

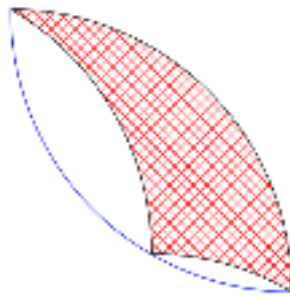
# The Unusual Properties of Tricurves

By

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## Abstract

The paper presents a type of curve-side triangle suitable for planar periodic and nonperiod tiling. While this allows some interesting single-shape tiling, the design may be of more interest because of some unusual mathematical properties. The simple shape has surprising complexity.

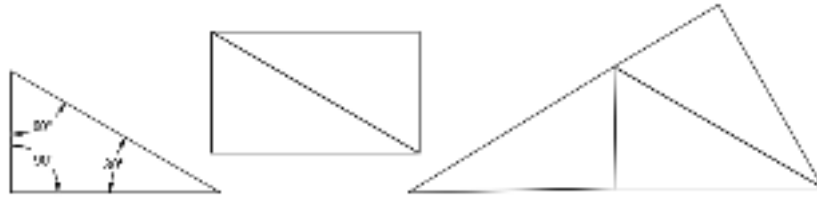
Some definitions and conventions

- All angle values are in degrees.
- All radii are assumed to be the same at one unit, with areas in square units.
- SQRT = square root of

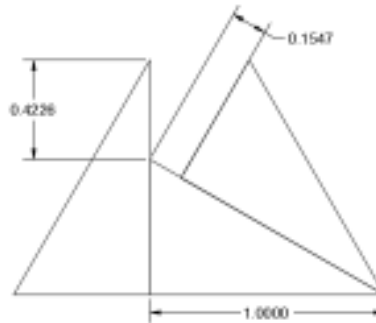
## 1. Intro/Context/Background/Scope

1.1 The paper addresses a family of single shapes that can tile both periodically and nonperiodically: not aperiodically. It deals only with the simplest shapes that can be constructed with straightedge and compass. Mirror images of the shape are allowed.

1.2 The triangle is of course the simplest of shapes. When tiling is done with only a single shape it usually requires same-length sides together, with agreeable angles, as shown below:



In this case, we don't care about the relative lengths of the sides since they don't intermix. But if we try to adjoin unequal sides, we often have odd lengths left over, that won't work with the tiling:



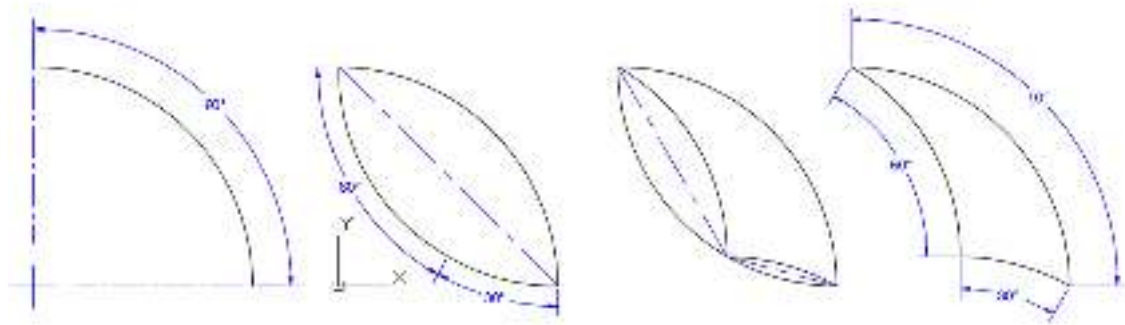
This is because of the Law of Sines, which says the side lengths won't be in the same (agreeable) ratios as the angles. If we start out making the side lengths in agreeable ratios, the angles will be wrong for tiling. The exception of course is the equilateral triangle.

