Math HW5

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5.2, 5.3

4.8

(a)

Work

First Derivative:

The only layer here is in the exponent of e; everything else is constant.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

$$A = -\frac{x^2}{2}, \quad f(x) = \frac{1}{\sqrt{2\pi}} e^{A}$$

$$A' = \frac{d}{dx}\Big(-\frac{x^2}{2}\Big) = -\frac{1}{2}\Big(2x^{2-1}\Big) = -\frac{2x^{2-1}}{2} = -x.$$

$$\frac{1}{\sqrt{2\pi}} e^{A}\Big(-x\Big) = \frac{-x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Second Derivative

Because the first derivative was multiplicative, we need the product to get the second derivative.
Product rule: $f''(x) = g'(x)f(x) + g(x)f'(x)$.
$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.
$f'(x) = \frac{-x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.
$g(x) = -x$.
$g'(x) = \frac{d}{dx}(-x) = -1$.

$$\begin{aligned} f''(x) &= -1\Big(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\Big) - x\Big(\frac{-x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\Big) \\ &= -\Big(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\Big) + \Big(\frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\Big) \\ &= \Big(\frac{x^2-1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\Big). \end{aligned}$$

Plug in 0 for each version

$$f(0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{0^2}{2}} = \frac{1}{\sqrt{2\pi}}.$$

$$f'(0) = \frac{0^2}{\sqrt{2\pi}} e^{-\frac{0^2}{2}} = 0.$$

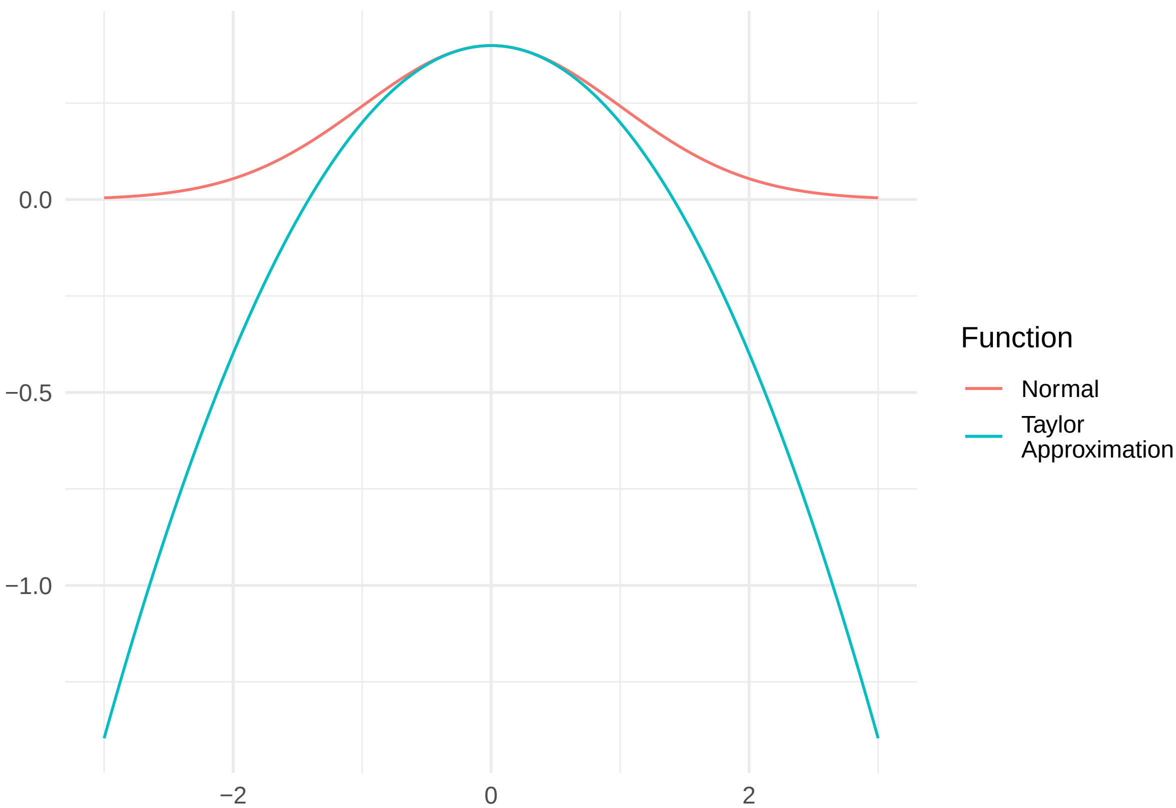
$$f''(0) = \frac{0^2 - 1}{\sqrt{2\pi}} e^{-\frac{0^2}{2}} = \frac{-1}{\sqrt{2\pi}}.$$

Taylor approximation - just plug in what we found:

$$f(x) \approx \frac{1}{\sqrt{2\pi}} + 0x + \frac{-1/\sqrt{2\pi}}{2} x^2.$$

$$f''(x) \approx \frac{-1}{\sqrt{8\pi}} x^2 + \frac{1}{\sqrt{2\pi}}.$$

(b)



Taylor series seems most accurate between about -1 and 1.

