THOMSON
DELMAR LEARNING

FUNDAMENTALS *of* SIGNALS & SYSTEMS

Benoit Boulet

DA VINCI ENGINEERING PRESS

FUNDAMENTALS OF SIGNALS AND SYSTEMS

LIMITED WARRANTY AND DISCLAIMER OF LIABILITY

THE CD-ROM THAT ACCOMPANIES THE BOOK MAY BE USED ON A SINGLE PC ONLY. THE LICENSE DOES NOT PERMIT THE USE ON A NETWORK (OF ANY KIND). YOU FURTHER AGREE THAT THIS LICENSE GRANTS PERMISSION TO USE THE PRODUCTS CONTAINED HEREIN, BUT DOES NOT GIVE YOU RIGHT OF OWNERSHIP TO ANY OF THE CONTENT OR PRODUCT CONTAINED ON THIS CD-ROM. USE OF THIRD-PARTY SOFTWARE CONTAINED ON THIS CD-ROM IS LIMITED TO AND SUBJECT TO LICENSING TERMS FOR THE RESPECTIVE PRODUCTS.

CHARLES RIVER MEDIA, INC. ("CRM") AND/OR ANYONE WHO HAS BEEN INVOLVED IN THE WRITING, CREATION, OR PRODUCTION OF THE ACCOMPANYING CODE ("THE SOFTWARE") OR THE THIRD-PARTY PRODUCTS CONTAINED ON THE CD-ROM OR TEXTUAL MATERIAL IN THE BOOK, CANNOT AND DO NOT WARRANT THE PERFORMANCE OR RESULTS THAT MAY BE OBTAINED BY USING THE SOFTWARE OR CONTENTS OF THE BOOK. THE AUTHOR AND PUBLISHER HAVE USED THEIR BEST EFFORTS TO ENSURE THE ACCURACY AND FUNCTIONALITY OF THE TEXTUAL MATERIAL AND PROGRAMS CONTAINED HEREIN. WE HOWEVER, MAKE NO WARRANTY OF ANY KIND, EXPRESS OR IMPLIED, REGARDING THE PERFORMANCE OF THESE PROGRAMS OR CONTENTS. THE SOFTWARE IS SOLD "AS IS" WITHOUT WARRANTY (EXCEPT FOR DEFECTIVE MATERIALS USED IN MANUFACTURING THE DISK OR DUE TO FAULTY WORKMANSHIP).

THE AUTHOR, THE PUBLISHER, DEVELOPERS OF THIRD-PARTY SOFTWARE, AND ANYONE INVOLVED IN THE PRODUCTION AND MANUFACTURING OF THIS WORK SHALL NOT BE LIABLE FOR DAMAGES OF ANY KIND ARISING OUT OF THE USE OF (OR THE INABILITY TO USE) THE PROGRAMS, SOURCE CODE, OR TEXTUAL MATERIAL CONTAINED IN THIS PUBLICATION. THIS INCLUDES, BUT IS NOT LIMITED TO, LOSS OF REVENUE OR PROFIT, OR OTHER INCIDENTAL OR CONSEQUENTIAL DAMAGES ARISING OUT OF THE USE OF THE PRODUCT.

THE SOLE REMEDY IN THE EVENT OF A CLAIM OF ANY KIND IS EXPRESSLY LIMITED TO REPLACEMENT OF THE BOOK AND/OR CD-ROM, AND ONLY AT THE DISCRETION OF CRM.

THE USE OF "IMPLIED WARRANTY" AND CERTAIN "EXCLUSIONS" VARIES FROM STATE TO STATE, AND MAY NOT APPLY TO THE PURCHASER OF THIS PRODUCT.

Fundamentals of Signals and Systems

Benoit Boulet

CHARLES RIVER MEDIA
Boston, Massachusetts

Cover Design: Tyler Creative

CHARLES RIVER MEDIA
25 Thomson Place
Boston, Massachusetts 02210
617-757-7900
617-757-7951 (FAX)
crm.info@thomson.com
www.charlesriver.com

This book is printed on acid-free paper.

Benoit Boulet. *Fundamentals of Signals and Systems.*

ISBN: 1-58450-381-5
eISBN: 1-58450-660-1

Library of Congress Cataloging-in-Publication Data
Boulet, Benoit, 1967-
Fundamentals of signals and systems / Benoit Boulet.— 1st ed.
p. cm.
Includes index.
ISBN 1-58450-381-5 (hardcover with cd-rom : alk. paper)
1. Signal processing. 2. Signal generators. 3. Electric filters. 4. Signal detection. 5. System analysis. I. Title.
TK5102.9.B68 2005
621.382'2—dc22

2005010054

07 7 6 5 4 3

CHARLES RIVER MEDIA titles are available for site license or bulk purchase by institutions, user groups, corporations, etc. For additional information, please contact the Special Sales Department at 800-347-7707.

Requests for replacement of a defective CD-ROM must be accompanied by the original disc, your mailing address, telephone number, date of purchase and purchase price. Please state the nature of the problem, and send the information to CHARLES RIVER MEDIA, 25 Thomson Place, Boston, Massachusetts 02210. CRM's sole obligation to the purchaser is to replace the disc, based on defective materials or faulty workmanship, but not on the operation or functionality of the product.

Contents

List of Lectures

Acknowledgments

I wish to acknowledge the contribution of Dr. Maier L. Blostein, emeritus professor in the Department of Electrical and Computer Engineering at McGill University. Our discussions over the past few years have led us to the current course syllabi for Signals & Systems I and II, essentially forming the table of contents of this textbook.

I would like to thank the many students whom, over the years, have reported mistakes and suggested useful revisions to my Signals & Systems I and II course notes.

The interesting and useful applets on the companion CD-ROM were programmed by the following students: Rafic El-Fakir (Bode plot applet) and Gul Pil Joo (Fourier series and convolution applets). I thank them for their excellent work and for letting me use their programs.

This page intentionally left blank

Preface

The study of signals and systems is considered to be a classic subject in the curriculum of most engineering schools throughout the world. The theory of signals and systems is a coherent and elegant collection of mathematical results that date back to the work of Fourier and Laplace and many other famous mathematicians and engineers. Signals and systems theory has proven to be an extremely valuable tool for the past 70 years in many fields of science and engineering, including power systems, automatic control, communications, circuit design, filtering, and signal processing. Fantastic advances in these fields have brought revolutionary changes into our lives.

At the heart of signals and systems theory is mankind's historical curiosity and need to analyze the behavior of physical systems with simple mathematical models describing the cause-and-effect relationship between quantities. For example, Isaac Newton discovered the second law of rigid-body dynamics over 300 years ago and described it mathematically as a relationship between the resulting force applied on a body (the input) and its acceleration (the output), from which one can also obtain the body's velocity and position with respect to time. The development of differential calculus by Leibniz and Newton provided a powerful tool for modeling physical systems in the form of differential equations implicitly relating the input variable to the output variable.

A fundamental issue in science and engineering is to predict what the behavior, or output response, of a system will be for a given input signal. Whereas science may seek to describe natural phenomena modeled as input-output systems, engineering seeks to design systems by modifying and analyzing such models. This issue is recurrent in the design of electrical or mechanical systems, where a system's output signal must typically respond in an appropriate way to selected input signals. In this case, a mathematical input-output model of the system would be analyzed to predict the behavior of the output of the system. For example, in the design of a simple resistor-capacitor electrical circuit to be used as a filter, the engineer would first specify the desired attenuation of a sinusoidal input voltage of a given frequency at the output of the filter. Then, the design would proceed by selecting the appropriate resistance R and capacitance C in the differential equation model of the filter in order to achieve the attenuation specification. The filter can then be built using actual electrical components.

A signal is defined as a function of time representing the evolution of a variable. Certain types of input and output signals have special properties with respect to linear time-invariant systems. Such signals include sinusoidal and exponential functions of time. These signals can be linearly combined to form virtually any other signal, which is the basis of the Fourier series representation of periodic signals and the Fourier transform representation of aperiodic signals.

The Fourier representation opens up a whole new interpretation of signals in terms of their frequency contents called the frequency spectrum. Furthermore, in the frequency domain, a linear time-invariant system acts as a filter on the frequency spectrum of the input signal, attenuating it at some frequencies while amplifying it at other frequencies. This effect is called the frequency response of the system. These frequency domain concepts are fundamental in electrical engineering, as they underpin the fields of communication systems, analog and digital filter design, feedback control, power engineering, etc. Well-trained electrical and computer engineers think of signals as being in the frequency domain probably just as much as they think of them as functions of time.

The Fourier transform can be further generalized to the Laplace transform in continuous-time and the z-transform in discrete-time. The idea here is to define such transforms even for signals that tend to infinity with time. We chose to adopt the notation $X(j\omega)$, instead of $X(\omega)$ or $X(f)$, for the Fourier transform of a continuous-time signal $x(t)$. This is consistent with the Laplace transform of the signal denoted as $X(s)$, since then $X(j\omega) = X(s)|_{s=j\omega}$. The same remark goes for the discrete-time Fourier transform: $X(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$.

Nowadays, predicting a system's behavior is usually done through computer simulation. A simulation typically involves the recursive computation of the output signal of a discretized version of a continuous-time system model. A large part of this book is devoted to the issue of system discretization and discrete-time signals and systems. The MATLAB software package is used to compute and display the results of some of the examples. The companion CD-ROM contains the MATLAB script files, problem solutions, and interactive graphical applets that can help the student visualize difficult concepts such as the convolution and Fourier series.

Undergraduate students see the theory of signals and systems as a difficult subject. The reason may be that signals and systems is typically one of the first courses an engineering student encounters that has substantial mathematical content. So what is the required mathematical background that a student should have in order to learn from this book? Well, a good background in calculus and trigonometry definitely helps. Also, the student should know about complex numbers and complex functions. Finally, some linear algebra is used in the development of state-space representations of systems. The student is encouraged to review these topics carefully before reading this book.

My wish is that the reader will enjoy learning the theory of signals and systems by using this book. One of my goals is to present the theory in a direct and straightforward manner. Another goal is to instill interest in different areas of specialization of electrical and computer engineering. Learning about signals and systems and its applications is often the point at which an electrical or computer engineering student decides what she or he will specialize in.

Benoit Boulet
March 2005
Montréal, Canada

This page intentionally left blank

1 Elementary Continuous-Time and Discrete-Time Signals and Systems

In This Chapter

((Lecture 1: Signal Models))

In this first chapter, we introduce the concept of a signal as a real or complex function of time. We pay special attention to sinusoidal signals and to real and complex exponential signals, as they have the fundamental property of keeping their "identity" under the action of a linear time-invariant (LTI) system. We also introduce the concept of a system as a relationship between an input signal and an output signal.

SYSTEMS IN ENGINEERING

The word *system* refers to many different things in engineering. It can be used to designate such tangible objects as software systems, electronic systems, computer systems, or mechanical systems. It can also mean, in a more abstract way, theoretical objects such as a system of linear equations or a mathematical input-output model. In this book, we greatly reduce the scope of the definition of the word system to the latter; that is, a system is defined here as a mathematical relationship between an input signal and an output signal. Note that this definition of system is different from what we are used to. Namely, the system is usually understood to be the engineering device in the field, and a mathematical representation of this system is usually called a system model.

