Math HW6

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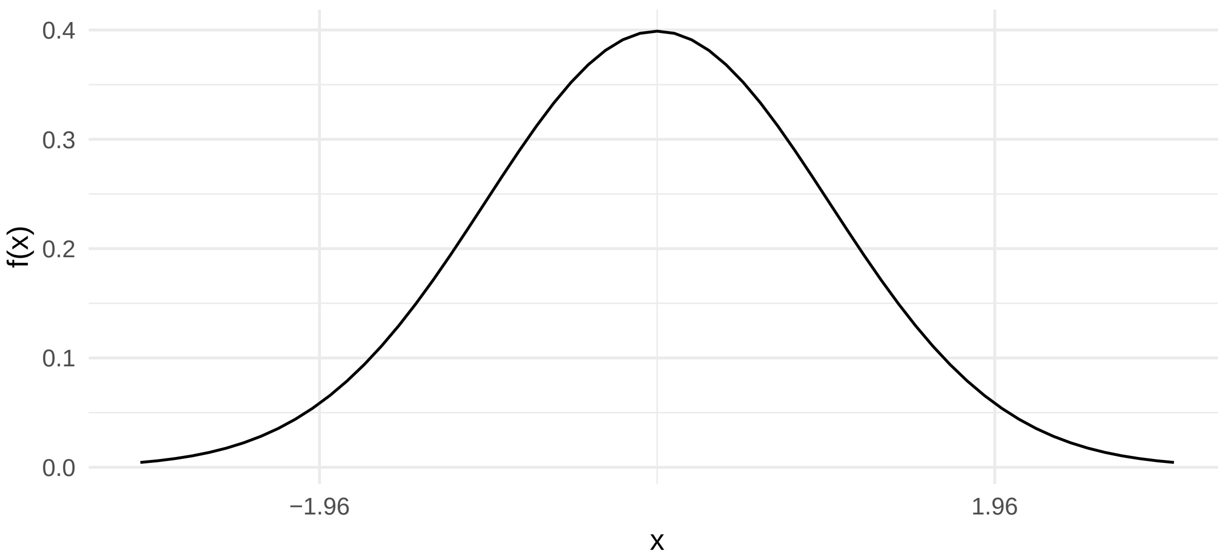
All work and code are shown in the appendix.

6.1

(a)

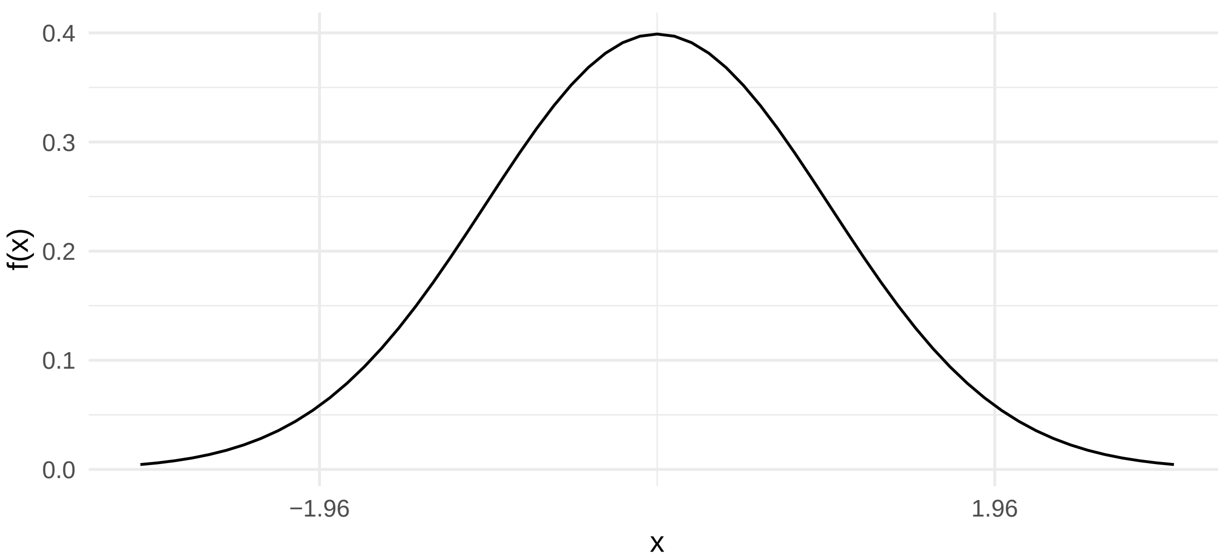
$$f(x) = \int_{-1.96}^{1.96} \frac{1}{\sqrt{2\pi}} e^{-0.5^2} dx$$

