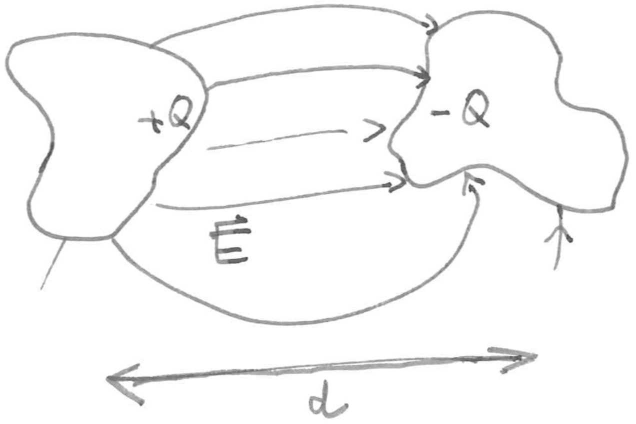
② Quantization of electrical circuits

Goal: Given an electrical circuit composed of inductors —eeee—, and capacitors —||—, find the system Hamiltonian.

Note: Josephson junctions —×— provide anharmonicity to the circuits and will be discussed in the following chapter. Resitors —ww— will be discussed later in the context of dissipation.

2.1 Lumped element representation of electrical circuits

Consider two metallic islands charged with ± Q.



$\vec{E}$ ... electric field

d ... characteristic length

In a static configuration, charges arrange on surface and electric field vanishes inside the metal. The time to reach static charge configuration $\tau \sim d/v$. If $Q(t)$ changes slowly compared to τ, electric field follows quasi-instantaneously. The energy required to move a unit charge from one to the other island is path-independent and given by voltage V.

$$\Rightarrow \quad V = \frac{Q}{C}$$

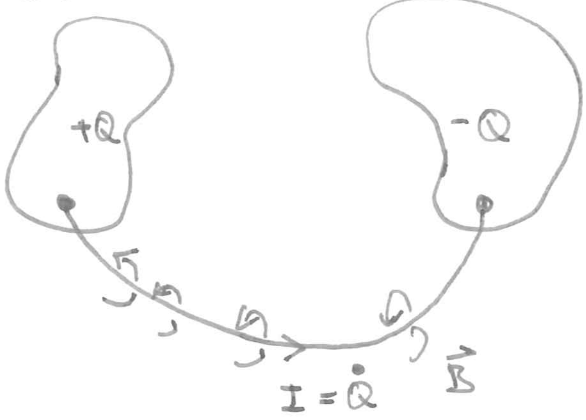
Capacitance = constant factor, which depends on geometry only.

... and dielectric medium

⇒ Total energy in electric field:

$$E_{el} = \int_0^Q dQ'\, V(Q') = \int_0^Q dQ' \frac{Q'}{C} = \frac{Q^2}{2C}$$

Now consider an additional wire connecting the two islands:



Similar arguments as before result in

$$\Phi = LI$$

magnetic flux

constant, geometry -depede

inductance

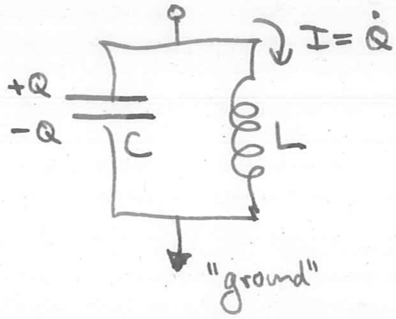
current

$$E_{mag} = \int_0^I dI'\, \Phi(I') = \frac{1}{2} L I^2 = \frac{\Phi^2}{2L}$$

Low frequency dynamics of this system are fully captured by two effective parameters L and C, the finite extent of which may be neglected.
The system is represented by a lumped element model, schematically

2.2 Quantization of the LC resonator

Consider



Electric field energy : $\frac{Q^2}{2C}$

Magnetic ~ : $\frac{\Phi^2}{2L}$

Faraday's law $\dot{\Phi} = -V$:

$$\Rightarrow \ddot{\Phi} = -\dot{V} = -\frac{\dot{Q}}{C} = -\frac{I}{C} = -\frac{\Phi}{LC} = -\omega_0^2 \Phi$$

$$\overset{\uparrow}{\omega_0 = \sqrt{\frac{1}{LC}}}$$

LC-resonator described by harmonic oscillator.
General procedure to find Hamilton function (see analytical mechanics):

