Hypothesis Testing and CI

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Originally, the final question was phrased in the following way: *Show that if a two-sided, equal tailed hypothesis test rejects the null, the confidence interval of the same significance level will not include 0.*

The idea was to provide an exercise that would show a relationship between confidence intervals and hypothesis tests for some unknown parameter θ. However, the question was somewhat lacking on detail, and thus I wanted to more fully demonstrate the idea.

A better way of phrasing the question would have been the following: *Let X be some estimator of unknown parameter θ, where X defined in R follows a symmetric distribution centered around θ. Show that if a two-sided, equal tailed hypothesis test rejects the null, the confidence interval of the same significance level will not include θ.*

There are a few instances in which this type of distribution might be useful.

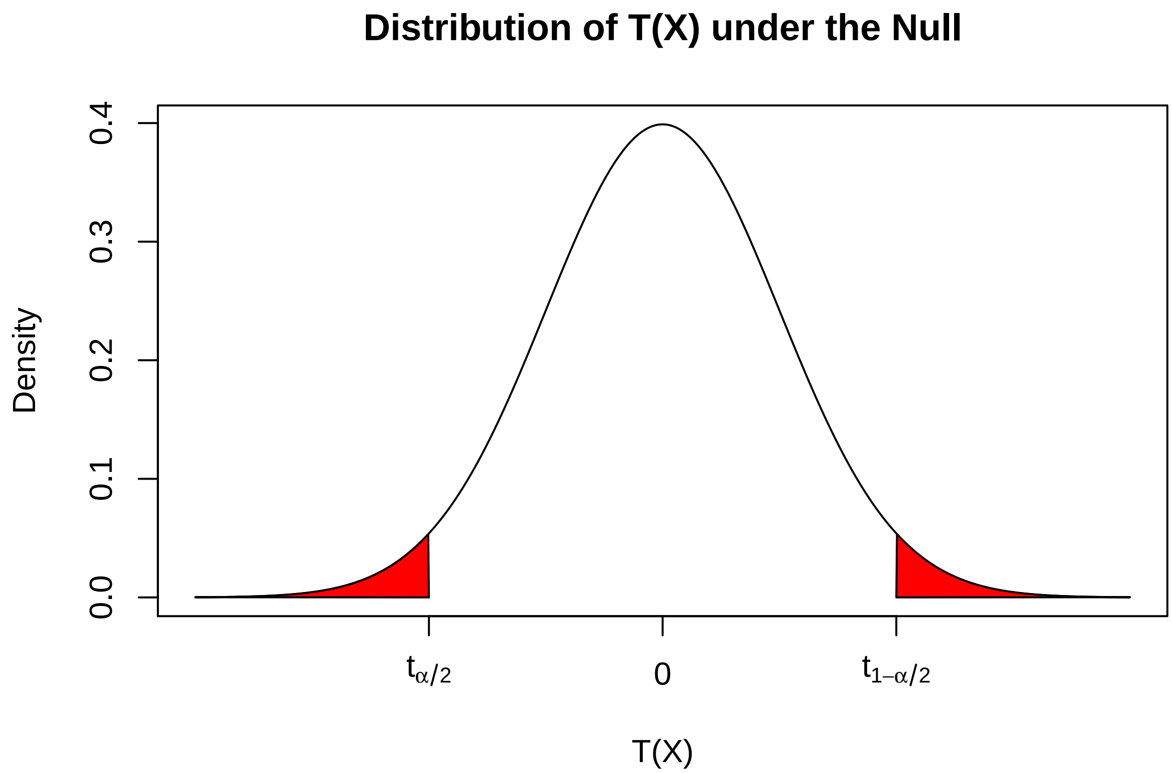
1. Some random variable X might actually follow a symmetric distribution, such as a normal or Laplace distribution, with variance σ^2.
2. Some collection of independent and identically distributed random variables $Y_1, Y_2, ...Y_n$ might follow some distribution p_Y with unknown mean θ and variance $\sigma^2 < \infty$. By the Central Limit Theorem, as $n \to \infty$, $\sum_{i=1}^{n} Y_i = \bar{Y_n} \sim N(\mu, \sigma^2/n)$, which is equivalent to saying that $\dfrac{\sqrt{n}(\bar{Y_n} - \mu)}{\sigma} \sim N(0,1)$ as $n \to \infty$ Note that p_Y can be any valid pmf or pdf so long as Y has finite variance.

Thus, in our example, while we can think of X as some random variable that happens to follow a symmetric distribution, we can also think of it as some $\bar{Y_n}$ where n is large enough such that invoking the CLT seems reasonable.

Returning back to the problem on hand, in hypothesis testing, we are trying to ascertain whether the data available provides sufficient evidence to reject the default proposed theory: the null hypothesis. As it related to this problem, the default theory is that our random variable X has a mean of θ, where θ is some fixed value. For example, θ could be 0:

From our random variable X, we generate some test statistic T(X). If X follows a normal distribution, a common test statistic is $T = \dfrac{X - \theta}{Var(X)}$, which will then follow a Normal(0,1) distribution. For now, we will assume this is our test statistic T.

This new random variable T that has been centered (subtracting θ) and scaled (dividing by Var(X), which is a constant) will still have a symmetric distribution but now centered at 0.



Above, the rejection region for a two-tailed $level - \alpha$ test is shaded red. That is, we reject that null hypothesis if we observe values of T greater than $t_{1-\alpha/2}$ or less than $t_\alpha/2$. Because of the symmetry of T's distribution, $t_{\alpha_2} = -t_{1-\alpha/2}$. Note that we are not talking about the t-distribution here (although it could be!). This is some generic symmetric distribution of the test statistic T, where $t_{1-\alpha/2}$ is the quantity such that $Pr(T \leq t_{1-\alpha/2}) = F_T(t_{1-\alpha/2}) = 1 - \alpha/2$.

