Real Analysis Chapter 1 Study Guide (for "Real Analysis, A First Course", 2nd Edition, Russell A. Gordon)

Number of Starred Exercises: **10**; Number of Notes: 10; Number of Other (non-starred) Exercises: 34; Minimum Number of Other (non-starred) Exercises to Do (to do at least 25% of them): **9**;

THE most important thing to get out of this chapter: The fundamental nature of the ***completeness axiom***; both in distinguishing the real numbers from the rational numbers, and in proving fundamental facts in real analysis.

Other matters of importance:

- The ordered field properties of real numbers
- The triangle inequality and other properties of absolute value
- The definition of bounded sets and the concepts of supremums and infimums.
- The Archimedean Property of real numbers
- The density of the rationals and irrationals in the real number system
- The notion of a one-to-one correspondence (bijection) and its relationship to the idea of the cardinality of a set.

Reading Guide:

1. *(Answer these before you begin your reading) If a person off the street asked you what the difference is between a rational number and an irrational number, what would you say? How would you define an irrational number? How do you know they exist? Is the previous question a nonsense question or not? Why or why not?
2. **Note:** the reference on page 2 to an article by B. Pourciau [20] was made possible by none other than yours truly…Dr. Kinney…well, that's maybe a bit of an exaggeration…Bruce Pourciau is a math professor at Lawrence University in Appleton, Wisconsin and he wrote that article while on sabbatical during the 1997-1998 school year…and, well,…I was his sabbatical replacement then, which was the year before I came to Bethel…though I suppose he may have written it without me replacing him anyway ☺
3. *Does your answer to the first question in #1 above agree with any of the descriptions given in the text at the bottom of page 3?
4. In the proof that $\sqrt{2}$ is irrational, why may we assume that p and q have no common divisors greater than 1?
5. For those of you who are scientifically inclined, ponder whether there really is a concept of "exact measurement" of continuous quantities (see the bottom of page 3). Does such an idea make sense? How does your answer to this inform your view of rational versus irrational numbers?
6. Come up with a few of your own fraction-to-decimal and decimal-to-fraction examples and work through the calculations similar to the ones in the book on pages 4 and 5. (don't make them trivial).

7. Look up Zeno's paradoxes…which one intrigues you the most? Why?
8. What does it mean for an operation to be defined on a set, as in the definition of a field on page 6? In other words, what is meant by the word "operation" (or, more precisely, "binary operation")? Can you define it more precisely? Hint: it's a certain type of function…but alas, then you may also want to more precisely define what a function is…
9. Try to recall how to define what a field in more brevity by using words from Algebraic Structures such as "ring", "commutative", "unity", and "unit". Of course, you might also want to try to recall how to define each of these words. Write down your thoughts.
10. Do you have any initial thoughts about the axiomatic definition of the real number system? Does this seem like a good way to go about "defining" a real number?
11. **Note:** Property 1 in Definition 1.2 is called the principle of trichotomy and Property 2 of Definition 1.2 is called transitivity (of a binary relation).
12. Use the ordered field properties to prove the property in the middle of page 8.
13. What would the conclusion of this property (in the middle of page 8) be if $z < 0$ instead? Can you prove the resulting statement…perhaps by using the original statement?
14. Does the piecewise formula for the absolute value function on page 10 make sense to you? Explain why it works. You might want to imagine that you are a teacher trying to explain it to a 6th grader.
15. Rewrite property (a) of Theorem 1.4 that would be more informative about the true nature of the relationship between an arbitrary real number *a* and the numbers $|a|$ and $-|a|$. If you're not sure what I mean here at first glance, think harder!
16. It's somewhat surprising how often property (d) of Theorem 1.4 comes up in proofs. Take note of it and think about it. Draw a picture to help you understand it. Use it to help you solve the inequality $|2x+7| < 4$. Draw graphs to help you understand the solution set of the inequality visually.
17. *Try to prove the Triangle Inequality (Theorem 1.5) ***without looking at the proof in the book***. If you've already read it, try it anyway. Hint: Use Theorem 1.4. Take note of the fact that the proof of the Triangle Inequality in the book is actually quite simple. Though if you didn't think about using Theorem 1.4, you'd probably have a hard time coming up with the proof (maybe you still had a hard time anyway).
18. Under what conditions on *a* and *b* is the triangle inequality a *strict* inequality? Under what conditions is it an *equality*? Can you generalize these to conditions on vectors **a** and **b** in $\mathbb{R}^2$? (you'll want to look up the more general triangle inequality for vectors first).
19. Is there a picture of vectors (like Figure 1.3) that would help you understand the Reverse Triangle Inequality? Try drawing one and see if you can figure out how to interpret it (Hint: you'll need to recall how to geometrically interpret the difference of two vectors).
20. **Note:** The discussion at the top of page 12 is pretty important because it will often come up in proofs. Read it carefully and think about it.
21. The proof of Theorem 1.7 is "tricky" yet somewhat typical of the style of many kinds of logical arguments in real analysis. Make sure you think about whether

you believe that it *really proves what you want to prove*. Write down any questions you may have about it. Also think carefully about the paragraph above Theorem 1.7 and write down any questions you have about it.

22. The equations for the maximum $x \vee \psi$ of two numbers and the minimum $x \wedge \psi$ of two numbers on page 12 are kind of wild. You probably never would have imagined that you could come up with such equations. Briefly spend time thinking about why they are true. See if you can verify them.
23. Before looking at Definition 1.8 on page 13, try to come up with your own definition of an ***interval*** of real numbers. Then compare your definition with the book's definition. How close did you get?
24. Write down the logical negation of the definition of an interval of real numbers. In other words, what does it mean for a subset *S* of ° to **not** be an interval. You might want to review parts of Appendix A.
25. The proof of Theorem 1.10 is a very nice trick. This theorem can also be proved by induction. See if you can give a nice proof by induction.
26. The material on page 15 through the top of page 17 is not essential for our course. However, if you find it interesting, go ahead and check the calculations and proofs and jot some notes about it. The Cauchy-Schwarz inequality actually is a very important inequality for more advanced Real Analysis. You should spend time thinking about it if you are thinking about going to graduate school in math someday.
27. *Write the logical negations of the definitions in Definition 1.14. Draw pictures to illustrate the definitions. Review parts of Appendix A if necessary.
28. What do you think the definition of "bounded" should be for a set of points $S \subseteq °^{\nu}$? Is there such a thing as an upper and/or lower bound for such a set?
29. *Determine the supremum and the infimum of some examples of sets that you generate. Try to make them nontrivial examples. (Keep generating these examples until you feel you have a reasonable handle on this concept). Is the infimum of a set of positive numbers necessarily a positive number? (Note: be careful about what the definition of a positive number is.)
30. Verify to your own level of satisfaction the fact that the statements near the bottom of page 22 lead to contradictions.
31. **Note:** For clarity, I think the statement of the Completeness Axiom on page 23 should include the phrase "in the real numbers'" at the end of it (this is implied by the author, but not stated). After all, every rational number is also a real number so the following statement is true, but could be misinterpreted: "Each nonempty set of rational numbers that is bounded above has a supremum". This is true, but the supremum might be an irrational number. If someone took this to implicitly imply that the supremum were a rational number, then it would be a false statement.
32. *Write down the statements of the Intermediate Value Theorem and the Mean Value Theorem from your Calculus text (try to write them down without looking them up first, then look them up if you need to). Why are these "existence results"?
33. **Note:** The equivalence of part 4 of Theorem 1.17 to part 1 (the Archimedean property) will be used very frequently.

34. **Note:** Theorem 1.18 is sometimes called the “density” property of the rationals and irrationals in the reals. The conclusion of this theorem can actually be strengthened to say that between any two distinct real numbers there are *infinitely* many rational numbers and *infinitely* many irrational numbers. Do you think you see how you could prove this stronger version? Explain.
35. *Mimic the proof of Theorem 1.19 to show that there exists a real number x such that $x^2 = 2$.
36. In the proof of Theorem 1.20, why is it true $0 \le \delta_v \le 9$ for each n?
37. A Philosophical Question: do you think the definition of what it means for two sets to have the **same size** (the same “cardinality”) on page 30 is the “best” definition that could be given, especially in the case where the two sets are infinite?
38. **Note:** a synonym for “countable” that you’ll often see in other math textbooks is “denumerable”.
39. See whether or not you can find a formula for the one-to-one correspondence between the set of positive integers and the set of all integers pictured as a pairing in the middle of page 31.
40. In the proof of Theorem 1.23, is it clear to you that f is a one-to-one function? Briefly explain.
41. In the proof of Theorem 1.23, explain why a is the smallest integer in the set $A \setminus \{f(1), f(2), K, f(p)\}$.
42. **Note:** the statement of Theorem 1.25 can be confusing if you misinterpret it (it might seem to be giving a conclusion that is assumed in the hypthesis). The “countable union” means that the *collection consisting of all the sets* in the union is itself a countable collection (set). This allows us to put the n = 1 to ∞ below and above the union symbol in the 6th line of the proof.
43. In the proof of Theorem 1.25, why is it important to assume that none of the sets have elements in common with any of the others?
44. Elaborate on why the Fundamental Theorem of Arithmetic implies that the pairing at the bottom of page 32 is the described one-to-one correspondence.
45. *Carefully prove Lemma 1.28 on page 34. Note: the collection of intervals in this lemma is called a **nested** collection.
46. Draw pictures to illustrate the idea of the proof of Theorem 1.29. In what fundamental way(s) does the completeness property of the real numbers enter into this proof?
47. *Are the real numbers as ``trivial" to understand as you may have once thought? Elaborate.
48. Think about the ways of defining the sine function in the middle of page 38. Do you understand these approaches? Try to figure out why they work and the advantages/disadvantages of each. How else can the sine function be defined for all real numbers? (Hint: see the top of page 39). Can you describe this other definition?
49. Give a definition of arctan(x) that is similar to the way arsin(x) is defined on page 39.