1) Define Lagrange function:

$$\mathcal{L} = T - V$$

$$= \underbrace{\frac{1}{2} C \dot{\Phi}^2}_{\text{"kinetic term"}} - \frac{1}{2L} \Phi^2 \leftarrow \text{"potential term"}$$

2) Legendre transformation
conjugate variable $Q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C \dot{\Phi}$

3) Hamilton function

$$H = \dot{\Phi} Q - \mathcal{L} = \frac{Q^2}{C} - \frac{Q^2}{2C} - \left(-\frac{\Phi^2}{2L}\right) = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

4) Quantize $Q \rightarrow \hat{Q}$, $\Phi \rightarrow \hat{\Phi}$

$$[\hat{\Phi}, \hat{Q}] = i\hbar$$

5) Express $\hat{H}$ in terms of annihilation and creation operators

$$\hat{\Phi} = \underbrace{\sqrt{\frac{\hbar\omega_0 L}{2}}}_{\equiv \Phi_{zpf}} \underbrace{(a+a^{\dagger})}_{\equiv 2\hat{X}} \quad , \quad \hat{Q} = \underbrace{\sqrt{\frac{\hbar\omega_0 C}{2}}}_{\equiv Q_{zpf}} \cdot \underbrace{i(a^{\dagger}-a)}_{\equiv 2\hat{P}}$$

$\hat{X}, \hat{P}$ ... quadrature amplitudes

to obtain

$$\hat{H} = \hbar\omega_0 \, a^{\dagger}a + \text{const} \quad , \quad [a, a^{\dagger}] = 1$$

//

This procedure is generally applicable to more complicated circuits as well, examples of which are discussed below.

2.3 Properties of the QHO

The ground state satisfies $a|0\rangle = 0$

Eigenstates of $a^{\dagger}a \equiv \hat{n}$ are called Fock states

$$a^{\dagger}a\,|n\rangle = n\,|n\rangle \quad , \quad n \in \{0,1,2,3,\dots\}$$

The number n corresponds to the number of elementary excitations, i.e. photons with frequency ω_0.

Applying a ($a^{\dagger}$) to a Fock state, decreases (increases) n by 1,

$$a^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle \quad , \quad a|n\rangle = \sqrt{n}\,|n-1\rangle$$

which follows from the commutation relation

$$a^{\dagger}a\,\underbrace{(a^{\dagger}|n\rangle)}_{\sim|n+1\rangle} = a^{\dagger}(1+a^{\dagger}a)\,|0\rangle = (n+1)\,\underbrace{(a^{\dagger}|n\rangle)}_{\sim|n+1\rangle}$$

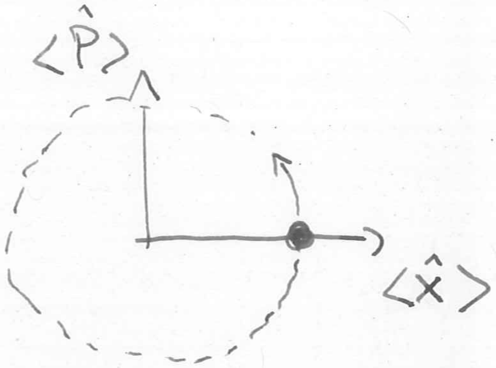
The zero-point-fluctuations (zpf) of $\hat{\Phi}$ and $\hat{Q}$ with respect to the ground state, are

$$\langle 0|\hat{\Phi}^2|0\rangle = \Phi_{zpf}^2 \quad , \quad \langle 0|\hat{Q}^2|0\rangle = Q_{zpf}^2 \, .$$

An important class of states are coherent states, defined as eigenstates of a

$$a|\alpha\rangle = \alpha|\alpha\rangle \quad , \quad \alpha \in \mathbb{C}$$

because they result in dynamics in correspondence to a classical harmonic oscillator



$$\hat{U}(t)|\alpha_0\rangle = |\alpha(t)\rangle = |\alpha_0 e^{-i\omega_0 t}\rangle$$

$$\langle\alpha(t)|\hat{\Phi}|\alpha(t)\rangle = 2\Phi_{zpf}\,\alpha_0\cos(\omega_0 t)$$

$$\langle\alpha(t)|\hat{Q}|\alpha(t)\rangle = 2Q_{zpf}\,\alpha_0\sin(\omega_0 t)$$

