Real Analysis Chapter 2 Study Guide (for "Real Analysis, A First Course", 2nd Edition, Russell A. Gordon)

Number of Starred Exercises: **6**; Number of Notes: 5; Number of Other (non-starred) Exercises: 29; Minimum Number of Other (non-starred) Exercises to Do (to do at least 25% of them): **8**;

The THREE most important thing to get out of this chapter: (1) A full understanding of the definition of the limit of a sequence and the use of this definition in proofs; (2) an understanding of the meaning, use, and significance of the Cauchy (pronounced "KOH-she") Convergence Criterion (Theorem 2.13), especially the fact that when a sequence is Cauchy, it converges; (3) an understanding of the meaning of the Bolzano-Weierstrass Theorem (Theorem 2.19), this theorem will be very useful for us later.

Other matters of importance:

- The algebraic properties of convergent sequences, the Squeeze Theorem, and the uses of these properties.
- If a sequence is monotone and bounded, it converges.
- The *definition* of what it means for a sequence to be a Cauchy sequence
- The use of induction in many proofs
- The idea and notation of subsequences
- The notions of a limit superior ("limsup") and a limit inferior ("liminf")

Reading Guide:

1. **Note:** the book's use of the notation$\{x_n\}$ to represent an arbitrary sequence is not universally used by mathematicians (a common alternative is (x_n)) and this notation can also sometimes be misinterpreted. Sequences are **NOT** *sets* (though you can speak of the set of values or "range" of the sequence, which is a set…see the bottom of page 50 through the top of page 51) because sequences *have an explicit ordering* and *can have repeated values* which we really want to *represent as different terms* in the sequence. On the other hand, the order of the elements in a *set* is of ***no importance*** (for example, {a,b} = {b, a}) and, if an element is repeated, it doesn't change the set (for example, {a, a, b} = {a, b}).
2. If indeed a sequence can be thought of as a function, does that mean it has a *graph*? If so, what would the graph of a sequence look like? How would you interpret the convergence of a sequence from a graphical viewpoint? Generate some examples and draw some pictures. Try writing out how you would explain convergence of a sequence visually to someone just learning calculus.
3. Negate the definitions in Definition 2.2
4. Generate some examples of sequences that are not bounded and/or not monotone.
5. This is not in the book, but it is possible to define what it means for a sequence to be *eventually increasing* or *eventually decreasing* or *eventually monotone*. Many theorems in the book (such as the very foundational Theorem 2.10) can be strengthened by *weakening* the hypotheses to include these "eventuallies" (so that

the theorems apply to more situations). What do you think I mean by this? What do you think the definition of an *eventually increasing sequence* should be?

6. Generate two sequences that converge to the number $e = 2.71828\ldots$ (you may have to look through your Calculus textbook for help here).
7. Think of a subject in Calculus where recursively-defined sequences are used for equation solving (Hint: think of the name (starting with the letter "N") of a person who thought a lot about Physics and gravity). Write down that recursive formula. How does it work to help solve equations? (draw a picture to help you understand it…read and try to understand the derivation of the recursive formula)
8. *Wrestle with the definition of a convergent sequence and the limit of such a sequence (page 52). Draw a picture with remarks showing what it means. What is the fact that $\varepsilon > 0$ must be *arbitrary* important for the definition? Write down the logical negation of the definition. Generate an example of an obviously false limit statement and determine an $\varepsilon > 0$ for which no N can be found (and explain why).
9. Verify that $\left|\dfrac{7n}{4n+5} - \dfrac{7}{4}\right| = \dfrac{35}{4(4\nu+5)}$ for any positive integer n.
10. *Make up your own limit example similar to the one starting at the bottom of page 52. Give the scratch-work necessary to find an appropriate value for N. Then give a polished formal proof.
11. **Note:** notice the use of "rough" estimates and of the Archimedean property (and hence the Completeness axiom) at the top of page 53. Notice the use of the triangle inequality and the trick of adding zero in a convenient way in the proof of Theorem 2.4 and Theorem 2.5. Also take note of the use of Theorem 1.7 in the proof of Theorem 2.4.
12. Check your negation of the definition of what it means for a sequence to not converge to a number L (see #8 above) with the book's negation on page 54. Comment on how close you were to getting it right. If you didn't get it right, then rewrite it without looking at the book.
13. Is the use of $\varepsilon = 1$ in the proof of Theorem 2.5 the only value of ε that could have been used? Why or why not? If not, how would the proof be affected?
14. **Note:** note that it is the contrapositive of Theorem 2.5 that is generally considered to be more "useful" for some specific sequences.
15. Negate the definition of what it means for a sequence of real numbers to "converge" to ∞ (I prefer to say "diverges" to ∞).
16. *Prove that the sequence $\{n^2\}$ diverges to ∞.
17. Work real hard at understanding the proof of Theorem 2.7(d). This proof illustrates some important real analysis proof techniques. Try to reprove it on your own without looking at the book's proof about 1 hour after you read it.
18. Draw a picture that graphically illustrates the Squeeze Theorem for Sequences.
19. Use induction to prove that $n^2 + 2\nu + 1 < 10\nu^2$ for all integers $n \geq 1$. Then use this fact and induction to prove that $n^2 < 10^\nu$ for all integers $n \geq 1$.
20. As in #17 above, work real hard at understanding the proof of Theorem 2.9. This proof illustrates some important real analysis proof techniques. Try to reprove it on your own without looking at the book's proof about 1 hour after you read it.

