Quantum Computing Project

Quantum Teleportation: Can it become the mode of transportation in the future?

Muzammil Tariq mt04305

Zainab Ali Vyajkora zv03573

Habib University

Spring 2021

Abstract

For the past few decades research has been going on on quantum computation. Several important properties have been identified. When it comes to applications of quantum computation none really exist yet, but ideas have been formulated for them. One such is teleportation. As quantum states can be connected to each other using un-explainable means these applications can indeed be physically created. This paper will explore the methods that go behind quantum teleportation and how data can be teleported using quantum properties. It will then explore the idea of dense coding which is similar to teleportation in the sense that it can also be used to transfer data using quantum properties.

Contents

1 Introduction

Often termed as the mode of transportation in science fiction movies is the concept of teleportation. It is usually portrayed as the disappearing and appearing of an object from one place to another, this gives the idea that no information is sent between this transition and the object just happened to get to the other place. In reality it does not follow this way as a lot of scientific limitations restrict it from being instantaneous, the foremost being that there is a transfer of classical data along with the movement of that object. For the past few decades a lot of work has been put into this which has certainly improved the viability of teleportation becoming an actual means of transport. The initial idea of quantum teleportation was set by Charles H. Bennett in 1993 and since then several scientists have been working on this. In recent works Michael A. Nielsen and Isaac L. Chuang have written their book "Quantum Computation and Quantum Information" which goes over the concept of teleportation in detail among other ideas around quantum computation.

Dense Coding is another such scheme through which classical information is sent over a quantum channel. The goal however, is different. To define dense coding in the simplest way, it is a method discovered to transfer 2 classical bits by transmitting only 1 qubit by making use of a special phenomenon called quantum entanglement. The idea was first realized in 1992, by Charles Bennett and Stephen Wiesner and several laboratory experiments and work to extend these protocols have been going on ever since.

2 Basic Terminologies

Before diving into how quantum teleportation happens or what dense coding is, let us establish some basics of the domain of quantum computation which enable us to achieve quantum teleportation.

2.1 Qubits

Qubits are the most basic unit of quantum information. They are just like bits from classical computation but with quantum properties and have a two-level quantum system with two basis states: $|0\rangle$ and $|1\rangle$

2.2 Superposition

The concept of superposition is that each qubit can be in more than one states at any instance such as being $|0\rangle$, $|1\rangle$ or both with different probabilities of each of them happening. A quantum gate called H gate is used to convert a qubit into a superposition.

2.3 Entanglement

The idea behind entanglement is that any two different quantum computers or systems will be connected to each other using ways which are incomprehensible when thought of in the domain of classical computation. So, if an entangled object is circle in nature then we would know that the object entangled to it would also be circular in nature.

2.4 Quantum Logic Gates

Quantum computation uses several logic gates which can be represented as matrices. Here are some of them:

$$\text{X gate} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{Y gate} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\text{Z gate} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\text{H gate} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$CX_{01}\ \text{gate} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Their functionalities are:

- X gate: Inverts the input
- Y gate: Performs a rotation by π radians around the y axis
- Z gate: Performs a rotation by π radians around the z axis
- H gate: Takes the state of the input to a superposition
- CX_{01} gate: This is a controlled X gate. The first qubit in the subscript tells us which qubit is the controlling one. If the controlling qubit is 1 then the X gate is applied to the target qubit.

2.5 Bell Pair

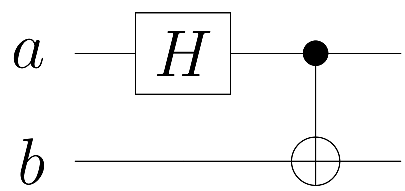


Figure 1: A Bell Pair

A bell pair is an interesting quantum circuit. Making a quantum circuit with an H gate before a CNOT gate gives the results some interesting properties. The figure above gives an example of a bell pair. The end results of these circuits are called bell states or EPR states or EPR pairs and are these:

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$|\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$|\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

When we replace a and b in Figure 1 with qubits we get these outputs:

Input	Output
$\|00\rangle$	$\frac{\|00\rangle+\|11\rangle}{\sqrt{2}}$
$\|01\rangle$	$\frac{\|01\rangle+\|10\rangle}{\sqrt{2}}$
$\|10\rangle$	$\frac{\|00\rangle-\|11\rangle}{\sqrt{2}}$
$\|11\rangle$	$\frac{\|01\rangle-\|10\rangle}{\sqrt{2}}$