50. *See if you can prove some of the statements in 1 – 4 in the middle of page 40 and the statements 1 – 4 in the middle of page 41. It's an "extra bonus" if you can prove them without resorting to using Calculus.
51. Prove the statement that saying that a nonempty bounded set has a maximum value if and only if it contains its supremum.
52. **Note:** I like to add the word "global" in front of the bold-faced words in parts a, b, and c of Definition 1.33.
53. **Note:** take careful note of the middle paragraph on page 42.
54. Does the collection of real-valued functions defined on an interval form a group? Does it form a ring? (careful…what are the group/ring operations?)

Deep Thoughts to Ponder (but not necessarily answer):

- Are there sets whose cardinality (loosely speaking, "number" of elements) is a "size of infinity" larger than the "size of infinity" of the cardinality of the real numbers? Do some research into the ***power set*** of a given set to find out.
- Since the set of irrational numbers and the set of real numbers are both uncountable, does that mean they can be put in one-to-one correspondence with each other? Can the real numbers be put in one-to-one correspondence with the points in the plane $\mathbb{R}^2$? With $\mathbb{R}^n$ for any $n \in \mathbb{Z}^+$? Do some research into the ***continuum hypothesis*** and the works of ***Georg Cantor*** to find out.

Real Analysis Chapter 2 Study Guide (for "Real Analysis, A First Course", 2nd Edition, Russell A. Gordon)

Number of Starred Exercises: **6**; Number of Notes: 5; Number of Other (non-starred) Exercises: 29; Minimum Number of Other (non-starred) Exercises to Do (to do at least 25% of them): **8**;

The THREE most important thing to get out of this chapter: (1) A full understanding of the definition of the limit of a sequence and the use of this definition in proofs; (2) an understanding of the meaning, use, and significance of the Cauchy (pronounced "KOH-she") Convergence Criterion (Theorem 2.13), especially the fact that when a sequence is Cauchy, it converges; (3) an understanding of the meaning of the Bolzano-Weierstrass Theorem (Theorem 2.19), this theorem will be very useful for us later.

Other matters of importance:

- The algebraic properties of convergent sequences, the Squeeze Theorem, and the uses of these properties.
- If a sequence is monotone and bounded, it converges.
- The *definition* of what it means for a sequence to be a Cauchy sequence
- The use of induction in many proofs
- The idea and notation of subsequences
- The notions of a limit superior ("limsup") and a limit inferior ("liminf")

Reading Guide:

1. **Note:** the book's use of the notation$\{x_n\}$ to represent an arbitrary sequence is not universally used by mathematicians (a common alternative is (x_n)) and this notation can also sometimes be misinterpreted. Sequences are **NOT** *sets* (though you can speak of the set of values or "range" of the sequence, which is a set…see the bottom of page 50 through the top of page 51) because sequences *have an explicit ordering* and *can have repeated values* which we really want to *represent as different terms* in the sequence. On the other hand, the order of the elements in a *set* is of ***no importance*** (for example, {a,b} = {b, a}) and, if an element is repeated, it doesn't change the set (for example, {a, a, b} = {a, b}).
2. If indeed a sequence can be thought of as a function, does that mean it has a *graph*? If so, what would the graph of a sequence look like? How would you interpret the convergence of a sequence from a graphical viewpoint? Generate some examples and draw some pictures. Try writing out how you would explain convergence of a sequence visually to someone just learning calculus.
3. Negate the definitions in Definition 2.2
4. Generate some examples of sequences that are not bounded and/or not monotone.
5. This is not in the book, but it is possible to define what it means for a sequence to be *eventually increasing* or *eventually decreasing* or *eventually monotone*. Many theorems in the book (such as the very foundational Theorem 2.10) can be *strengthened* by *weakening* the hypotheses to include these "eventuallies" (so that

the theorems apply to more situations). What do you think I mean by this? What do you think the definition of an *eventually increasing sequence* should be?

6. Generate two sequences that converge to the number $e = 2.71828\ldots$ (you may have to look through your Calculus textbook for help here).
7. Think of a subject in Calculus where recursively-defined sequences are used for equation solving (Hint: think of the name (starting with the letter "N") of a person who thought a lot about Physics and gravity). Write down that recursive formula. How does it work to help solve equations? (draw a picture to help you understand it…read and try to understand the derivation of the recursive formula)
8. *Wrestle with the definition of a convergent sequence and the limit of such a sequence (page 52). Draw a picture with remarks showing what it means. What is the fact that $\varepsilon > 0$ must be *arbitrary* important for the definition? Write down the logical negation of the definition. Generate an example of an obviously false limit statement and determine an $\varepsilon > 0$ for which no N can be found (and explain why).
9. Verify that $\left|\dfrac{7n}{4n+5} - \dfrac{7}{4}\right| = \dfrac{35}{4(4\nu+5)}$ for any positive integer n.
10. *Make up your own limit example similar to the one starting at the bottom of page 52. Give the scratch-work necessary to find an appropriate value for N. Then give a polished formal proof.
11. **Note:** notice the use of "rough" estimates and of the Archimedean property (and hence the Completeness axiom) at the top of page 53. Notice the use of the triangle inequality and the trick of adding zero in a convenient way in the proof of Theorem 2.4 and Theorem 2.5. Also take note of the use of Theorem 1.7 in the proof of Theorem 2.4.
12. Check your negation of the definition of what it means for a sequence to not converge to a number L (see #8 above) with the book's negation on page 54. Comment on how close you were to getting it right. If you didn't get it right, then rewrite it without looking at the book.
13. Is the use of $\varepsilon = 1$ in the proof of Theorem 2.5 the only value of ε that could have been used? Why or why not? If not, how would the proof be affected?
14. **Note:** note that it is the contrapositive of Theorem 2.5 that is generally considered to be more "useful" for some specific sequences.
15. Negate the definition of what it means for a sequence of real numbers to "converge" to ∞ (I prefer to say "diverges" to ∞).
16. *Prove that the sequence $\{n^2\}$ diverges to ∞.
17. Work real hard at understanding the proof of Theorem 2.7(d). This proof illustrates some important real analysis proof techniques. Try to reprove it on your own without looking at the book's proof about 1 hour after you read it.
18. Draw a picture that graphically illustrates the Squeeze Theorem for Sequences.
19. Use induction to prove that $n^2 + 2\nu + 1 < 10\nu^2$ for all integers $n \geq 1$. Then use this fact and induction to prove that $n^2 < 10^\nu$ for all integers $n \geq 1$.
20. As in #17 above, work real hard at understanding the proof of Theorem 2.9. This proof illustrates some important real analysis proof techniques. Try to reprove it on your own without looking at the book's proof about 1 hour after you read it.

21. **Note:** Theorem 2.10 is very important basic theorem in Real Analysis (even though it seems limited in scope because it only applies to monotone sequences). Take note of the use of the Completeness Axiom in the proof of this theorem.
22. As in #17 and #20 above, work real hard at understanding the proof of Theorem 2.10. This proof illustrates some important real analysis proof techniques. Try to reprove it on your own without looking at the book's proof about 1 hour after you read it.
23. The sequence $\{a_n\}$ generated at the bottom of page 61 is called the *sequence of partial sums* for an *infinite series*. This particular sequence of partial sums converges to $\ln(2)$ (and we write $\sum_{k=1}^{\infty}\frac{1}{k2^k} = \lambda\nu(2)$). Do some research to figure out why. Write down what you find.
24. *Verify the details of the calculations in the middle of page 62.
25. It is possible to come up with examples of recursively defined sequences which do not converge, yet when you do a calculation like that done on the bottom of page 62, you can get an answer for L. See if you can come up with such an example. Why is there no contradiction to the fact that your sequence does not converge, yet you can get an answer for L? (Hint: think about assumptions you might be making in finding L).
26. *Negate the definition of a Cauchy sequence. Also, for the example right after the definition of a Cauchy sequence, why can you not show this sequence converges without reference to Theorem 2.13 at the top of page 64? (You can wait until you get to that spot before answering this question.)
27. Prove Theorem 2.12.
28. **Note:** the proofs of parts of Theorem 2.14 on page 65 are interesting, but kind of specialized. What is more important for us is that you know the facts in the statement of Theorem 2.14, especially parts 1 – 6. Parts 7 and 8 are related to a question earlier in this study guide (I'll let you decide where).
29. Fill in any missing details in the proof of the Nested Intervals Theorem (Theorem 2.15). How is the Completeness Axiom used? (this is a bit tricky to determine).
30. Try to write the definition of a subsequence of a given sequence (page 69) in a way that you could explain to someone off the street.
31. Come up with examples of non-convergent sequences that have convergent subsequences (and describe the convergent subsequences).
32. *Prove part (b) of Theorem 2.17.
33. Fill in any details from the proof of Theorem 2.18 that are confusing to you.
34. Does the recursively-defined sequence $x_{n+1} = 4\,\xi_v(1-\xi_v),\ \xi_1 = 0.1$ have a convergent subsequence? Explain why or why not. (Use *Mathematica* or your calculator to help you find some of the terms of this sequence.)
35. Create a picture or a chart showing how the Completeness Axiom and other definitions and facts lead to the statement and truth of the Bolzano-Weierstrass Theorem.
36. Create your own examples of sets with limit inferiors and limit superiors. See if you can explain why the limit inferior and/or limit superior are what you think they are for your examples.

