Longest Paths in DAGs

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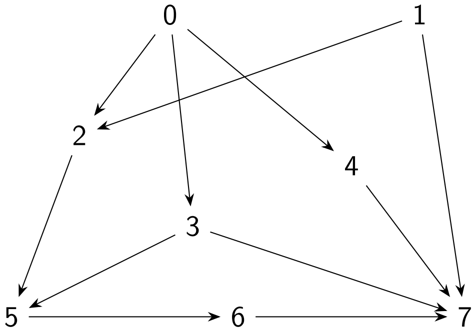
Mathematics for Data Science 1
Week 11

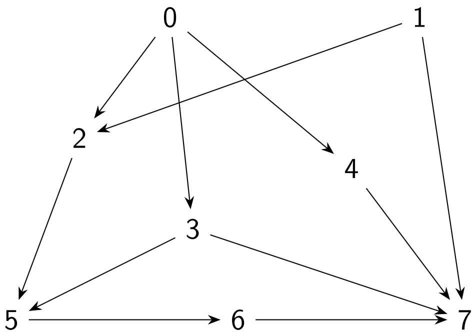
Directed Acyclic Graphs

- $G = (V, E)$, a directed graph without directed cycles



Directed Acyclic Graphs

- $G = (V, E)$, a directed graph without directed cycles
- Topological sorting
  - Enumerate $V = \{0, 1, \ldots, n-1\}$ such that for any $(i, j) \in E$, i appears before j
  - Feasible schedule



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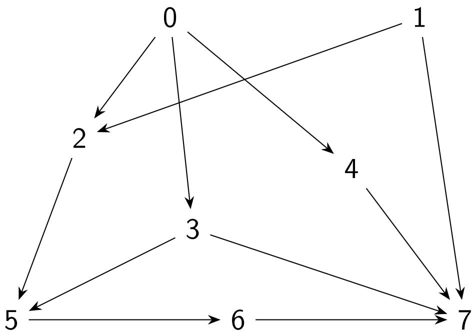

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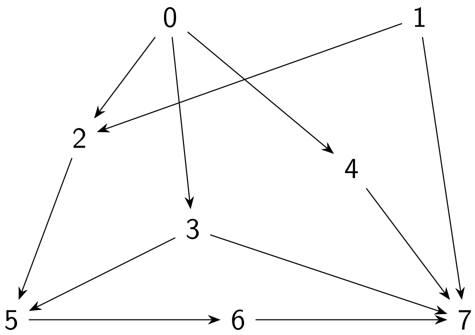

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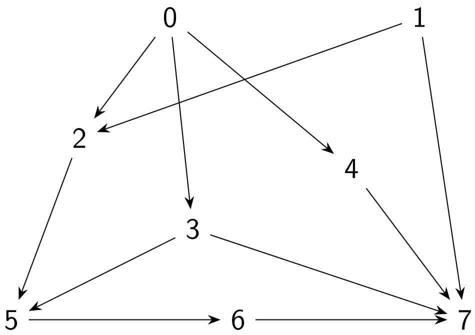

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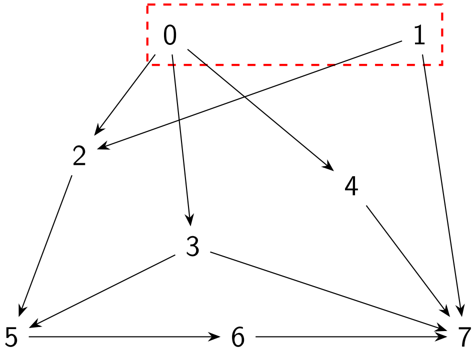

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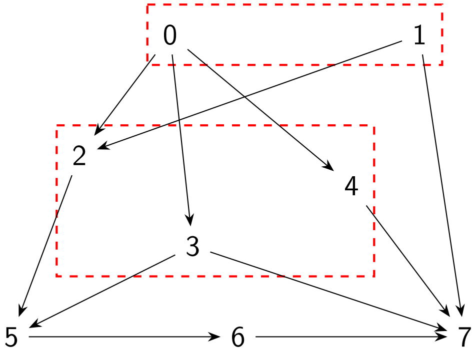