Table 1: Inputs and Outputs of a Bell Pair

In general terms this can be written down as a formula like this:

$$|\beta_{xy}\rangle = \frac{|0, y\rangle + (-1)^x |1, \bar{y}\rangle}{\sqrt{2}}$$

In layman terms a bell pair is an extremely entangled quantum state consisting of 2 qubits. So, if 2 people Alice and Bob each hold a qubit part of a bell pair then that bell pair can be written as: $|\sigma^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$
Now if we were to measure the qubit in possession of Alice (the one with subscript A) then the result will be completely random and when we measure the qubit in possession of Bob (the one with subscript B) then that result will be completely random too, but if we measure both of them at the same time then we notice that the results are correlated and they are exactly the same. This concept is used in teleportation to send data from one qubit to another without any physical movement between them.

2.6 No Cloning Theorem

The no cloning theorem exists in the quantum mechanics realm and it states that identical copies of an arbitrary unknown quantum state can not be created. It was established in 1982 by William Wooters, Wojciech Zurek, and Dennis Dieks. The proof of the theorem is as such:

Suppose there is a quantum state with the qubit $|\psi\rangle$ which we have to copy. It can be written as $|\psi\rangle = a|0\rangle + b|1\rangle$ where a and b are unknown coefficients. We take another quantum state with the qubit $|\sigma\rangle$ to copy the information from $|\psi\rangle$ into $|\sigma\rangle$. Now say there is a copier U which copies the data of the qubit from the first state to the second state. It works and should work for all possible input states in the following way:

$$U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$$

$$U|\sigma\rangle|0\rangle = |\sigma\rangle|\sigma\rangle$$

Now we know that the state $a|\psi\rangle + b|\sigma\rangle$ is also a valid state which should also get copied over by U. When we use U to clone $a|\psi\rangle + b|\sigma\rangle$ it becomes $a|\psi\rangle|\psi\rangle + b|\sigma\rangle|\sigma\rangle$ by the procedure:

$$U[a|\psi\rangle + b|\sigma\rangle]|0\rangle = aU|\psi\rangle|0\rangle + bU|\sigma\rangle|0\rangle = a|\psi\rangle|\psi\rangle + b|\sigma\rangle|\sigma\rangle$$

What we want this to generate is:

$$(a|\psi\rangle + b|\sigma\rangle)|0\rangle \xrightarrow{\text{U}} (a|\psi\rangle + b|\sigma\rangle)(a|\psi\rangle + b|\sigma\rangle)$$

$$= a^2|\psi\rangle|\psi\rangle + ab|\psi\rangle|\sigma\rangle + ab|\sigma\rangle|\psi\rangle + b^2|\sigma\rangle|\sigma\rangle$$

As we can see these 2 final states are not equal to each other hence by contradiction we have proved that identical copies of an arbitrary unknown quantum state can not be created.

3 Quantum Teleportation

3.1 How it works

Suppose two people Alice and Bob live far away and Alice has to transfer some information, say the qubit $|\sigma\rangle$ to Bob. Prior to this they met with each other some time ago and established an EPR Pair between each other. To further add to this, Alice does not know which state the qubit $|\sigma\rangle$ is in and she can only send classical information to Bob. In order to send the qubit $|\sigma\rangle$ Alice will take both the qubits in her possession, the qubit $|\sigma\rangle$ and her half or the EPR pair, and measure them. According to the no cloning theorem she can not measure $|\sigma\rangle$ alone so she has to use her half of the EPR pair to do the measurement. Her measurement will be one of these 4 classical states: 00,01,10,11. She sends this information to Bob who then performs some computation on his half of the EPR pair which will result in the state of the qubit $|\sigma\rangle$ teleported to Bob's qubit.

