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Quantitative Finance Homework

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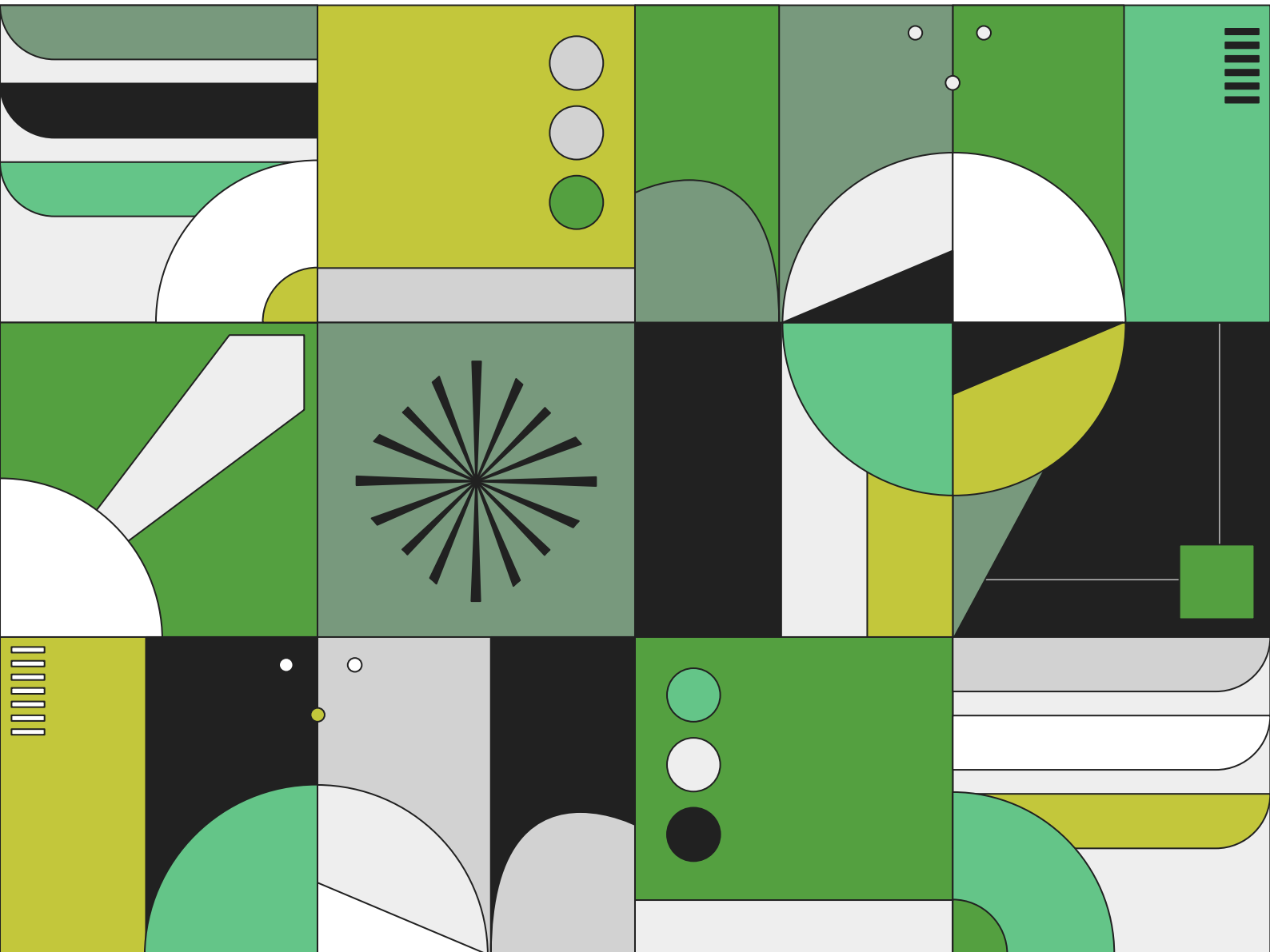


Assignment I

Dataset: Prices 2020-2022 (three years): Apple, Amazon, Meta, General Motors, Allianz, HSBC.