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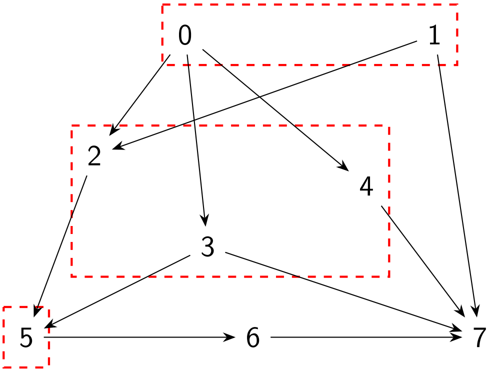

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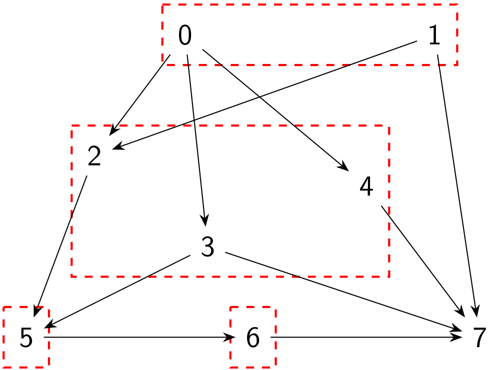

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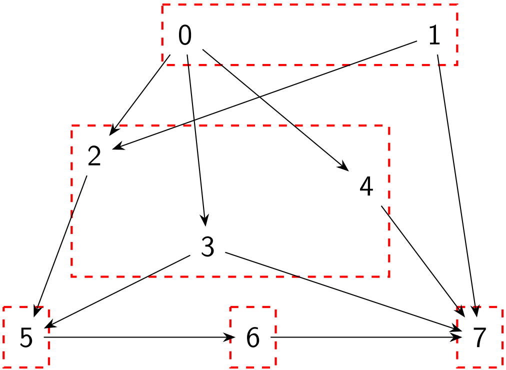
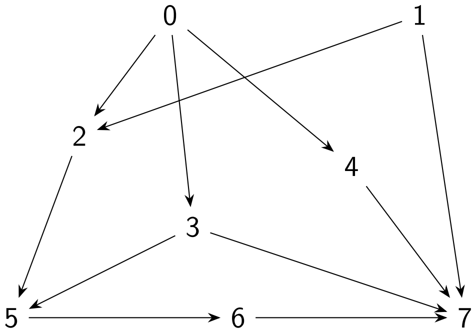


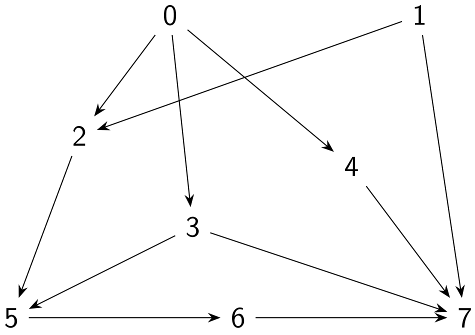
Longest Path

- Find the longest path in a DAG



Longest Path

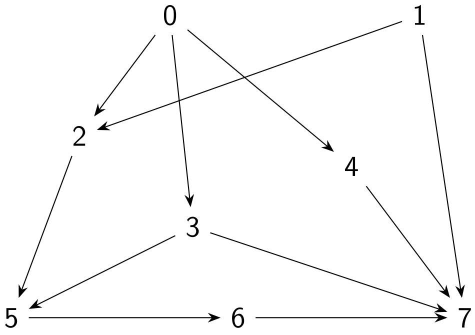
- Find the longest path in a DAG
- If $\text{indegree}(i) = 0$, $\text{longest-path-to}(i) = 0$



Longest Path

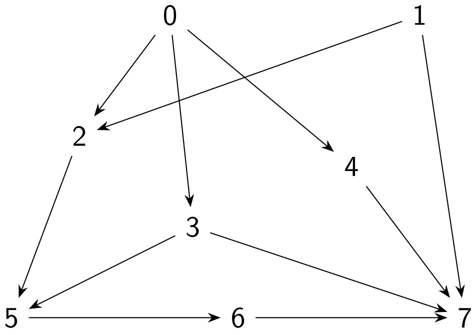
- Find the longest path in a DAG
- If $\text{indegree}(i) = 0$,
  $\text{longest-path-to}(i) = 0$
- If $\text{indegree}(i) > 0$, longest path to i is 1 more than longest path to its incoming neighbours

  $\text{longest-path-to}(i) =$
  $1 + \max\{\text{longest-path-to}(j) \mid (j, i) \in E\}$



Longest Path

- longest-path-to(i) $=$ $1 + \max\{\text{longest-path-to}(j) \mid (j,i) \in E\}$



