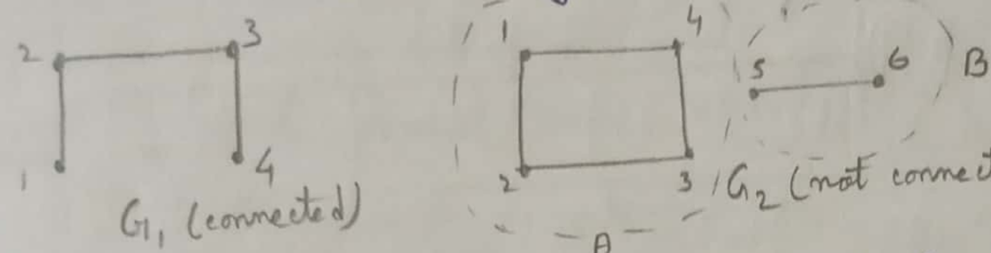
$S = \{1, 2, 3, 5\}$. Then $G[S]$



[basically restricting the graph to the chosen S]

Connected Graph:

A graph is connected if every ~~vertex~~ pair of vertices are joined by a path.

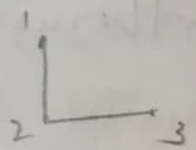


G_1 (connected)

G_2 (not connected)

* A maximal connected subgraph of G is called a connected component.

Example: In G_2, A and B are components. ~~of~~



is not a connected component of G_2 (not maximal)

* A connected graph has 1 connected component.

Isomorphism in Graphs:

* An isomorphism from a graph G to a graph H is a bijection f that maps V(G) to V(H) and E(G) to E(H) and each edge of G with end-points u and v is mapped to an edge of H with end-points f(u) and f(v).

* G and H is isomorphic if ∃ an isomorphism between G and H.

(preserves the adjacency).

$f_1 : V(G) \rightarrow V(H)$ and $\forall \{u, v\} \in E(G)$,

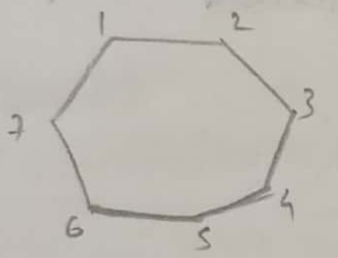
$f_2 : E(G) \rightarrow E(H)$ $\{f_1(u), f_1(v)\} \in E(H)$.

* An isomorphism from G to itself is called an automorphism.

$Aut(G)$ — set of all isomorphisms of G.
(read up!)

- **Distance**: (Metric on a graph)

Let G be a graph and $u, v \in V(G)$.
The distance $d(u,v)$ between two vertices u and v is the length of the shortest path joining them (if any), otherwise $d(u,v) = \infty$. (when the graph is not connected - finite)



$d(1,4) = 3$

→ $d(u,u) = 0$
→ $d(u,v) = d(v,u)$
→ $d(u,v) \leq d(u,w) + d(w,v)$

diameter
* Let G be a connected graph. Then the diameter of G, denoted as $diam(G)$, is

$$\boxed{diam(G) = \max\{d(u,v) : u, v \in V(G)\}}$$

eccentricity
* For $v \in V(G)$, the eccentricity $e(v)$ is defined as

$$\boxed{e(v) = \max\{d(u,v) : u \in V(G)\}}$$

radius

* The radius of G, $r(G)$ is the minimum eccentricity of G.

$\therefore$ $\boxed{r(G) = \min\{e(v);\ v \in G\}}$

Also, $\boxed{diam(G) = \max\{e(v);\ v \in G\}}$

$\therefore$ $\boxed{r(G) \leq diam(G) \leq 2r(G)}$

for complete graph, $r(G) = diam(G) = 1$

* Proof of $diam(G) \leq 2r(G)$:

Theorem:

A graph is bipartite iff all its cycles are even.

Proof: $\Rightarrow$ Suppose, G is bipartite.

Let V_1 and V_2 be a bipartition of $V(G)$.

Let $C_k = v_1 v_2 \cdots v_k v_1$ be a cycle in G.

WLOG, let $v_1 \in V_1$.

Then all odd-subscripted vertices are in V_1 and all even-subscripted vertices are in V_2.

Since $v_1 \in V_1$, $v_k \in V_2$.

$\therefore$ k is even.

$\therefore C_k$ is an even cycle.

Since C_k is an arbitrary cycle in G, all the cycles in G is even.

$\Leftarrow$ Suppose, all the cycles in G is even.

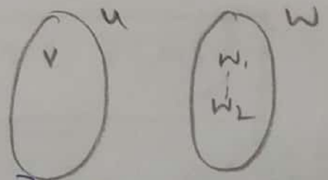
Suppose, G is connected.

Let, $v \in V(G)$.

Define, $U = \{ u \in V(G) \mid d(u,v) \text{ is even} \}$

$W = \{ w \in V(G) \mid d(w,v) \text{ is odd} \}$

Also, $V(G) = U \sqcup W$.



Let $w_1, w_2 \in W$ and $\{w_1, w_2\}$ be an edge.

Let p_1 be a shortest path from v to w_1 and p_2 be a shortest path from v to w_2.

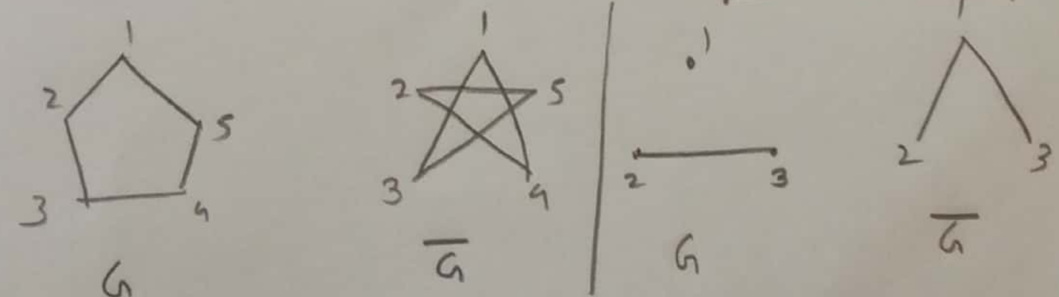
Let z be the last common point between p_1 and p_2.

Complement of a Graph:

• **Def^n:** Let G be a simple graph. The complement of G, $\bar{G}$ (or G^c) is the graph with vertex set $V(G)$ and two vertices are adjacent in $\bar{G}$ iff they are not adjacent in G.

$$\boxed{(\bar{\bar{G}}) = G}$$

• **Example:**



- A graph is called <u>self-complementary</u> if it is ~~to~~ isomorphic to its complement.