FUNCTIONS OF TIME AS SIGNALS

Signals are functions of time that represent the evolution of variables such as a furnace temperature, the speed of a car, a motor shaft position, or a voltage. There are two types of signals: *continuous-time* signals and *discrete-time* signals.

Continuous-time signals are functions of a continuous variable (time).

Example 1.1: The speed of a car $v(t)$ as shown in Figure 1.1.

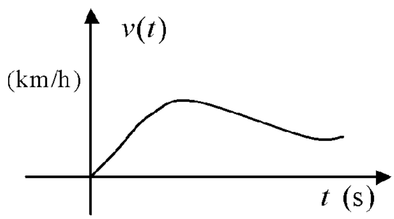

FIGURE 1.1 Continuous-time signal representing the speed of a car.

Discrete-time signals are functions of a discrete variable; that is, they are defined only for integer values of the independent variable (time steps).

Example 1.2: The value of a stock $x[n]$ at the end of month n, as shown in Figure 1.2.

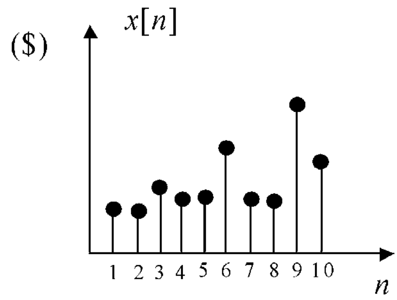

FIGURE 1.2 Discrete-time signal representing the value of a stock.

Note how the discrete values of the signal are represented by points linked to the time axis by vertical lines. This is done for the sake of clarity, as just showing a set of discrete points "floating" on the graph can be confusing to interpret.

Continuous-time and discrete-time functions map their *domain* $\mathcal{T}$ (time interval) into their *co-domain* $\mathcal{V}$ (set of values). This is expressed in mathematical notation as $f:\mathcal{T}\rightarrow\mathcal{V}$. The *range* of the function is the subset $\mathcal{R}\{f\}\subseteq\mathcal{V}$ of the co-domain, in which each element $v\in\mathcal{R}\{f\}$ has a corresponding time t in the domain $\mathcal{T}$ such that $v=f(t)$. This is illustrated in Figure 1.3.

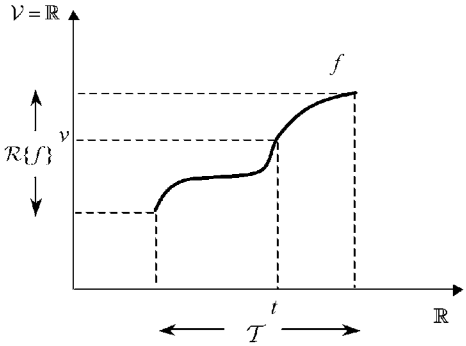

FIGURE 1.3 Domain, co-domain, and range of a real function of continuous time.

If the range $\mathcal{R}\{f\}$ is a subset of the real numbers $\mathbb{R}$, then f is said to be a real signal. If $\mathcal{R}\{f\}$ is a subset of the complex numbers $\mathbb{C}$, then f is said to be a complex signal. We will study both real and complex signals in this book. Note that we often use the notation $x(t)$ to designate a continuous-time signal (not just the value of x at time t) and $x[n]$ to designate a discrete-time signal (again for the whole signal, not just the value of x at time n).

For the car speed example above, the domain of $v(t)$ could be $\mathcal{T}=[0,+\infty)$ with units of seconds, assuming the car keeps on running forever, and the range is $\mathcal{V}=[0,+\infty)\subset\mathbb{R}$, the set of all non-negative speeds in units of kilometers per hour.

For the stock trend example, the domain of $x[n]$ is the set of positive natural numbers $\mathcal{T}=\{1,2,3,\ldots\}$, the co-domain is the non-negative reals $\mathcal{V}=[0,+\infty)\subset\mathbb{R}$, and the range could be $\mathcal{R}\{x\}=[0,100]$ in dollar unit.

An example of a complex signal is the complex exponential $x(t)=e^{j10t}$, for which $\mathcal{T}=\mathbb{R}$, $\mathcal{V}=\mathbb{C}$, and $\mathcal{R}\{x\}=\{z\in\mathbb{C}:|z|=1\}$; that is, the set of all complex numbers of magnitude equal to one.

TRANSFORMATIONS OF THE TIME VARIABLE

Consider the continuous-time signal $x(t)$ defined by its graph shown in Figure 1.4 and the discrete-time signal $x[n]$ defined by its graph in Figure 1.5. As an aside, these two signals are said to be of *finite support,* as they are nonzero only over a finite time interval, namely on $t\in[-2,2]$ for $x(t)$ and when $n\in\{-3,\ldots,3\}$ for $x[n]$. We will use these two signals to illustrate some useful transformations of the time variable, such as time scaling and time reversal.

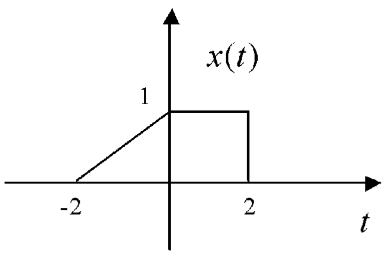

FIGURE 1.4 Graph of continuous time signal $x(t)$.

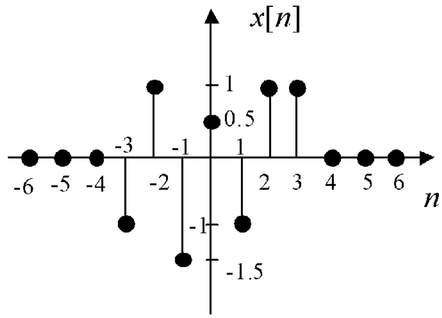

FIGURE 1.5 Graph of discrete-time signal $x[n]$.

Time Scaling

Time scaling refers to the multiplication of the time variable by a real positive constant α. In the continuous-time case, we can write

$$y(t)=x(\alpha t). \tag{1.1}$$

Case $0 < \alpha < 1$: The signal $x(t)$ is *slowed down* or *expanded* in time. Think of a tape recording played back at a slower speed than the nominal speed.

Example 1.3: Case $\alpha = \frac{1}{2}$ shown in Figure 1.6.

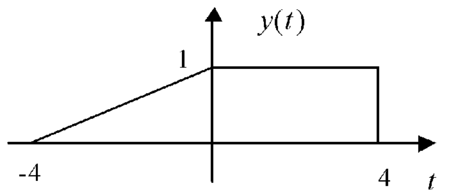

FIGURE 1.6 Graph of expanded signal $y(t) = x(0.5t)$.

Case $\alpha > 1$: The signal $x(t)$ is *sped up* or *compressed* in time. Think of a tape recording played back at twice the nominal speed.

Example 1.4: Case $\alpha = 2$ shown in Figure 1.7.

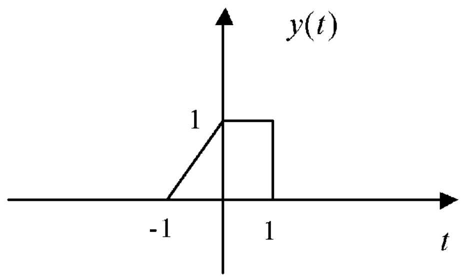

FIGURE 1.7 Graph of compressed signal $y(t) = x(2t)$.

For a discrete-time signal $x[n]$, we also have the time scaling

$$y[n] = x[\alpha n], \tag{1.2}$$

but only the case $\alpha > 1$, where α is an integer, makes sense, as $x[n]$ is undefined for fractional values of n. In this case, called *decimation* or *downsampling*, we not only get a time compression of the signal, but the signal can also lose part of its information; that is, some of its values may disappear in the resulting signal $y[n]$.

Example 1.5: Case $\alpha = 2$ shown in Figure 1.8.

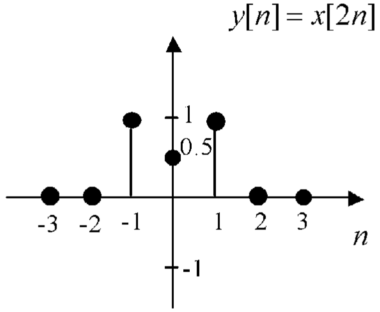

FIGURE 1.8 Graph of compressed signal $y[n] = x[2n]$.

In Chapter 12, *upsampling*, which involves inserting $m - 1$ zeros between consecutive samples, will be introduced as a form of time expansion of a discrete-time signal.

Time Reversal

A time reversal is achieved by multiplying the time variable by –1. The resulting continuous-time and discrete-time signals are shown in Figure 1.9 and Figure 1.10, respectively.

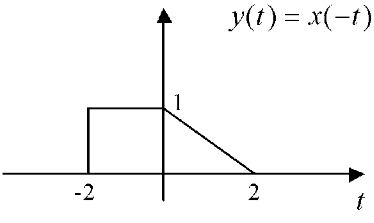

FIGURE 1.9 Graph of time-reversed signal $y(t) = x(-t)$.

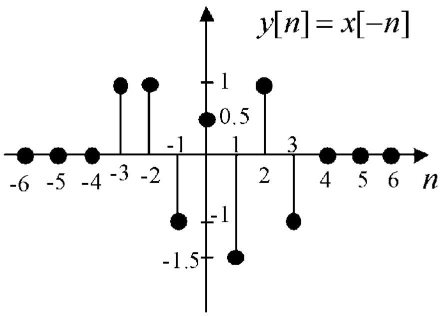

FIGURE 1.10 Graph of time-reversed signal $y[n] = x[-n]$.

Time Shift

A time shift delays or advances the signal in time by a continuous-time interval $T \in \mathbb{R}$:

$$y(t) = x(t + T). \tag{1.3}$$

For T positive, the signal is advanced; that is, it starts at time $t = -4$, which is before the time it originally started at, $t = -2$, as shown in Figure 1.11. For T negative, the signal is delayed, as shown in Figure 1.12.

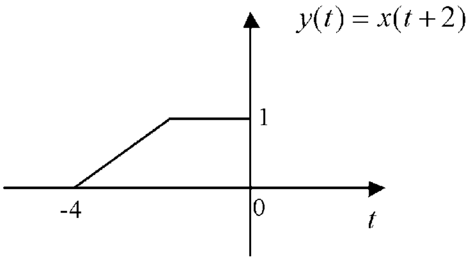

FIGURE 1.11 Graph of time-advanced signal $y(t) = x(t + 2)$.

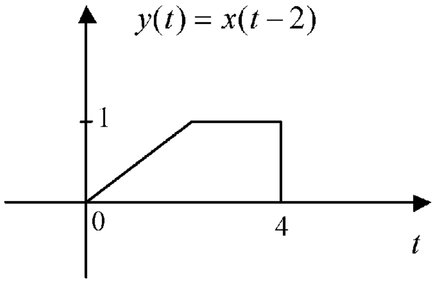

FIGURE 1.12 Graph of time-delayed signal $y(t) = x(t - 2)$.

