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**Definition 1: (Metric)**

Let $X$ be a non-empty set. A metric $d$ on $X$ is a func$^n$ $d: X \times X \longrightarrow \mathbb{R}$ satisfying the following properties —

i) $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$

ii) $d(x,y) = d(y,x)$ $\forall x, y \in X$

iii) $d(x,y) \leq d(x,z) + d(z,y)$ $\forall x, y, z \in X$

**Example 2: (Discrete metric)**

Let $X$ be a non-empty set. Then the discrete metric $d$ on $X$ is defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

**Example 3: (Euclidean metric)**

$X = \mathbb{R}/\mathbb{C}$ and $d(x,y) = |x-y|$ $\forall x, y \in \mathbb{R}/\mathbb{C}$

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**Sequence Spaces:**

A real sequence $\{x_n\}_{n \in \mathbb{N}}$ is a function $f: \mathbb{N} \longrightarrow \mathbb{R}$ defined by $f(n) = x_n$ $\forall n \in \mathbb{N}$.

• Let $1 \le p < \infty$.

Define, $l^p = \left\{ \{x_n\} \mid x_n \in \mathbb{R} \ \forall n \in \mathbb{N}, \ \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$

(p-summable)

[ $\sum_{n=1}^{\infty} |x_n|^p < \infty$ means the sequence $\{S_n\}$ of partial sum is convergent, where $S_n = \sum_{i=1}^{n} |x_i|^p$ ]

• Define $\| \cdot \|_p : l^p \to \mathbb{R}$ such that

(norm) $\| \{x_n\} \|_p = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$ [well-defined because of non-negativity of $\{S_n\}$]

* $l_p \neq \phi$ $\{0_n\}_{n \in \mathbb{N}} \in l_p$

* $l_p$ forms a vector space

{ length of a seq^n in $l_p$ (vaguely) }

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•• **Theorem 4:**

For any two sequences $\{x_n\}, \{y_n\} \in l^p$, the function $d_p$, defined by ~~and~~ $d_p(\{x_n\}, \{y_n\}) = \| \{x_n - y_n\} \|_p$ is a metric on $l^p$.

Proof:

•• **Theorem 5:** (Hölder's Inequality)

Let $p, q \in (0, \infty)$ st $\frac{1}{p} + \frac{1}{q} = 1$.

Let $\{x_n\} \in l^p$, $\{y_n\} \in l^q$. Then —

$$\sum_{n=1}^{\infty} |x_n y_n| \le \| \{x_n\} \|_p \cdot \| \{y_n\} \|_q$$

($l^1$ norm of product seq^n) [ $l^p$ is the dual of $l^q$ vector space ]

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Theorem 6: (Young's Inequality)

Let $p, q \in (1, \infty)$ st $\frac{1}{p} + \frac{1}{q} = 1$. Then for any $a, b \geq 0$, $\boxed{a^{1/p} b^{1/q} \leq \frac{a}{p} + \frac{b}{q}}$

Proof: For $t \in [1, \infty)$, define

$$f(t) = k(t-1) - t^k + 1, \text{ where } 0 < k < 1.$$

Now, $f'(t) = k(1 - t^{k-1}) \geq 0 \quad \forall t \in [1, \infty)$.

$\Rightarrow f(t)$ is increasing.

$\Rightarrow f(t) \geq f(1) = 0 \quad \forall t \in [1, \infty)$.

WLOG, let $a \geq b > 0$.

Put $t = a/b$, $k = 1/p$.

Then $f(a/b) = \frac{1}{p}\left(\frac{a}{b} - 1\right) - \left(\frac{a}{b}\right)^{1/p} + 1 \geq 0$

$b^{1/q}$.

$$\Rightarrow \frac{a}{p} \cdot \frac{1}{b} - \frac{1}{p} - \frac{a^{1/p}}{b^{1/p}} + 1 \geq 0$$

$$\Rightarrow \frac{a}{p} \cdot \frac{1}{b} \cancel{\phantom{p}} + 1 \geq \frac{a^{1/p}}{b^{1/p}}$$

$$\Rightarrow \left(\frac{a}{p} \cdot \frac{1}{b} + 1\right) \geq a^{1/p} \cdot b^{1/q}$$

- Proof of Hölder's Inequality;

WLOG, take $\{x_n\}$ and $\{y_n\}$ to be non-zero sequences.

