Kuratowski's Theorem

Suppose G is a plane graph. Then any graph obtained from G by removing some of its vertices and/or edges must also be a planar graph. If H is not planar then any graph containing a subgraph isomorphic to H can not be planar

Let G be a graph. A subdivision of G is a graph obtained from G by inserting vertices into some of the edges. Clearly any subdivision of a planar graph is again planar. Moreover any subdivision of a nonplanar graph is again non planar.

$\underline{\underline{Th^m}}$ (Kuratowski)

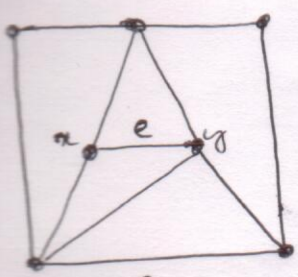
A graph G is planar if and only if it has no subgraph ~~obtained from G~~ isomorphic to a subdivision of K_5 or $K_{3,3}$

Given an edge $e = \{u, v\}$ of a simple connected graph G, the graph $G * e$ is obtained from G by contracting the edge e, that is, to get $G * e$ we identify the vertices u and v and remove all resulting loops and ~~multiple ed~~ multiple (duplicate) edges.

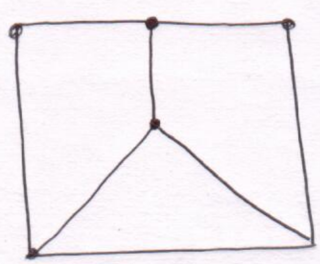
A graph obtained by a sequence of edge contractions is said to be a contraction of G.

A graph H is a minor of G if it is a subgraph of a graph obtained from G by a sequence of edge contraction.

Example



G

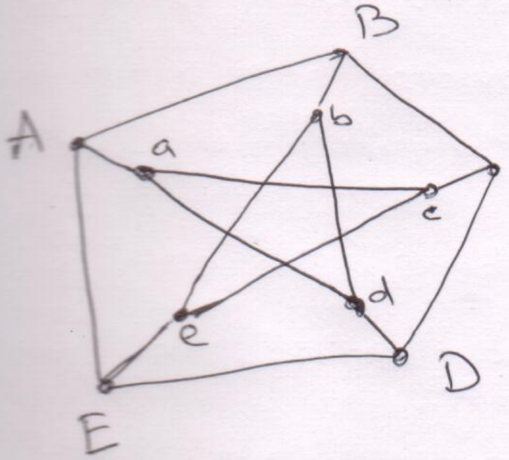


$G * e$

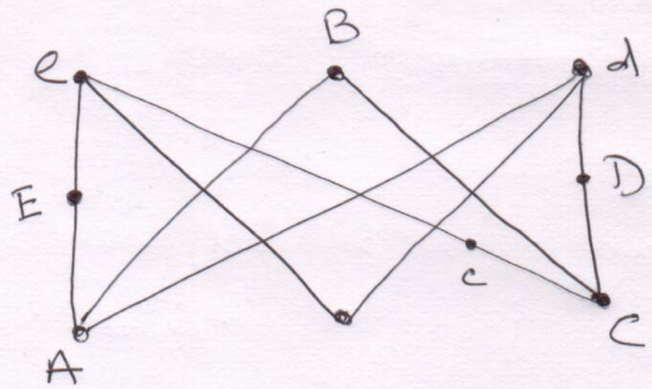
Th^m (Wagner)

A graph is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a minor.

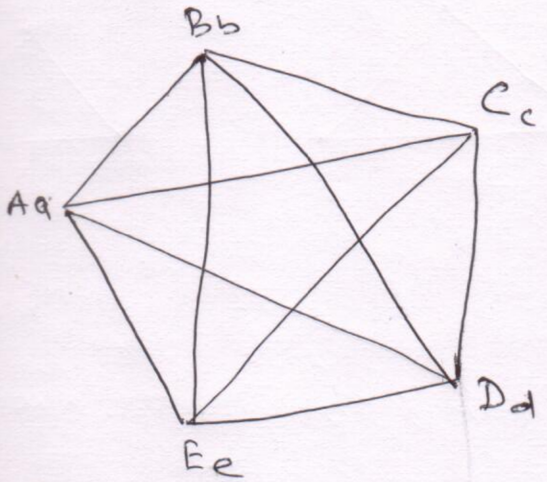
Ex Petersen graph



Petersen graph



A subgraph of the Petersen graph which is a subdivision of $K_{3,3}$



A minor of the Petersen graph

Dual of a Plane graph

Let G be a plane graph. One can form a new graph H in the following way:

Corresponding to each face f of G, take a vertex f^*, and an edge e^* corresponding to each edge e of G. Then edge e^* joins vertices f^* and g^* in H if and only if edge e is common to the boundary of faces f and g in G. (It is possible that f may be the same as g.) Graph H is called the dual of G.

