Math HW7

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All work and code are shown in the Appendix.

7.1a: $f(a, b, c, d, e, f) = a \times b \times c \times d \times e \times f$

7.3a: 5.

7.6

(a): $\nabla f(xy) = \begin{bmatrix} y - 2x + 2 \\ x - 2y + 1 \end{bmatrix}$.

(b): Critical point = $(\frac{5}{3}, \frac{4}{3})$.

(c): $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$.

(d): At $(\frac{5}{3}, \frac{4}{3})$, the matrix is negative-definite, and at a local maximum.

7.7: The pair (111.11, 133.33) maximizes the function.

7.10

(a): 8.33.

(b): 27.

(c): 15.

(d): -6.

(e): -0.298.

Appendix

7.1a

Work shown in answer.

7.3a

Work:
Plug in 3 and 1:

```
(3^2 - 3 * 1 + (6 * 1)^2) / (3 - 3 * 1)
```

```
[1] Inf
```

So that doesn't work. Need to find some way to cancel out the denominator.

$$x^2 - xy + 6y^2,$$

Set y equal to 1:

$$= x^2 - (x \cdot 1) + (6 \cdot 1)^2,$$
$$= x^2 - x + 6,$$
$$= (x - 3)(x + 2).$$

Plug y back in:

$$= (x - 3y)(x + 2y).$$

Check work:

```
from __future__ import division
from sympy import *
x, y = symbols('x y')
print(expand((x - 3*y) * (x + 2*y)))
```

```
x**2 - x*y - 6*y**2
```

plug back into original expression:

$$\frac{(x - 3y)(x + 2y)}{x - 3y},$$
$$= x + 2y.$$

Plugging in x and y values:

```
3 + 2 * 1
```

```
[1] 5
```

7.6

(a)

Work:
Gradient = first partial derivative. Need to find one for x and one for y, and construct a matrix where row

1 is the first partial derivative with respect to x and row 2 is the first partial derivative with respect to y.
With respect to x:

$$\frac{\partial}{\partial x}(-x^2 + xy - y^2 + 2x + y),$$
$$= \frac{\partial}{\partial x}(-x^2) + \frac{\partial}{\partial x}(x) \cdot \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial x}(2) \cdot \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(y),$$
$$y - 2x + 2.$$

With respect to y:

$$\frac{\partial}{\partial y}(-x^2 + xy - y^2 + 2x + y),$$
$$= \frac{\partial}{\partial y}(-x^2) + \frac{\partial}{\partial y}(x) \cdot \frac{\partial}{\partial y}(y) - \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial y}(2) \cdot \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial y}(y),$$
$$= x - 2y + 1.$$

Therefore, the gradient is:

$$\nabla f(xy) = \begin{bmatrix} y - 2x + 2 \\ x - 2y + 1 \end{bmatrix}.$$

(b)

Work:

start by solving for x:

$$-2x + y + 2 = 0,$$
$$y = 2x - 2.$$

Plug into the bottom element:

$$x - 2y + 1 = 0,$$
$$x - 2(2x - 2) + 1 = 0,$$
$$x - 4x - 3 = 0,$$
$$3x = 5,$$
$$x = \frac{5}{3}.$$

Back to the top element:

$$y - 2x + 2 = 0,$$
$$-y = 2x + 2,$$
$$-y = -2(\frac{5}{3}) + 2,$$
$$y = \frac{4}{3}$$

(c)

Work:
The gradient is:

$$\nabla f(xy) = \begin{bmatrix} y - 2x + 2 \\ x - 2y + 1 \end{bmatrix}.$$

Hessian:

$$\begin{aligned} H(f(x,y)) &= \begin{bmatrix} \frac{\partial}{\partial x}(y - 2x + 2) & \frac{\partial}{\partial y}(y - 2x + 2) \\ \frac{\partial}{\partial x}(x - 2y + 1) & \frac{\partial}{\partial y}(x - 2y + 1) \end{bmatrix}, \\ &= \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}. \end{aligned}$$

(d)