Similarly, a time shift delays or advances a discrete-time signal by an integer discrete-time interval N:

$$y[n] = x[n + N]. \tag{1.4}$$

For N positive, the signal is advanced by N time steps, as shown in Figure 1.13. For N negative, the signal is delayed by $|N|$ time steps.

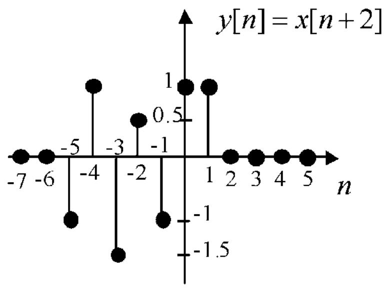

FIGURE 1.13 Graph of time-advanced signal $y[n] = x[n + 2]$.

PERIODIC SIGNALS

Intuitively, a signal is periodic when it repeats itself. This intuition is captured in the following definition: a continuous-time signal $x(t)$ is periodic if there exists a positive real T for which

$$x(t) = x(t+T), \quad \forall t \in \mathbb{R}. \tag{1.5}$$

A discrete-time signal $x[n]$ is periodic if there exists a positive integer N for which

$$x[n] = x[n+N], \quad \forall n \in \mathbb{Z}. \tag{1.6}$$

The smallest such T or N is called the *fundamental period* of the signal.

Example 1.6: The square wave signal in Figure 1.14 is periodic. The fundamental period of this square wave is $T = 4$, but 8, 12, and 16 are also periods of the signal.

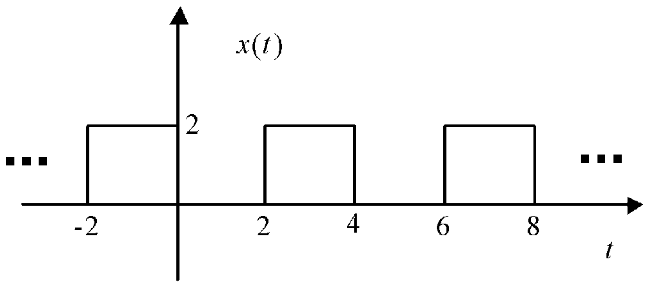

FIGURE 1.14 A continuous-time periodic square wave signal.

Example 1.7: The complex exponential signal $x(t) = e^{j\omega_0 t}$:

$$x(t+T) = e^{j\omega_0 (t+T)} = e^{j\omega_0 t} e^{j\omega_0 T}. \tag{1.7}$$

The right-hand side of Equation 1.7 is equal to $x(t) = e^{j\omega_0 t}$ for $T = \frac{2\pi k}{\omega_0}$, $k = \pm 1, \pm 2, \ldots$, so these are all periods of the complex exponential. The fundamental period is $T = \frac{2\pi}{\omega_0}$.

It may become more apparent that the complex exponential signal is periodic when it is expressed in its real/imaginary form:

$$x(t) = e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t). \tag{1.8}$$

where it is clear that the real part, $\cos(\omega_0 t)$, and the imaginary part, $\sin(\omega_0 t)$, are periodic with fundamental period $T = \frac{2\pi}{\omega_0}$.

Example 1.8: The discrete-time signal $x[n] = (-1)^n$ in Figure 1.15 is periodic with fundamental period $N = 2$.

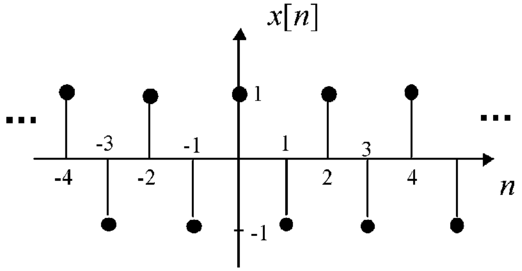

FIGURE 1.15 A discrete-time periodic signal.

EXPONENTIAL SIGNALS

Exponential signals are extremely important in signals and systems analysis because they are invariant under the action of linear time-invariant systems, which will be discussed in Chapter 2. This means that the output of an LTI system subjected to an exponential input signal will also be an exponential with the same exponent, but in general with a different real or complex amplitude.

Example 1.9: Consider the LTI system represented by a first-order differential equation initially at rest, with input $x(t) = e^{-2t}$:

$$\frac{dy(t)}{dt} + y(t) = x(t). \tag{1.9}$$

Its output signal is given by $y(t) = -e^{-2t}$. (Check it!)

Real Exponential Signals

Real exponential signals can be defined both in continuous time and in discrete time.

Continuous Time

We can define a general real exponential signal as follows:

$$x(t) = Ce^{\alpha t}, \quad 0 \neq C, \alpha \in \mathbb{R}. \tag{1.10}$$

We now look at different cases depending on the value of parameter α.

Case α = 0: We simply get the constant signal $x(t) = C$.

Case α > 0: The exponential tends to infinity as $t \to +\infty$, as shown in Figure 1.16, where $C > 0$. Notice that $x(0) = C$.

Case α < 0: The exponential tends to zero as $t \to +\infty$; see Figure 1.17, where $C < 0$.

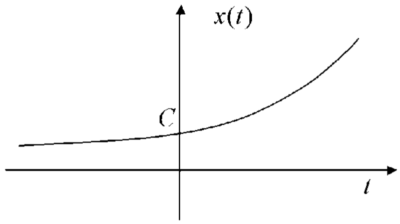

FIGURE 1.16 Continuous-time exponential signal growing unbounded with time.

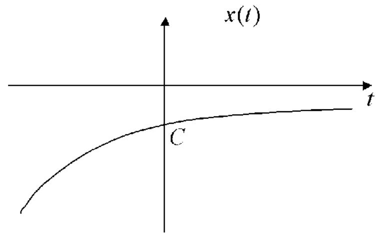

FIGURE 1.17 Continuous-time exponential signal tapering off to zero with time.

Discrete Time

We define a general real discrete-time exponential signal as follows:

$$x[n] = C\alpha^n, \quad C,\alpha \in \mathbb{R}. \tag{1.11}$$

There are six cases to consider, apart from the trivial cases α = 0 or C = 0: α = 1, α > 1, $0 < \alpha < 1$, α < –1, α = –1, and $-1 < \alpha < 0$. Here we assume that C > 0, but for C negative, the graphs would simply be flipped images of the ones given around the time axis.

Case α = 1: We get a constant signal $x[n] = C$.

Case α > 1: We get a positive signal that grows exponentially, as shown in Figure 1.18.

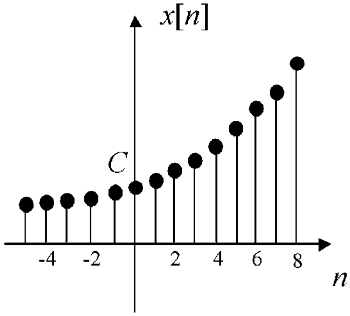

FIGURE 1.18 Discrete-time exponential signal growing unbounded with time.

Case $0 < \alpha < 1$: The signal $x[n] = C\alpha^n$ is positive and decays exponentially, as shown in Figure 1.19.

Case $\alpha < -1$: The signal $x[n] = C\alpha^n$ alternates between positive and negative values and grows exponentially in magnitude with time. This is shown in Figure 1.20.

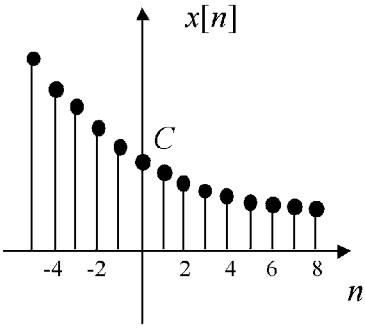

FIGURE 1.19 Discrete-time exponential signal tapering off to zero with time.

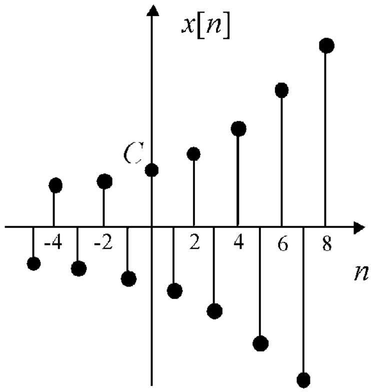

FIGURE 1.20 Discrete-time exponential signal alternating and growing unbounded with time.

Case $\alpha = -1$: The signal alternates between C and $-C$, as seen in Figure 1.21.

Case $-1 < \alpha < 0$: The signal alternates between positive and negative values and decays exponentially in magnitude with time, as shown in Figure 1.22.

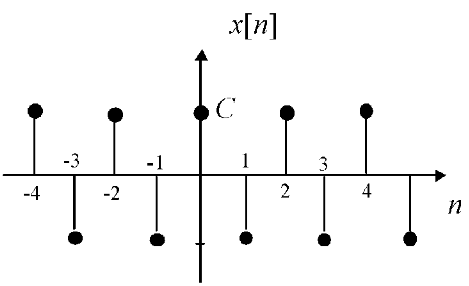

FIGURE 1.21 Discrete-time exponential signal reduced to an alternating periodic signal.

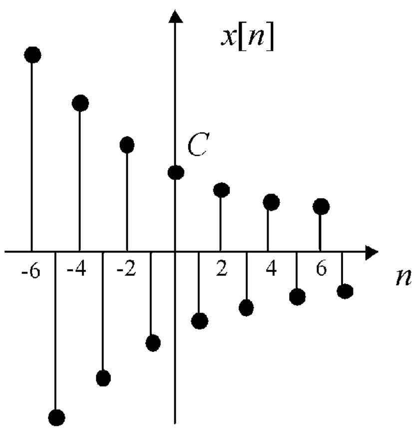

FIGURE 1.22 Discrete-time exponential signal alternating and tapering off to zero with time.

((Lecture 2: Some Useful Signals))

Complex Exponential Signals

Complex exponential signals can also be defined both in continuous time and in discrete time. They have real and imaginary parts with sinusoidal behavior.

Continuous Time

The continuous-time complex exponential signal can be defined as follows:

$$x(t) := Ce^{at}, \quad C, a \in \mathbb{C}, \tag{1.12}$$

where $C = Ae^{j\theta}$, $A, \theta \in \mathbb{R}, A > 0$ is expressed in polar form, and $a = \alpha + j\omega_0$, $\alpha, \omega_0 \in \mathbb{R}$ is expressed in rectangular form. Thus, we can write

$$\begin{aligned} x(t) &= Ae^{j\theta}e^{(\alpha + j\omega_0)t} \\ &= Ae^{\alpha t}e^{j(\omega_0 t + \theta)} \end{aligned} \tag{1.13}$$

If we look at the second part of Equation 1.13, we can see that $x(t)$ represents either a circular or a spiral trajectory in the complex plane, depending whether α is zero, negative, or positive. The term $e^{j(\omega_0 t + \theta)}$ describes a unit circle centered at the origin counterclockwise in the complex plane as time varies from $t = -\infty$ to $t = +\infty$, as shown in Figure 1.23 for the case $\theta = 0$. The times t_k indicated in the figure are the times when the complex point $e^{j\omega_0 t_k}$ has a phase of $\pi/4$.

