HW 4

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All work and code are shown in the appendix.

4.1

Answer:
As n approaches infinity, the shape would approach a circle.

4.2

(a)

Answer:
32

(b)

Answer:
Limit is 0. Approaches but never quite gets there from the left nor right.

(c)

Answer:
Positive infinity. The larger the absolute value of z, the larger the quotient.

(d)

Answer:
The limit is 0.

(e)

Answer:

$$\frac{3}{5}$$

(f)

Answer:
The limit is 1.

(g)

Answer:
∞

(h)

Answer:
The limit from the right is ∞ while the limit from the left is $-\infty$, therefore there is no limit.

(i)

Answer:
≈ 7.388611

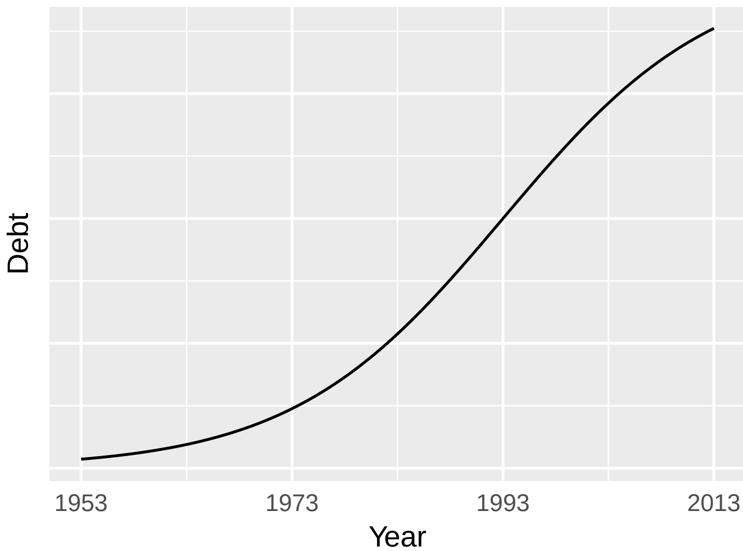
4.3

(a)

Answer:
When Barack Obama says that the deficit is changing at a certain pace, we can think of that pace as being the slope of the tangent line at a given point along the x- and y-axes, with the x-axis representing time and the y-axis representing debt. We can really only approximate the rate at which debt is falling with the tangent line (i.e. instantaneous rate of change). The tangent line is just the secant line between two points very very close together along the line (line being the deficit throughout the years), and we could achieve greater and greater resolution by taking the derivative as many times as needed. Given the function $f(t)$ represents the national debt for a given year (t), then in our case we could get the average rate of change for that year by taking the first derivative: $f'(t)$. However, that's not enough resolution: to make a claim about the *current rate* at which the deficit is falling, you'd need the instantaneous rate of change. To get that, you'd take the second derivative, $f''(t)$. This gives you the value of the tangent line for a given t. We also know that this has to be a negative number because the deficit is going down (i.e. tangent line is a negative slope, indicating that the slopoe of the overall curve is decreasing).

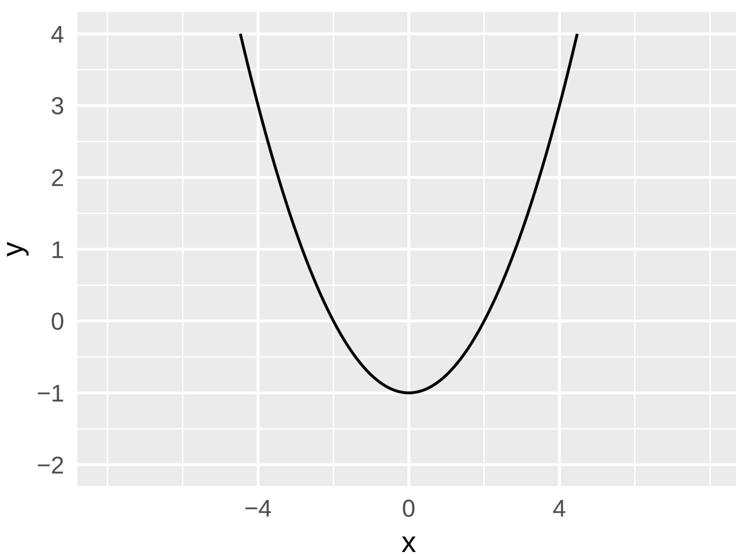
(b)

Answer:
This trend isn't accurate, but is true in the sense that 2013 would indeed be the year with the fastest falling deficit:



4.4

A rough approximation:



4.5

(a)

Answer:
$= 6x^5 + 25x^4 - 4x.$

(c)

Answer:
$\frac{1}{z} - \frac{1}{z^2} + \ln{(3)} 3^z$.