Work:

$$f_{xx}(\frac{5}{3}, \frac{4}{3}) \cdot f_{yy}(\frac{5}{3}, \frac{4}{3}) - f_{xy}(\frac{5}{3}, \frac{4}{3})^2 > 0,$$
$$-2(-2) - 1^2 > 0,$$
$$3.$$

We therefore know the matrix is either a positive-definite or negative-definite matrix.

$$\begin{aligned} &f_{xx}(\frac{5}{3}, \frac{4}{3}) + f_{yy}(\frac{5}{3}, \frac{4}{3}), \\ &= -2 + (-2), \\ &= -4. \end{aligned}$$

At $(\frac{5}{3}, \frac{4}{3})$, the matrix is negative-definite, and at a local maximum.

7.7

Work:
First, find gradient of the function (partial derivative with respect to x, then y). With respect to x:

$$\begin{aligned} &\frac{\partial}{\partial x}(150x^{1/3}y^{2/3}), \\ &= 150y^{2/3}\frac{\partial}{\partial x}(x^{1/3}), \\ &= 150y^{2/3}\frac{1}{3}(x^{-2/3}), \\ &= 150y^{2/3}\frac{1}{3}\frac{1}{x^{2/3}}, \\ &= \frac{150y^{2/3}}{3x^{2/3}}, \\ &= \frac{50y^{2/3}}{x^{2/3}}. \end{aligned}$$

Partial respect to y is identical steps except isolating the y instead:

$$\frac{100x^{1/3}}{y^{1/3}}.$$

Which makes the gradient:

$$\nabla f(x,y) = \begin{bmatrix} \frac{50y^{2/3}}{x^{2/3}} \\ \frac{100x^{1/3}}{y^{1/3}} \end{bmatrix}.$$

Step 2 is to find the gradient of the constraint fuction 300x + 500y = 100,000. Partial with respect to x:

$$\frac{\partial}{\partial x}(300x + 500y),$$

treating y as constant, derivative of constant (y) = 0:

$$= \frac{\partial}{\partial x}(300x),$$

$$= 300\frac{\partial}{\partial x}(x),$$
$$= 300.$$

Partial with respect to y = 500. Therefore,

$$\nabla g(x,y) = \begin{bmatrix} 300 \\ 500 \end{bmatrix}.$$

Step 3, set each $\nabla f(x,y) = \lambda \nabla g(x,y)$, and Step 4 is to add the constraint to the system:

$$\frac{50y^{2/3}}{x^{2/3}} = 300\lambda$$

$$\frac{100x^{1/3}}{y^{1/3}} = 500\lambda$$
$$300x + 500y = 100,000$$

If you can make the first two equal to lambda, they will be equal to each other:

$$\frac{50y^{2/3}}{x^{2/3}} = 300\lambda,$$

$$\frac{50y^{2/3}}{300x^{2/3}} = \lambda,$$
$$\frac{y^{2/3}}{6x^{2/3}} = \lambda.$$
$$\frac{100x^{1/3}}{y^{1/3}} = 500\lambda,$$

$$\frac{100x^{1/3}}{500y^{1/3}} = \lambda,$$

$$\frac{x^{1/3}}{5y^{1/3}} = \lambda.$$

Now it's a lot easier to solve for x and y:

$$\frac{y^{2/3}}{6x^{2/3}} = \frac{x^{1/3}}{5y^{1/3}},$$

$$y^{2/3} = \frac{x^{1/3}}{5y^{1/3}} 6x^{2/3},$$
$$5y^{1/3}(y^{2/3}) = x^{1/3}(6x^{2/3}),$$
$$5y = 6x,$$
$$\frac{5}{6}y = x.$$

$$300(\frac{5}{6}y) + 500y = 100,000,$$
$$\frac{1500y}{6} + 500y = 100,000,$$
$$250y + 500y = 100,000,$$
$$750y = 100,000,$$
$$y = 133.33.$$