For a fixed $i \in \mathbb{N}$, define —

$$a = \left(\frac{|x_i|}{\|\{x_n\}\|_p}\right)^p, \quad b = \left(\frac{|y_i|}{\|\{y_n\}\|_q}\right)^q.$$

Then by theorem 6, we have —

$$\frac{|x_i y_i|}{\|\{x_n\}\|_p \, \|\{y_n\}\|_q} \leq \frac{1}{p}\left(\frac{|x_i|}{\|\{x_n\}\|_p}\right)^p + \frac{1}{q}\left(\frac{|y_i|}{\|\{y_n\}\|_q}\right)^q$$

$$\Rightarrow \frac{\sum_{i=1}^{n} |x_i y_i|}{\|\{x_n\}\|_p \, \|\{y_n\}\|_q} \leq \frac{1}{p} \cdot \frac{\sum_{i=1}^{n} |x_i|^p}{(\|\{x_n\}\|_p)^p} + \frac{1}{q} \cdot \frac{\sum_{i=1}^{n} |y_i|^q}{(\|\{y_n\}\|_q)^q}$$

[taking summation]

Now, $|x_i| \leq \|\{x_n\}\|_p \quad \forall i \in [n]$

$\therefore$ RHS $\leq 1$

$$\therefore \sum_{i=1}^{n} |x_i y_i| \leq \|\{x_n\}\|_p \, \|\{y_n\}\|_q \qquad \square.$$

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- Remark 7: In particular, for $p = q = 2$, the Hölder's inequality is known as the Cauchy-Schwartz inequality.

$$\left[\langle \{x_n\}, \{y_n\} \rangle \leq \|\{x_n\}\|_2 \, \|\{y_n\}\|_2\right]$$

.. Theorem 8: (Minkowsky's Inequality)

For $\{x_n\}, \{y_n\} \in l^p$,

$$\|\{x_n + y_n\}\|_p \le \|\{x_n\}\|_p + \|\{y_n\}\|_p .$$

Proof:

Case I: for $p=1$, we have

?? [some finite sum, read up]

$$|x_n + y_n| \le |x_n| + |y_n| \quad \forall n \in \mathbb{N}.$$

case II: For $1 < p < \infty$.

$$\sum_{i=1}^{m} |x_i + y_i|^p = \sum_{i=1}^{m} |x_i + y_i|^{p-1} |x_i + y_i|$$

$$\le \sum_{i=1}^{m} |x_i + y_i|^{p-1} |x_i| + \sum_{i=1}^{m} |x_i + y_i|^{p-1} |y_i| \quad \text{--- (1)}$$

Define, $z_n = \begin{cases} x_n + y_m & \text{if } 1 \le n \le m \\ 0 & \text{otherwise} \end{cases}$

Clearly, $z_n \in l^p, l^q$ where $1/p + 1/q = 1$

By Hölder's inequality,

$$\sum_{i=1}^{\infty} |z_i|^{p-1} |x_i| \le \|\{x_n\}\|_p \, \|\{z_n^{p-1}\}\|_q$$

Now, $\|\{z_n^{p-1}\}\|_q = \left( \sum_{i=1}^{m} |x_i + y_i|^{(p-1)q} \right)^{1/q}$

$$= \left( \sum_{i=1}^{m} |x_i + y_i|^p \right)^{1/q}$$

**Proof of Theorem 4:** ( $d_p = \|\{x_n - y_n\}\|_p$ is a metric ).

If $\{y_n\} \in \ell^p$, then $\{-y_n\} \in \ell^p$.

positivity

Then $d_p(\{x_n\}, \{y_n\}) \geq 0 \;\; \forall \; \{x_n\}, \{y_n\} \in \ell^p$.

[by Minkowski's inequality] ??

