Portfolio Construction using Black-Litterman Model and Factors

Michele Vannucci

August 2025

Contents

Introduction **2**

1 Asset and Factor Selection **3**
1.1 Investment Universe . . . 3
1.1.1 Data and Preprocessing . . . 4
1.2 Investable Factor Assets . . . 5
1.2.1 The Fama-French Model . . . 5
1.2.2 Extra Factors . . . 5
1.2.3 Factor Data and Preprocessing . . . 7
1.2.4 Selection Rules for Proxy ETFs . . . 8
1.2.5 Performance Analysis and Rolling Factor Loadings . . . 10
1.3 Final Asset Set and Covariance Estimation . . . 14

2 The Black-Litterman Model **18**
2.1 Model Formulation . . . 18
2.2 Model Implementation . . . 20
2.2.1 Equilibrium Return . . . 20
2.2.2 Views Definition . . . 23
2.2.3 Views Uncertainty . . . 24
2.2.4 Posterior Returns and Unconstrained BL Allocation . . . 25

3 Constrained Portfolio Optimization **29**
3.1 Optimization Methods . . . 29
3.2 Optimization Results . . . 32
3.2.1 Performance Metrics . . . 33
3.2.2 Allocation Comparison . . . 34
3.2.3 Risk Contributions . . . 37
3.2.4 Performance Statistics . . . 38
3.2.5 Portfolio Exposures . . . 39

Conclusions **41**

A Fama-French Factors **42**

B Computation of the Diagonal Elements of Ω According to Idzorek **43**

Introduction

The Black–Litterman (BL) model extends the traditional mean–variance framework by addressing its sensitivity to input estimates of expected returns. By combining market-implied equilibrium returns with subjective investor views, the BL model produces posterior expectations that are more stable and can be tailored to reflect specific opinions on asset or factor performance. The project develops a full BL allocation framework on a universe of U.S. exchange-traded funds and factor proxies, chosen to ensure both diversification and investability. The analysis proceeds in three stages. The first concerns data retrieval and preprocessing, where asset and factor returns are collected, aligned, and transformed into weekly excess returns. The second introduces the BL model itself, where equilibrium returns are combined with investor views and their associated confidence levels to produce posterior expected returns. The third stage applies constrained optimization techniques to the posterior distribution, enforcing realistic long-only, fully invested portfolios under three alternative criteria: mean–variance, maximum Sharpe ratio, and Expected Shortfall risk budgeting. The optimized portfolios are then evaluated and compared through multiple lenses: allocations and tilts relative to the market, risk contributions, ex-ante performance statistics, and factor exposures. This multi-step approach makes it possible to highlight the distinct properties of each optimization method while assessing how the BL model reconciles equilibrium information with subjective views in a practical, constrained setting.

For clarity of notation, bold symbols denote vectors and matrices, the transposition operator is indicated with a prime symbol ($'$) and the identity matrix is written as $\mathbb{I}$. Detailed data-handling procedures and accurate algorithm descriptions are documented in the accompanying Jupyter Notebook, so that the present report can focus on the conceptual framework and results.

Technique	**Description**
Factor regression	OLS with Newey–West adjustment
Idzorek method	Confidence calibration for $\boldsymbol{\Omega}$
Black–Litterman posterior	Equilibrium returns + investor views
Mean–variance optimization	Quadratic program with constraints
Maximum Sharpe optimization	Dinkelbach algorithm (fractional programming)
Risk budgeting (ES)	SLSQP with ES-based contributions
Risk contributions	Variance and ES decomposition
Performance metrics	Return, volatility, Sharpe, diversification ratio

Table 1: Numerical techniques implemented in the project.

Chapter 1

Asset and Factor Selection

This chapter defines the investable universe employed in the empirical analysis. The objective is to construct a diversified set of U.S.-listed exchange-traded funds (ETFs) spanning multiple asset classes and systematic factor exposures, providing broad coverage of distinct sources of return.

In addition to these conventional asset classes, a subset of ETFs is selected to act as investable proxies for theoretical factor portfolios such as those in the Fama–French five-factor model. This integration enables the Black–Litterman framework to incorporate factor-tilted positions directly.

The chapter is organized as follows: Section 1.1 presents the baseline asset universe, detailing the asset classes and their respective representative ETFs. Section 1.1.1 describes the preprocessing steps applied to the raw data to ensure consistency across series. Section 1.2 introduces the factor-augmented investable universe and outlines the selection of ETF proxies for each targeted factor exposure.

1.1 Investment Universe

The portfolio universe is composed exclusively of U.S. ETFs, with the primary objective of achieving a well-diversified and low-correlation asset pool across major asset classes. The target universe size is 20 assets, of which 4 will be introduced in the following section to represent factor exposures. The 16 assets listed in Table 1.1 span commodities, precious metals, volatility, multiple fixed-income segments, broad and factor-tilted equities, sector-specific equities, and real estate. This selection is designed to capture distinct sources of risk premia and to maintain resilience across market regimes: for example, volatility futures (`VIXM`), gold (`GLD`), and long-dated Treasuries (`TLT`) offer potential hedging characteristics during equity drawdowns, while sector and factor tilts allow for targeted exposures within equities. The table also reports the benchmark ETF (`SPY`) and the 3-month T-bill rate (`TB3MS`) used as the risk-free asset; these serve as references for performance evaluation and excess return computation, but are not treated as investable portfolio components for the optimization steps.

Ticker	Asset Class	Description
Real Assets		
DBC	Broad Commodities	Invesco DB Commodity Index Tracking Fund
GLD	Precious Metals (Gold)	SPDR Gold Shares
Volatility		
VIXM	Volatility Futures	ProShares VIX Mid-Term Futures ETF
Fixed-Income		
IEF	U.S. Treasuries (7–10Y)	iShares U.S. Treasury 7–10Y
TLT	U.S. Treasuries (20Y+)	iShares 20+ Year Treasury Bond ETF
TIP	Treasury Inflation-Protected Securities	iShares TIPS Bond ETF
LQD	Investment-Grade Corporate Bonds	iShares iBoxx $ Investment Grade Corporate Bond ETF
HYG	High-Yield Corporate Bonds	iShares iBoxx $ High Yield Corporate Bond ETF
Broad / Factor-Tilted Equities		
QQQ	U.S. Large-Cap Growth	Invesco QQQ Trust
SPHQ	Quality Equity Factor	Invesco S&P 500 Quality ETF
Real Estate		
VNQ	Real Estate (REITs)	Vanguard Real Estate Index Fund ETF Shares
Sector Equities		
XLE	Energy	The Energy Select Sector SPDR Fund
XLF	Financials	The Financial Select Sector SPDR Fund
XLP	Consumer Staples	The Consumer Staples Select Sector SPDR Fund
XLU	Utilities	The Utilities Select Sector SPDR Fund
XLV	Health Care	The Health Care Select Sector SPDR Fund
Benchmark		
SPY	U.S. Broad-Equity Benchmark	SPDR S&P 500 ETF
Risk-free rate		
TB3MS	Risk-Free (3M T-Bill)	3-Month Treasury Bill Secondary Market Rate, Discount Basis

Table 1.1: Baseline asset pool, covering major asset classes—real assets, volatility, fixed-income, equities (broad and sector), and real estate—to ensure robust diversification.

1.1.1 Data and Preprocessing

All ETF price series are obtained as daily adjusted close prices from the `yFinance` Python API. The daily adjusted close price on day t are resampled to a weekly frequency by selecting Friday observations (or the last available trading day in case of market holidays) and computing the corresponding simple return:

$$r_t^{(\text{week})} = \frac{P_t^{(\text{Fri})}}{P_{t-1}^{(\text{Fri})}} - 1.$$

This is done because daily data tend to be too noisy. The risk-free rate series (`TB3MS`) is obtained from the FRED database [1], and it is expressed in annualized form and reported monthly. To align it with the weekly return frequency, the series is first converted to a weekly rate via forward filling and then annualized-to-weekly:

$$r_t^{\text{week}} \;=\; \left(1 + y_t^{\text{ann}}\right)^{1/52} - 1,$$

where y_t^{ann} is the quoted annualized yield in decimal form.

All asset and risk-free series are then aligned on a common weekly calendar. The analysis period is truncated to the range from 02 December 2015 to 25 April 2025, yielding 490 weekly observations. Weekly excess returns are then computed as

$$r_{i,t}^{(\text{excess})} = r_{i,t}^{(\text{week})} - r_{\text{week},t}^{(\text{rf})}$$

for each asset i and week t.

1.2 Investable Factor Assets

Incorporating factor portfolios as investable assets enables the direct application of the Black–Litterman framework to instruments that are actively traded in the market. Theoretical factors are defined as zero-cost, long–short constructs and therefore cannot be held directly. To bridge this gap, they are approximated using long-only ETFs whose underlying holdings exhibit similar exposures. This proxy-based approach facilitates data acquisition and simplifies trading implementation.

This section describes the Fama–French factor model, the custom factors introduced in this study, the preprocessing of factor data, and the methodology adopted to select suitable ETF proxies for each targeted exposure.

1.2.1 The Fama-French Model

The Capital Asset Pricing Model (CAPM) provides the cornerstone of modern asset-pricing theory by stating that the expected excess return of an asset is proportional to a single source of systematic risk—the excess return on the market portfolio over the risk-free rate. Subsequent empirical work, however, documented return differentials that the one-factor structure fails to reconcile, most prominently the persistent outperformance of small-capitalization and value stocks (Fama & French, 1992).

The Fama–French three-factor model [2] augments the traditional CAPM by adding size and value factors to the market excess return. Continued investigation into cross-sectional return patterns led to a further expansion in 2015 [3], when the model was extended to five factors by including proxies for profitability and investment, as summarized in Table 1.2.

Taken together, the five factors—market, size, value, profitability, and investment—yield a substantially richer description of the cross-section of returns, enabling portfolio designs that target a diversified set of systematic premia rather than the market alone. Formal definitions and construction details for all factors are available in the French database [4]; their mathematical expressions are also reproduced in Appendix A.

1.2.2 Extra Factors

In addition to the five canonical Fama-French factors, new systematic premia may be included to address particular investment objectives or to exploit documented market anomalies. Two such premia are employed: the *Momentum* factor, with returns obtained directly from the French database [4], and a bespoke *Betting-Against-Beta* factor, with returns sourced from the AQR website [5].

Code	Name	Description
MKT	Market Excess Return	Traditional CAPM factor: long the market portfolio, short the risk-free asset.
SMB	Small Minus Big	*Size factor*: long small-cap stocks, short large-cap stocks.
HML	High Minus Low	*Value factor*: long high book-to-market (value) stocks, short low book-to-market (growth) stocks.
RMW	Robust Minus Weak	*Profitability factor*: long firms with robust operating profitability, short those with weak profitability.
CMA	Conservative Minus Aggressive	*Investment factor*: long firms following conservative investment policies, short those with aggressive capital expenditures.

Table 1.2: Fama–French five-factor model components.

Momentum

The Momentum (MOM) factor is formed by first partitioning the universe of U.S. equities into two size groups—*small* and *large*—using the NYSE median market equity as the breakpoint. Within each size group, securities are sorted into three prior-return buckets (low, medium, high) according to their cumulative returns over months 2–12, with cut-offs at the 30th and 70th NYSE percentiles. MOM is the average return on the two high prior return portfolios minus the average return on the two low prior return portfolios:

$$\texttt{MOM} = \frac{1}{2}(Small\,High + Big\,High) - \frac{1}{2}(Small\,Low + Big\,Low)$$

This spread captures the well-documented continuation effect whereby past winners tend to outperform and past losers tend to underperform, a phenomenon generally attributed to behavioural under- and over-reaction to information. By construction, MOM is largely orthogonal to size, value, profitability, investment, and volatility exposures, thereby providing an additional and diversifying source of expected excess return.

Betting-Against-Beta

The Betting Against Beta (BAB) factor, introduced by Frazzini and Pedersen (2014) [6], is based on the observation that many investors face leverage constraints and thus overweight high-beta securities instead of levering low-beta ones. This behavior flattens the security market line compared to CAPM predictions. BAB captures the resulting anomaly by going long leveraged portfolios of low-beta assets and short portfolios of high-beta assets, producing a market-neutral strategy that earns a persistent return premium across markets and asset classes.

In this project the Betting-Against-Beta factor is proxied (BAB_p) by the difference of excess return of a low-volatility ETF over a high-beta ETF,

$$\texttt{BAB\_p} = r_{\texttt{SPLV}} - r_{\texttt{SPHB}},$$

where $r_{\texttt{SPLV}}$ and $r_{\texttt{SPHB}}$ denote, respectively, excess returns on the Invesco S&P 500 Low Volatility ETF and the Invesco S&P 500 High Beta ETF. The long position in SPLV provides exposure to stocks with below-average historical volatility, while the short position in SPHB captures stocks exhibiting above-average market sensitivity. The resulting spread exploits the "low-risk" anomaly, under which low-beta stocks have historically delivered returns in excess of their CAPM-implied expectations. BAB therefore introduces an orthogonal volatility-driven premia that tends to perform comparatively well in turbulent markets and imparts a defensive tilt to the overall factor mix.