## 2. The concept

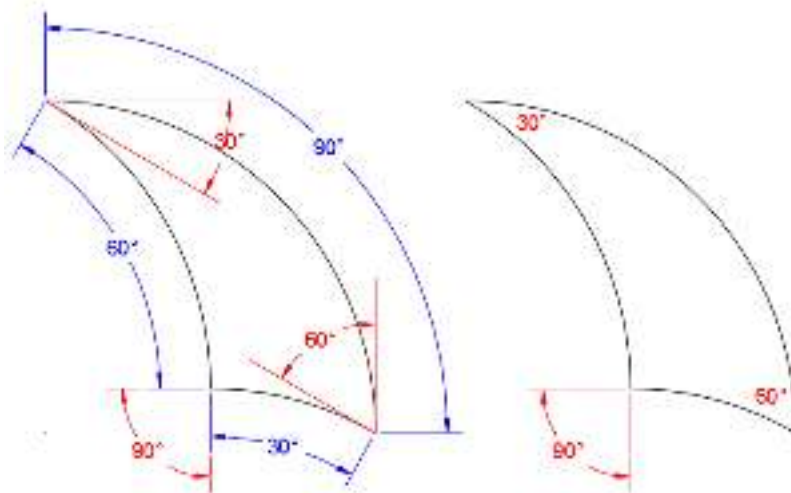
**2.1 Approach.** In this paper I focus on single tiling shapes with curved sides, where all the sides have the same radius. In order for the tiling to work, of course there must be as much concave arc as convex arc. This paper focuses on those with three curved sides, although there are some shapes that have more than three curved sides.

**2.2 Basic construction.** The tricurve can be constructed as shown below starting with a 90-degree arc. The arc is then mirrored about the line between its endpoints, and the new arc is divided into the two parts

such as 30 and 60 degrees. Each of these smaller arcs is in turn mirrored about a line between its endpoints, forming the two final concave portions of the tricurve.



This shape does not obey the Law of Sines, as shown below, since each corner angle (in red) is the same as the opposite "side" arc (in blue). In all cases the corner angle is measured at the joint of the two arcs.

