Analysis 2 - Hausaufgabe 6

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Aufgabe 1

Da der Definitionsbereich von f offen und konvex ist, muss nur noch überprüft werden, ob rot $\vec{v} = \vec{0}$

$$\begin{aligned}
\text{rot } \vec{v} &= \text{rot } \begin{pmatrix} f(x,y,z) \\ x^2 + yz^2 \\ y^2 z \end{pmatrix} \\
&= \begin{pmatrix} 2yz - 2zy \\ -(0 - \frac{\partial f}{\partial z}) \\ 2x - \frac{\partial f}{\partial y} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \frac{\partial f}{\partial z} \\ 2x - \frac{\partial f}{\partial y} \end{pmatrix}
\end{aligned}$$

Damit $\text{rot}(\vec{v}) = \vec{0}$ muss gelten, dass f kein z enthält (da sonst $\frac{\partial f}{\partial z}$ nicht 0 ist), und dass es nur ein y enthält. Somit muss $f(x,y,z) = 2xy + K(x)$, wobei $K(x)$ eine von y und z unabhängige Konstante ist.

Daher muss gelten:

$$\begin{aligned}
&\text{grad } u = -\vec{v} \\
\Leftrightarrow &\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{pmatrix} = - \begin{pmatrix} 2xy + K(x) \\ x^2 + yz^2 \\ y^2 z \end{pmatrix} \begin{array}{l} \text{I} \\ \text{II} \\ \text{III} \end{array}
\end{aligned}$$

Daraus lassen sich zeilenweise Gleichungen aufstellen.

$$\text{I: } \frac{\partial u}{\partial x} = -2xy + K(x)$$
$$\text{II: } \frac{\partial u}{\partial y} = -x^2 - yz^2$$
$$\text{III: } \frac{\partial u}{\partial z} = -y^2 z$$

Wir integrieren I nach x:

$$u = -x^2 y - \int K(x)\partial x + C_1(y,z)$$

Wir integrieren II nach y:

$$u = -x^2 y - \frac{y^2 z^2}{2} + C_2(x,z)$$

Wir integrieren III nach z:

$$u = -\frac{y^2 z^2}{2} + C_3(x,y)$$

Man sieht, dass u folgende Gestalt hat:

$$u = -x^2 y - \int K(x)\partial x - \frac{y^2 z^2}{2} + C$$

Bleibt das Ergebnis noch zu überprüfen:

$$\begin{aligned} \text{grad} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{pmatrix} &= \begin{pmatrix} -2xy - K(x) \\ -x^2 - yz^2 \\ -y^2 z \end{pmatrix} \\ &= -\begin{pmatrix} 2xy + K(x) \\ x^2 + yz^2 \\ y^2 z \end{pmatrix} = -\vec{v} \end{aligned}$$

Somit ist u das Potential von $\vec{v}$, falls dieses existiert (nach geeigneter Wahl von f).

Aufgabe 2

Sei $\vec{v}(x,y,z) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ und $\vec{w}(x,y,z) = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$.

