Integer cooperative game theory

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Introduction

Cooperative game theory

Cooperative game

A **cooperative game** is an ordered pair (N, v), where $N = \{1, \dots, n\}$ is a finite set of players and $v\colon 2^N \to \mathbb{R}$ is the characteristic function. Further, $v(\emptyset) = 0$.

- $S \subseteq N$ ... coalition
- $v(S)$ ... values of coalition

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Goal: *To distribute the value of the game among players*

- **Payoff vector** $\mathbf{x} \in \mathbb{R}^n$
  - x_i represents payoff of player i
- Vector $\mathbf{x} \in \mathbb{R}^n$ is **efficient**, if $\sum_{i \in N} x_i = v(N)$
  - Usually, we distribute $v(N)$
- Vector $\mathbf{x} \in \mathbb{R}^n$ is **individually rational**, if $x_i \geq v(i)$
  - players prefer x_i over $v(i)$

But firstly...

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Imputation set

For cooperative game (N, v), we define the following sets:

- **Preimputation**: $\mathcal{I}^*(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N)\}$
- **Imputation**: $\mathcal{I}(v) = \{x \in \mathcal{I}^*(v) \mid \forall i \in N : x_i \geq v(i)\}$

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Possible fair solution...

The Shapley value

For a cooperative game (N, v), the **Shapley value** $\phi(v)$ of player i is

$$\phi_i(v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} \left(v(S \cup i) - v(S)\right)$$

Excess

Let (N, v) be an essential game, i.e., $v(N) \geq \sum_{i \in N} v(i)$, $x \in \mathcal{I}(v)$ and $\emptyset \neq S \subseteq N$. The *excess* $e(S, x, v)$ *of coalition* S *in game* (N, v) *with respect to payoff vector* x is defined as $e(S, x, v) = v(S) - x(S)$.

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- We further need to define the *vector of excesses* for every $x \in \mathbb{R}^n$ as $\Theta(x) = (e(S_1, x, v), \ldots, e(S_{2^n-2}, x, v))$, where the excesses are ordered in decreasing order.

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Nucleolus

For a cooperative game (N, v), the *nucleolus* $\eta(v)$ is defined as

$$\eta(v) = \{x \in \mathcal{I}(v) \mid \forall y \in \mathcal{I}(v) : \Theta(x) \preceq_{lex} \Theta(y)\},$$

where $\preceq_{lex}$ is the lexicographical ordering of vectors.

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- Does not ensure the integrality of the solution concepts
- Introduce new solution concepts based on the already known ones, ensuring integrality

Multi point sets...

Multi point sets...

Imputation set

For integer cooperative game $G_I \in \mathcal{G}_I^n$, the *Integer imputation set* set is defined as $\mathcal{I}_{\mathbb{Z}}(G_I) = \mathcal{I}(G_I) \cap \mathbb{Z}^n$.

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Solution concepts

For integer cooperative game $G_I \in \mathcal{G}_I^n$, the following solution concepts are defined:

- *Integer core*: $\mathcal{C}_{\mathbb{Z}}(G_I) = \mathcal{C}(G_I) \cap \mathbb{Z}^n$,
- *Integer D-Core*: $DC_{\mathbb{Z}}(G_I) = \mathcal{I}_{\mathbb{Z}}(G_I) \setminus dom(\mathcal{I}_{\mathbb{Z}}(G_I))$,
- ...

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- $\ldots$

These solution concepts were already studied in the past...

Results

Floor Shapley value

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For an integer cooperative game $G_I \in \mathcal{G}_I^n$, the **Floor Shapley value** $\lfloor\phi\rfloor(G_I)$ is given by $\lfloor\phi\rfloor(G_I) = \lfloor\phi(G_I)\rfloor$.