Solving for x:

$$\frac{5}{6}y = x,$$
$$\frac{5}{6}133.33 = x,$$
$$111.11 = x.$$

Step 6 is to compare f at the critical points. But plugging in 0s does not make equations equal to zero because we can't divide by zero (e.g. plugging zero into $y^{1/3}$ would equal zero). But you can try different points to see how the outcomes of those equations compare to the current one. The current one:

```
150 * 111.11^(1/3) * 133.33^(2/3)
```

```
[1] 18820.34
```

Plug in a higher x value (x = 120) and solve for y:

```
from sympy import solveset, S
from sympy.abc import y
from sympy import Symbol, Eq
print(solveset(Eq(300 * 120 + 500*y, 100000),y))
```

```
{128}
```

Plug in (120, 128) to production function:

```
150 * 120^(1/3) * 128^(2/3)
```

```
[1] 18791.36
```

We lose money. What if you decease x? Make x = 100 and solve for y:

```
from sympy import solveset, S
from sympy.abc import y
from sympy import Symbol, Eq
print(solveset(Eq(300 * 100 + 500*y, 100000),y))
```

```
{140}
```

```
150 * 100^(1/3) * 140^(2/3)
```

```
[1] 18771.97
```

Because decreasing x and increasing y (and vice versa) decreases the output of the function, we can be confident that (111.11, 133.33) maximizes the function.

7.10

(a)

Work:
We already have everything we need:
$s(X) = \sqrt{4} = 2$
$s(Y) = \sqrt{9} = 3$

$$Corr(X, Y) = \frac{Cov(X, Y)}{s(X) \cdot s(Y)},$$
$$= \frac{5}{6},$$
$$= 8.33.$$

(b)

Work:
First remove constant since V(X + C) = V(X):

$$V(A) = V(3X - 3Y),$$

$$V(A) = V(3X - 3Y),$$

Weighted difference, so use the weighted sum of two variances formula (which is the same as for when you need a difference also):

$$V(aX - bY) = a^2V(X) + b^2V(Y) + 2ab\text{Cov}(X, Y),$$
$$= 3^2V(X) + (-3)^2V(Y) + 2(3)(-3)\text{Cov}(X, Y),$$

Then plug in the information the question has given:

$$= 3^2(4) + (-3)^2(9) + 2(3)(-3)(5),$$

```
3^2 * 4 + (-3)^2 * 9 + 2 * 3 * (-3) * 5
```

```
[1] 27
```

(c)

Work:
First remove constant since V(X + C) = V(X):

$$V(B) = V(-2X + Y),$$

Weighted difference, so use the weighted sum of two variances formula (which is the same as for when you need a difference also):

$$V(aX - bY) = a^2V(X) + b^2V(Y) + 2ab\text{Cov}(X, Y),$$
$$= -2^2V(X) + (1)^2V(Y) + 2(-1)(1)\text{Cov}(X, Y),$$

Then plug in the information the question has given:

$$= (-2)^2(4) + 1^2(9) + 2(-1)(1)(5).$$

```r
(-2)^2 * 4 + 1^2 * 9 + 2 * (-1) * (1) * 5
```

```
[1] 15
```

(d)

Work:
When both variables are replaced by linear functions of other variables, you use:

$$\mathrm{Cov}(aX + bY, cW + dZ) = ac\mathrm{Cov}(X, W) + ad\mathrm{Cov}(X, Z) + bc\mathrm{Cov}(Y, W) + bd\mathrm{Cov}(Y, Z)$$

$$= 3(-2)\mathrm{Cov}(X, W) + 3(1)\mathrm{Cov}(X, Z) + (-3)(-2)\mathrm{Cov}(Y, W) + (-3)(1)\mathrm{Cov}(Y, Z),$$
$$= 3(-2)(4) + 3(1)(5) + (-3)(-2)(5) + (-3)(1)(9),$$

```r
3*(-2)*(4) + 3*(1)*(5) + (-3)*(-2)*(5) + (-3)*(1)*(9)
```

```
[1] -6
```

(e)

Work:

$$\frac{-6}{\sqrt{27}\sqrt{15}}.$$

```r
-6/(sqrt(27) * sqrt(15))
```

```
[1] -0.2981424
```