$$
\begin{aligned}
&\operatorname{div}\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\right) = \left(\operatorname{rot}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) \cdot \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot \operatorname{rot}\left(\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\right) \\
\Leftrightarrow\ &\operatorname{div}\left(\begin{pmatrix} v_2 w_3 - v_3 w_2 \\ -v_1 w_3 + v_3 w_1 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}\right) = \begin{pmatrix} \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \\ -\frac{\partial v_3}{\partial x} + \frac{\partial v_1}{\partial z} \\ \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial w_3}{\partial y} - \frac{\partial w_2}{\partial z} \\ -\frac{\partial w_3}{\partial x} + \frac{\partial w_1}{\partial z} \\ \frac{\partial w_2}{\partial x} - \frac{\partial w_1}{\partial y} \end{pmatrix} \\
\Leftrightarrow\ &\frac{\partial}{\partial x}(v_2 w_3 - v_3 w_2) + \frac{\partial}{\partial y}(-v_1 w_3 + v_3 w_1) + \frac{\partial}{\partial z}(v_1 w_2 - v_2 w_1) \\
&= \left(\frac{\partial}{\partial y} v_3\right) w_1 - \left(\frac{\partial}{\partial z} v_2\right) w_1 - \left(\frac{\partial}{\partial x} v_3\right) w_2 + \left(\frac{\partial}{\partial z} v_1\right) w_2 + \left(\frac{\partial}{\partial x} v_2\right) w_3 - \left(\frac{\partial}{\partial y} v_1\right) w_3 \\
&\quad - v_1 \left(\frac{\partial}{\partial y} w_3\right) + v_1 \left(\frac{\partial}{\partial z} w_2\right) + v_2 \left(\frac{\partial}{\partial x} w_3\right) - v_2 \left(\frac{\partial}{\partial z} w_1\right) - v_3 \left(\frac{\partial}{\partial x} w_2\right) + v_3 \left(\frac{\partial}{\partial y} w_1\right) \\
\Leftrightarrow\ &\frac{\partial}{\partial x} v_2 w_3 - \frac{\partial}{\partial x} v_3 w_2 - \frac{\partial}{\partial y} v_1 w_3 + \frac{\partial}{\partial y} v_3 w_1 + \frac{\partial}{\partial z} v_1 w_2 - \frac{\partial}{\partial z} v_2 w_1 \\
&= \left(\frac{\partial}{\partial y} v_3\right) w_1 - \left(\frac{\partial}{\partial z} v_2\right) w_1 - \left(\frac{\partial}{\partial x} v_3\right) w_2 + \left(\frac{\partial}{\partial z} v_1\right) w_2 + \left(\frac{\partial}{\partial x} v_2\right) w_3 - \left(\frac{\partial}{\partial y} v_1\right) w_3 \\
&\quad - v_1 \left(\frac{\partial}{\partial y} w_3\right) + v_1 \left(\frac{\partial}{\partial z} w_2\right) + v_2 \left(\frac{\partial}{\partial x} w_3\right) - v_2 \left(\frac{\partial}{\partial z} w_1\right) - v_3 \left(\frac{\partial}{\partial x} w_2\right) + v_3 \left(\frac{\partial}{\partial y} w_1\right) \\
\overset{\text{prod. R.}}{\Leftrightarrow}\ &\left(\frac{\partial}{\partial x} v_2\right) w_3 + \left(\frac{\partial}{\partial x} w_3\right) w_2 - \left(\left(\frac{\partial}{\partial x} v_3\right) w_2 + \left(\frac{\partial}{\partial x} w_2\right) v_3\right) - \left(\left(\frac{\partial}{\partial y} v_1\right) w_3 + \left(\frac{\partial}{\partial y} w_3\right) v_1\right) \\
&\quad + \left(\frac{\partial}{\partial y} v_3\right) w_1 + \left(\frac{\partial}{\partial y} w_1\right) v_3 + \left(\frac{\partial}{\partial z} v_1\right) w_2 + \left(\frac{\partial}{\partial z} w_2\right) v_1 - \left(\left(\frac{\partial}{\partial z} v_2\right) w_1 + \left(\frac{\partial}{\partial z} w_1\right) v_2\right) \\
&= \left(\frac{\partial}{\partial y} v_3\right) w_1 - \left(\frac{\partial}{\partial z} v_2\right) w_1 - \left(\frac{\partial}{\partial x} v_3\right) w_2 + \left(\frac{\partial}{\partial z} v_1\right) w_2 + \left(\frac{\partial}{\partial x} v_2\right) w_3 - \left(\frac{\partial}{\partial y} v_1\right) w_3 \\
&\quad - v_1 \left(\frac{\partial}{\partial y} w_3\right) + v_1 \left(\frac{\partial}{\partial z} w_2\right) + v_2 \left(\frac{\partial}{\partial x} w_3\right) - v_2 \left(\frac{\partial}{\partial z} w_1\right) - v_3 \left(\frac{\partial}{\partial x} w_2\right) + v_3 \left(\frac{\partial}{\partial y} w_1\right) \\
\Leftrightarrow\ &\left(\frac{\partial}{\partial x} v_2\right) w_3 + \left(\frac{\partial}{\partial x} w_3\right) w_2 - \left(\frac{\partial}{\partial x} v_3\right) w_2 - \left(\frac{\partial}{\partial x} w_2\right) v_3 - \left(\frac{\partial}{\partial y} v_1\right) w_3 - \left(\frac{\partial}{\partial y} w_3\right) v_1 \\
&\quad + \left(\frac{\partial}{\partial y} v_3\right) w_1 + \left(\frac{\partial}{\partial y} w_1\right) v_3 + \left(\frac{\partial}{\partial z} v_1\right) w_2 + \left(\frac{\partial}{\partial z} w_2\right) v_1 - \left(\frac{\partial}{\partial z} v_2\right) w_1 - \left(\frac{\partial}{\partial z} w_1\right) v_2 \\
&= \left(\frac{\partial}{\partial y} v_3\right) w_1 - \left(\frac{\partial}{\partial z} v_2\right) w_1 - \left(\frac{\partial}{\partial x} v_3\right) w_2 + \left(\frac{\partial}{\partial z} v_1\right) w_2 + \left(\frac{\partial}{\partial x} v_2\right) w_3 - \left(\frac{\partial}{\partial y} v_1\right) w_3 \\
&\quad - v_1 \left(\frac{\partial}{\partial y} w_3\right) + v_1 \left(\frac{\partial}{\partial z} w_2\right) + v_2 \left(\frac{\partial}{\partial x} w_3\right) - v_2 \left(\frac{\partial}{\partial z} w_1\right) - v_3 \left(\frac{\partial}{\partial x} w_2\right) + v_3 \left(\frac{\partial}{\partial y} w_1\right)
\end{aligned}
$$

