4.3 Input-output relations for (quantum) electrical circuits

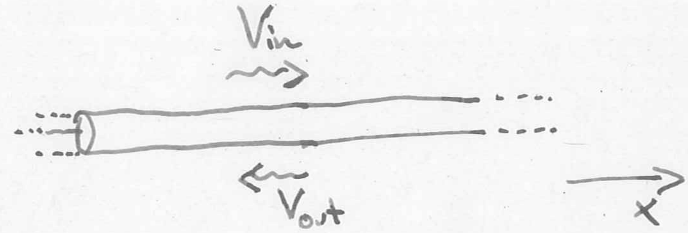
Goal: Find relations between input and output signals scattering off an electrical circuit.

Signals propagate as EM waves through transmission lines realized e.g. as coaxial cables, waveguides, coplanar waveguides, etc.

Consider transmission line with

$v = \frac{1}{\sqrt{lc}}$ ... wave velocity

$Z_0 = \sqrt{\frac{l}{c}}$ ... characteristic line impedance



Evaluate Euler-Lagrange equation for Lagrange function in chapter 2.7 results in

$$v^2 \Phi''(x,t) - \ddot{\Phi}(x,t) = 0 \qquad \text{wave equation}$$

voltage:

$$V(x,t) = -\dot{\Phi}$$

current:

$$I(x,t) = \frac{1}{l}\Phi'(x,t)$$

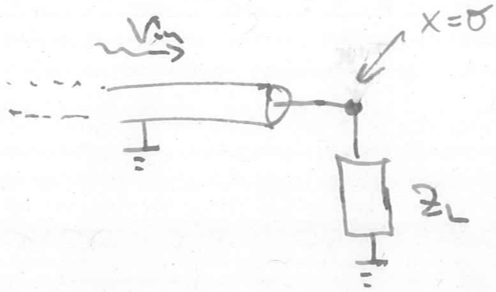
Solution to wave equation:

$$V(x,t) = V_{in}\left(t - \frac{x}{v}\right) + V_{out}\left(t + \frac{x}{v}\right)$$

$$I(x,t) = \frac{1}{Z_0}\left(V_{in} - V_{out}\right)$$

Waves V_{in} (V_{out}) travel in positiv (negativ) x-direction along the TL.

Now consider additional circuit element(s) with impedance $Z_L \in \mathbb{Z}$ terminating the line at position $x = 0$:



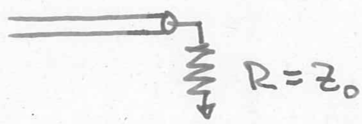
at $x = 0$: $\quad Z_L \equiv \dfrac{V(0,t)}{I(0,t)} = Z_0 \dfrac{V_{out} + V_{in}}{V_{in} - V_{out}}$

$$\Rightarrow \quad \frac{V_{out}}{V_{in}} \equiv \underset{\substack{\uparrow \\ \text{reflection} \\ \text{coefficient}}}{S_{11}} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$Z_L = \underset{\substack{\uparrow \\ \text{resistance}}}{R} + j \underset{\substack{\uparrow \\ \text{reactance}}}{X} = 1 / \underset{\substack{\uparrow \\ \text{admittance}}}{Y_L}$$

Example 1: $Z_L = R = \overset{\overset{R}{\Downarrow}}{Z_0}$ (= 50Ω typically)

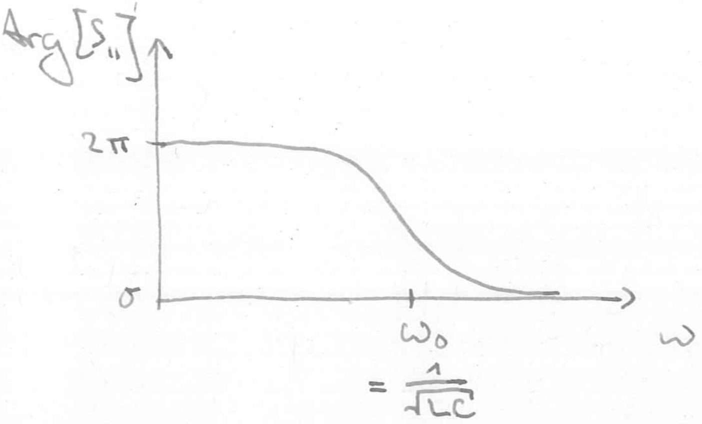
$$\Rightarrow \quad S_{11} = 0$$



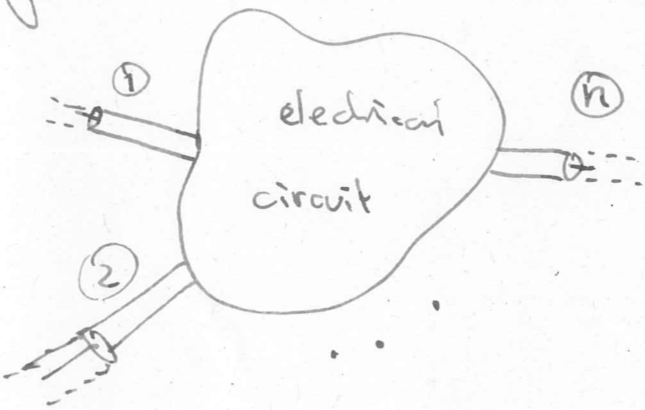
A resistor matched to the line impedance acts like a perfect absorber for the incoming waves, ($\hat{=}$ black body !)

