Real Analysis Chapter 5 Study Guide (for "Real Analysis, A First Course", 2nd Edition, Russell A. Gordon)

Number of Starred Exercises: **3**; Number of Notes: 6; Number of Other (non-starred) Exercises: 40; Minimum Number of Other (non-starred) Exercises to Do (to do at least 25% of them): **10**

The most important things to get out of this chapter: (1) The definition of what it means for a function to be Riemann integrable. (2) An equivalent condition for Riemann integrability (Theorem 5.10). (3) Conditions for Riemann integrability (Theorems 5.11 and 5.12) and non-Riemann integrability (the contrapositive of Theorem 5.7). (4) The Fundamental Theorem of Calculus (FTC) and its importance. (5) The Mean Value Theorems for Integrals.

Other matters of importance:

1. Basic Properties of Integrals.
2. Evaluating Integrals.
3. Enjoying Integrals (no joke).

Reading Guide:

1. *Before reading Chapter 5, try to write down the definition of a definite integral $\int_a^b f(x)dx$ as you recall it from Calculus. Try not to peek at any sources.
2. Think about how you might try to teach a young child about the concept of area, especially for an irregularly shaped object. It's not an easy thing to do, is it? Volume might actually be easier. You could just focus on the volume of water displaced in a rectangular tub when you place the object in the tub. Why is describing the meaning of area so difficult? Think about it and write down some thoughts.
3. Do you think it's possible for an area under the graph of a function to "not exist"? Think about some of the strange and wild functions we've looked at.
4. Write an equation expressing a tagged partition of the interval [3, 7] into 8 equal sized sub-intervals with the right-hand endpoints being the tags. Write the form of a general Riemann sum of an arbitrary function *f* for such a tagged partition. If $f(x) = \xi^3$, find the value of this Riemann sum.
5. *Spend some time working on memorizing Definition 5.4. Once you think you have it memorized, try to write it down without looking and then compare what you wrote with what's in the book. Try to write it down again a day later without looking and compare. Do it again a day later. Maybe even on a fourth day. (You can write these on different pages in your journal).
6. Compare Definition 5.4 with your answer to #1 above and what you find in calculus textbooks. Notice any differences?
7. Look up Bernhard Riemann on the Internet and write down a few things you find out about him.

8. To illustrate how difficult Definition 5.4 is to use. Try to use it to prove that the function $h(x) = \xi$ is Riemann integrable on [0,1]. This will be an extra credit problem on homework (see Exercise #10 in Section 5.1).
9. How are the oscillation and the variation of a monotone function on a closed interval [a, b] related?
10. Write down the contrapositive of the statement in Theorem 5.7. This contrapositive is a true statement as well (since the contrapositive of a statement is equivalent to the statement).
11. Make an outline of the overall strategy used in the proof of Theorem 5.7.
12. Draw detailed pictures that help you understand the proof of Theorem 5.7.
13. Fill in any details that help you understand the proof of Theorem 5.7.
14. Outline and describe the overall strategy used in the proof of Theorem 5.8 (the part of the proof that's in the book). Then fill in details to help you understand.
15. Prove the converse of the statement that is proved for Theorem 5.8.
16. Explain why it is sufficient to consider the case in which $P_1 = \{\alpha, \beta\}$ in the proof of Lemma 5.9.
17. Explain why $\left|\sum_{\imath=1}^{\pi} f(t_i)(x_i - \xi_{\imath-1}) - \phi(\varpi)(\beta - \alpha)\right| = \left|\sum_{\imath=1}^{\pi} (\phi(\tau_\imath) - \phi(\varpi))(\xi_\imath - \xi_{\imath-1})\right|$ in the proof of Lemma 5.9.
18. Do Exercise #6 from Section 5.2 (given an area interpretation of Theorem 5.10 and explain how it shows the two geometric approaches to the area under a curve are equivalent).
19. Outline the proof of Theorem 5.10 and fill in confusing details from this proof (this might be one of the hardest proofs in the whole book). Think about a strategy for how you might prove the converse of the statement actually proved here.
20. **Note:** carefully read and make a note of the theme of the paragraph after the proof of Theorem 5.10. You now have two ways to prove that a given function is Riemann integrable. The second way (using the condition in Theorem 5.10) may often be easier. You might try testing this to see if the proof that $h(x) = \xi$ is Riemann integrable over [0, 1] is any easier now (the extra credit problem #10 from Section 5.1)
21. Try proving Theorem 5.11 without looking at the book's proof (Hints: use Theorem 5.10 and also make use of the Uniform Continuity Theorem…Theorem 3.28).
22. Prove Theorem 5.12. Again, Theorem 5.10 may be helpful. Note that a monotone function is not necessarily continuous (and a continuous function is not necessarily monotone).
23. Prove any part of Theorem 5.13 that interests you.
24. *Try to recall and precisely write down the Fundamental Theorem of Calculus (FTC) without looking at any references. You may also recall that there are actually two parts to this theorem in most calculus books. Try to write down both statements.
25. Prove Theorem 5.14 (by this point, this should be one of the easier proofs in Chapter 5).