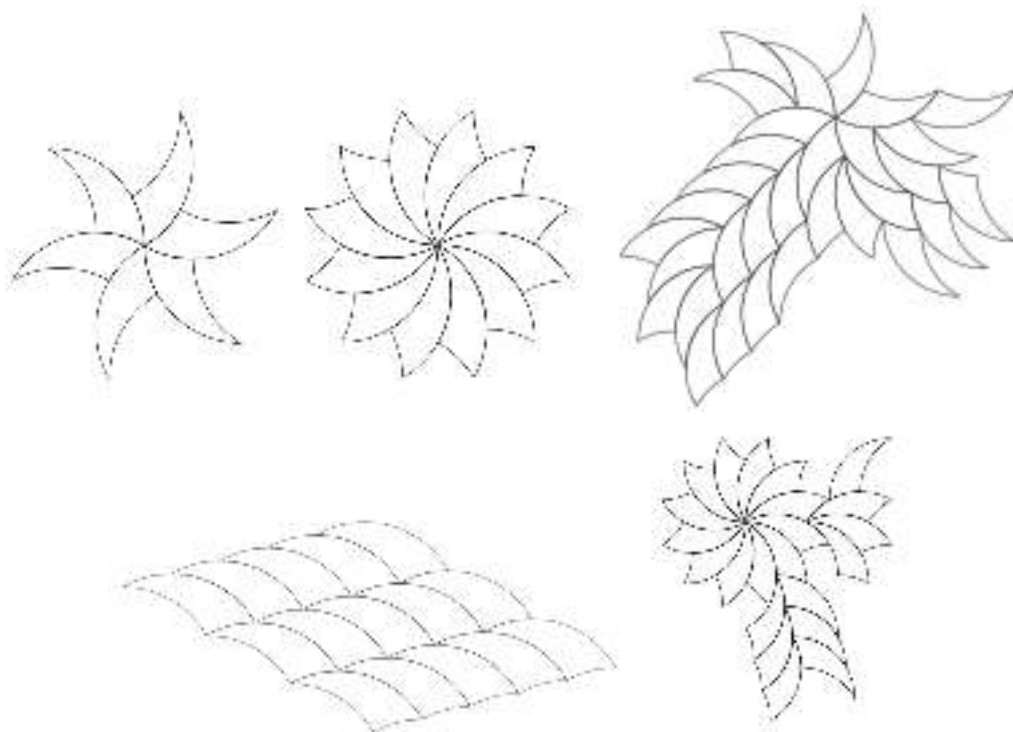


For each of the two smaller angles, the angle is the same whether it is measured between the two tangents or the two perpendiculars of the arc ends; and for the rest of this paper these two angles will simply be shown in the inside corners. But the largest, 90-degree angle must be measured between the perpendiculars of the two arc ends where they join, and not between the tangents. In this case of the 90 degrees these are the same, but for all other angles they are not the same, as will be seen. To make this clear, the largest, corner angle will always be shown outside of the corner, between the perpendiculars.

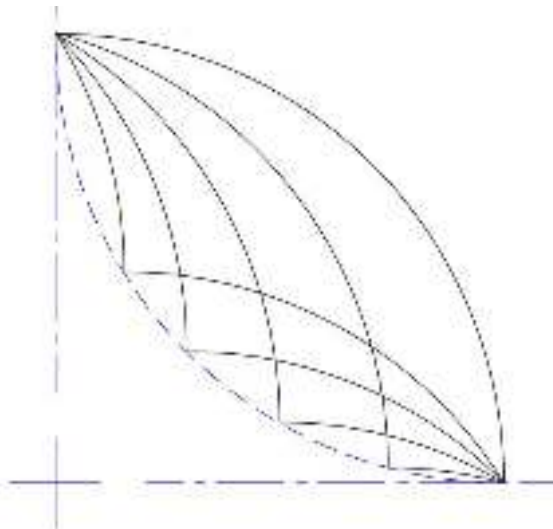
Note the three angles do not add up to 180 degrees as they would for a normal triangle. Instead the two smaller angles add up to the large angle, just as the two arcs add up to the large arc.

This is a 30-60-90 Tricurve. Since its corner angles and arcs are in the same agreeable proportions, it makes for some interesting tiling.

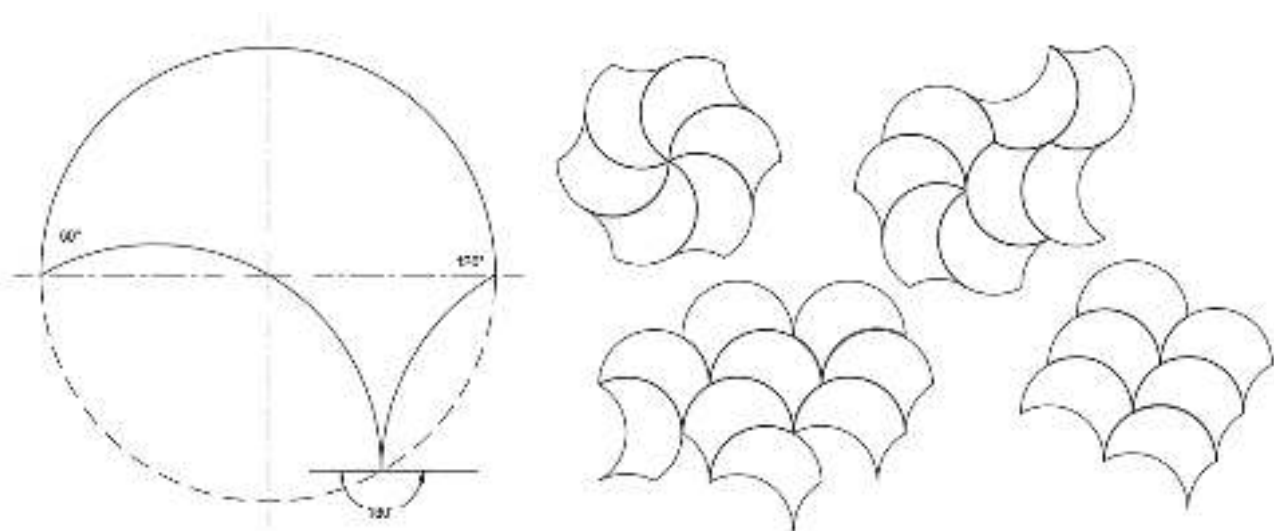
**2.3 Tiling.** Below are some examples of periodic and nonperiodic tiling with the 30/60/90 tricurve. Recall I am allowing the mirror image of the shape to be used.