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Theorem 16

The Floor Shapley value $\lfloor\phi\rfloor(v_I)$ satisfies the following properties for all integer games $(N, v_I), (N, w_I) \in \mathcal{G}_I^n$:

1. Axiom of near efficiency: $v_I(N) - n \leq \sum_{i \in N} \lfloor\phi\rfloor_i(v_I) \leq v_I(N)$,
2. Axiom of symmetry: $\forall i, j \in N(\forall S \subseteq N \setminus \{i, j\} : v_I(S \cup i) = v_I(S \cup j)) \Rightarrow \lfloor\phi\rfloor_i(v_I) = \lfloor\phi\rfloor_j(v_I)$,
3. Axiom of null player: $\forall i \in N(\forall S \subseteq N : v_I(S) = v_I(S \cup i)) \implies \lfloor\phi\rfloor_i(v_I) = 0$,
4. Axiom of near additivity: $\lfloor\phi\rfloor(v_I + w_I) = \lfloor\phi(v_I) + \phi(w_I)\rfloor$.

Efficient Floor Shapley value

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- Redistribution of the remaining value...

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Efficient Floor Shapley value

For an integer cooperative game $G_I = (N, v_I) \in \mathcal{G}_I^n$, the **Efficient Floor Shapley value** $\phi^E(G_I)$ is defined as follows:

1. Compute the Floor Shapley value $\lfloor\phi\rfloor(G_I)$ and the Shapley value $\phi(G_I)$.
2. Compute the weights $w_i = \phi_i(G_I) - \lfloor\phi\rfloor_i(G_I)$ for all $i \in N$.
3. Sort the weights in descending order such that if multiple players have the same weight, then their ordering is uniformly random.
4. Each player receives his Floor Shapley value. Additionally, the top k players, where $k = v_I(N) - \sum_{i \in N} \lfloor\phi\rfloor_i(v_I) = w(N)$, receive one extra unit.

Efficient Floor Shapley value - properties

Theorem 17

The Efficient Floor Shapley value ϕ^E satisfies the following properties for all integer games $(N, v_I) \in \mathcal{G}_I^n$:

1. Axiom of efficiency: $\sum_{i \in N} \phi_i^E(v_I) = v_I(N)$,
2. Axiom of expected symmetry: $\forall i, j \in N (\forall S \subseteq N \setminus \{i, j\} : v_I(S \cup i) = v_I(S \cup j)) \Rightarrow \mathbb{E}[\phi_i^E(v_I)] = \mathbb{E}[\phi_j^E(v_I)]$,
3. Axiom of null player: $\forall i \in N (\forall S \subseteq N : v_I(S) = v_I(S \cup i)) \implies \phi_i^E(v_I) = 0$.

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- No axiom of additivity

Probabilistic Efficient Floor Shapley value

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Probabilistic Efficient Floor Shapley value

For an integer cooperative game $G_I \in \mathcal{G}_I^n$, the **Probabilistic Efficient Floor Shapley value** $\phi^{\mathbb{E}}(v)$ is defined as follows:

1. Compute the Floor Shapley value $\lfloor \phi \rfloor (G_I)$ and the Shapley value $\phi(G_I)$.
2. Compute the remainders $\tilde{p}_i = \phi_i(G_I) - \lfloor \phi \rfloor_i(G_I)$ for all $i \in N$.
3. Compute the probabilities $p_i = \frac{\tilde{p}_i}{\sum_{j \in N} \tilde{p}_j}$ for all $i \in N$.
4. Each player receives his Floor Shapley value and additionally, each unit of the remainder with probability p_i, i.e., each unit of $\tilde{p}(N) = \sum_{j \in N} \tilde{p}_j$ is given to player i with probability p_i.

Probabilistic Efficient Floor Shapley value - properties

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Theorem 18

The Probabilistic Efficient Floor Shapley value $\phi^{\mathbb{E}}$ satisfies the following properties for all integer games $(N, v_I), (N, w_I) \in \mathcal{G}_I^n$:

1. The expected value is the same as the Shapley value: $\mathbb{E}\left[\phi^{\mathbb{E}}(v_I)\right] = \phi(v_I)$,
2. Axiom of efficiency: $\sum_{i \in N} \phi_i^{\mathbb{E}}(v_I) = v_I(N)$,
3. Axiom of expected symmetry: $\forall i, j \in N (\forall S \subseteq N \setminus \{i, j\} : v_I(S \cup i) = v_I(S \cup j)) \Rightarrow \mathbb{E}\left[\phi_i^{\mathbb{E}}(v_I)\right] = \mathbb{E}\left[\phi_j^{\mathbb{E}}(v_I)\right]$,
4. Axiom of null player: $\forall i \in N(\forall S \subseteq N : v_I(S) = v_I(S \cup i)) \implies \phi_i^{\mathbb{E}}(v_I) = 0$,
5. Axiom of expected additivity: $\mathbb{E}\left[\phi^{\mathbb{E}}(v_I + w_I)\right] = \mathbb{E}\left[\phi^{\mathbb{E}}(v_I)\right] + \mathbb{E}\left[\phi^{\mathbb{E}}(w_I)\right]$.

