A collection of finite nonempty sets $S_1, \cdots, S_n$ has a system of distinct representatives if there exist n distinct elements $x_1, \cdots, x_n$ such that $x_i \in S_i$ for $1 \le i \le n$.

Th^m A collection $\{S_1, \cdots, S_n\}$ of finite nonempty sets has a system of distinct representatives if and only if for each integer k with $1 \le k \le n$ the union of any k-sets contains atleast k-elements.

Proof: Assume first that $\{S_1, \cdots, S_n\}$ has a system of distinct representatives. Then for each integer k with $1 \le k \le n$, the union of any k of these sets contains at least k elements.

Conversely, suppose that $\{S_1, \cdots, S_n\}$ is a collection of n sets such that for each integer k with $1 \le k \le n$, the union of any k of these sets contains at least k-elements. We now consider the bipartite graph G with bipartite sets

$$U = \{S_1, \cdots, S_n\} \quad \text{and} \quad W = S_1 \cup S_2 \cup \cdots \cup S_n$$

such that a vertex $S_i (1 \le i \le n)$ in U is adjacent to a vertex $w \in W$ if $w \in S_i$. Let X be any subset of U with $|X| = k$, $1 \le k \le n$. Since the union of any k sets in U contains at least k elements, it follows that $|N(X)| \ge |X|$. Thus G satisfies Hall's condition. So G contains a matching from U to a subset of W. This matching pairs of the sets $S_1, \cdots, S_n$ with n distinct elements in $S_1 \cup S_2 \cup \cdots \cup S_n$, producing a system of distinct representatives for $\{S_1, \cdots, S_n\}$.

A component of a graph is odd or even according to whether its order is odd or even. The number of odd components in G is denoted by $\omega_o(G)$.

$\underline{\text{Th}}^m$ A nontrivial graph G contains a perfect matching iff $\omega_o(G-S) \le |S|$ for every proper subset S of $V(G)$.

__Proof__ First suppose that G contains a perfect matching M. Let S be a proper subset of $V(G)$. If $G-S$ has no odd components, then $\underset{0}{\underset{\shortparallel}{\omega_o(G-S)}} \le |S|$. Thus we may assume that $\omega_o(G-S) = k \geq 1$. Let $G_1, \dots, G_k$ be ~~odd~~ the odd components of $G-S$. (There may be some even components of $G-S$ as well.) For each component G_i of $G-S$, there is atleast one edge M joining a vertex of G_i and a vertex of S. Thus, $\omega_o(G-S) \le |S|$.

Conversely, let G be a graph such that $\omega_o(G-S) \le |S|$ for every proper subset S of $V(G)$. In particular, $\omega_o(G-\phi) \le |\phi| = 0$, implying that every component of G is even and so G itself has even order. We now show that G has a perfect matching by employing induction on the order (even) of G.

Since K_2 is the only graph of order 2 having no odd components and K_2 has a perfect matching, the base case is verified.

For a given even integer $n \geq 4$, assume that all graphs H of even order less than n and satisfying $\omega_o(H-S) \le |S|$ for every proper subset S of $V(H)$ contains a perfect matching.

Now let G be a graph of order n satisfying $\omega_0(G-S) \le |S|$ for every proper subset S of $V(G)$. As above, every component of G has even order. We show that G has a perfect matching.

For a vertex v of G that is not a cut vertex and $R = \{v\}$ it follows that $\omega_0(G-R) = |R| = 1$. Hence there are nonempty proper subset T of $V(G)$ for which $\omega_0(G-T) = |T|$. Among all such sets T, let S be one of maximum cardinality. Suppose that $\omega_0(G-S) = |S| = k \geq 1$ and let $G_1, \dots, G_k$ be the odd components of $G-S$.

Claim $\omega(G-S) = k$, i.e. $G_1, \dots, G_k$ are the only components of $G-S$. Assume to the contrary, that $G-S$ has an even component G_0. Let v_0 be a vertex of G_0 that is not a cut vertex of G_0. Let $S_0 = S \cup \{v_0\}$

Since G_0 has even order

$$\omega_0(G-S_0) \geq \omega_0(G-S) + 1 = k+1,$$

which is impossible. Thus $G_1, \dots, G_k$ are the only components of $G-S$.

For each integer i $(1 \le i \le k)$, let S_i denote the set of those vertices in S adjacent to at least one vertex of G_i. Since G has only even components, each set S_i is nonempty.

Claim For each integer l with $1 \le l \le k$, the union of any l of the sets $S_1, \dots, S_k$ contains at least l vertices.