Left



Right

Midpoint



Trapezoidal

(b)

Left

Answer:
$A \approx \frac{0.392}{\sqrt{2\pi}}\left(\sum_{i=0}^{9} e^{-0.5^2\left(0.392i-1.96\right)^2}\right).$

Work:
The left Riemann Sum formula is:

$$A \approx \sum_{i=0}^{n-1} f\Big(a + \frac{(b-a)i}{n}\Big).$$

We want to find the sums for 10 partitions between -1.96 and 1.96. Which means that n = 10, a = -1.96, and = 1.96. We also plug in the pdf for f. The Reimann Sum formula gets substituted in for x in f(x):

$$A \approx \sum_{i=0}^{n-1} \frac{1}{\sqrt{2\pi}} e^{-0.5\left(a+\frac{(b-a)i}{n}\left(\frac{b-a}{n}\right)\right)^2},$$

$$\approx \sum_{i=0}^{10-1} \frac{1}{\sqrt{2\pi}} e^{-0.5\left(-1.96+\frac{(1.96-(-1.96))i}{10}\left(\frac{1.96-(-1.96)}{10}\right)\right)^2},$$

$$\approx \sum_{i=0}^{9} \frac{1}{\sqrt{2\pi}} e^{-0.5\left((0.392i-1.96)0.392\right)^2},$$

0.392 does not depend on index, and neither does the constant $\frac{1}{\sqrt{(2\pi)}}$:

$$\approx \frac{1}{\sqrt{2\pi}}(0.392)\left(\sum_{i=0}^{9} e^{-0.5\left((0.392i-1.96)0.392\right)^2}\right),$$

$$\approx \frac{0.392}{\sqrt{2\pi}}\left(\sum_{i=0}^{9} e^{-0.5^2\left(0.392i-1.96\right)^2}\right).$$

```r
# calculation:
(1.96 + 1.96) / 10
```

```
[1] 0.392
```

Right:

Answer:
$A \approx \frac{0.392}{\sqrt{2\pi}}\left(\sum_{i=1}^{10} e^{-0.5^2\left(0.392i-1.96\right)^2}\right).$

Work:
Do the same thing here except the index is from 1 to n instead of from 0 to $n-1$.

$$A \approx \left(\sum_{i=1}^{10} \frac{1}{\sqrt{2\pi}} e^{-0.5\left(-1.96+\frac{1.96-(-1.96)}{10}i\left(\frac{1.96-(-1.96)}{10}\right)\right)^2}\right),$$

$$\approx \sum_{i=1}^{10} \frac{1}{\sqrt{2\pi}} e^{-0.5\left((0.392i-1.96)0.392\right)^2},$$

$$\approx \frac{0.392}{\sqrt{2\pi}}\left(\sum_{i=1}^{10} e^{-0.5^2\left(0.392i-1.96\right)^2}\right).$$

Midpoint:

Answer:
$A \approx 0.392\left(\sum_{i=1}^{10} \frac{1}{\sqrt{2\pi}} e^{-0.5\left(-2.16+0.392i\right)^2}\right).$

Work:

$$A \approx \sum_{i=0}^{n-1} f\left(a + \frac{(b-a)(i-0.5)}{n}\right).$$

We want to find the sums for 10 partitions between -1.96 and 1.96. Which means that n = 10, a = -1.96, and = 1.96. We also plug in the pdf for f. The Reimann Sum formula gets substituted in for x in f(x):

$$A \approx \left(\sum_{i=0}^{n-1} \frac{1}{\sqrt{2\pi}} e^{-0.5\left(a+\frac{(b-a)(i-0.5)}{n}\left(\frac{b-a}{n}\right)\right)^2}\right),$$

$$\approx \left(\sum_{i=1}^{10} \frac{1}{\sqrt{2\pi}} e^{-0.5\left(-1.96+\frac{1.96-(-1.96)}{10}\left(\frac{(i-0.5)}{1}\right)\left(\frac{1.96-(-1.96)}{10}\right)\right)^2}\right),$$

$$\approx \left(\sum_{i=1}^{10} \frac{1}{\sqrt{2\pi}} e^{-0.5\left(-1.96+0.392(i-0.5)(0.392)\right)^2}\right),$$

$$\approx 0.392\left(\sum_{i=1}^{10} \frac{1}{\sqrt{2\pi}} e^{-0.5\left(-1.96+0.392i-0.196)\right)^2}\right),$$

$$\approx 0.392\left(\sum_{i=1}^{10} \frac{1}{\sqrt{2\pi}} e^{-0.5\left(-2.16+0.392i)\right)^2}\right),$$

$$\approx 0.392\left(\sum_{i=1}^{10} \frac{1}{\sqrt{2\pi}} e^{-0.5\left(-2.16+0.392i)\right)^2}\right).$$

```
# Calculations
0.392 * 0.5
```

```
[1] 0.196
```

```
-1.96 - 0.196
```

```
[1] -2.16
```