3.2 The Algorithm

The following algorithm describes the process of the teleportation protocol and is accompanied by the circuit in figure 2

Algorithm 1: The Quantum Teleportation Algorithm

---

1. Alice's first qubit starts out in the input state $|\psi\rangle$
2. Alice's qubits go through a CNOT gate
3. Alice's first qubit goes through a hadamard gate
4. Both of Alice's qubits are measured
5. **if** *measurement is 01* **then**
   Apply X gate to Bob's qubit

   if *measurement is 10* **then**
   Apply Z gate to Bob's qubit

   if *measurement is 11* **then**
   Apply X gate and then Z gate to Bob's qubit

   else
   Continue
6. Bob's qubit is now in the state $|\psi\rangle$

3.3 The Protocol

The description of quantum teleportation can be made into a quantum circuit and explained using equations, this would explain it much better. Figure 2 shows the protocol that is followed for teleportation. The state that Alice has to teleport to Bob is $|\sigma\rangle = \alpha|0\rangle + \beta|1\rangle$, where α and β are unknown amplitudes, this is what puts the qubit $|\sigma\rangle$ in an unknown state and helps verify the workings of the protocol. The input to the circuit will be: $|\sigma_0\rangle = |\sigma\rangle|\beta_{00}\rangle$ which translates to:

$$|\sigma_0\rangle = |\sigma\rangle|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)\right]$$

The circuit for quantum teleportation protocol is:

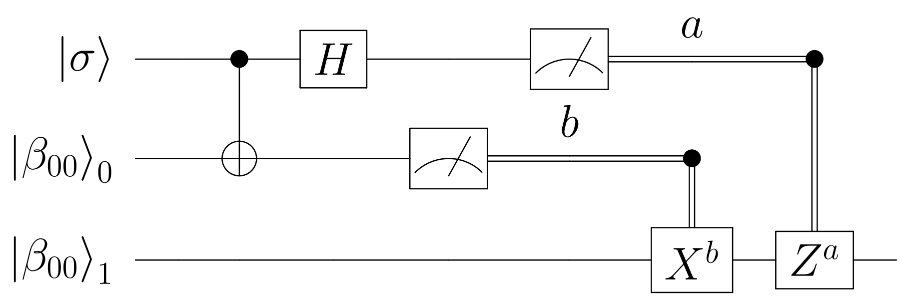


Figure 2: The Quantum Teleportation Protocol

In this circuit the first two qubits belong to Alice and the last qubit belongs to Bob. The second qubit of Alice and first qubit of Bob were entangled to each other in a bell pair prior to becoming part of this circuit. First the qubits belonging to Alice go through a CNOT gate which converts the state $|\sigma_0\rangle$ into:

$$|\sigma_1\rangle = \frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|10\rangle + |01\rangle)\right]$$

Then Alice sends her second qubit through a Hadamard gate which converts the state $|\sigma_1\rangle$ into:

$$|\sigma_2\rangle = \frac{1}{2}\left[\alpha(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + \beta(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)\right]$$

We can rewrite $|\sigma_2\rangle$ to make it easy to understand by rearranging terms like this:

$$|\sigma_2\rangle = \frac{1}{2}\left[|00\rangle\left(\alpha\,|0\rangle + \beta\,|1\rangle\right) + |01\rangle\left(\alpha\,|1\rangle + \beta\,|0\rangle\right) + |10\rangle\left(\alpha\,|0\rangle - \beta\,|1\rangle\right) + |11\rangle\left(\alpha\,|1\rangle - \beta\,|0\rangle\right)\right]$$

The expression $|\sigma_2\rangle$ can be divided into these 4 terms:

$$|00\rangle\left(\alpha\,|0\rangle + \beta\,|1\rangle\right)$$

$$|01\rangle\left(\alpha\,|1\rangle + \beta\,|0\rangle\right)$$

$$|10\rangle\left(\alpha\,|0\rangle - \beta\,|1\rangle\right)$$

$$|11\rangle\left(\alpha\,|1\rangle - \beta\,|0\rangle\right)$$

This makes it easier to understand that before doing the final measurement the state of the qubits will be either of these 4. Last step in the protocol is to measure Alice's qubits and give this measurement to Bob, this is where classical communication is involved. Depending on the measurement of Alice's qubits Bob will then convert his qubit to the state that needs to be teleported.