Use the dataset to address the following exercises:

Exercise 1

- Compute the mean and the variance of the daily returns $\frac{p(t)}{p(t-1)} - 1$ of the stocks.
- Evaluate the normality of stock returns (use a standard test).
- Apply the Mean-Variance criterion to rank the assets.
- Use a mean-variance utility $u(x) = E[\dot{x}] - \frac{a}{2}\sigma^2(\dot{x})$ (compute the expected utility as in the slides) to rank the assets for a= 0.05, 0.5.

Exercise 2

- Compute the mean, variance, covariance of the daily returns $\frac{p(t)}{p(t-1)} - 1$ of the stocks.
- Compute the minimum variance portfolio (and its expected return and variance) for two frontiers with three assets (Apple, Amazon, Meta) and (General Motors, Allianz, HSBC). Use the formula in the slides. What is the best combination (of the three assets) in terms of Variance and of the Mean-Variance criterion of the minimum variance portfolio?
- Compute the minimum variance portfolio (and the variance) of the Frontier with all the assets (mean, variance and covariance estimated from the time series) and assuming zero correlation among the assets (mean and variance estimated from the time series).

You can provide comments on the results.

Exercise I

- Compute the mean and the variance of the daily returns $\frac{p(t)}{p(t-1)} - 1$ of the stocks.

1.1 The mean and the variances of the six stocks are as follows:

- AAPL
  - Mean: 0.001023
  - Variance: 0.000540
- AMZN
  - Mean: 0.000141
  - Variance: 0.000605
- ALLIANZ
  - Mean: 0.000256
  - Variance: 0.000381
- GM:
  - Mean: 0.000357
  - Variance: 0.000948
- HSBC
  - Mean: 0.000061
  - Variance: 0.000455
- META
  - Mean: -0.000248
  - Variance: 0.000942

Here, observing the results, we can see that the mean for AAPL is the highest which means that it provides the best results in terms of the returns.

On the other hand, we can see that the Variance for META and General Motors is quite high compared to the other stocks.

Exercise I

- Evaluate the normality of stock returns (use a standard test).

1.1. So, to compute the normality, I opted for the Jarque Bera test and doing so got me the following results:

	AAPL	***AMZN***	***ALLIANZ***	***GM***	***HSBC***	***META***
t-score	514.616	489.573	7223.394	746.3597	357.645	6611.0769
p-value	1.7888E-112	4.9021E-107	0.00000	8.5E-163	2.18E-78	0.000000

Here, our null hypothesis is that the stocks are normally distributed, and the alternate hypothesis is that the stocks are not normally distributed.

Looking at the p-value, we can see that the numbers are very small and are enough for us to conclude that the values are not normally distributed. Therefore, we reject the null hypothesis that the stocks are normally distributed.

Exercise I

- Apply the Mean-Variance criterion to rank the assets.

1.1 We can judge the stocks by their expected return per risk.

- AAPL
  - Mean: 0.001023
  - Variance: 0.000540
  - Expected return per risk: 1.891473

- AMZN
  - Mean: 0.000141
  - Variance: 0.000605
  - Expected return per risk: 0.232775

- ALLIANZ
  - Mean: 0.000256
  - Variance: 0.000381
  - Expected Return per risk: 0.672173

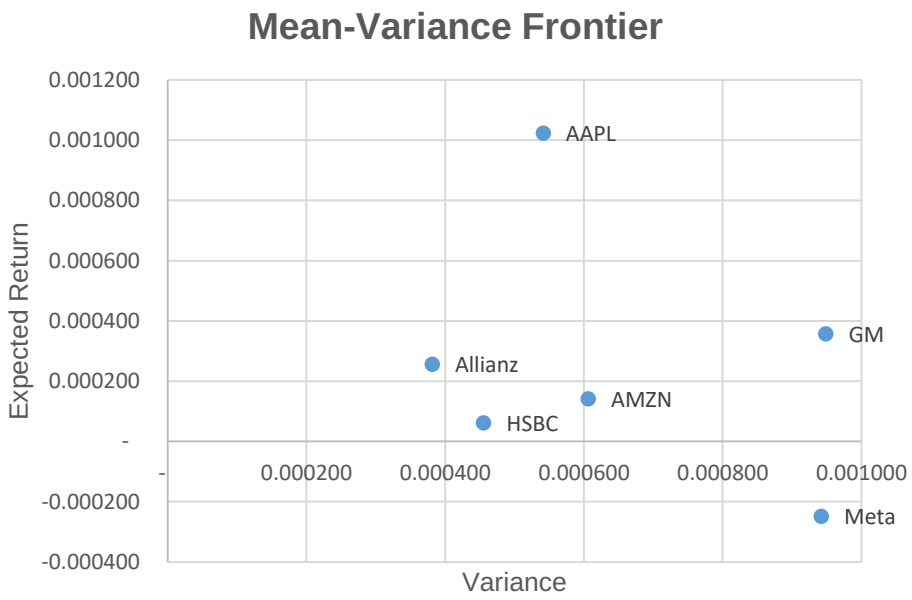
- GM:
  - Mean: 0.000357
  - Variance: 0.000948
  - Expected return per risk: 0.376593