Aufgabe 3

(a)

$$\begin{aligned}
\text{div rot } \vec{v} &= \text{div rot } \begin{pmatrix} x^2 - y^2 \\ y^2 - z^2 \\ z^2 - x^2 \end{pmatrix} \\
&= \text{div } \begin{pmatrix} 2z \\ 2x \\ 2y \end{pmatrix} \\
&= 0 + 0 + 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\text{rot grad } f &= \text{rot grad } e^{xy} \\
&= \text{rot } \begin{pmatrix} ye^{xy} \\ xe^{xy} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 - 0 \\ 0 - 0 \\ e^{xy} + xye^{xy} - e^{xy} - xye^{xy} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

(b)

$$\begin{aligned}
\text{div grad } f &= \text{div grad } e^{xy} \\
&= \text{div } \begin{pmatrix} ye^{xy} \\ xe^{xy} \\ 0 \end{pmatrix} \\
&= y^2 e^{xy} + x^2 e^{xy}
\end{aligned}$$

$$\begin{aligned}
\Delta f &= \text{div grad } f \\
&\overset{\text{siehe oben}}{=} y^2 e^{xy} + x^2 e^{xy}
\end{aligned}$$

(c)

$$\begin{aligned}
fg &= xe^{xy} + y^2e^{xy} + z^3e^{xy} \\
f\vec{v} &= \begin{pmatrix} x^2e^{xy} - y^2e^{xy} \\ y^2e^{xy} - z^2e^{xy} \\ z^2e^{xy} - x^2e^{xy} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{grad } fg &= \text{grad } xe^{xy} + y^2e^{xy} + z^3e^{xy} \\
&= \begin{pmatrix} e^{xy} + xye^{xy} + y^3e^{xy} + yz^3e^{xy} \\ x^2e^{xy} + 2ye^{xy} + y^2xe^{xy} + xz^3e^{xy} \\ 3z^2e^{xy} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{div } f\vec{v} &= \text{div } \begin{pmatrix} x^2e^{xy} - y^2e^{xy} \\ y^2e^{xy} - z^2e^{xy} \\ z^2e^{xy} - x^2e^{xy} \end{pmatrix} \\
&= 2xe^{xy} + yx^2e^{xy} - y^3e^{xy} + 2ye^{xy} + y^2xe^{xy} - z^2xe^{xy} + 2ze^{xy}
\end{aligned}$$

$$\begin{aligned}
\text{rot } f\vec{v} &= \text{rot } \begin{pmatrix} x^2e^{xy} - y^2e^{xy} \\ y^2e^{xy} - z^2e^{xy} \\ z^2e^{xy} - x^2e^{xy} \end{pmatrix} \\
&= \begin{pmatrix} z^2xe^{xy} - x^3e^{xy} + 2ze^{xy} \\ -z^2ye^{xy} + 2xe^{xy} + x^2ye^{xy} \\ y^3e^{xy} - z^2ye^{xy} - (x^3e^{xy} - 2ye^{xy} - y^2xe^{xy}) \end{pmatrix}
\end{aligned}$$

Bitte nochmal nachrechnen.