

ANALYSIS 2 - HAUSAUFGABE 7

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Aufgabe 1

Da der Definitionsbereich von f offen und konvex ist, muss nur noch überprüft werden, ob $\operatorname{rot} \vec{v} = \vec{0}$

$$\begin{aligned}\operatorname{rot} \vec{v} &= \operatorname{rot} \begin{pmatrix} f(x, y, z) \\ x^2 + yz^2 \\ y^2z \end{pmatrix} \\ &= \begin{pmatrix} 2yz - 2zy \\ -(0 - \frac{\partial f}{\partial z}) \\ 2x - \frac{\partial f}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \frac{\partial f}{\partial z} \\ 2x - \frac{\partial f}{\partial y} \end{pmatrix}\end{aligned}$$

Damit $\operatorname{rot}(\vec{v}) = \vec{0}$ muss gelten, dass f kein z enthält (da sonst $\frac{\partial f}{\partial z}$ nicht 0 ist), und dass es nur ein y enthält. Somit muss $f(x, y, z) = 2xy + K(x)$, wobei $K(x)$ eine von y und z unabhängige Konstante ist.

Daher muss gelten:

$$\begin{aligned}\operatorname{grad} u &= -\vec{v} \\ \Leftrightarrow \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{pmatrix} &= - \begin{pmatrix} 2xy + K(x) \\ x^2 + yz^2 \\ y^2z \end{pmatrix} \quad \begin{matrix} \text{I} \\ \text{II} \\ \text{III} \end{matrix}\end{aligned}$$

Daraus lassen sich zeilenweise Gleichungen aufstellen.

$$\text{I: } \frac{\partial u}{\partial x} = -2xy + K(x)$$

$$\text{II: } \frac{\partial u}{\partial y} = -x^2 - yz^2$$

$$\text{III: } \frac{\partial u}{\partial z} = -y^2z$$

Wir integrieren I nach x:

$$u = -x^2y - \int K(x) \partial x + C_1(y, z)$$

Wir integrieren II nach y:

$$u = -x^2y - \frac{y^2z^2}{2} + C_2(x, z)$$

Wir integrieren III nach z:

$$u = -\frac{y^2z^2}{2} + C_3(x, y)$$

Man sieht, dass u folgende Gestalt hat:

$$u = -x^2y - \int K(x) \partial x - \frac{y^2z^2}{2} + C$$

Bleibt das Ergebnis noch zu überprüfen:

$$\begin{aligned} \text{grad} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{pmatrix} &= \begin{pmatrix} -2xy - K(x) \\ -x^2 - yz^2 \\ -y^2z \end{pmatrix} \\ &= - \begin{pmatrix} 2xy + K(x) \\ x^2 + yz^2 \\ y^2z \end{pmatrix} = -\vec{v} \end{aligned}$$

Somit ist u das Potential von \vec{v} , falls dieses existiert (nach geeigneter Wahl von f).

Aufgabe 2

Sei $\vec{v}(x, y, z) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ und $\vec{w}(x, y, z) = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$.