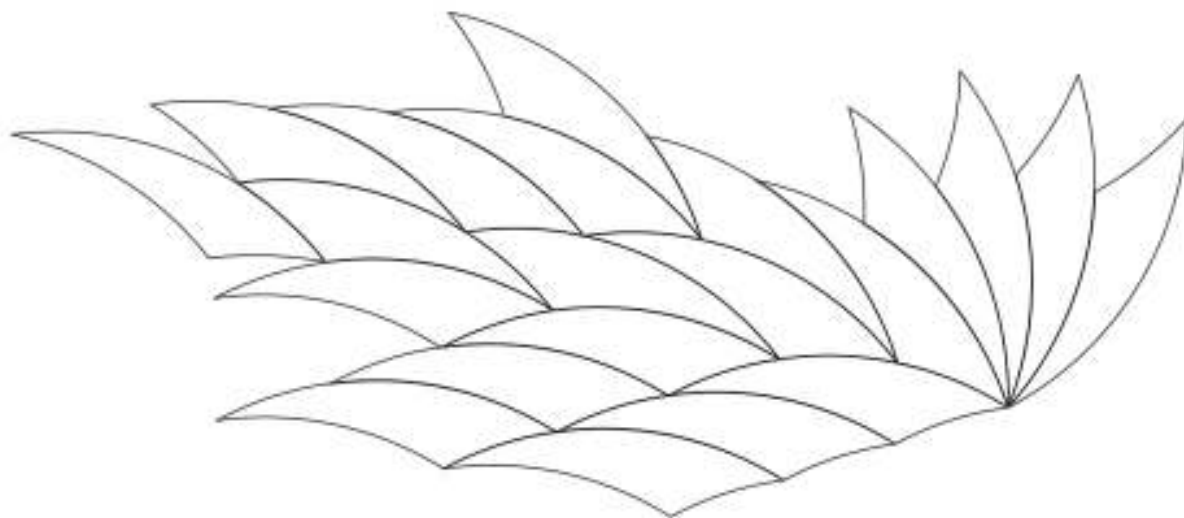
**2.4 Variations.** A tricurve can be laid out for a wide range of large (convex) and small (concave) angles. The large angle can be any angle up to 180 degrees; the small angles can be any angles that are complimentary. However, for the shape to work for tiling beyond simple periodic patterns, all the angles should be some simple fraction of 360 degrees, such as multiples of 12 and 15. The above large angle of 90 degrees can be “split” in many practical ways as indicated here:



With a large angle of 180 degrees the two small angles could be 90/90, 30/150, 120/60, 135-45, etc. The Tricurve with angles of 180/120/60 is shown below, along with tiling options:



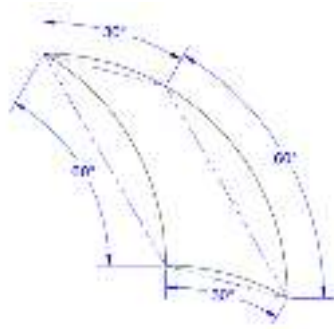
For a large angle of 60 degrees the two small angles could be 30/30, 15/45, 12/48, etc. An example of tiling with a 60/40/20 tricurve is shown below.



