Edge Coloreing

An edge coloreing of a loopless graph G is a function $\pi: E(G) \to S$, where S is a set of distinct colores; and it is proper if no two adjacent edges receive the same colore.

Thus a proper edge coloreing π of G is a function $\pi: E(G) \to S$ such that $\pi(e) \neq \pi(e')$ whenever edges e and e' are adjacent in G.

The minimum k for which a loopless graph G has a proper edge coloring is called the edge chromatic number or chromatic index of G. It is denoted by $\chi'(G)$.

A graph G is k-edge chromatic if $\chi'(G) = k$.

Since $\Delta(G)$ edges incident at a vertex u of maximum degree, we have for any loopless graph $\chi'(G) \geq \Delta(G)$.

Thm If G is a loopless bipartite graph then $\chi'(G) = \Delta(G)$.

Proof ~~The Proof is~~ we proceed by induction on the number of edges m of G.

The result is true for $m=1$.

Assume the result for bipartite graphs of size at most $m-1$.

Let G have m edges. Let $e=\{u,v\} \in E(G)$. Then $G-e$ has (since $\Delta(G-e) \leq \Delta(G)$) a proper Δ-edge coloring, say C. Out of these Δ colors, suppose that one particular color is not represent at both u and v. Then the edge $\{u,v\}$ can be colored with this color and a proper Δ-edge-coloring of G is obtained.

In other case (i.e. in the case for which each of the Δ colors is represented either at u or at v), since the degrees of u and v in $G-e$ are at most $\Delta-1$, there exist a color out of the Δ-colors that is not represented at u, and similarly there exists a color not represented at v. Thus, if the color j is not represented at u in C, then j is represented at v in C, and if the color i is not represented at v in C, then i is represented at u in C.

Since G is bipartite and u and v are not in the same parts of the bipartition, there can not exist a $u-v$ path in $G-e$ in which the color alternate between i and j

Let P be a maximal path in $G-e$ starting from u in which the colors of the edges alternate between i and j. Interchange the color i and j in P. This would still yield an edge coloring of $G-e$ using Δ colors in which the color i is not represented at both u and v. Now color the edge $\{u,v\}$ by the color i. This results in a proper Δ-edge-coloring of G.

Th^m $\chi'(K_n) = \begin{cases} n-1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$

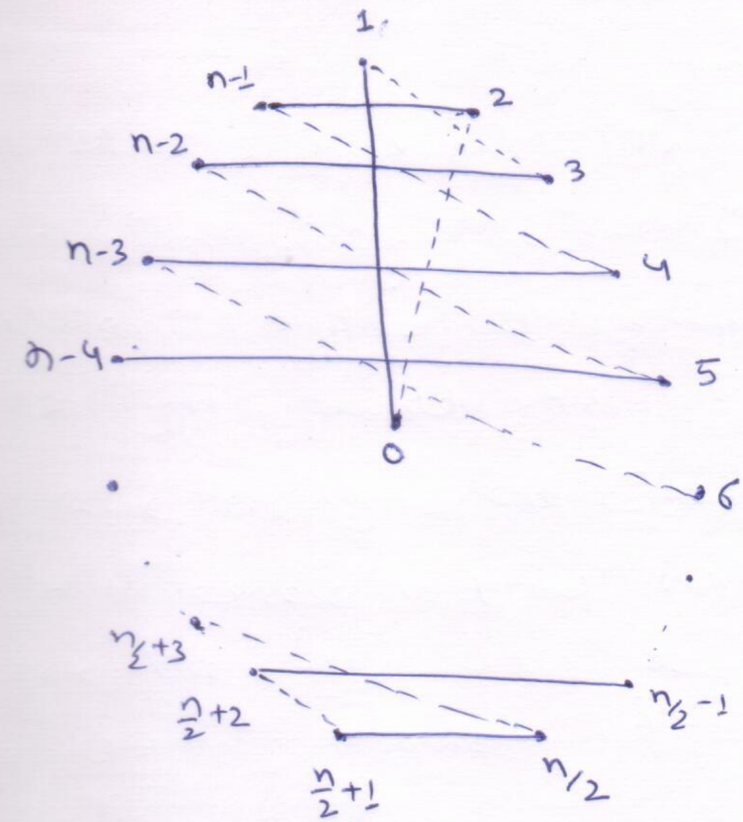
Proof Since K_n is regular of degree $n-1$, $\chi'(K_n) \geq n-1$.

Case I n is even

We now show that $\chi'(K_n) \leq n-1$ by exhibiting a proper $(n-1)$-edge coloring of K_n.

Label the n-vertices of K_n as $0, 1, \ldots, n-1$.

Draw a circle with center at 0 and place the remaining $n-1$ numbers on the circumference of the circle so that they form a regular $(n-1)$-gon.



Then the $\frac{n}{2}$ edges $\{0,1\}, \{2, n-1\}, \cdots \{\frac{n}{2}, \frac{n}{2}+1\}$ form a ~~matching of~~ perfect matching of K_n. Rotation of these edges through the angle $\frac{2\pi}{n-1}$ in succession gives $(n-1)$ edge-disjoint 1-factors (perfect matching) of K_n. This would account for $\frac{n}{2}(n-1)$ edges, or all edges of K_n. Each 1-factor can be assigned a color. Thus $\chi'(K_n) \leq n-1$.

<u>Case II</u> n is odd.

Take a new vertex and make it adjacent to all the n vertices of K_n. This gives K_{n+1}. By case I, $\chi'(K_{n+1}) = n$. Hence $\chi'(K_n) \leq n$. However, K_n

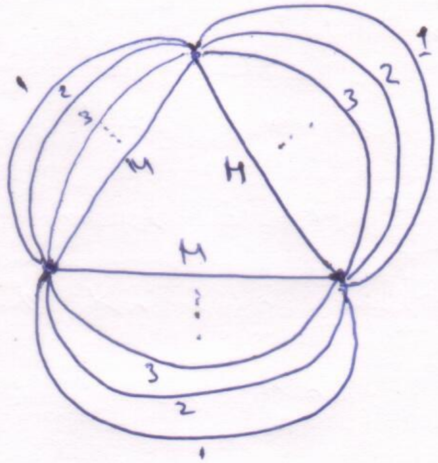
can not be edge colored properly with $n-1$ colors. This is because the size of any matching of K_n can contain no more than $\frac{n-1}{2}$ edges and hence $n-1$ matchings of K_n can contain no more than $\frac{(n-1)^2}{2}$ edges. But K_n has $\frac{n(n-1)}{2}$ edges. Thus $\chi'(K_n) \geq n$ and hence $\chi'(K_n) = n$.

Th^m (Vizing-Gupta)
For any simple graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$.

Actually, Vizing proved a more general result than the above. Let G be any loopless graph and let M denote the maximum number of edges joining two vertices in G. Then the generalized Vizing theorem states that $\Delta(G) \leq \chi'(G) \leq \Delta(G)+M$.

This theorem is best possible in that there are graphs with $\chi'(G) = \Delta(G)+M$

Ex



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Since any two edges of G are adjacent, $\chi'(G) = 3M = \Delta + M$.

Def^n Graphs for which $\chi' = \Delta$ are called class I graphs, and those for whic $\chi' = 1 + \Delta$ are called class II graphs.