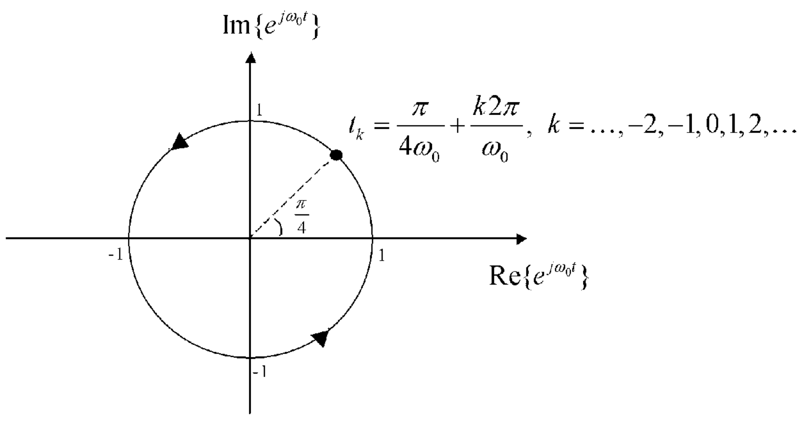

FIGURE 1.23 Trajectory described by the complex exponential.

Using Euler's relation, we obtain the signal in rectangular form:

$$x(t) = Ae^{\alpha t}\cos(\omega_0 t + \theta) + jAe^{\alpha t}\sin(\omega_0 t + \theta), \tag{1.14}$$

where $\mathrm{Re}\{x(t)\} = Ae^{\alpha t}\cos(\omega_0 t + \theta)$ and $\mathrm{Im}\{x(t)\} = Ae^{\alpha t}\sin(\omega_0 t + \theta)$ are the real part and imaginary part of the signal, respectively. Both are sinusoidal, with time-varying amplitude (or envelope) $Ae^{\alpha t}$. We can see that the exponent $\alpha = \mathrm{Re}\{a\}$ defines the type of real and imaginary parts we get for the signal.

For the case $\alpha = 0$, we obtain a complex periodic signal of period $T = \frac{2\pi}{\omega_0}$ (as shown in Figure 1.23 but with radius A) whose real and imaginary parts are sinusoidal:

$$x(t) = A\cos(\omega_0 t + \theta) + jA\sin(\omega_0 t + \theta). \tag{1.15}$$

The real part of this signal is shown in Figure 1.24.

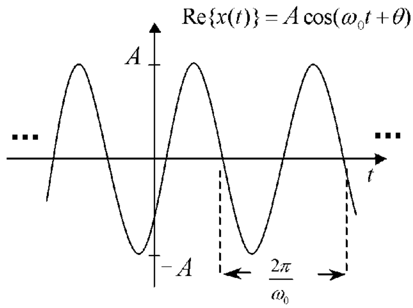

FIGURE 1.24 Real part of periodic complex exponential for $\alpha = 0$.

For the case $\alpha < 0$, we get a complex periodic signal multiplied by a decaying exponential. The real and imaginary parts are *damped sinusoids* that are signals that can describe, for example, the response of an *RLC* (resistance-inductance-capacitance) circuit or the response of a mass-spring-damper system such as a car suspension. The real part of $x(t)$ is shown in Figure 1.25.

For the case $\alpha > 0$, we get a complex periodic signal multiplied by a growing exponential. The real and imaginary parts are *growing sinusoids* that are signals that can describe the response of an unstable feedback control system. The real part of $x(t)$ is shown in Figure 1.26.

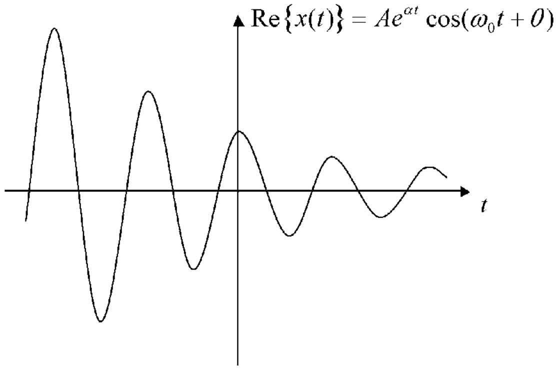

FIGURE 1.25 Real part of damped complex exponential for $\alpha < 0$.

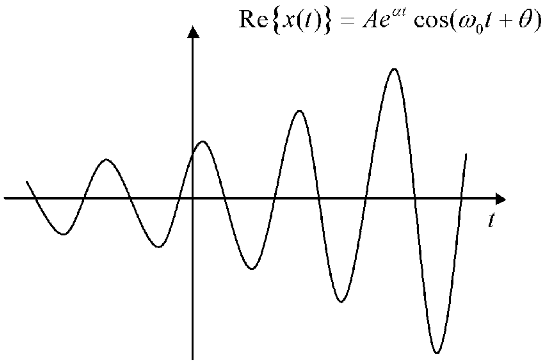

FIGURE 1.26 Real part of growing complex exponential for $\alpha > 0$.

ON THE CD

The MATLAB script given below and located on the CD-ROM in D:\Chapter1\complexexp.m, where D: is assumed to be the CD-ROM drive, generates and plots the real and imaginary parts of a decaying complex exponential signal.

```
%% complexexp.m generates a complex exponential signal and plots
%% its real and imaginary parts.
% time vector
t=0:.005:1;
% signal parameters
A=1;
```

```
theta=pi/4;
C=A*exp(j*theta);
alpha=-3;
w0=20;
a=alpha+j*w0;
% Generate signal
x=C*exp(a*t);
%plot real and imaginary parts
figure(1)
plot(t,real(x))
figure(2)
plot(t,imag(x))
```

Discrete Time

The discrete-time complex exponential signal can be defined as follows:

$$x[n] = Ca^n, \tag{1.16}$$

where $C, a \in \mathbb{C}, \quad C = Ae^{j\theta}, A, \theta \in \mathbb{R}, A > 0 \quad a = re^{j\omega_0}, r, \omega_0 \in \mathbb{R}, r > 0$.

Substituting the polar forms of C and a in Equation 1.16, we obtain a useful expression for $x[n]$ with time-varying amplitude:

$$\begin{aligned} x[n] &= Ae^{j\theta} r^n e^{j\omega_0 n} \\ &= Ar^n e^{j(\omega_0 n + \theta)}, \end{aligned} \tag{1.17}$$

and using Euler's relation, we get the rectangular form of the discrete-time complex exponential:

$$x[n] = Ar^n \cos(\omega_0 n + \theta) + jAr^n \sin(\omega_0 n + \theta). \tag{1.18}$$

Clearly, the magnitude r of a determines whether the envelope of $x[n]$ grows, decreases, or remains constant with time.

For the case $r = 1$, we obtain a complex signal whose real and imaginary parts have a sinusoidal envelope (they are sampled cosine and sine waves), *but the signal is not necessarily periodic*! We will discuss this issue in the next section.

$$x[n] = A\cos(\omega_0 n + \theta) + jA\sin(\omega_0 n + \theta) \tag{1.19}$$

Figure 1.27 shows the real part of a complex exponential signal with $r = 1$.

For the case $r < 1$, we get a complex signal whose real and imaginary parts are damped sinusoidal signals (see Figure 1.28).

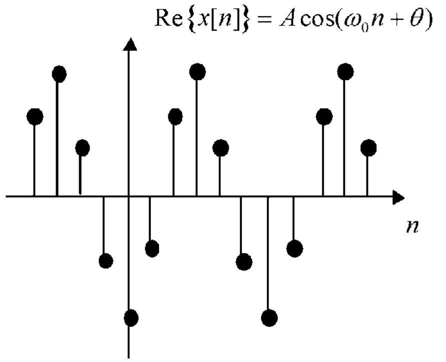

FIGURE 1.27 Real part of discrete-time complex exponential for $r = 1$.

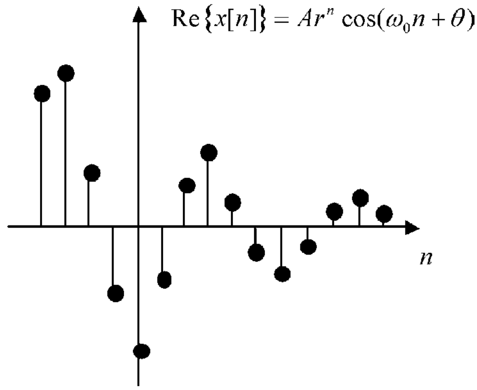

FIGURE 1.28 Real part of discrete-time damped complex exponential for $r < 1$.

For the case r > 1, we obtain a complex signal whose real and imaginary parts are growing sinusoidal sequences, as shown in Figure 1.29.

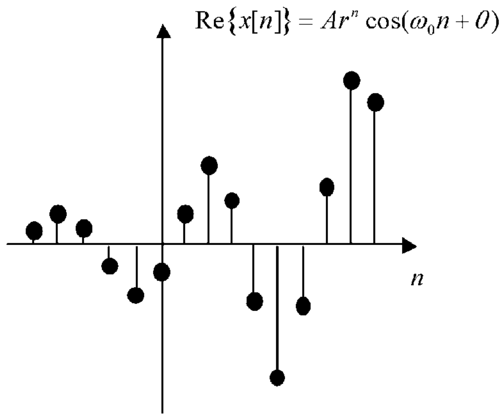

FIGURE 1.29 Real part of growing complex exponential for $r > 1$.

ON THE CD

The MATLAB script given below and located on the CD-ROM in D:\Chapter1\complexDTexp.m generates and plots the real and imaginary parts of a decaying discrete-time complex exponential signal.

```
%% complexDTexp.m generates a discrete-time
%% complex exponential signal and plots
%% its real and imaginary parts.
% time vector
n=0:1:20;
% signal parameters
A=1;
theta=pi/4;
C=A*exp(j*theta);
r=0.8;
w0=0.2*pi;
a=r*exp(j*w0);
% Generate signal
x=C*(a.^n);
%plot real and imaginary parts
figure(1)
stem(n,real(x))
figure(2)
stem(n,imag(x))
```

PERIODIC COMPLEX EXPONENTIAL AND SINUSOIDAL SIGNALS

In our study of complex exponential signals so far, we have found that in the cases $\alpha = \mathrm{Re}\{a\} = 0$ in continuous time and $r = |a| = 1$ in discrete time, we obtain signals whose trajectories lie on the unit circle in the complex plane. In particular, their real and imaginary parts are sinusoidal signals. We will see that in the continuous-time case, these signals are always periodic, but that is not necessarily the case in discrete time. Periodic complex exponentials can be used to define sets of harmonically related exponentials that have special properties that will be used later on to define the Fourier series.