21. **Note:** Theorem 2.10 is very important basic theorem in Real Analysis (even though it seems limited in scope because it only applies to monotone sequences). Take note of the use of the Completeness Axiom in the proof of this theorem.
22. As in #17 and #20 above, work real hard at understanding the proof of Theorem 2.10. This proof illustrates some important real analysis proof techniques. Try to reprove it on your own without looking at the book's proof about 1 hour after you read it.
23. The sequence $\{a_n\}$ generated at the bottom of page 61 is called the *sequence of partial sums* for an *infinite series*. This particular sequence of partial sums converges to $\ln(2)$ (and we write $\sum_{k=1}^{\infty}\frac{1}{k2^k} = \lambda\nu(2)$). Do some research to figure out why. Write down what you find.
24. *Verify the details of the calculations in the middle of page 62.
25. It is possible to come up with examples of recursively defined sequences which do not converge, yet when you do a calculation like that done on the bottom of page 62, you can get an answer for *L*. See if you can come up with such an example. Why is there no contradiction to the fact that your sequence does not converge, yet you can get an answer for *L*? (Hint: think about assumptions you might be making in finding *L*).
26. *Negate the definition of a Cauchy sequence. Also, for the example right after the definition of a Cauchy sequence, why can you not show this sequence converges without reference to Theorem 2.13 at the top of page 64? (You can wait until you get to that spot before answering this question.)
27. Prove Theorem 2.12.
28. **Note:** the proofs of parts of Theorem 2.14 on page 65 are interesting, but kind of specialized. What is more important for us is that you know the facts in the statement of Theorem 2.14, especially parts 1 – 6. Parts 7 and 8 are related to a question earlier in this study guide (I'll let you decide where).
29. Fill in any missing details in the proof of the Nested Intervals Theorem (Theorem 2.15). How is the Completeness Axiom used? (this is a bit tricky to determine).
30. Try to write the definition of a subsequence of a given sequence (page 69) in a way that you could explain to someone off the street.
31. Come up with examples of non-convergent sequences that have convergent subsequences (and describe the convergent subsequences).
32. *Prove part (b) of Theorem 2.17.
33. Fill in any details from the proof of Theorem 2.18 that are confusing to you.
34. Does the recursively-defined sequence $x_{n+1} = 4\xi_v(1-\xi_v)$, $\xi_1 = 0.1$ have a convergent subsequence? Explain why or why not. (Use *Mathematica* or your calculator to help you find some of the terms of this sequence.)
35. Create a picture or a chart showing how the Completeness Axiom and other definitions and facts lead to the statement and truth of the Bolzano-Weierstrass Theorem.
36. Create your own examples of sets with limit inferiors and limit superiors. See if you can explain why the limit inferior and/or limit superior are what you think they are for your examples.

37. Fill in any details of the proofs of parts c, d, and i of Theorem 2.21 that are confusing to you.
38. Pick one or two of the properties of Theorem 2.21 that are not proved in the book and see if you can prove it/them. Give it your very best shot.
39. Come up with an example that shows that the equation near the bottom of the first paragraph on page 73 may not be valid.
40. Fill in any details from the proof of Theorem 2.22 that are confusing to you.

Deep Thoughts to Ponder (but not necessarily answer):

- The concept of a Cauchy sequence of rational numbers can be used to construct the real numbers ° from the rational numbers § . The basic idea is this, define two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ of rational numbers to be equivalent if the sequence $\{x_n - y_n\}$ of differences in their values converges to zero (in other words, we are defining an equivalence relation on the set of all Cauchy sequences of rational numbers). The real number system is defined to be the equivalence classes of this equivalence relation. From this point, you need to define how to add and multiply these equivalence classes to form a field, how to define an order on these equivalence classes to make it an ordered field, and show that the Completeness property (no longer an axiom in this approach) holds.