

ANALYSIS 2 - HAUSAUFGABE 11

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Aufgabe 1

(i)

$$\vec{x}: [0, 2\pi[\times [0, 3] \rightarrow \mathbb{R}^3$$
$$\vec{x}(u, v) = \begin{pmatrix} 4 \cos u \\ v \\ 4 \sin u \end{pmatrix}$$

(ii)

$$\vec{y}: [0, 2\pi[\times [0, 1] \rightarrow \mathbb{R}^3$$
$$\vec{y}(u, v) = \begin{pmatrix} \sqrt{v} \cos u \\ \sqrt{v} \sin u \\ v \end{pmatrix}$$

Aufgabe 2

Wir parametrisieren die Oberfläche:

$$\vec{x}: [0, \pi[\times [0, 2\pi[\rightarrow \mathbb{R}^3$$
$$\vec{x}(u, v) = \begin{pmatrix} a \sin(u) \cos(v) \\ b \sin(u) \sin(v) \\ c \cos(u) \end{pmatrix}$$

dO ist somit:

$$\begin{aligned}
\left| \frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} \right| du dv &= \left| \begin{pmatrix} a \cos u \cos v \\ b \cos u \sin v \\ -c \sin u \end{pmatrix} \times \begin{pmatrix} -a \sin u \sin v \\ b \sin u \cos v \\ 0 \end{pmatrix} \right| du dv \\
&= \left| \begin{pmatrix} bc \sin^2 u \cos v \\ ac \sin^2 u \sin v \\ ab \sin u \cos u \cos^2 v + ab \sin u \cos u \sin^2 v \end{pmatrix} \right| du dv \\
&= \sin(u) \left| \begin{pmatrix} bc \cos v \sin u \\ ac \sin v \sin u \\ ab \cos u \cos^2 v + ab \cos u \sin^2 v \end{pmatrix} \right| du dv \\
&= \sin(u) \left| \begin{pmatrix} bc \cos v \sin u \\ ac \sin v \sin u \\ ab \cos u (\cos^2 v + \sin^2 v) \end{pmatrix} \right| du dv \\
&= \sin(u) \left| \begin{pmatrix} bc \cos v \sin u \\ ac \sin v \sin u \\ ab \cos u \end{pmatrix} \right| du dv \\
&= \sin(u) \sqrt{(bc \cos v \sin u)^2 + (ac \sin v \sin u)^2 + (ab \cos u)^2} du dv
\end{aligned}$$

Das gesuchte Integral ist somit:

$$\begin{aligned}
&\int_0^{2\pi} \int_0^\pi f(\vec{x}(u, v)) \cdot \sin(u) \sqrt{(bc \cos v \sin u)^2 + (ac \sin v \sin u)^2 + (ab \cos u)^2} du dv \\
&= \int_0^{2\pi} \int_0^\pi \sin(u) \sqrt{\frac{(a \sin(u) \cos(v))^2}{a^4} + \frac{(b \sin(u) \sin(v))^2}{b^4} + \frac{(c \cos(u))^2}{c^4}} \\
&\quad \cdot \sqrt{(bc \cos v \sin u)^2 + (ac \sin v \sin u)^2 + (ab \cos u)^2} du dv \\
&= \int_0^{2\pi} \int_0^\pi \sin(u) \cdot \sqrt{\frac{\sin^2(u) \cos^2(v)}{a^2} + \frac{\sin^2(u) \sin^2(v)}{b^2} + \frac{\cos^2(u)}{c^2}} \\
&\quad \cdot \sqrt{(bc \cos v \sin u)^2 + (ac \sin v \sin u)^2 + (ab \cos u)^2} du dv \\
&= \int_0^{2\pi} \int_0^\pi \sin(u) \cdot \sqrt{\frac{\sin^2(u) \cos^2(v)}{a^2} + \frac{\sin^2(u) \sin^2(v)}{b^2} + \frac{\cos^2(u)}{c^2}} \\
&\quad \cdot \sqrt{a^2 b^2 c^2 \left(\frac{\cos v \sin u}{a} \right)^2 + a^2 b^2 c^2 \left(\frac{\sin v \sin u}{b} \right)^2 + a^2 b^2 c^2 \left(\frac{\cos u}{c} \right)^2} du dv \\
&= \int_0^{2\pi} \int_0^\pi abc \sin u \sqrt{\frac{\sin^2(u) \cos^2(v)}{a^2} + \frac{\sin^2(u) \sin^2(v)}{b^2} + \frac{\cos^2(u)}{c^2}} \sqrt{\left(\frac{\cos v \sin u}{a} \right)^2 + \left(\frac{\sin v \sin u}{b} \right)^2 + \left(\frac{\cos u}{c} \right)^2} du dv \\
&= \int_0^{2\pi} \int_0^\pi abc \sin u \cdot \left(\frac{\sin^2(u) \cos^2(v)}{a^2} + \frac{\sin^2(u) \sin^2(v)}{b^2} + \frac{\cos^2(u)}{c^2} \right) du dv \\
&= abc \int_0^\pi \sin(u) \int_0^{2\pi} \frac{\sin^2(u) \cos^2(v)}{a^2} + \frac{\sin^2(u) \sin^2(v)}{b^2} + \frac{\cos^2(u)}{c^2} dv du
\end{aligned}$$