- HSBC
  - Mean: 0.000061
  - Variance: 0.000454
  - Expected return per risk: 0.135297

- META
  - Mean: -0.000248
  - Variance: 0.000940
  - Expected return per risk: -0.263786

Continued…

Exercise I



Judging by these numbers, we can see that AAPL stocks provide the best results in terms of the risk we take. On the other hand, META stocks provide the worst results for the risk taken.

We can rank the assets as belows:

- AAPL
- ALLIANZ
- AMZN
- GM
- HSBC
- META

	AAPL	AMZN	ALLIANZ	GM	HSBC	META
Kurtosis	4.040	3.939	14.905	4.870	3.342	14.264
Skewness	0.095	0.108	0.205	0.033	0.225	-1.294

In the above table, we can see that the above stocks have a value greater than 3 for Kurtosis. This means that all of them have a positive kurtosis, which indicate fat tails in the distribution.

Also, we can see that all of the stocks (except Meta) are positively skewed. This means that these stocks are not perfectly symmetrical and have outliers toward the left.

Exercise I

- Use a mean-variance utility $u(x) = E[\dot{x}] - \frac{a}{2}\sigma^2(\dot{x})$ (compute the expected utility as in the slides) to rank the assets for a= 0.05, 0.5.

1.1. Mean-variance utility → $u(x)$ = $E[x]$ - (a/2)σ^2(x):

- AAPL:
  - Utility score for a = 0.05→ 0.00102384
  - Utility score for a = 0.5 → 0.00102378
- AMZN:
  - Utility score for a = 0.05 → 0.000141096
  - Utility score for a = 0.5 → 0.000141014
- ALLIANZ:
  - Utility score for a = 0.05 → 0.000256284
  - Utility score for a = 0.5 → 0.000256252
- GM:
  - Utility score for a = 0.05 → 0.00035721
  - Utility score for a = 0.5 → 0.00035701
- HSBC:
  - Utility score for a = 0.05 → 6.159E-05
  - Utility score for a = 0.5 → 6.15434E-05
- META:
  - Utility score for a = 0.05 → -0.000248515
  - Utility score for a = 0.5 → -0.000248714

Continued…

So, this is how they would rank be ranked:

- For a = 0.05:
    1. AAPL: 0.00102384
    2. ALLIANZ: 0.000256284
    3. GM: 0.00035721
    4. AMZN: 0.000141096
    5. HSBC: 6.159E-05
    6. META: -0.000248515

For a person whose coefficient of risk aversion is 0.05, the person would prefer the stocks in the above order. Here, we can see that the person with a risk aversion with 0.05, he/she would prefer AAPL stocks the most and META stocks the least.

- For a = 0.5:
    1. AAPL: 0.00102378
    2. ALLIANZ: 0.000256252
    3. GM: 0.00035701
    4. AMZN: 0.000141014
    5. HSBC: 6.15434E-05
    6. META: -0.000248714

Now, with a person whose risk aversion is 0.5, we can see that he/she would prefer AAPL more and here again, we can see that META is at the bottom since they have a negative utility in both the cases.

Exercise II

- Compute the mean, variance, covariance of the daily returns $\frac{p(t)}{p(t-1)} - 1$ of the stocks.