If e is a bridge of G embedded in face f of G, the e^* is a loop at f^*.

The dual H is a planar graph, and there exists a natural way of embedding H in the plane.

Vertex f^*, corresponding to face f, is placed in face f of G. Edge e^*, joining f^* and g^*, is drawn so that so that e^* crosses e once and only once and crosses no other edges.

We denote the dual of a plane graph G by G^*

The definition of dual implies that

$m(G^*) = m(G)$, i.e. Number of edges of G and G^* are equal.

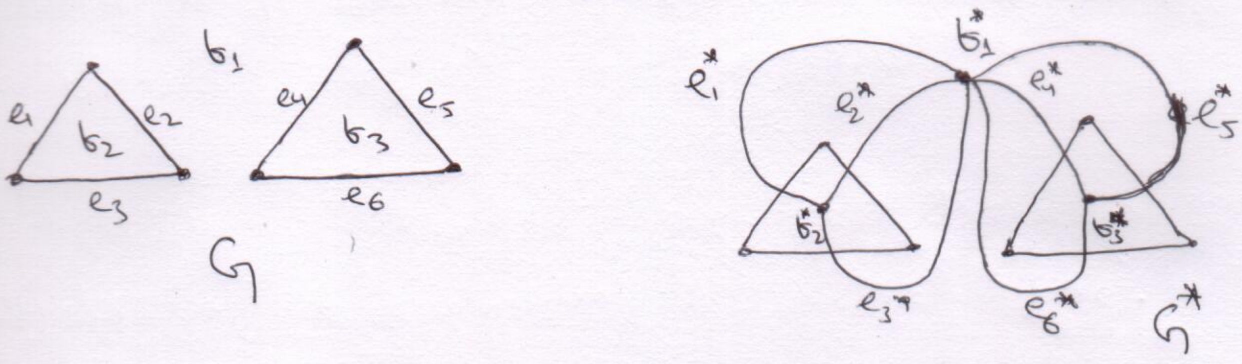
$n(G^*) = f(G)$, i.e. Number of vertices in G^* is equal to the number of faces in G.

From the construction of G^*, it follows that

(i) an edge e of a plane graph G is a bridge of G if and only if, and a loop if and only if e^* is a cut edge of G^*.

(2) G^* is connected whether G is connected or not.

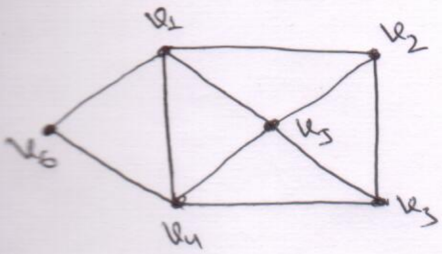
Ex



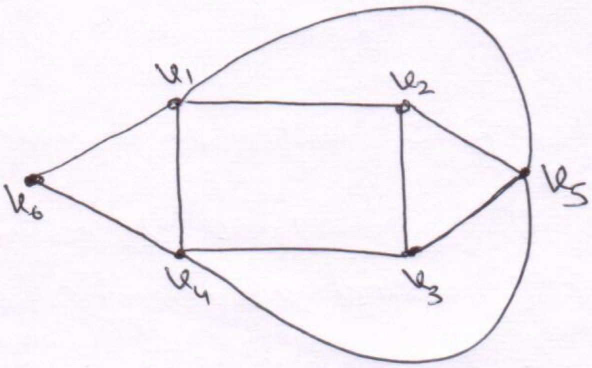
A disconnected graph and its connected dual G^*

It is easy to check that G^{**} is isomorphic to G if and only if G is connected.

Ex



G



H

G and H are isomorphic plane graph but G^* is not isomorphic to H^*

A graph G is called self dual if $G \cong G^*$.