3.4 Analysis

Now, if we analyze the first term in these 4 terms it has Alice's qubits in the state $|00\rangle$, meanwhile Bob's qubit is in the state $\alpha\,|0\rangle + \beta\,|1\rangle$. If we step back then we see that this was the state $|\sigma\rangle$ we started with originally. This shows us that if Alice gets the state $|00\rangle$ when she measures her qubits then Bob's qubits are in the state $|\sigma\rangle$. Same goes for the other 3 possible states that Alice can measure, measuring each will mean that Bob's qubits will be in the state associated with it. One thing to note here is that in order for Bob to know the state his qubits belong to he must know the measurement outcome of Alice's qubits, this puts this teleportation protocol at a disadvantage as it includes transfer of classical data. This stops teleportation from being faster than light. Now, every time Alice measures her state and reports it to Bob he can modify his state accordingly to recover the original input $|\sigma\rangle$. If the measurement is $|00\rangle$ then he needs to do nothing, if the measurement is $|01\rangle$ then he can apply an X gate to retrieve the original input, if it is $|10\rangle$ then he can apply the Z gate to retrieve the original input, if it is $|11\rangle$ then he can apply the X gate and then the Z gate to retrieve the original input. It is important to note that even though this process makes it looks like that the circuit creates a copy of the state that is being teleported(which is $|\sigma\rangle$ in our example) in reality the original state that entered the circuit as an input has ended up being either of the basic computational state $|0\rangle$ or $|1\rangle$.

3.4.1 Timing Analysis

If we do a timing analysis of the protocol we notice that it is not instantaneous like the concept of teleportation makes us believe. Assuming that all quantum operations take negligible times we can see that there is a classical operation involved that is sending Bob the readings of Alice's qubits. This classical operation will take $\frac{distance}{speed}$ time units where assuming that the speed is the speed of light. To put this into perspective this means teleporting from earth to moon will take approximately 1.3 seconds and teleporting from earth to mars will take approximately 17 minutes. This shows that even if this protocol is made common as a way of transport the teleportation will still not be instantaneous as classical operations are involved.

4 Quantum Teleportation as a Primitive Function

In this section, we will describe in detail how the quantum computation protocol can be seen as a basic primitive function which can be built upon to achieve greater results.

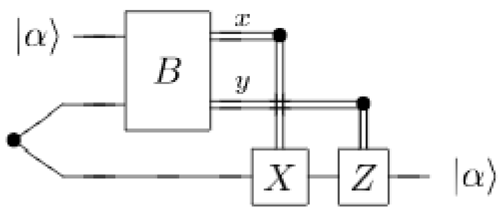


Figure 3: The Quantum Teleportation Circuit

In Figure 3 the input state $|\alpha\rangle$ is $\alpha|0\rangle + \beta|1\rangle$ while the next two rows represent the entangled state $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ of which one qubit is sent to Alice while one remains in Bob's possession. Hence the state being fed into the next step for measurement in the Bell basis is

$$(\alpha|0\rangle + \beta|1\rangle)(\tfrac{|00\rangle+|11\rangle}{\sqrt{2}})$$
$$= \frac{\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle}{\sqrt{2}}$$

This state goes through a CNOT and then a Hadamard gate and then measured by Alice in the Bell basis, which is covered in detail in section 3.3 Depending on the resulting classical bits from the measurement x and y, Bob takes his qubit and applies the X or Z gates, or both to recover the original state as explained in section 3.4.

This basic circuit involving 2 qubits can further be combined together to expand the teleportation mechanism. Figure 4 shows a circuit which accomplishes this by making use of a state $|X\rangle = \dfrac{(|00\rangle + |11\rangle)|00\rangle + (|01\rangle + |10\rangle)|11\rangle}{\sqrt{2}}$ along with two arbitrary single qubit states $|\alpha\rangle = a|0\rangle + b|1\rangle$ and $|\beta\rangle = c|0\rangle + d|1\rangle$. It is noticeable that the state $|X\rangle$ can be constructed by combining together, two EPR pairs.