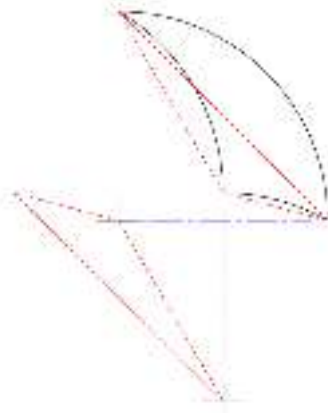
## 2.5 Food for Thought:

2.5.1 One might be tempted to try to use three same-radius curved sides in other proportions, and not bound by the mirror image of the large radius. But this presents that same challenge as a normal triangle for tiling--that of finding agreeable angles and side lengths—along with the new challenge of keeping concave and convex arc lengths the same.

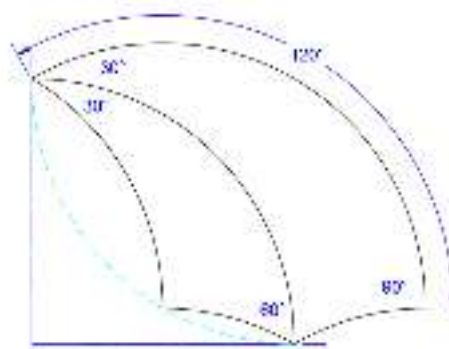
2.5.2 Since the tricurve's large arc is the sum of the two smaller arcs, the tricurve can be thought of as a sort of curved-side parallelogram, as shown below:



2.5.3 Due the construction of the tricurve, the three arc centers form the corners of a triangle identical to that form by the corners of the tricurve:



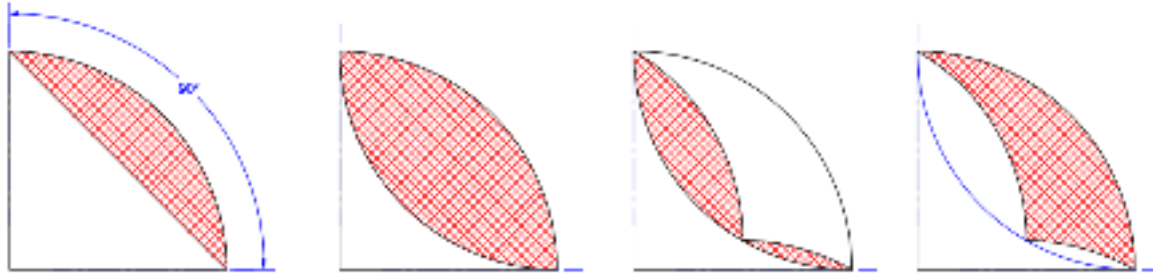
2.5.4 While the focus here is on single shapes for tiling, the tricurves of same radius but different angles could of course be put next to each other. Below is the above 30/60/90 tricurve with a 30/90/120 tricurve constructed next to it, sharing the 90-degree side:



2.5.5 The tricurve shape is very simple; in some ways for tiling it is simpler than a normal triangle. Note that curved side do not along congruent sides to be adjoining. But the shape is also more complex than it first appears, as explained below.

### 3. Areas

The area of a normal triangle is easy to calculate, whereas the area of a Tricurve is a little more complex. The key part is the segment: the area bounded by a chord and its associated arc. This segment area is the difference between the sector area, and the triangular area bounded by the chord and the two radial lines. This is shown below left for a 90-degree angle/arc.



The segment area is then doubled, for the area bounded by the arc and its mirror image about its chord. The same is done for each of the two smaller arcs (30 and 60 degrees, above). The area of the Tricurve is the area of the large double-segment minus the areas of the two smaller double-segments. See Appendix B for calculations.

The area calculations start out with some complexity, and include sectors areas calculated using  $\pi$ , and so one would not expect to get tricurve areas that are easily recognized values. But in several cases we get area values that are surprisingly simple.