Continuous Time

In continuous time, complex exponential and sinusoidal signals of constant amplitude are all periodic.

Periodic Complex Exponentials

Consider the complex exponential signal $e^{j\omega_0 t}$. We have already shown that this signal is periodic with fundamental period $T = \frac{2\pi}{\omega_0}$. Now let us consider *harmonically related complex exponential signals*:

$$\phi_k(t) := e^{jk\omega_0 t}, \quad k = \ldots,-2,-1,0,1,2,\ldots, \tag{1.20}$$

that is, complex exponentials with fundamental frequencies that are integer multiples of ω_0. These harmonically related signals have a very important property: they form an *orthogonal set*. Two signals $x(t)$, $y(t)$ are said to be orthogonal over an interval $[t_1, t_2]$ if their inner product, as defined in Equation 1.21, is equal to zero:

$$\int_{t_1}^{t_2} x(t)^* y(t)dt = 0, \tag{1.21}$$

where $x^*(t)$ is the complex conjugate of $x(t)$. This notion of orthogonality is a generalization of the concept of perpendicular vectors in three-dimensional Euclidean space $\mathbb{R}^3$. Two such perpendicular (or orthogonal) vectors $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ have an inner product equal to zero:

$$u^T v = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{i=1}^{3} u_i^T v_i = 0. \tag{1.22}$$

We know that a set of three orthogonal vectors can span the whole space $\mathbb{R}^3$ by forming linear combinations and therefore would constitute a basis for this space. It turns out that harmonically related complex exponentials (or *complex harmonics*) can also be seen as orthogonal vectors forming a basis for a space of vectors that are actually signals over the interval $[t_1, t_2]$. This space is infinite-dimensional, as there are infinitely many complex harmonics of increasing frequencies. It means that infinite linear combinations of the type $\sum_{k=-\infty}^{\infty} \alpha_k \phi_k(t)$ can basically represent any function of time in the signal space, which is the basis for the Fourier series representation of signals.

We now show that any two distinct complex harmonics $\phi_k(t) = e^{jk\omega_0 t}$ and $\phi_m(t) = e^{jm\omega_0 t}$, where $m \neq k$ are indeed orthogonal over their common period $T = \frac{2\pi}{\omega_0}$:

$$\begin{aligned} \int_0^{\frac{2\pi}{\omega_0}} \phi_k(t)^* \phi_m(t)dt &= \int_0^{\frac{2\pi}{\omega_0}} e^{-jk\omega_0 t} e^{jm\omega_0 t} dt = \int_0^{\frac{2\pi}{\omega_0}} e^{j(m-k)\omega_0 t} dt \\ &= \frac{1}{j(m-k)\omega_0} \left[\underbrace{e^{j(m-k)2\pi}}_{=1} - 1 \right] = 0. \end{aligned} \tag{1.23}$$

However, the inner product of a complex harmonic with itself evaluates to $T = \frac{2\pi}{\omega_0}$:

$$\int_0^{\frac{2\pi}{\omega_0}} \phi_k(t)^* \phi_k(t) dt = \int_0^{\frac{2\pi}{\omega_0}} e^{-jk\omega_0 t} e^{jk\omega_0 t} dt = \int_0^{\frac{2\pi}{\omega_0}} dt = \frac{2\pi}{\omega_0}. \tag{1.24}$$

Sinusoidal Signals

Continuous-time sinusoidal signals of the type $x(t) = A\cos(\omega_0 t + \theta)$ or $x(t) = A\sin(\omega_0 t + \theta)$ such as the one shown in Figure 1.30 are periodic with (fundamental) period $T = \frac{2\pi}{\omega_0}$, frequency $f_0 = \frac{\omega_0}{2\pi}$ in Hertz, angular frequency ω_0 in radians per second, and amplitude $|A|$. It is important to remember that in sinusoidal signals, or any other periodic signal, the shorter the period, the higher the frequency. For instance, in communication systems, a 1-MHz sine wave carrier has a period of 1 microsecond (10^{-6}s), while a 1-GHz sine wave carrier has a period of 1 nanosecond (10^{-9}s).

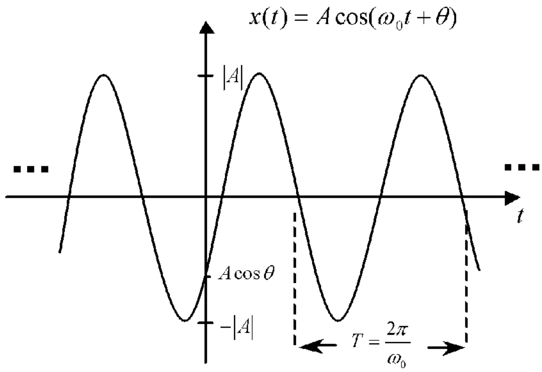

FIGURE 1.30 Continuous-time sinusoidal signal.

The following useful identities allow us to see the link between a periodic complex exponential and the sine and cosine waves of the same frequency and amplitude.

$$A\cos(\omega_0 t + \theta) = \frac{A}{2} e^{j\theta} e^{j\omega_0 t} + \frac{A}{2} e^{-j\theta} e^{-j\omega_0 t} = \mathrm{Re}\{Ae^{j(\omega_0 t + \theta)}\}, \tag{1.25}$$

$$A\sin(\omega_0 t + \theta) = \frac{A}{2j} e^{j\theta} e^{j\omega_0 t} - \frac{A}{2j} e^{-j\theta} e^{-j\omega_0 t} = \mathrm{Im}\{Ae^{j(\omega_0 t + \theta)}\}. \tag{1.26}$$

Discrete Time

In discrete time, complex exponential and sinusoidal signals of constant amplitude are not necessarily periodic.

Complex Exponential Signals

The complex exponential signal $Ae^{j\omega_0 n}$ is not periodic in general, although it seems like it is for any ω_0. The intuitive explanation is that the signal values, which are points on the unit circle in the complex plane, do not necessarily fall at the same locations as time evolves and the circle is described counterclockwise. When the signal values do always fall on the same points, then the discrete-time complex exponential is periodic. A more detailed analysis of periodicity is left for the next subsection on discrete-time sinusoidal signals, but it also applies to complex exponential signals.

The discrete-time complex harmonic signals defined by

$$\phi_k[n] := e^{jk\frac{2\pi}{N}n}, \quad k = 0,\ldots,N-1 \tag{1.27}$$

are periodic of (not necessarily fundamental) period N. They are also orthogonal, with the integral replaced by a sum in the inner product:

$$\begin{aligned}\sum_{n=0}^{N-1}\phi_k[n]^*\phi_m[n] &= \sum_{n=0}^{N-1} e^{-jk\frac{2\pi}{N}n} e^{jm\frac{2\pi}{N}n} = \sum_{n=0}^{N-1} e^{j(m-k)\frac{2\pi}{N}n} \\ &= \frac{1-e^{j(m-k)\frac{2\pi}{N}N}}{1-e^{j(m-k)\frac{2\pi}{N}}} = \frac{1-\overbrace{e^{j(m-k)2\pi}}^{=1}}{1-e^{j(m-k)\frac{2\pi}{N}}} = 0, \quad m \neq k.\end{aligned} \tag{1.28}$$

Here there are only N such distinct complex harmonics. For example, for $N = 8$, we could easily check that $\phi_0[n] = \phi_8[n] = 1$. These signals will be used in Chapter 12 to define the discrete-time Fourier series.

Sinusoidal Signals

Discrete-time sinusoidal signals of the type $x[n] = A\cos(\omega_0 n + \theta)$ are *not always periodic*, although the *continuous envelope* of the signal $A\cos(\omega_0 t + \theta)$ *is* periodic of period $T = \frac{2\pi}{\omega_0}$. A periodic discrete-time sinusoid such as the one in Figure 1.31 is such that the signal values, which are samples of the continuous envelope, always repeat the same pattern over any period of the envelope.

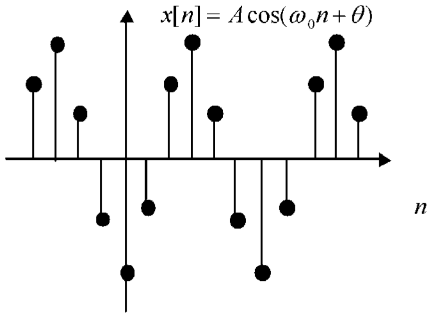

FIGURE 1.31 A periodic discrete-time sinusoidal signal.

Mathematically, we saw that $x[n]$ is periodic if there exists an integer $N > 0$ such that

$$x[n] = x[n+N] = A\cos(\omega_0 n + \omega_0 N + \theta). \tag{1.29}$$

That is, we must have $\omega_0 N = 2\pi m$ for some integer m, or equivalently:

$$\frac{\omega_0}{2\pi} = \frac{m}{N}; \tag{1.30}$$

that is, $\frac{\omega_0}{2\pi}$ must be a rational number (the ratio of two integers.) Then, the fundamental period $N > 0$ can also be expressed as $m\frac{2\pi}{\omega_0}$, assuming m and N have no common factor. The fundamental frequency defined by

$$\Omega_0 := \frac{2\pi}{N} = \frac{\omega_0}{m} \tag{1.31}$$

is expressed in radians. When the integers m and N have a common integer factor, that is, $m = m_0 q$ and $N = N_0 q$, then N_0 is the fundamental period of the sinusoid. These results hold for the complex exponential signal $e^{j(\omega_0 n + \theta)}$ as well.

FINITE-ENERGY AND FINITE-POWER SIGNALS

We defined signals as very general functions of time, although it is of interest to define classes of signals with special properties that make them significant in engineering. Such classes include signals with finite energy and signals of finite power.

The instantaneous power dissipated in a resistor of resistance R is simply the product of the voltage across and the current through the resistor:

$$p(t) = v(t)i(t) = \frac{v^2(t)}{R}, \tag{1.32}$$

and the *total energy* dissipated during a time interval $[t_1, t_2]$ is obtained by integrating the power

$$E_{[t_1,t_2]} = \int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} \frac{v^2(t)}{R} dt. \tag{1.33}$$

The *average power* dissipated over that interval is the total energy divided by the time interval:

$$P_{[t_1,t_2]} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{v^2(t)}{R} dt. \tag{1.34}$$

Analogously, the total energy and average power over $[t_1, t_2]$ of an arbitrary integrable continuous-time signal $x(t)$ are defined as though the signal were a voltage across a one-ohm resistor:

$$E_{[t_1,t_2]} := \int_{t_1}^{t_2} |x(t)|^2 \, dt, \tag{1.35}$$

$$P_{[t_1,t_2]} := \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 \, dt. \tag{1.36}$$

The total energy and total average power of a signal defined over $t \in \mathbb{R}$ are defined as

$$E_\infty := \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt, \tag{1.37}$$

$$P_\infty := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt. \tag{1.38}$$

The total energy and average power over $[n_1, n_2]$ of an arbitrary discrete-time signal $x[n]$ are defined as

$$E_{[n_1,n_2]} := \sum_{n=n_1}^{n_2} |x[n]|^2, \tag{1.39}$$

$$P_{[n_1,n_2]} := \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2. \tag{1.40}$$

Notice that $n_2 - n_1 + 1$ is the number of points in the signal over the interval $[n_1, n_2]$. The total energy and total average power of signal $x[n]$ defined over $n \in \mathbb{Z}$ are defined as

$$E_\infty := \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2, \tag{1.41}$$

$$P_\infty := \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2. \tag{1.42}$$

The class of continuous-time or discrete-time *finite-energy signals* is defined as the set of all signals for which $E_\infty < +\infty$.