(d)

Answer:
$\Big((x+3)^7\Big)\Big(12x^3 - 4x\Big) + \Big(7(x+3)^6\Big)\Big(3x^4 - 2x^2 - 8\Big)$.

(f)

Answer:
$\frac{1 - \ln(z)}{z^2}$.

(g)

Answer:
$\frac{-1}{(1+e^{-x})^2} \times e^{-x} \times -1 \times \frac{e^{-x}}{(1+e^{-x})^2}$.

4.6

(a)

Answer:
$f^{(n)}(x) = (-1)^n \frac{(n-2)!}{x^{n-1}}$.

Each derivativei ends in a fraction. The denominator is x to the power of the number of the derivative minus 1 ((n^{th} derivative)$^{n-1}$). The numerator is the factorial of the n^{th} derivative minus 2, with positive values for the even derivatives and negative values for the odd derivatives (accomplished by multiplying by $(-1)^n$.

(b)

Answer:
$g^{(n)}(x) = (x+n)e^x$.

Every derivative ends in x + (number of the derivative) times e to the power of x.

Appendix

4.1

N/A

4.2

(a)

Work:

$$\lim_{x \to 5} 2x^2 - 5x + 7,$$
$$= 2(5^2) - 5(5) + 7,$$
$$= 32.$$

```
# chck work
2*(5)^2 - (5*5) + 7
```

```
[1] 32
```

(b)

Work:
$\lim_{y \to \infty} \dfrac{1}{y^6}$

```
# check work
1 / 2^6
1 / 4^6
1 / 20^6
1 / -2^6
1 / -4^6
1 / -20^6
```

```
[1] 0.015625
[1] 0.0002441406
[1] 0.000000015625
[1] -0.015625
[1] -0.0002441406
[1] -0.000000015625
```

(c)

Work:
$\lim_{z \to 0} \dfrac{1}{z^6}$

```
# check work
1 / 1^6
1 / 0.5^6
1 / 0.01^6
1 / (-1)^6
1 / (-0.5)^6
1 / (-0.01)^6
```

```
[1] 1
[1] 64
[1] 1000000000000
[1] 1
[1] 64
[1] 1000000000000
```

(d)

Work:

```
# check work
2*2 / 6 * (2^2)
2*5 / 6 * (5^2)
2*10 / 6 * (10^2)
2*1 / 6 * (1^2)
2*0.1 / 6 * (0.1^2)
2*0.0001 / 6 * (0.001^2)
```

```
[1] 2.666667
[1] 41.66667
[1] 333.3333
[1] 0.3333333
[1] 0.0003333333
[1] 0.00000000003333333
```

Could also solve this way:

$$\lim_{x \to 0} \frac{2x+3}{5x^2},$$

$$\lim_{x \to 0} \frac{2x}{5x^2} + \frac{3}{5x^2},$$

$$\lim_{x \to 0} \frac{2}{5x} + \frac{3}{5x^2},$$

$$\lim_{x \to 0} \frac{2}{5x} + \lim_{x \to 0} \frac{3}{5x^2},$$

$$\frac{2}{5} \lim_{x \to 0} \frac{1}{x} + \frac{3}{5} \lim_{x \to 0} \frac{1}{x^2},$$

$$\frac{2}{5}\lim_{x\to 0}\frac{1}{x}+\frac{3}{5}\lim_{x\to 0}\frac{1}{x^2},$$

The limits to $\dfrac{1}{x}$ and $\dfrac{1}{x^2}$ are 0.

(e)

Work:
We know that the limit of $\frac{1}{x^y} = 0$, so we can reduce many of the terms of the expression to 0s by making them into the form of $\frac{1}{x^y} = 0$.

$$\lim_{y\to\infty}\frac{3y^7+4y^6-2y^5-8y^3-7y+1}{2y^7+y^3-8}\times\frac{\frac{1}{y^7}}{\frac{1}{y^7}},$$

$$=\lim_{y\to\infty}\frac{3y^7+\frac{4}{y}-\frac{2}{y^2}-\frac{8}{y^4}-\frac{7}{y^6}+\frac{1}{y^7}}{2+\frac{1}{4y^4}-\frac{8}{y^7}},$$

$$=\frac{3}{5}.$$

(f)