37. Fill in any details of the proofs of parts c, d, and i of Theorem 2.21 that are confusing to you.
38. Pick one or two of the properties of Theorem 2.21 that are not proved in the book and see if you can prove it/them. Give it your very best shot.
39. Come up with an example that shows that the equation near the bottom of the first paragraph on page 73 may not be valid.
40. Fill in any details from the proof of Theorem 2.22 that are confusing to you.

Deep Thoughts to Ponder (but not necessarily answer):

- The concept of a Cauchy sequence of rational numbers can be used to construct the real numbers ° from the rational numbers § . The basic idea is this, define two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ of rational numbers to be equivalent if the sequence $\{x_n - y_n\}$ of differences in their values converges to zero (in other words, we are defining an equivalence relation on the set of all Cauchy sequences of rational numbers). The real number system is defined to be the equivalence classes of this equivalence relation. From this point, you need to define how to add and multiply these equivalence classes to form a field, how to define an order on these equivalence classes to make it an ordered field, and show that the Completeness property (no longer an axiom in this approach) holds.

Real Analysis Chapter 3 Study Guide (for "Real Analysis, A First Course", 2nd Edition, Russell A. Gordon)

Number of Starred Exercises: **4**; Number of Notes: 16; Number of Other (non-starred) Exercises: 43; Minimum Number of Other (non-starred) Exercises to Do (to do at least 25% of them): **11**

The most important things to get out of this chapter: (1) A full understanding of the definition of the limit of a function and the use of this definition in proofs. (2) A full understanding of the definition of continuity and how it is related to the limit concept. (3) Fundamental consequences of continuity (the Intermediate Value Theorem (IVT) and the Extreme Value Theorem (EVT)). (4) The definition of uniform continuity and how it differs from the definition of continuity. (5) Functions which are continuous on a closed and bounded interval are also uniformly continuous on that closed and bounded interval.

Other matters of importance:

- The relationship between limits of functions and limits of sequences and how this is used in proofs
- How to prove that a limit does not exist by using sequences
- The algebraic properties of limits of functions and the squeeze theorem for functions
- The relationship between one-sided limits and "ordinary" two-sided limits
- The definitions of "infinite" limits (a special kind of divergence)
- Continuous functions are "nice"
- Most functions encountered in Calculus are continuous where they are defined
- Monotone functions are "nice"
- Gaining an ability to plow through many difficult proofs

Reading Guide:

1. *In one paragraph and with pictures, summarize your understanding of the limit of a function $f(x)$ at this point (before you read Chapter 3).
2. Does the function $f(x)=\varepsilon^{-\frac{1}{\xi^2}}$ have a limit as $x \to 0$? If so, what is its value? Can you explain your answer?
3. Rewrite Definition 3.1 in a less precise way, perhaps using phrases like "arbitrarily close" and "sufficiently close".
4. Rewrite Definition 3.1 with mostly symbols (like $\forall$, $\exists$, etc...).
5. Why is it not necessary for *c* to be in *I* in Definition 3.1? Is there an example that justifies this?
6. *Write the negation of Definition 3.1.
7. Create an example of a function and a value of *L* which is not the correct value of the limit of the function as $x \to \chi$. Then find an $\varepsilon > 0$ such that there are no values of δ that will result in Definition 3.1 being satisfied (keep your example simple).

8. Explain why the fact that $\varepsilon > 0$ is arbitrary is essential in Definition 3.1.

9. *Use a technique similar to the book's example at the bottom for page 84 to show that $\lim\limits_{x \to 2} \xi^2 = 4$. Draw a picture to illustrate the idea of your proof.

10. Note: take note of how a sequence $\{x_n\}$ converging to *c* is constructed in the proof of Theorem 3.2. The method used to construct this sequence is quite common in the book's proofs from here on out. Study it carefully like the author emphasizes.

11. Prove that the function *h* defined near the bottom of pate 85 does have a limit at 0.

12. Prove Theorem 3.3(d) using two methods. First, use Definition 3.1. Second, use Theorem 3.2. Looking at the proof of Theorem 2.7(d) may be helpful.

13. Prove that $L \le M$ in Theorem 3.4 by showing that the assumption $L > M$ leads to a contradiction.

14. Draw a picture illustrating the Squeeze Theorem for Functions.

15. Use one-sided limits to help you determine the value, if any, of $\lim\limits_{x \to 0} \dfrac{\xi}{|\xi|}$. Explain.

16. Note: make a note of the strange notation sometimes used for one-sided limits.

17. Write a definition for what you think the notation $\lim\limits_{x \to \infty} \phi(\xi) = \infty$ should mean.

18. Before reading the paragraph after Definition 3.9, compare Definition 3.9 with Definition 3.1. How do they seem to be related? What kinds of differences are there?

19. Note: carefully read the last paragraph of page 93 and think about the examples that go onto the top of page 94 to understand some of the subtleties involved with the function and its domain when you define continuity.

20. Based on thinking about the formulation of continuity at a point given in statement 2 of Theorem 3.10, draw pictures that illustrate different ways a function can fail to be continuous at a point *c*.

21. Note: in the proof of Theorem 3.11 for quotients, it's not necessary to use ε's and δ's. The truly hard work has already been done in the proof of Theorem 3.3.

22. Before reading the proof of Theorem 3.13, try to prove it on your own (if you've already read it, try to prove it an hour later without looking at the book's proof).

23. Compare the paragraph at the bottom of page 95 to your work on #20 above. Did you omit any possible ways a function can fail to be continuous at *c*?

24. Draw a graph of $f(x) = \lfloor \xi \rfloor + \lfloor -\xi \rfloor$ to illustrate the idea in the first paragraph on page 96.

25. Draw some graphs of discontinuous functions that either have no extreme values or do not attain every intermediate value over an interval [a, b] (see the first paragraph on page 99).

26. Prove the fact at the bottom of page 99 (hint: use something similar to the "common technique" for generating a sequence $\{x_n\}$ in the proof of Theorem 3.2 (see #10 above)).

27. In the proof of the IVT (Theorem 3.16), explain why $c_n \to \chi$, $c_n \in \Sigma$ for all *n*, and $f(c_n) < \varpi\phi o\rho\alpha\lambda\lambda\nu$ imply that $f(c) \le \varpi$.

28. In the proof of the IVT, explain why $\{d_n\}$ is a sequence in $[a,b] \setminus S$ and why $f(d_n) \ge \varpi\phi o\rho\alpha\lambda\lambda\nu$.

29. From a topological viewpoint, the IVT can be interpreted as saying that continuous functions map intervals onto intervals (though we need to think of one-point sets as being "intervals" for this to be a completely valid observation). Since intervals are "connected" sets, this can be generalized to saying that continuous functions map connected sets onto connected sets. They don't "break" the set into pieces.

30. Note: take note of the technique in the first paragraph on page 101 that allows us to apply the IVT to prove that a certain equation has a solution. This trick is common.

31. Note: both paragraphs of the proof of the EVT (Theorem 3.17) require the use of the Bolzano-Weierstrass Theorem. Note this means we *do not care* whether the *originally constructed* sequence $\{x_n\}$ or $\{d_n\}$ converges or not. All we care about is the existence of *some* sequence that converges (in this case, as in many cases, the convergent sequence is a subsequence of $\{x_n\}$ or $\{d_n\}$, in the frist and second paragraphs, respectively).

32. Note: notice once more the slick ways of constructing sequences in the proof of the EVT (see #10 above).

33. A topological interpretation of the EVT would say that continuous functions map closed and bounded intervals onto closed and bounded intervals. There's no way a continuous function could map a closed and bounded interval onto an open and bounded interval or onto an unbounded interval (draw some pictures of this to convince yourself). However, continuous functions *can* map open and bounded intervals onto unbounded intervals and can also map open intervals onto closed intervals (draw pictures to convince yourself). The distinctions in properties between closed and open intervals are sometimes very subtle.

34. Note: Theorem 3.18 codifies the topological interpretations in #29 and #33 above).

35. Prove the statement at the bottom of page 102.

36. Think back to Algebraic Structures. Was the converse of Lagrange's Theorem true? If not, can you think of any "partial" converses of Lagrange's Theorem that were true? What did these statements say?

37. Note: "proof by contraposition" (in the proof of Theorem 3.20) means "a proof of the contrapositive". Also note that the negation of the definition of an interval (look up Definition 1.8) is used in the proof of this theorem.

38. In the proof of Theorem 3.21, why does the fact that *f* is strictly increasing on *I* imply that $f_{inv}(u) < f_{inv}(v)$ when *u* < *v*? Also take note of the fact that Theorem 3.20 is used to prove the inverse function is continuous. In other words, take note of the fact that Theorem 3.20 might be useful for you in proofs that certain monotone functions are continuous.

39. Note: make sure you are clear about the fact that continuous functions satisfy the intermediate value property, but there are some discontinuous functions which satisfy it too (see the example at the bottom of page 104 which goes to the top of page 105).