Trapezoidal

Answer:
$A \approx \left(\frac{0.196}{\sqrt{2\pi}}\right) \sum_{i=0}^{9} \frac{e^{-0.5(0.392i-1.96)^2} + e^{-0.5(0.392i-0.564)^2}}{2}.$

Work:

$$A \approx \sum_{i=0}^{n-1} \frac{b-a}{n} \frac{\frac{1}{\sqrt{2\pi}} e^{\left(a+\frac{(b-a)i}{n}\right)^2} + \frac{1}{\sqrt{2\pi}} e^{\left(a+\frac{(b-a)i+1}{n}\right)^2}}{2} \frac{b-a}{n},$$

$$\approx \sum_{i=0}^{10-1} \frac{1.96-(-1.96)}{10} \frac{\frac{1}{\sqrt{2\pi}} e^{\left((-1.96)+\frac{(1.96-(-1.96))i}{10}\right)^2} + \frac{1}{\sqrt{2\pi}} e^{\left((-1.96)+\frac{(1.96-(-1.96))i+1}{10}\right)^2}}{2},$$

$$\approx \sum_{i=0}^{9} (0.392) \frac{\frac{1}{\sqrt{2\pi}} e^{-0.5(0.392i-1.96)^2} + \frac{1}{\sqrt{2\pi}} e^{-0.5(0.392i-0.564)^2}}{2},$$

$$\approx \frac{0.392}{2}\left(\frac{1}{\sqrt{2\pi}}\right) \sum_{i=0}^{9} \frac{e^{-0.5(0.392i-1.96)^2} + e^{-0.5(0.392i-0.564)^2}}{2},$$

$$\approx \left(\frac{0.196}{\sqrt{2\pi}}\right) \sum_{i=0}^{9} \frac{e^{-0.5(0.392i-1.96)^2} + e^{-0.5(0.392i-0.564)^2}}{2}.$$

```
0.392 /2
```

```
[1] 0.196
```

(c)

Left

Answer:
0.947

Work:

Just plug in to x each level of n from 0 - 9:

$$A \approx \frac{0.392}{\sqrt{2\pi}} \left(\sum_{i=0}^{9} e^{-0.5^2\left(0.392i - 1.96\right)^2} \right).$$

```r
e <- exp(1) # get euler's number
(0.392 / (sqrt(2 * pi))) * (e^(-0.5 * ((0.392 * 0) - 1.96)^2)   +
                              e^(-0.5 * ((0.392 * 1) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 2) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 3) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 4) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 5) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 6) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 7) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 8) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 9) - 1.96)^2)
                            )
```

```
[1] 0.947
```

Right

Answer:
0.947

** Work:**

Same thing, except runs from 1-10 instead of 0-9. Not surprising that it gives you the same answer:

```r
e <- exp(1) # get euler's number
(0.392 / (sqrt(2 * pi))) * (e^(-0.5 * ((0.392 * 1) - 1.96)^2)   +
                              e^(-0.5 * ((0.392 * 2) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 3) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 4) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 5) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 6) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 7) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 8) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 9) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 10) - 1.96)^2)
                            )
```

```
[1] 0.947
```

Midpoint

Answer:
0.951

Work:

$$A \approx 0.392\left(\sum_{i=1}^{10}\frac{1}{\sqrt{2\pi}}e^{-0.5\left(-2.16+0.392i\right)^2}\right).$$

```
e <- exp(1) # get euler's number
(0.392 / (sqrt(2 * pi))) * (e^(-0.5 * ((0.392 * 1) - 2.16)^2)   +
                              e^(-0.5 * ((0.392 * 2) - 2.16)^2) +
                              e^(-0.5 * ((0.392 * 3) - 2.16)^2) +
                              e^(-0.5 * ((0.392 * 4) - 2.16)^2) +
                              e^(-0.5 * ((0.392 * 5) - 2.16)^2) +
                              e^(-0.5 * ((0.392 * 6) - 2.16)^2) +
                              e^(-0.5 * ((0.392 * 7) - 2.16)^2) +
                              e^(-0.5 * ((0.392 * 8) - 2.16)^2) +
                              e^(-0.5 * ((0.392 * 9) - 2.16)^2) +
                              e^(-0.5 * ((0.392 * 10) - 2.16)^2)
                            )
```

```
[1] 0.951
```