Thus, in summary, if we reject the null hypothesis, this implies that our observed value of T is one such that $|T| > t_{1-\alpha/2}$.

How can we construct a $(1-\alpha)\%$ confidence interval for θ? By definition, we want to find some $C(X)$ such that $Pr(\theta \in C(X)) = 1 - \alpha$. Furthermore, we're choosing a symmetric confidence interval (of the form $X \pm a$).

$$Pr(\theta \in C(X)) = Pr(L(X) \leq \theta \leq U(X)) = Pr(X-a \leq \theta \leq X+a) = Pr(X-\theta-a \leq 0 \leq X-\theta+a) = Pr(T-a/\sigma \leq 0 \leq T+a/$$

What values of a make $Pr(-a/\sigma \leq T \leq a/\sigma) = 1 - \alpha$? Recalling our previous discussion of the distribution of T, $Pr(-t_{\alpha/2} \leq T \leq t_{\alpha/2}) = 1 - \alpha$ by definition! Thus, solving for a gives that $a = \sigma t_{1-\alpha/2}$. Thus, we have that $X \mp \sigma t_{1-\alpha/2}$ is our $1 - \alpha$ CI for θ.

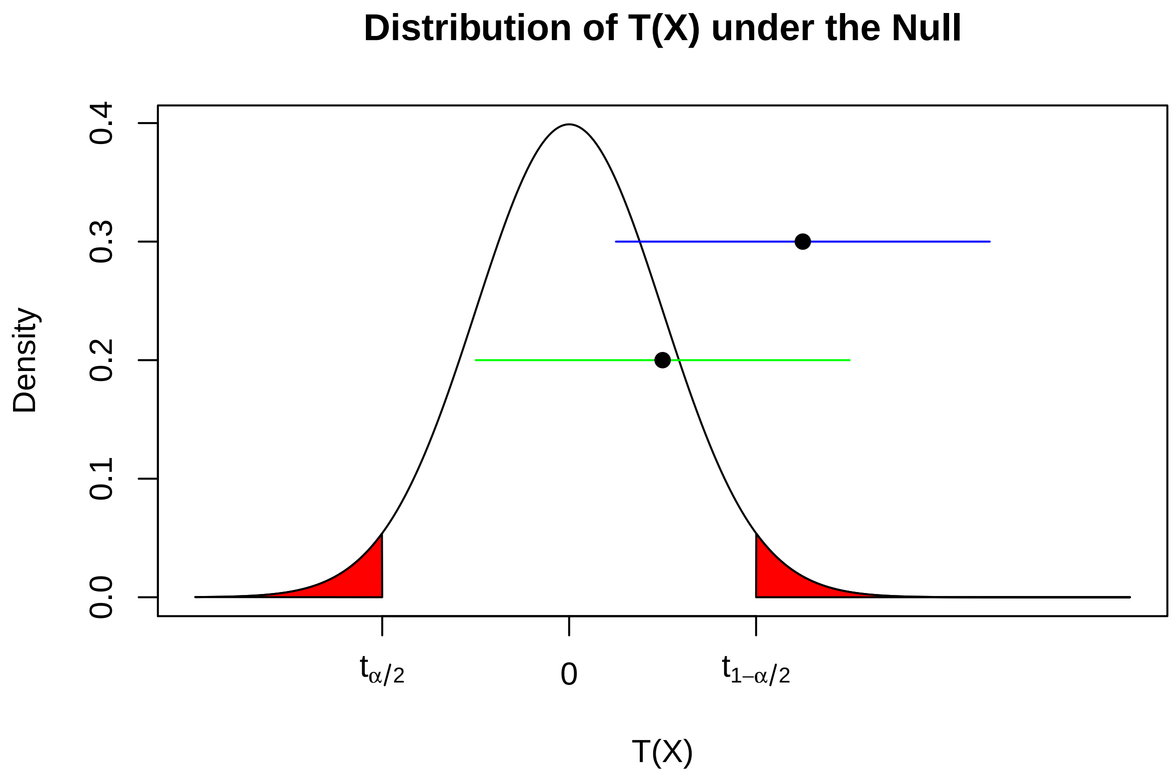
How can these two facts (the rejection criteria and the confidence interval) be combined to give the conclusion from the problem?

Returning to the rejection criteria, we determined that if we reject the null hypothesis, then we must have observed some T such that $|T| > t_{1-\alpha/2}$. Substituting in X, this gives that $\mid \dfrac{X-\theta}{\sigma} \mid > t_{1-\alpha/2}$, so $\mid X - \theta \mid > \sigma t_{1-\alpha/2}$.

Let us first consider the case in which $X - \theta > \sigma t_{1-\alpha/2}$. This implies that $X - \sigma t_{1-\alpha/2} > \theta$, which is another way of saying that the lower bound of our interval is greater than θ, and thus the interval does not contain θ.

In the second case, we have that $X - \theta < -\sigma t_{1-\alpha/2}$, or that $X + \sigma t_{1-\alpha/2} < \theta$. In this case, the upper bound $(X + \sigma t_{1-\alpha/2})$ is less than theta, so clearly the interval does not contain theta.

The larger point, which you can confirm visually is that if we are rejecting the null hypothesis, our observed X must lie far out in the tails of its distribution, away from θ, and thus the confidence interval we would create is not wide enough to include θ.



Above, this is illustrated in terms of T. The two points represent two observed values of T(X). In the lower one, the null hypothesis is not rejected (it does not lie in the red rejection region) and thus the confidence interval contains 0. On the other hand, in the the upper one, the null hypothesis is rejected and thus the confidence interval does not contain 0.