40. What are the other three cases in the proof of Theorem 3.24?

41. Note: note the fact that derivatives always satisfy the intermediate value property (when they exist), even though they might not always be continuous (think about why the function $f(x) = \xi^2$ σιν$(1/\xi)$ ωηεν $\xi \neq 0$ and $f(0) = 0$ provides one such example).

42. Find a function which is defined everywhere but not locally bounded at one point.

43. Explain why $\delta = 0.001$ and $M = 1000$ work to prove that $g(x) = 1/\xi$ is locally bounded at 0.002.

44. Ponder how a function which is locally bounded at every point in an interval may not be bounded on that interval.

45. Draw pictures which help you understand examples 1 and 2 on pages 110 and 111. Make up your own example and draw pictures for it.

46. *Negate the definition of a uniformly continuous function on an interval. Can you find an example of a function that is continuous on an interval but which seems to be not uniformly continuous on that interval? (hint: infinite intervals are allowed).

47. For a given $\varepsilon > 0$, how should we choose $\delta > 0$ to prove that the sine function is uniformly continuous on ° ?

48. Prove that $h(x) = 1/\xi$ is uniformly continuous on the interval $(0.002, 5)$.

49. Compare the book's negation of the definition of uniform continuity on page 112 with your own from #46 above.

50. Prove that x^3 is not uniformly continuous on ° .

51. Can you find a function which is bounded on ° but which seems to be not uniformly continuous on ° ?

52. **Note:** in the proof of Theorem 3.28, once again take note of the trick of how to generate sequences that have certain properties and take note of the use of the Bolzano-Weierstrass Theorem.

53. Fill in any details in the proof of Theorem 3.29 which are confusing to you.

54. **Note:** Theorem 3.30 gives a nice and simple criterion for determining whether a function *f* which is continuous on an open interval (*a*, *b*) is uniformly continuous on that interval.

55. **Note:** even though the content of Section 3.5 gets quite technical, it's main theme is not so hard to grasp: the main theme is "monotone functions are relatively nice" or, at the very least, "monotone functions are not super-nasty". See the top of page 116 for more explanation of this.

56. **Note:** when you see equations like $f(x-) =$ συπ$\{\phi(\tau) : \tau \in [\alpha, \xi)\}$ and $f(x+) =$ ινφ$\{\phi(\tau) : \tau \in (\xi, \beta]\}$ in Theorem 3.31, you should immediately say to yourself: "hey, in this situation the set $\{f(t) : t \in [\alpha, \xi)\}$ must be bounded above for each $x \in [\alpha, \beta]$ and the set $\{f(t) : t \in (\xi, \beta]\}$ must be bounded below for each $x \in [\alpha, \beta]$". Draw pictures of monotone (and possibly discontinuous) functions to help you believe this.

57. Come up with an example of a function defined on an interval [*a*, *b*] where the set of discontinuities is uncountable.

58. Prove the result in part (d) of Theorem 3.35 for sums (the proof for differences would be similar).

59. Can you think of a statement and equation similar to the statement and equation of Theorem 3.36 for definite integrals? Write down that statement.

60. Note: the proofs of Theorems 3.35 and 3.36 are difficult, but taking the time to understand them will pay dividends for helping you understand proofs about definite integrals in Chapter 5.

61. Fill in any details of the proofs of Theorems 3.35 and 3.36 that you find to be confusing. It might also be helpful to make notes to yourself about the overall strategies of the proofs.

62. Draw a picture to help you understand the argument that $f(x)=\sqrt{\xi}\chi o\sigma(\pi / \xi)$ if $x \neq 0$ and $f(0)=0$ is not of bounded variation on [0, 1].

63. Draw pictures to help you understand the proof of Theorem 3.37. In particular, for graphs of various (relatively simple) functions *f*, see if you can draw the graphs of f_1 and f_2 defined in the proof.

Deep Thoughts to Ponder (but not necessarily answer):

- Why is the “definition” of continuous function as being one whose graph can be drawn without picking up your pencil not a sufficient definition for Real Analysis?
- Do the ideas of limits and continuity seem easier or harder than they used to? Do you feel it has been beneficial to you to study this chapter (either personally or for your future job…perhaps in teaching)? Do you have more of an appreciation for the subtleties that are glossed over in calculus and other pre-calculus mathematics?

Real Analysis Chapter 4 Study Guide (for "Real Analysis, A First Course", 2nd Edition, Russell A. Gordon)

Number of Starred Exercises: **3**; Number of Notes: 2; Number of Other (non-starred) Exercises: 46; Minimum Number of Other (non-starred) Exercises to Do (to do at least 25% of them): **12**

The most important things to get out of this chapter: an understanding of the meaning and use of the definition of the derivative; an understanding of the meaning, use, and importance of the mean value theorem; an understanding of how everything in the chapter fits together; experience with typical counter-examples.

Other matters of importance:

- In the proofs encountered in this chapter, take note of how often a new function is "created" as an aid to complete a proof. This is done so that the previous theorems may more easily be applied to the new situations. Make a note of this each time you see it.
- Recall how to take derivatives.
- Recalling how to use derivatives in optimization problems and L'Hopital's Rule problems.

Reading Guide:

1. Our author is not very careful about the conceptual difference between velocity and speed or between position and distance. What extra assumption can you make about the particle in the first paragraph that makes these concepts equivalent?
2. Think of other applied situations (besides velocity, acceleration, decay rates, and growth rates) where the limit of a difference quotient gives you the quantity of interest in the application. Write them down.
3. Make a *Mathematica* animation (you can write your code by hand in your journal) that illustrates how a smooth curve looks more and more like a straight line as you zoom in near any point on the smooth curve.
4. Why are the limits near the top of page 131 equivalent?
5. Fill in any steps which help you understand the calculation of $f'(4)$ and $g'(c)$ underneath Definition 4.1.
6. Use the definition of the derivative and one-sided limits to show that $f(x)=|\xi|$ is not differentiable at $x = 0$. Next, show this same result using Theorem 4.2.
7. Write the general form of the equation of the tangent line to the graph of an arbitrary differentiable function $y = \phi(\xi)$ at an arbitrary point $(c, f(c))$ where *c* is in the domain of *f*.
8. *Prove Theorem 4.3. This is ultimately an "S" exercise anyway (see Exercise #8).

9. *Draw a picture (of rectangles) that illustrates the ideas and calculations in the paragraph before the product rule. Can you "see" the product rule visually in this picture?

10. Prove the product rule without looking at the book's proof. Here's the key trick: note that $f(v)g(v) - \phi(\xi)\gamma(\xi) = \phi(\varpi)\gamma(\varpi) - \phi(\varpi)\gamma(\xi) + \phi(\varpi)\gamma(\xi) - \phi(\xi)\gamma(\xi)$.

11. Prove the quotient rule from the definition of the derivative. Here's one key trick: note that $f(v)g(x) - \phi(\xi)\gamma(\varpi) = \phi(\varpi)\gamma(\xi) - \phi(\xi)\gamma(\xi) + \phi(\xi)\gamma(\xi) - \phi(\xi)\gamma(\varpi)$.

12. Note: the proof of the Chain Rule is the first instance where we "create" a new function to aid us in completing a proof. This will be done a number of times in the future.

13. Fill in any details in the proof of the Chain Rule that confuse you. For example, why is *F* continuous at $g(c)$? Why is $F \circ g$ continuous at *c*? Why are the equalities at the top of page 135 true?

14. Write the "Leibniz notation" form of the Chain Rule (look it up if you need to). Describe how to think of it in terms of rates of change.

15. Make sure you understand the notation at the bottom of page 135 and in Theorem 4.8. Think about examples if that helps. Work through the proof of Theorem 4.8 and try to understand the tricky notation there. Carefully think about the example after this proof.

16. Does the statement of Theorem 4.8 contain a redundancy? If *f* is strictly monotone on *I* and differentiable at *c*, does it automatically follow that $f'(c) \neq 0$?

17. Do you remember all the derivative rules in Theorem 4.9? Do you think you could write them all down without looking? What about the derivatives of arccos(*x*), arccsc(*x*), and arccot(*x*)?

18. Prove Theorem 4.10.

19. Draw a picture that accurately illustrates the example in the middle of page 140.

20. Does it seem like a philosophical problem that a special case of a theorem (like Rolle's Theorem) can be used to prove the theorem (like the Mean Value Theorem)? Did you know the Pythagorean Theorem can actually be used to prove its converse? That is, the truth of the statement "Given a right triangle of side lengths *a* and *b* and hypoteneuse of length *c*, it follows that $c^2 = \alpha^2 + \beta^2$" can be used to prove the statement "Given a triangle with sides of lengths *a*, *b*, and *c* satisfying the relation $c^2 = \alpha^2 + \beta^2$, it follows that the triangle is a right triangle with the side of length *c* opposite the right angle". The proof of this would make a good in-class presentation.

21. In the hypotheses of Rolle's Theorem and the Mean Value Theorem (MVT), couldn't we just assume that $f:[a,b] \to$ ° is differentiable on [*a*, *b*] and leave it at that? What's the point of saying that *f* is continuous on [*a*, *b*] and differentiable on (*a*, *b*)? After all, we defined differentiability at endpoints by saying to just consider the appropriate one-sided limits. Wouldn't Theorem 4.3 play a role in justifying the simpler hypothesis?

22. Where is the Completeness Axiom used in the proof of the Mean Value Theorem? You may have to backtrack in the text. You may find it useful to look at the *Mathematica* notebooks from the first week of class. Describe what you find.