The reason for this is that the sectors areas cancel out: the large sector has the same area as the subtracted two smaller sectors. For a unit radius, the area is the sum of the sines of the two small angles, minus the sine of the large angle. In terms of the two small angles, the area is the sum of the sines minus the sine of the sums. And we are dealing largely with “popular” angles (multiples of 30 and 45 degrees) with well-known sines based on simpler values.

This is shown for the main cases in the table below. In Part A the large angle is zero (with sine of zero), and the tricurve is symmetrical with the two small angles the same. Therefore, the area is twice the sine of the small angle. In Part B the large angle and one of the small angles have the same sine and so cancel out, and the area is simply the sine of the 2<sup>nd</sup> small angle.

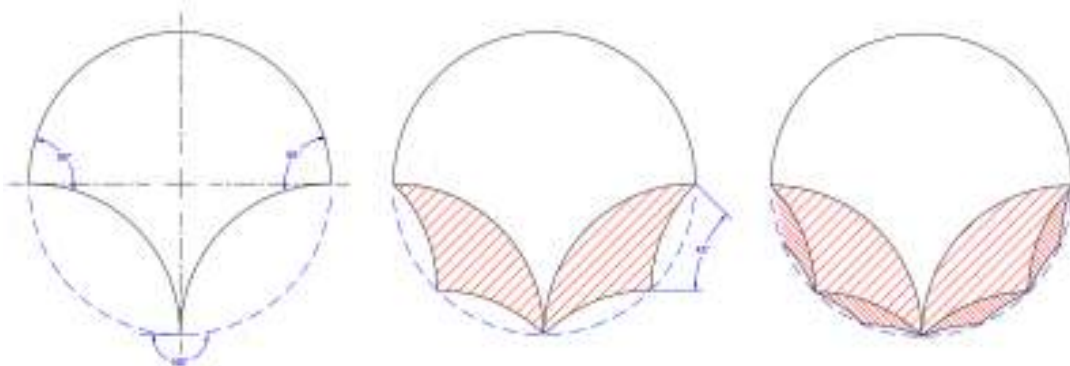
	large angle	1st small angle	2nd small angle	Area = $2x\sin$ :
Part A	180, with sine of zero	with same sines		
		90	90	2
		120	60	SQRT 3
		135	45	SQRT 2
		150	30	1
Part B	same sines: cancel out			Area = sine:
	165	15	150	1/2
	150	30	120	half of SQRT 3
	135	45	90	1
	120	60	60	half of SQRT 3
	105	75	30	1/2
	112.5	67.5	45	half of SQRT 2
	157.5	22.5	135	half of SQRT 2

Naturally the angles for many other tricurves could be shown, but the area would then be more complex combinations of sines, or combinations of more complex sines. Note that following the pattern of Part B, a tricurve could be designed with an area equal to the sine of any angle between 0 and 180.

#### 4. Finding Pi

For a large angle of 180 degrees, it is obvious the tricurve it taking up part of a circle, with two double-segment areas left open. These spaces in turn can be filled with tricurves. Compared to an asymmetrical tricurve, a symmetrical tricurve is not as interesting for tiling purposes—but for this topic it is helpful.

Consider the symmetrical 180/90/90 tricurve below left. The two open 90-degree spaces on the lower corners can be partially filled with 90/45/45 tricurves. This leaves four empty spaces as shown middle. These four spaces can in turn be filled with 45/22.5/22.5 tricurves as shown on the right, leaving eight empty spaces.



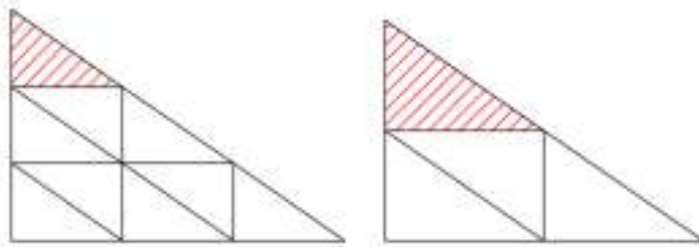
Thus the area of the circle can be filled with—and that area calculated as-- an infinite series of symmetrical tricurves described by their large angles:

The combined area of one 180 + two 90's + four 45's + eight 22.5's + etc.....See Appendix B for calculations. With a radius of one unit, the area of this circle is PI, so this is an infinite series for calculating PI. I don't know how this compares with the many other infinite series for calculating PI.



