HW3MuyangShi

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Note: the cpp source code to this document can be found on my Github, listed as `EM.cpp`, here.

Problem 1

(a)

Define $Q(\theta|\theta^{(t)})$ to be the expectation of the joint log likelihood for the complete data $X = (Y, \Delta)$, conditioned on the observed data $Y = y$,

$$\begin{aligned}
Q(\theta|\theta^{(t)}) &= \mathbb{E}\left\{\log L(\theta|X)|y, \theta^{(t)}\right\} \\
&= \mathbb{E}\left\{\log f_X(x|\theta)|y, \theta^{(t)}\right\} \\
&= \int \left[\log f_X(x|\theta)\right] f_{\delta|y}(\delta|y, \theta^{(t)}) d\delta \\
(\star) &= (1-p)\mu\exp(-\mu y) \cdot \frac{(1-p)\mu\exp(-\mu y)}{p\lambda\exp(-\lambda y) + (1-p)\mu\exp(-\mu y)} \\
&\quad + p\lambda\exp(-\lambda y) \cdot \frac{p\lambda\exp(-\lambda y)}{p\lambda\exp(-\lambda y) + (1-p)\mu\exp(-\mu y)}
\end{aligned}$$

So for the E-step we just compute $(\star)$.

(b)

We maximize $Q(\theta|\theta^{(t)})$ with respect to $\theta = (p, \lambda, \mu)$; so, differentiating with respect to p, λ, μ yields:

$$\begin{aligned}
\frac{dQ(\theta|\theta^{(t)})}{dp} &= -\mu\exp(\mu y) \cdot \frac{(1-p)\mu\exp(-\mu y)}{p\lambda\exp(-\lambda y) + (1-p)\mu\exp(-\mu y)} \\
&\quad + (1-p)\mu\exp(-\mu y) \\
&\quad \cdot \frac{-\mu\exp(-\mu y) \cdot [p\lambda\exp(-\lambda y) + (1-p)\mu\exp(-\mu y)] - (1-p)\mu\exp(-\mu y)[\lambda\exp(-\mu y) - \mu\exp(-\mu y)]}{[p\lambda\exp(-\lambda y) + (1-p)\mu\exp(-\mu y)]^2} \\
&\quad + \lambda\exp(-\lambda y) \cdot \frac{p\lambda\exp(-\lambda y)}{p\lambda\exp(-\lambda y) + (1-p)\mu\exp(-\mu y)} \\
&\quad + p\lambda\exp(-\lambda y) \cdot \frac{\lambda\exp(-\lambda y) \cdot [p\lambda\exp(-\lambda y) + (1-p)\mu\exp(-\mu y)] - p\lambda\exp(-\lambda y)[\lambda\exp(\lambda y) - \mu\exp(-\mu y)]}{[p\lambda\exp(-\lambda y) + (1-p)\mu\exp(-\mu y)]^2}
\end{aligned}$$

$$
\begin{aligned}
\frac{dQ(\theta|\theta^{(t)})}{d\lambda} &= (1-p)\mu\exp(-\mu y)\cdot\frac{-(1-p)\mu\exp(-\mu y)[p\exp(-\lambda y)+p\lambda\exp(-\lambda y)(-y)]}{[p\lambda\exp(-\lambda y)+(1-p)\mu\exp(-\mu y)]^2} \\
&+ [p\exp(-\lambda y)+p\lambda\exp(-\lambda y)(-y)]\cdot\frac{p\lambda\exp(-\lambda y)}{p\lambda\exp(-\lambda y)+(1-p)\mu\exp(-\mu y)} \\
&+ p\lambda\exp(-\lambda y)\cdot \\
&\left(\frac{[p\exp(-\lambda y)+p\lambda\exp(-\lambda y)(-y)][p\lambda\exp(-\lambda y)+(1-p)\mu\exp(-\mu y)]}{[p\lambda\exp(-\lambda y)+(1-p)\mu\exp(-\mu y)]^2}\right. \\
&\left.-\ \frac{p\lambda\exp(-\lambda y)[p\exp(-\lambda y)+p\lambda\exp(-\lambda y)(-y)]}{[p\lambda\exp(-\lambda y)+(1-p)\mu\exp(-\mu y)]^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dQ(\theta|\theta^{(t)})}{d\mu} &= [(1-p)\exp(-\mu y)+(1-p)\mu\exp(-\mu y)(-y)]\cdot\frac{(1-p)\mu\exp(-\mu y)}{p\lambda\exp(-\lambda y)+(1-p)\mu\exp(-\mu y)} \\
&+ (1-p)\mu\exp(-\mu y)\cdot \\
&\left(\frac{[(1-p)\exp(-\mu y)+(1-p)\mu\exp(-\mu y)(-y)][p\lambda\exp(-\lambda y)+(1-p)\mu\exp(-\mu y)]}{[p\lambda\exp(-\lambda y)+(1-p)\mu\exp(-\mu y)]^2}\right. \\
&\left.-\ \frac{(1-p)\mu\exp(-\mu y)[(1-p)\exp(-\mu y)+(1-p)\mu\exp(-\mu y)(-y)]}{[p\lambda\exp(-\lambda y)+(1-p)\mu\exp(-\mu y)]^2}\right) \\
&+ p\lambda\exp(-\lambda y)\cdot\frac{-p\lambda\exp(-\lambda y)[(1-p)\exp(-\mu y)+(1-p)\mu\exp(-\mu y)(-y)]}{[p\lambda\exp(-\lambda y)+(1-p)\mu\exp(-\mu y)]^2}
\end{aligned}
$$

Setting these derivatives equal to zero and solving for $\hat{p}, \hat{\mu}, \hat{\lambda}$ completes the M step; we set $\theta^{(t+1)}$ to be these $\hat{p}, \hat{\mu}, \hat{\lambda}$.

(Note: to perform the EM, we'd now be returning to the E-step, unless a stopping criteria, e.g. $||\theta^{(t+1)} - \theta^{(t)}||_2 < \epsilon$, has been met.

Problem 2

Problem 3

Problem 4

Problem 5

Problem 6