23. Mimic the example on page 142 to prove that $|\sin(a) - \alpha\nu(\beta)| \leq |\alpha - \beta|$ for all *a* and *b* in ° . In fact, $|\sin(x)| < |\xi|$ φορ αλλ $\xi \neq 0$. Why does this mean that *x* = 0 is the only fixed point of the function $f(x) = \alpha\nu(\xi)$ under iteration of this function via the recursive equation $x_{n+1} = \phi(\xi_\nu)$?

24. *Is the converse of Theorem 4.13, part (a) true? How about part (b)? Explain.

25. Prove Corollary 4.14. Hint: create a new function to apply Theorem 4.13 to.

26. See if you can find a statement of “The Racetrack Principle” on the Internet or in a Calculus text. Once you’ve found it, state it, and prove it with the MVT.

27. Use Theorem 4.13 part c) to prove that if *f* is a function satisfying the property that $f'(x) = f(x)$ for all $x \in$ ° , it follows that $f(x) = X\varepsilon^{\xi}$ for some constant *C*. Hint: create a new function to apply the theorem to.

28. Use Theorem 4.13 part c) to prove that if *f* and *g* are functions satisfying the conditions $f'(x) = g(x)$ for all $x \in$ ° and $g'(x) = -f(x)$ for all $x \in$ ° , it then follows that the function $(f(x))^2 + (\gamma(\xi))^2$ is constant. If $f(0) = 0$ and $g(0) = 1$, what is the value of the constant? What are *f* and *g* in this case?

29. Prove the First Derivative test, part (a). This is also a parenthesized problem… you could do it either here or in another section of your journal.

30. Prove the Second Derivative test, part (a). This is also a parenthesized problem… you could do it either here or in another section of your journal.

31. Which test is more “powerful”, the First or Second Derivative test? That is, which test applies to more situations…even when the functions considered are all twice differentiable? Come up with an example where one of the tests “works” and one doesn’t “work”…meaning one gives you the correct conclusion about the nature of a critical point but the other one doesn’t tell you anything.

32. Carry out more details in the proof of the Cauchy Mean Value Theorem than the book does.

33. How is the statement of Theorem 4.18 different than the assumptions used in the equations in the middle of page 144?

34. Come up with your own examples to apply L’Hopital’s Rule to. Work through the use of L’Hoptial’s Rule in your examples.

35. Verify the formula for $f'(x)$ in the second paragraph on page 147. Be careful when you verify $f'(0)$ …you should use the definition of the derivative for that one!

36. Verify any confusing details in the proof of Theorem 4.19.

37. Write *Mathematica* code that illustrates the proof of Theorem 4.19. You can write your code by hand in your journal.

38. Some functions that are increasing and concave down have horizontal asymptotes and some do not. Write down two functions, one of which is increasing and concave down with a horizontal asymptote, the other of which is increasing and concave down with no horizontal asymptote. Is there a way to distinguish between these two functions in terms of the behavior of their derivatives? You may want to think about integrals to answer this question.

39. **Note:** by Definition 4.20, constant functions are both concave up and concave down.

40. Before looking at the proofs of Theorems 4.23 and 4.24, find formulas for $T_c(x)$ and $S_{ab}(x)$ in Definition 4.22.
41. Fill in any details in the proof of Theorem 4.23 that are confusing.
42. Fill in any details in the proof of Theorem 4.24 that are confusing.
43. Why isn't *f* assumed to be twice differentiable on *I* in the statements of Theroems 4.23 and 4.24?
44. Prove that the absolute value function lies below its secant lines on any interval.
45. Look up the definition of a convex *set* and compare and contrast it with the definition of a convex function in Definition 4.25.
46. Prove that Definition 4.25 and 4.22(b) are equivalent.
47. Draw pictures that illustrate how discontinuous functions on open intervals fail to be convex (even when they are "mostly" concave up).
48. Finish off the final details in the proof of Lemma 4.27.
49. Draw an example of a discontinuous convex function on a closed interval.
50. Fill in any details in the proof of Theorem 4.30 that are confusing.
51. Fill in any details in the proof of Theorem 4.31 that are confusing.

Deep Thoughts to Ponder (but not necessarily answer):

- Does the Mean Value Theorem have any physical interpretations? (Hint: the answer is "yes"… ☺). Think of some.
- Does it seem like a contradiction that $f(x) = \xi^3$ has $f'(0) = 0$ yet *f* is strictly increasing on every interval? Can you resolve this seeming paradox visually in your mind?
- Do you think it is possible for a function f to satisfy the conditions that $f(0) = \phi'(0) = \phi''(0) = \phi'''(0) = \ldots = 0$ yet f is not constant on any interval containing 0? If so, can you come up with such a function?

Real Analysis Chapter 5 Study Guide (for "Real Analysis, A First Course", 2nd Edition, Russell A. Gordon)

Number of Starred Exercises: **3**; Number of Notes: 6; Number of Other (non-starred) Exercises: 40; Minimum Number of Other (non-starred) Exercises to Do (to do at least 25% of them): **10**

The most important things to get out of this chapter: (1) The definition of what it means for a function to be Riemann integrable. (2) An equivalent condition for Riemann integrability (Theorem 5.10). (3) Conditions for Riemann integrability (Theorems 5.11 and 5.12) and non-Riemann integrability (the contrapositive of Theorem 5.7). (4) The Fundamental Theorem of Calculus (FTC) and its importance. (5) The Mean Value Theorems for Integrals.

Other matters of importance:

1. Basic Properties of Integrals.
2. Evaluating Integrals.
3. Enjoying Integrals (no joke).

Reading Guide:

1. *Before reading Chapter 5, try to write down the definition of a definite integral $\int_a^b f(x)dx$ as you recall it from Calculus. Try not to peek at any sources.
2. Think about how you might try to teach a young child about the concept of area, especially for an irregularly shaped object. It's not an easy thing to do, is it? Volume might actually be easier. You could just focus on the volume of water displaced in a rectangular tub when you place the object in the tub. Why is describing the meaning of area so difficult? Think about it and write down some thoughts.
3. Do you think it's possible for an area under the graph of a function to "not exist"? Think about some of the strange and wild functions we've looked at.
4. Write an equation expressing a tagged partition of the interval [3, 7] into 8 equal sized sub-intervals with the right-hand endpoints being the tags. Write the form of a general Riemann sum of an arbitrary function *f* for such a tagged partition. If $f(x) = \xi^3$, find the value of this Riemann sum.
5. *Spend some time working on memorizing Definition 5.4. Once you think you have it memorized, try to write it down without looking and then compare what you wrote with what's in the book. Try to write it down again a day later without looking and compare. Do it again a day later. Maybe even on a fourth day. (You can write these on different pages in your journal).
6. Compare Definition 5.4 with your answer to #1 above and what you find in calculus textbooks. Notice any differences?
7. Look up Bernhard Riemann on the Internet and write down a few things you find out about him.

8. To illustrate how difficult Definition 5.4 is to use. Try to use it to prove that the function $h(x) = \xi$ is Riemann integrable on [0,1]. This will be an extra credit problem on homework (see Exercise #10 in Section 5.1).
9. How are the oscillation and the variation of a monotone function on a closed interval [a, b] related?
10. Write down the contrapositive of the statement in Theorem 5.7. This contrapositive is a true statement as well (since the contrapositive of a statement is equivalent to the statement).
11. Make an outline of the overall strategy used in the proof of Theorem 5.7.
12. Draw detailed pictures that help you understand the proof of Theorem 5.7.
13. Fill in any details that help you understand the proof of Theorem 5.7.
14. Outline and describe the overall strategy used in the proof of Theorem 5.8 (the part of the proof that's in the book). Then fill in details to help you understand.
15. Prove the converse of the statement that is proved for Theorem 5.8.
16. Explain why it is sufficient to consider the case in which $P_1 = \{\alpha, \beta\}$ in the proof of Lemma 5.9.
17. Explain why $\left|\sum_{\iota=1}^{\pi} f(t_i)(x_i - \xi_{\iota-1}) - \phi(\varpi)(\beta - \alpha)\right| = \left|\sum_{\iota=1}^{\pi} (\phi(\tau_\iota) - \phi(\varpi))(\xi_\iota - \xi_{\iota-1})\right|$ in the proof of Lemma 5.9.
18. Do Exercise #6 from Section 5.2 (given an area interpretation of Theorem 5.10 and explain how it shows the two geometric approaches to the area under a curve are equivalent).
19. Outline the proof of Theorem 5.10 and fill in confusing details from this proof (this might be one of the hardest proofs in the whole book). Think about a strategy for how you might prove the converse of the statement actually proved here.
20. **Note:** carefully read and make a note of the theme of the paragraph after the proof of Theorem 5.10. You now have two ways to prove that a given function is Riemann integrable. The second way (using the condition in Theorem 5.10) may often be easier. You might try testing this to see if the proof that $h(x) = \xi$ is Riemann integrable over [0, 1] is any easier now (the extra credit problem #10 from Section 5.1)
21. Try proving Theorem 5.11 without looking at the book's proof (Hints: use Theorem 5.10 and also make use of the Uniform Continuity Theorem…Theorem 3.28).
22. Prove Theorem 5.12. Again, Theorem 5.10 may be helpful. Note that a monotone function is not necessarily continuous (and a continuous function is not necessarily monotone).
23. Prove any part of Theorem 5.13 that interests you.
24. *Try to recall and precisely write down the Fundamental Theorem of Calculus (FTC) without looking at any references. You may also recall that there are actually two parts to this theorem in most calculus books. Try to write down both statements.
25. Prove Theorem 5.14 (by this point, this should be one of the easier proofs in Chapter 5).