1.2. The mean and the variances of the six stocks are as follows:

- AAPL
  - Mean: 0.001023
  - Variance: 0.000540
- AMZN
  - Mean: 0.000141
  - Variance: 0.000605
- ALLIANZ
  - Mean: 0.000266
  - Variance: 0.000376
- GM:
  - Mean: 0.000357
  - Variance: 0.000948
- HSBC
  - Mean: 0.000062
  - Variance: 0.000454
- META
  - Mean: -0.000248
  - Variance: 0.000940

The covariances among the six stocks are as follows:

	AAPL	**AMZN**	**META**	**ALLIANZ**	**GM**	**HSBC**
AAPL	0.00054058					
AMZN	0.00037939	0.000605383				
META	0.00044815	0.000467196	0.000940775			
ALLIANZ	9.0218E-05	4.87448E-05	8.05382E-05	0.000381		
GM	0.00031148	0.000239797	0.000343457	0.000126	0.00094733	
HSBC	0.00016208	0.000125621	0.000158163	7.92E-05	0.00034616	0.00045465

Exercise II

- Compute the minimum variance portfolio (and its expected return and variance) for two frontiers with three assets (Apple, Amazon, Meta) and (General Motors, Allianz, HSBC). Use the formula in the slides. What is the best combination (of the three assets) in terms of Variance and of the Mean-Variance criterion of the minimum variance portfolio?

2. The following are the expected return and the variance of the two portfolios:

○ Portfolio 1 (Apple, Amazon, Meta):

	Weights	Expected Return	Variance
AAPL	0.56513	0.00102	0.00054
AMZN	0.40016	0.00014	0.00061
Meta	0.03471	-0.00025	0.00094

- **Expected return of Portfolio 1** → 0.000626453
- **Variance of Portfolio 1** → 0.000472871

○ Portfolio 2 (Allianz, GM, HSBC):

	Weights	Expected Return	Variance
Allianz	0.54701	0.00026	0.00038
GM	0.03280	0.00036	0.00095
HSBC	0.42020	0.00006	0.00046

- **Expected return of Portfolio 2** → 0.00017779
- **Variance of the Portfolio 2**→ 0.000245697

We can see here that even though the second portfolio has the smaller variance, it also has a very low return. On the other hand, the first portfolio offers a higher return, but it also poses a lot more risk compared to the first portfolio.

Exercise II

- Compute the minimum variance portfolio (and the variance) of the Frontier with all the assets (mean, variance and covariance estimated from the time series) and assuming zero correlation among the assets (mean and variance estimated from the time series).

1.2. Let's calculate the portfolio return and variance, putting the correlation term to 0:

	Weights	Variance	Returns
AAPL	0.177088	0.000541	0.001023
AMZN	0.158132	0.000606	0.000141
Meta	0.101757	0.000942	-0.000248
Allianz	0.251409	0.000381	0.000256
GM	0.101053	0.000948	0.000357
HSBC	0.210557	0.000455	0.000061

- **Portfolio expected return** → 0.000291842
- **Portfolio variance** → 0.0000957

Here, we can see that with no correlation among the stocks in the portfolio, we are seeing a return that is not much different the previous results we obtained.

However, this time we can see that the variance has drastically reduced. This is a positive for us as we have perfectly demonstrated the effects of a portfolio when there is no correlation among the assets.

Assignment II

Exercise 1

Given a binomial model, consider 6 months horizon. Every three months the asset price can go up of 13% and down of 13%. The strike price is 10$ and the price today is 10$, the annual risk-free interest rate is 4% (pay attention apply the equivalent interest rate according to the compound interest rate law):

- What is the price of a Call Option today?
- What is the price of the Put Option today?
- Verify the Put-Call parity (it is on the slides)

Exercise 2

Given a binomial model, consider a 4 years horizon. Every year the asset price can go up of 7% and down of 7%. The strike price is 11$ and the price today is 9$, the annual risk-free interest rate is 3%. Compute the price of a Call Option today. Compute the Delta-hedging startegy in t=0.

Exercise 3 (non compulsory)

Consider an American Put option with exercise price K=14$, maturity T=3 months, the underlying asset price evolves as a binomial model with S(0)=15$, u=1.4, d=0.78 over three periods (up and down each month); annual risk-free interest rate 1% (pay attention apply the equivalent interest rate according to the compound interest rate law). Compute the price of the derivative contract today.

You can provide comments to the results.

