Real Analysis Chapter 8 Study Guide…specifically, Sections 8.1 through 8.3 (for "Real Analysis, A First Course", 2nd Edition, Russell A. Gordon)

Number of Starred Exercises: **1**; Number of Notes: 5; Number of Other (non-starred) Exercises: 39; Minimum Number of Other (non-starred) Exercises to Do (to do at least 25% of them): **10**

The most important things to get out of this section: (1) The basic definitions and facts about open and closed sets. (2) The definition of a compact set and its use. (3) The characterizations of continuous functions in terms of open and closed sets. (4) How continuous functions behave with respect to compact sets. (5) Practice with basic examples and proofs related to this.

Other matters of importance: Understanding the significance of the Heine-Borel Theorem

Reading Guide:

1. Ponder the following sentence from the first paragraph of page 291: "In order to discuss more unusual functions, it is helpful to be aware of subsets of $\mathbb{R}$ that are not intervals." Have we thought about unusual functions in this course? Do you remember me saying that "Real Analysis, though dealing with a more familiar subject than Algebraic Structures, is sometimes trickier because statements that seem reasonable are not always true"? Knowing certain counterexamples is extremely important in this course, especially if you are either going to teach math in the future or go to graduate school.
2. Think deeply about the second paragraph on page 291 that goes on to the next page. Do you remember being taught similar ideas in Linear Algebra? (Think in terms of vector spaces and different examples of vector spaces). Page through your linear algebra book and write down any "spaces" you see in there that are not Euclidean spaces (not spaces whose points can be represented with coordinates in the usual way we think about them). You should be able to find some (especially in chapters 4 and 6 of the linear algebra book).
3. Do your best to draw a picture of the set A defined at the bottom of page 292. You won't be able to draw it perfectly, but technology will probably be of help.
4. Write a paragraph of explanation to convince yourself as to why the set B defined at the bottom of page 292 contains no intervals.
5. In one of the parts (a), (b), or (c) of Definition 8.1 on page 293, the point x is not necessarily an element of the set E, in the other two parts, it is. In which one part ((a), (b), or (c)) is x not necessarily an element of E? Why? Can you think of an example for this part where x is not in E? Can you think of an example for this part where x is in E?
6. *Think deeply about Definition 8.1 and the examples and observations that follow on page 293. Write some sentences that help you understand the definitions and try to explain why they make sense (intuitively). Make up your own examples

where you identify interior, isolated, and limit points. Make up your own examples of open sets, closed sets, and sets that are neither open nor closed. Make sure you consider sets besides just intervals…perhaps the set of rationals and the set of irrationals would be good to consider. If you are looking for more "exotic" examples, perhaps you'll want to look up information about the Cantor set with respect to these definitions. Write down what you find out.

7. Draw pictures that help you think about how you might go about proving the 10 observations on page 293. Then, if you feel confident, attempt actually writing out three or four of these proofs. Compare your proofs with a friend's proofs (maybe agree on which ones you want to try ahead of time).
8. Thoroughly study the proof of Theorem 8.2 on page 294 until you completely understand it. Then, take a 10 minute break and prove it yourself without looking at the book's proof.
9. Thoroughly study the proof of Theorem 8.3 on page 294 until you completely understand it. Then, take a 10 minute break and prove it yourself without looking at the book's proof.
10. Thoroughly study the proof of Theorem 8.4 on pages 294-295 until you completely understand it. Then, take a 10 minute break and prove it yourself without looking at the book's proof.
11. Thoroughly study the proof of Theorem 8.5 on page 295 until you completely understand it. Then, take a 10 minute break and prove it yourself without looking at the book's proof.
12. Fill in details of the proof of Theorem 8.6 on page 296 that are confusing to you. For example, you may want to do the verification that $x \in I \subseteq O$ and that $\alpha_x, \beta_x \notin O$ for each $x \in O$.
13. Draw pictures and/or write down details that help you understand the example discussed on the bottom of page 296 to the top of page 297.
14. Make up your own examples for *E* to calculate the interior, derived set, and closure of *E*. Try to make the examples interesting and enlightening.
15. Thoroughly study the proof of Theorem 8.8 on page 297 until you completely understand it. Then, take a 10 minute break and prove it yourself without looking at the book's proof.
16. **Note:** As you start studying Section 8.2, realize that if *K* is a subset of $\mathbb{R}$ and if $f : K \to \mathbb{R}$, then $f(K) = \{f(x) : x \in K\}$ is the range of *f* (also called the *image* of *K* under *f*).
17. Pick a few of the open covers of the interval (0,1) at the bottom of page 301 and verify that the union of the elements in the open cover contains the interval (0,1). Besides the 7th and 8th examples, are there any others of these that contain finite subcovers of the interval (0,1)?
18. Verify the details of the paragraph in the middle of page 302 to show that the interval (0,1) is not a compact set. Draw pictures and/or write down words and equations that would help you verify that an arbitrary open interval (a,b) is not a compact set.
19. Write down thoughts that help you understand the details of showing that the set *E* defined in the bottom half of page 302 is compact.
20. Fill in details of the proof of Theorem 8.11 on page 303 that are confusing to you.