4.9

(a)

Start with

$$\ln\left(\prod_{i=1}^{N} p^{y_i}(1-p)^{1-y_i}\right),$$

$$\sum_{i=1}^{N}\ln\left(p^{y_i}(1-p)^{1-y_i}\right),$$

$$\sum_{i=1}^{N}\Bigg(\ln\left(p^{y_i}\right)+\ln\left((1-p)^{1-y_i}\right)\Bigg),$$

$$\sum_{i=1}^{N}\Bigg((y_i)\ln\left(p\right)+\ln\left((1-p)^{1-y_i}\right)\Bigg),$$

$$\sum_{i=1}^{N}\Bigg((y_i)\ln\left(p\right)+(1-y_i)\ln(1-p)\Bigg),$$

$$\sum_{i=1}^{N}(y_i)\ln\left(p\right)+\sum_{i=1}^{N}(1-y_i)\ln(1-p),$$

$$\ln\left(p\right)\sum_{i=1}^{N}(y_i)+\ln(1-p)\sum_{i=1}^{N}(1-y_i),$$

$$\ln\left(p\right)\sum_{i=1}^{N}(y_i)+\ln(1-p)\left(\sum_{i=1}^{N}1\right)-\left(\sum_{i=1}^{N}y_i\right),$$

$$\ln\left(p\right)\sum_{i=1}^{N}(y_i)+\ln(1-p)\left(N-\sum_{i=1}^{N}y_i\right).$$

(b)

$$\frac{d}{dp}\left(\ln\left(p\right)\sum_{i=1}^{N}(y_i)+\ln(1-p)\left(N-\sum_{i=1}^{N}y_i\right)\right),$$

$$\frac{d}{dp}\left(\ln\left(p\right)\sum_{i=1}^{N}(y_i)\right)+\frac{d}{dp}\left(\ln(1-p)\left(N-\sum_{i=1}^{N}y_i\right)\right),$$

$$\frac{d}{dp}\Bigg(\ln\left(p\right)\Bigg)\frac{d}{dp}\left(\sum_{i=1}^{N}(y_i)\right)+\frac{d}{dp}\Bigg(\ln(1-p)\Bigg)\frac{d}{dp}\left(N-\sum_{i=1}^{N}y_i\right),$$

$$\frac{d}{dp}\Bigg(\ln\left(p\right)\Bigg)\frac{d}{dp}\left(\sum_{i=1}^{N}(y_i)\right)+\frac{d}{dp}\Bigg(\ln(1-p)\Bigg)\frac{d}{dp}\left(N-\sum_{i=1}^{N}y_i\right),$$

$$\frac{1}{p}\left(\sum_{i=1}^{N}(y_i)\right)+\frac{d}{dp}\Bigg(\ln(1-p)\Bigg)\left(N-\sum_{i=1}^{N}y_i\right).$$

Gotta use the chain rule:

$$\frac{d}{dp}\Bigg(\ln(1-p)\Bigg) = \left(\frac{1}{1-p}\right)\frac{d}{dp}(1-p) = \frac{\frac{d}{dp}(1) - \frac{d}{dp}(p)}{1-p} = \frac{0-1}{1-p} = -\frac{-1}{1-p}.$$

$$\frac{1}{p}\left(\sum_{i=1}^{N}(y_i)\right) + \frac{-1}{1-p}\left(N - \sum_{i=1}^{N} y_i\right),$$

$$\frac{1}{p}\left(\frac{\sum_{i=1}^{N}(y_i)}{1}\right) + \frac{-1}{1-p}\left(\frac{N - \sum_{i=1}^{N} y_i}{1}\right),$$

$$\frac{\sum_{i=1}^{N}(y_i)}{p} - \frac{N - \sum_{i=1}^{N} y_i}{1-p}.$$

(c)