Also, if $\{x_n\} = \{y_n\}$, then

$$d_p(\{x_n\}, \{y_n\}) = 0$$

Conversely, if $d_p(\{x_n\}, \{y_n\}) = 0$

$$\Rightarrow \|\{x_n - y_n\}\|_p = 0$$

$$\Rightarrow \sum_{i=1}^{\infty} |x_i - y_i|^p = 0$$

symmetry

$$d_p(\{x_n\}, \{y_n\}) = \|\{x_n - y_n\}\|_p$$
$$= \|\{y_n - x_n\}\|_p$$
$$= d_p(\{x_n\}, \{y_n\})$$

triangle inequality

$$d_p(\{x_n\}, \{y_n\}) = \|\{x_n - y_n\}\|_p$$
$$= \|\{x_n - z_n + z_n - y_n\}\|_p$$
$$\leq \|\{x_n - z_n\}\|_p + \|\{z_n - y_n\}\|_p \quad \text{[by Minkowski's]}$$
$$= d_p(\{x_n\}, \{y_n\}) + d_p(\{x_n\}, \{y_n\})$$

∎

∴ Example 9: (space of bounded sequence)

$l^\infty = \{\{x_n\} \mid x_n \in \mathbb{R}\ \forall n \in \mathbb{N},\ \sup_{n\in\mathbb{N}} |x_n| < \infty\}$.

Define, $d_\infty : l^\infty \times l^\infty \to \mathbb{R}$ by

$d_\infty(\{x_n\},\{y_n\}) = \sup_{n\in\mathbb{N}} |x_n - y_n|$

[$l^\infty, d_\infty$ forms a Banach space]

non-negativity,

$d_\infty(\{x_n\},\{y_n\}) = \sup_{n\in\mathbb{N}} |x_n - y_n| \geq 0$ [$\because |x_n - y_n| \geq 0\ \forall n \in \mathbb{N}$]

Let $\{x_n\} = \{y_n\}$ then $x_n = y_n\ \forall n \in \mathbb{N}$

$\therefore |x_n - y_n| = 0\ \forall n \in \mathbb{N}$

$\Rightarrow \sup_{n\in\mathbb{N}} |x_n - y_n| = 0$

$\Rightarrow d_\infty(\{x_n\},\{y_n\}) = 0$

Conversely, let $d_\infty(\{x_n\},\{y_n\}) = 0$

$\Rightarrow \sup_{n\in\mathbb{N}} |x_n - y_n| = 0$

$\Rightarrow |x_n - y_n| = 0\ \forall n \in \mathbb{N}$ [$\because |x_n - y_n| \geq 0\ \forall n \in \mathbb{N}$]

$\Rightarrow x_n = y_n\ \forall n \in \mathbb{N}$

$\Rightarrow \{x_n\} = \{y_n\}$

symmetry,

$d_\infty(\{x_n\},\{y_n\}) = \sup_{n\in\mathbb{N}} |x_n - y_n|$

$= \sup_{n\in\mathbb{N}} |y_n - x_n|$

$= d_\infty(\{y_n\},\{x_n\})$.

triangle inequality

$d_\infty(\{x_n\},\{z_n\}) + d_\infty(\{y_n\},\{z_n\}) = \sup_{n\in\mathbb{N}} |x_n - z_n| + \sup_{n\in\mathbb{N}} |z_n - y_n|$

$\geq \sup_{n\in\mathbb{N}} |x_n - z_n + z_n - y_n|$

??

$= \sup_{n\in\mathbb{N}} |x_n - y_n| = d_\infty(\{x_n\},\{y_n\})$ $\square$.

• Example 10: ($p$-adic metric)

Let $p$ be a prime number.

Then any $x \in \mathbb{Q} \setminus \{0\}$ can be written as

$$x = p^k \cdot \frac{r}{s}, \quad \begin{array}{l} k \in \mathbb{Z} \text{ is unique} \\ r \in \mathbb{Z},\ p \nmid r \\ s \in \mathbb{Z},\ p \nmid s. \end{array}$$

$p^k / r$

Define $|x|_p = \begin{cases} p^{-k} & \text{if } x \in \mathbb{Q} \setminus \{0\} \\ 0 & \text{if } x = 0 \end{cases}$

$k$ is called the order of $x$, ($Ord_p(x)$) st —

$\rightarrow Ord_p(xy) = Ord_p(x) + Ord_p(y)$.

$\rightarrow Ord_p(x/y) = Ord_p(x) - Ord_p(y)$.

Claim: $d : \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{R}$ defined by $d(x,y) = |x-y|_p$ is a metric.

[ Completion of $\mathbb{Q}$ wrt Euclidean metric gives $\mathbb{R}$
Completion of $\mathbb{Q}$ wrt $p$-adic metric gives ______
~~* ___ does not have Ar~~
* ______ is a non-Archimidean field ]

$\rightarrow$ Let $x = a/b$, $y = c/d$.