Exercise I

Exercise 1

Given a binomial model, consider 6 months horizon. Every three months the asset price can go up of 13% and down of 13%. The strike price is 10$ and the price today is 10$, the annual risk-free interest rate is 4% (pay attention apply the equivalent interest rate according to the compound interest rate law):

- What is the price of a Call Option today?
- What is the price of the Put Option today?
- Verify the Put-Call parity (it is on the slides)

1.1.

T0	T1	T2 Spot	T2 Strike	T2 Decision	
		12.769	10	2.769	
	11.3				
	1.474				Price of Option
10		9.831	10	0	Price of Stock
0.785					
	8.7				
	0				
		7.569	10	0	

Here we can see that the price of the Call Option today is $0.785.

T0	T1	T2 Spot	T2 Strike	T2 Decision	
		12.769	10	0	
	11.3				
	0.0579				Price of Option
10		9.831	10	0.169	Price of Stock
0.347					
	8.7				
	0.923				
		7.569	10	2.431	

Let's now talk about the situation where we are on the opposite side i.e., a Put Option. In that situation, the price of the Put Option is $0.347 today.

As the time-period of prices going up or down is three months per year, we have discounted the risk-free rate with ¼ in the formula.

Exercise II

Exercise 2

Given a binomial model, consider a 4 years horizon. Every year the asset price can go up of 7% and down of 7%. The strike price is 11$ and the price today is 9$, the annual risk-free interest rate is 3%. Compute the price of a Call Option today. Compute the Delta-hedging startegy in t=0.

1.2. Here, in this question, the price of the Call Option today is $0.184.

	T0	T1	T2	T3		T4	
					Spot Price	Strike	Decision
					11.797	11	0.7974
				0.55			
				11.03			
			0.383				
			10.3041		10.25	11	0
		0.27		0			
		9.63		9.582			
Delta	0.184		0				
0.129	9		8.96		8.912	11	0
		0		0			
		8.37		8.33			
			0				
			7.784		7.75	11	0
Price of Option				0			
Price of Stock				7.239			
					6.732	11	0

After calculating the delta hedging strategy in T0, we can get to the outcome that we have to short 0.12 units of stocks to hedge against the risk of 0.12 for the delta hedging strategy of a Call Option.

Exercise III

Exercise 3 (non compulsory)

Consider an American Put option with exercise price K=14$, maturity T=3 months, the underlying asset price evolves as a binomial model with S(0)=15$, u=1.4, d=0.78 over three periods (up and down each month); annual risk-free interest rate 1% (pay attention apply the equivalent interest rate according to the compound interest rate law). Compute the price of the derivative contract today.

You can provide comments to the results.

After calculating the price of the American Put Option and taking the decision of whether to exercise or hold it each step, we have come to the price of the derivative on the present day.

T0		T1		T2			T4		
						Spot Price	Strike Price	Decision
						41.16	14	0
				29.4	0			
				0	0			
		21	0.506			22.932	14	0
		0	0.506					
15	2.377			16.38	0.787			
				0	0.787			
		11.7	3.416			12.776	14	1.223
		2.3	3.416					
				9.126	4.874			
				4.874	4.862			
Price of Stock						7.118	14	6.881
Price of American Option								

Being an American Option, we know that it can be called at any moment. Hence, we calculated the price of the stock at each moment and assessed at each step whether it would be feasible to call the Option or not.

The price of the Put Option today is $2.377.

One noteworthy point here is that as the risk-free rate was annual and the period was three months, we had to discount the price by 1/12 each time.

Assignment III

Dataset: prices of the last five years of: Intesa San Paolo, Eni, UniCredit, General Motors, Stellantis.

Exercise 1

- Compute the mean and the variance of the daily log-returns ($\log\left(\frac{p(t)}{p(t-1)}\right)$) of the stocks through the sample. In particular, compute the variance through two methods: sample variance, EWMA with λ=0.95.
- Given these estimates, compute the VaR at 0.01 of each asset at one day and ten days. under the assumption of Normality and i.i.d. returns.
- Under the hypothesis of Normality compute the one day VaR at 0.01 of the portfolio made up of one stock of Unicredit and one stock of Intesa Sanpaolo.