1.2.3 Factor Data and Preprocessing

The tickers below serve as candidate ETFs for the empirical factor proxies, grouped by factor. Selection was guided chiefly by the availability of at least ten years of history. Because very few U.S. equity funds explicitly target profitability or investment styles, only a single ETF is used to represent the RMW and CMA factors.

- SMB **(Size)** — IJR, VB, SLYV, SCHA, IWM.
  All five funds target the small-capitalisation universe but follow distinct index constructions—S&P 600 (IJR, SLYV), CRSP Small-Cap (VB), Schwab U.S. Small-Cap (SCHA), and Russell 2000 (IWM).
- HML **(Value)** — VLUE, SPVU, VTV, IVE, IVLU.
  Each tracks a systematic value index that emphasises low price-to-fundamental ratios across large- and mid-caps.
- MOM **(Momentum)** — SPMO, QMOM, MTUM.
  These vehicles apply 6–12-month total-return screens to isolate the cross-sectional momentum effect.
- RMW **(Profitability)** — QUAL
  Constituents are weighted by strong return-on-equity, stable earnings, and low leverage, closely mirroring the robust-minus-weak construct.
- CMA **(Investment)** — SYLD.
  This fund favours firms with disciplined capital spending, high free cash flow, and active buy-backs—attributes that should translate into a pronounced positive loading on the CMA factor.
- BAB **(Betting-Aganist-Beta)** — BAB_p

Collectively, these candidates offer robust and liquid exposure to the intended factors.

The preprocessing of proxy ETF prices follows exactly the procedure outlined in Section 1.1.1 for the baseline asset universe. The only difference is that Fama–French daily factor returns, retrieved from the French Data Library [4], need to be converted from percentages to decimal form (divided by 100) before being aggregated to a Friday-weekly frequency:

$$r_{\text{week}} = \prod_{d \in \text{week}} \bigl(1 + r_d\bigr) - 1.$$

As all Fama–French factors are defined as zero-cost portfolios, excess returns need not be computed: subtracting the risk-free rate from both the long and short legs cancels out by construction.

1.2.4 Selection Rules for Proxy ETFs

A regression of the weekly excess returns of each ETF is performed using the function

$$r_t^{\mathrm{ETF}} = \alpha + \sum_{k \in \text{factors}} \beta_k \cdot k_t + \varepsilon_t \tag{1.1}$$

where factors $= \{$MKT, SMB, HML, MOM, RMW, CMA, BAB$\}$, α is the systematic drift, β_k measures the sensitivity of the asset's returns to factor and ε_t is the remaining idiosyncratic shock. An ideal ETF candidate should satisfy all the following requirements in order to accurately represents its target factor:

- Factor $\beta > 0.8$: ensures a strong sensitivity to the intended factor.
- Cross-factor $|\beta| < 0.3$: limits unintended exposures to other factors.
- Market $\beta \in [0.8, 1.2]$: confirms appropriate sensitivity to the benchmark.
- $\alpha \approx 0$: indicates the absence of persistent unexplained performance.
- $R^2 > 0.8$: requires that at least 80% of return variability is explained by the factor model.

A Newey–West t-statistic is also performed to verify the statistical significance of the obtained parameters, taking into account autocorrelation and heteroskedasticity. This distribution is asymptotically $\mathcal{N}(0, 1)$, hence the rule of thumb $|t| > 2$ corresponds to rejecting the null coefficient ($\beta = 0$ or $\alpha = 0$) at approximately the 5% two–sided significance level, since the exact normal critical value is $z_{0.975} = 1.96$.

Table 1.3 summarizes the seven–factor regression estimates, with all the useful information needed for a solid factor selection. The regression results show that the market factor captures the bulk of systematic variation: every ETF displays a market loading in the target band $0.8 \leq \beta_{\mathtt{MKT}} \leq 1.2$, with the only exceptions of IVLU ($\beta_{\mathtt{MKT}} = 0.73$) and BAB_p ($\beta_{\mathtt{MKT}} = -0.56$). The latter case is not surprising: unlike the other long-only ETFs, BAB_p is a long–short construction that is specifically designed to reduce systematic market exposure. Its negative market beta reflects the fact that the strategy profits from tilting against high-beta stocks and therefore carries an implicit short position on the broad market.

Most factor loadings fall short of the $\beta > 0.8$ requirement, a threshold that is admittedly arbitrary; in such cases the decision relies on statistical precision and cross-contamination as shown by the Newey–West t-statistics.

Size (SMB) Three funds—IJR, SLYV, and IWM—exceed $\beta_{\mathtt{SMB}} = 0.8$ with $R^2 > 0.98$. Despite a statistically significant negative intercept ($\alpha = -1.82\%$ p.a., $|t_\alpha| = 2.54$), IWM is preferred because (i) it delivers a strong and precisely estimated size loading ($\beta_{\mathtt{SMB}} = 0.88$, $|t_\beta| = 57.93$), (ii) explanatory power is extremely high ($R^2 = 0.99$), and (iii) the intercept is economically small relative to the annual volatility of weekly returns (23.37% p.a.)[1].

Value (HML) Among the value candidates, only SPVU delivers a substantial loading ($\beta_{\texttt{HML}} = 0.68$, $|t_\beta| = 16.84$) with an intercept indistinguishable from zero ($\alpha = -1.01\%$, $|t_\alpha| = 0.57$). It is therefore retained as the HML proxy.

Momentum (MOM) QMOM shows the strongest momentum beta ($\beta = 0.68$) but also exhibits sizable non-target exposures. MTUM, with a more moderate $\beta_{\texttt{MOM}} = 0.38$ and a highly significant $|t_\beta| = 13.89$, is chosen instead.

Profitability (RMW) and Investment (CMA) No available ETF achieves $\beta > 0.16$ for these factors, so both are excluded from further analysis.

Betting Against Beta (BAB) BAB_p achieves the intended exposure with $\beta_{\texttt{BAB}} = 0.76$ ($|t_\beta| = 7.21$). The intercept is negative but not statistically significant ($\alpha = -5.34\%$ p.a.; $|t_\alpha| = 1.40$), while the moderate fit ($R^2 = 0.69$) reflects the long–short structure. Despite a non-negligible profitability tilt, the strength of the target loading justifies retaining BAB_p as the BAB proxy.

Asset	Factor	$\beta_{\texttt{MKT}}$	β_{factor}	NW $\lvert t_\beta \rvert$	Cross-β	α (p.a.)	NW $\lvert t_\alpha \rvert$	R^2
IJR	SMB	1.01	0.90	49.38	0.24 (HML)	−1.78%	1.94	0.98
IWM	SMB	1.02	0.88	57.93	0.13 (HML)	−1.82%	2.54	0.99
VB	SMB	1.03	0.58	24.39	0.18 (HML)	−1.68%	1.64	0.98
SCHA	SMB	1.04	0.76	39.39	0.16 (HML)	−2.14%	2.61	0.99
SLYV	SMB	1.00	0.93	43.58	0.35 (HML)	−1.60%	1.55	0.98
VLUE	HML	0.98	0.32	13.01	0.13 (SMB)	−3.33%	2.24	0.95
IVLU	HML	0.73	0.31	6.67	0.10 (BAB)	−1.08%	0.35	0.70
VTV	HML	0.88	0.30	16.30	0.17 (CMA)	−1.12%	0.99	0.96
IVE	HML	0.90	0.26	13.05	0.15 (CMA)	−1.26%	1.09	0.96
SPVU	HML	0.98	0.68	16.84	0.17 (MOM)	−1.01%	0.57	0.90
MTUM	MOM	1.08	0.38	13.04	−0.16 (RMW)	−0.58%	0.32	0.93
QMOM	MOM	1.13	0.65	13.89	0.45 (SMB)	0.02%	0.01	0.83
SPMO	MOM	1.00	0.25	7.35	−0.29 (SMB)	2.68%	1.30	0.85
QUAL	RMW	0.99	0.17	8.03	−0.09 (SMB)	−1.31%	1.50	0.97
SYLD	CMA	1.04	0.07	1.01	0.52 (SMB)	−1.37%	0.61	0.92
BAB_p	BAB	−0.56	0.76	7.21	0.42 (RMW)	−5.34%	1.40	0.69

Table 1.3: Seven-factor Newey–West regressions results. The Cross-β column reports the largest loading (by absolute value) among the non-target factors. Newey–West t-statistics are shown in absolute value.

Table 1.4 summarizes the non-target loadings. Cross-factor contamination is mild: IWM shows only a small value footprint (0.13) and minor exposures to profitability and investment; SPVU adds negligible size and a moderate negative momentum coefficient (−0.15); MTUM has slight negative loadings on size and profitability; and BAB_p carries the largest spillover, a positive tilt to profitability (0.42) along with a negative size exposure (−0.34). Nevertheless, BAB_p is retained because no cleaner investable proxy

[1] Annualized standard deviation of the IWM time series.

for the BAB factor is explored and its inclusion is essential to capture the low–beta anomaly. This choice should be viewed as a rough approximation, suitable for the current application but open to refinement in future implementations where more specialized instruments or construction methods may be employed.

	SMB	HML	MOM	RMW	CMA	BAB
IWM	—	0.13	0.03	−0.10	−0.06	0.05
SPVU	0.08	—	−0.15	−0.06	−0.07	0.07
MTUM	−0.13	0.04	—	−0.15	−0.05	0.00
BAB_p	−0.34	−0.14	0.11	0.42	0.11	—

Table 1.4: Non-target factor loadings for the selected ETFs. A dash marks the primary factor the ETF is intended to proxy, so that entry is omitted here.

The proxy set is finalized as IWM (SMB), SPVU (HML), MTUM (MOM), BAB_p (BAB). Once the minimum loading threshold on the target style was relaxed, all four funds satisfied every remaining screen, and each target coefficient is still highly significant ($|t_\beta| > 7$).

1.2.5 Performance Analysis and Rolling Factor Loadings

This section presents the performance and rolling statistics of the selected factor proxies.

Profit & Loss Figure 1.1 compares the cumulative excess returns of each series against the benchmark, chosen to be SPY. The benchmark has undergone the same preprocessing steps described in Section 1.1.1 to ensure consistency.

The most visually striking feature is the divergent trajectory of the BAB_p factor, but this is actually expected. By construction, BAB_p is a market-neutral, zero-net-investment portfolio (long SPLV, short SPHB), so its cumulative return would be expected to fluctuate around zero absent persistent differences in performance between the two legs. This behaviour is roughly observed until 2020, after which SPHB consistently outperformed SPLV, producing a sustained drawdown and persistent underperformance.

The other factors, proxied by long-only ETFs, display more conventional equity-like trends. All show a marked drawdown during the COVID-19 outbreak in early 2020, followed by dispersion in cumulative performance. Most factors underperformed the benchmark in the post-2020 period, with MTUM being the notable exception: it not only kept pace with SPY but also outperformed from 2020 through 2022, before converging back towards benchmark-like behaviour.

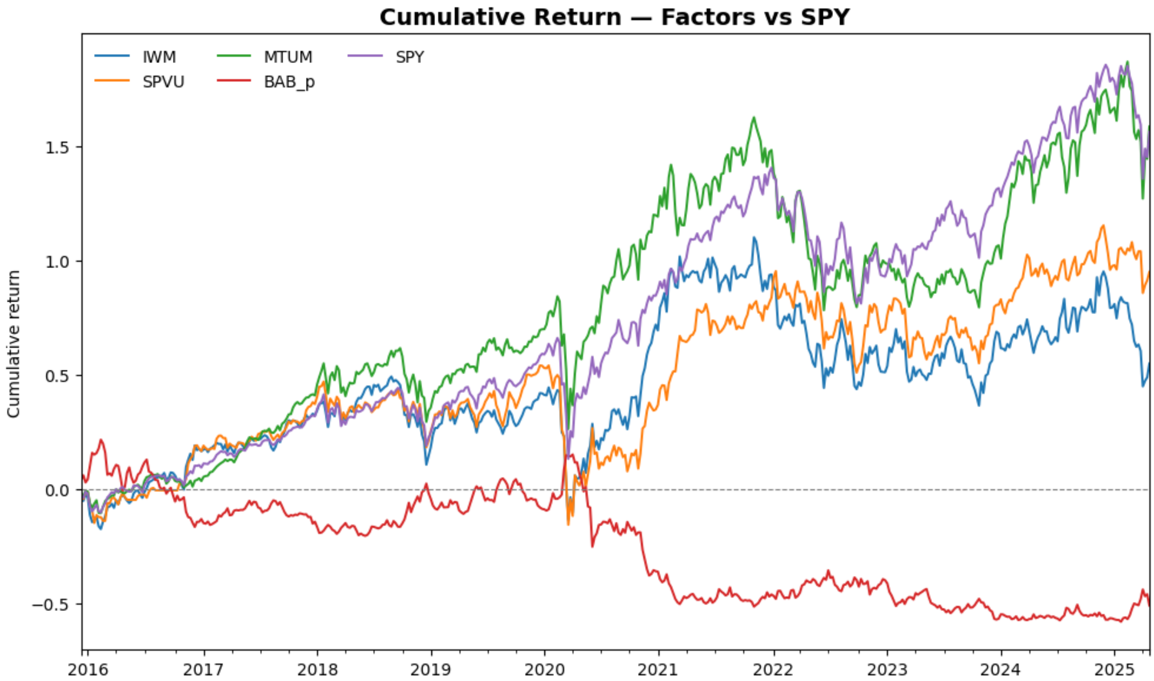


Figure 1.1: Cumulative excess returns of the factor proxies versus the `SPY` benchmark. Colors correspond to the selected assets: `IWM` (blue), `SPVU` (yellow), `MTUM` (green), `BAB_p` (red), and the benchmark `SPY` (violet).