Example 2 : $Z_L =$  $= \frac{1}{j\omega C} + j\omega L$

$|S_{11}|^2 = 1$, no absorption.



Generalization to multiport circuits possible



Scattering matrix

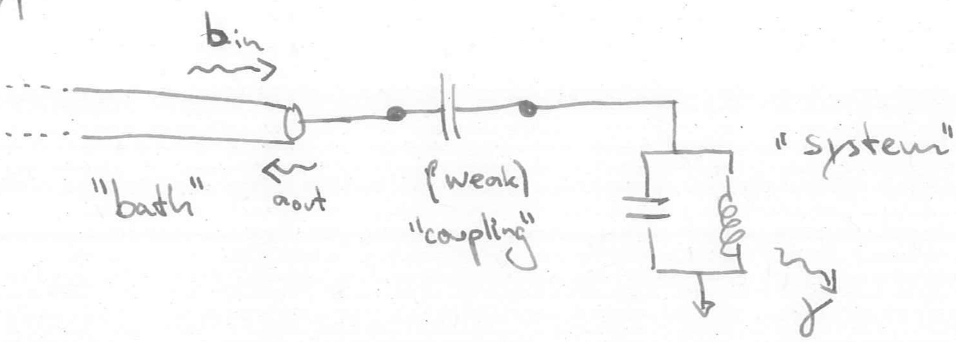
$$S_{ij} = \frac{V_{out,i}}{V_{in,j}} \Bigg|_{\substack{V_{in,k} = 0 \\ \forall k \neq j}}$$

- Further reading: D. Pozar, Microwave engineering
  Chapter 4 : Network analysis

- Software tools for simulations exist.

Let's reconsider the problem using a quantum-mechanical description:

Typical situation



Compare Gardiner/Zoller 85, Walls/Milburn for details

$$H_{sys} = \hbar\omega_0 a^\dagger a + \ldots$$

$$H_{coupl} = \hbar\sqrt{\frac{\kappa}{2\pi}}\int d\omega \left(a b_\omega^\dagger + a^\dagger b_\omega\right)$$

$$H_{bath} = \hbar\int d\omega\, \omega\, b_\omega^\dagger b_\omega$$

Equation of motion for the system operator

$$\dot{a} = -\frac{i}{\hbar}\left[a, H_{sys}\right] \underbrace{-\frac{\kappa+\gamma}{2}a}_{\text{decay into bath}} + \underbrace{\sqrt{\kappa}\, b_{in}}_{\text{drive}}$$

Photon flux towards sample $b_{in}^\dagger b_{in} = n_{in}$

Boundary condition relates b_{in} and a to output field

$$b_{out}(t) = \sqrt{\kappa}\, a(t) - b_{in}(t)$$

Typical situation: Classical coherent input field

$$\langle b_{in}(t)\rangle = \beta_{in}\, e^{-i\omega t}$$

Measure $\langle b_{out} \rangle$

$$\frac{\langle b_{out} \rangle}{\langle b_{in} \rangle} = S_{11} \qquad \text{as before.}$$

Example : Harmonic oscillator

ansatz $\langle a \rangle = \alpha e^{-i\omega t}$

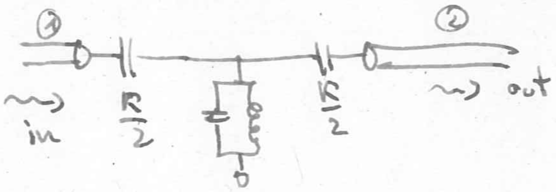
$$-i\omega\alpha = -i\omega_0\alpha - \frac{\kappa+\gamma}{2}\alpha + \sqrt{\kappa}\,\beta_{in}$$

see problem set 5

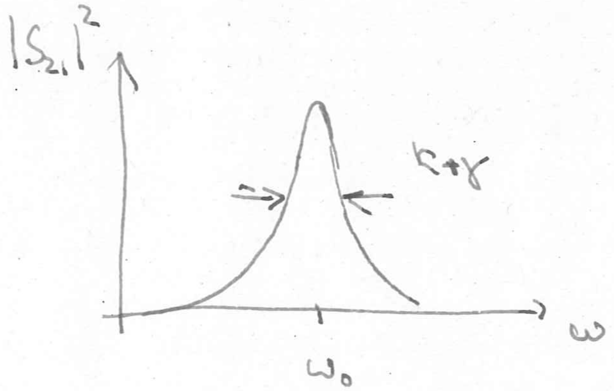
$$\alpha = \frac{\sqrt{\kappa}\,\beta_{in}}{\frac{\kappa+\gamma}{2} + i(\omega_0-\omega)}$$

$$S_{11} = \frac{\langle b_{out} \rangle}{\langle b_{in} \rangle} = \frac{\kappa}{\kappa+\gamma}\,\frac{1}{\frac{1}{2} + i\frac{(\omega_0-\omega)}{\kappa+\gamma}} - 1$$