The Fock state representation of $|\alpha\rangle$ reads

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

compare
Problem set 3

2.4 Experimental realization of lumped LC resonator

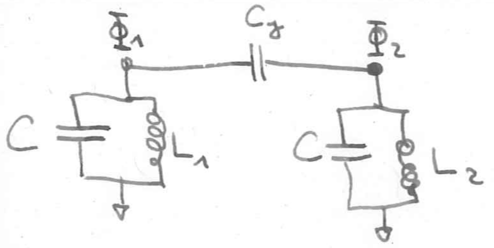
see slides

2.5 Dipole moment and coupling to an electric field

see slides

2.6 Quantization of coupled resonator systems

We consider the circuit



with Lagrange function

$$\mathcal{L} = \frac{1}{2} C \dot{\Phi}_1^2 + \frac{1}{2} C \dot{\Phi}_2^2 + \frac{1}{2} C_g \left(\dot{\Phi}_2 - \dot{\Phi}_1\right)^2 - \frac{\Phi_1^2}{2L_1} - \frac{\Phi_2^2}{2L_2}$$

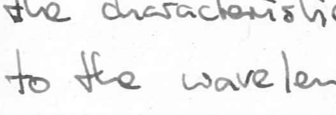
Applying the canonical transformation described in sec. 2.2 results in the Hamiltonian

$$H = \hbar\omega_1 a^\dagger a + \hbar\omega_2 b^\dagger b + \hbar J \left(a - a^\dagger\right)\left(b - b^\dagger\right).$$

(Derivation, see problem set 3.)

2.7 Transmission line (distributed) resonators

A finite length transmission line, e.g. a coaxial waveguide,



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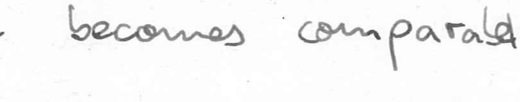


can be represented by a series of inductances and capacitances. Such a distributed model is required when the characteristic length $d \sim \lambda$ becomes comparable to the wavelength $\lambda = v/f$.

In the continuum limit Φ becomes a position-dependent $\Phi(x)$ and the Lagrange function is

$$\mathcal{L} = \int_0^d dx \left\{ \frac{c}{2} \dot{\Phi}(x)^2 - \frac{1}{2l} \left(\partial_x \Phi(x)\right)^2 \right\},$$

c, l are capacitance and inductance per unit length

Taking into account the boundary condition of vanishing current at the two open ends, allows us to express $\Phi(x)$ in terms of the normal modes

$$\Phi(x) = \sum_{n=1}^{\infty} \phi_n \cos(k_n x) \quad , \quad k_n = n\frac{\pi}{d}$$

resulting in

$$\mathcal{L} = \frac{1}{2}\sum_n C\dot{\phi}_n^2 - \frac{1}{L_n}\phi_n^2 \quad \text{with} \quad C = \frac{cd}{2}, \quad L_n = \frac{2ld}{\pi^2 n^2}$$

Introducing $q_n = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_n}$ we obtain

$$H = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{q_n^2}{C} + \frac{\phi_n}{L_n}\right)$$

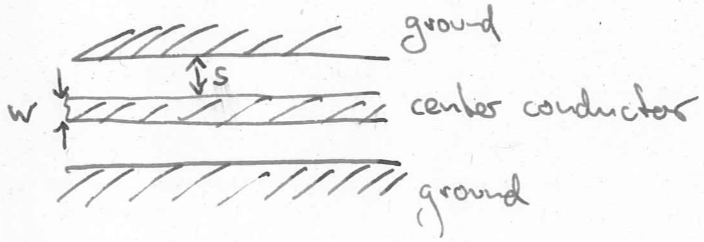
, sum of harmonic modes

$$= \sum_{n=1}^{\infty} \omega_n a_n^{+} a_n$$

with $= \omega_n = \frac{1}{\sqrt{L_n C}} = n\frac{\pi v}{d}$,

$v = \frac{1}{\sqrt{lc}}$ ... phase velocity.

Most common implementation in planar superconducting circuits is based on coplanar waveguides (CPW).



compare slides