$$0 = \frac{\sum_{i=1}^{N}(y_i)}{p} - \frac{N - \sum_{i=1}^{N} y_i}{1-p},$$

$$\frac{\sum_{i=1}^{N}(y_i)}{p} = \frac{N - \sum_{i=1}^{N} y_i}{1-p},$$

$$\frac{\sum_{i=1}^{N}(y_i)}{p} = \frac{N - \sum_{i=1}^{N} y_i}{1-p},$$

$$\sum_{i=1}^{N} y_i(1-p) = p\left(N - \sum_{i=1}^{N} y_i\right),$$

$$\sum_{i=1}^{N} y_i - \sum_{i=1}^{N} y_i(p) = (p)N - (p)\sum_{i=1}^{N} y_i,$$

$$\sum_{i=1}^{N} y_i = (p)N,$$

$$\frac{\sum_{i=1}^{N} y_i}{N} = p.$$

In the maximum likelihood framework, the proportion observed in the data tracks the probability of in the population, assuming assumptions hold. In this case, we have a sample where we add up all of the y values, the divide by the number of values. This gives us the sample proportion (or mean), which is an estimate of the probability.

(d)

If $p = \frac{\sum_{i=1}^{N} y_i}{N}$, and we're given a function that gives us p, then all we have to do is substitute the function for p, then substitute x_i into that function.

$$p = \frac{1}{1 + e^{-(0.2+0.5x_i)}}.$$

Very Conservative:

$$p = \frac{1}{1 + e^{-(0.2+0.53)}} = 0.85.$$

```r
## calculations
# get euler's number
exp(1)
```

```
[1] 2.718282
```

```r
# get p for Very conservative:
1 / (1 + exp(1)^-(0.2 + (0.5*3)))
```

```
[1] 0.8455347
```

Moderate:

$$p = \frac{1}{1 + e^{-(0.2+0.5(0))}} = 0.55.$$

```r
## calculations
# get p for Moderate:
1 / (1 + exp(1)^-(0.2 + (0.5*0)))
```

```
[1] 0.549834
```

Very Liberal:

$$p = \frac{1}{1 + e^{-(0.2+0.5(-3))}} = 0.21.$$

```r
## calculations
# get p for Very Liberal:
1 / (1 + exp(1)^-(0.2 + (0.5*(-3))))
```

```
[1] 0.214165
```

(e)

Predicted probabilities (?):

$$p' = \frac{d}{dx}\left(\frac{1}{1 + e^{-(0.2+0.5x_i)}}\right).$$

Need chain rule...

$$p = \frac{1}{A}, \quad A = 1 + e^B, \quad B = -(0.2 + 0.5x_i).$$

$$p' = \frac{d}{dp}\left(\frac{1}{A}\right) = -\frac{1}{A^2}.$$

$$A' = \frac{d}{dA}(1 + e^B) = \frac{d}{dA}(1) + \frac{d}{dA}(e^B) = 0 + e^B = e^B.$$

$$B' = \frac{d}{dB}[-(0.2 + 0.5x_i)] = \frac{d}{dB}(-0.2) + (-0.5)\frac{d}{dB}(x_i) = 0 - 0.5 \times 1 = -0.5.$$

$$\frac{(-1)(e^B)(-0.5)}{A^2} = \frac{(0.5)e^B}{A^2},$$

$$\frac{(0.5)e^B}{A^2} = \frac{(0.5)e^B}{(1 + e^B)^2},$$

$$\frac{(0.5)e^B}{(1 + e^B)^2} = \frac{0.5e^{-(0.2+0.5x_i)}}{(1 + e^{-(0.2+0.5x_i)})^2}.$$

Plug in zero:

$$\frac{0.5e^{-(0.2+0.5(0))}}{(1+e^{-(0.2+0.5(0))})^2} = 0.124.$$

```
# calculation
0.5 * exp(1)^-(0.2+(0.5 * 0)) / (1 + exp(1)^-(0.2+(0.5 * 0)))^2
```

```
[1] 0.1237583
```

(f)

This funciton gives you the instantaneous rate of change for a given x value. This means that for a moderate voter (x = 0), the instantaneous rate of change in probability of voting for the incumbant is 0.124. In other words, it is the slope of the line at a given point of x.

5.1

(a)

Take derivative:

$$f(x) = 3x^4 - 4x^3 - 36x^2,$$
$$f'(x) = (4)3x^{4-1} - (3)4x^{3-1} - (2)36x^{2-1} = 12x^3 - 12x^2 - 72x.$$
$$\begin{aligned} 0 &= 12x^3 - 12x^2 - 72x \\ &= 12(x^3 - x^2 - 6x) \\ &= 12x(x^2 - x - 6) \\ &= 12x(x-3)(x+2). \end{aligned}$$

Critical points are when x equals 3, -2, or 0. Using second derivitive test, use the critical points rules:

$$f''(x) = (3)12x^{3-1} - (2)12x^{2-1} - 72x^{1-1} = 36x^2 - 24x - 72.$$

For x = -2:

$$36(-2)^2 - 24(-2) - 72 = 120$$

```
# calculation
36 * (-2)^2 - 24 * (-2) - 72
```

```
[1] 120
```