Then $Ord_p(x+y) = Ord_p\left(\frac{ad+bc}{bd}\right)$

$= Ord_p(ad+bc) - Ord_p(bd)$

$Ord_p(x+y) \geq \min(Ord_p(ad), Ord_p(bc)) - Ord_p(bd)$

$= \min(Ord_p(a) + Ord(d), Ord_p(b) + Ord_p(c))$

$- Ord_p(b) - Ord_p(d)$

$= \min(Ord_p(a) - Ord_p(b), Ord_p(c) - Ord(d))$

$= \min(Ord_p(a/b), Ord_p(c/d))$.

~~$\min(\mathrm{Ord}_p(a/b), \mathrm{Ord}_p(c/d))$~~

$\therefore |x+y|_p = p^{-\mathrm{Ord}_p(x+y)}$

$\leq \max\{\underbrace{p^{-\mathrm{Ord}_p(x)}}_{|x|_p}, \underbrace{p^{-\mathrm{Ord}_p(y)}}_{|y|_p}\}$

$\leq |x|_p + |y|_p$

[stronger than triangle inequality].

[metrics satisfying stronger triangle inequality are called non-Archimedean metric].

**Example 11.** $\mathcal{F}(S)$ be the set of finite subsets of a non-empty set $S$.

Define, $d : \mathcal{F}(S) \times \mathcal{F}(S) \longrightarrow \mathbb{R}$ by

$d(A, B) = \text{Card}(A \Delta B)$.

Then $d$ is a metric on $\mathcal{F}(S)$.

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**Example 12:** $a, b \in \mathbb{R}$, $a < b$.

$C[a,b] = \{ f : [a,b] \longrightarrow \mathbb{R} \mid f \text{ is continuous} \}$.

Define, $\|f\|_\infty = \sup\limits_{x \in [a,b]} |f(x)|$

and $d_\infty : C[a,b] \times C[a,b] \longrightarrow \mathbb{R}$ by

$d_\infty(f_1, f_2) = \|f_1 - f_2\|_\infty$.

(compare with $d_\infty$ in Example 9)

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**Example 13:**

Define, $d : C[a,b] \times C[a,b] \longrightarrow \mathbb{R}$ by

$d(f_1, f_2) = \int_a^b |f_1(x) - f_2(x)| \, dx$

**Exercise 1:**

Suppose $n \in \mathbb{N}$ and $\forall i \in [n]$, $(X_i, d_i)$ be a metric space. The following functions are metrics on $\prod_{i=1}^{n} X_i$ ——

1) $M_1(a,b) = \sum_{i=1}^{n} d_i(a_i, b_i)$

11) $M_p(a,b) = \left( \sum_{i=1}^{n} d_i(a_i, b_i)^p \right)^{1/p}$ where $1 \le p < \infty$

111) $M_\infty(a,b) = \max \{ d_i(a_i, b_i) : i \in \mathbb{N} \}$.

* In particular, if $X_i = \mathbb{R}$ ~~$\forall i \in \mathbb{N}$~~, $d_i = |\cdot|$ $\forall i \in [n]$. then $\prod_{i=1}^{n} X_i = \mathbb{R}^n$, $M_2$ is the usual Euclidean metric.

* If $n = 2$, $M_1$ is defined by
$M_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$
[Taxicab metric]

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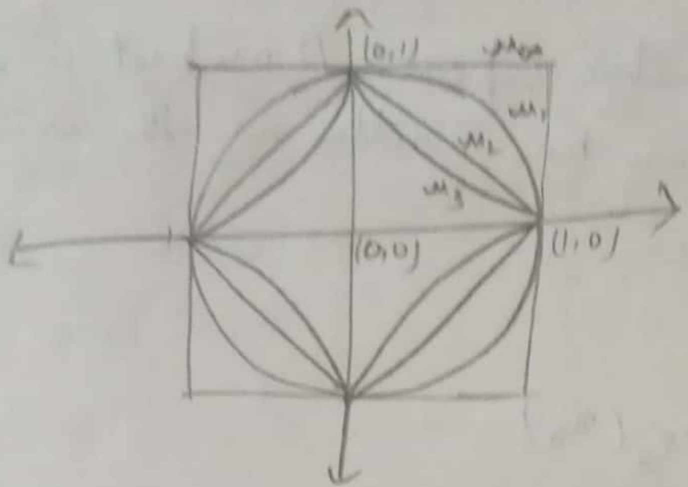
**Definition: (Open Sphere)**

Let $(X, d)$ be a metric space. An open sphere centred at $x_0 \in X$ with radius $\varepsilon > 0$ is defined by $S_\varepsilon(x_0) = \{ x \in X : d(x_0, x) < \varepsilon \}$.