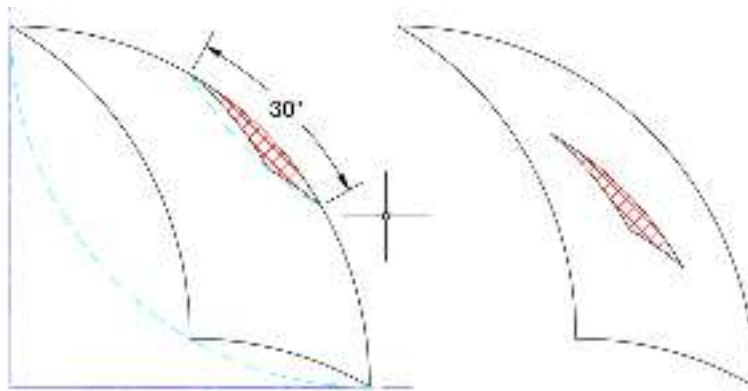
## 5. Similar triangles

With a normal triangle, it is easy to construct a smaller similar triangle: one simply makes all sides proportionally smaller as shown below. There is no problem fitting the smaller triangles in the larger. A normal triangle can be scaled up and down in size, and the angles stay the same and sides scale proportionally.



With a tricurve one could try to make similar triangles in one of two ways. Firstly, one could simply scale the shape up or down in size—but then of course the radii would be different and the original and new shapes would not fit together. Secondly, one could keep the radius the same, and scale up or down the angles and corresponding arcs lengths—but again the original and new shape would not fit together.

The drawing below shows our 30/60/90 Tricurve with a shaded “similar” 10/20/30 Tricurve with all arcs one-third the original length. It’s not the same shape, because the corner angles are now also one-third size. This kind of situation happens for all tricurve when one tries to make similar triangles. In the normal triangle above the small triangle with one-third the dimensions has one-ninth of the original area; however the smaller tricurve below has an area smaller by a factor of about 23.



It is not hard to construct the above smaller tricurve if one already knows the one-third size of any two of the three arcs. As was done above, a 90-degree angle can be made one of the known sides and easily trisected; then an arc of interest could be made one of the other sides, and trisected. But I have not yet found a way to construct a one-third tricurve using elementary construction methods (i.e., compass and straight edge). If that were possible, then it would be possible to trisect an angle in the classical sense.

## 6. Conclusions

The tricurve design started as an attempt to produce unusual tiles shapes. While the resulting tiling options are interesting, it seems it seems the math behind the shape are even more so.

This is intended as an informal paper. I am freely presenting the idea, for what it is worth; and I am soliciting feedback from any interested readers. I welcome any input, and look forward to discussions on this. The best way to reach me is by email.

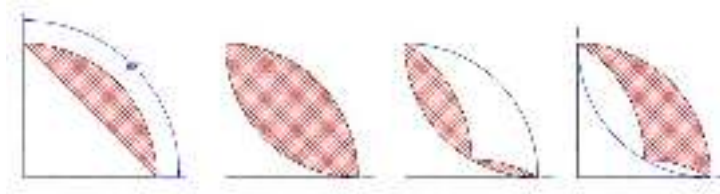
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## Appendix A

### Area Calculation for Basic Tricurve



For any single angle  $a$ ,

Area  $A_{\text{segment}} = A_{\text{sector}} - A_{\text{triangle}}$  where the triangle is bounded by the chord and the two radial legs

$$A_{\text{sector}} = \pi \times r^2 \times (a/360) = r^2 \pi (a/360)$$

$$A_{\text{triangle}} = \frac{1}{2} r \times (r \sin a) = r^2 (\sin a) / 2$$

$$A_{\text{segment}} = r^2 \pi (a/360) - r^2 (\sin a) / 2$$

For the double segment (abbreviated DS) this area is of course doubled:

$$A_{\text{DS}} = 2[r^2 \pi (a/360)] - 2r^2 (\sin a) / 2 = 2r^2 \pi (a/360) - r^2 (\sin a)$$

Note the area based on the sector is kept separate from the area based on the triangle, rather than combining the terms under  $r^2$ .