26. **Note:** take care to note the fact pointed out in the last sentence of page 177 which goes to the top of page 178. Can you explain this comment?
27. Prove part (a) of Theorem 5.15.
28. **Note:** in the proof of part (b) of Theorem 5.15, it's nice to see that, once we finish the hard work of showing that *f* is Riemann integrable over [a, b], it's relatively easy to apply Theorem 5.14 to show that the equation in part (b) of Theorem 5.15 is true.
29. **Note:** in many (if not most) calculus textbooks, part (b) of Theorem 5.17 is labeled FTC, Part I and part (a) of Theorem 5.17 is labeled FTC, Part II.
30. Use part (a) of the FTC to prove part (b) of the FTC.
31. The proof of the FTC is rather ingenious (or, for that matter, the thinking up of the statement of the FTC is rather ingenious), but its proof isn't really quite as difficult as some of the other proofs earlier in Chapter 5 (even though it looks kind of daunting at first glance). Is this surprising? Should it be surprising? After studying the proof, do you feel like you could reproduce most of it without looking? Do you think you could have come up with it on your own (assuming you had the correct statement of the FTC)?
32. How do the statements in the last couple of sentences on page 180 that go to the top of page 181 "jive" with the statement of the FTC, part (a)? Don't these statements contradict each other? Can you resolve this paradox?
33. Meditate (seriously…☺) on the paragraph in the middle of page 181. *Ponder it's cosmic significance*. Think about it's "physical" meaning. *Enjoy it*. This is *deep* stuff. It's one of humanities greatest discoveries.
34. Start thinking about how to prove Theorems 5.18 and 5.19 (these will eventually be exercises to hand in anyway).
35. In the proof of Theorem 5.20, explain whey the proof is complete when the author writes "This completes the proof".
36. Draw pictures to help you understand the proof of Theorem 5.21(b).
37. Why can the condition in the first sentence of the paragraph after the proof of Theorem 5.21 be described as "the final word on the class of Riemann integrable functions"? Does its statement "jive" with, for example, Theorem 5.12? Explain.
38. Write down a formula for $f \circ g$ before the statement of Theorem 5.22. Do you see why it is not Riemann integrable?
39. In the proof of Theorem 5.22, why does *f* have a bound on [*c*, *d*]? Why is *f* uniformly continuous on [*c*, *d*]?
40. Verify the inequalities on the very bottom line of page 185.
41. Explain the second inequality (at the beginning of the third line of equations/inequalities) near the top of page 186.
42. How is the statement of Theorem 5.23 related to the concept of the average value of a Riemann integrable function? Rewrite Theorem 5.23 in an equivalent way using this relation. Draw a picture if it is helpful, especially in the case where $f(x)$ is always nonnegative.
43. Make up your own examples for *f* and *g* and find a value of *c* that works to satisfy the conclusion of the Generalized Mean Value Theorem for Integrals.

44. **Note:** take note of and enjoy the fact that both the Extreme Value Theorem and the Intermediate Value Theorem are used in the proof of Theorem 5.24 (the Completeness Axiom in action again!).
45. **Note:** the "inverse image" or "preimage" of a point in the range is the set of all points that get mapped to it. For example, if *y* is in the range of $f:[a,b]\to \circ$, then the inverse image of *y* is $\{x \in [\alpha,\beta] : \phi(\xi) = \psi\}$. This set is often denoted by $f^{-1}(\psi)$ or $f^{-1}(\{\psi\})$, even when the function *f* is not invertible (not one-to-one).
46. The proof of Theorem 5.26 is trivial. Do you believe this statement? Write out a brief proof for each part of the theorem.
47. Can you imagine how a proof of Theorem 5.27 might go? Perhaps by using induction on the number of subintervals in the partition? You don't have to do it. Just think about it.
48. Draw a picture to illustrate Theorem 5.28. Note that the φ and the ψ have to be "close together" in order for $\int_{\alpha}^{\beta}(\psi - \varphi) < \varepsilon$.
49. Draw pictures to help you understand the proof of Theorem 5.28. Hopefully this proof, though somewhat lengthy, is actually fairly easy for you to understand after having gone through the "gauntlet" that is Chapter 5.

Deep Thoughts to Ponder (but not necessarily answer):

- Integrals are a deep subject. Take some time to enjoy how deep and rich the theory is…especially the FTC. God has structured the universe in ways we can understand and "measure" (more advanced integration theory is related to something called measure theory). This is really cool!
- Is the fact that some (even bounded) functions are not Riemann integrable somehow problematic? "Should" they be? Do you feel informed enough to even have an opinion on such questions?