Example 1.10: The discrete-time signal $x[n] := \begin{cases} 1, 0 \le n \le 10 \\ 0, \text{otherwise} \end{cases}$, for which $E_\infty = 11$ is a finite-energy signal.

The class of continuous-time or discrete-time *finite-power signals* is defined as the set of all signals for which $P_\infty < +\infty$.

Example 1.11: The constant signal $x(t) = 4$ has infinite energy, but a total average power of 16:

$$P_\infty := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 4^2\, dt = \lim_{T\to\infty} \frac{4^2}{2T} 2T = 16. \tag{1.43}$$

The total average power of a periodic signal can be calculated over one period only as $P_\infty = \frac{1}{T}\int_0^T |x(t)|^2\, dt$.

Example 1.12: For $x(t) = Ce^{j\omega_0 t}$, the total average power is computed as

$$P_\infty = \frac{1}{T}\int_0^T |Ce^{j\omega_0 t}|^2\, dt = \frac{|C|^2}{T}\int_0^T dt = \frac{|C|^2}{T}\left[T - 0\right] = |C|^2. \tag{1.44}$$

Note that $e^{j\omega_0 t}$ has unit power.

EVEN AND ODD SIGNALS

A continuous-time signal is said to be *even* if $x(t) = x(-t)$, and a discrete-time signal is even if $x[n] = x[-n]$. An even signal is therefore symmetric with respect to the vertical axis.

A signal is said to be *odd* if $x(t) = -x(-t)$ or $x[n] = -x[-n]$. Odd signals are symmetric with respect to the origin. Another way to view odd signals is that their portion at positive times can be flipped with respect to the vertical axis, then with respect to the horizontal axis, and the result corresponds exactly to the portion of the signal at negative times. It implies that $x(0) = 0$ or $x[0] = 0$.

Figure 1.32 shows a continuous-time even signal, whereas Figure 1.33 shows a discrete-time odd signal.

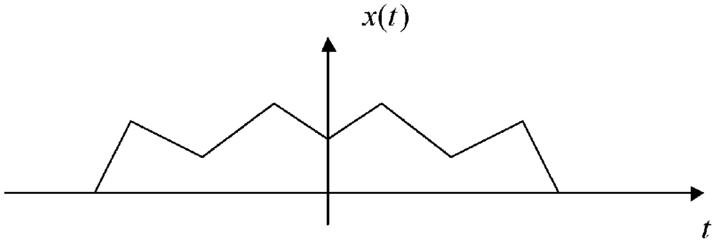

FIGURE 1.32 Even continuous-time signal.

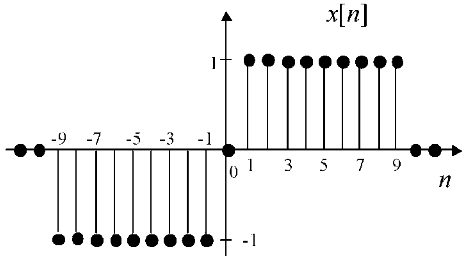

FIGURE 1.33 Odd discrete-time signal.

Any signal can be decomposed into its *even part* and its *odd part* as follows:

$$x(t) = x_e(t) + x_o(t) \tag{1.45}$$

Even part: $x_e(t) := \frac{1}{2}[x(t) + x(-t)]$ (1.46)

Odd part: $x_o(t) := \frac{1}{2}[x(t) - x(-t)]$ (1.47)

The even part and odd parts of a discrete-time signal are defined in the exact same way.

DISCRETE-TIME IMPULSE AND STEP SIGNALS

One of the simplest discrete-time signals is the *unit impulse* $\delta[n]$, also called the Dirac delta function, defined by

$$\delta[n] := \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \tag{1.48}$$

Its graph is shown in Figure 1.34.

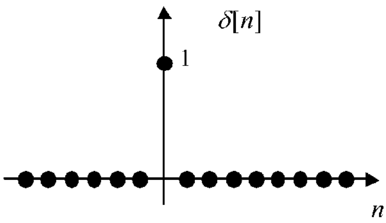

FIGURE 1.34 Discrete-time unit impulse.

The discrete-time *unit step* signal $u[n]$ is defined as follows:

$$u[n] := \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \tag{1.49}$$

The unit step is plotted in Figure 1.35.

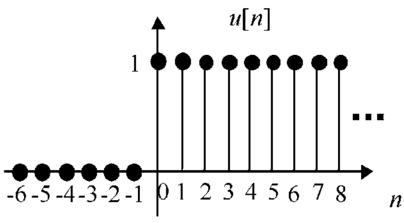

FIGURE 1.35 Discrete-time unit step signal.

The unit step is the running sum of the unit impulse:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k], \tag{1.50}$$

and conversely, the unit impulse is the *first-difference* of a unit step:

$$\delta[n] = u[n] - u[n-1]. \tag{1.51}$$

Also, the unit step can be written as an infinite sum of time-delayed unit impulses:

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]. \tag{1.52}$$

The *sampling property* of the unit impulse is an important property in the theory of sampling and in the calculation of convolutions, both of which are discussed in later chapters. The sampling property basically says that when a signal $x[n]$ is multiplied by a unit impulse occurring at time n_0, then the resulting signal is an impulse at that same time, but with an amplitude equal to the signal value at time n_0:

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]. \tag{1.53}$$

Another way to look at the sampling property is to take the sum of Equation 1.53 to obtain the signal sample at time n_0:

$$\sum_{k=-\infty}^{+\infty} x[k]\delta[k-n_0] = x[n_0]. \tag{1.54}$$

((Lecture 3: Generalized Functions and Input-Output System Models))

GENERALIZED FUNCTIONS

Continuous-Time Impulse and Step Signals

The continuous-time *unit step* function $u(t)$, plotted in Figure 1.36, is defined as follows:

$$u(t) := \begin{cases} 1, t > 0 \\ 0, t \le 0 \end{cases} \tag{1.55}$$

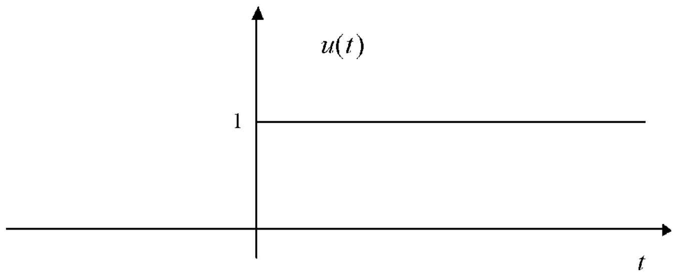

FIGURE 1.36 Continuous-time unit step signal.

Note that since $u(t)$ is discontinuous at the origin, it cannot be formally differentiated. We will nonetheless define the derivative of the step signal later and give its interpretation.

One of the uses of the step signal is to apply it at the input of a system in order to characterize its behavior. The resulting output signal is called the *step response* of the system. Another use is to truncate some parts of a signal by multiplication with time-shifted unit step signals.

Example 1.13: The finite-support signal $x(t)$ shown in Figure 1.37 can be written as $x(t) = e^t[u(t) - u(t-1)]$ or as $x(t) = e^t u(t)u(-t+1)$.

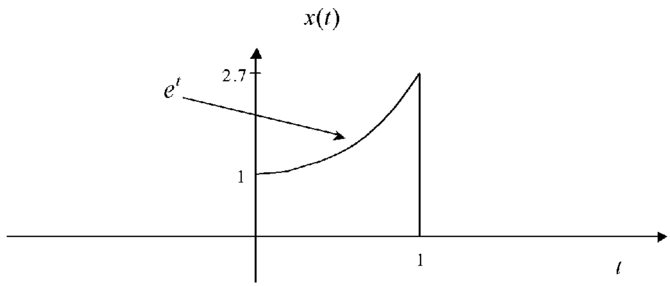

FIGURE 1.37 Truncated exponential signal.

The running integral of $u(t)$ is the *unit ramp* signal $tu(t)$ starting at $t = 0$, as shown in Figure 1.38:

$$\int_{-\infty}^{t} u(\tau)d\tau = tu(t) \tag{1.56}$$

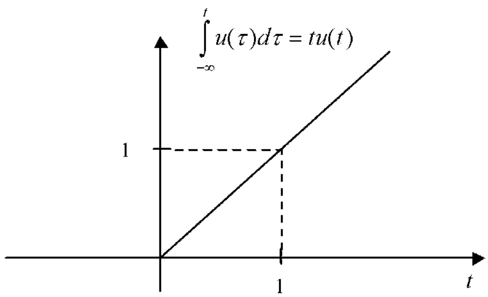

FIGURE 1.38 Continuous-time unit ramp signal.

Successive integrals of $u(t)$ yield signals with increasing powers of t:

$$\int_{-\infty}^{t}\int_{-\infty}^{\tau_{k-1}}\cdots\int_{-\infty}^{\tau_1} u(\tau)d\tau d\tau_1\cdots d\tau_{k-1} = \frac{1}{k!}t^k u(t) \tag{1.57}$$

The *unit impulse* $\delta(t)$, a generalized function that has infinite amplitude over an infinitesimal support at $t = 0$, can be defined as follows. Consider a rectangular pulse function of unit area shown in Figure 1.39, defined as:

$$\delta_\Delta(t) := \begin{cases} \frac{1}{\Delta}, & 0 < t < \Delta \\ 0, & \text{otherwise} \end{cases} \tag{1.58}$$

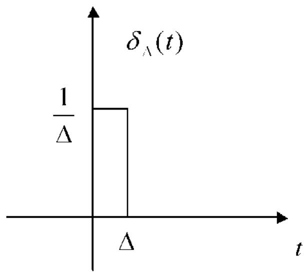

FIGURE 1.39 Continuous-time rectangular pulse signal.

The running integral of this pulse is an approximation to the unit step, as shown in Figure 1.40.

$$u_\Delta(t) := \int_{-\infty}^{t} \delta_\Delta(\tau)d\tau = \frac{1}{\Delta}tu(t) - \frac{1}{\Delta}(t-\Delta)u(t-\Delta) \qquad (1.59)$$

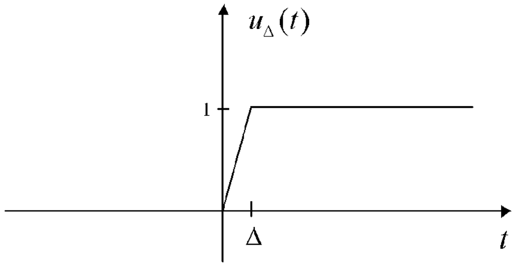

FIGURE 1.40 Integral of rectangular pulse signal approximating the unit step.

