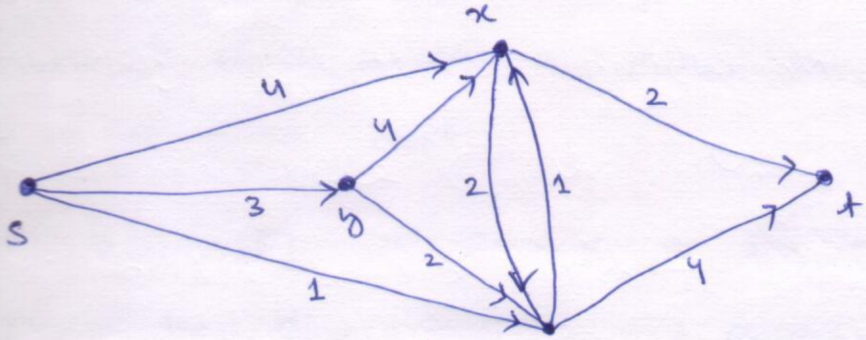
Networks



N

A network N is a weakly connected simple digraph in which every arc a of N has been assigned a non-negative integer $c(a)$, called capacity of a.

A vertex s of a network N is called a source if it has indegree 0 while a vertex t of N is called a sink if it has outdegree 0. Any other vertex of N is called an intermediate vertex.

We will assume first that any network N we consider has exactly one source and exactly one sink.

Given any vertex u of the network N, we denote the set of arcs going into u and going out of u by $I(u)$ and $O(u)$, respectively.

A flow in a network N from the source s to the sink t is a function f which assigns a nonnegative to each of the arcs in N such that

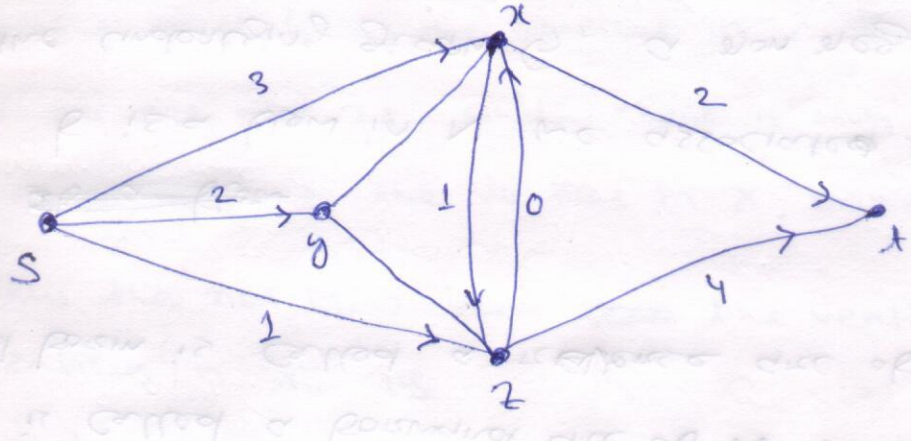
(i) $f(a) \leq c(a)$ for each arc a

(ii) the total flow into the sink t equals the total flow out of the source s

(iii) for any intermediate vertex x, the total flow into x equals the total flow out of x.

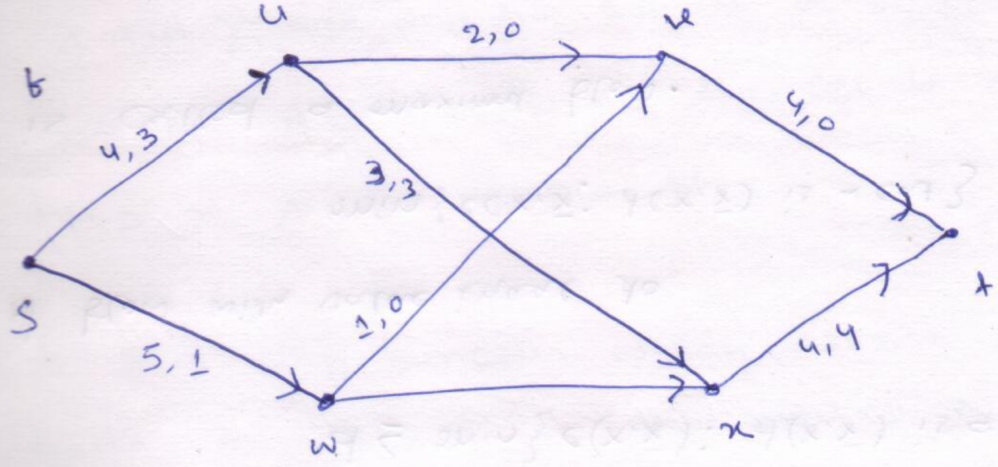
That means for the source s and the sink t $\sum_{a \in O(s)} f(a) = \sum_{a \in I(t)} f(a)$

and for any intermediate vertex x, $\sum_{a \in O(x)} f(a) = \sum_{a \in I(x)} f(a)$



The number $d = \sum_{a \in O(s)} f(a) = \sum_{a \in I(t)} f(a)$, where s and t are the source and sink of the network N, is called the value of the flow f.