For x = 3

$$36(3)^2 - 24(3) - 72 = 180$$

```
36*3^2 - 24 * 3 - 72
```

```
[1] 180
```

For x = 0:

$$36(0)^2 - 24(0) - 72 = -72$$

```
36*0^2 - 24 * 0 - 72
```

```
[1] -72
```

According to the second derivative test, a critical point is a local maximum if $f''(x) < 0$, a local minimum if $f''(x) > 0$, and is a saffle point when $f''(x) = 0$. This means critical points x = 3 and -2 are local minimums while x = 0 is a local maximum.
to find global max and min, first plug in the boundary points into the original function.
Lower boundary:

$$f(-4) = 3(-4)^4 - 4(-4)^3 - 36(-4)^2 = 448$$

```
# calculation
3*(-4)^4 - 4*(-4)^3 - 36*(-4)^2
```

```
[1] 448
```

Upper boundary:

$$f(4) = 3(4)^4 - 4(4)^3 - 36(4)^2 = -64$$

```
# calculation
3*(4)^4 - 4*(4)^3 - 36*(4)^2
```

```
[1] -64
```

Finally, compare these to the local minimum and maximum points:
local maximum:

$$f(0) = 3(0)^4 - 4(0)^3 - 36(0)^2 = 0.$$

```
3*(0)^4 - 4*(0)^3 - 36*(0)^2
```

```
[1] 0
```

Local minima:

$$f(3) = 3(3)^4 - 4(3)^3 - 36(3)^2 = -189.$$
$$f(-2) = 3(-2)^4 - 4(-2)^3 - 36(-2)^2 = -64.$$

```
3*(3)^4 - 4*(3)^3 - 36*(3)^2
```

```
[1] -189
```

```
3*(-2)^4 - 4*(-2)^3 - 36*(-2)^2
```

```
[1] -64
```

Output of function is lowest at x = (3), therefore this is the global minimum. Output of function is highest at x = -4, therefore this is global maximum.

(b)

*lower boundary approaches 0.

$$\begin{aligned} g'(x) &= \frac{d}{dx}(x\ln(x) - x), \\ &= \frac{d}{dx}\big(x\ln(x)\big) - \frac{d}{dx}(x), \\ &= (x)\frac{d}{dx}\ln(x) - 1, \\ &= (x)\frac{1}{x} + \ln(x) - 1, \\ &= 1 + \ln(x) - 1, \end{aligned}$$

$$= \ln(x).$$

Set derivative equal to zero:

$$0 = \ln(x) \text{ when } x = 1.$$

Is critical point positive or negative?

$$g''(x) = \frac{x}{dx}\Big(\ln(x)\Big) = \frac{1}{x},$$

$$g''(x) = \frac{x}{dx}\Big(\ln(1)\Big) = \frac{1}{1} = 1.$$

Positve, therefore local minimum. Now compare critical point $x = 1$ with boundary point in original function:

$$f(1) = (1)\ln(1) - 1 = -1.$$

```
1 * log(1) - 1
```

```
[1] -1
```

$$f(3) = (3)\ln(3) - 3 = 2.30$$

```
(3) * log(3) - 3
```

```
[1] 0.2958369
```

Critical point $x = 1$ is global minimum (-1) and $x = 3$ is global maximum (2.30).

5.2

(a)

Take the derivative:

$$f'(x) = 3x^2 - 15x + 12,$$
$$= 3(x^2 - 5 + 4),$$
$$= 3(x - 1)(x - 4).$$

We have critical points 1 and 4.
First derivative test: left of x is positive:

$$f'(0) = 3(0)^2 - 15(0) + 12 = 12$$

```
3*(0)^2 - 15*(0) + 12
```

```
[1] 12
```

Right of x is negative:

$$= 3(2)^2 - 15(2) + 12 = -6$$

```
3*(2)^2 - 15*(2) + 12
```

```
[1] -6
```

Critical point x = 1 is a local maximum.
Critical point 4, to the left is negative:

$$f'(3) = 3(3)^2 - 15(3) + 12 = -6$$

```r
3*(3)^2 - 15*(3) + 12
```

```
[1] -6
```

And right is positive:

$$f'(5) = 3(5)^2 - 15(5) + 12 = 12$$

```r
3*(5)^2 - 15*(5) + 12
```

```
[1] 12
```

