M204 : Metric Spaces (Even Semester 2020-21), Practice problems

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1. Let X denote the set of all sequences of real numbers. For $X = \{x_n\}_{n=1}^{\infty}$ and $Y = \{y_n\}_{n=1}^{\infty}$, define
$$d(X,Y) = \sum_{n=1}^{\infty} \frac{1}{2^n}\Big[\frac{|x_n - y_n}{1+|x_n - y_n|}\Big].$$
Prove or disrpove : (X, d) is a metric space.

2. If d is a metric on a set X then so are d_1, d_2, where
$$d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}, \qquad d_2(x,y) = \min\{1, d(x,y)\}.$$

3. Let $X = \mathbb{N}$ and define $d(m,n) = |\frac{1}{m} - \frac{1}{n}|$ for all $m, n \in \mathbb{N}$. Prove or disprove : (X, d) is a metric space.

4. Let $X = \mathbb{R}$ and define $d(x,y) = |\tan^{-1}(x) - \tan^{-1}(y)|$ for all $x, y \in X$. Prove or disprove : (X, d) is a metric space.

   Let (X, d) is a metric space. The metric d is said to be *bounded* if $d(X)$ is a bounded subset of (X, d). Similarly, d is said to be *unbounded* if $d(X)$ is not a bounded subset of (X, d).

5. Show that every infinite set X admits an unbounded metric d on it.

6. Show that every metric d of a metric space (X, d) is equivalent to a bounded metric on X.

7. Let (X, d) be a metric space. Consider a function $f : \mathbb{R} \to \mathbb{R}$. If $(X, f \circ d)$ is metric space, then what are the conditions that f requires to satisfy?

8. Let A be a closed subset of a metric space (X, d). Show that for all $a \in A$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in A such that $x_n \xrightarrow{d} a$ as $n \to \infty$.

   Let A be a subset of a metric space (X, d). The *boundary* of A, denoted by ∂A, is the set of all points of $x \in X$ such that x is neither an interior point of A nor an exterior point of A. Equivalently, $x \in \partial A$ if and only if for all $\epsilon > 0$, $S_\epsilon(x) \cap A \neq \emptyset$ and $S_\epsilon(x) \cap (X \setminus A) \neq \emptyset$.

9. Show that $\partial A = \partial(X \setminus A)$.

10. Prove or disprove : $\partial\mathbb{Q} = \mathbb{R}$, $\partial\mathbb{N} = \mathbb{N}$, $\partial\mathbb{Z} = \mathbb{Z}$.

11. Show that in a discrete metric space X, $\partial A = \emptyset$ for all $A \subseteq X$. Is the converse true?

12. If A is a open subset of a metric space (X, d), then prove or disprove : $\overline{(X \setminus \partial A)} = X$.

13. Let A be an open subset of $\mathbb{R}$ equipped with the Euclidean metric. Show that for each $x, y \in \mathbb{R}$, there exists $a, b \in A$ such that $x = a - b$.

    Let A be a subset of a metric space (X, d). An element $a \in A$ is an *isolated point* if there exists $r > 0$ such that $S_r(a) \cap A = \{a\}$.

14. A point x of a metric space (X, d) is an isolated point of X if and only if $\{x\}$ is open in X.

    Let (X, d) be a metric space. A subset A is said to be *dense* in X if and only if $\overline{A} = X$.

15. Suppose (X, d) is a metric space without any isolated point and Y is a dense subset of X. Show that for any open subset U of X, $U \cap X$ is infinite and hence Y has no isolated point.

    A metric space (X, d) is said to be *separable* if X has at least one countable dense subset.

16. Let (X, d) be a metric space and $Y \subset X$ such that (Y, d) is separable and $\overline{Y} = X$. Show that (X, d) is separable.
17. Show that (ℓ^p, d_p) is separable for all $1 \leq p \leq \infty$.
18. Prove or disprove (ℓ^∞, d_∞) is not separable.