Longest Path

- longest-path-to$(i) =$
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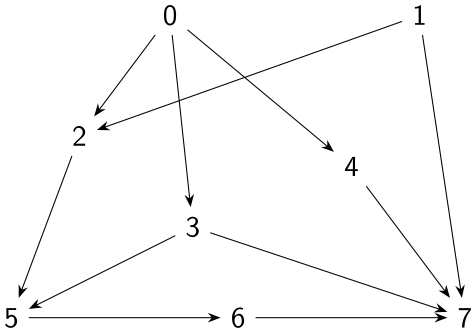
- To compute longest-path-to(i), need longest-path-to(k), for each incoming neighbour k



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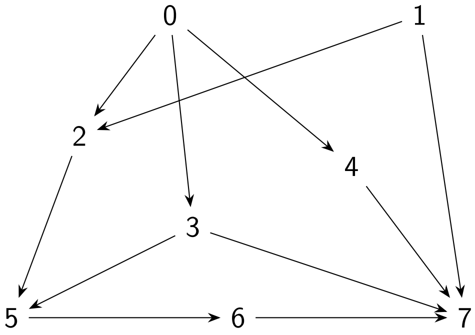
- To compute longest-path-to(i), need longest-path-to(k), for each incoming neighbour k
- If graph is topologically sorted, k is listed before i



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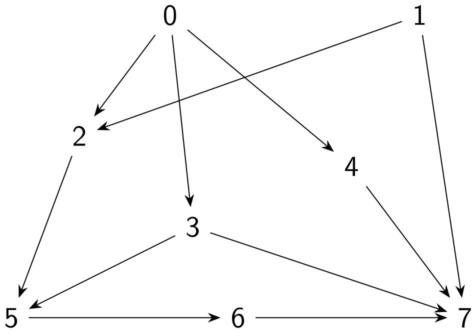
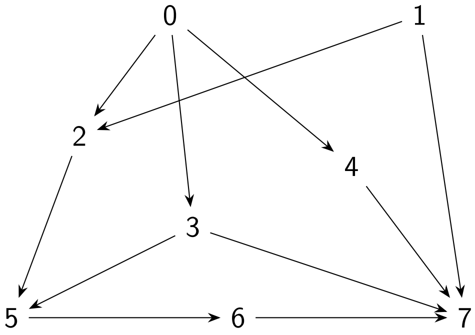
- To compute longest-path-to(i), need longest-path-to(k), for each incoming neighbour k
- If graph is topologically sorted, k is listed before i
- Hence compute longest-path-to() in topological order



Longest Path

- Let $i_0, i_1, \ldots, i_{n-1}$ be a topological ordering of V



Longest Path

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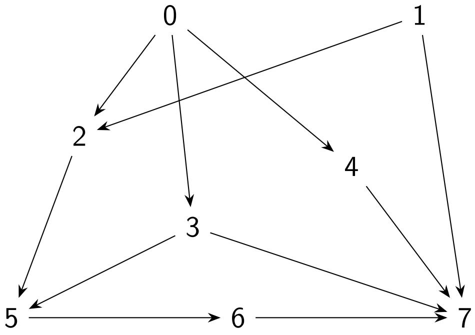
- All neighbours of i_k appear before it in this list



Longest Path

- Let $i_0, i_1, \ldots, i_{n-1}$ be a topological ordering of V
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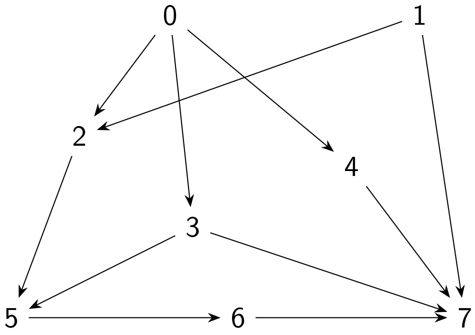
- From left to right, compute $\text{longest-path-to}(i_k)$ as $1 + \max\{\text{longest-path-to}(i_j) \mid (i_j, i_k) \in E\}$