Critical point x = 4 is a local minimum.
Compare critical points against boundary points:

$$f(x) = x^3 - \frac{15}{2}x^2 + 12x + 8$$

At lower boundary:

$$f(0) = 0^3 - \frac{15}{2}0^2 + 12(0) + 8 = 8.$$

```r
0^3 - {15}/{2} * 0^2 + 12*(0) + 8
```

```
[1] 8
```

At upper boundary:

$$f(6) = 6^3 - \frac{15}{2}6^2 + 12(6) + 8 = 26.$$

```r
6^3 - {15}/{2} * 6^2 + 12*(6) + 8
```

```
[1] 26
```

At critical point x = 1:

$$f(1) = 1^3 - \frac{15}{2}1^2 + 12(1) + 8 = 13.5.$$

```r
1^3 - {15}/{2} * 1^2 + 12*(1) + 8
```

```
[1] 13.5
```

At critical point x = 4:

$$f(4) = 4^3 - \frac{15}{2}4^2 + 12(4) + 8 = 0.$$

```r
4^3 - {15}/{2} * 4^2 + 12*(4) + 8
```

```
[1] 0
```

Critical points x = 4 and x = 6 are the locations of the global minimum and maximum, respectively.

(b)

Starting at 2:

```r
# make some functions
f_p <- function (x) {3*x^2 - 15*x + 12}
f_pp <- function (x) {6*x - 15}
f_nr <- function (x) {x - (3*x^2 - 15*x + 12) / (6*x - 15)}
# make a matrix with iteration 0
iter_0 <- matrix(c(0, 2, f_p(2), f_pp(2), f_nr(2)), nrow = 1, ncol=5)
```

```r
colnames(iter_0) <- c("Iteration", "x", "fp(x)", "fpp(x)", "fNR")
# see what iteration 0 was, plug into functions
iter_1 <- rbind(iter_0, matrix(c(1, f_nr(2), f_p(f_nr(2)), f_pp(f_nr(2)), f_nr(f_nr(2))), n
# see what iteration 1 was, etc...
iter_2 <- rbind(iter_1, matrix(c(2, 0.8, f_p(0.8), f_pp(0.8), f_nr(0.8)), nrow = 1, ncol=5

iter_3 <- rbind(iter_2, matrix(c(3, 0.9882353, f_p(0.9882353), f_pp(0.9882353), f_nr(0.988

iter_4 <- rbind(iter_3, matrix(c(4, 0.9999542, f_p(0.9999542), f_pp(0.9999542), f_nr(0.999

iter_5 <- rbind(iter_4, matrix(c(5, 1, f_p(1), f_pp(1), f_nr(1)), nrow = 1, ncol=5))
# Probably an easier way to do this, but gets the job done...
knitr::kable(iter_5)
```

Iteration	x	fp(x)	fpp(x)	fNR
0	2.0000000	-6.0000000	-3.000000	0.0000000
1	0.0000000	12.0000000	-15.000000	0.8000000
2	0.8000000	1.9200000	-10.200000	0.9882353
3	0.9882353	0.1062975	-9.070588	0.9999542
4	0.9999542	0.0004122	-9.000275	1.0000000
5	1.0000000	0.0000000	-9.000000	1.0000000
Converges af	ter 5, at ro	ot 1.		

Starting at 5:

```r
# make a matrix with iteration 0
iter_0 <- matrix(c(0, 5, f_p(5), f_pp(5), f_nr(5)), nrow = 1, ncol=5)
colnames(iter_0) <- c("Iteration", "x", "fp(x)", "fpp(x)", "fNR")
# see what iteration 0 was, plug into functions
iter_1 <- rbind(iter_0, matrix(c(1, f_nr(5), f_p(f_nr(5)), f_pp(f_nr(5)), f_nr(f_nr(5))), n
# see what iteration 1 was, etc...
iter_2 <- rbind(iter_1, matrix(c(2, 4.011765, f_p(4.011765), f_pp(4.011765), f_nr(4.011765

iter_3 <- rbind(iter_2, matrix(c(3, 4.000046, f_p(4.000046), f_pp(4.000046), f_nr(4.000046

iter_4 <- rbind(iter_3, matrix(c(4, 4, f_p(4), f_pp(4), f_nr(4)), nrow = 1, ncol=5))
# Meh, only took 5 seconds this time
knitr::kable(iter_4)
```

Iteration	x	fp(x)	fpp(x)	fNR
0	5.000000	12.0000000	15.000000	4.200000
1	4.200000	1.9200000	10.200000	4.011765
2	4.011765	0.1063002	9.070590	4.000046
3	4.000046	0.0004140	9.000276	4.000000
4	4.000000	0.0000000	9.000000	4.000000
Converges af	ter 4, at r	oot 4.		