Rolling α and β Using ordinary least squares (OLS) estimation on Equation 1.1, for a single factor f_t the slope and intercept are computed as

$$\beta = \frac{\mathrm{Cov}(r_{\text{asset}}, f)}{\mathrm{Var}(f)}, \qquad \alpha = \bar{r}_{\text{asset}} - \beta\,\bar{f},$$

where bars denote sample means over the chosen time period.

Rolling alpha and beta plots are produced by re-estimating these quantities over a moving window (104 weeks) to monitor the stability of factor exposures and abnormal returns over time.

Figures 1.2–1.5 report 104–week rolling estimates of the target factor loading β (left axis) and intercept α (right axis, annualized) for the selected proxies.

IWM vs SMB (Fig. 1.2): The `SMB` loading has remained consistently above 0.9 since mid–2020, stabilizing in the 1.3–1.5 range, which is well above the minimum threshold for a valid small–cap proxy. The intercept exhibits periods of negative drift in 2022–2023 but has been positive more recently; its magnitude remains within ±0.2% annualized, implying only a modest economic impact.

SPVU vs HML (Fig. 1.3): The `HML` loading shows more pronounced time-variation, with values oscillating between 0.25 and 1.3, and a distinct regime shift around the 2020 COVID-19 episode. This instability suggests `SPVU`'s value exposure is somewhat sensitive to macro or sector composition changes. The rolling intercept is mostly positive but small, with only two exceptions at the beginning of 2021 and 2022.

MTUM vs MOM (Fig. 1.4): The momentum beta has varied substantially over the sample, dipping near zero in 2023 before recovering sharply to ≈ 0.75 in 2025.

This volatility in exposure reflects index turnover effects and shifting momentum leadership. The intercept has been positive in several sub-periods, with spikes up to 0.3% annualized, but its persistence is low.

BAB_p vs BAB (Fig. 1.5): The loading on `BAB` fluctuates considerably over time, ranging from about 0.5 in 2021 to peaks above 1.8 in 2020 and 2023. Despite this variability, it generally remains well above the minimum threshold, confirming that `BAB_p` provides meaningful exposure to the low–beta factor. The rolling intercept, annualized, alternates between positive and negative phases: it was strongly negative around 2021, improved toward zero in 2023, and has since oscillated modestly between −0.2% and +0.1%. These shifts indicate that while `BAB_p` captures the broad low–beta premium, it is also affected by episodic drifts linked to market conditions.

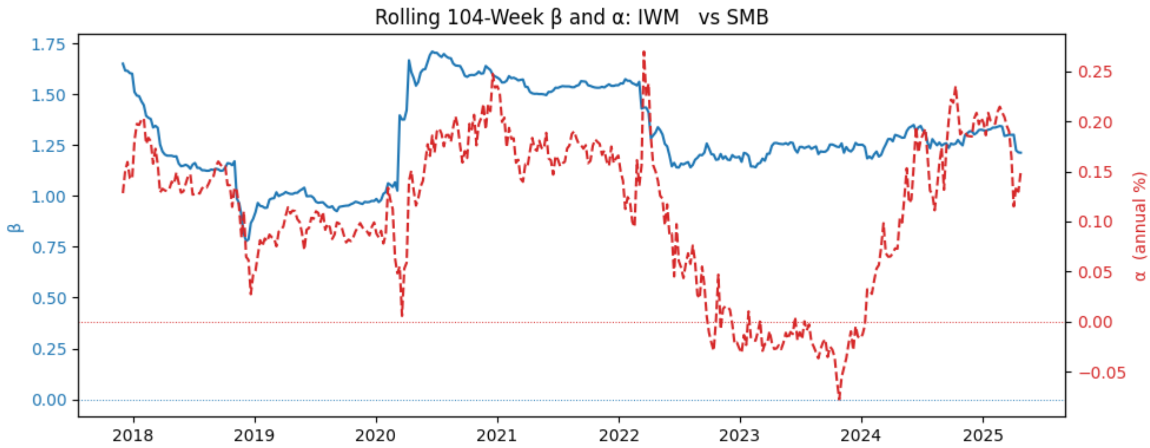


Figure 1.2: 104–week rolling $\beta_{\texttt{SMB}}$ (blue, left axis) and $\alpha_{\texttt{SMB}}$ (red, right axis, annualized). Blue and red dashed lines represent the zero for β and α scales respectively.

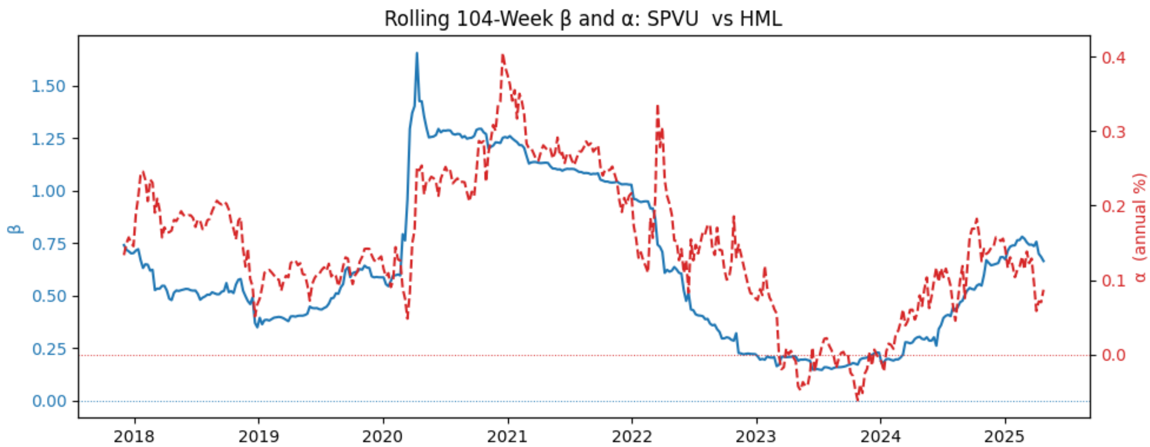


Figure 1.3: 104–week rolling $\beta_{\texttt{HML}}$ (blue, left axis) and $\alpha_{\texttt{HML}}$ (red, right axis, annualized). Blue and red dashed lines represent the zero for β and α scales respectively.

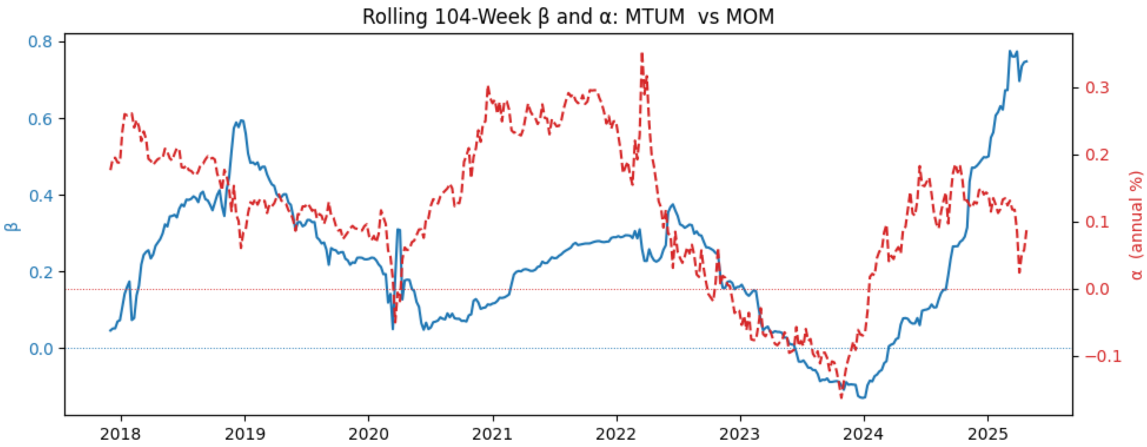


Figure 1.4: 104–week rolling $\beta_{\texttt{MOM}}$ (blue, left axis) and $\alpha_{\texttt{MOM}}$ (red, right axis, annualized). Blue and red dashed lines represent the zero for β and α scales respectively.

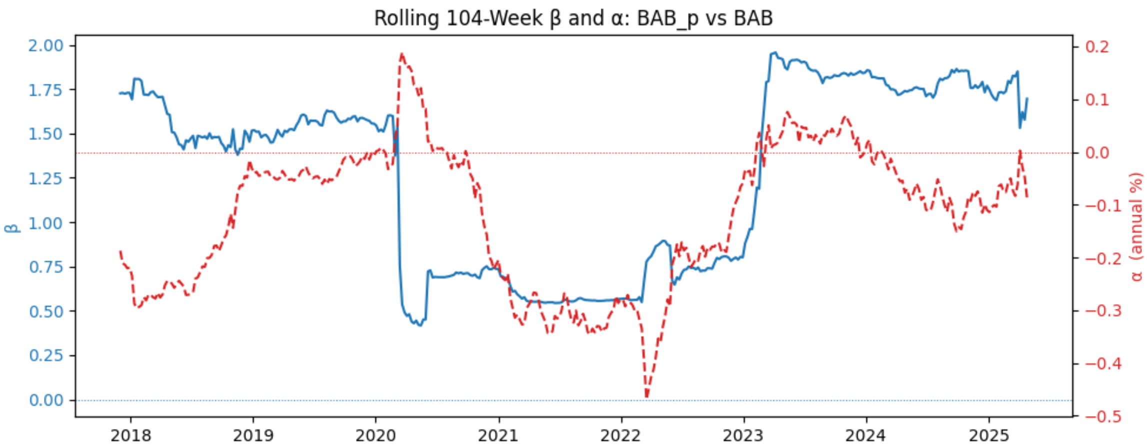


Figure 1.5: 104–week rolling $\beta_{\texttt{BAB\_p}}$ (blue, left axis) and $\alpha_{\texttt{BAB\_p}}$ (red, right axis, annualized). Blue and red dashed lines represent the zero for β and α scales respectively.

Summary Performance Statistics Table 1.5 reports annualized return, volatility, Sharpe ratio, and maximum drawdown for the factor proxies over the sample period, alongside the SPY benchmark.

	Annual Ret	Annual Vol	Sharpe	Max DD
IWM	4.74%	23.37%	0.20	−42.05%
SPVU	7.33%	22.49%	0.33	−45.28%
MTUM	10.60%	20.27%	0.52	−32.16%
BAB_p	−7.31%	21.15%	−0.34	−65.60%
SPY	10.50%	17.37%	0.60	−31.88%

Table 1.5: Annualized performance statistics for factor proxies and benchmark.

The statistics confirm the observations from the cumulative return plots and rolling regressions:

- `MTUM` shows the highest risk-adjusted performance (Sharpe 0.52), only marginally below the benchmark `SPY` (Sharpe 0.60), and the smallest maximum drawdown among the long-only proxies.

- `SPVU` and `IWM` both underperform the benchmark in absolute and risk-adjusted terms, with large drawdowns exceeding 40%.

- `BAB_p` delivers negative performance over the sample (−7.31% p.a., Sharpe −0.34), driven largely by a sustained drawdown post–2020, consistent with the divergence between `SPLV` and `SPHB` noted earlier. Its high volatility and deep maximum drawdown (−65.60%) are typical of unhedged long–short spreads under prolonged relative underperformance of the long leg.

1.3 Final Asset Set and Covariance Estimation

The final investable universe consists of the 16 baseline assets from Section 1.1 and the four factor-proxy ETFs selected in section 1.2.4. Table 1.6 lists all tickers, grouped by asset class. The asset numbering reflects the order used throughout the optimization workflow, and will become relevant when specifying views in the Black–Litterman framework. Throughout the whole project, the constant $N = 20$ is used to fix the number of assets.

