6) Characterization of Quantum Processes

Previous chapters: Implementation of quantum gates and measurements.

In practice, such operations are subject to errors, e.g due to ...
... decoherence of qubits during the operation.
... imperfect control pulses.

Question: How to experimentally verify the performance of a quantum operation?

6.1 State tomography

Goal: Infer density matrix ρ of an unknown state from a set of measurements.

⚠ Single quantum measurement cannot distinguish between non-orthogonal states, e.g $|0\rangle$ and $(|0\rangle + |1\rangle)/\sqrt{2}$. Furthermore, outcome of measurements is probabilistic.

$\Rightarrow$ Need ability to prepare quantum system repeatedly in ρ to perform multiple experiments and average.

Elements of ρ are not directly measurable. Need to express in terms of measurable observables.

For single qubit:

$$\rho = \frac{\mathbb{1}}{2} + \frac{\langle \sigma_x \rangle}{2} \sigma_x + \frac{\langle \sigma_y \rangle}{2} \sigma_y + \frac{\langle \sigma_z \rangle}{2} \sigma_z \quad (*)$$

Exercise: Show that (*) holds, by explicitly verifying $\langle \sigma_i \rangle = \mathrm{Tr}[\rho\, \sigma_i]$

Measurement of expectation values $\langle \sigma_i \rangle$ provides estimate of ρ.

Typically, apparatus measures in z-basis

0 1 1 1 0 ... 0 1

repeat n times

$$\Rightarrow \langle \sigma_z \rangle \approx \frac{1}{n} \sum_{i=1}^{n} (-1)^{i}$$

How to measure in x & y basis? Insert $-\boxed{R_y(\frac{\pi}{2})}-$ or $-\boxed{R_x(\frac{-\pi}{2})}-$

Basis rotation pulses prior to measurement.

How to ensure that ρ estimated according to (*) is physical, i.e. $\mathrm{Tr}[\rho] = 1$ and $\rho \geq 1$?

Consider the (extreme) case, in which we measure 5 times in each basis and obtain "0" in all $3 \times 5 = 15$ individual shots. For a state $|0\rangle$ this would e.g. occur with a (small) probability $\left(\frac{1}{2}\right)^5 \times \left(\frac{1}{2}\right)^5 \approx 0.1\%$.

$\Rightarrow \langle \sigma_x \rangle = \langle \sigma_y \rangle = \langle \sigma_z \rangle = 1$ our estimated expectation values.

$$\stackrel{(*)}{\Longrightarrow} \quad \rho = \frac{1}{2} \begin{pmatrix} 2 & 1-i \\ 1+i & 0 \end{pmatrix} \quad ↯ \quad \text{negative eigenvalue!}$$

Avoid unphysical density matrix by asking:
"What is the most likely physical dens. mat. given the observed measurements?"

$\hat{=}$ Maximum likelihood estimation

Assuming independence of measurements and Gaussian distribution underlying the observables:

$$\mathcal{L}_{\log} = \sum_i \left[\langle \sigma_i \rangle - \mathrm{Tr}\left[\rho \sigma_i \right] \right]^2 / v_i$$

negative log-likelihood function

measured average of observable

dens. matrix

Minimize $\mathcal{L}_{\log}$ w.r.t elements of ρ subject to $\rho \geq 0$.

Constrained optimization. Solve e.g. using semi-definit programming.

Generalize to multiple qubits:

$$\rho = \frac{1}{2^n} \sum_{\{i_1, \dots, i_n\}} \langle \sigma_{i_1} \dots \sigma_{i_n} \rangle \, \sigma_{i_1} \dots \sigma_{i_n}$$

with $\sigma_i \in \{ \mathbb{1}, \sigma_x, \sigma_y, \sigma_z \}$

Full state tomo requires measurement of $4^n - 1$ Pauli expectation values.

How close is measured state to target state?

Measure of distance: Fidelity F

$$F = \sqrt{\langle \psi_t | \rho | \psi_t \rangle} \qquad \text{for pure target state}$$

$$= \mathrm{tr} \sqrt{\sqrt{\rho}\, \rho_t \sqrt{\rho}} \qquad \text{for mixed } \rho_t .$$

For mathematical properties and motivation of this metric, see Nielsen & Chuang 9.2

6.2 Process tomography

$$\rho \rightarrow \mathcal{E}(\rho) \qquad \text{quantum operation}$$

Example 1 : $\mathcal{E}(\rho) = U \rho U^{+}$ unitary

~ 2 : $\mathcal{E}(\rho) = M_j \rho M_j^{+}$ measurement with outcome j

More generally

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^{+} \qquad \text{operator sum representation}$$

Example : Decay with prob. γ

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\gamma} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

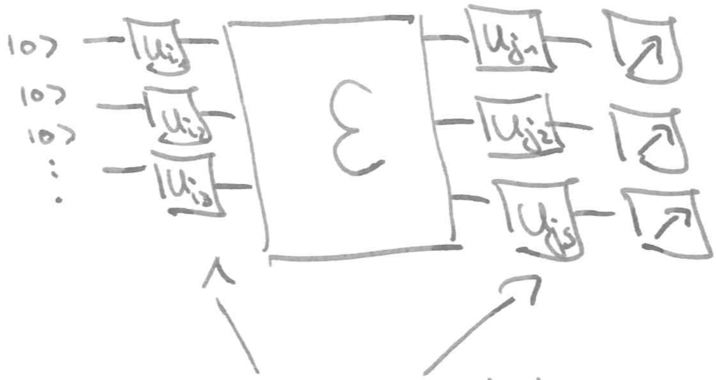
$$\mathcal{E}(|0\rangle\langle 0|) = |0\rangle\langle 0|, \quad \mathcal{E}(|1\rangle\langle 1|) = (1-\gamma)|1\rangle\langle 1| + \gamma |0\rangle\langle 0|$$

Properties of $\mathcal{E}$:

- $\mathrm{tr}[\mathcal{E}(\rho)] \;\hat{=}\;$ „prob that process $\mathcal{E}$ occurs"
- completely positive
- convex-linear map

How to characterize a process?

=> Perform state tomo on output states for complete set of (known) input states.



basis rotation pulses

Different representations:

- χ-matrix
- Pauli transfer matrix $\mathcal{R}$

$$\vec{r}_{out} = \mathcal{R}\,\vec{r}_{in}$$

output Bloch vector

process matrix

multi-qubit Bloch vector with elements $\langle \sigma_{i_1} \sigma_{i_2} \dots \rangle$

$\{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$

example see slide.