Trapezoidal

Answer:
0.947

Work:

$$A \approx \left(\frac{0.196}{\sqrt{2\pi}}\right)\sum_{i=0}^{9}\frac{e^{-0.5(0.392i-1.96)^2}+e^{-0.5(0.392i-0.564)^2}}{2}.$$

```
e <- exp(1) # get euler's number
(0.196 / (sqrt(2 * pi))) * (e^(-0.5 * ((0.392 * 0) - 1.96)^2)       +
                              e^(-0.5 * (0.392 * (0 + 1) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 1) - 1.96)^2)     +
                              e^(-0.5 * (0.392 * (1 + 1) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 2) - 1.96)^2)     +
                              e^(-0.5 * (0.392 * (2 + 1) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 3) - 1.96)^2)     +
                              e^(-0.5 * (0.392 * (3 + 1) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 4) - 1.96)^2)     +
                              e^(-0.5 * (0.392 * (4 + 1) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 5) - 1.96)^2)     +
                              e^(-0.5 * (0.392 * (5 + 1) - 1.96)^2) +
                              e^(-0.5 * ((0.392 * 6) - 1.96)^2)     +
```

```
  e^(-0.5 * (0.392 * (6 + 1) - 1.96)^2) +
  e^(-0.5 * ((0.392 * 7) - 1.96)^2)     +
  e^(-0.5 * (0.392 * (7 + 1) - 1.96)^2) +
  e^(-0.5 * ((0.392 * 8) - 1.96)^2)     +
  e^(-0.5 * (0.392 * (8 + 1) - 1.96)^2) +
  e^(-0.5 * ((0.392 * 9) - 1.96)^2)     +
  e^(-0.5 * (0.392 * (9 + 1) - 1.96)^2)
)
```

```
[1] 0.947
```

6.2

(a)

Answer:

$= \frac{x^{101}}{101} + 3e^x - 7\frac{4^x}{1.39} + c.$

Work:

$$\begin{aligned}
F(x) &= \int x^{100} + 3e^x - 7(4^x)dx, \\
&= \int x^{100}dx + \int 3e^x dx - \int 7(4^x)dx, \\
&= \frac{x^{100+1}}{100+1} + \int 3e^x dx - \int 7(4^x)dx, \\
&= \frac{x^{101}}{101} + 3\int e^x dx - 7\int (4^x)dx,
\end{aligned}$$

The constant will cancel out, so leave it out.

$$\begin{aligned}
&= \frac{x^{101}}{101} + 3e^x - 7\frac{4^x}{\ln(4)} + c, \\
&= \frac{x^{101}}{101} + 3e^x - 7\frac{4^x}{1.39} + c.
\end{aligned}$$

(b)

Answer:

`87.7`

Work:

$$\begin{aligned}
F(x) &= \int_1^9 5\sqrt{x}\ dx + \frac{3}{x^4}dx, \\
&= \int_1^9 5\sqrt{x}\ dx + \int \frac{3}{x^4}dx,
\end{aligned}$$

$$= 5\int_1^9 \sqrt{x}\; dx + 3\int \frac{1}{x^4} dx,$$

$$= 5\int_1^9 x^{1/2}\; dx + 3\int x^{-4} dx,$$

$$= 5\frac{x^{1/2+1}}{\frac{1}{2}+1} + 3\frac{x^{-4+1}}{-4+1},$$

$$= 5\frac{x^{3/2}}{\frac{3}{2}} + 3\frac{x^{-3}}{-3},$$

$$= 5\Big(x^{3/2}\frac{2}{3}\Big) + 3\Big(\frac{-1}{3}x^{-3}\Big),$$

$$= (x^{3/2})\frac{10}{3} + (3)\frac{1}{x^3}.$$

From here, plug in the bounds and take the difference:

```
(9^(3/2) * 10/3 - 1/9^3) - (1^(3/2) * 10/3 - 1/1^3)
```

```
[1] 87.7
```

(c)

Answer:

6

Work:

$$F(x) = \int_2^\infty \frac{12}{x^2} dx,$$

$$= \lim_{k\to\infty} \int_2^k \frac{12}{x^2} dx,$$

$$= 12 \lim_{k\to\infty} \int_2^k x^{-2} dx,$$

$$= 12 \lim_{k\to\infty} \int_2^k \frac{x^{-2+1}}{-2+1} dx,$$

$$= 12 \lim_{k\to\infty} \frac{-1}{x}.$$

Plug in the bounds (plugging for the $-1/x$ since it approaches zero the higher x gets):

```
12 * (0 + 1/2)
```

```
[1] 6
```

(d)

Answer:
$2y^2$.

