Colorcing

A (vertex) coloring of a graph is an assignment of colors to its veretices so that no two adjacent vertices have same color.

A k-coloring of a graph G uses k colors

The chromatic number $\chi(G)$ is defined as the minimum number k for which G has a k-coloring.

A graph is k-colorable if $\chi(G) \leq k$ and is k-chromatic if $\chi(G) = k$.

$\underline{\underline{Th^m}}$ (1) If the graph G has n vertices, then $\chi(G) \leq n$

(2) If H is a subgraph of G then $\chi(H) \leq \chi(G)$.

(3) $\chi(K_n) = n$ for all $n \geq 1$.

(4) If the graph G has $G_1, G_2, \cdots, G_l$ as its connected components, then $\chi(G) = \max_{1 \leq i \leq l} \chi(G_i)$.

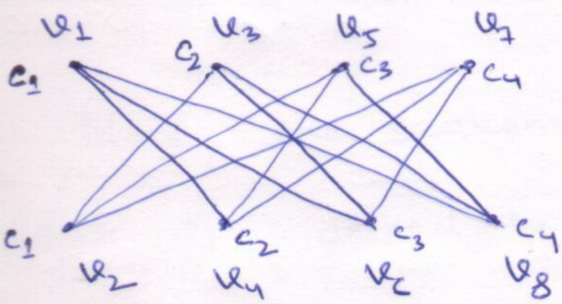
$\underline{\underline{Ex}}$ Let G be a nonempty graph. Then $\chi(G) = 2$ if and only if G is bipartite.

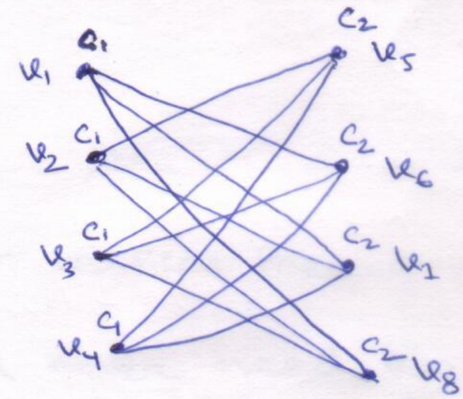
$\underline{\underline{Ex}}$ ~~The~~ Let G be a graph. Then $\chi(G) \geq 3$ if and only if G has an odd cycle.

Greedy Algorithm

Order the vertices of G, say $v_1, v_2, \dots, v_n$ and color them one by one; give v_1 color c_1, then give v_2 color c_1 if $v_1v_2 \notin E(G)$ and color c_2 otherwise and so on. Color each vertex with the smallest color it can have at this stage. This ~~produce~~ algorithm does produce a coloring.

Example





Proposition:

For any graph G, $\chi(G) \leq \Delta(G) + 1$.

Proof: Use greedy algorithm.

Lemma: Let $K = \max_H \delta(H)$, where the maximum is taken over all induced subgraphs of G. Then $\chi(G) \leq K+1$.

Proof The graph G itself has a vertex of degree at most K. Let v_n be such a vertex and take $H_{n-1} = G \setminus \{v_n\}$. By assumption H_{n-1} has a vertex of degree at most K. Let v_{n-1} be one of them and take $H_{n-2} = H_{n-1} - \{v_{n-1}\}$ $= G - \{v_n, v_{n-1}\}$. Continuing in this way, we can enumerate all the vertices.

Now, the sequence $v_1, v_2, \ldots, v_n$ is such that each v_j is joined to at most K vertices preceding it. Hence the greedy algorithm will never need $K+2$ colors to color the vertices.

Th^m (Brooks)

Let G be a connected graph. Suppose G is neither complete nor an odd cycle. Then $\chi(G) \leq \Delta(G)$.

Proof

If G is not $\Delta(G)$ regular then for any induced subgraph H of G, $\delta(H) \leq \Delta(G) - 1$. So by above lemma $\chi(G) \leq \Delta(G)$.

So assume that G is $\Delta(G)$ regular. First suppose that G has a cut vertex v and let G' be a subgraph consisting of a component of $G-v$ together with its edges to v (ie. the induced subgraph of G with vertex set consists of v and the vertices of a component of $G-v$). The degree of v in G' is less than $\Delta(G)$. The method above provides a $\Delta(G)$ coloring of G'. By permuting the names of colors in the subgraphs resulting in this way from components of $G-v$, we can make a $\Delta(G)$ coloring of G.