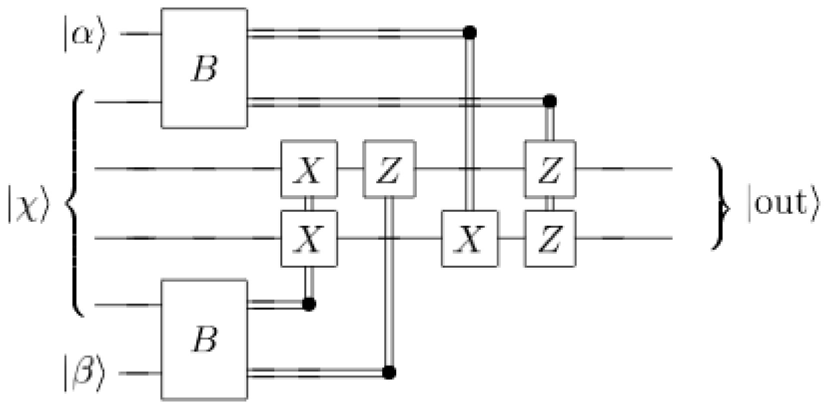


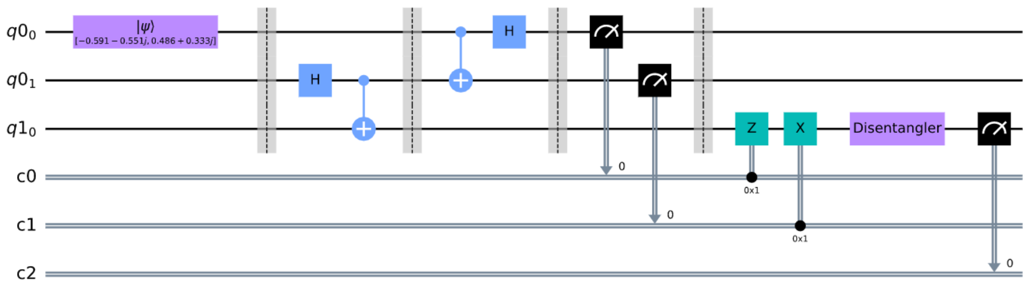
Figure 4: The Quantum Teleportation Circuit for Teleporting 2 Qubits Through a CNOT Gate [2]

As can be seen from the circuit above, the CNOT gate for each of the two EPR pairs is introduced from the single qubit in the other half of the involved EPR pair. It can be observed that as we move from left to right , the $|\beta\rangle$ state is recovered before the $|\alpha\rangle$ state by application of the required X and Z gates. The output state is therefore $|out\rangle = CNOT\,|\beta\rangle\,|\alpha\rangle$. [2]

5 Implementations

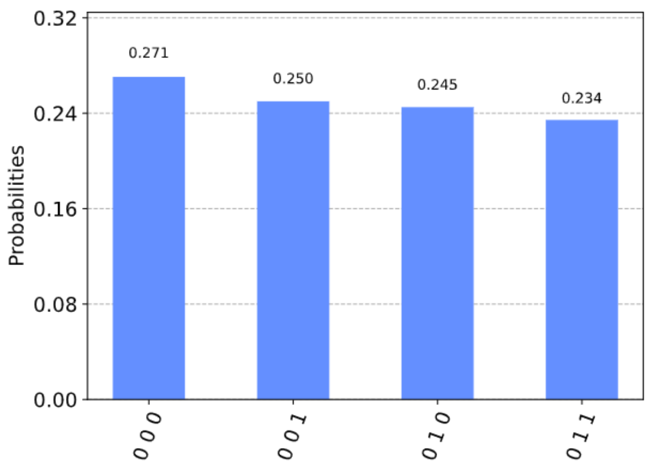
5.1 Quantum Teleportation

Firstly, we start off with giving Alice's first qubit a random state (before which it was in the state $|0\rangle$). Then we create a bell pair between Alice's second qubit and Bob's qubit and apply the quantum teleportation protocol on it. Lastly we perform corrections on the required qubits. At this point we are done with the entire circuit and we just apply the inverse of the initialization process that was done on Alice's first qubit on Bob's qubit, if his qubit is in the state $|0\rangle$ then we have successfully teleported Alice's first qubit to Bob's qubit. The following circuit is made in python using qiskit:



5.1.1 Simulation

The circuit is first ran as a simulation using the qasm_simulator. This simulation of the circuit should always give us the correct results. The histogram below shows that we have 100% certainty that the teleportation has worked properly as Bob's qubit (the leftmost qubit) is measured to be always in the state $|0\rangle$