Correlation Structure The sample correlation matrix (Figure 1.6) reveals several notable patterns. Fixed-income ETFs `TLT` and `IEF` (long and intermediate Treasuries respectively) exhibit very high correlation, reflecting their common exposure to U.S. government bond yields. Equity style and factor tilts such as `MTUM` (Momentum) and `SPHQ` (Quality) are also strongly correlated with the broad growth benchmark `QQQ`. The financial sector ETF `XLF` shows high correlation with the value-tilted `SPVU`.

Conversely, the volatility futures ETF `VIXM` and the proxy `BAB_p` tend to be negatively correlated with most other assets, consistent with their defensive or hedging characteristics. Some asset, such as `IEF`, `TLT`, `TIP` and `GLD`, display generally low correlation, underscoring their potential diversification benefit within the portfolio.

Covariance Estimation Given that the time series span a ten-year period, it is important to verify whether structural changes occurred, particularly around the COVID-19 pandemic in early 2020. To investigate this, the dataset was split into two sub-periods using 28 February 2020 as the cut-off date. For each sub-period, the correlation matrix was computed and the average off-diagonal correlation was calculated as

$$\bar{\rho} = \frac{2}{N(N-1)} \sum_{i<j} \rho_{ij}.$$

The resulting values are reported in Table 1.7. The 50% increase in the average off-diagonal correlation indicates a material change in the dependence structure.

#	Ticker	Asset Class / Factor Proxy
1	IWM	Size (`SMB`)
2	SPVU	Value (`HML`)
3	MTUM	Momentum (`MOM`)
4	BAB	Betting Against Beta (`BAB_p`)
5	DBC	Broad Commodities
6	GLD	Gold
7	HYG	High-Yield Corporate Bonds
8	IEF	U.S. Treasuries (7–10Y)
9	LQD	Investment-Grade Corporate Bonds
10	QQQ	U.S. Large-Cap Growth
11	SPHQ	Quality Equity Factor
12	TIP	Treasury Inflation-Protected Securities
13	TLT	U.S. Treasuries (20Y+)
14	VIXM	Volatility Futures
15	VNQ	Real Estate (REITs)
16	XLE	Energy
17	XLF	Financials
18	XLP	Consumer Staples
19	XLU	Utilities
20	XLV	Health Care

Table 1.6: Final 20-asset investable universe, including baseline assets and selected factor proxies. The asset number in the first column will become useful when implementing views in the Black-Litterman model.

$\bar{\rho}_{\text{pre}}$	$\bar{\rho}_{\text{post}}$	$\Delta\bar{\rho}$
0.20	0.30	0.10

Table 1.7: Average off-diagonal correlation before and after the COVID-19 outbreak. The change is computed as $\Delta\bar{\rho} = \bar{\rho}_{\text{post}} - \bar{\rho}_{\text{pre}}$.

Figure 1.7 illustrates the change in pairwise correlations on an asset-by-asset basis. For visual clarity, the color scale in this figure differs from that of the initial correlation matrix: the range is fixed to ± 0.65, so red tones indicate increases in correlation and blue tones indicate decreases. Notable shifts include an increase in the correlation of `LQD` with `IWM`, `SPVU`, and `XLF`, and an increased decorrelation between `BAB_p` and `IEF`, `TLT`, `LQD`, and `XLU`, but also for `LQD` and `VIXM`.

To account for the observed non-stationarity in the covariance structure, an exponentially weighted moving average (EWMA) estimator was considered as an alternative to the full-sample covariance. In this approach, more recent observations receive greater weight, with exponential decay controlled by a smoothing parameter $\alpha \in (0, 1)$. The mean vector is updated recursively as

$$\boldsymbol{\mu}_t = \alpha\, \mathbf{r}_t + (1 - \alpha)\, \boldsymbol{\mu}_{t-1},$$

where $\mathbf{r}_t$ is the vector of observed asset returns at time t, and the corresponding

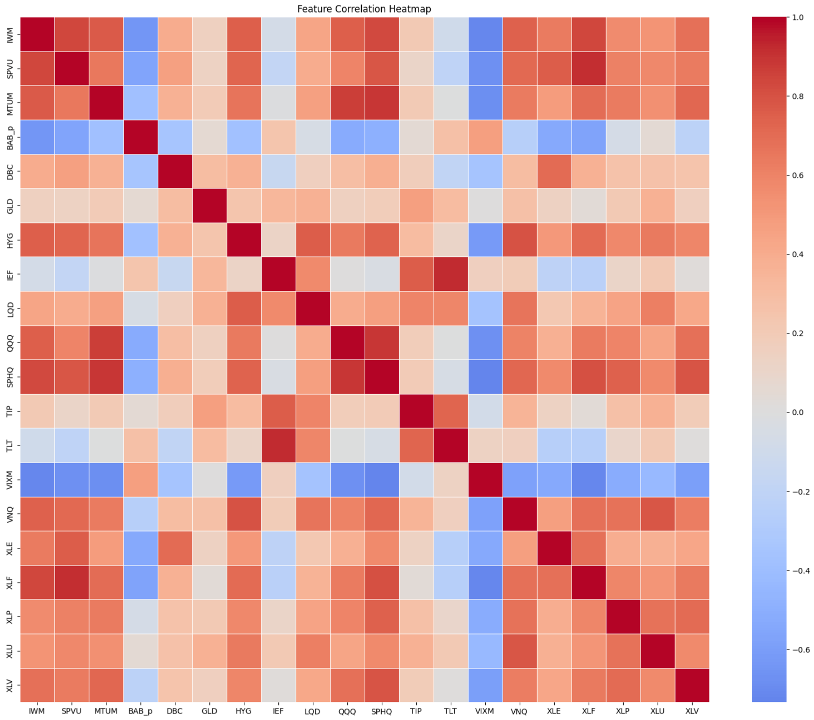


Figure 1.6: Correlation matrix of the 20-asset universe. Color intensity reflects the magnitude of pairwise correlations: red tones denote positive correlation, blue tones denote negative correlation (anti-correlation).

covariance matrix is given by

$$\mathbf{\Sigma}_t = \alpha\,(\mathbf{r}_t - \boldsymbol{\mu}_t)(\mathbf{r}_t - \boldsymbol{\mu}_t)' + (1-\alpha)\,\mathbf{\Sigma}_{t-1}.$$

Following standard practice in financial risk modelling, a smoothing parameter of $\alpha = 0.266$ was adopted, consistent with the RiskMetrics daily decay after converting to weekly frequency[2] [7].

While the EWMA estimator is more responsive to recent changes in market conditions, its statistical properties can be problematic in high-dimensional settings with strong cross-asset correlations. In this case, the estimated matrix was found to be nearly singular: the smallest eigenvalue was $\lambda_{\min} \approx 1.6 \cdot 10^{-8}$, the largest eigenvalue was $\lambda_{\max} \approx 2.4 \cdot 10^{-2}$, and the ratio $\lambda_{\min}/\lambda_{\max} \approx 6.9 \cdot 10^{-7}$.

While the EWMA covariance matrix remains positive semi-definite, it is ill-conditioned, making inversion-based optimization unstable. A more robust approach is therefore adopted: the Ledoit–Wolf shrinkage estimator [8], which combines the sample covariance matrix with a structured target to reduce estimation error and improve conditioning. The estimator takes the form

$$\mathbf{\Sigma}_{\mathrm{LW}} = (1-\delta)\mathbf{\Sigma} + \delta\frac{\mathrm{Tr}(\mathbf{\Sigma})}{N}\mathbb{I}_N,$$

where $\mathbb{I}_N$ is the $(N \times N)$ identity matrix, $\delta \in [0,1]$ is the shrinkage intensity chosen to minimize the mean-squared error between $\mathbf{\Sigma}_{\mathrm{LW}}$ and the (unobserved)

[2]RiskMetrics specifies a daily decay of $\lambda_{\mathrm{day}} = 0.94$. Pandas' EWMA uses the smoothing factor $\alpha = 1 - \lambda$. To preserve the same effective memory at weekly sampling, set $\lambda_{\mathrm{week}} = \lambda_{\mathrm{day}}^5 \approx 0.7339$, hence $\alpha_{\mathrm{week}} = 1 - \lambda_{\mathrm{week}} \approx 0.2661$

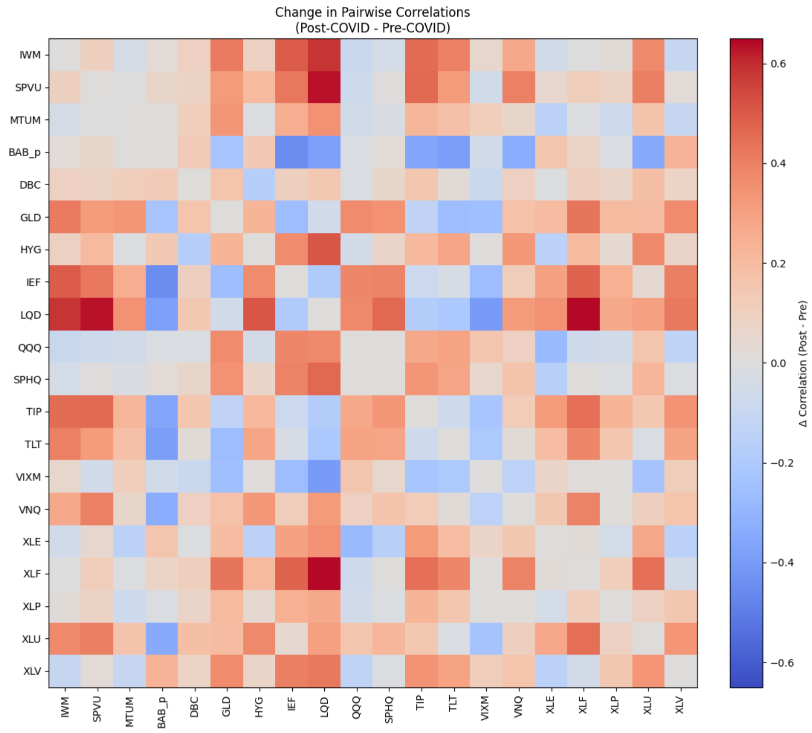


Figure 1.7: Change in pairwise correlations between the pre- and post-COVID sub-periods. Red tones indicate increases, blue tones indicate decreases. The color scale is fixed to the range ±0.65 for improved readability.

true covariance matrix. This optimal δ has a closed-form expression under mild assumptions and is estimated directly from the data. This target preserves the average variance while dampening noisy cross-asset co-movements, yielding a well-conditioned, positive-definite $\mathbf{\Sigma}_{\mathrm{LW}}$. The improvement obtained using this method is clear from Table 1.8, where the minimum and maximum eigenvalues, as well as their ratio, are reported together with the EWMA ones.

	$\lambda_{\min}$	$\lambda_{\max}$	$\lambda_{\min}/\lambda_{\max}$
EWMA	$1.6 \cdot 10^{-8}$	$2.4 \cdot 10^{-2}$	$6.9 \cdot 10^{-7}$
Ledoit-Wolf	$3.5 \cdot 10^{-5}$	$8.1 \cdot 10^{-3}$	$4.3 \cdot 10^{-3}$

Table 1.8: Extreme eigenvalues and condition ratio for the EWMA and Ledoit–Wolf covariance matrices.

All downstream methods and optimization procedures consistently employ the Ledoit–Wolf covariance estimator. For notational simplicity, the subscript “LW” is omitted in the subsequent sections, so that $\mathbf{\Sigma}$ will denote the Ledoit–Wolf shrunk covariance matrix throughout.

Chapter 2

The Black-Litterman Model

Modern Portfolio Theory (MPT)—introduced by Markowitz in 1952 [9]—frames portfolio choice as the maximization of expected return for a specified level of risk, thereby formalizing the principle of diversification. Its practical appeal is tempered, however, by several well-known weaknesses. The model rests on unrealistic assumptions of frictionless and perfectly efficient markets populated by fully rational investors; empirical evidence on transaction costs, behavioural biases and persistent return anomalies contradicts these assumptions. In addition, optimal weights are acutely sensitive to sampling noise in the estimated mean vector and, through matrix inversion, to noise in the covariance matrix; numerical experiments and out-of-sample studies demonstrate that minimal perturbations in the inputs can overturn the efficient-frontier solution and impair realised performance[10, 11]. Unconstrained optimization also tends to produce corner portfolios, concentrating capital in a handful of assets, while the exclusive reliance on historical estimates ignores the documented time-variation in risk premia and omits forward-looking information. Fischer Black and Robert Litterman of Goldman Sachs (1990–1992) introduced a Bayesian portfolio–allocation framework that mitigates several drawbacks of classical mean–variance optimization [12]. Expected returns are first anchored to a diversified benchmark through the equilibrium returns implied by market-capitalization weights; these *prior* estimates are then updated with the investor's subjective *views*. Each view is supplied together with a confidence level, so the prior is shifted toward the views in direct proportion to their stated reliability. In this way, forward-looking information is incorporated without relying on the assumption of perfectly efficient markets.