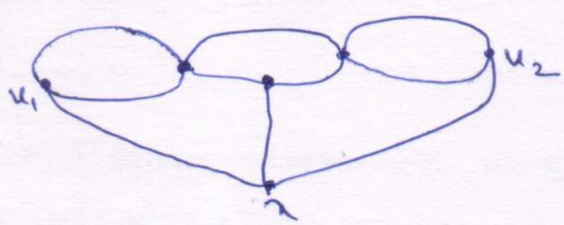
We may thus assume that G is 2 connected. In every vertex ordering, the last vertex has k earlier neighbours. The greedy algorithm idea may still work of we arrange the two neighbours of v_n get the same color.

In particular, suppose that some vertex v_n has neighbours v_1, v_2 such that, $v_1 v_2 \notin E(G)$ and $G-\{v_1, v_2\}$ is connected. In this case, we index the vertices of a spanning tree of $G-\{v_1, v_2\}$ using $3, \dots, n$ such that labels increase along paths to the root v_n. As before each vertex before v_n has at most $\Delta(G)-1$ lower index neighbour. The greedy algorithm uses at most $\Delta(G)-1$ colors on neighbours of v_n since v_1 and v_2 recieve the same color.

Hence it ~~is sufficient~~ suffices to show that every 2-connected $\Delta(G)$-regular graph with $\Delta(G) \geq 3$ has such a triple v_1, v_2, v_n. Choose a vertex x.

If $\kappa(G-x) \geq 2$, let v_1 be x and let v_2 be a vertex with distance 2 from x. Such a vertex v_2 exists because G is regular and not a complete graph. Let v_n be a common neighbour of v_1 and v_2.

If $\kappa(G-x) = 1$, let $v_n = x$. Since G has no cut-vertex, x has a neighbour in every leaf block of $G-x$. Neighbours v_1, v_2 of x in two such blocks are nonadjacent. Also, $G - \{x, v_1, v_2\}$ is connected, since blocks have no cut-vertices. Since $\Delta(G) \geq 3$, the vertex x has another neighbour, and $G - \{v_1, v_2\}$ is connected.



Defn A clique in a graph is a set of pairwise adjacent vertices. The maximum order of a clique in G is called the clique number of G, and we denote it by $\omega(G)$.

For any graph G, $\chi(G) \geq \omega(G)$.

Defn A graph is perfect if $\chi(H) = \omega(H)$ for every induced subgraph H of G.

The Perfect graph theorem:

A graph is perfect if and only if its complement is perfect.

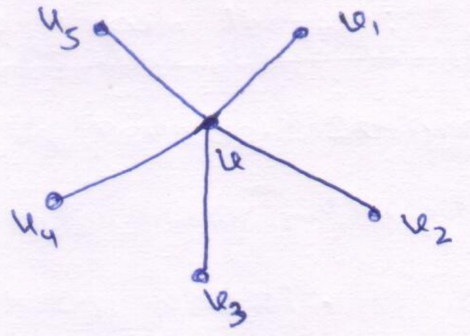
Thm Every planar graph is 5-colorable.

Proof We proceed by induction on the number of vertices. For any planar graph having $n \le 5$ vertices, the result follows trivially since the graph is n-colorable.

We assume that all planar graphs with $n-1$ vertices, are 5-colorable.

Let G be a planar graph on $n > 5$ vertices. Then G has a vertex v of degree at most 5. By induction hypothesis $G - v$ is 5-colorable. Consider an assignment of colors to the vertices of $G - v$ such that a 5-coloring results, where the colors are denoted by c_i, $1 \le i \le 5$. If some color, say c_j is not used in the coloring of the vertices adjacent to v, then by assigning the colors c_j to v, a 5-coloring of G results.

Otherwise 5 colors are used for the vertices of G adjacent to v. Now label the vertex adjacent with v and colored c_i by v_i, $1 \leq i \leq 5$.



Let G_{13} denote the subgraph of $G-v$ induced by those vertices colored c_1 and c_3. If v_1 and v_3 belong to different components of G_{13}, then a 5-coloring of $G-v$ may be accomplished by interchanging the colors of the vertices in the component of G_{13} containing v_1. In this 5-coloring, no vertex adjacent with v is colored c_1. So by coloring v with the color c_1, a 5-coloring of G results.

If v_1 and v_3 belong to the same component of G_{13} then there exists a path in G between v_1 and v_3 all whose vertices are colored c_1 or c_3. This path together with the path v_1-v-v_3 produce a cycle which necessarily enclose the vertex v_2 or both the vertex v_4 and v_5. In any case, there exist no path joining v_2 and v_4, all of whose veretices are colored c_2 or c_4.

Let G_{24} denote the subgraph of $G-v$ induced by the vertices colored C_2 and C_4. Then v_2 and v_4 belong to different components of G_{24}. Interchanging the colors of the vertices in the component of G_{24} containing v_2, a 5-coloring of $G-v$ is produced in which no vertex adjacent with v is colored C_2. We then obtain a coloring of G by assigning to v the color C_2.

Four Color Theorem:

Every planar graph is 4-colorable.