③ Superconducting Qubits

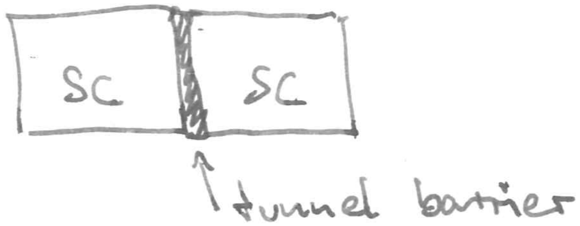
3.1 Brief introduction to Superconductivity

3.2 The Josephson effect

→ see slides

3.3 Analogy between Josephson junction & inductor

The Josephson equations, describing the voltage and current across a Josephson junction



read

$$\boxed{\begin{aligned} I &= I_0 \sin\delta \\ V &= \frac{\hbar}{2e}\dot{\delta} \equiv \varphi_0 \dot{\delta} \end{aligned}}$$

, I_0 ... critical current

, φ_0 ... reduced flux quantum

We may define an effective flux variable $\tilde{\Phi} = \varphi_0 \delta$, proportional to the phase variable δ.

By furthermore introducing the Josephson inductance $L_J = \varphi_0 / I_0$, we obtain

assume $\delta \ll 1$, i.e. $I \ll I_0$

$$I = \frac{\varphi_0}{L_J} \cdot \sin(\tilde{\Phi}/\varphi_0) = \frac{\tilde{\Phi}}{L_J} \underbrace{- \frac{1}{6}\frac{\tilde{\Phi}^3}{L_J \varphi_0^2} + \ldots}_{\text{nonlinear relation between } \tilde{\Phi} \text{ \& } I.}$$

$$V = \dot{\tilde{\Phi}}$$

In this form, the Josephson equations appear analogous to the relations between V, I and the magnetic flux in a linear inductor.
However, the current depends nonlinearly on the flux Φ. In many instances, the Josephson junction thus behaves like an inductor with finite nonlinearity. Furthermore, the current flows without dissipation as a result of the vanishing electrical resistance in the superconductor.
Due to this analogy we introduce a circuit element for the Josephson junction, schematically represented as

Combining this element with inductors and capacitors

allows us to construct quantum electrical circuits with finite anharmonicity, as discussed in the following.

3.4 Hamiltonian of the Josephson junction

Let us assume both the temperature T and the frequency of resonant modes ω are well below the gap energy $\hbar\omega, k_B T \ll 2\Delta$ and the total number of Cooper pairs $N = N_L + N_R$ on the two islands of a JJ is constant.

The ability to tunnel allows the system to be in any configuration with $m \in \mathbb{Z}$ Cooper pairs having crossed the barrier. Here, the configuration with $m=0$ we refer to as the equilibrium state.

Each of these configurations can be associated with a quantum-mechanical basis state $|m\rangle$.

An effective Hamiltonian describing this system is given by

$$\hat{H}_{JJ} = -\frac{E_J}{2} \sum_{m=-\infty}^{+\infty} \left(|m\rangle\langle m+1| + |m+1\rangle\langle m| \right)$$

The parameter E_J is called the Josephson energy and is a measure of the ability to tunnel. The larger E_J, the larger the ability of the system to lower its energy by tunneling.

The eigen states of this Hamiltonian are given by the (unnormalized) „plane waves"

$$|\delta\rangle = \sum_{n=-\infty}^{+\infty} e^{in\delta} |n\rangle ,$$

as follows from

$$\hat{H}_{JJ} |\delta\rangle = -\frac{E_J}{2} \sum_{\substack{m \\ m'}} \Big(|m\rangle \underbrace{\langle m+1|m'\rangle}_{\delta_{m+1,m'}} + |m+1\rangle\langle m|m'\rangle\Big) e^{im'\delta}$$

$$= -\frac{E_J}{2} \sum_m \left(e^{i(m+1)\delta} + e^{i(m-1)\delta}\right) |m\rangle$$

$$= -E_J \cos(\delta) \, |\delta\rangle$$

With the charge operator $\hat{n} = \sum_m m\, |m\rangle\langle m|$ we can derive an expression for the current operator

$$\hat{I} = 2e \frac{d\hat{n}}{dt} = 2e \frac{i}{\hbar} [\hat{H}_{JJ}, \hat{n}]$$

$$= -i \frac{e}{\hbar} E_J \sum_m \left(|m\rangle\langle m+1| - |m+1\rangle\langle m|\right)$$