Work:
Do some factoring

$$\lim_{z\to 3}\frac{z^2-5z+6}{z-3},$$

$$=\lim_{z\to 3}\frac{(z-2)(z-3)}{z-3},$$

$$=\lim_{z\to 3}z-2=1$$

```
# check work
import sympy
from sympy import *
x, y, z = symbols('x y z')
init_printing(use_unicode=True)
expr = Limit((z - 2)*(z - 3) / (z - 3), z, 3)
# find the limit
print(expr.doit())
```

```
1
```

(g)

Work:
Need limit from the right:

```
1 / (5.1 - 5)
1 / (5.01 - 5)
1 / (5.00001 - 5)
```

```
[1] 10
[1] 100
[1] 100000
```

(h)

Work:
Need limit from the right:

```
1 / (7.1 - 7)
1 / (7.01 - 7)
1 / (7.00001 - 7)
```

```
[1] 10
[1] 100
[1] 100000
```

and left:

```
1 / (6.9 - 7)
1 / (6.99 - 7)
1 / (6.99999- 7)
```

```
[1] -10
[1] -100
[1] -100000
```

(i)

Work:
Aha! This is just Euler's number, which is equal to 2.7182… Therefore:

$$\lim_{z\to\infty}\left(1+\frac{1}{z}\right)^{2z},$$

Becaause the exponenents are mutliplicative, put the z on the inside to get euler's number:

$$= \lim_{z\to\infty}\left(\left(1+\frac{1}{z}\right)^{z}\right)^{2},$$

$$= \lim_{z\to\infty}(e)^2$$

$$= \lim_{z\to\infty}(2.7182)^2$$

$$\approx 7.388611$$

```r
# calculations
2.7182^2
```

```
[1] 7.388611
```

4.3

(a)

N/A

(b)

```r
sigmoid = function(x) {
   1 / (1 + exp(-x))
}
x <- seq(-5, 5, 0.01)
p1 <- ggplot(data_frame())

p1 + stat_function(aes(-4:3), fun = sigmoid) + ylab("Debt") +
  theme(
  axis.text.y = element_blank(),
  axis.ticks = element_blank())+
  scale_x_continuous(name = "Year", breaks = c(-4, -2, 0, 2),
                     labels = c("1953", '1973', '1993', '2013'),
                     limits = c(-4, 2))
```

4.4

A rough approximation:

```r
x <- seq(-5, 5, 0.001)
dat <- data.frame(x,y=(2*x)^2)
f <- function(x) (x*2)^2
ggplot(dat, aes(x,y)) +
    geom_line()  +
  theme(
  axis.ticks = element_blank())+
  scale_x_continuous(name = "x", breaks = c(
    -1, 0, 1),
    labels = c(-4, 0, 4),
    limits = c(-2, 2)) +
    scale_y_continuous(name = "y", breaks = c(
    -1, 0, 1, 2, 3, 4, 5),
    labels = c(-2, -1, 0,1, 2, 3, 4),
    limits = c(-1, 5))
```

4.5

(a)

Work:

$$\text{if } f(x) = x^6 + 5x^5 - 2x^2 + 8, \text{then}$$
$$f'(x) = \frac{d}{dx}\Big(x^6 + 5x^5 - 2x^2 + 8\Big),$$

Use the addition/subtraction shortcut:

$$= \frac{d}{dx}(x^6) + \frac{d}{dx}(5x^5) - \frac{d}{dx}(2x^2) + \frac{d}{dx}(8),$$
$$= 6x^{6-1} + 5(5x^{5-1}) - 2(2x^{2-1}) + 0,$$
$$= 6x^5 + 25x^4 - 4x.$$

(c)

Work:

$$\text{if } h(z) = ln(z) + \frac{1}{z} + 3^z, \text{ then}$$
$$h'(z) = \frac{d}{dz}\big(\ln(z)\big) + \frac{d}{dz}\big(\frac{1}{z}\big) + \frac{d}{dz}(3^z),$$
$$\frac{1}{z} + \frac{d}{dz}\big(\frac{1}{z}\big) + \frac{d}{dz}(3^z),$$
$$\frac{1}{z} + \frac{-1}{z^{1+1}} + \frac{d}{dz}(3^z),$$
$$\frac{1}{z} - \frac{1}{z^2} + \frac{d}{dz}(3^z),$$
$$\frac{1}{z} - \frac{1}{z^2} + \ln(3)3^z.$$

(d)

Work:

$$j(x) = (x+3)^7(3x^4 - 2x^2 - 8).$$
$$j'(x) = \frac{d}{dx}(x+3)^7\frac{d}{dx}(3x^4 - 2x^2 - 8).$$

