4.5 General properties of quantum measurements

4.6 Circuit QED and the Jaynes-Cummings Hamiltonian

$\Big\}$ see slides

4.7 Dispersive limit $\Delta = \omega_{ge} - \omega_r \gg g$

We consider the Hamiltonian

$$H/\hbar = \underbrace{\omega_r a^+ a + \omega_{ge} b^+ b + g(ab^+ + a^+ b)}_{\equiv h_2} \underbrace{- \frac{\alpha}{2} b^{+2} b^2}_{\equiv h_4}$$

Our goal is to find the eigen modes with field operators $\tilde{a}, \tilde{b}$, which diagonalize the quadratic part h_2 of the Hamiltonian:

$$h_2 = (a^+ \; b^+) \underbrace{\begin{pmatrix} \omega_r & g \\ g & \omega_{ge} \end{pmatrix}}_{u^+ u} \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{u^+ u = \mathbb{1}}$$

$$= (\tilde{a}^+ \; \tilde{b}^+) \begin{pmatrix} \tilde{\omega}_r & 0 \\ 0 & \tilde{\omega}_{ge} \end{pmatrix} \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = u \begin{pmatrix} a \\ b \end{pmatrix}$$

In the limit of $\varepsilon = g/\Delta \ll 1$ it turns out that

$$a \approx \sqrt{1-\varepsilon^2}\, \tilde{a} - \varepsilon \tilde{b}$$

$$b \approx \sqrt{1-\varepsilon^2}\, \tilde{b} + \varepsilon \tilde{a}$$

$$H/\hbar = \tilde{\omega}_r \tilde{a}^\dagger \tilde{a} + \tilde{\omega}_{ge} \tilde{b}^\dagger \tilde{b} - \frac{\alpha}{2} b^{\dagger 2} b^2$$

$\tilde{\omega}_r \approx \omega_r - \frac{g^2}{\Delta}$ $\qquad$ $\tilde{\omega}_{ge} \approx \omega_{ge} + \frac{g^2}{\Delta}$

Replace

$b \to b = \sqrt{1-\varepsilon^2}\,\tilde{b} + \varepsilon \tilde{a}$

Nonlinear part of the Hamiltonian becomes:

$$h_4 = -\frac{\alpha}{2} b^{\dagger 2} b^2 = -\frac{\alpha}{2}\Big[4\varepsilon^2 \tilde{b}^\dagger \tilde{b} \tilde{a}^\dagger \tilde{a} + \underbrace{(1-\varepsilon^2)^2}_{\mathcal{O}(1)} \tilde{b}^{\dagger 2} \tilde{b}^2 + \dots \Big]$$

combinatorial factor $2\binom{2}{1}$

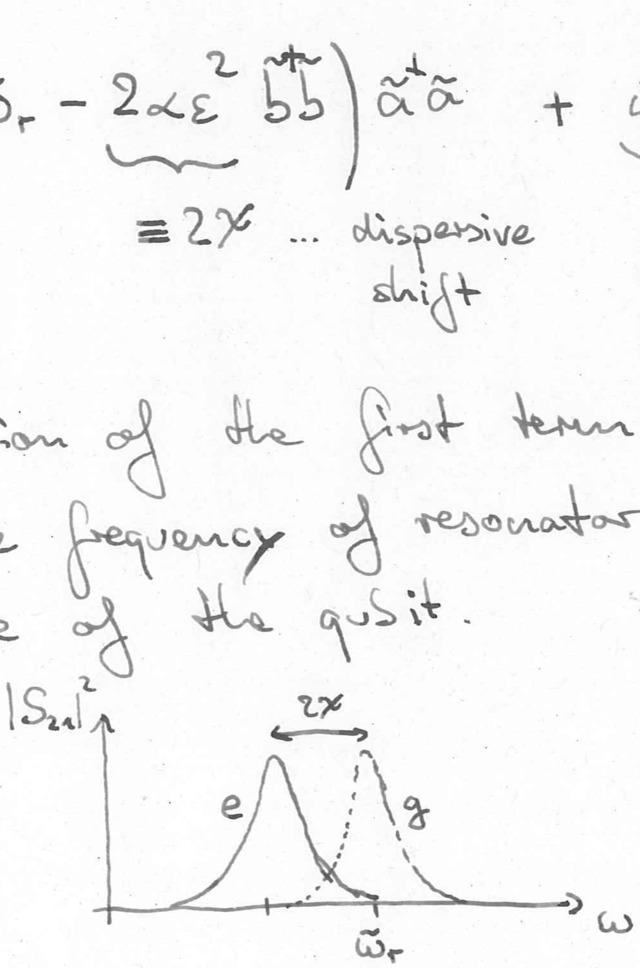
terms with $\# a \neq \# a^\dagger$ and $\mathcal{O}(\varepsilon^4)$

Rewriting the Hamiltonian:

$$H/\hbar \approx \Big(\tilde{\omega}_r - \underbrace{2\alpha\varepsilon^2}_{\equiv 2\chi} \tilde{b}^\dagger \tilde{b}\Big) \tilde{a}^\dagger \tilde{a} + \underbrace{\tilde{\omega}_{ge} \tilde{b}^\dagger \tilde{b} - \frac{\alpha}{2} \tilde{b}^{\dagger 2} \tilde{b}^2}_{\text{qubit part}}$$

$\equiv 2\chi$ ... dispersive shift

Interpretation of the first term:
Resonance frequency of resonator depends on the state of the qubit.



Probe this resonator frequency shift to measure the state of the qubit.

Remarks:

- Alternative derivation using Schrieffer-Wolff transformation

$$H \longrightarrow \tilde{H} = e^{S} H e^{-S} = H + [S, H] + \frac{1}{2}[S,[S,H]] + \ldots$$

with
$$S = \sum_i \varepsilon_i \left(a\, |i+1\rangle\langle i| + a^\dagger |i\rangle\langle i+1| \right), \quad \varepsilon_i = \frac{g_i}{\Delta_{i,i+1}}$$

transmon eigenstates $i \in \{g, e, \ldots\}$

(see e.g. Koch et al., PRA 2007)

- To determine dispersive shift one can also use 2nd order perturbation theory or numerical diagonalization of the full Hamiltonian, e.g.

$$2\hbar\chi = E_{e1} - E_{g1} - E_{e0}$$

eigen energies of perturbed states

- Schrieffer-Wolff trafo leads to approximate

$$\chi \approx -\alpha \frac{g^2}{\Delta(\Delta-\alpha)}$$

which also holds if $|\Delta|$ is not much larger than α.