$$\begin{aligned}
 \operatorname{div} \left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right) &= \left(\operatorname{rot} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right) \cdot \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot \operatorname{rot} \left(\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right) \\
 \Leftrightarrow \operatorname{div} \left(\begin{pmatrix} v_2 w_3 - v_3 w_2 \\ -v_1 w_3 + v_3 w_1 \\ v_1 w_2 - v_2 w_1 \end{pmatrix} \right) &= \begin{pmatrix} \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \\ -\frac{\partial v_3}{\partial x} + \frac{\partial v_1}{\partial z} \\ \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial w_3}{\partial y} - \frac{\partial w_2}{\partial z} \\ -\frac{\partial w_3}{\partial x} + \frac{\partial w_1}{\partial z} \\ \frac{\partial w_2}{\partial x} - \frac{\partial w_1}{\partial y} \end{pmatrix} \\
 \Leftrightarrow \frac{\partial}{\partial x}(v_2 w_3 - v_3 w_2) + \frac{\partial}{\partial y}(-v_1 w_3 + v_3 w_1) + \frac{\partial}{\partial z}(v_1 w_2 - v_2 w_1) \\
 &= \left(\frac{\partial}{\partial y} v_3 \right) w_1 - \left(\frac{\partial}{\partial z} v_2 \right) w_1 - \left(\frac{\partial}{\partial x} v_3 \right) w_2 + \left(\frac{\partial}{\partial z} v_1 \right) w_2 + \left(\frac{\partial}{\partial x} v_2 \right) w_3 - \left(\frac{\partial}{\partial y} v_1 \right) w_3 \\
 &\quad - v_1 \left(\frac{\partial}{\partial y} w_3 \right) + v_1 \left(\frac{\partial}{\partial z} w_2 \right) + v_2 \left(\frac{\partial}{\partial x} w_3 \right) - v_2 \left(\frac{\partial}{\partial z} w_1 \right) - v_3 \left(\frac{\partial}{\partial x} w_2 \right) + v_3 \left(\frac{\partial}{\partial y} w_1 \right) \\
 \Leftrightarrow \frac{\partial}{\partial x} v_2 w_3 - \frac{\partial}{\partial x} v_3 w_2 - \frac{\partial}{\partial y} v_1 w_3 + \frac{\partial}{\partial y} v_3 w_1 + \frac{\partial}{\partial z} v_1 w_2 - \frac{\partial}{\partial z} v_2 w_1 \\
 &= \left(\frac{\partial}{\partial y} v_3 \right) w_1 - \left(\frac{\partial}{\partial z} v_2 \right) w_1 - \left(\frac{\partial}{\partial x} v_3 \right) w_2 + \left(\frac{\partial}{\partial z} v_1 \right) w_2 + \left(\frac{\partial}{\partial x} v_2 \right) w_3 - \left(\frac{\partial}{\partial y} v_1 \right) w_3 \\
 &\quad - v_1 \left(\frac{\partial}{\partial y} w_3 \right) + v_1 \left(\frac{\partial}{\partial z} w_2 \right) + v_2 \left(\frac{\partial}{\partial x} w_3 \right) - v_2 \left(\frac{\partial}{\partial z} w_1 \right) - v_3 \left(\frac{\partial}{\partial x} w_2 \right) + v_3 \left(\frac{\partial}{\partial y} w_1 \right) \\
 \stackrel{\text{prod. R.}}{\Leftrightarrow} &\left(\frac{\partial}{\partial x} v_2 \right) w_3 + \left(\frac{\partial}{\partial x} w_3 \right) w_2 - \left(\left(\frac{\partial}{\partial x} v_3 \right) w_2 + \left(\frac{\partial}{\partial x} w_2 \right) v_3 \right) - \left(\left(\frac{\partial}{\partial y} v_1 \right) w_3 + \left(\frac{\partial}{\partial y} w_3 \right) v_1 \right) \\
 &\quad + \left(\frac{\partial}{\partial y} v_3 \right) w_1 + \left(\frac{\partial}{\partial y} w_1 \right) v_3 + \left(\frac{\partial}{\partial z} v_1 \right) w_2 + \left(\frac{\partial}{\partial z} w_2 \right) v_1 - \left(\left(\frac{\partial}{\partial z} v_2 \right) w_1 + \left(\frac{\partial}{\partial z} w_1 \right) v_2 \right) \\
 &= \left(\frac{\partial}{\partial y} v_3 \right) w_1 - \left(\frac{\partial}{\partial z} v_2 \right) w_1 - \left(\frac{\partial}{\partial x} v_3 \right) w_2 + \left(\frac{\partial}{\partial z} v_1 \right) w_2 + \left(\frac{\partial}{\partial x} v_2 \right) w_3 - \left(\frac{\partial}{\partial y} v_1 \right) w_3 \\
 &\quad - v_1 \left(\frac{\partial}{\partial y} w_3 \right) + v_1 \left(\frac{\partial}{\partial z} w_2 \right) + v_2 \left(\frac{\partial}{\partial x} w_3 \right) - v_2 \left(\frac{\partial}{\partial z} w_1 \right) - v_3 \left(\frac{\partial}{\partial x} w_2 \right) + v_3 \left(\frac{\partial}{\partial y} w_1 \right) \\
 \Leftrightarrow &\left(\frac{\partial}{\partial x} v_2 \right) w_3 + \left(\frac{\partial}{\partial x} w_3 \right) w_2 - \left(\frac{\partial}{\partial x} v_3 \right) w_2 - \left(\frac{\partial}{\partial x} w_2 \right) v_3 - \left(\frac{\partial}{\partial y} v_1 \right) w_3 - \left(\frac{\partial}{\partial y} w_3 \right) v_1 \\
 &\quad + \left(\frac{\partial}{\partial y} v_3 \right) w_1 + \left(\frac{\partial}{\partial y} w_1 \right) v_3 + \left(\frac{\partial}{\partial z} v_1 \right) w_2 + \left(\frac{\partial}{\partial z} w_2 \right) v_1 - \left(\frac{\partial}{\partial z} v_2 \right) w_1 - \left(\frac{\partial}{\partial z} w_1 \right) v_2 \\
 &= \left(\frac{\partial}{\partial y} v_3 \right) w_1 - \left(\frac{\partial}{\partial z} v_2 \right) w_1 - \left(\frac{\partial}{\partial x} v_3 \right) w_2 + \left(\frac{\partial}{\partial z} v_1 \right) w_2 + \left(\frac{\partial}{\partial x} v_2 \right) w_3 - \left(\frac{\partial}{\partial y} v_1 \right) w_3 \\
 &\quad - v_1 \left(\frac{\partial}{\partial y} w_3 \right) + v_1 \left(\frac{\partial}{\partial z} w_2 \right) + v_2 \left(\frac{\partial}{\partial x} w_3 \right) - v_2 \left(\frac{\partial}{\partial z} w_1 \right) - v_3 \left(\frac{\partial}{\partial x} w_2 \right) + v_3 \left(\frac{\partial}{\partial y} w_1 \right)
 \end{aligned}$$