26. **Note:** take care to note the fact pointed out in the last sentence of page 177 which goes to the top of page 178. Can you explain this comment?
27. Prove part (a) of Theorem 5.15.
28. **Note:** in the proof of part (b) of Theorem 5.15, it's nice to see that, once we finish the hard work of showing that *f* is Riemann integrable over [a, b], it's relatively easy to apply Theorem 5.14 to show that the equation in part (b) of Theorem 5.15 is true.
29. **Note:** in many (if not most) calculus textbooks, part (b) of Theorem 5.17 is labeled FTC, Part I and part (a) of Theorem 5.17 is labeled FTC, Part II.
30. Use part (a) of the FTC to prove part (b) of the FTC.
31. The proof of the FTC is rather ingenious (or, for that matter, the thinking up of the statement of the FTC is rather ingenious), but its proof isn't really quite as difficult as some of the other proofs earlier in Chapter 5 (even though it looks kind of daunting at first glance). Is this surprising? Should it be surprising? After studying the proof, do you feel like you could reproduce most of it without looking? Do you think you could have come up with it on your own (assuming you had the correct statement of the FTC)?
32. How do the statements in the last couple of sentences on page 180 that go to the top of page 181 "jive" with the statement of the FTC, part (a)? Don't these statements contradict each other? Can you resolve this paradox?
33. Meditate (seriously…☺) on the paragraph in the middle of page 181. *Ponder it's cosmic significance*. Think about it's "physical" meaning. *Enjoy it*. This is *deep* stuff. It's one of humanities greatest discoveries.
34. Start thinking about how to prove Theorems 5.18 and 5.19 (these will eventually be exercises to hand in anyway).
35. In the proof of Theorem 5.20, explain whey the proof is complete when the author writes "This completes the proof".
36. Draw pictures to help you understand the proof of Theorem 5.21(b).
37. Why can the condition in the first sentence of the paragraph after the proof of Theorem 5.21 be described as "the final word on the class of Riemann integrable functions"? Does its statement "jive" with, for example, Theorem 5.12? Explain.
38. Write down a formula for $f \circ g$ before the statement of Theorem 5.22. Do you see why it is not Riemann integrable?
39. In the proof of Theorem 5.22, why does *f* have a bound on [*c*, *d*]? Why is *f* uniformly continuous on [*c*, *d*]?
40. Verify the inequalities on the very bottom line of page 185.
41. Explain the second inequality (at the beginning of the third line of equations/inequalities) near the top of page 186.
42. How is the statement of Theorem 5.23 related to the concept of the average value of a Riemann integrable function? Rewrite Theorem 5.23 in an equivalent way using this relation. Draw a picture if it is helpful, especially in the case where $f(x)$ is always nonnegative.
43. Make up your own examples for *f* and *g* and find a value of *c* that works to satisfy the conclusion of the Generalized Mean Value Theorem for Integrals.

44. **Note:** take note of and enjoy the fact that both the Extreme Value Theorem and the Intermediate Value Theorem are used in the proof of Theorem 5.24 (the Completeness Axiom in action again!).
45. **Note:** the "inverse image" or "preimage" of a point in the range is the set of all points that get mapped to it. For example, if *y* is in the range of $f:[a,b]\to {}^\circ$, then the inverse image of *y* is $\{x \in [\alpha,\beta]: \phi(\xi) = \psi\}$. This set is often denoted by $f^{-1}(\psi)$ or $f^{-1}(\{\psi\})$, even when the function *f* is not invertible (not one-to-one).
46. The proof of Theorem 5.26 is trivial. Do you believe this statement? Write out a brief proof for each part of the theorem.
47. Can you imagine how a proof of Theorem 5.27 might go? Perhaps by using induction on the number of subintervals in the partition? You don't have to do it. Just think about it.
48. Draw a picture to illustrate Theorem 5.28. Note that the φ and the ψ have to be "close together" in order for $\int_\alpha^\beta (\psi - \varphi) < \varepsilon$.
49. Draw pictures to help you understand the proof of Theorem 5.28. Hopefully this proof, though somewhat lengthy, is actually fairly easy for you to understand after having gone through the "gauntlet" that is Chapter 5.

Deep Thoughts to Ponder (but not necessarily answer):

- Integrals are a deep subject. Take some time to enjoy how deep and rich the theory is…especially the FTC. God has structured the universe in ways we can understand and "measure" (more advanced integration theory is related to something called measure theory). This is really cool!
- Is the fact that some (even bounded) functions are not Riemann integrable somehow problematic? "Should" they be? Do you feel informed enough to even have an opinion on such questions?

Real Analysis Chapter 6 Study Guide (for “Real Analysis, A First Course”, 2nd Edition, Russell A. Gordon)

Number of Starred Exercises: **3**; Number of Notes: 5; Number of Other (non-starred) Exercises: 18; Minimum Number of Other (non-starred) Exercises to Do (to do at least 25% of them): **5**

The most important things to get out of this chapter: (1) The definition of convergence of an infinite series. (2) The realization that the terms of a series converging to zero is ***not*** sufficient to guarantee that the series converges. (3) Knowledge of convergence tests and ability to apply them. (4) Absolute versus nonabsolute (or “conditional”) convergence.

Other matters of importance:

1. Thoroughly understanding geometric series
2. Rearrangements and products

Reading Guide:

1. *Before starting to read Chapter 6, write down what you think it should mean for an infinite series $\sum_{k=1}^{\infty} a_k = \alpha_1 + \alpha_2 + \alpha_3 + \Lambda$ of numbers to converge (that is, what does it mean for it to equal a particular real number?)
2. *Compare and contrast your answer to #1 with Definition 6.1.
3. **Note:** infinite series are not literal sums. By definition, a true sum must terminate at some point. Newton and Leibniz did imagine that they were sums, however, and the notation has stuck. It usually doesn’t cause problems to pretend they are sums.
4. **Note:** make sure you are careful to distinguish the sequence of partial sums of a series with the sequence of terms of the series (see the second paragraph on pag 211). Also take note of the first sentence of the second paragraph on page 211. Chapter 2 will be very useful for Chapter 6!
5. How is the main point of the second paragraph on page 212 (the one that starts with the word “Since”) similar to something that happens with integrals? (Hint: think in terms of evaluating integrals using antiderivatives.)
6. Write down the contrapositive of the statement in Theorem 6.2. This is a true statement too (see the bottom of page 212).
7. What other “famous” series (which has a name) can be used to verify that the converse of Theorem 6.2 is false?
8. **Note:** an actual closed-form expression for $s_n = \sum_{\kappa=1}^{\nu} \frac{1}{\sqrt[3]{\kappa}}$ is not found at the bottom of page 212. This does not matter. We can still prove that $\{s_n\}$ diverges.
9. *Prove Theorem 6.4.

10. Prove Theorem 6.5.
11. Prove Theorem 6.6(a).
12. About an hour after reading the proof of the Limit Comparison Test. Come back to it and try to prove it yourself without looking at the book's proof.
13. **Note:** be careful to heed the lessons/warnings in the book after the Comparison and Limit Comparison Tests (about how to use them).
14. Create your own example like that in the book near the bottom of page 218 where you need to do a bit of manipulation with a summation to get the index starting at 0 so that you can apply the formula in Theorem 6.8.
15. **Note:** the formulas at the bottom of page 219 could generate a mini-research project to find out why they are true.
16. Do you understand why the proof of Theorem 6.10 is a simple consequence of Theorem 6.3 and the inequality in the proof? Fill in some details if you are not sure.
17. Recall how the ideas of limsups and liminfs were justified in class by referring to the Ratio Test on page 223. Compare this Theorem with the Ratio Test in your Calculus textbook (hopefully there's a ratio test there). Try to think of an example where the limsup or the liminf is really needed rather than just an ordinary limit. Also note how these ideas compare with the comments in the book at the bottom of page 224.
18. Try to do the same thing as in #17 for the Root Test on page 224.
19. Prove the Root Test.
20. Draw a picture of a number line and the sequence of partial sums for an alternating series to help you understand the statement and proof of the Alternating Series Test (Theorem 6.14).
21. Use *Mathematica* and the bound in the Alternating Series test to estimate the sum of the series at the bottom of page 225 to within 10^{-6}.
22. Consider the sequence of equations
    $0 = (1-1) + (1-1) + (1-1) + \cdots = 1 + (-1+1) + (-1+1) + \cdots = 1 + 0 + 0 + \cdots = 1$.
    Where is the mistake? How does this illustrate what can go "wrong" with infinite series.
23. Compare the definition of a permutation of the positive integers near the bottom of page 228 with the definition of a permutation in your Algebraic Structures book. Are they the same kind of idea? Can you use array and cycle notation to represent the permutation at the bottom of page 228?
24. Verify the details in the proof of Theorem 6.16 near the bottom of page 229.
25. Draw a picture to help you understand the proof of Theorem 6.17.
26. Write and draw anything you need to in order to help you understand Theorem 6.18 and its proof. (though you won't need to know this theorem for the final exam).

Deep Thoughts to Ponder (but not necessarily answer):

- How would you define what it means for an infinite product to converge?

- I have claimed that infinite sums are not true sums. Do you think it's possible to have a philosophical perspective where they are true sums?