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- From left to right, compute longest-path-to(i_k) as
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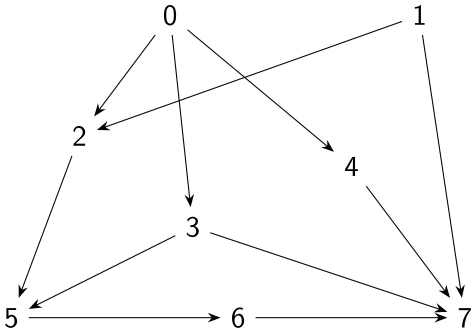
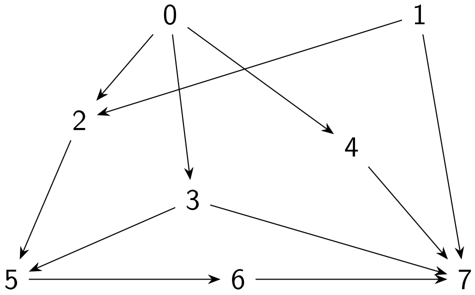
- Overlap this computation with topological sorting



Longest path algorithm

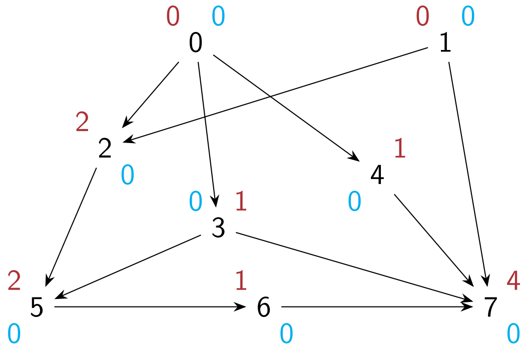
- Compute indegree of each vertex



Longest path algorithm

- Compute indegree of each vertex
  - Scan each column of the adjacency matrix
- Initialize $textlongest - path - to$ to 0 for all vertices

Indegree, Longest path



Topological order

Longest path to

Longest path algorithm

- Compute indegree of each vertex
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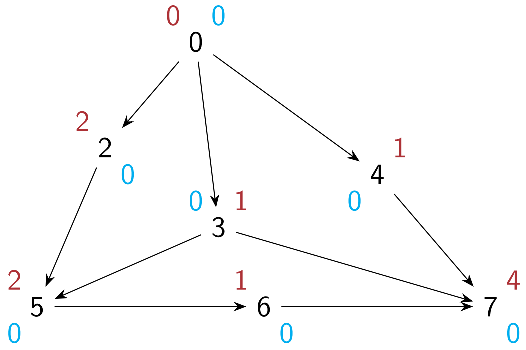
- List a vertex with indegree 0 and remove it from the DAG

Indegree, Longest path



Topological order	1
Longest path to	0

Longest path algorithm

- Compute indegree of each vertex
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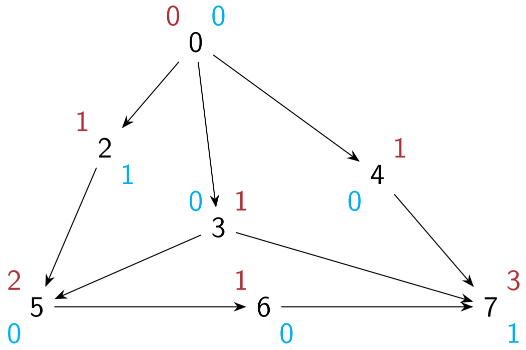
- Update indegrees, longest path

Indegree, Longest path



Topological order	1
Longest path to	0

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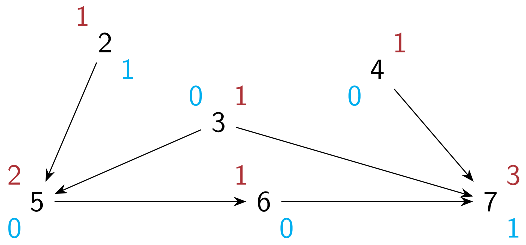
Indegree, Longest path



Topological order	1	0
Longest path to	0	0

Longest path algorithm

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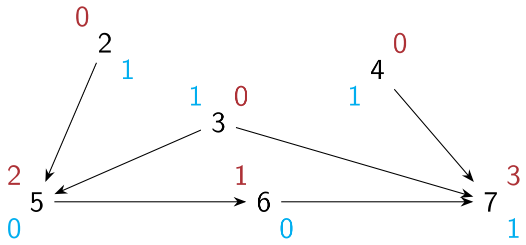
Indegree, Longest path



Topological order	1	0
Longest path to	0	0

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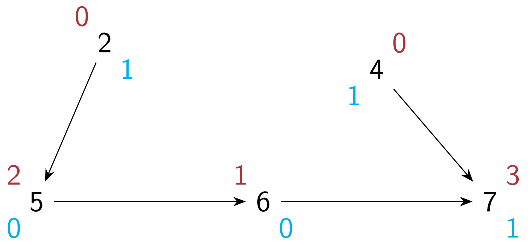

