Chromatic Polynomials

For a graph G and a given set of λ colors, the function $f(G;\lambda)$ is defined to be the number of ways of vertex coloring G properly using the λ colors

Hence $f(G;\lambda) = 0$ when G has no proper λ-coloring.

Clearly, the minimum λ for which $f(G;\lambda) > 0$ is the chromatic number $\chi(G)$ of G.

- $f(K_n;\lambda) = \lambda(\lambda-1) \cdots (\lambda-n+1)$ for $\lambda \geq n$
- $f(\bar{K}_n;\lambda) = \lambda^n$

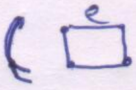
Let $e = \{u, v\}$ be an edge of G. The graph $G \cdot e$ is obtained from G by contracting the edge e.

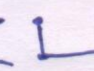
$\underline{\underline{Th^m}}$ Let G be any graph. Then $f(G;\lambda) = f(G-e;\lambda) - f(G\cdot e;\lambda)$ for any edge e of G.

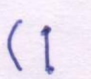
$\underline{\underline{\text{Proof}}}$ $f(G-e;\lambda)$ denotes the number of proper colorings of $G-e$ using λ colors. Hence it is the sum of the number of proper colorings of $G-e$ in which u and v receive the same color and the number of proper colorings of $G-e$ in which u and v receive distinct colors. The former number is $f(G\cdot e;\lambda)$ and the latter number is $f(G;\lambda)$. $\square$

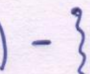
Note that if G and H are disjoint, then

$$f(G\cup H;\lambda) = f(G;\lambda)\, f(H;\lambda)$$

<u>Example</u> $f(C_4;\lambda) = ($  $)$

$= ($  $) - ($  $)$

$= ($  $) - ($  $) - ($  $)$

$= ($  $) - \{($  $) - ($  $)\} - ($ 

$$= (\lambda(\lambda-1))^2 - \{\lambda^2(\lambda-1) - \lambda(\lambda-1)\} - \lambda(\lambda-1)(\lambda-2)$$

$$= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda$$

$$\boxed{f(C_4;\lambda) = \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda}$$

$f(G;\lambda)$ is called the chromatic polynomial of G

$\underline{\text{Th}^m}$ For a simple graph G of order n and size m, $t(G;\partial)$ is a monic polynomial of degree n in ∂ with integer co-efficients and constant term zero. In addition, its co-efficients alternate in sign and the co-efficient of ∂^{n-1} is $-m$.

Proof We proceed by induction on number of edges.

If $m=0$, G is $\bar{K}_n$ and $t(\bar{K}_n,\partial)=\partial^n$ and the statement of the theorem is trivially true in this case.

Suppose now that the theorem holds for all graphs with fewer than m edges, where $m\geq 1$.

Let G be any simple graph of order n and size m and let e be an edge of G. Both $G-e$ and $G\cdot e$ (after removal of multiple edges, if necessary) are simple graphs with atmost $m-1$ edge, and hence by induction hypothesis,

$$t(G-e;\partial) = \partial^n - a_0\partial^{n-1} + a_1\partial^{n-2} + \cdots + (-1)^{n-1}a_{n-2}\partial,$$

and

$$t(G\cdot e\partial) = \partial^{n-1} - b_1\partial^{n-2} + \cdots + (-1)^{n-2}b_{n-2}\partial$$

where $a_0,\ldots a_{n-2}; b_1,\ldots, b_{n-2}$ are nonnegative integers (so that the co-efficients alternate in sign), and a_0 is the number of edges in $G-e$, which is $m-1$.

Then $t(G;\partial) = t(G-e;\partial) - t(G\cdot e,\partial)$

$$= \partial^n - (a_0+1)\partial^{n-1} + (a_1+b_1)\partial^{n-2} - \cdots + (-1)^{n-1}(a_{n-2}+b_{n-2})\partial$$

Since $a_0 + 1 = m$, $f(G;\lambda)$ has all the stated properties.

Th^m A simple graph G on n-vertices is a tree if and only if $f(G;\lambda) = \lambda(\lambda-1)^{n-1}$.

Proof: Let G be a tree. We prove that $f(G;\lambda) = \lambda(\lambda-1)^{n-1}$ by induction on n. If $n=1$ the result is trivial.

Assume ~~that~~ the result for trees with at most $(n-1)$ vertices, $n \geq 2$.