Real Analysis Chapter 7 Study Guide (for "Real Analysis, A First Course", 2nd Edition, Russell A. Gordon)

Number of Starred Exercises: **2**; Number of Notes: 6; Number of Other (non-starred) Exercises: 41; Minimum Number of Other (non-starred) Exercises to Do (to do at least 25% of them): **11**

The most important things to get out of this chapter: (1) The differences between pointwise and uniform convergence of a sequence of functions. (2) Counterexamples that show things can "go wrong" for pointwise convergence. (3) Theorems that guarantee that things "go right" for uniform convergence. (4) Application of these ideas to power series and Taylor series.

Other matters of importance:

1. Weierstrass M-test
2. Abel's Theorem
3. Weierstrass Approximation Theorem
4. Existence of an everywhere continuous nowhere differentiable function

Reading Guide:

1. **Note:** Do you remember seeing calculations like those on page 241 and the top of page 242 in Calculus? These are some "fun" but "naïve" power series calculations. Newton, the Bernoulli's, Lagrange, Laplace, and Euler were great believers in these kinds of calculations. They used them to do powerful things.
2. Take note of the conceptual distinction between f and $f(x)$. Is this confusing to you? Take time to try to describe the distinction in your own words if it is confusing. Ask me questions about it to clarify if you need to…maybe your confusion will be cleared away by reading further in this section (even right away…see the bottom of page 242 and the top of page 243).
3. After reading the definition of pointwise convergence, try to restate it on your own without looking.
4. *Spend lots of time sketching graphs and doing calculations to verify many (though not necessarily all) of the things described in the examples 1 – 7 on pages 243 to 244.
5. Try to come up with a sequence of bounded functions which converge pointwise to an unbounded function.
6. Try to define what it should mean for a series (infinite sum) of functions to converge pointwise to a function on an interval without looking at Definition 7.2. After you are done, compare your definition with Definition 7.2.
7. In the example from the bottom of page 245 to the top of page 246, why doesn't the series converge for other values of x? After all, the function $f(x) = (x - 1)/(30 - 5x)$ is defined for other values of x, isn't it? Is this a paradox? Can you resolve it?

8. Draw some pictures to help you understand what the book is trying to get at in explaining why the inequality near the bottom of page 247 is not sufficient to prove that the continuity of *f* follows from the continuity of the f_n's.
9. *After reading and thinking about the definition of uniform convergence, wait one minute, and then rewrite it on your own. Rewrite it again later without looking and see if you get a match.
10. The picture in Figure 7.1 and the comments at the bottom of page 248 are key to understanding the definition of uniform convergence. Draw pictures of particular examples (similar to those on pages 243 and 244) that help you understand this. Draw at least one example where the convergence is uniform and at least one example where the convergence is not uniform.
11. Draw a picture similar to Figure 7.1 (with the f_n's added) to illustrate the proof in the first paragraph on page 249.
12. Negate Definition 7.1 before looking at the second paragraph on page 249. Then compare your negation with the one in that paragraph.
13. Make sure you understand the proof in the third paragraph on page 249. Why is what the book has done sufficient to prove non-uniform convergence?
14. Prove Theorem 7.4. Draw a picture similar to Figure 7.1 to illustrate ideas in your proof.
15. **Note:** Take the example in the paragraph after Theorem 7.4 to heart. Especially the part about it being easier than using the negation of Definition 7.3!
16. Draw pictures to illustrate the ideas in the example from the bottom of page 249 to the top of page 250.
17. Prove the converse of the statement already proved in the proof of Theorem 7.5.
18. Prove the Weierstrass *M*-test before reading it's proof (Hint: use the Cauchy Criterion you just read about).
19. **Note:** the Weierstrass *M*-test is very useful!
20. Try to come up with an example where you could use the Weierstrass *M*-test. Outline how you would use it.
21. Fill in the minor details needed to verify that the equality $\lim_{x\to\chi}\phi(\xi)=\phi(\chi)$ is equivalent to the equality $\lim_{x\to\chi}\lambda\mu_{\nu\to\infty}\phi_\nu(\xi)=\lambda\mu_{\nu\to\infty}\lambda\mu_{\xi\to\chi}\phi_\nu(\xi)$ at the bottom of page 252.
22. Try to prove Theorem 7.8 before you look at its proof.
23. After you study the proof of Theorem 7.8, wait 15 minutes, and then prove it without looking.
24. **Note:** carefully think about the comments in the two paragraphs after Corollary 7.9 on page 253.
25. Prove Dini's Theorem (7.10).
26. Explain why the two equations near the top of page 254 are equivalent.
27. What theorem is a reference you could give as a reason for the inequality $\left|\int_\alpha^\beta(f_n-\phi_\mu)\right|\le\int_\alpha^\beta|\phi_\nu-\phi_\mu|$ in the proof of Theorem 7.11.
28. Fill in a few missing details to explain how what the book has done in the first paragraph of the proof of Theorem 7.11 proves that the sequence of numbers $\{\int_a^b f_n\}$ is Cauchy.

29. Near the bottom of the proof of Theorem 7.11 on page 254, verify that $\left|S(f,{}^tP) - \Sigma(\phi_\theta, {}^\tau\Pi)\right| < \varepsilon(\beta-\alpha)$ for any tagged partition of [a, b] which satisfies $\left\|{}^tP\right\| < \delta$, where δ is chosen so that for all tagged partitions satisfying $\left\|{}^tP\right\| < \delta$, we have $\left|S(f_q,{}^tP) - \int_\alpha^\beta \phi_\theta\right| < \varepsilon$.
30. Explain why the two equations near the top of page 255 are equivalent.
31. Verify that Example (7) of Section 7.1 provides a counterexample to the "anticipated theorem" near the top of page 255.
32. Why can Theorem 7.12 not be applied to Example (7) of Section 7.1?
33. Think about Theorem 7.13 in the context of the examples on pages 241 and 242. Also, look in your calculus book (in the sections about power series and Taylor series) for where this theorem is used. Write down some of those examples. Compare what you've thought about and done here with the statement of Theorem 7.17 in the next section.
34. Draw a picture to illustrate a typical interval of convergence that comes from Theorem 7.15.
35. **Note:** take note of how the Root Test and Weierstrass *M*-test are used in the proof of Theorem 7.15. Do you think the Ratio Test could also be used instead of the Root Test?
36. Think about Exercise 15 in Section 6.3 to verify the statement in the paragraph at the bottom of page 259.
37. Prove Lemma 7.16. (Hint: The Root and/or Ratio Tests might be helpful)
38. Use induction to give a more rigorous proof of the equation in the proof of Theorem 7.18.
39. **Note:** The main point of the paragraph after Corollary 7.19 is pretty amazing. The derivative of a given function at a given point provides *local* information about the function. However, if the function can be represented by a power series centered at that point, then this local information produces *global* information (on the interval of convergence of the power series). A function that has a power series representation over a certain interval is often called an **analytic** function on that interval. Most of the functions encountered in Calculus are analytic. Analytic functions are even more significant in the context of complex-valued functions of a complex variable, where the concept of analyticity is equivalent to the concept of differentiability (existence of one derivative ends up implying the existence of infinitely many derivatives in that context!). This is not true for real-valued functions of a real variable. The function *f* defined by $f(x) = \varepsilon^{-1/\xi^2}$ for $x \neq 0$ and $f(0) = 0$ provides a counterexample. This function is infinitely differentiable at 0, yet only equals its power series representation centered at 0 when $x = 0$ (its power series representation there is trivial: $0 + 0x + 0x^2 + 0x^3 + \ldots$)…see Exercise 22 on page 265.
40. Write the equations in the middle of page 263 out in "long form" (without the summation sign).
41. Write the equations at the bottom of page 263 out in "long form".
42. Verify the details in the proof of Theorem 7.20 (Taylor's Formula with Integral Remainder).

43. Referring to the proof of Theorem 7.21, prove that the sequence $\{(x-c)^{\nu}/\nu!\}$ converges to 0 for any fixed *x* and any fixed *c*.
44. Verify that the Maclaurin series for e^x converges for all real numbers *x*.
45. Prove Lemma 7.22.
46. Verify the details in the proof of Theorem 7.23.
47. In the proof of Theorem 7.24, elaborate on how to choose *A* so that $F(c) = \phi(\xi)$.
48. Do the computations to see that $A = \dfrac{\phi^{(\nu+1)}(\zeta)}{(\nu+1)!}$.
49. How is the statement of Theorem 7.24 (Taylor's Formula with Derivative Remainder) a generalization of the statement of the Mean Value Theorem from Chapter 4?

Deep Thoughts to Ponder (but not necessarily answer):

- Take some time to ponder the intricacies of Calculus and Real Analysis: the definitions of various kinds of limits, continuity, derivatives, integrals….various amazing theorems: the IVT, EVT, MVT, FTC…and all the facts you've just been learning about sequences and series of functions. Think about how these things are related. Hopefully you can now see it as an amazing whole. Imagine yourself teaching Calculus in the future. Think about how you might bring in some of the more rigorous ideas you learned in Real Analysis.
- It's also amazing that, though infinite series are not truly sums, we can usually pretend they are and get away with it (same comment with integrals). What about an infinite product? How would you define such a thing?

