Bootcamp Assessment 2019

Name: ______________________________

Please show your work for all problems, even if you do not arrive at a solution!

1. For which values of $p > 0$ does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge? __________________________

2. If $X = \begin{bmatrix} -2 & 2 \\ -1 & -1 \end{bmatrix}$ and $Y = \begin{bmatrix} 3 & -8 \\ 4 & -1 \end{bmatrix}$, does $XY = YX$? __________________________

3. Male verbal GRE scores are normally distributed, with a mean of 149 and a standard deviation of 9. Female verbal GRE scores are also normally distributed, with a mean of 149 and a standard deviation of 8. 55% of the students who take the GRE are female.

   What is the probability that a randomly chosen student is female, given that their verbal GRE score is 170? ____________________________

4. Let $U \sim Unif(0, 1)$ and define $X = -\ln U$.

   What is the the distribution of X? ____________________________

5. Let $X_1, \ldots, X_n$ be iid samples from $Pois(\lambda)$. Show that both $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$ are unbiased estimators for λ.

6. Skewness is a measure of the asymmetry of a probability distribution of a random variable about its mean. We define the skewness of a r.v. X as $\mathsf{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$. A random variable that is right-skewed will have positive skewness, and a variable that is left-skewed will have negative skewness.

   If $X \sim Exp(\lambda)$, what is the skewness of X? ____________________________

7. Let $X_1, \ldots, X_n$ be iid $Beta(\theta, 1)$ random variables, where $\theta > 0$. Find the MLE of $\frac{1}{\theta}$.

   MLE = ________________________

8. The joint pdf of X and Y is

$$f(x,y) = \frac{e^{-yx^2/2}}{\sqrt{2\pi/y}} \cdot ye^{-y}, \quad x \in \mathbb{R}, y > 0$$

(a) Find the conditional density $f_{X|Y}(x|y)$ of X given $Y = y$. ______________________
Hint: consider decomposing the joint into the product of conditional and marginal densities.

(b) What is $\mathsf{E}[X|Y]$? ______________________

(c) What is $\mathsf{Var}(X|Y)$? ______________________

(d) What is $\mathsf{Var}(X)$? ______________________

9. What is your favorite distribution? ______________________

10. **Optional: if you are considering STA 721, you should attempt this problem.** Let $\mathbf{Z} = (Z_1, \ldots, Z_m)'$ be a vector of m iid $N(0,1)$ random variables. Then $\mathsf{E}[\boldsymbol{Z}] = \mathbf{0}$ and $Cov(\boldsymbol{Z}) = \boldsymbol{I}_m$. Recall that we say a random n-dimensional vector $\boldsymbol{Y}$ is distributed multivariate normal if it has the same distribution as $\boldsymbol{AZ} + \boldsymbol{b}$ where $\boldsymbol{A}$ is some fixed $n \times m$ matrix and $\boldsymbol{b}$ is some fixed $n \times 1$ vector. That is, $\boldsymbol{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{V})$, where $\boldsymbol{\mu} = \boldsymbol{b}$ and $\boldsymbol{V} = \boldsymbol{AA'}$.

    However, it is not clear that if we know $\boldsymbol{b}$ and $\boldsymbol{AA'}$ then $\boldsymbol{Y}$ is uniquely or well defined. Notice that the covariance matrix of $\boldsymbol{Y}$ is $\boldsymbol{V} = \boldsymbol{AA'}$. But because matrix square roots are not unique, we could have some matrix $\boldsymbol{B}$ such that $\boldsymbol{V} = \boldsymbol{BB'}$ but $\boldsymbol{B} \neq \boldsymbol{A}$.

    With this information, prove that if $\boldsymbol{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{V})$ and $\boldsymbol{W} \sim N_n(\boldsymbol{\mu}, \boldsymbol{V})$, then $\boldsymbol{Y}$ and $\boldsymbol{W}$ follow the same distribution.