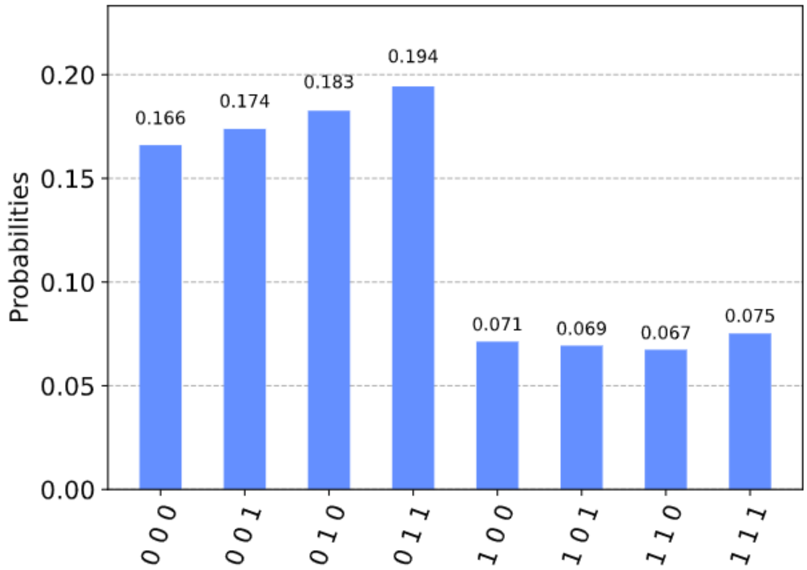


5.1.2 Real Quantum Computer

After running the circuit on a simulator we run it on a real quantum computer. On an actual quantum computer we are bound to receive a bunch of errors. We run the circuit on the least busy IBM quantum computer available using this code:

```
#Finding out which real life IBMQ backend provider can be used
provider = IBMQ.get_provider(hub='ibm-q')
backend = least_busy(provider.backends(simulator=False))
```

After running this we get the histogram below as the result which shows that there are some errors generated as the state $|1\rangle$ is measured in Bob's qubit (the leftmost one).



5.1.3 The Code

The following code is used to construct the circuit and run it on a real quantum computer

```
Alice = QuantumRegister(2)
Bob = QuantumRegister(1)
AliceC = ClassicalRegister(1)
BobC = ClassicalRegister(1)
Classical = ClassicalRegister(1)
#Constructing the circuit
qc = QuantumCircuit(Alice,Bob,AliceC,BobC,Classical)
psi = Initialize(random_state(1))
qc.append(psi,[0])
qc.barrier()
qc.h(Alice[1])
qc.cx(Alice[1], Bob)
qc.barrier()
qc.cx(Alice[0], Alice[1])
qc.h(Alice[0])
qc.barrier()
qc.measure(Alice[0], AliceC)
```

```
qc.measure(Alice[1], BobC)
qc.barrier()
qc.cx(Alice[1], Bob)
qc.cz(Alice[0], Bob)
reverse = psi.gates_to_uncompute()
qc.append(reverse, [2])
qc.measure(Bob, Classical)
#Running it on the backend
job_exp = execute(qc, backend=backend, shots=1024)
```

6 Dense Coding

6.1 Introduction

Dense coding, or superdense coding is a protocol in quantum communication, the goal of which is to communicate a larger number of classical bits while using a smaller number of qubits. It can be understood as the opposite of the more popular quantum teleportation. The first realisation of this protocol was in 1992, when Charles Bennett and Stephen Wiesner explained how to transmit two classical bits of information, while only transmitting one quantum bit from sender to receiver. [1]

6.2 How It Works

Like quantum teleportation, dense coding also makes use of maximally entangled states; the Bell pair or EPR state. Suppose there are two parties, Alice and Bob, who are at a large distance from each other, and Alice wants to transmit two bits of classical information to Bob, by only sending one qubit. This can be made possible with the help of an entangled state that may be prepared by a third party, after which Alice and Bob are sent one qubit each from the entangled state

$$|\Psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Alice then performs some operations on her qubit to encode the bit string that she wishes to send and sends it over to Bob. Now, Bob receives the entangles qubit and must apply quantum gates on this qubit along with his own qubit, to decode the string sent by Alice.

6.3 The Protocol

To understand the process thoroughly, we can look at the circuit in Figure 5 and a demonstration of an example communication through this circuit.