Exercise 2

Estimate VaR at 0.01 and 0.05 (again logarithmic return) through historical simulation (simple and weighted historical simulation) for each of the five asset with daily observations (5 years).

You can provide comments to the results.

Exercise I

- Compute the mean and the variance of the daily log-returns ($\log\left(\frac{p(t)}{p(t-1)}\right)$) of the stocks through the sample. In particular, compute the variance through two methods: sample variance, EWMA with λ=0.95.

3.1.Calculating mean of the stocks:

Stellantis	UniCredit	GM	ENI	Intesa Sanpaolo
0.000124	0.000013	- 0.000041	0.000091	0.000066

Calculating variance through sample variance:

Stellantis	UniCredit	GM	ENI	Intesa Sanpaolo
0.000119	0.000128	0.000132	0.000073	0.000084

GM has the highest sample variance, indicating higher variability in its returns. Stellantis and UniCredit have relatively lower sample variances compared to GM. ENI and Intesa Sanpaolo have the lowest sample variance among the assets.

Calculating variance through EWMA with λ=0.95:

Stellantis	UniCredit	GM	ENI	Intesa Sanpaolo
0.000172	0.000048	0.000066	0.000022	0.000038

Stellantis has the highest EWMA variance, suggesting higher volatility when considering recent returns more heavily. UniCredit and GM show a decrease in variance compared to sample variance, indicating that recent observations have a dampening effect on volatility. ENI and Intesa Sanpaolo also exhibit a decrease in variance with EWMA.

Continued…

Exercise II

Looking at the above results, we can see that there is a clear difference in the variance when companred through the sample variance and the equally weighted moving averages method. The choice of variance calculation method (sample variance vs. EWMA) affects the volatility measures, with EWMA giving more weight to recent observations.

The results from EWMA result in lower volatility measures compared to sample variance, especially when recent observations indicate lower volatility. On the other hand, the results from sample variance may be more influenced by historical outliers, leading to potentially higher volatility measures.

Exercise I

- Given these estimates, compute the VaR at 0.01 of each asset at one day and ten days.

  under the assumption of Normality and i.i.d. returns.

3.1. Computing the VaR at 0.01 of each individual asset at one day and ten days.

○ **Stellantis**:

- Expected Return: 0.00012
- Standard Deviation: 0.010892
- 1-day VaR: 0.0255
- 10-day VaR: 0.081

The 1-day VaR for Stellantis at a 1% significance level suggests that there is a 1% chance of a daily loss exceeding 2.55%. While the 10-day VaR at a 1% significance level indicates that there is a 1% chance of a cumulative loss exceeding 8.1% over ten days.

Similarly, we get the results from the other given assets.

○ **UniCredit**:

- Expected Return: 0.0000133
- Standard Deviation: 0.011316
- 1-day VaR: 0.0263
- 10-day VaR: 0.0833

The 1-day VaR at a 1% significance level suggests that there is a 1% chance of a daily loss exceeding 2.63%. However, as we can see that the 10-day VaR at a 1% significance level indicates that there is a 1% chance of a cumulative loss exceeding 8.33% over ten days.

Continued…

Exercise I

3.1. Continuing the results from the previous question, we are now going to observe the rest of the assets.

- **ENI**:
  - Expected Return: 0.000091
  - Standard Deviation: 0.0085
  - 1-day VaR: 0.0199
  - 10-day VaR: 0.0636

The 1-day VaR at a 1% significance level suggests that there is a 1% chance of a daily loss exceeding 1.99%. Whereas, the 10-day VaR at a 1% significance level indicates that there is a 1% chance of a cumulative loss exceeding 6.36% over ten days.

- **Intesa Sanpaolo:**
  - Expected Return: 0.000066
  - Standard Deviation: 0.00913
  - 1-day VaR: 0.0213
  - 10-day VaR: 0.0679

Going through the results in the same fashion, we can observe that the 1-day VaR at a 1% significance level suggests that there is a 1% chance of a daily loss exceeding 2.13%. On the other hand, the 10-day VaR at a 1% significance level indicates that there is a 1% chance of a cumulative loss exceeding 6.79% over ten days.

