$\underline{\underline{Th^m}}$ Let G be a 2-connected graph. Then every two distinct vertices lie on a common cycle of G.

$\underline{\underline{Proof}}$ Assume to the contrary that there are pairs of vertices of G that do not lie on a common cycle. Among all such pairs, let u, v be a pair for which $d(u,v)$ is minimum. Now $d(u,v) \neq 1$, for otherwise $uv \in E(G)$. Since G contains no bridges so uv lies on a cycle of G. Therefore $d(u,v) = k \geq 2$.

Let $P: u = v_0 v_1 \ldots v_{k-1} v_k = v$ be a shortest u-v path in G. Since $d(u, v_{k-1}) = k-1 < k$, there is a cycle C containing u and v_{k-1}. By assumption v is not on C. Since v_{k-1} is not a cut vertex of G and u and v are distinct from v_{k-1}, there is a u-v path Q that does not contain v_{k-1}.

Since u is on C, there is a first vertex x of Q that is on C. Let Q' be the v-x subpath of Q and let P' be a v_{k-1}-x path on C that contains u. (If $x \neq u$, then the path P' is unique). However, the cycle C' produced by proceeding from v to its neighbour v_{k-1}, along P' to x, and then along Q' to v contains both u and v, a contradiction.

$\underline{\underline{\text{Th}^m}}$ If G is a k-connected graph, $k \geq 2$ then every k vertices of G lie on a common cycle.

Proof Let $S = \{v_1, \ldots, v_k\}$ be a set of k vertices of G. Among all cycles in G, let C be one containing a maximum number l of vertices of S. Then $l \leq k$. If $l = k$, then the result follows, so we may assume that $l < k$. Since G is k-connected, G is 2-connected and so $l \geq 2$. We may further assume that $v_1, v_2, \ldots, v_l$ lie on C. Let u be a vertex of S that does not lie on C. We consider two cases.

Case I The cycle contains exactly l vertices, say $C = v_1 v_2 \ldots v_l v_1$.

By Corollary above, G contains a u-v_i path P_i for each i with $1 \leq i \leq l$ such that every two paths $P_1, P_2, \ldots, P_l$ have only u in common. Replace the edge $v_1 v_2$ on C by P_1 and P_2. Produce a cycle containing at least $l+1$ vertices of S. This is a contradiction.

Case 2 The cycle contains at least $l+1$ vertices.

Let v_0 be the vertex on C that does not belong to S. Since $2 < l+1 \leq k$, it follows by corollary above, that G contains a $u-u_i$ path P_i for each i with $0 \le i \le l$ such that every two paths $P_0, \ldots, P_l$ have only u in common.

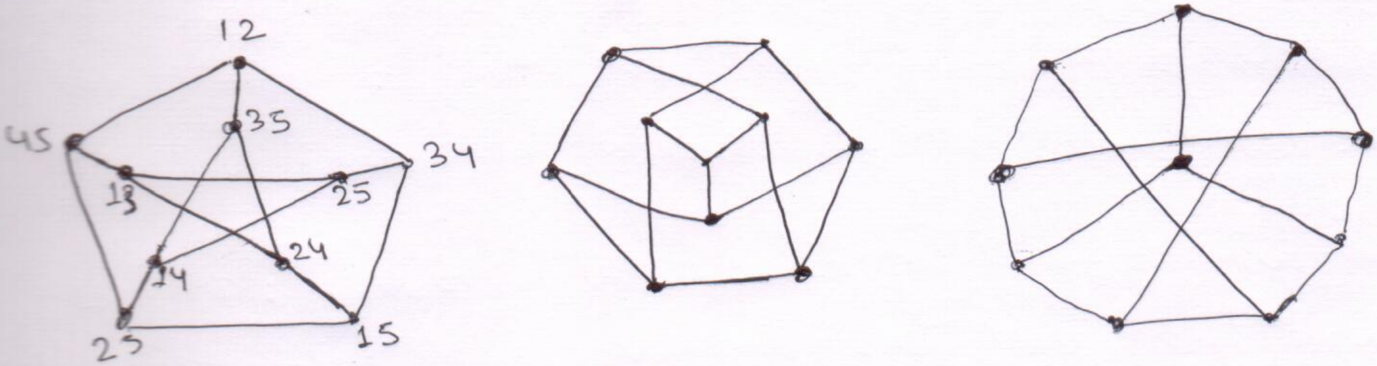
For each i, $(0 \le i \le l)$ let u_i be the first vertex of P_i that belongs to C and let P_i' be the $u-u_i$ subpath of P_i. Suppose that the vertices u_i $(0 \le i \le l)$ are encountered in the order $u_0, u_1, \ldots, u_l$ as we proceed about C in some direction. For some i with $0 \le i \le l$ and $u_{l+1} = u_0$, there is a $u_i - u_{i+1}$ path P on C, none of whose internal vertices belong to S. Replacing P on C by P_i' and P_{i+1}' produce a cycle containing atleast $l+1$ vertices of S. Again this is a contradiction.

$\underline{\text{Th}}^m$ For distinct vertices u, and v in a graph G, the minimum cardinality of a set X of edges of G such that u and v lie in distinct components of $G-X$ equals the maximum number of pairwise edge disjoint $u-v$ paths in G.

$\underline{\text{Th}}^m$ A non trivial graph G is K-edge connected if and only if G contains K pairwise edge-disjoint $u-v$ paths for each pair u, v of distinct vertices of G.

Petersen graph

The Petersen graph is the simple graph whose vertices are the 2-element subsets of a 5-element set and whose edges are the pairs of disjoint 2 element subsets.



Note that if two vertices are non adjacent in the Petersen graph, then they have exactly one common neighbour.

Def^n The girth of a graph with a cycle is the length of its shortest cycle. A graph with no cycle has infinite girth.

Ex The Petersen graph has girth 5.

Ex The Petersen graph is not Hamiltonian.