As Δ tends to 0, the pulse $\delta_\Delta(t)$ gets taller and thinner but keeps its unit area, which is the key property here, while $u_\Delta(t)$ approaches a unit step function. At the limit,

$$\delta(t) := \lim_{\Delta \to 0} \delta_\Delta(t) \qquad (1.60)$$

$$u(t) = \lim_{\Delta \to 0} u_\Delta(t) \qquad (1.61)$$

Note that $\delta_\Delta(t) = \frac{d}{dt}u_\Delta(t)$, and in this sense we can write $\delta(t) = \frac{d}{dt}u(t)$ at the limit, so that the impulse is the derivative of the step. Conversely, we have the important relationship stating that the unit step is the running integral of the unit impulse:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \qquad (1.62)$$

Graphically, $\delta(t)$ is represented by an arrow "pointing to infinity" at $t = 0$ with *its length equal to the area of the impulse*, as shown in Figure 1.41.

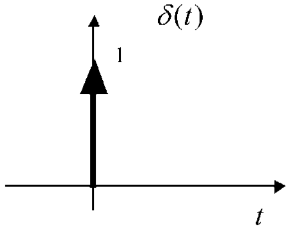

FIGURE 1.41 Unit impulse signal.

We mentioned earlier that the key property of the pulse $\delta_\Delta(t)$ is that its area is invariant as $\Delta \to 0$. This means that the impulse $\delta(t)$ packs significant "punch," enough to make a system react to it, even though it is zero at all times except at $t = 0$. The output of a system subjected to the unit impulse is called the *impulse response*.

Note that with the definition in Equation 1.60, the area of the impulse lies to the right of $t = 0$, so that integrating $A\delta(t)$ from $t = 0$ yields $\int_0^\infty A\delta(t)dt = A$. Had we defined the impulse as the limit of the pulse $\tilde{\delta}_\Delta(t) := \frac{1}{\Delta}[u(t+\Delta) - u(t)]$ whose area lies to the left of $t = 0$, we would have obtained $\int_0^\infty A\tilde{\delta}(t)dt = 0$. In order to "catch the impulse" in the integral, the trick is then to integrate from the left of the y-axis, but infinitesimally close to it. This time is denoted as $t = 0^-$. Similarly, the time $t = 0^+$ is to the right of $t = 0$ but infinitesimally close to it, so that for our definition of $\delta(t)$ in Equation 1.60, the above integral would have evaluated to zero: $\int_{0^+}^\infty A\delta(t)dt = 0$.

The following example provides motivation for the use of the impulse signal.

Example 1.14: Instantaneous discharge of a capacitor.

Consider the simple RC circuit depicted in Figure 1.42, with a constant voltage source V having fully charged a capacitor C through a resistor R_1. At time $t = 0$, the switch is thrown from position S_2 to position S_1 so that the capacitor starts discharging through resistor R. What happens to the current $i(t)$ as R tends to zero?

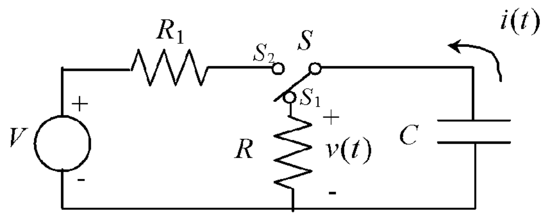

FIGURE 1.42 Simple *RC* circuit for analysis of capacitor discharge.

The capacitor is charged to a voltage V and a charge $Q = CV$ at $t = 0^-$. When the switch is thrown from S_2 to S_1 at $t = 0$, we have:

$$i(t) = \frac{v(t)}{R}, \tag{1.63}$$

$$i(t) = -C\frac{dv(t)}{dt}. \tag{1.64}$$

Combining Equation 1.63 and Equation 1.64, we get

$$RC\frac{dv(t)}{dt} + v(t) = 0. \tag{1.65}$$

The solution to this differential equation is

$$v(t) = Ve^{-t/RC}u(t), \tag{1.66}$$

and the current is simply

$$i(t) = \frac{V}{R}e^{-t/RC}u(t), \tag{1.67}$$

If we let R tend to 0, $i(t)$ tends to a tall, sharp pulse whose area remains constant at $Q = CV$, the initial charge in the capacitor (as $Q = \int_0^\infty i(t)dt$). We get an impulse. Of course if you tried this in reality, that is, shorting a charged capacitor, it would probably blow up, thereby demonstrating that the current flowing through the capacitor went "really high" in a very short time, burning the device.

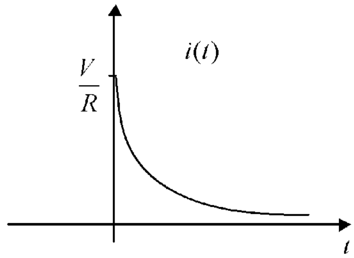

FIGURE 1.43 Capacitor discharge current in *RC* circuit.

Some Properties of the Impulse Signal

Sampling Property

The pulse function $\delta_\Delta(t)$ can be made narrow enough so that $x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t)$, and at the limit, for an impulse at time t_0,

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0) \tag{1.68}$$

so that

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0) \tag{1.69}$$

This last equation is often cited as the correct definition of an impulse, since it implicitly defines the impulse through what it does to any continuous function under the integral sign, rather than using a limiting argument pointwise, as we did in Equation 1.60.

Time Scaling

Time scaling of an impulse produces a change in its area. This is shown by calculating the integral in the sampling property with the time-scaled impulse. For $\alpha \in \mathbb{R}, \alpha \neq 0$:

$$
\begin{aligned}
\int_{-\infty}^{+\infty} x(t)\,\delta(\alpha t)dt &= \frac{1}{\alpha}\int_{-\infty}^{+\infty} x(\frac{\tau}{\alpha})\,\delta(\tau)d\tau \\
&= \begin{cases} \dfrac{1}{\alpha}\displaystyle\int_{-\infty}^{+\infty} x(\frac{\tau}{\alpha})\,\delta(\tau)d\tau, & \alpha > 0 \\ \dfrac{1}{\alpha}\displaystyle\int_{+\infty}^{-\infty} x(\frac{\tau}{\alpha})\,\delta(\tau)d\tau, & \alpha < 0 \end{cases} \\
&= \frac{1}{|\alpha|}\int_{-\infty}^{+\infty} x(\frac{\tau}{\alpha})\,\delta(\tau)d\tau \\
&= \frac{1}{|\alpha|}x(0)
\end{aligned}
\tag{1.70}
$$

Hence,

$$
\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t). \tag{1.71}
$$

Note that the equality sign in Equation 1.71 means that both of these impulses have the same effect under the integral in the sampling property.

Time Shift

The *convolution* of signals $x(t)$ and $y(t)$ is defined as

$$
x(t) * y(t) := \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau = \int_{-\infty}^{\infty} y(\tau)x(t-\tau)d\tau \tag{1.72}
$$

The convolution of signal $x(t)$ with the time-delayed impulse $\delta(t-T)$ delays the signal by T:

$$
\delta(t-T) * x(t) = \int_{-\infty}^{\infty} \delta(\tau - T)x(t-\tau)d\tau = x(t-T) \tag{1.73}
$$

Unit Doublet and Higher Order "Derivatives" of the Unit Impulse

What is $\delta'(t) := \frac{d\delta(t)}{dt}$, the *unit doublet*? That is, what does it do for a living? To answer this question, we look at the following integral, integrated by parts:

$$\int_{-\infty}^{\infty}\delta'(t)x(t)dt = \left[x(0)\delta(t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\delta(t)\frac{d}{dt}x(t)dt$$
$$= 0 - \frac{dx(0)}{dt} = -\frac{dx(0)}{dt}. \qquad (1.74)$$

Thus, the unit doublet samples the *derivative of the signal* at time $t = 0$ (modulo the minus sign.) For higher order derivatives of $\delta(t)$, we have

$$\int_{-\infty}^{\infty}\delta^{(k)}(t)x(t)dt = (-1)^k \frac{d^k x(0)}{dt^k}. \qquad (1.75)$$

Why is $\delta'(t)$ called a "doublet?" A possible representation of this generalized function comes from differentiating the pulse $\delta_\Delta(t)$, which produces two impulses, one negative and one positive. Then by letting $\Delta \to 0$, we get a "double impulse" at $t = 0$, as shown in Figure 1.44. Note that the resulting "impulses" are not regular impulses since their area is infinite.

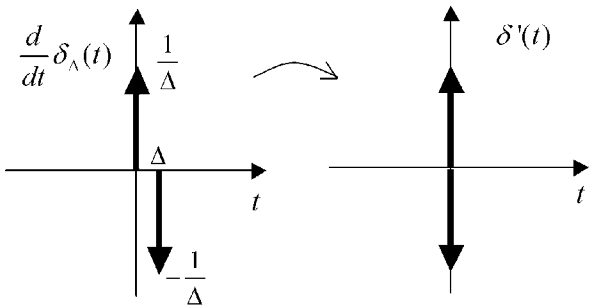

FIGURE 1.44 Representation of the unit doublet.

SYSTEM MODELS AND BASIC PROPERTIES

Recall that we defined signals as functions of time. In this book, a *system* is also simply defined as a mathematical relationship, that is, a function, between an input signal $x(t)$ or $x[n]$ and an output signal $y(t)$ or $y[n]$. Without going into too much d etail, recall that functions map their domain (set of input signals) into their co-domain (set of output signals, of which the range is a subset) and have the special property that any input signal in the domain of the system has a single associated output signal in the range of the system.

Input-Output System Models

The mathematical relationship of a system H between its input signal and its output signal can be formally written as $y = Hx$ (the time argument is dropped here, as this representation is used both for continuous-time and discrete-time systems). Note that this is not a multiplication by H—rather, it means that system (or function) H is applied to the input signal. For example, system H could represent a very complicated nonlinear differential equation linking $y(t)$ to $x(t)$.

A system is often conveniently represented by a block diagram, as shown in Figure 1.45 and Figure 1.46.

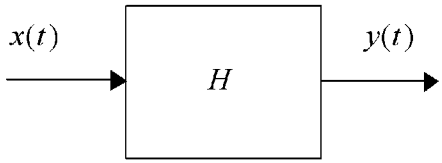

FIGURE 1.45 Block diagram representation of a continuous-time system *H*.

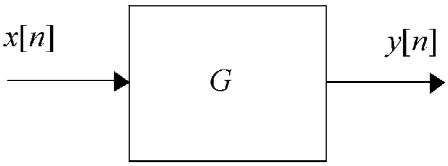

FIGURE 1.46 Block diagram representation of a discrete-time system *G*.