Continued…

Exercise I

3.1. Lastly, we will now observe the results for General Motors.

- **GM**:
    - Expected Return: -0.00004
    - Standard Deviation: 0.01147
    - 1-day VaR: 0.0267
    - 10-day VaR: 0.0840

As 1-day VaR at a 1% significance level suggests that there is a 1% chance of a daily loss exceeding 2.67%. Whereas the 10-day VaR at a 1% significance level indicates that there is a 1% chance of a cumulative loss exceeding 8.4% over ten days.

Observing the results, we know that stocks with higher volatility tend to have higher VaR. GM, UniCredit, and Stellantis have approximately similar 1-day VaR values, suggesting comparable short-term risk. Over 10 days, all these stocks see an increase in risk, with GM having the highest 10-day VaR.

We can also interpret that despite GM having a negative expected return, its risk level (VaR) is comparable to those with positive expected returns. Also, note that ENI stands out with relatively lower VaR values, indicating potentially lower risk compared to the other stocks.

Exercise I

- Under the hypothesis of Normality compute the one day VaR at 0.01 of the portfolio made up of one stock of Unicredit and one stock of Intesa Sanpaolo.

3.1. Now, let's look at this question. We have to calculate the VaR of the portfolio of two assets, namely UniCredit and Intesa Sanpaolo.

UniCredit	
Expected Return	0.00000215
St. Dev	0.01135131

Intesa SanPaolo	
Expected Return	0.00007154
St. Dev	0.009146825

After calculating their return and standard deviation, let's now compute the covariance in between them.

	UniCredit	***Intesa Sanpaolo***
UniCredit	0.00012875	0.00008533
Intesa Sanpaolo	0.00008533	0.00008360

After that, we will assign weights inside the portfolio. Then, we will find the portfolio standard deviation.

Total Portfolio	
Weight of UniCredit	0.5
Weight of Intesa Sanpaolo	0.5
Portfolio Variance	0.0000958
Portfolio St. Dev	0.0097853

Then, finally we will chip in the formula to find the VaR at 1%.

Days	1
Expected Return	0.00003684
Confidence Interval	99%
Z-Score	2.326
VaR	**0.0227976**

The VaR of 2.27976% at a 1% confidence level indicates the potential downside risk of the portfolio over a one-day period.

Exercise II

Exercise 2

Estimate VaR at 0.01 and 0.05 (again logarithmic return) through historical simulation (simple and weighted historical simulation) for each of the five asset with daily observations (5 years).

3.2. So, let's compute the VaR with the Simple and Weighted Historical Simulation method.

Stellantis	
VaR Historical Simulation at 1%	-0.0308
VaR Historical Simulation at 5%	-0.0158
VaR Weighted Historical Simulation at 1%	-2.36%
VaR Weighted Historical Simulation at 5%	-1.44%

GM	
VaR Historical Simulation at 1%	-0.0280
VaR Historical Simulation at 5%	-0.0175
VaR Weighted Historical Simulation at 1%	-3.30%
VaR Weighted Historical Simulation at 5%	-1.99%

ENI	
VaR Historical Simulation at 1%	-0.0251
VaR Historical Simulation at 5%	-0.0119
VaR Weighted Historical Simulation at 1%	-2.18%
VaR Weighted Historical Simulation at 5%	-1.49%

Intesa Sanpaolo	
VaR Historical Simulation at 1%	-0.0262
VaR Historical Simulation at 5%	-0.0134
VaR Weighted Historical Simulation at 1%	-2.57%
VaR Weighted Historical Simulation at 5%	-1.38%

Continued…

Exercise II

UniCredit	
VaR Historical Simulation at 1%	-0.0311
VaR Historical Simulation at 5%	-0.0170
VaR Weighted Historical Simulation at 1%	-2.77%
VaR Weighted Historical Simulation at 5%	-1.48%

3.2. Looking at the above results, we can infer that Stellantis and UniCredit have similar VaR results for both HS and WHS at 1% and 5% confidence levels.

On the other hand, GM tends to have higher negative VaR values, indicating higher potential downside risk, especially with WHS.

ENI shows relatively lower VaR values, indicating potentially lower risk compared to the other banks. Intesa Sanpaolo falls within the range of Stellantis and UniCredit but with a slightly lower risk based on WHS.

As expected, increasing the confidence level from 1% to 5% results in less negative VaR values, reflecting a broader range of potential outcomes with a higher level of confidence.