Ex



A network with a flow

Let f be a flow on the network $N = (V, A)$ and for any proper subset X of the vertex set V of N, let $\bar{X}$ denote the complement of X in V, ie $\bar{X} = V - X$

Take $X = \{s, u, x\}$ so $\bar{X} = \{v, w, t\}$

Then the arcs from the vertices in X to the vertices in $\bar{X}$ are uv, sw and xt while there is only one arc from the vertices in $\bar{X}$ to the vertices in X, namely wx.

Thus the net flow from ~~the~~ the vertices in X to the vertices in $\bar{X}$ is

$$f(uv) + f(sw) + f(x,t) - f(wx) = 0 + 1 + 4 - 1 = 4$$

which is the value of the flow.

The value of the flow is $f(su) + f(sw) = 3 + 1 = 4$

$f(vt) + f(x,t) = 0 + 4 = 4$

If X and Y are any two subsets of vertices of the network N, we let $A(X, Y)$ denote the set of arcs from the vertices X to the vertices Y.

If g is any function which assigns non-negative integers to the arcs of the network N (for example, g could be the capacity function c or ~~the~~ a flow f), then for any to subsets of vertices X, Y of N, we define

$$g(X, Y) = \sum_{a \in A(X,Y)} g(a)$$

In other words, $g(X,Y)$ is the sum of the values of the function g on each arc from a vertex in X to a vertex in Y.

A cut is a set of arcs $A(X, \bar{X})$, where the source s is in X and the sink t is in $\bar{X}$.

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For $X = \{s, u, x\}$, $A(X, \bar{X})$ is a cut and

$$c(X, \bar{X}) = \sum_{a \in A(X,\bar{X})} c(a) = c(uv) + c(sv) + c(x,t) = 2+5+4 = 11$$

$\underline{\underline{Th^m}}$ Let f be a flow on a network $N = (V, A)$ and let the value of f be d. If $A(X, \bar{X})$ is a cut in N then

$$d = f(X, \bar{X}) - f(\bar{X}, X)$$

and

$$d \leq c(X, \bar{X})$$

In other words, the total flow out of X minus the total flow into X ie the net flow out of X equals d, the value of the flow, and this never exceeds the total capacity of the arcs from X to $\bar{X}$.

$\underline{\underline{\text{Proof}}}$ From the definition of flow, for the source s, we have

$$f(\{s\}, V) = d \quad \text{and} \quad f(V, \{s\}) = 0,$$

while, for any vertex u different from both s and the sink t,

$$f(\{u\}, V) = \sum_{a \in O(u)} f(a) = \sum_{a \in I(u)} f(a) = f(V, \{u\})$$

ie. $f(\{u\}, V) - f(V, \{u\}) = 0$ for $u \neq s, t$.

Thus for each cut $A(X, \bar{X})$, we have

$$\sum_{x \in X} f(\{x\}, V) - f(V, \{x\}) = f(\{s\}, V) - f(V, \{s\}) + 0 = d + 0 = d$$

$$\text{ie. } f(X, V) - f(V, X) = d$$

However

$$f(X,V) = f(X, X \cup \bar{X}) = f(X,X) + f(X,\bar{X})$$

and similarly

$$f(V,X) = \cancel{f(X\cup\bar{X})}\ f(X,X) + f(\bar{X},X)$$

Thus $d = f(X,V) - f(V,X) = f(X,\bar{X}) - f(\bar{X},X)$.

Moreover, since for each arc a of N, we have $f(a) \le c(a)$

Thus we have $f(X,\bar{X}) \le c(X,\bar{X})$ and so

$$d = f(X,\bar{X}) - f(\bar{X},X) \le f(X,\bar{X}) \le c(X,\bar{X}). \quad //$$

The value of any flow is less than or equal to the capacity of the arcs from X to $\bar{X}$ for any cut $A(X,\bar{X})$

Thus, if f is a flow with value d, we have

$$d \le \min\{c(X,\bar{X}) : A(X,\bar{X}) \text{ is a cut}\}.$$

A flow with value equal to

$$\min\{c(X,\bar{X}) : A(X,\bar{X}) \text{ is a cut}\}$$

is called a maximal flow.