**Example:** $X = \mathbb{R}^2$

$$S_1((0,0)) = \begin{cases} |x| + |y| < 1 & \text{for } M_1 \\ |x|^2 + |y|^2 < 1 & \text{for } M_2 \\ |x|^3 + |y|^3 < 1 & \text{for } M_3 \\ \max\{|x|, |y|\} < 1 & \text{for } M_\infty \end{cases}$$




## Open Set:

**Defn:** Let $S \subseteq (X,d)$. $S$ is open if $\forall\, s \in S$, $\exists\, \varepsilon > 0$ st $S_\varepsilon(s) \subseteq S$.

**Theorem:** In a metric space $(X,d)$, $X$ and $\phi$ are open.

**Theorem:** In a metric space $(X,d)$, every open sphere is an open set.

**Proof:** Let $\varepsilon > 0$ be given and $x_0 \in X$.

Consider $S_\varepsilon(x_0)$.

Let $x \in S_\varepsilon(x_0)$ be arbitrary.

~~Put $r < \frac{1}{2}\min\{d(x,x_0), \varepsilon - d(x,x_0)\}$~~ Put $r = \varepsilon - d(x,x_0)$ (why?)

**Claim:** $S_r(x) \subset S_\varepsilon(x_0)$.

Let $y \in S_r(x)$ be arbitrary.

$\therefore\ d(y,x) < r = \varepsilon - d(x,x_0)$

$\Rightarrow d(y,x) + d(x,x_0) < \varepsilon$

$\Rightarrow d(y,x_0) < \varepsilon \Rightarrow y \in S_\varepsilon(x_0)$ $\square$

Theorem:

Let $(X, d)$ be a metric space. A subset $G$ of $X$ is open iff $G$ can be written as the union of open spheres.

Proof:

$\Leftarrow$ Suppose $G = \bigcup_{\lambda \in \Lambda} S_{\varepsilon_\lambda}(x_\lambda)$

Take $x \in G$

$\therefore \exists \lambda' \in \Lambda$ st $x \in S_{\varepsilon_{\lambda'}}(x_{\lambda'})$.

Since $S_{\varepsilon_{\lambda'}}(x_{\lambda'})$ is open, $\exists r > 0$ st

$S_r(x) \subseteq S_{\varepsilon_{\lambda'}}(x_{\lambda'}) \subseteq G$.

$\therefore G$ is open.

$\Rightarrow$ Suppose $G$ is open.

Take $x \in G$. Then $\exists \varepsilon_x > 0$ st $S_{\varepsilon_x}(x) \subseteq G$.

$\therefore \bigcup_{x \in G} S_{\varepsilon_x}(x) \subseteq G$.

Also, $G \subseteq \bigcup_{x \in G} S_{\varepsilon_x}(x)$ [by def$^n$]

$\therefore G = \bigcup_{x \in G} S_{\varepsilon_x}(x)$ $\square$

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Theorem: (i)

Let $(X, d)$ be a metric space. Then any union of open subsets of $X$ is open in $X$.

(ii) Any finite intersection of open subsets of $X$ is open in $X$.

Proof:

(ii) Let $G_1, G_2, \ldots, G_n$ be open in $X$.

Take $x \in \bigcap_{i=1}^{n} G_i = G$.

Then $x \in G_i \quad \forall i \in [n]$.

$\therefore \exists\, r_i > 0$ st $S_{r_i}(x) \subseteq G_i$ [$\because G_i$'s are open]

Let $r = \min\{r_i \mid i \in [n]\}$.

Then if $z \in S_r(x)$

$\Rightarrow d(x, z) < r \leq r_i \quad \forall i \in [n]$

$\Rightarrow z \in S_{r_i}(x) \quad \forall i \in [n]$
$\subseteq G_i$

$\Rightarrow z \in G_i \quad \forall i \in [n]$

$\Rightarrow z \in \bigcap_{i=1}^{n} G_i$

$\therefore S_r(x) \subset \bigcap_{i=1}^{n} G_i = G$

$\square$

## Limit Point:

- **Defⁿ**: $S \subseteq (X, d)$. An element $x \in X$ is said to be a limit point of $S$ if $\forall\, \varepsilon > 0$, $\boxed{S_\varepsilon(x) \setminus \{x\} \cap S \neq \phi}$.