Da das Integral von 0 bis 2π sowohl beim Sinus als auch beim Sinus π ergibt, gilt:

$$\begin{aligned}
 &= \pi \cdot abc \int_0^\pi \sin(u) \left(\frac{\sin^2(u)}{a^2} + \frac{\sin^2(u)}{b^2} + \frac{\cos^2(u)}{c^2} \right) du \\
 &= \pi \cdot abc \int_0^\pi \frac{\sin^3(u)}{a^2} + \frac{\sin^3(u)}{b^2} + \frac{\cos^2(u) \sin(u)}{c^2} du
 \end{aligned}$$

$$\text{Es gilt: } \int_0^\pi \sin^3(x) dx = \frac{4}{3} \text{ und } \int_0^\pi \cos^2(x) \sin(x) dx = \frac{2}{3}$$

$$\begin{aligned}
 &= \pi \cdot abc \left(\frac{4}{3a^2} + \frac{4}{3b^2} + \frac{2}{3c^2} \right) \\
 &= \frac{2\pi}{3} abc \left(\frac{2}{a^2} + \frac{2}{b^2} + \frac{1}{c^2} \right)
 \end{aligned}$$

Aufgabe 3

Die Parametrisierung für die Fläche S ist wie folgt:

$$\begin{aligned}
 \vec{y}: [0, 2\pi[\times [0, 1] &\rightarrow \mathbb{R}^3 \\
 \vec{y}(u, v) &= \begin{pmatrix} \sqrt{v} \cos u \\ \sqrt{v} \sin u \\ v \end{pmatrix}
 \end{aligned}$$

dO ist somit:

$$\begin{aligned}
 \frac{\partial \vec{y}}{\partial u} \times \frac{\partial \vec{y}}{\partial v} du dv &= \begin{pmatrix} -\sqrt{v} \sin u \\ \sqrt{v} \cos u \\ 0 \end{pmatrix} \times \begin{pmatrix} \frac{1}{2} v^{-\frac{1}{2}} \cos u \\ \frac{1}{2} v^{-\frac{1}{2}} \sin u \\ 1 \end{pmatrix} du dv \\
 &= \begin{pmatrix} \sqrt{v} \cos u \\ \sqrt{v} \sin u \\ -\frac{1}{2} \sin^2 u - \frac{1}{2} \cos^2 u \end{pmatrix} du dv \\
 &= \begin{pmatrix} \sqrt{v} \cos u \\ \sqrt{v} \sin u \\ -\frac{1}{2} (\sin^2 u + \cos^2 u) \end{pmatrix} du dv \\
 &= \begin{pmatrix} \sqrt{v} \cos u \\ \sqrt{v} \sin u \\ -\frac{1}{2} \end{pmatrix} du dv
 \end{aligned}$$