System Block Diagrams

Systems may be interconnections of other systems. For example, the discrete-time system $y[n] = Gx[n]$ shown as a block diagram in Figure 1.47 can be described by the following system equations:

$$
\begin{aligned}
v[n] &= G_1 x[n] \\
w[n] &= G_2 v[n] \\
z[n] &= G_3 x[n] \\
s[n] &= w[n] - z[n] \\
y[n] &= G_4 s[n]
\end{aligned}
\tag{1.76}
$$

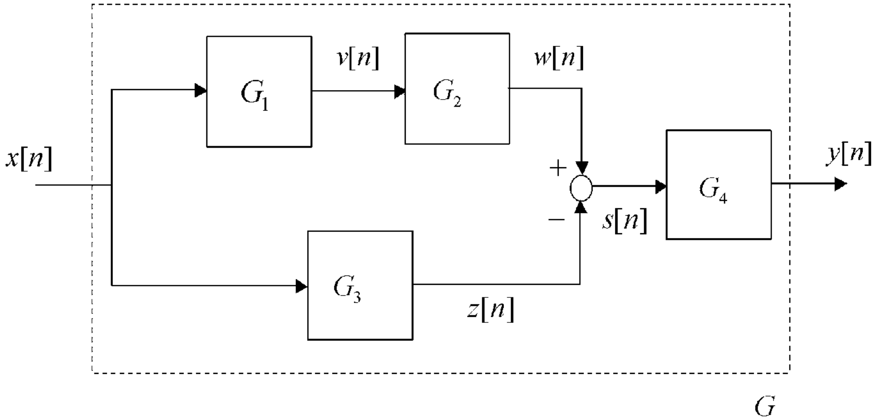

FIGURE 1.47 A discrete-time system composed of an interconnection of other systems.

We now look at some basic system interconnections, of which more complex systems are composed.

Cascade Interconnection

The cascade interconnection shown in Figure 1.48 is a successive application of two (or more) systems on an input signal:

$$y = G_2 \underbrace{(G_1 x)}_{y_1} \tag{1.77}$$

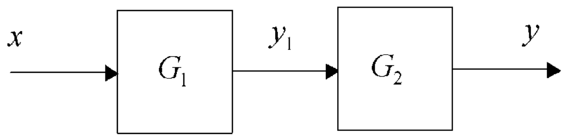

FIGURE 1.48 Cascade interconnection of systems.

Parallel Interconnection

The parallel interconnection shown in Figure 1.49 is an application of two (or more) systems to the same input signal, and the output is taken as the sum of the outputs of the individual systems.

$$y = G_1 x + G_2 x \tag{1.78}$$

Note that because there is no assumption of linearity or any other property for systems G_1, G_2, we are not allowed to write, for example, $y = (G_1 + G_2)x$. System properties will be defined later.

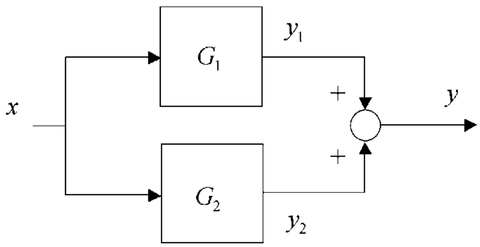

FIGURE 1.49 Parallel interconnection of systems.

Feedback Interconnection

The feedback interconnection of two systems as shown in Figure 1.50 is a feedback of the output of system G_1 to its input, through system G_2. This interconnection is quite useful in feedback control system analysis and design. In this context, signal e is the error between a desired output signal and a direct measurement of the output. The equations are

$$\begin{aligned} e &= x - G_2 y \\ y &= G_1 e \end{aligned} \tag{1.79}$$

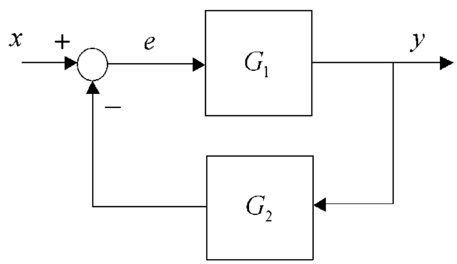

FIGURE 1.50 Feedback interconnection of systems.

Example 1.15: Consider the car cruise control system in Figure 1.51, whose task is to keep the car's velocity close to its setpoint. The system G is a model of the car's dynamics from the throttle input to the speed output, whereas system C is the controller, whose input is the velocity error e and whose output is the engine throttle.

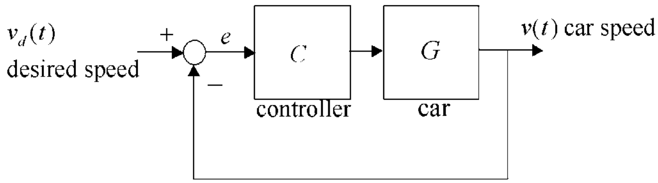

FIGURE 1.51 Feedback interconnection of a car cruise control system.

((Lecture 4: Basic System Properties))

Basic System Properties

All of the following system properties apply equally to continuous-time and discrete-time systems.

Linearity

A system S is *linear* if it has the *additivity* property *and* the *homogeneity* property. Let $y_1 := Sx_1$ and $y_2 := Sx_2$.

$$\text{Additivity: } y_1 + y_2 = S(x_1 + x_2) \tag{1.80}$$

That is, the response of S to the combined signal $x_1 + x_2$ is the sum of the individual responses y_1 and y_2.

$$\text{Homogeneity: } ay_1 = S(ax_1), \quad \forall a \in \mathbb{C} \tag{1.81}$$

Homogeneity means that the response of S to the scaled signal ax_1 is a times the response $y_1 = Sx_1$. An important consequence is that the response of a linear system to the 0 signal is the 0 signal. Thus, the system $y(t) = 2x(t) + 3$ is nonlinear because for $x(t) = 0$, we obtain $y(t) = 3$.

The linearity property (additivity and homogeneity combined) is summarized in the important *Principle of Superposition*: *the response to a linear combination of input signals is the same linear combination of the corresponding output signals.*

Example 1.16: Consider the ideal operational-amplifier (op-amp) integrator circuit shown in Figure 1.52.

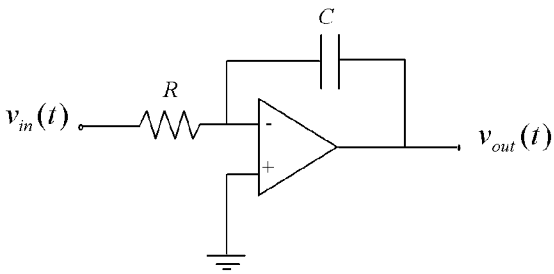

FIGURE 1.52 Ideal op-amp integrator circuit.

The output voltage of this circuit is given by a running integral of the input voltage:

$$v_{out}(t) = \frac{1}{RC}\int_{-\infty}^{t} v_{in}(\tau)d\tau \tag{1.82}$$

If $v_{in}(t) = av_1(t) + bv_2(t)$, then

$$v_{out}(t) = \frac{1}{RC}\int_{-\infty}^{t} v_{in}(\tau)d\tau = \frac{a}{RC}\int_{-\infty}^{t} v_1(\tau)d\tau + \frac{b}{RC}\int_{-\infty}^{t} v_2(\tau)d\tau \tag{1.83}$$

and hence this circuit is linear.

Time Invariance

A system S is *time-invariant* if its response to a time-shifted input signal $x[n - N]$ is equal to its original response $y[n]$ to $x[n]$, but also time shifted by N: $y[n - N]$. That is, if for $y[n] := Sx[n]$, $y_1[n] := Sx[n - N]$, the equality $y_1[n] = y[n - N]$ holds for any integer N, then the system is time-invariant.

Example 1.17: $y(t) = \sin(x(t))$ is time-invariant since $y_1(t) = \sin(x(t - T)) = y(t - T)$.

On the other hand, the system $y[n] = nx[n]$ is not time-invariant (it is *time-varying*) since $y_1[n] = nx[n-N] \neq (n-N)x[n-N] = y[n-N]$.

The time-invariance property of a system makes its analysis easier, as it is sufficient to study, for example, the impulse response or the step response starting at time $t = 0$. Then, we know that the response to a time-shifted impulse would have the exact same shape, except it would be shifted by the same interval of time as the impulse.

Memory

A system is *memoryless* if its output y at time t or n depends only on the input at that same time.

Examples of memoryless systems:

$$y[n] = x[n]^2$$

$$y(t) = \frac{x(t)}{1 + x(t)}$$

Resistor: $v(t) = Ri(t)$.

Conversely, a system has *memory* if its output at time t or n depends on input values at some other times.

Examples of systems with memory:

$$y[n] = x[n+1] + x[n] + x[n-1]$$

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

Causality

A system is *causal* if its output at time t or n depends only on past or current values of the input.

An important consequence is that if $y_1 := Sx_1, y_2 := Sx_2$ and $x_1(\tau) = x_2(\tau)$, $\forall \tau \in (-\infty, t]$, then $y_1(\tau) = y_2(\tau), \forall \tau \in (-\infty, t]$. This means that a causal system subjected to two input signals that coincide up to the current time t produces outputs that also coincide up to time t. This is not the case for noncausal systems because their output up to time t depends on future values of the input signals, which may differ by assumption.

Examples of causal systems:
A car does not anticipate its driver's actions, or the road condition ahead.

$$y[n] = \sum_{k=-\infty}^{n} x[k-N], \quad N \geq 1$$

$$\frac{dy(t)}{dt} + ay(t) = bx(t) + cx(t-T) \qquad T > 0$$

Example of a *noncausal* system:

$$y[n] = \sum_{k=-\infty}^{n} x[n-k]$$

Bounded-Input Bounded-Output Stability

A system S is *bounded-input bounded-output (BIBO) stable* if for any bounded input x, the corresponding output y is also bounded. Mathematically, the continuous-time system $y(t) = Sx(t)$ is BIBO stable if

$$\begin{aligned} &\forall K_1 > 0, \exists K_2 > 0 \text{ such that} \\ &|x(t)| < K_1, -\infty < t < \infty \quad \Rightarrow \quad |y(t)| < K_2, -\infty < t < \infty \end{aligned} \tag{1.84}$$

In this statement, $\Rightarrow$ means *implies*, $\forall$ means *for every*, and $\exists$ means *there exists*.

In other words, if we had a system S that we claimed was BIBO stable, then for any positive real number K_1 that someone challenges us with, we would have to find another positive real number K_2 such that, for *any input signal* $x(t)$ bounded in magnitude by K_1 at all times, the corresponding output signal $y(t)$ of S would also be bounded in magnitude by K_2 at all times. This is illustrated in Figure 1.53.

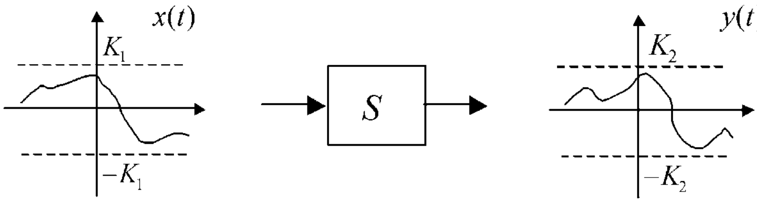

FIGURE 1.53 BIBO stability of a system.