This chapter is split into two sections: Section 2.1 only focuses on the theoretical framework of the Black-Litterman model, while Section 2.2 explains how the main quantities are actually calculated, setting a starting point for the optimization step.

2.1 Model Formulation

The model formulation begins with the vector of excess returns for the N investable assets, which is assumed to be normally distributed:

$$\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}),$$

where $\mathbf{\Sigma}$ is the covariance matrix, and $\boldsymbol{\mu}$ is unknown. In a Bayesian setting the mean itself is modeled as a random variable,

$$\boldsymbol{\mu} \sim \mathcal{N}(\boldsymbol{\pi}, \tau\mathbf{\Sigma}), \tag{2.1}$$

with *equilibrium return* $\boldsymbol{\pi}$ and uncertainty scaled by the positive constant τ.

Reverse Optimization If estimation risk is ignored ($\tau = 0$), the prior mean collapses to $\boldsymbol{\pi}$. Solving an unconstrained mean–variance problem

$$\mathbf{w} = \operatorname*{argmax}_{\mathbf{w}} \Big\{ \mathbf{w}'\boldsymbol{\pi} - \lambda\, \mathbf{w}'\mathbf{\Sigma}\mathbf{w} \Big\} \tag{2.2}$$

links the equilibrium returns to the market portfolio weights $\mathbf{w}_{\mathrm{mkt}}$ through

$$\boldsymbol{\pi} = 2\lambda_{\mathrm{mkt}}\mathbf{\Sigma}\mathbf{w}_{\mathrm{mkt}}, \tag{2.3}$$

where λ_{mkt} denotes the market's aggregate risk aversion and $\mathbf{w}_{\mathrm{mkt}}$ is typically proxied by capitalization weights. This reverse optimization procedure recovers the returns implied by observable market holdings.

Incorporating Views The implementation of subjective information is one of the most distinctive features of the Black–Litterman framework. Since an investor's views can differ from the equilibrium implied by the market model (2.1), their influence is introduced via a $K \times N$ "pick" matrix $\mathbf{P}$:

$$\mathbf{P}\boldsymbol{\mu} \sim \mathcal{N}(\mathbf{Q}, \mathbf{\Omega}),$$

where $\mathbf{Q}$ collects the K view values and $\mathbf{\Omega}$ quantifies their individual uncertainties. The uncertainty matrix is assumed to be diagonal, reflecting independence among views, so all off-diagonal elements are zero. Each row of $\mathbf{P}$ represents a view: the assets not affected by that particular view have zero entries.

The Black–Litterman update accommodates both *absolute* and *relative* views. Absolute views fix the expected level of an asset's excess return, while relative views specify a spread between two or more assets. In the posterior, a relative view tilts the portfolio according to how the stated spread compares with the corresponding equilibrium spread (e.g., $\boldsymbol{\pi}_A - \boldsymbol{\pi}_B$, where $\boldsymbol{\pi}$ denotes the equilibrium excess return): if the view spread is smaller than the equilibrium spread, the optimizer favours the first asset; if it is larger, it favours the second. Absolute views, by contrast, shift weights depending on whether the stated level lies above or below the corresponding equilibrium return.

Posterior Distributions Applying Bayes' rule yields

$$\boldsymbol{\mu}|\mathbf{Q};\mathbf{\Omega} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathrm{BL}}, \mathbf{\Sigma}^{\boldsymbol{\mu}}_{\mathrm{BL}}),$$

with

$$\boldsymbol{\mu}_{\mathrm{BL}} = \big[(\tau\mathbf{\Sigma})^{-1} + \mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{P}\big]^{-1}\big[(\tau\mathbf{\Sigma})^{-1}\boldsymbol{\pi} + \mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{Q}\big], \tag{2.4}$$

$$\mathbf{\Sigma}^{\boldsymbol{\mu}}_{\mathrm{BL}} = \big[(\tau\mathbf{\Sigma})^{-1} + \mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{P}\big]^{-1}. \tag{2.5}$$

The corresponding distribution for future returns is therefore

$$\mathbf{X}|\mathbf{Q};\boldsymbol{\Omega} \sim \mathcal{N}\big(\boldsymbol{\mu}_{\mathrm{BL}}, \boldsymbol{\Sigma}_{\mathrm{BL}}\big).$$

In [13] Meucci proposes equivalent, but computationally robust formulas:

$$\boldsymbol{\mu}_{\mathrm{BL}} = \boldsymbol{\pi} + \tau\boldsymbol{\Sigma}\mathbf{P}'\big(\tau\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}' + \boldsymbol{\Omega}\big)^{-1}\big(\mathbf{Q} - \mathbf{P}\boldsymbol{\pi}\big), \tag{2.6}$$

$$\boldsymbol{\Sigma}_{\mathrm{BL}} = (1+\tau)\boldsymbol{\Sigma} - \tau^2\boldsymbol{\Sigma}\mathbf{P}'\big(\tau\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}' + \boldsymbol{\Omega}\big)^{-1}\mathbf{P}\boldsymbol{\Sigma}. \tag{2.7}$$

These representations avoid large matrix inversions and enhance numerical stability while leaving all model relationships unchanged. It is evident that the posterior return is just a modification of the equilibrium return 2.3 due to the effect of the views. The posterior covariance $\boldsymbol{\Sigma}_{\mathrm{BL}}$ embeds both return co-movement and parameter uncertainty, which can artificially inflate perceived risk when views are weak or uninformative, converging to $(1+\tau)\boldsymbol{\Sigma}$ in the worst case. For portfolio optimization, this would unduly penalize allocations without adding informational value, so the robust $\boldsymbol{\Sigma}$ is preferred for consistency and comparability. Table 2.1 summarizes all the variables used to define the posterior quantities, specifying their shape and according to the convention that an object with dimensions $(M \times 1)$ is a column vector.

Symbol	**Dimension**	**Meaning**
N	scalar	Number of assets
K	scalar	Number of views
$\boldsymbol{\Sigma}$	$N \times N$	Covariance matrix of asset excess returns
$\mathbf{w}_{\mathrm{mkt}}$	$N \times 1$	Market-capitalization weight vector
λ_{mkt}	scalar	Market risk-aversion coefficient
$\boldsymbol{\pi}$	$N \times 1$	Implied equilibrium excess returns
τ	scalar	Scaling factor for prior–mean uncertainty
$\mathbf{P}$	$K \times N$	Pick matrix
$\mathbf{Q}$	$K \times 1$	Expected excess return for each view
$\boldsymbol{\Omega}$	$K \times K$	Diagonal covariance of view errors
$\boldsymbol{\mu}_{\mathrm{BL}}$	$N \times 1$	Posterior excess-return vector

Table 2.1: Black–Litterman notation summary.

2.2 Model Implementation

This section explains how the Black-Litterman variables are computed. It is further divided into four subsection, each one focusing on a specific variable: equilibrium return, views, views uncertainty and posterior return.

2.2.1 Equilibrium Return

The construction of the equilibrium return vector requires three inputs: market weights, the risk-aversion coefficient, and the asset covariance matrix. The following subsections address the computation of the first two.

Market weights The computation of equilibrium returns in the Black–Litterman framework requires a set of benchmark portfolio weights, $\mathbf{w}_{\text{mkt}}$, which are taken to represent the allocation of the representative investor. In this study, five broad asset classes are defined: Equities, Fixed Income, Real Assets, Factors, and Volatility. Each portfolio constituent is assigned to one of these categories in order to reflect the heterogeneity of typical global portfolios.

As a reference for the class-level partition, the Global Market Portfolio (GMP) 2024, as reported by State Street Global Advisors [14], is employed. The GMP provides an aggregate snapshot of investable capital across asset classes and therefore serves as a natural benchmark for equilibrium allocation. The GMP composition is showed in Figure 2.1, while the asset classes used for the analysis are reported in Table 2.2. It should be noted that the GMP allocation is published as a point-in-time snapshot and is subject to change thereafter. In this study, the June 2024 release is used solely to define the basket partition of asset classes. This choice is intended as a representative approximation of global capital allocation, sufficient for structuring the equilibrium portfolio while acknowledging that the true composition evolves over time.

Asset Class	**Market Cap (USD trillions)**	**Share [%]**
Equities	78.00	51.64
Fixed Income	57.23	38.62
Real Assets	12.95	8.74
Total	**148.18**	**100**

Table 2.2: Global Market Portfolio allocation (2024) for the asset classes considered in this study. The outer market weights are derived from the reported market capitalizations.

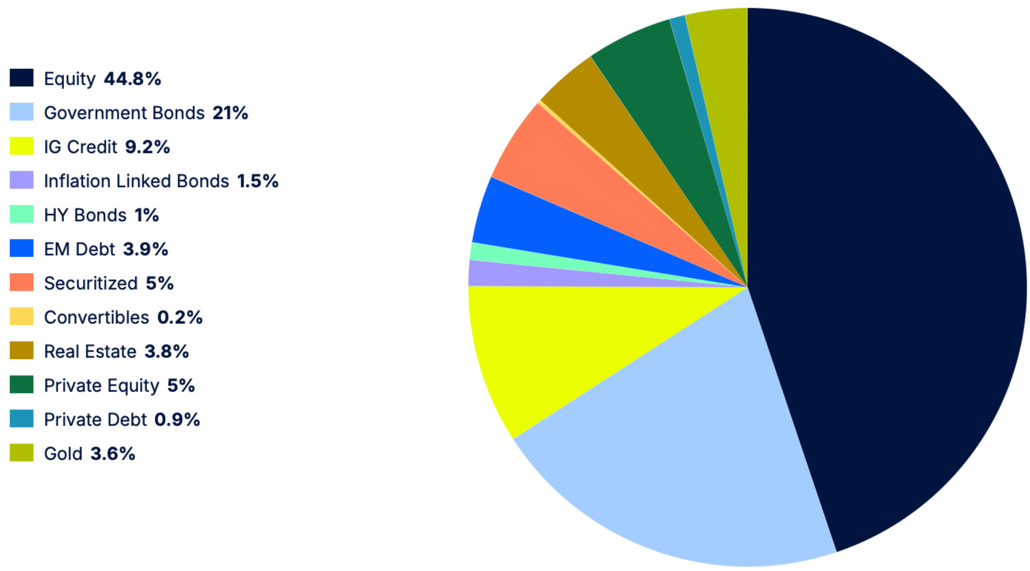


Figure 2.1: Global Market Portfolio composition. The total is valued at USD 175 trillion as of the end of June 2024.

For each asset, market capitalization is computed as suggested in [15]:

$$\text{Market Cap} = \text{Shares Outstanding} \times \text{Price}.$$

where Shares Outstanding represents the total number of issued shares held by shareholders, including insiders and institutional investors, but excluding treasury shares repurchased by the company. When information on shares outstanding is not available through Yahoo Finance, Assets Under Management (AUM) are used as a proxy as it represents the scale of investable capital effectively allocated by investors.

The Factors and Volatility categories are assigned zero aggregate weight in the benchmark portfolio. This choice reflects the fact that factor exposures and volatility products are synthetic constructs rather than securities with a direct investable market capitalization. Their role in the analysis is therefore limited to tilts relative to the benchmark, rather than equilibrium holdings.

Finally, within each basket, asset weights are computed proportionally to their respective market capitalizations. Each basket is then normalized to sum to 100%, and the basket-level weights are scaled according to the GMP allocation. The resulting $\mathbf{w}_{\text{mkt}}$ vector provides a coherent and economically grounded representation of the market equilibrium portfolio, which is subsequently used to derive implied equilibrium returns. All weights are reported in Table 2.3.

Market risk-aversion coefficient The λ_{mkt} parameter is obtained from the same unconstrained mean-variance optimization problem expressed in Equation 2.2. The general solution is

$$\boldsymbol{\mu} = 2\lambda\boldsymbol{\Sigma}\mathbf{w}$$

Left-multiplying both sides by $\mathbf{w}'$ and defining

$$\sigma^2_{\text{mkt}} := \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \quad \text{and} \quad \mu_{\text{mkt}} := \mathbf{w}'\boldsymbol{\mu}$$

gives the expression for the market risk-aversion coefficient:

$$\lambda_{\text{mkt}} = \frac{\mu_{\text{mkt}}}{\sigma^2_{\text{mkt}}}$$

In practice, the SPDR S&P 500 ETF Trust (`SPY`) is used as a proxy for the U.S. equity market. The required inputs are replaced by the following estimators:

$$\hat{\mu}_{\text{mkt}} = \frac{1}{T}\sum_{t=1}^{T}(r_t - r_{f,t}) \quad \hat{\sigma}^2_{\text{mkt}} = \frac{1}{T-1}\sum_{t=1}^{T}\Big[(r_t - r_{f,t}) - \hat{\mu}_{\text{mkt}}\Big]^2$$

where r_t denotes the weekly total return of `SPY`, $r_{f,t}$ is the weekly risk-free rate, and T is the number of observations. From SPY weekly excess returns over the 2015–2025 sample, the estimate is

$$\lambda_{\text{mkt}} = 1.91$$

which will be used in the computation of equilibrium returns.