Closest lattice Shapley

- Different approach using integer programming

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Closest Lattice Shapley value

Let $||\bullet||$ be a vector norm. The **Closest Lattice Shapley (CLS)** value of integer cooperative game $G_I \in \mathcal{G}_I^n$ is given by

$$\phi^{\mathcal{W}}(G_I) = \min_{x \in \mathcal{W}_{\mathbb{Z}}(G_I)} ||\phi(G_I) - x||.$$

Closest lattice Shapley - properties

- Does it even exist?

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Claim 19

The CLS value exists for all integer games $G_I \in \mathcal{G}_I^n$.

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- What about the properties?

Theorem 20

The CLS value $\phi^{\mathcal{W}}(v_I)$ satisfies the following properties for all integer games $(N, v_I) \in \mathcal{G}_I^n$:

1. Axiom of efficiency: $\sum_{i \in N}(\phi^W)_i(v_I) = v_I(N)$,
2. Axiom of null player:
   $\forall i \in N (\forall S \subseteq N : v_I(S) = v_I(S \cup i)) \implies (\phi^W)_i(v_I) = 0$.

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   $\forall i \in N (\forall S \subseteq N : v_I(S) = v_I(S \cup i)) \implies (\phi^W)_i(v_I) = 0$.

- No symmetry and additivity

Closest lattice Shapley - properties

- What other properties...

- What other properties...

Claims 21 - 24

For integer game $G_I \in \mathcal{G}_I^n$, the CLS value $\phi^{\mathcal{W}}(G_I)$:

- is not unique in general,
- is not always an extreme point of the Weber set $\mathcal{W}(G_I)$,
- depends on the choice of the norm,
- is different from the Efficient Floor Shapley value.

Integer nucleolus

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For an integer cooperative game $G_I \in \mathcal{G}_I^n$, the **integer nucleolus** $\eta_{\mathbb{Z}}(G_I)$ is defined as

$$\eta_{\mathbb{Z}}(G_I) = \{x \in \mathcal{I}_{\mathbb{Z}}(G_I) \mid \forall y \in \mathcal{I}_{\mathbb{Z}}(G_I) : \Theta(x) \preceq_{lex} \Theta(y)\} .$$

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- When does it exist?

Theorem 25 - 26

For an integer cooperative game $G_I \in \mathcal{G}_I^n$, it holds

$$\mathcal{I}_{\mathbb{Z}}(G_I) \neq \emptyset \implies \eta_{\mathbb{Z}}(G_I) \neq \emptyset,$$

additionally it holds that

$$\eta_{\mathbb{Z}}(G_I) \neq \emptyset \iff \eta(G_I) \neq \emptyset.$$

Integer nucleolus - properties

We lose the uniqueness...

We lose the uniqueness...

Theorem 27

For an integer cooperative game $G_I \in \mathcal{G}_I^n$ the nucleolus is not necessarily a single point solution concept.

Integer classes

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- Classes - games whose characteristic function satisfies certain properties

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- Classes - games whose characteristic function satisfies certain properties
- Introduced new integer based classes, for given c:
  - c-tight - $\forall S \subseteq N : 0 \leq v(S) \leq c \wedge v(N) = c$
  - c-bounded - $\forall S \subseteq N : 0 \leq v(S) \leq c$

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  - c-bounded - $\forall S \subseteq N : 0 \leq v(S) \leq c$
- Experiments:
  - Counted games from standard classes (e.g. convex games) that are also c-bounded/tight
  - Derived exact formulae for positive and k-games

Library

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  - I/O - possibility to load and save games
- Future plans:
  - More solution concepts - nucleolus, kernel, etc.
  - More generators - might not be complete uniformly random
  - Better documentation
  - Include integer results

Thank you for your attention