Transitive Closure

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Mathematics for Data Science 1
Week 11

Transitive closure of a relation

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- p is an ancestor of q if we can find a sequence of people $r_0, r_1, \ldots, r_n$ such that
  - $p = r_0$
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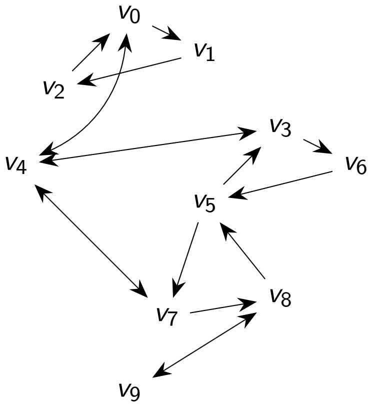
  - $p = r_0$
  - For each $i \in \{0, 1, \ldots, n-1\}$, $(r_i, r_{i+1}) \in R$
  - $q = r_n$
- This is called the transitive closure of R, written R^+
  - $R^+ \subseteq S \times S$ is also a relation
  - R^+ is derived from $R \subseteq S \times S$

Computing transitive closure

- Represent $R \subseteq S \times S$ as a (directed) graph $G = (V, E)$
  - $V = S$
  - $(u, v) \in E$ if and only if $(u, v) \in R$



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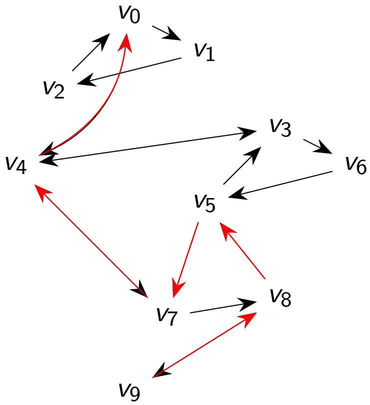

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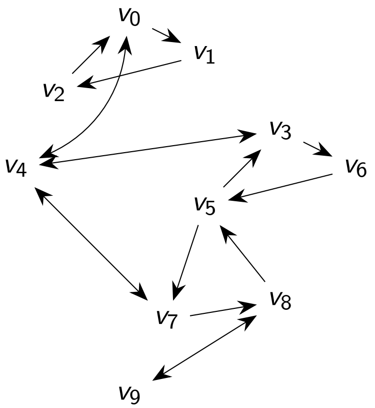

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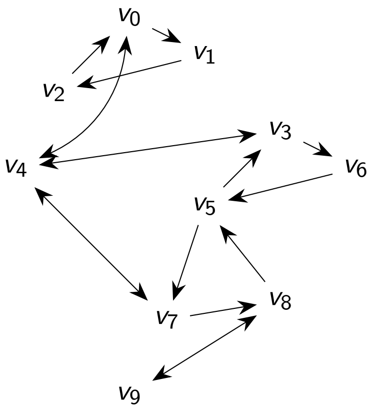
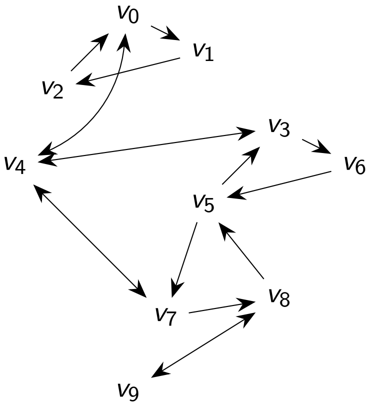
- Perform BFS/DFS from all verties to compute R^+



Using the adjacency matrix

- Another strategy



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- Consider the adjacency matrix A for G

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1	0	0	1	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0
3	0	0	0	0	1	0	1	0	0	0
4	1	0	0	1	0	0	0	1	0	0
5	0	0	0	1	0	0	0	1	0	0
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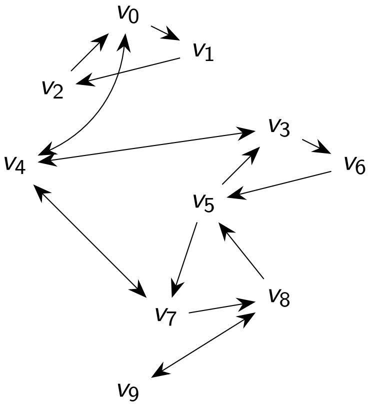

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3	0	0	0	0	1	0	1	0	0	0
4	1	0	0	1	0	0	0	1	0	0
5	0	0	0	1	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	0	0
7	0	0	0	0	1	0	0	0	1	0
8	0	0	0	0	0	1	0	0	0	1
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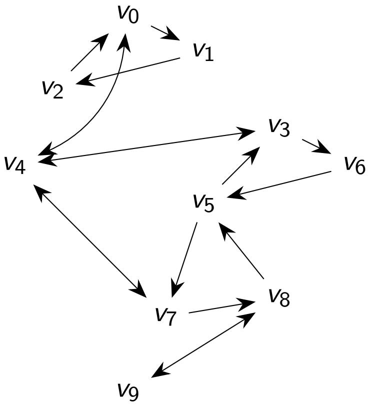
- $A[i,j] = 1$ — path of length 1 from i to j



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4	1	0	0	1	0	0	0	1	0	0
5	0	0	0	1	0	0	0	1	0	0
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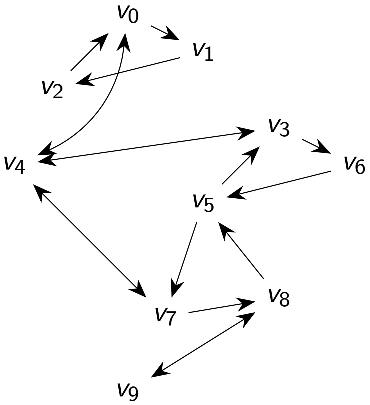

- $A[i,j] = 1$ — path of length 1 from i to j

- Want $A^+[i,j] = 1$ — path of length ≥ 1 from i to j

Paths of length 2

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A

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2	1	0	0	0	0	0	0	0	0	0
3	0	0	0	0	1	0	1	0	0	0
4	1	0	0	1	0	0	0	1	0	0
5	0	0	0	1	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	0	0
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8	0	0	0	0	0	1	0	0	0	1
9	0	0	0	0	0	0	0	0	1	0

A^2

	0	1	2	3	4	5	6	7	8	9
0	1	0	1	1	0	0	0	1	0	0
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3	0	0	0	0	1	0	1	0	0	0
4	1	0	0	1	0	0	0	1	0	0
5	0	0	0	1	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	0	0
7	0	0	0	0	1	0	0	0	1	0
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0	1	0	1	1	0	0	0	1	0	0
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5	0	0	0	1	0	0	0	1	0	0
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- Sufficient to check paths upto length $n-1$

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- This calculation can be described directly using matrix multiplication
- $A^2 = A \times A$, $A^3 = A^2 \times A$, $\ldots$, $A^{\ell+1} = A^\ell \times A$

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- A typical example is the ancestor relation derived from the parent relation
- If we represent a relation as a graph, transitive closure corresponds to reachability
- Reachability between all pairs of vertices can be checked using repeated BFS/DFS starting from each vertex
- Alternatively, we can perform repeated matrix multiplication on the adjacency matrix A, observing that the length of a path is at most $n-1$