Equilibrium return Now the equilibrium returns vector can be easily computed using Equation 2.3. It's entries will be displayed and commented later, together with the posterior returns in Table 2.4.

Ticker	Asset Class	Market Cap AUM (USD bn)	w_{int} [%]	w_{class} [%]	w_{mkt} [%]
Equities	QQQ	185.55	59.03	52.64	31.07
	SPHQ	14.59	4.64		2.44
	XLE	15.24	4.85		2.55
	XLF	42.28	13.45		7.08
	XLP	16.87	5.37		2.83
	XLU	12.68	4.03		2.12
	XLV	27.12	8.63		4.54
Fixed Income	LQD	31.21	35.10	38.62	13.56
	HYG	15.11	16.99		6.56
	IEF	13.72	15.43		5.96
	TLT	9.61	10.80		4.17
	TIP	19.26	21.66		8.37
Real Assets	GLD	79.32	69.83	8.74	6.10
	VNQ	31.90	28.08		2.45
	DBC	2.38	2.09		0.18
Factors	IWM	–	0.00	0.00	0.00
	SPVU	–	0.00		0.00
	MTUM	–	0.00		0.00
	BAB_p	–	0.00		0.00
Volatility	VIXM	–	0.00	0.00	0.00

Table 2.3: Market capitalization (or AUM), internal weights, and prior weights by asset class and ticker. Class weights are shown once per category. All weights are expressed in percentages.

2.2.2 Views Definition

For the present analysis, the following views were imposed[1]:

View 1: *Gold will deliver a +7.0% excess return over the risk-free rate over the next year.* (Confidence level 60%)

View 2: *IPS ETF will outperform cash, reflecting a rising real yields by 2% in the next 12 months.* (Confidence level 45%)

View 3: *10-yr Treasuries is expected to underperform High-yield credit by 5% in the next year.* (Confidence level 55%)

View 4: *The US large-cap growth and real estate will outperform IG credit by 1.2% in the next year.* (Confidence level 50%)

[1] All the values and confidence levels are made up, and are not based on any actual observation. They were only chosen for didactical purposes.

View 5: *Financial and Healthcare cyclicals are expected to outperform by 3.5% the basket of Energy, Staples & Utilities cyclicals in the next year.* (Confidence level 65%)

Absolute views (e.g., Views 1 and 2) are represented in the pick matrix by placing a +1 in the position corresponding to the asset of interest, following the numbering convention in Table 1.6. Relative views involving exactly two assets (e.g., View 3) are encoded with a +1 for the asset expected to outperform and a −1 for the other.

For relative views that include multiple assets on either side (e.g., Views 4 and 5), the pick matrix does not assign equal weights within each group. Instead, it uses the corresponding market weights, normalized so that the coefficients on the long side sum to +1 and those on the short side sum to −1. This preserves the capitalization structure of the assets within each basket, as suggested by Idzorek [16].

The $\mathbf{P}$ matrix encoding these views rounded to the second decimal place is:

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0.85 & 0 & 0 & 0 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.43 & 0.62 & -0.25 & -0.33 & 0.38 & 0 & 0 \end{bmatrix}$$

The magnitudes of the views are collected in the $\mathbf{Q}$ vector, while the confidence level of each view is stored in the vector $\mathbf{C}$, which will determine the diagonal entries of $\mathbf{\Omega}$ (see Section 2.2.3). The view and confidence vectors are respectively :

$$\mathbf{Q} = \begin{bmatrix} 0.070 \\ 0.020 \\ 0.050 \\ 0.012 \\ 0.035 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 0.60 \\ 0.45 \\ 0.55 \\ 0.50 \\ 0.65 \end{bmatrix}$$

Note that the views are expressed in annual terms, whereas the data have been resampled to weekly frequency. Accordingly, each entry of $\mathbf{Q}$ is divided by the number of weeks in a year (52) to ensure consistency of units.

2.2.3 Views Uncertainty

Two quantities in the Black–Litterman model still require proper definition: the scalar τ and the uncertainty matrix $\mathbf{\Omega}$. The parameter τ controls the confidence in the equilibrium prior: the smaller its value, the stronger the confidence in the equilibrium returns. The literature offers various approaches for setting τ, but in practice it typically requires an ad-hoc calibration.

Meucci in [13] proposes an alternative "market formulation" of the BL model, in which views are expressed directly on the market rather than on returns. This reformulation removes the scalar τ from the posterior distributions altogether.

Idzorek in [16] presents another solution that addresses the τ issue while providing an intuitive interpretation of the diagonal elements of $\mathbf{\Omega}$. His approach considers views one at a time, and exploits the approximation

$$\mathbf{Tilt}_k \approx \left(\mathbf{w}_{k,\,100\%} - \mathbf{w}_{\text{mkt}}\right) \cdot C_k,$$

where C_k denotes the confidence level in view k, $\mathbf{w}_{k,100\%}$ is the portfolio that would be held if that view were known with complete certainty, and $\mathbf{w}_{\text{mkt}}$ is the market-capitalization portfolio. Under this framework, a view's portfolio weight can be interpreted as the deviation from the certainty-view portfolio, scaled by the view's confidence level.

Idzorek provides a step-by-step procedure to compute the diagonal elements of $\mathbf{\Omega}$, reported in Appendix B. A key advantage of this method is that τ is treated as a constant, and in the posterior distributions it only appears through the ratio $\mathbf{\Omega}/\tau$. Consequently, changing τ merely rescales $\mathbf{\Omega}$ without affecting other quantities in the model.

2.2.4 Posterior Returns and Unconstrained BL Allocation

Posterior returns Once the equilibrium returns, views, and view uncertainties are specified, the posterior return vector $\boldsymbol{\mu}_{\text{BL}}$ is obtained from Equation 2.6. These posterior returns combine the market-implied equilibrium with the subjective views, weighted by their associated confidence levels. They represent the key output of the Black–Litterman model and provide the inputs for portfolio optimization.

In Table 2.4, posterior returns are compared with their equilibrium counterparts, and the resulting tilts ($\boldsymbol{\mu}_{\text{BL}} - \boldsymbol{\pi}$) are reported. Overall, tilts are modest (all below 3%), which indicates that the specified views remain broadly consistent with the equilibrium prior. The largest adjustment occurs for `GLD`, where the view was deliberately exaggerated to illustrate its effect on both posterior returns and subsequent allocations. Due to the correlation structure among assets, securities not directly subject to views are nonetheless affected. A clear example is provided by `BAB_p` and `VIXM`, whose equilibrium and posterior returns are both negative, reflecting their strong negative correlation with most other assets. Since their market weights were set to zero, their equilibrium returns are entirely determined by the interaction of the covariance matrix with the remainder of the portfolio. The same mechanism applies to the other zero-weighted assets (`IWM`, `SPVU`, and `MTUM`), although in their case correlations with the investable universe lead to positive equilibrium returns.

Unconstrained BL Allocation Given the Black–Litterman posterior excess return $\boldsymbol{\mu}_{\text{BL}}$ and the covariance matrix $\mathbf{\Sigma}$, the unconstrained mean–variance (MV) portfolio solves Equation 2.2, where the return vector is replaced by the posterior returns vector $\boldsymbol{\mu}_{\text{BL}}$. The analytical solution is

$$\mathbf{w}^* = \frac{1}{2\lambda}\mathbf{\Sigma}^{-1}\boldsymbol{\mu}_{\text{BL}} \tag{2.8}$$

This is the simplest MV allocation: it is fully invested only up to a scaling and it does not impose budget or box constraints (short positions and leverage can arise). It is presented here as a concise benchmark, inspired by the expository treatments of Idzorek and Meucci, to illustrate how BL posterior returns propagate into optimal weights. In Chapter 3, constraints $\sum_i \mathrm{w}_i = 1$ and $0 \le \mathrm{w}_i \le 1$ are enforced and the MV solution is compared with alternative optimization criteria.

Ticker	$\boldsymbol{\mu}_{\mathrm{BL}}$	$\boldsymbol{\pi}$	$\boldsymbol{\mu}_{\mathrm{BL}} - \boldsymbol{\pi}$
IWM	9.95	8.21	1.73
SPVU	8.74	7.06	1.69
MTUM	8.54	7.51	1.02
BAB_p	−4.38	−4.09	−0.28
DBC	3.30	2.89	0.41
GLD	4.84	1.98	2.86
HYG	4.81	3.50	1.31
IEF	0.82	0.55	0.27
LQD	4.55	2.98	1.57
QQQ	8.47	8.08	0.39
SPHQ	7.57	6.62	0.94
TIP	1.61	1.01	0.60
TLT	1.98	1.09	0.89
VIXM	−10.45	−9.38	−1.07
VNQ	9.63	7.62	2.01
XLE	6.75	6.63	0.12
XLF	9.20	7.19	2.00
XLP	4.77	4.41	0.36
XLU	6.88	5.52	1.36
XLV	6.90	5.51	1.38

Table 2.4: Posterior and equilibrium returns, together with their differences, expressed in percentage terms.

To study the sensitivity to risk exposure, three representative levels of risk aversion are considered:

Kelly	Average	Trustee
0.005	1.12	3.0

Qualitatively, since $\mathbf{w}^* \propto \lambda^{-1}$:

- **Kelly (very low λ).** Large position sizes and stronger tilt toward assets with higher BL Sharpe ratios; allocations can become concentrated and more volatile.
- **Average (λ moderate).** Intermediate risk taking; tilts remain visible but less extreme, often closer to market-like exposures.
- **Trustee (high λ).** Smaller absolute weights, greater diversification and reduced active bets; allocations tend to move toward lower-variance combinations.

For each profile, the results are reported in Table 2.5, together with equal and market weights for reference. Equal weights correspond to an identical allocation across assets ($\mathrm{w}_i = 5\%$). A first observation is that the unconstrained optimization leaves unchanged the positions of assets with zero market weight. This occurs because, in the absence of market capitalization, prior expected returns are determined entirely by covariances with the remaining assets, leading to posterior returns that do not generate active allocations in the unconstrained solution. Although some factor proxies exhibit sizeable positive posterior returns (e.g., IWM, $\approx 10\%$), the optimizer assigns them zero weights owing to their collinearity with equity ETFs already included in the portfolio. Once correlations are accounted for, these exposures do not provide incremental diversification benefits, and their contribution is absorbed by correlated assets. As expected, the Kelly profile produces extreme allocations, with highly leveraged long positions and a few shorts, yielding a markedly imbalanced portfolio. The Average and Trustee profiles deliver more moderate allocations, although the Average profile still implies gross leverage exceeding 270%. The impact of the views is clearly visible across investor types: outperforming assets receive substantially higher weights than their prior market shares (notably GLD, HYG, TIP, XLF, and XLV), while underperforming assets are shorted in the unconstrained solution (IEF, XLE, XLP, and XLU). Other holdings are adjusted accordingly but without dramatic shifts. Finally, it is interesting to note that, despite being among the outperforming assets, QQQ has a negative tilt.

A visual comparison is provided in Figure 2.2, where normalized weights are shown to improve interpretability. Since the only difference between investor types is the risk-aversion parameter λ, which scales overall leverage but not relative allocations, the normalized weights coincide across profiles.