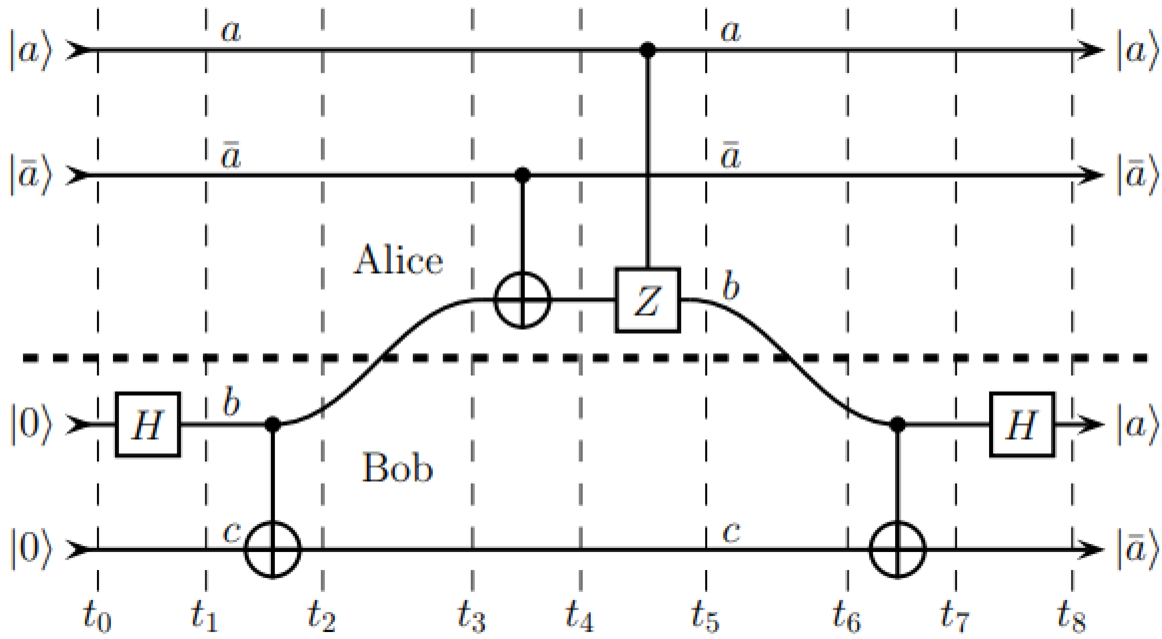


Figure 5: The Dense Coding Protocol [3]

The first part of the circuit, until t_2 represents the preparation of an entangled state after which one of the qubits is sent to Alice for encoding. Now, in order to communicate the desired classical bits, Alice must first prepare her qubit to encode the exact two bits she wishes to send. She does nothing is she wishes to send '00', If she wishes to send '01' then she applies the phase flip Z to her qubit. If she wishes to send '10' then she applies the quantum X gate, to her qubit. If she wishes to send '11' then she applies the ZX gates to her qubit. Note that $|a\rangle$ and $|\bar{a}\rangle$ determine the choice of gates to be applied according to the input bit string. After applying one of these sets of gates, the resulting state is one of the four strings corresponding to one of the encoded states:

$$'00' : |\Psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$'01' : |\Psi\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$'10' : |\Psi\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}}$$

$$'11' : |\Psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

As seen in section 2.5, these states are known as Bell states or EPR pairs and form the Bell basis. After getting her qubit into one of these states, Alice sends it over to Bob by some quantum physical means where Bob can measure the qubit in the Bell basis to determine which of the four possible bit strings was sent by Alice.

To understand how Bob would recover the state, let us consider an example. Suppose Alice wants to send the bit string '11' to Bob. We start with the state

$$\frac{1}{\sqrt{2}} |00\rangle + |11\rangle$$

After Alice performs the necessary operations, we would have

$$\frac{1}{\sqrt{2}} |01\rangle - |10\rangle$$

Now, when Bob receives this, he applies a CNOT gate with control on Alice's qubit 'b' to obtain

$$\frac{1}{\sqrt{2}} |01\rangle - |11\rangle$$
$$\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) |1\rangle$$

After this he would apply a Hadamard gate on 'b', which is Alice's qubit through which he recovers the state $|11\rangle$ that Alice had wished to send.

6.4 Conclusion

Through the use of entangled states, we have demonstrated how it is possible to teleport quantum bits. Due to the no-cloning theorem, it is impossible to create perfect copies of the information to be transmitted, however, entangled states or Bell pairs provide a way to communicate information without means of a physical transmission in case of quantum teleportation. Furthermore, we can perceive single qubit operations as the most basic or linear form of circuits which can be used to build a larger circuit, and even to construct a full quantum computer. Dense coding, on the other hand, is also shown as a means to transfer more classical bits using less qubits by the use of entangled states which would not have been possible through any classical channel. Both quantum teleportation, and dense coding are active areas of research and over the years, several experiments have been performed to extend the idea to multiple senders and receivers.

References

[1] Michael A. Nielsen, Isaac L. Chuang. 2010. *Quantum Computation and Quantum Information*. 10th Anniversary edition. Cambridge: Cambridge University Press

[2] Daniel Gottesman, Isaac L. Chuang. 2 August 1999. *Quantum teleportation is a universal computational primitive*

[3] Robert B. Griffiths. 8 February 2012. *Dense Coding, Teleportation, No Cloning*