21. Prove that a compact set is bounded (the part left out in the proof of Theorem 8.11).
22. Thoroughly study the proof of Theorem 8.12 on page 303 until you completely understand it. Then, take a 10 minute break and prove it yourself without looking at the book's proof.
23. Prove Theorem 8.13
24. Look up DeMorgan's Laws and write them down. Next, prove DeMorgan's Laws
25. Thoroughly study the proof of Theorem 8.14 on pages 303-304 until you completely understand it. Then, take a 10 minute break and prove it yourself without looking at the book's proof.
26. Fill in details of the proof of the Heine-Borel Theorem (Theorem 8.15) on page 304 that are confusing to you. Take note of the use of the Completeness Axiom. There are spaces for which the Heine-Borel Theorem is not true (where compactness is not equivalent to being closed and bounded in that space).
27. Draw pictures or write down and argument to help you convince yourself that the set *E* defined at the bottom of page 304 is closed and bounded.
28. Thoroughly study the half proof of Theorem 8.16 on page 305 until you completely understand it. Then, take a 10 minute break and prove it yourself without looking at the book's proof.
29. Prove the converse of the statement proved for Theorem 8.16.
30. **Note:** by "universe" on page 307, the author means "the 'big' set that every set under discussion is a subset of". For the most part, in real analysis, the "universe" is the set of real numbers $\mathbb{R}$. Sometimes this is called the "universe of discourse" or "domain of discourse".
31. **Note:** take note of the fact that the word "in" is boldfaced in both parts of Definition 8.17…this is no accident, but an important part of these definitions.
32. Draw pictures of examples that illustrate the truth of the statements in Theorem 8.18. Then pick a couple of these properties and prove them.
33. Given two sets *A* and *B*, a subset $C \subseteq B$, and a function $f : A \to B$ (whether one-to-one or not), the symbol $f^{-1}(C)$ (called the "preimage" or "inverse image" of *C* under *f* …also see the bottom of page 310) represents the set of all elements of *A* that have outputs in *C*, i.e. $f^{-1}(C) = \{x \in A : f(x) \in C\}$ (we write this whether the inverse function of *f* exists or not). Write the sets described in Theorem 8.19 in terms of this notation (also use interval notation for *C*).
34. Fill in details of the proof of Theorem 8.19 on pages 308-309 that are confusing to you.
35. **Note:** Stop to ponder the fact that every function defined on the integers (or any "discrete" set) is continuous on that set, according to Definition 8.20. Our intuitive definition of continuity as meaning you can draw the graph without picking up your pencil has really been debunked!
36. Fill in the details of the proof that the function *g* defined at the bottom of page 309 is continuous on the set *E*.
37. Prove Theorem 8.21.
38. Write the sets in statements (2) and (3) of Theorem 8.22 in terms of inverse image notation (see #33 above).

39. Fill in the details of the proofs of the implications proved in the book for Theorem 8.22.
40. Prove the missing implication for Theorem 8.22.
41. **Note:** the paragraph at the very bottom of page 310 is very important to ponder and recall (and use) if you ever go to graduate school in mathematics, statistics, or physics (or even economics…I once knew an economics graduate student who needed to know very advanced real analysis for his Ph.D. in economics).
42. Thoroughly study the proof of Theorem 8.23 on page 311 until you completely understand it. Then, take a 10 minute break and prove it yourself without looking at the book's proof.
43. Explain how to relate the statement of Theorem 8.23 to the Extreme Value Theorem (can one be used to prove the other?). The Heine-Borel Theorem is important to consider.
44. Thoroughly study the proof of Theorem 8.24 on pages 311-312 until you completely understand it. Then, take a 10 minute break and prove it yourself without looking at the book's proof.
45. Close out your reading experience in this course by drawing pictures that help you understand the ideas on pages 312-314, including the Tietze Extension Theorem. Also, look up the Tietze Extension Theorem online and take time to appreciate the fact that it can be stated in a more general point-set topology manner. Also note that it generalizes another important result called the Urysohn Lemma.

Deep Thoughts to Ponder (but not necessarily answer):

- Perhaps this chapter re-emphasizes the fact that the real number system is not as simple as you may have once thought (see the Chapter 1 study guide). Spend time thinking about this. Give praise to God that he has created such structures and given our minds the capacity to think about them.
- If you have the stomach to consider yet another topological topic, you might want to explore the topic of the connectedness of a set of real numbers (look up "connected space"…also study Section 8.4) and how continuous functions behave relative to connected sets. You would also want to relate what you learn to the Intermediate Value Theorem.
- If you are thinking about graduate school, you should also study Section 8.5 at some point (and perhaps study Chapter 6 of your linear algebra textbook at the same time).