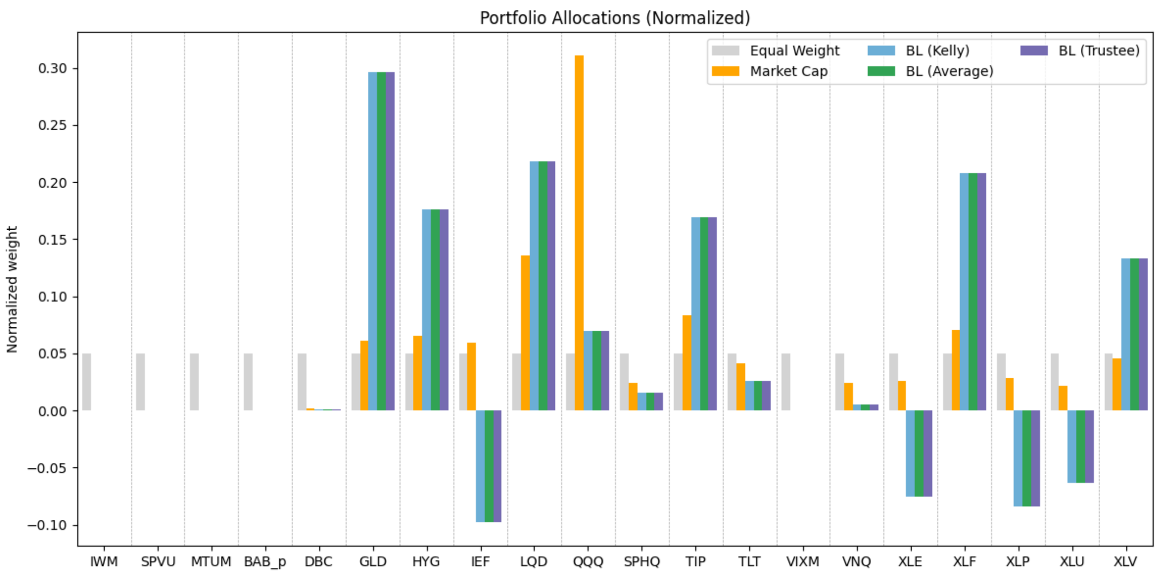


Figure 2.2: Comparison of normalized portfolio weights. Grey bars represent equal weights, yellow bars market weights, blue bars Black–Litterman Kelly weights, green bars Black–Litterman Average weights, and violet bars Black–Litterman Trustee weights.

Ticker	Equal [%]	Mkt [%]	BL Weights [%]			Weight tilts [%]		
	w_{eq}	w_{mkt}	Kelly	Average	Trustee	Kelly	Average	Trustee
IWM	5.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
SPVU	5.00	0.00	-0.00	-0.00	-0.00	-0.00	0.00	0.00
MTUM	5.00	0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00
BAB_p	5.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
DBC	5.00	0.18	69.77	0.11	0.31	69.59	0.13	-0.07
GLD	5.00	6.10	18123.10	29.66	80.91	18117.00	74.80	24.10
HYG	5.00	6.56	10765.83	17.62	48.06	10759.26	41.50	11.38
IEF	5.00	5.96	-5990.17	-9.80	-26.74	-5996.13	-32.70	-15.94
LQD	5.00	13.56	13352.55	21.85	59.61	13339.00	46.05	8.70
QQQ	5.00	31.07	4264.52	6.98	19.04	4233.44	-12.03	-23.96
SPHQ	5.00	2.44	931.86	1.53	4.16	929.42	1.72	-0.89
TIP	5.00	8.37	10359.86	16.96	46.25	10351.49	37.88	8.90
TLT	5.00	4.17	1591.36	2.60	7.10	1587.19	2.93	-1.52
VIXM	5.00	0.00	0.00	0.00	0.00	0.00	0.00	-0.00
VNQ	5.00	2.45	336.78	0.55	1.50	334.33	-0.95	-1.89
XLE	5.00	2.55	-4623.68	-7.57	-20.64	-4626.24	-23.19	-10.26
XLF	5.00	7.08	12722.73	20.82	56.80	12715.65	49.72	14.12
XLP	5.00	2.83	-5121.09	-8.38	-22.86	-5123.92	-25.69	-11.36
XLU	5.00	2.12	-3848.62	-6.30	-17.18	-3850.74	-19.31	-8.54
XLV	5.00	4.54	8162.30	13.36	36.44	8157.76	31.90	9.06
Sum	100.00	100.00	61097.11	272.75	101.83	60997.11	172.75	1.83

Table 2.5: Equal, market, and Black–Litterman weights (Kelly, Average, Trustee profiles), together with weight tilts relative to the market benchmark ($\mathbf{w}_{\mathrm{BL},i} - \mathbf{w}_{\mathrm{mkt}}$, with i denoting the investor type). All figures are expressed in percentage terms.

Chapter 3

Constrained Portfolio Optimization

Once the posterior excess returns $\boldsymbol{\mu}_{\mathrm{BL}}$ are obtained, the portfolio optimization problem can be addressed through different methods. In the first part of this chapter three approaches are introduced: mean–variance optimization, maximum Sharpe ratio, and minimum Expected Shortfall. A detailed description of the numerical algorithms can be found in the accompanying Notebook. Among the three, the mean–variance problem is the only one that depends on the risk-aversion coefficient. Throughout this section, the analysis refers to the Average investor, with $\lambda = 1.12$.

All methods are solved under the same set of constraints: the budget constraint, requiring the portfolio weights to sum to one, and the box constraint, requiring weights to remain non-negative:

$$\sum_{i=1}^{N} \mathrm{w}_i = 1, \tag{3.1}$$

$$0 \leq \mathrm{w}_i \leq 1 \quad \forall\, i = 1, 2, \ldots, N, \tag{3.2}$$

where N denotes the number of assets. The upper bound $\mathrm{w}_i \leq 1$ is actually redundant when the budget and non-negativity constraints are satisfied, but it is retained as a diagnostic to capture small numerical violations.

3.1 Optimization Methods

Mean-Variance This is the simplest optimization method, and it consists in the maximization of the utility function:

$$U(\mathbf{w}) = \mathbf{w}'\boldsymbol{\mu}_{\mathrm{BL}} - \lambda \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$$

This is the formulation originally proposed by Markowitz in his Modern Portfolio Theory paper [9]. The intuition is rather basic: the objective is to maximize the portfolio return while limiting its risk (represented by the variance). Varying the risk aversion parameter it is possible to draw the *efficient frontier*, the curve of portfolios that satisfy the optimization conditions. Since here λ is fixed, the result of this procedure leads to a single point lying on the efficient frontier.

Even thought the current implementation employs the Black-Litterman posterior distribution instead of the plain portfolio returns, this method still suffers from the critical drawbacks already pointed out in Chapter 2.

After the introduction of constraints 3.1 and 3.2 the problem becomes a Quadratic Program (QP), which can't be solved using directly Equation 2.8, but must be solved numerically. Several techniques and algorithms exist to tackle such problems. The Python library `CVXOPT` provides a reliable and efficient solution of a small portfolio like the one under study. Numerical quality of the solution is assessed through the solver's optimality conditions, while feasibility is verified against the budget and box constraints.

Maximum Sharpe Ratio This approach aims at finding the portfolio weights that maximize Sharpe ratio. Therefore, the utility function in this case is

$$\mathrm{SR}(\mathbf{w}) = \frac{\mathbf{w}'\boldsymbol{\mu}_{\mathrm{BL}}}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}}$$

The geometric interpretation of this method can be easily inferred from the mean-variance one. The Capital Market Line (CML) is the straight line from the risk-free asset tangent to the efficient frontier, and the tangency point is the maximum-Sharpe portfolio (which is fact also called *tangency portfolio*). Figure 3.1 helps to visualize both the efficient frontier and the CML. For this reason, this optimization procedure is independent from the risk-aversion parameter.

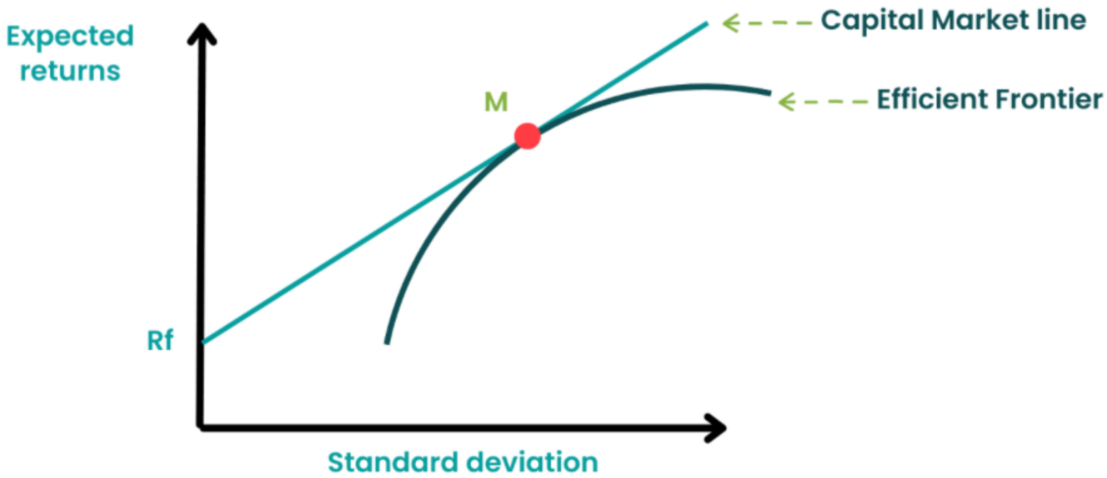


Figure 3.1: Efficient frontier and Capital Market Line. The image is taken from [17].

This approach shares most of the limitations of the mean-variance optimization, in particular the extreme sensitivity to input parameters. In this case, a closed form solution exists after applying the budget constraint:

$$\mathbf{w}^* = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{\mathrm{BL}}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_{\mathrm{BL}}}$$

Then again, if the box constraint (3.2) is applied as well, the analytical solution no longer exists. This is not even quadratic programming, due to the presence of the square root in the denominator of the utility function. Luckily, this problem belongs to the class of *fractional programs*, which benefit from several numerical approaches. D. Palomar in [18] suggests some alternatives like the bisection method, the Schaible Transform method and the Dinkelbach algorithm, which is the one that has been implemented for this project. This approach allows to turn the original problem into an iterative routine of simpler convex problems of the form

$$\underset{\mathbf{w}}{\text{maximize}}\, \mathbf{w}'\boldsymbol{\mu}_{\mathrm{BL}} - y_{(k)}\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}$$

subject to

$$\sum_i \mathrm{w}_i = 1, \quad \mathrm{w}_i > 0 \;\; \forall\, i = 1, 2, \ldots, N$$

where the parameter

$$y_{(k)} = \frac{\mathbf{w}'_{(k)}\, \boldsymbol{\mu}_{\mathrm{BL}}}{\sqrt{\mathbf{w}'_{(k)}\boldsymbol{\Sigma}\,\mathbf{w}_{(k)}}}$$

is be updated at each iteration k. The iterations stop when the the absolute difference between two consecutive values of y is below a fixed tolerance set to $\epsilon_{tol} = 10^{-6}$. In other words,

$$r = |y_{(k+1)} - y_{(k)}| < \epsilon_{tol}$$

The Dinkelbach residual r provides a natural convergence diagnostic.

Risk Budgeting Using risk-based portfolios is becoming increasingly attractive, as they allow for explicit control of the distribution of risk across portfolio components. Their properties and advantages have been explored in recent works such as [19, 20, 21], with the latter being most closely related to the Expected Shortfall risk budgeting approach adopted here. The purpose of this optimization is to determine a portfolio allocation such that each asset contributes to the portfolio's total Expected Shortfall (ES) in predefined proportions, referred to as the risk budgets. This method falls under the broader class of risk-based portfolio construction techniques, where allocations are not chosen to maximize return for a given risk, but instead to control the distribution of risk contributions across assets. The risk measure adopted here is the Expected Shortfall at a given confidence level $1 - c$. The marginal ES-contribution for asset i is

$$\mathrm{RC}_i(\mathbf{w}) = \mathrm{w}_i \left(-\boldsymbol{\mu}_{\mathrm{BL},i} + \varphi_{\mathrm{ES}} \cdot \frac{(\boldsymbol{\Sigma}\mathbf{w})_i}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}} \right) \tag{3.3}$$

where

$$\varphi_{\mathrm{ES}} = \frac{\phi(z_c)}{c}, \quad z_c = \Phi^{-1}(1 - c)$$

is the tail risk multiplier under the Normal assumption, with $\phi(\cdot)$ and $\Phi^{-1}(\cdot)$ denoting the standard normal pdf and quantile function (inverse cdf), respectively. This factor converts portfolio volatility into Expected Shortfall at the tail probability c. Following the Basel Fundamental Review of the Trading Book (FRTB) standard, the confidence level is set to $1 - c = 97.5\%$ [22].
The goal of risk budgeting is to choose $\mathbf{w}$ such that:

$$\frac{\mathrm{RC}_i(\mathbf{w})}{\sum_j \mathrm{RC}_j(\mathbf{w})} = \beta_i$$

where $\boldsymbol{\beta}$ is chosen to match the market weight vector $\mathbf{w}_{\mathrm{mkt}}$. This leads to a non-linear constrained optimization problem:

$$\min_{\mathbf{w}} \sum_{i} \left(\frac{\mathrm{RC}_i(\mathbf{w})}{\sum_j \mathrm{RC}_j(\mathbf{w})} - \beta_i \right)^2$$

subject to

$$\sum_{i=0}^{N} \mathrm{w}_i = 1, \quad 0 \leq \mathrm{w}_i \leq 1 \;\; \forall\, i = 1, 2, \ldots, N$$

The above is solved numerically using Sequential Least Squares Quadratic Programming (SLSQP), which handles both equality and bound constraints. The optimization's numerical quality is measured through the deviation of realized risk contributions from the target budgets, while feasibility is checked against the portfolio constraints.