Work:

$$F(x) = \frac{d}{dy}\int_{-3}^{y^2} \sqrt{x}\; dx,$$

$$= \sqrt{y^2}\ \frac{d}{dy} y^2,$$

Power rule...

$$= \sqrt{y^2}\ 2y,$$
$$= 2y^2.$$

(e)

Answer:
$-e^{\sqrt{z}+\ln(z)}\Big(\frac{1}{2\sqrt{z}} + \frac{1}{z}\Big).$

Work:

$$F(x) = \frac{d}{dz}\int_{\sqrt{z}+\ln(z)}^{10} e^x dx,$$
$$= -1 \times \frac{d}{dz}\int_{\sqrt{z}+\ln(z)}^{10} e^x dx,$$
$$= -\frac{d}{dz}\int_{10}^{\sqrt{z}+\ln(z)} e^x dx,$$
$$= -e^{\sqrt{z}+\ln(z)}\frac{d}{dy}\sqrt{z} + \ln(z),$$
$$= -e^{\sqrt{z}+\ln(z)}\Big(\frac{1}{2\sqrt{z}} + \frac{1}{z}\Big).$$

6.3a

Answer:

$\frac{(5x^{10} - 25x^4 + 15x)^8}{40} + c.$

Work:

$$\int (10x^9 - 20x^3 + 3)(5x^10 - 25x^4 + 15x)^7 dx,$$
$$\int (10x^9 - 20x^3 + 3)(5x^{10} - 25x^4 + 15x)^7 dx.$$
$$u = 5x^{10} - 25x^4 + 15x,$$
$$\frac{du}{dx} = \frac{d}{dx}(5x^{10} - 25x^4 + 15x)$$
$$\frac{du}{dx} = \frac{d}{dx}(5x^{10}) - \frac{d}{dx}(25x^4) + \frac{d}{dx}(15x),$$
$$\frac{du}{dx} = 5(10x^9) - 25(4x^3) + 15,$$

$$\frac{du}{dx} = 50x^9 - 100x^3 + 15,$$
$$\frac{1}{5}\frac{du}{dx} = (50x^9 - 100x^3 + 15)\frac{1}{5},$$
$$\frac{1}{5}\frac{du}{dx} = 10x^{10} - 20x^4 + 5x,$$
$$\frac{1}{5}du = (10x^{10} - 20x^4 + 5x)dx.$$

Substitute in u and $1/5\ du$:

$$\int (10x^9 - 20x^3 + 3)(5x^{10} - 25x^4 + 15x)^7 dx,$$

$$\int (u)^7 (\frac{1}{5}du),$$
$$\frac{1}{5}\int (u)^7 (du),$$
$$\frac{1}{5}\int \frac{u^8}{8} + c,$$
$$\frac{u^8}{40} + c,$$
$$\frac{(5x^{10} - 25x^4 + 15x)^8}{40} + c.$$

6.4a

Answer:

`20.9`

$$\int_1^4 x\sqrt{x+5}\ dx,$$

We want to get to:

$$\int u\ dv = uv - \int v\ du.$$

Use some u-substution:

$$u = x,\ dv = \sqrt{x+5}\ dx.$$
$$\frac{du}{dx} = 1,\ \text{therefore}\ du = dx.$$
$$w = x + 5,$$
$$\frac{dw}{dx}(x+5) = 1 + 0 = 1,$$
$$dw = dx.$$
$$\int \sqrt{w}\ dw,$$
$$\int w^{\frac{1}{2}}\ (1),$$
$$\frac{w^{\frac{1}{2}+1}}{\frac{1}{2}+1},$$

$$\frac{w^{\frac{3}{2}}}{\frac{3}{2}},$$

$$\frac{2}{3}w^{\frac{3}{2}},$$

$$\frac{2}{3}(x+5)^{\frac{3}{2}} = v.$$

Plug back into:

$$\int u\ dv = uv - \int v\ du,$$

with
$u = x$.
$dv = \sqrt{x+5}\ dx$.
$du = dx = dw$.
$w = x + 5$.
$v = \frac{2}{3}(x+5)^{\frac{3}{2}}$.