Would take too long to FOIL. To get derivative of first half, break it into layers:

$$A^7 = (x+3)^7, \ A = x + 3,$$

Take derivative of A^7 with power rule:

$$h'(x) = \frac{d}{dx}(A^7) = 7(A^{7-1}) = 7A^6.$$

Lastly, substitute for A:

$$7A^6 = 7(x+3)^6.$$

For second half:

$$\begin{aligned}
&\frac{d}{dx}(3x^4 - 2x^2 - 8),\\
&= \frac{d}{dx}(3x^4) - \frac{d}{dx}(2x^2) - \frac{d}{dx}(8),\\
&= 3\frac{d}{dx}(x^4) - 2\frac{d}{dx}(x^2) - 0,\\
&= 3(4x^{4-1}) - 2(2x^{2-1}),\\
&= 12x^{4-1} - 4x^{2-1},\\
&= 12x^3 - 4x.
\end{aligned}$$

Plugging the derivatives back into the function:

$$j'(x) = \frac{d}{dx}(x+3)^7 \frac{d}{dx}(3x^4 - 2x^2 - 8),$$

$$= \Big((x+3)^7\Big)\Big(12x^3 - 4x\Big) + \Big(7(x+3)^6\Big)\Big(3x^4 - 2x^2 - 8\Big).$$

(f)

Work:

$$l(z) = \frac{\ln(z)}{z},$$

$$l'(z) = \frac{\frac{d}{dz}\Big(\ln(z)\Big)}{\frac{d}{dz}(z)},$$

$$= \frac{\frac{d}{dz}\Big(\ln(z)\Big)}{\frac{d}{dz}(z)}$$

$$= \frac{1/z}{1}$$

But remember quotient rule:

$$\frac{g(z)f'(z) - f(z)g'(z)}{g(z)^2}$$

Subbing back in:

$$= \frac{(z \times 1/z) - (\ln(z) \times 1)}{z^2},$$

Clean it up a bit:

$$= \frac{1 - \ln(z)}{z^2}.$$

(g)

Work:

$$m(x) = \frac{1}{1+e^{-x}}.$$

Identify any layers:

$$m(x) = \frac{1}{A},\ A = 1 + e^{B},\ B = -x.$$

take the derivative of each layer:

First layer:

$$m'(x) = \frac{d}{dA}\left(\frac{1}{A}\right) = \frac{-1}{A^2}.$$

Substitute:

$$m'(x) = \frac{-1}{A^2} = (a + e^{B})^2 = \frac{-1}{(1+e^{-x})^2}.$$

Second layer:

$$\frac{d}{dB}\Big(1 + e^{B}\Big) = e^{B} = e^{-x}.$$

Third layer:

$$\frac{d}{dx}(-x) = -1.$$

Combine:

$$m'(x) = \frac{-1}{(1+e^{-x})^2} \times e^{-x} \times -1 \times \frac{e^{-x}}{(1+e^{-x})^2}.$$

4.6

(a)

Work:

First derivative:

$$\begin{aligned} f'(x) &= x\frac{d}{dx}\Big(\ln(x)\Big) + \ln(x)\frac{d}{dx}(x), \\ &= x\frac{1}{x} + \ln(x), \\ &= 1 + \ln(x). \end{aligned}$$

Second Derivative:

$$f''(x) = \frac{d}{dx}\Big(1 + \ln(x)\Big) = \frac{1}{x}.$$

Third Dervivative:

$$f'''(x) = \frac{d}{dx}\Big(\frac{1}{x}\Big) = \frac{-1}{x^2}.$$

Fourth Derivative:

$$f''''(x) = \frac{d}{dx}\Big(\frac{-1}{x^2}\Big) = \frac{d}{dx}(-x^{-2}) = 2x^{-3} = \frac{2}{x^3}.$$

Fifth Derivative:

$$f'''''(x) = \frac{d}{dx}(2x^{-3} = -6x^{-4} = \frac{-6}{x^4}).$$

Formula:

$$f^{(n)}(x) = (-1)^n \frac{(n-2)!}{x^{n-1}}.$$

(b)

Work:

First derivative:

$$g'(x) = x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x) = xe^x + e^x = (x+1)e^x.$$

Second Derivative:

$$g''(x) = (x+1)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x+1) = (x+1)e^x + e^x = (x+2)e^x$$

Third Dervivative:

$$g'''(x) = (x+2)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x+2) = (x+2)e^x + e^x = (x+3)e^x.$$

Formula:

$$g^{(n)}(x) = (x+n)e^x.$$