For the whole tricurve, the area is the large-angle double segment minus the two small-angle double segments. This can be expressed in terms of small angles  $a$  and  $b$ , and their total  $(a+b)$  which is the large angle.

$$\begin{aligned} A_{\text{tricurve}} &= A_{\text{DS}(a+b)} - A_{\text{DS}a} - A_{\text{DS}b} \\ &= [2r^2 \pi ((a+b)/360) - r^2 (\sin (a+b))] - [2r^2 \pi (a/360) - r^2 (\sin a)] - [2r^2 \pi (b/360) - r^2 (\sin b)] \end{aligned}$$

Double-Sector areas and triangle areas can be grouped together

$$\begin{aligned} &= [2r^2 \pi ((a+b)/360) - 2r^2 \pi (a/360) - 2r^2 \pi (b/360)] + [r^2 (\sin a) + r^2 (\sin b) - r^2 (\sin (a+b))] \\ &= 2[r^2 \pi ((a+b)/360) - r^2 \pi (a/360) - r^2 \pi (b/360)] + [r^2 (\sin a) + r^2 (\sin b) - r^2 (\sin (a+b))] \\ &= 2[A_{\text{sector}(a+b)} - (A_{\text{sector}(a)} + A_{\text{sector}(b)})] + [r^2 (\sin a) + r^2 (\sin b) - r^2 (\sin (a+b))] \end{aligned}$$

But  $A_{\text{sector}(a+b)} = A_{\text{sector}(a)} + A_{\text{sector}(b)}$  so the first part of this equation is zero.

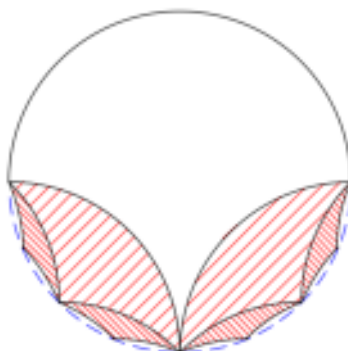
$$A_{\text{tricurve}} = r^2 (\sin a) + r^2 (\sin b) - r^2 (\sin (a+b)) = r^2 [(\sin a) + (\sin b) - (\sin (a+b))]$$

$r^2$  is one unit, squared = one square unit

Therefore the area  **$A_{\text{tricurve}} = \sin a + \sin b - \sin (a+b)$**  in square units

## Appendix B

### Calculation of an Infinite Series for Pi



See section 4 of paper. The area of the circle—and thus the value of Pi—can be approximated with a series of tricurves. These tricurves are symmetrical, with the two small angles the same at half the large angle. The tricurve is described and defined by its large angle. The area of any tricurve in the series is then found as

$$A_{\text{tricurve}} = 2\sin(a/2) - \sin a \quad \text{in square units, where } a \text{ is the large angle.}$$

The circle is filled and pi approximated by the sequence as follows:

- one 180-degree symmetrical tricurve with area=  $2(\sin 90) - \sin 180$ ,
- + two 90-degree symmetrical tricurve with area=  $2(\sin 45) - \sin 90$ ,
- + four 45-degree symmetrical tricurve with area=  $2(\sin 22.5) - \sin 45$ ,
- + eight  $22\frac{1}{2}$ -degree symmetrical tricurve with area=  $2(\sin 11\frac{1}{4}) - \sin 22.5$ ,
- + etc....doubling the number and halving the large angle with each step, in an infinite series.