$$\int_1^4 x\ \sqrt{x+5}\ dx = x\ \left(\frac{2}{3}(x+5)^{\frac{3}{2}}\right) - \int_1^4 \frac{2}{3}(x+5)^{\frac{3}{2}}\ dx,$$

$$\int_1^4 x\ \sqrt{x+5}\ dx = x\ \left(\frac{2}{3}(x+5)^{\frac{3}{2}}\right) - \frac{2}{3}\int_1^4 (x+5)^{\frac{3}{2}}\ dx.$$

Before we can start plugging in numbers, gotta take care of the remaining integral:

$$\int (x+5)^{\frac{3}{2}}\ dx,$$

$$\int (x+5)^{\frac{3}{2}}\ dx,$$

$$u = x + 5,$$

$$\int (u)^{\frac{3}{2}}\ dx,$$

$$\int \frac{(u)^{\frac{3}{2}+1}}{\frac{3}{2}+1}\ dx,$$

$$\frac{(u)^{\frac{5}{2}}}{\frac{5}{2}},$$

$$\frac{2}{5}(u)^{\frac{5}{2}},$$

$$\frac{2}{5}(x+5)^{\frac{5}{2}}.$$

So in the end you have:

$$\int_1^4 x\ \sqrt{x+5}\ dx = x\ \left(\frac{2}{3}(x+5)^{\frac{3}{2}}\right) - \frac{2}{3}\Big(\frac{2}{5}(x+5)^{\frac{5}{2}}\Big).$$

Plug in the bounds:

```
((2/3 * 4 * ((4+5)^(3/2))) - 2/3 * (2/5 * (4 + 5)^(5/2))) -

  ((2/3 * 1 * ((1+5)^(3/2))) - 2/3 * (2/5 * (1 + 5)^(5/2)))
```

```
[1] 20.9
```

6.5a

Answer:

291

Work: Second theorem:

$$\int_a^b f(x)\ dx = F(x)\Big|_a^b = F(b) - F(a).$$

$$F(x) = \int_{\ln(8)}^{\sqrt[3]{167}} x^2 + e^x\ dx,$$

$$= \int_{\ln(8)}^{\sqrt[3]{167}} x^2 + \int_{\ln(8)}^{\sqrt[3]{167}} e^x\ dx,$$

$$= \frac{x^3}{3} + e^x + c,$$

$$= \left(\frac{x^3}{3} - \frac{x^3}{3}\right) + \Big((e^x + c) - (e^x + c)\Big),$$

the constant cancels out

$$= \left(\frac{(\sqrt[3]{167})^3}{3} - \frac{\ln(8)^3}{3}\right) + \left(e^{\sqrt[3]{167}} - e^{\ln(8)}\right),$$

$$= \left(\frac{(\sqrt[3]{167})^3}{3} - \frac{\ln(8)^3}{3}\right) + \left(e^{\sqrt[3]{167}} - e^{\ln(8)}\right),$$

$$= 291.$$

```
e <- exp(1)
167 / 3 - (log(8))^3 / 3 + e^(167^(1/3)) - e^(log(8))
```

```
[1] 291
```

6.6

(a)

Answer:
A function is a PDF if the function is never less than 0 and the total area under the curve (over the domain of the function) is equal to 1.
Step 1: show that function is never less than 0:

```
(3 * sqrt(0) / 2)
(3 * sqrt(1) / 2)
(3 * sqrt(100) / 2)
```

```
[1] 0
[1] 1.5
[1] 15
```

Function is never less than 0!
Step 2: show that the total area under the curve over the domain is equal to 1.
To get the area under the curve, integrate the function (it'd be a definite integral):

$$\int_0^1 \frac{3\sqrt{x}}{2} dx,$$

$$\frac{3}{2}\int_0^1 x^{1/2}\ dx,$$

$$\frac{3}{2}\int_0^1 \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\ dx,$$

$$\frac{3}{2}\int_0^1 \frac{x^{3/2}}{\frac{3}{2}}\ dx,$$

$$\frac{3}{2}\left(\frac{2}{3}x^{3/2}\right)\ dx.$$

Plug in the values of the domain and get the area under the curve within the domain of the function:

```
3/2 * (2/3 * 1^(3/2)) - 3/2 * (2/3 * 0^(3/2))
```

```
[1] 1
```

Because this function satisfies both criteria, it is a PDF.

(b)