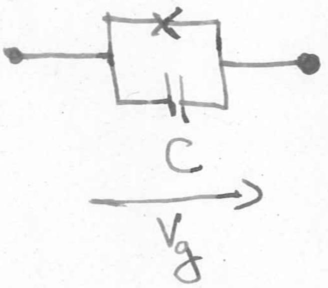
The states $|\delta\rangle$ turn out to be eigenstates of $\hat{I}$ as well

$$\hat{I} |\delta\rangle = \overbrace{E_J \frac{2e}{\hbar}}^{=I_0} \sin(\delta) \, |\delta\rangle$$

By identifying the relation $I_0 = E_J \frac{2e}{\hbar}$ between the critical current I_0 and the Josephson energy, we recover the 1. Josephson relation. This indicates that the phenomenological Hamiltonian H_{JJ} captures the physics of the JJ with regard to tunneling.

3.5 Cooper pair box Hamiltonian

In the derivation above we did not include the electrostatic energy of Cooper pairs. Let's do so now. Electrostatic energy may arise both from the finite capacitance across the two islands C, and an external voltage V_g. Here, V_g is ofte referred to as "gate voltage" and may either result from controlled or uncontrolled external fields. In the case of uncontrolled sources of V_g, this parameter can be subject to noise.



$$H_{el} = (2e)^2 \frac{\hat{n}^2}{2C} + 2e\hat{n}V_g$$

$n_g = \frac{CV_g}{2e}$ ... offset charge

$$= \frac{4e^2}{2C}\left(\hat{n} - n_g\right)^2 + \text{const}$$

Together with the tunneling term we obtain

$$\boxed{\hat{H}_{CPB} = 4\, E_c\, (\hat{n} - n_g)^2 - E_J \cos\hat{\delta}\,,}$$

the Hamiltonian of the Cooper pair box.

charging energy $E_c = \frac{e^2}{2C}$

offset charge

The wave function of a state $|\psi\rangle$ in the basis of $\{|\delta\rangle\}$ is $\psi(\delta) \equiv \langle\delta|\psi\rangle$. The charge operator in this basis acts like a derivative of this wave function with respect to δ:

$$\hat{n}\,\psi(\delta) = \langle\delta|\hat{n}|\psi\rangle = i\frac{d}{d\delta}\psi(\delta)$$

Exercise: Show this equality.

Let us sketch the exact solution of the $\hat{H}_{CPB}$ Hamiltonian by writing the Schrödinger equation in the phase basis $|\delta\rangle$:

$$\left[4E_c\left(i\frac{d}{d\delta} - n_g\right)^2 - E_J\cos\delta\right]\psi(\delta) = E\,\psi(\delta)$$

equivalent to a quasi momentum

Taking the boundary condition $\psi(\delta) = \psi(\delta + 2\pi)$ into account, this is equivalent to the SE of a quantum rotor (pendulum).

Details in Koch et al., PRA (2018)

Exact solution with ansatz $\psi(\varphi) = e^{in_g\delta}\,g(\delta/2)$

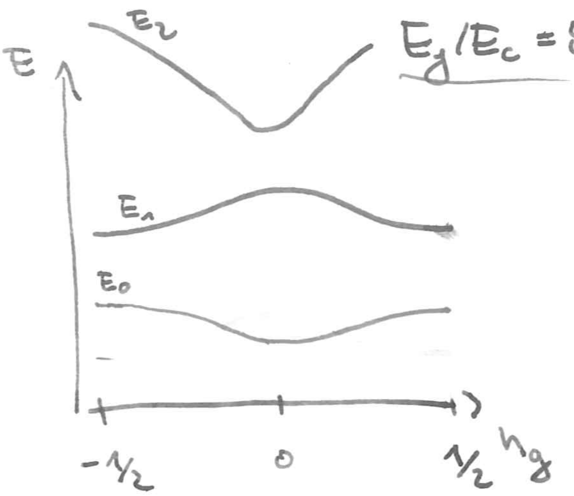
$$\Rightarrow \quad g''(x) + \left(\frac{E}{E_c} + \frac{E_J}{E_c}\cos(2x)\right)g(x) = 0$$

