# Math Final

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## 7.11

### (a)

$$\frac{\partial}{\partial x_1} = \alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i2}^2 + \beta_4 x_{i2}^3 + \beta_5 x_{i3} + \beta_6 x_{i4} + \beta_7 x_{i3} x_{i4},$$

Derivative of a constant is zero, so drop $\alpha$:

$$= \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i2}^2 + \beta_4 x_{i2}^3 + \beta_5 x_{i3} + \beta_6 x_{i4} + \beta_7 x_{i3} x_{i4},$$

Break up across addition and subtraction, then bring out constants:

$$= \beta_1 \frac{\partial}{\partial x_1} x_{i1} + \beta_2 \frac{\partial}{\partial x_1} x_{i2} + \beta_3 \frac{\partial}{\partial x_1} x_{i2}^2 + \beta_4 \frac{\partial}{\partial x_1} x_{i2}^3 + \beta_5 \frac{\partial}{\partial x_1} x_{i3} + \beta_6 \frac{\partial}{\partial x_1} x_{i4} + \beta_7 \frac{\partial}{\partial x_1} x_{i3} \frac{\partial}{\partial x_1} x_{i4},$$

All other independent variables are treated as constants:

$$= \frac{\partial}{\partial x_1} \beta_1 x_{i1} + 1(0) + 1(0) + 1(0) + 1(0) + 1(0) + 1(0)(0),$$

Bring the constant out:

$$= \beta_1 \frac{\partial}{\partial x_1} x_{i1} = \beta_1(1) = \beta_1.$$

### (b)

As long as the $x$ variable is not exponentiated or multiplied, the coefficient is the partial derivative. This is not true for exponentiated terms. Take for example:

$$\frac{d}{dx}(a + \beta x^2),$$

$$= \frac{d}{dx}(a) + \frac{d}{dx}(\beta x^2),$$

$$= \beta \frac{d}{dx}(x^2),$$

$$= \beta(2x^{2-1}),$$

$$= 2\beta x.$$

This derivative does not yield the coefficient by itself, so you cannot interpret the coefficient as a straightforward linear relationship between $x$ and $y$ in the case of exponentiated terms. When the slope is exponential, the partial derivative gives you the instantaneous rate of change at each level of x which, unlike in the case of (a), will be different at each level of $x$.

Multiplicative relationships also cannot be interpreted as a straightfoward linear relationship. Consider a model with two predictors, $x_1$ and $x_2$, and their interaction:

$$\frac{\partial}{\partial x_1}(a + \beta_1 x_1 + \beta_2 x_2 + \beta_{12}(x_1 \times x_2) = \beta 1 + \beta_{12} x_2.$$

Here, the first derivative with respect to $x_1$ is a function of $x_2$. If there is an interaction, this partial derivative will change when evaluated across differing $x_2$ values. The key thing to notice is that because this function incorporates information about $x_2$, it does not describe an exclusive relationship between $x_1$ and $y$ like function (a) does. Therefore, when an interaction term is present, the first partial derivaitve does not yield the regression coefficient $\beta_1$. However, if this derivative does not change across differing levels of $x_2$, there is no interaction. In cases where there is no interaction (i.e. in cases where $\beta_{12}x_2$ comes out to zero), this function *would* describe the linear relationship between $x_1$ and $y$ in the way function (a) does. This would also be true whenever $x_2 = 0$. However, mathematically, this derivative does not directly represent the coefficient one sees in regression output for $x_1$ in a main-effects only model, nor does it yield the coefficient for the interaction.

To get the coefficient for the interaction, you need to take the *cross-partial* derivative with respect to both $x_1$ and $x_2$:

$$\frac{\partial^2}{\partial x_1 x_2}(a + \beta_1 x_1 + \beta_2 x_2 + \beta_{12}(x_1 \times x_2) = \beta_{12}.$$

Importantly, this coefficient does not describe an exclusive relationship between $x_1$ and $y$ since it is a function of both $x_1$ and $x_2$. The cross-partial derivative introduces a third slope, which is the slope between $x_1$ and $y$ spanning across all levels of $x_2$. If this slope is zero, you have no interaction.

In sum, the coefficient seen in regression output describes a linear relationship between $x$ and $y$ in a linear main-effects model but the meaning of this coefficient changes when $x$ is exponentiated or multiplied by another independent variable.

## (c)

$$\frac{\partial}{\partial x_2} = \alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i2}^2 + \beta_4 x_{i2}^3 + \beta_5 x_{i3} + \beta_6 x_{i4} + \beta_7 x_{i3} x_{i4},$$

Derivative of a constant is zero, so drop $\alpha$:

$$= \frac{\partial}{\partial x_2}(\beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i2}^2 + \beta_4 x_{i2}^3 + \beta_5 x_{i3} + \beta_6 x_{i4} + \beta_7 x_{i3} x_{i4}),$$

Break up across addition and subtration, then again across multiplication:

$$= \frac{\partial}{\partial x_2}\beta_1 x_{i1} + \frac{\partial}{\partial x_2}\beta_2 x_{i2} + \frac{\partial}{\partial x_2}\beta_3 x_{i2}^2 + \frac{\partial}{\partial x_2}\beta_4 x_{i2}^3 + \frac{\partial}{\partial x_2}\beta_5 x_{i3} + \frac{\partial}{\partial x_2}\beta_6 x_{i4} + \frac{\partial}{\partial x_2}\beta_7 x_{i3} x_{i4},$$

The independent variables all get treated as constants:

$$\begin{aligned} &= 0 + \frac{\partial}{\partial x_2}\beta_2 x_{i2} + \frac{\partial}{\partial x_2}\beta_3 x_{i2}^2 + \frac{\partial}{\partial x_2}\beta_4 x_{i2}^3 + 0 + 0 + 0 + 0, \\ &= \beta_2 \frac{\partial}{\partial x_2} x_{i2} + \beta_3 \frac{\partial}{\partial x_2} x_{i2}^2 + \beta_4 \frac{\partial}{\partial x_2} x_{i2}^3, \\ &= \beta_2 + 2\beta_3 x_{i2} + 3\beta_4 x_{i2}^2. \end{aligned}$$

## (d)

# 9.10

# 10.9