Das gesuchte Integral ist somit:

$$\begin{aligned}
 \int_0^1 \int_0^{2\pi} \vec{v}(\vec{y}(u, v)) \cdot \begin{pmatrix} \sqrt{v} \cos u \\ \sqrt{v} \sin u \\ -\frac{1}{2} \end{pmatrix} du dv &= \int_0^1 \int_0^{2\pi} \begin{pmatrix} \sqrt{v} \sin u \\ -\sqrt{v} \cos u \\ v^2 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{v} \cos u \\ \sqrt{v} \sin u \\ -\frac{1}{2} \end{pmatrix} du dv \\
 &= \int_0^1 \int_0^{2\pi} \left(v \sin u \cos u - v \sin u \cos u - \frac{1}{2} v^2 \right) du dv \\
 &= \int_0^1 \int_0^{2\pi} -\frac{1}{2} v^2 du dv \\
 &= \int_0^1 -\frac{1}{2} v^2 u \Big|_0^{2\pi} dv \\
 &= \int_0^1 -\pi v^2 dv \\
 &= -\frac{\pi}{6} v^3 \Big|_0^1 \\
 &= -\frac{\pi}{6} \\
 &\stackrel{\text{per Def. immer pos.}}{=} \frac{\pi}{6}
 \end{aligned}$$

Aufgabe 4

Die Winkel der Breiten betragen somit: $\theta_1 = 66.5^\circ = \frac{2\pi \cdot 66.5}{360} = 1.16$ und $\theta_2 = 113.5^\circ = \frac{2\pi \cdot 113.5}{360} = 1.98$

Die Parametrisierung ist also:

$$\begin{aligned}
 \vec{z} &: [1.16, 1.98] \times [0, 2\pi[\rightarrow \mathbb{R}^3 \\
 \vec{z}(u, v) &= 6378 \cdot \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix}
 \end{aligned}$$

dO ist somit:

$$\begin{aligned}
\left| \frac{\partial \vec{y}}{\partial u} \times \frac{\partial \vec{y}}{\partial v} \right| du dv &= 6378^2 \left| \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ -\sin u \end{pmatrix} \times \begin{pmatrix} -\sin u \sin v \\ \sin u \cos v \\ 0 \end{pmatrix} \right| du dv \\
&= 6378^2 \left| \begin{pmatrix} \sin^2 u \cos v \\ -\sin^2 u \sin v \\ \cos u \sin u \cos^2 v + \sin u \cos u \sin^2 v \end{pmatrix} \right| du dv \\
&= 6378^2 \sqrt{(\sin^2 u \cos v)^2 + (\sin^2 u \sin v)^2 + (\cos u \sin u \cos^2 v + \sin u \cos u \sin^2 v)^2} du dv \\
&= 6378^2 \sqrt{\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + (\cos u \sin u \cos^2 v + \sin u \cos u \sin^2 v)^2} du dv \\
&= 6378^2 \sqrt{\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + (\sin u \cos u)^2} du dv \\
&= 6378^2 \sqrt{\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \sin^2 u \cos^2 u} du dv \\
&= 6378^2 \sqrt{\sin^4 u (\cos^2 v + \sin^2 v) + \sin^2 u \cos^2 u} du dv \\
&= 6378^2 \sqrt{\sin^4 u + \sin^2 u \cos^2 u} du dv \\
&= 6378^2 \sqrt{\sin^2 u (\sin^2 u + \cos^2 u)} du dv \\
&= 6378^2 \sqrt{\sin^2 u} du dv \\
&= 6378^2 \sin u du dv
\end{aligned}$$

Das Integral ist somit:

$$\begin{aligned}
6379^2 \cdot \int_0^{2\pi} \int_{1.18}^{1.98} \sin u du dv &= 6379^2 \cdot \int_0^{2\pi} -\cos u \Big|_{1.18}^{1.98} dv \\
&= 6379^2 (-\cos 1.98 + \cos 1.18) \cdot \int_0^{2\pi} 1 dv \\
&= 6379^2 (-\cos 1.98 + \cos 1.18) \cdot \left(v \Big|_0^{2\pi} \right) \\
&= 6379^2 (-\cos 1.98 + \cos 1.18) 2\pi
\end{aligned}$$