## Closed set:

A set $F \subseteq (X, d)$ is closed if it contains each of its limit point.

**Theorem:**

$\phi$ and $X$ are closed subsets of $(X, d)$.

## Induced metric:

$Y \subseteq (X, d)$ $\quad d_Y(y_1, y_2) = d(y_1, y_2)$ $\forall\, y_1, y_2 \in Y$

* Let $X = \mathbb{R}$, $d = |\cdot|$

Then $Y = [0, 1) \subseteq \mathbb{R}$ is neither closed nor open

But $(Y, d_Y)$ is both open and closed.

(subset of a metric space with induced metric ~~can be~~ is always a metric space)

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**Theorem**: $F \subseteq X$ is closed iff $X \setminus F$ is open.

**Proof**; Suppose $F$ is closed.

Let $y \in X \setminus F$.

$\therefore y$ is not a limit pt. of $F$.

Then $\exists\, \varepsilon > 0$ st $S_\varepsilon(y) \setminus \{y\} \cap F = \phi$.

$\Rightarrow S_\varepsilon(y) \setminus \{y\} \subseteq X \setminus F$

$\Rightarrow S_\varepsilon(y) \subseteq X \setminus F$

$\therefore X \setminus F$ is open.

Conversely, let $X \setminus F$ is open.

Let $x$ be a limit point of $F$, but $x \notin F$.

$\Rightarrow x \in X \setminus F$.

$\Rightarrow \exists\, \varepsilon > 0$ st $S_\varepsilon(x) \subseteq X \setminus F$ [since $X \setminus F$ is open]

$\Rightarrow S_\varepsilon(x) \setminus \{x\} \cap F = \phi$

$\Rightarrow x$ is not a limit pt of $F$. $\rightarrow\leftarrow$

$\therefore x \in F$.

$\therefore F$ contains all of its limit points.

$\therefore F$ is closed. $\square$

---

Theorem:

Let $x_0 \in X$ and $\varepsilon > 0$. Then the closed sphere centered at $x_0$ with radius $\varepsilon$,

$S_\varepsilon[x_0] = \{x \in X \mid d(x, x_0) \leq \varepsilon\}$ is a closed set.

Proof: It is equivalent to show that $X \setminus S_\varepsilon[x_0]$ is open.

Let $y \in X \setminus S_\varepsilon[x_0]$.

Then $d(x_0, y) > \varepsilon$

Define, $r = d(x_0, y) - \varepsilon > 0$.

Claim; $S_r(y) \subseteq X \setminus S_\varepsilon[x_0]$.

Let $z \in S_r(y)$.

~~Then $d(z,y) < r < d(x_0,y)$~~

Now, $d(z,x_0) + d(z,y) \geq d(x_0,y)$

---

Theorem; In a metric space —

(i) Arbitrary intersection of closed sets is closed.

(ii) Finite union of closed sets is closed.

Proof;

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## Sequence :

(convergence)

**Def$^n$:** A sequence $\{x_n\} \subseteq X$ is said to converge to a point ~~in~~ $x \in X$ if $\forall \varepsilon > 0$ $\exists n_\varepsilon \in \mathbb{N}$ st $x_n \in S_\varepsilon(x)$ $\forall n \geq n_\varepsilon$.

(cauchy)

**Def$^n$:** $\{x_n\} \subseteq X$ is said to be Cauchy if $\forall \varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ st $d(x_m, x_n) < \varepsilon$ $\forall m, n \geq n_0$.

**Theorem:** [Every convergent sequence is Cauchy.]

**Proof:** $x_n \to x$, $\varepsilon > 0$ be given

~~[illegible]~~ $\exists n_0 \in \mathbb{N}$ $\ni$ $\forall n \geq n_0$, $d(x_n, x) < \varepsilon/2$.

Take $m \geq n_0$.

Then $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon$

**Example:** $([0,1), |\cdot|)$

$x_n = 1 - \frac{1}{n}$, $x_n \to 1 \notin [0,1)$

$\{x_n\}$ is cauchy.

(complete)

**Def$^n$:** A metric space $(X, d)$ is complete if every Cauchy sequence $\{x_n\} \subseteq X$ converges in $X$.

* [$l^p$ is complete]