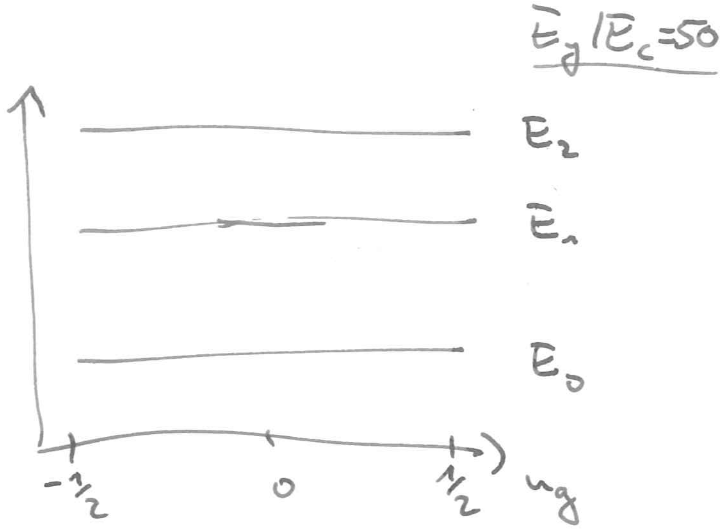
independent of n_g!

- Solution of differential equation given by Mathieu function

- n_g ... determines boundary condition of $g(x)$

$\Rightarrow E_n$ ... eigenenergies depend on n_g

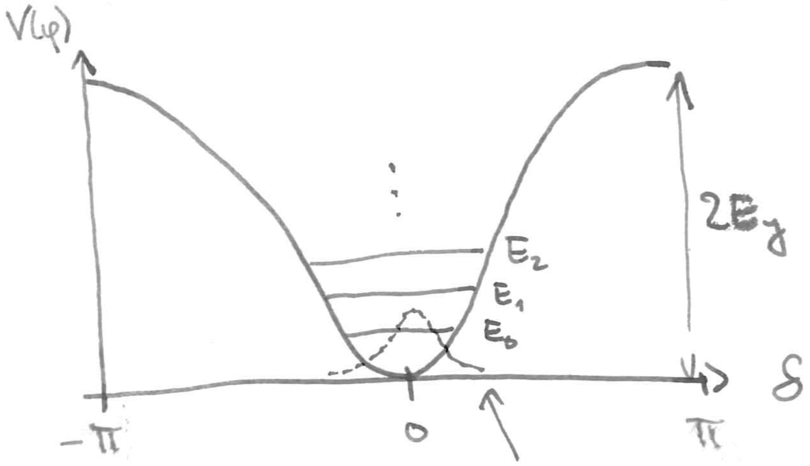




Charge dispersion : $\varepsilon = \Delta E(n_g=0) - \Delta E(n_g=1/2) \sim e^{-\sqrt{8E_J/E_C}}$

3.6 The transmon limit $E_J/E_C \gg 1$

Quantum rotor $\hat{=}$ "Particle moving in a cosine potential"



$\psi_0(\delta)$ ... groundstate wave function

- characteristic energy scale of low lying levels $\sqrt{8E_JE_C} \ll E_J$

- $\langle \hat{\delta}^2 \rangle \ll 1$ for $\psi_0(\varphi)$

• Taylor expansion justified:

$$-E_J \cos(\hat{\delta}) = \text{const} + E_J \frac{\hat{\delta}^2}{2} - E_J \frac{\hat{\delta}^4}{24} + \ldots$$

• Harmonic approximation

$$H \approx 4E_c \hat{n}^2 + \tfrac{1}{2} E_J \hat{\delta}^2 \qquad E_J = \frac{\varphi_0^2}{L_J}$$

$$\delta = \Phi/\varphi_0$$

$$= \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L}$$

$$= \hbar\Omega\,(a^\dagger a + 1/2) \qquad \hat{\delta} = \delta_{zpf}\,(a + a^\dagger)$$

$$\delta_{zpf} = \sqrt{\frac{2E_c}{E_J}} \ll 1$$

• Quartic correction

linear scaling in E_c

$$V = -\frac{1}{24} E_J \hat{\delta}^4 = \ldots = -\frac{E_c}{2}\left(a^{\dagger 2} a^2 + 2a^\dagger a\right)$$

neglect terms $a^{\dagger n} a^m$ with $n \neq m$.

anharmonicity

frequency shift

• Treat ✗ like a nonlinear inductor $\mathcal{L} \to H$!

Discussion see slides

In conclusion, we found that the macroscopic object, consisting of a Josephson junction shunted with a capacitor, is simply described by the Hamiltonian of an anharmonic oscillator

$$\hat{H}_{CPB} \underset{E_J \gg E_c}{\approx} \hbar\omega_{01}\, a^\dagger a - \frac{\hbar\alpha}{2}\, a^{\dagger 2} a^2$$

Typical values:
$\omega_{01}/2\pi = 5\,\text{GHz}$
$\alpha/2\pi \approx 250\,\text{MHz}$