Topological order	1	0	3
Longest path to	0	0	1

Longest path algorithm

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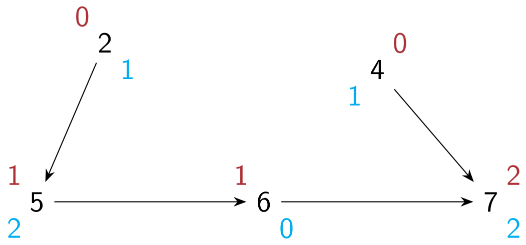

Topological order	1	0	3
Longest path to	0	0	1

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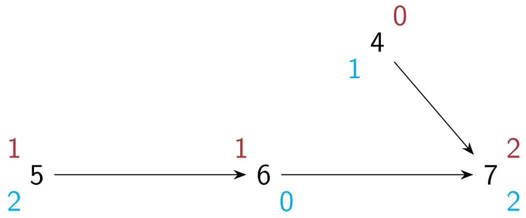

Topological order	1	0	3	2
Longest path to	0	0	1	1

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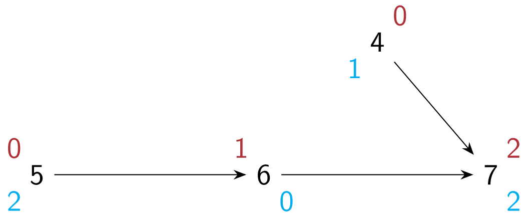

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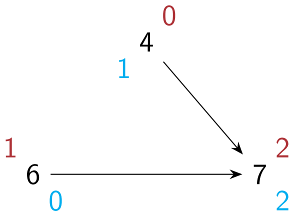

Topological order	1	0	3	2	5
Longest path to	0	0	1	1	2

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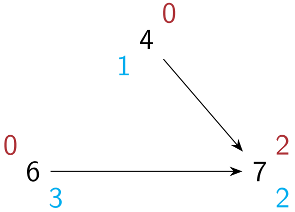
Indegree, Longest path



Topological order	1	0	3	2	5
Longest path to	0	0	1	1	2

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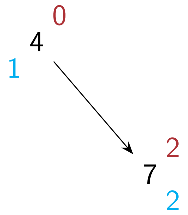

Topological order	1	0	3	2	5	6
Longest path to	0	0	1	1	2	3

Longest path algorithm

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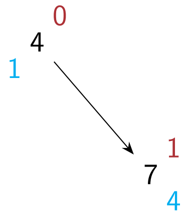
Indegree, Longest path



Topological order	1	0	3	2	5	6
Longest path to	0	0	1	1	2	3

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Indegree, Longest path

1
7
4

Topological order	1	0	3	2	5	6	4
Longest path to	0	0	1	1	2	3	1

Longest path algorithm

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- Repeat till all vertices are listed

Indegree, Longest path

0
7
4

Topological order	1	0	3	2	5	6	4
Longest path to	0	0	1	1	2	3	1

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Indegree, Longest path

Topological order	1	0	3	2	5	6	4	7
Longest path to	0	0	1	1	2	3	1	4

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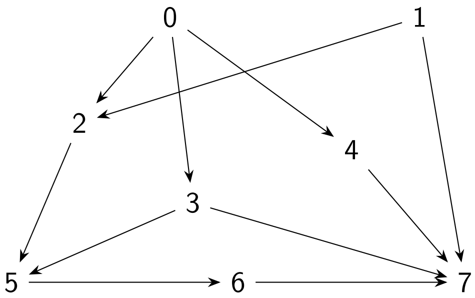

Topological order	1	0	3	2	5	6	4	7
Longest path to	0	0	1	1	2	3	1	4

Summary

- Directed acyclic graphs are a natural way to represent dependencies
- Topological sort gives a feasible schedule that represents dependencies
- In parallel with topological sort, we can compute the longest path

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Summary

- Directed acyclic graphs are a natural way to represent dependencies
- Topological sort gives a feasible schedule that represents dependencies
- In parallel with topological sort, we can compute the longest path
- Notion of longest path makes sense even for graphs with cycles
  - No repeated vertices in a path, so path has at most $n-1$ edges
- However, computing longest paths in arbitrary graphs is much harder than for DAGs
  - No better strategy known than exhaustively enumerating paths