Real Analysis Chapter 8 Study Guide…specifically, Sections 8.1 through 8.3 (for "Real Analysis, A First Course", 2nd Edition, Russell A. Gordon)

Number of Starred Exercises: **1**; Number of Notes: 5; Number of Other (non-starred) Exercises: 39; Minimum Number of Other (non-starred) Exercises to Do (to do at least 25% of them): **10**

The most important things to get out of this section: (1) The basic definitions and facts about open and closed sets. (2) The definition of a compact set and its use. (3) The characterizations of continuous functions in terms of open and closed sets. (4) How continuous functions behave with respect to compact sets. (5) Practice with basic examples and proofs related to this.

Other matters of importance: Understanding the significance of the Heine-Borel Theorem

Reading Guide:

1. Ponder the following sentence from the first paragraph of page 291: "In order to discuss more unusual functions, it is helpful to be aware of subsets of $\mathbb{R}$ that are not intervals." Have we thought about unusual functions in this course? Do you remember me saying that "Real Analysis, though dealing with a more familiar subject than Algebraic Structures, is sometimes trickier because statements that seem reasonable are not always true"? Knowing certain counterexamples is extremely important in this course, especially if you are either going to teach math in the future or go to graduate school.
2. Think deeply about the second paragraph on page 291 that goes on to the next page. Do you remember being taught similar ideas in Linear Algebra? (Think in terms of vector spaces and different examples of vector spaces). Page through your linear algebra book and write down any "spaces" you see in there that are not Euclidean spaces (not spaces whose points can be represented with coordinates in the usual way we think about them). You should be able to find some (especially in chapters 4 and 6 of the linear algebra book).
3. Do your best to draw a picture of the set A defined at the bottom of page 292. You won't be able to draw it perfectly, but technology will probably be of help.
4. Write a paragraph of explanation to convince yourself as to why the set B defined at the bottom of page 292 contains no intervals.
5. In one of the parts (a), (b), or (c) of Definition 8.1 on page 293, the point x is not necessarily an element of the set E, in the other two parts, it is. In which one part ((a), (b), or (c)) is x not necessarily an element of E? Why? Can you think of an example for this part where x is not in E? Can you think of an example for this part where x is in E?
6. *Think deeply about Definition 8.1 and the examples and observations that follow on page 293. Write some sentences that help you understand the definitions and try to explain why they make sense (intuitively). Make up your own examples

where you identify interior, isolated, and limit points. Make up your own examples of open sets, closed sets, and sets that are neither open nor closed. Make sure you consider sets besides just intervals…perhaps the set of rationals and the set of irrationals would be good to consider. If you are looking for more “exotic” examples, perhaps you’ll want to look up information about the Cantor set with respect to these definitions. Write down what you find out.

7. Draw pictures that help you think about how you might go about proving the 10 observations on page 293. Then, if you feel confident, attempt actually writing out three or four of these proofs. Compare your proofs with a friend’s proofs (maybe agree on which ones you want to try ahead of time).
8. Thoroughly study the proof of Theorem 8.2 on page 294 until you completely understand it. Then, take a 10 minute break and prove it yourself without looking at the book’s proof.
9. Thoroughly study the proof of Theorem 8.3 on page 294 until you completely understand it. Then, take a 10 minute break and prove it yourself without looking at the book’s proof.
10. Thoroughly study the proof of Theorem 8.4 on pages 294-295 until you completely understand it. Then, take a 10 minute break and prove it yourself without looking at the book’s proof.
11. Thoroughly study the proof of Theorem 8.5 on page 295 until you completely understand it. Then, take a 10 minute break and prove it yourself without looking at the book’s proof.
12. Fill in details of the proof of Theorem 8.6 on page 296 that are confusing to you. For example, you may want to do the verification that $x \in I \subseteq O$ and that $\alpha_x, \beta_x \notin O$ for each $x \in O$.
13. Draw pictures and/or write down details that help you understand the example discussed on the bottom of page 296 to the top of page 297.
14. Make up your own examples for *E* to calculate the interior, derived set, and closure of *E*. Try to make the examples interesting and enlightening.
15. Thoroughly study the proof of Theorem 8.8 on page 297 until you completely understand it. Then, take a 10 minute break and prove it yourself without looking at the book’s proof.
16. **Note:** As you start studying Section 8.2, realize that if *K* is a subset of $\mathbb{R}$ and if $f : K \to \mathbb{R}$, then $f(K) = \{ f(x) : x \in K \}$ is the range of *f* (also called the *image* of *K* under *f*).
17. Pick a few of the open covers of the interval (0,1) at the bottom of page 301 and verify that the union of the elements in the open cover contains the interval (0,1). Besides the 7^{th} and 8^{th} examples, are there any others of these that contain finite subcovers of the interval (0,1)?
18. Verify the details of the paragraph in the middle of page 302 to show that the interval (0,1) is not a compact set. Draw pictures and/or write down words and equations that would help you verify that an arbitrary open interval (a,b) is not a compact set.
19. Write down thoughts that help you understand the details of showing that the set *E* defined in the bottom half of page 302 is compact.
20. Fill in details of the proof of Theorem 8.11 on page 303 that are confusing to you.

21. Prove that a compact set is bounded (the part left out in the proof of Theorem 8.11).
22. Thoroughly study the proof of Theorem 8.12 on page 303 until you completely understand it. Then, take a 10 minute break and prove it yourself without looking at the book's proof.
23. Prove Theorem 8.13
24. Look up DeMorgan's Laws and write them down. Next, prove DeMorgan's Laws
25. Thoroughly study the proof of Theorem 8.14 on pages 303-304 until you completely understand it. Then, take a 10 minute break and prove it yourself without looking at the book's proof.
26. Fill in details of the proof of the Heine-Borel Theorem (Theorem 8.15) on page 304 that are confusing to you. Take note of the use of the Completeness Axiom. There are spaces for which the Heine-Borel Theorem is not true (where compactness is not equivalent to being closed and bounded in that space).
27. Draw pictures or write down and argument to help you convince yourself that the set *E* defined at the bottom of page 304 is closed and bounded.
28. Thoroughly study the half proof of Theorem 8.16 on page 305 until you completely understand it. Then, take a 10 minute break and prove it yourself without looking at the book's proof.
29. Prove the converse of the statement proved for Theorem 8.16.
30. **Note:** by "universe" on page 307, the author means "the 'big' set that every set under discussion is a subset of". For the most part, in real analysis, the "universe" is the set of real numbers $\mathbb{R}$. Sometimes this is called the "universe of discourse" or "domain of discourse".
31. **Note:** take note of the fact that the word "in" is boldfaced in both parts of Definition 8.17…this is no accident, but an important part of these definitions.
32. Draw pictures of examples that illustrate the truth of the statements in Theorem 8.18. Then pick a couple of these properties and prove them.
33. Given two sets *A* and *B*, a subset $C \subseteq B$, and a function $f : A \to B$ (whether one-to-one or not), the symbol $f^{-1}(C)$ (called the "preimage" or "inverse image" of *C* under *f* …also see the bottom of page 310) represents the set of all elements of *A* that have outputs in *C*, i.e. $f^{-1}(C) = \{x \in A : f(x) \in C\}$ (we write this whether the inverse function of *f* exists or not). Write the sets described in Theorem 8.19 in terms of this notation (also use interval notation for *C*).
34. Fill in details of the proof of Theorem 8.19 on pages 308-309 that are confusing to you.
35. **Note:** Stop to ponder the fact that every function defined on the integers (or any "discrete" set) is continuous on that set, according to Definition 8.20. Our intuitive definition of continuity as meaning you can draw the graph without picking up your pencil has really been debunked!
36. Fill in the details of the proof that the function *g* defined at the bottom of page 309 is continuous on the set *E*.
37. Prove Theorem 8.21.
38. Write the sets in statements (2) and (3) of Theorem 8.22 in terms of inverse image notation (see #33 above).

39. Fill in the details of the proofs of the implications proved in the book for Theorem 8.22.
40. Prove the missing implication for Theorem 8.22.
41. **Note:** the paragraph at the very bottom of page 310 is very important to ponder and recall (and use) if you ever go to graduate school in mathematics, statistics, or physics (or even economics…I once knew an economics graduate student who needed to know very advanced real analysis for his Ph.D. in economics).
42. Thoroughly study the proof of Theorem 8.23 on page 311 until you completely understand it. Then, take a 10 minute break and prove it yourself without looking at the book's proof.
43. Explain how to relate the statement of Theorem 8.23 to the Extreme Value Theorem (can one be used to prove the other?). The Heine-Borel Theorem is important to consider.
44. Thoroughly study the proof of Theorem 8.24 on pages 311-312 until you completely understand it. Then, take a 10 minute break and prove it yourself without looking at the book's proof.
45. Close out your reading experience in this course by drawing pictures that help you understand the ideas on pages 312-314, including the Tietze Extension Theorem. Also, look up the Tietze Extension Theorem online and take time to appreciate the fact that it can be stated in a more general point-set topology manner. Also note that it generalizes another important result called the Urysohn Lemma.

Deep Thoughts to Ponder (but not necessarily answer):

- Perhaps this chapter re-emphasizes the fact that the real number system is not as simple as you may have once thought (see the Chapter 1 study guide). Spend time thinking about this. Give praise to God that he has created such structures and given our minds the capacity to think about them.
- If you have the stomach to consider yet another topological topic, you might want to explore the topic of the connectedness of a set of real numbers (look up "connected space"…also study Section 8.4) and how continuous functions behave relative to connected sets. You would also want to relate what you learn to the Intermediate Value Theorem.
- If you are thinking about graduate school, you should also study Section 8.5 at some point (and perhaps study Chapter 6 of your linear algebra textbook at the same time).