Assume to the contrary, that this is not the case. Then there is an integer j such that the union ~~of~~ S' of j of the sets $S_1, \dots, S_k$ has fewer than j elements. Suppose $S_1, S_2, \dots, S_j$ have this property. Thus

$$S' = S_1 \cup S_2 \cup \dots \cup S_j \quad \text{and} \ |S'| < j$$

Then $G_1, G_2, \dots, G_j$ are atleast some of the components of $G - S'$ and so $\omega_o(G - S') \geq j > |S'|$, which contradicts the hypothesis. Thus, for each integer l, $1 \leq l \leq k$, the union of any l ~~[illegible]~~ of the sets $S_1, \dots, S_k$ contains atleast l-vertices.

~~Hence~~ So, there is a set $\{v_1, \dots, v_k\}$ of k distinct vertices of S such that $v_i \in S_i$ for $1 \leq i \leq k$. Since every component G_i of $G - S$ contains a vertex u_i such that $u_i v_i$ is an edge of G, it follows that $\{u_i v_i : 1 \leq i \leq k\}$ is a matching of G.

We now show that for each non trivial components G_i of $G - S$ $(1 \leq i \leq k)$ the graphs $G_i - u_i$ contains a perfect matching. Let W be a proper subset of $V(G_i - u_i)$. We claim that

$$\omega_o(G_i - u_i - W) \leq |W|.$$

Assume to the contrary, that $\omega_o(G_i - u_i - |W|) > |W|$. Since G_i has odd order, $G_i - u_i$ has even order and so $\omega_o(G_i - u_i - W)$ and $|W|$ are both even or both odd. Hence $\omega_o(G_i - u_i - W) \geq |W| + 2$.

Let $X = S \cup W \cup \{u_i\}$. Then

$$|X| = |S| + |W| + 1 = |S| + (|W|+2) - 1$$

$$\leq \omega_0(G-S) + \omega_0(G_i - u_i - W) - 1$$

$$= \omega_0(G-X) \leq |X|$$

which implies that $\omega_0(G-X) = |X|$ and contradicts the defining property of S. Thus $\omega_0(G_i - u_i - W) \leq |W|$

Therefore by induction hypothesis, for each nontrivial component G_i of $G-S$, the graph $G_i - u_i$ ~~for~~ $(1 \leq i \leq k)$ has a perfect matching. The collection of perfect matchings of $G_i - u_i$ for all nontrivial graphs G_i of $G-S$ together with the edges in $\{u_i v_i : 1 \leq i \leq k\}$ produce a perfect matching.

Thm Every bridgeless (3-regular) cubic graph contains a perfect matching.

Proof Let G be a bridgeless cubic ~~gr~~graph, and let S be a proper subset of $V(G)$ with $|S| = k$. We show that $\omega_0(G-S) \leq |S|$. This is true if $G-S$ has no odd components; so we assume that $G-S$ has $l \geq 1$ odd components, say $G_1, \ldots, G_l$.

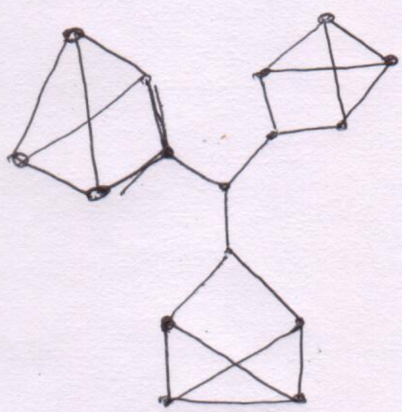
Let E_i $(1 \leq i \leq l)$ denote the set of edges joining the vertices of G_i and the vertices of S. Since G is cubic, every vertex of G_i has degree 3 in G. Because the sum of the degrees in G of the vertices of G_i is odd and the sum of the degrees in G_i ~~is eve~~

of the vertices of G_i is even, it follows that $|E_i|$ is odd. Because G is bridgeless, $|E_i| \neq 1$ and so $|E_i| \geq 3$ for $1 \leq i \leq l$. This implies that there are at least $3l$ edges joining the vertices of $G-S$ and the vertices of S. Since $|S| = k$, at most $3k$ edges join the vertices of $G-S$ and the vertices of S. Thus

$$3\omega_o(G-S) = 3l \leq 3k = 3|S|$$

and so $\omega_o(G-S) \leq |S|$. Thus G has a perfect matching.

Ex



A cubic graph with no perfect matching.