Let G be a tree with n vertices and e be an pendant edge of G. We have $f(G;\lambda) = f(G-e;\lambda) - f(G \cdot e;\lambda)$.

Since $G-e$ is a forest with two components trees of orders $n-1$ and 1, and hence $f(G-e;\lambda) = \lambda \cdot \lambda(\lambda-1)^{n-2} = \lambda^2(\lambda-1)^{n-2}$

Also $G \cdot e$ is tree with $n-1$ vertices, so $f(G \cdot e;\lambda) = \lambda(\lambda-1)^{n-2}$

Thus $f(G;\lambda) = \lambda^2(\lambda-1)^{n-2} - \lambda(\lambda-1)^{n-2} = \lambda(\lambda-1)^{n-2}(\lambda-1) = \lambda(\lambda-1)^{n-1}$.

Conversely, assume that G is a simple graph with $f(G;\lambda) = \lambda(\lambda-1)^{n-1} = \lambda^n - (n-1)\lambda^{n-1} + \cdots + (-1)^{n-1}\lambda$.

Thus G has n-vertices and $n-1$ edges. Further the last term $(-1)^{n-1}\lambda$, ensure that G is connected.

Hence G is a tree.

$\underline{\text{Def}^n}$ A graph G is triangle-free if G contains no K_3.

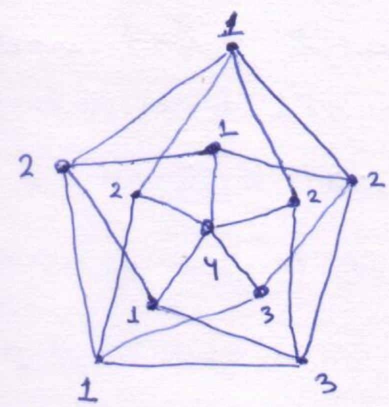
$\underline{\text{Th}^m \text{ (Mycielski)}}$

For every positive integer k, there exists a triangle free graph with chromatic number k.

$\underline{\text{Proof}}$ Since no graph with chromatic number 1 or 2 contains a triangle, the theorem is obviously true for $k=1$ and $k=2$

To verify the theorem for $k \geq 3$, we proceed by induction on k. Since $\chi(C_5)=3$ and C_5 is triangle free, the statement is true for $k=3$

Assume that there exists a triangle-~~free~~ graph with chromatic number k, where $k \geq 3$. We show that there exists a triangle-free $(k+1)$ chromatic graph. Let H be a triangle free graph with $\chi(H)=k$, where $V(H)=\{v_1, \ldots, v_n\}$. We construct a graph G from H by adding $n+1$ new vertices $u, u_1, \ldots, u_n$, joining u to each vertex u_i $(1 \leq i \leq n)$ and joining u_i to each neighbour of v_i in H.



Grötzsch graph

— A 4 chromatic triangle free graph

We claim that G is triangle-free $(k+1)$-chromatic graph. First we show that G is triangle free. Since $\{u_1, \dots, u_n\}$ is an independent set of vertices of G and u is adjacent to no vertex of H, it follows that u belongs to no triangle in G. Hence if there is a triangle T in G, then two of the three vertices must belong to H and the third vertex must belong to S, say $V(T) = \{u_i, v_j, v_k\}$. Since u_i is adjacent to v_j and v_k, it follows that v_i is adjacent to v_j and v_k. Since v_j and v_k are adjacent, H contains a triangle which is a contradiction. Thus G is triangle free.

Next we show that $\chi(G) = k+1$. Since H is a subgraph of G and $\chi(H) = k$, it follows that $\chi(G) \geq k$. Let a k-coloring of H is given and assign to u_i the same color that is assign to v_i, $1 \leq i \leq n$. Assigning the color $k+1$ to u produce a $(k+1)$-coloring of G and so $\chi(G) \leq k+1$. Hence eithe $\chi(G) = k$ or $k+1$. Suppose $\chi(G) = k$, then there is a k-coloring of G with colors $1, 2, \dots, k$, where u is assigned to color k, say. Necessarily non of the vertices $u_1, \dots, u_n$ is assigned the color k; that is, each vertex of S is assigned one of the color $1, \dots, k-1$. Since $\chi(H) = k$, one or more vertices of H are assigned the color k. For each vertex v_i of H colored k, recolor it with the color assigned to u_i. This produces a $(k-1)$ coloring of H, which is impossible. Thus $\chi(G) = k+1$.