Ex: Add a 2nd port and show



$$S_{21} = \frac{\kappa}{\kappa+\gamma}\,\frac{1}{1 + i\frac{(\omega_0-\omega)}{\kappa+\gamma}} \xrightarrow[\substack{\omega=\omega_0 \\ \gamma\to 0}]{} 1$$

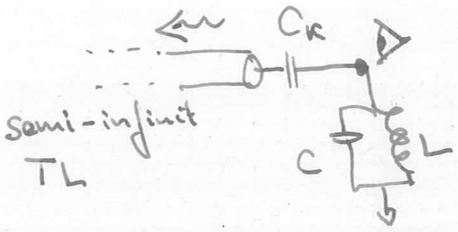


Lorentzian line shape

See slides for experimental discussion.

4.4 Dissipation in quantum systems

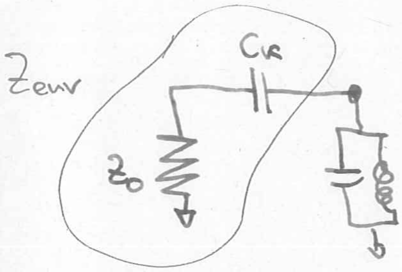
Q: How to calculate decay rate κ in the previous example and in general?



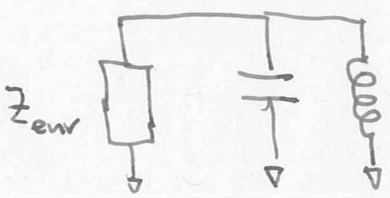
Once a photon has decayed into the TL, it will never come back.

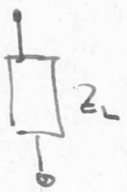
$\Rightarrow$ like a black body!
(or 50 Ω resistor)

$\Longleftrightarrow$



$\Longleftrightarrow$





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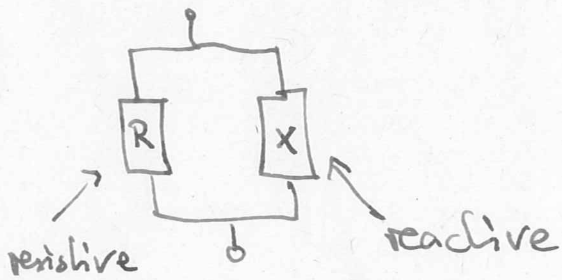


$R = 1/\mathrm{Re}[Y_{env}]$ ... causes loss.

$X = 1/\mathrm{Im}[Y_{env}]$ ... causes a frequency shift

For $Z_{env} = Z_0 + \frac{1}{j\omega C_\kappa}$:

$X \approx \frac{1}{j\omega C_\kappa}$ $\Rightarrow$ $C \rightarrow \tilde{C} = C + C_\kappa$

$R \approx \frac{1}{Z_0 \omega^2 C_\kappa^2}$ $\qquad \kappa = \frac{1}{R\tilde{C}} = \frac{C_\kappa^2 Z_0 \omega^2}{\tilde{C}}$

$C_\kappa \ll C \frac{Z_{res}}{Z_0}$

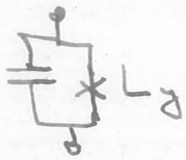
//

Generalization see Nigg & Girvin, PRL (2012):

$$Q^{-1} \equiv \frac{\kappa}{\omega} = \mathrm{Re}\left[Y_{env}\right]_{\omega=\omega_0} Z_{res}$$

with $Z_{res} = \sqrt{L/C}$

Also applicable to transmon circuits



Q: How does the quantum state of a system evolve in the presence of dissipation?

Schrödinger equation assumes closed system

$$\dot{\rho} = \frac{i}{\hbar}\left[\rho, H\right] + ?$$

density matrix

need extension to account for dissipation.

Let's consider a specific example:

$H = \hbar\omega_0 a^\dagger a$ and initial state $\rho(t) = |n\rangle\langle n|$

Fock state with n photons.

What is the state after infinitesimal time step dt, if decay event is possible?

$$\rho(t+dt) = (1 - \underbrace{\kappa n\, dt}_{\text{probability for decay}})\,|n\rangle\langle n| + \kappa n\, dt\, |n-1\rangle\langle n-1|$$

$$\rho(t+dt) = \rho(t) - \Big[\frac{\kappa}{2}\underbrace{(a^\dagger a\,\rho(t) + \rho(t)\,a^\dagger a)}_{\text{symmetrized}} - \kappa\, a\rho a^\dagger\Big]\,dt$$

$$\xRightarrow{dt \to 0}$$

$$\dot{\rho} = \underbrace{-\frac{\kappa}{2}\left(a^\dagger a\rho + \rho a^\dagger a\right) + \kappa\, a\rho a^\dagger}_{\text{Lindblad term}} + \frac{i}{\hbar}[\rho, H]$$

Master equation

Similar terms for other decoherence mechanics such as dephasing.