Aufgabe 3

(a)

$$\begin{aligned}\operatorname{div} \operatorname{rot} \vec{v} &= \operatorname{div} \operatorname{rot} \begin{pmatrix} x^2 - y^2 \\ y^2 - z^2 \\ z^2 - x^2 \end{pmatrix} \\ &= \operatorname{div} \begin{pmatrix} 2z \\ 2x \\ 2y \end{pmatrix} \\ &= 0 + 0 + 0 \\ &= 0\end{aligned}$$

$$\begin{aligned}\operatorname{rot} \operatorname{grad} f &= \operatorname{rot} \operatorname{grad} e^{xy} \\ &= \operatorname{rot} \begin{pmatrix} ye^{xy} \\ xe^{xy} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 - 0 \\ 0 - 0 \\ e^{xy} + xye^{xy} - e^{xy} - xye^{xy} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\end{aligned}$$

(b)

$$\begin{aligned}\operatorname{div} \operatorname{grad} f &= \operatorname{div} \operatorname{grad} e^{xy} \\ &= \operatorname{div} \begin{pmatrix} ye^{xy} \\ xe^{xy} \\ 0 \end{pmatrix} \\ &= y^2 e^{xy} + x^2 e^{xy}\end{aligned}$$

$$\begin{aligned}\Delta f &= \operatorname{div} \operatorname{grad} f \\ &\stackrel{\text{siehe oben}}{=} y^2 e^{xy} + x^2 e^{xy}\end{aligned}$$

(c)

$$fg = xe^{xy} + y^2e^{xy} + z^3e^{xy}$$

$$f\vec{v} = \begin{pmatrix} x^2e^{xy} - y^2e^{xy} \\ y^2e^{xy} - z^2e^{xy} \\ z^2e^{xy} - x^2e^{xy} \end{pmatrix}$$

$$\begin{aligned} \text{grad } fg &= \text{grad } xe^{xy} + y^2e^{xy} + z^3e^{xy} \\ &= \begin{pmatrix} e^{xy} + xye^{xy} + y^3e^{xy} + yz^3e^{xy} \\ x^2e^{xy} + 2ye^{xy} + y^2xe^{xy} + xz^3e^{xy} \\ 3z^2e^{xy} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{div } f\vec{v} &= \text{div} \begin{pmatrix} x^2e^{xy} - y^2e^{xy} \\ y^2e^{xy} - z^2e^{xy} \\ z^2e^{xy} - x^2e^{xy} \end{pmatrix} \\ &= 2xe^{xy} + yx^2e^{xy} - y^3e^{xy} + 2ye^{xy} + y^2xe^{xy} - z^2xe^{xy} + 2ze^{xy} \end{aligned}$$

$$\begin{aligned} \text{rot } f\vec{v} &= \text{rot} \begin{pmatrix} x^2e^{xy} - y^2e^{xy} \\ y^2e^{xy} - z^2e^{xy} \\ z^2e^{xy} - x^2e^{xy} \end{pmatrix} \\ &= \begin{pmatrix} z^2xe^{xy} - x^3e^{xy} + 2ze^{xy} \\ -z^2ye^{xy} + 2xe^{xy} + x^2ye^{xy} \\ y^3e^{xy} - z^2ye^{xy} - (x^3e^{xy} - 2ye^{xy} - y^2xe^{xy}) \end{pmatrix} \end{aligned}$$

Bitte nochmal nachrechnen.