3.2 Optimization Results

This section presents the empirical results of the constrained portfolio optimization procedures. The analysis is structured as follows. First, the numerical performance of the solvers is assessed in terms of feasibility, convergence quality, and robustness. Second, the resulting allocations are compared across methods. Third, the distribution of risk contributions is examined, followed by an evaluation of portfolio-level performance statistics such as Sharpe ratio, volatility, and diversification ratio. Finally, the factor exposures of the optimized portfolios are analyzed to provide an economic interpretation of the allocations. The performance of each optimization method is evaluated along three dimensions:

- **Feasibility**: solutions are checked against the budget and box constraints. Reported quantities are the residual of the budget constraint and the maximum violation of the lower and upper bounds on the portfolio weights.
- **Numerical quality**: method-specific diagnostics are used to verify optimality. For quadratic programmes, the primal–dual objective gap is reported[1] ; for Dinkelbach iterations, the outer residual, inner optimality condition, and constraint violations are monitored; for risk budgeting, the value of the objective function is reported, which measures the squared distance between realized and target risk contributions.
- **Robustness**: stability is assessed by re-solving after small perturbations of $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Weight sensitivity is summarized using the ℓ_1 and ℓ_∞[2] norms of deviations from the baseline solution, based on five independent perturbations for each method.

[1] The primal–dual gap is defined as $|p^* - d^*|$, where p^* is the value of the optimization problem at the candidate solution and d^* is the value of the associated dual problem.

[2] For a perturbation $\Delta\mathbf{w}$, the ℓ_1 norm is defined as $|\Delta\mathbf{w}|_1 = \sum_i |\Delta w_i|$, measuring the aggregate deviation across assets, while the $\ell\infty$ norm is defined as $|\Delta\mathbf{w}|_\infty = \max_i |\Delta w_i|$, measuring the largest single-asset deviation.

More details about each metric definition are reported in the accompanying Notebook.

The second part of the chapter presents the optimization results, including both the allocations and the performance metrics.

3.2.1 Performance Metrics

Mean–Variance The solver terminated successfully with an optimal solution after 8 iterations. Feasibility checks confirm that both the budget and box constraints are satisfied, with no violations observed. The primal and dual objectives coincide up to a duality gap of $2.7 \cdot 10^{-10}$, well below the prescribed tolerance, ensuring numerical optimality. In robustness assessment, the iteration count remained stable at 8, and no feasibility or bound violations occurred. Weight sensitivity, measured as deviations from the baseline allocation, had median values of 0.11 (aggregate) and 0.03 (largest single asset). These small deviations indicate that the solution is numerically stable with respect to small input perturbations.

Maximum Sharpe Ratio The Dinkelbach algorithm converged in 5 outer iterations, solving 189 inner subproblems (39 iterations in the final inner loop). Feasibility was preserved throughout, with no budget or box violations. Numerical quality is confirmed by the convergence Dinkelbach residual of $1.8 \cdot 10^{-8}$, the inner optimality condition at $9.8 \cdot 10^{-9}$, and zero inner constraint violation, all well below the stopping tolerances. Under perturbations of $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, the algorithm remained stable: the same outer iteration count was required, and feasibility was consistently preserved. Weight sensitivity showed median deviations of 0.12 (aggregate) and 0.05 (largest single asset). These diagnostics suggest that the maximum Sharpe optimization is computationally more demanding than the mean–variance case, but still delivers robust and feasible solutions.

Risk Budgeting The SLSQP routine converged in 28 iterations. All constraints were satisfied within numerical precision, with no budget or box violations. The objective value at termination was essentially zero, indicating that the realized risk contributions matched the target budgets to within tolerance. Robustness checks confirmed the stability of the solution: the iteration count was unchanged, and feasibility was consistently preserved. Portfolio weights exhibited very limited sensitivity, with median deviations of 0.005 (aggregate) and 0.002 (largest single asset). Compared with mean–variance and maximum Sharpe optimization, the risk-budgeting problem proved computationally inexpensive and numerically stable.

Table 3.1 summarizes the performance diagnostics across the three optimization methods. All algorithms converged successfully within tight numerical tolerances and satisfied the portfolio constraints. As expected, the maximum Sharpe ratio optimization was the most computationally demanding, while the risk-budgeting approach was both the fastest and the most stable. Robustness checks indicate that the resulting allocations are not overly sensitive to small perturbations of $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, with median deviations in portfolio weights remaining below 0.12 in ℓ_1 norm and 0.05 in ℓ_∞ norm. Overall, the diagnostics confirm that all optimization routines deliver feasible, numerically reliable, and stable solutions.

Method	Iterations	Gap / Residual	Feasibility	$\|\|\Delta\mathbf{w}\|\|_1$	$\|\|\Delta\mathbf{w}\|\|_\infty$
Mean–Variance	8	$2.7 \cdot 10^{-10}$	0.00	0.11	0.03
Max Sharpe	5 / 189	$1.9 \cdot 10^{-8}$	0.00	0.12	0.05
Risk Budgeting	28	≈ 0.00	0.00	0.005	0.002

Table 3.1: Performance diagnostics across optimization methods. Feasibility refers to the maximum budget/box violation; robustness is measured by the median ℓ_1 and ℓ_∞ deviations of portfolio weights under input perturbations. For the maximum Sharpe method both inner and outer number of iterations is reported.

3.2.2 Allocation Comparison

Allocation results are reported in Table 3.2 and Figure 3.2, in numerical and graphical form, respectively. Since all optimizations were performed under budget and box constraints, the resulting portfolios are fully invested and contain only non-negative weights.

Asset	$\mathbf{w}_{\text{eq}}$	$\mathbf{w}_{\text{mkt}}$	$\mathbf{w}_{\text{mv}}$	$\mathbf{w}_{\text{ms}}$	$\mathbf{w}_{\text{rb}}$
IWM	5.00	0.00	13.22	0.22	0.00
SPVU	5.00	0.00	0.00	0.06	0.00
MTUM	5.00	0.00	6.41	0.18	0.00
BAB_p	5.00	0.00	0.00	0.12	0.00
DBC	5.00	0.18	0.00	0.04	0.25
GLD	5.00	6.10	25.12	25.10	7.85
HYG	5.00	6.56	0.00	12.99	6.84
IEF	5.00	5.96	0.00	0.19	15.79
LQD	5.00	13.56	0.00	19.32	13.53
QQQ	5.00	31.07	14.85	8.91	16.01
SPHQ	5.00	2.44	0.00	0.16	1.50
TIP	5.00	8.37	0.00	7.17	18.49
TLT	5.00	4.17	0.00	2.14	5.67
VIXM	5.00	0.00	0.00	1.17	0.00
VNQ	5.00	2.45	24.41	0.06	1.17
XLE	5.00	2.55	0.00	0.02	1.61
XLF	5.00	7.08	15.99	13.75	4.31
XLP	5.00	2.83	0.00	0.04	2.39
XLU	5.00	2.12	0.00	0.03	1.32
XLV	5.00	4.54	0.00	8.33	3.27

Table 3.2: Portfolio weights (in %) across optimization methods: mean-variance (mv), maximum Sharpe ratio (ms) and risk budgeting (rb). Equal and market weights are reported as reference.

The concentration effect of the mean–variance method is clearly visible: only 6 out of 20 assets receive positive allocations. Unlike the unconstrained case, however, some assets with zero market weight still enter the portfolio in both the

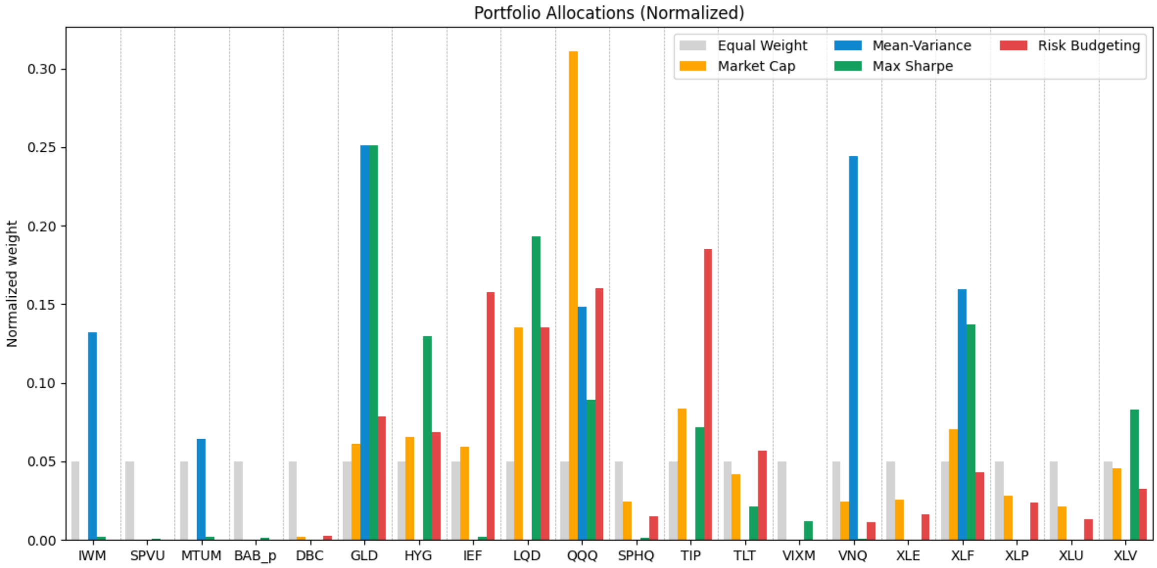


Figure 3.2: Allocation comparison among the different optimization methods. Grey: equal weights; yellow: market weights; blue: mean-variance; green:max sharpe; red: risk budgeting.

mean–variance and maximum Sharpe solutions. By contrast, the risk-budgeting approach assigns zero weight to these assets. In this project, the Expected Shortfall risk-budgeting formulation was implemented with the Black–Litterman posterior mean $\boldsymbol{\mu}_{\mathrm{BL}}$ explicitly included in the marginal contributions (see Equation 3.3). This ensures that the resulting allocation remains consistent with the BL framework and does not disregard the posterior return information. It should be noted, however, that in many practical applications the mean component is deliberately omitted from the ES decomposition. The rationale is twofold. First, the objective of risk budgeting is to control the distribution of risk, and including expected returns mixes a return estimate into a risk measure. Second, mean estimates are typically far less reliable than volatility and covariance estimates, so excluding them avoids introducing additional noise into the optimization. When $\boldsymbol{\mu}$ is excluded, the ES risk-budgeting solution becomes a purely risk-based allocation, independent of BL views, and therefore similar in spirit to equal-risk-contribution portfolios. Retaining $\boldsymbol{\mu}_{\mathrm{BL}}$ in the formulation may lead to allocations that appear counterintuitive, such as relatively large weights in assets with weak expected performance but low marginal ES (e.g., `IEF`). These outcomes are not artifacts of the implementation, but structural implications of enforcing risk budgets when both return and risk components are taken into account. Weight tilts with respect to the market are reported as lollipop plots in Figure 3.3. Across all three optimization methods, `QQQ` exhibits a consistently negative tilt relative to the market portfolio. This is particularly noteworthy because `QQQ` held the largest prior market weight, and the reduction can be interpreted as a correction of concentration risk: in the equilibrium portfolio, a single large-cap growth index carries disproportionate weight, amplifying exposure to equity market beta. Once the optimizations are applied—whether targeting variance, Sharpe ratio, or risk budgets—the models tend to diversify away from this concentration. Importantly, this occurs even in the mean–variance portfolio, which is otherwise rather concentrated, confirming that the underweight in QQQ is a structural effect rather than a peculiarity of a single method. Other tilt patterns are also consistent with the mechanics of each optimization. For instance, assets such as GLD, HYG, and LQD receive strong positive tilts under the maximum Sharpe portfolio, reflecting their role in improving the risk–return trade-off once posterior expectations are considered. In contrast, the risk-budgeting allocation shows smaller tilts overall, but tends to favor lower-volatility instruments such as Treasuries (IEF, TIP) in order to meet its target distribution of Expected Shortfall contributions. Meanwhile, some sector ETFs with negligible prior market weights (IWM, MTUM) remain effectively excluded from the risk-budgeting solution, while receiving minor positive allocations in the mean–variance and maximum Sharpe portfolios. Overall, the tilt patterns show that the optimization methods systematically reduce exposure to the dominant market component. The maximum Sharpe and risk-budgeting allocations reallocate towards assets that improve diversification or balance risk contributions more evenly, while the mean–variance solution remains relatively concentrated despite satisfying the formal optimization criterion.

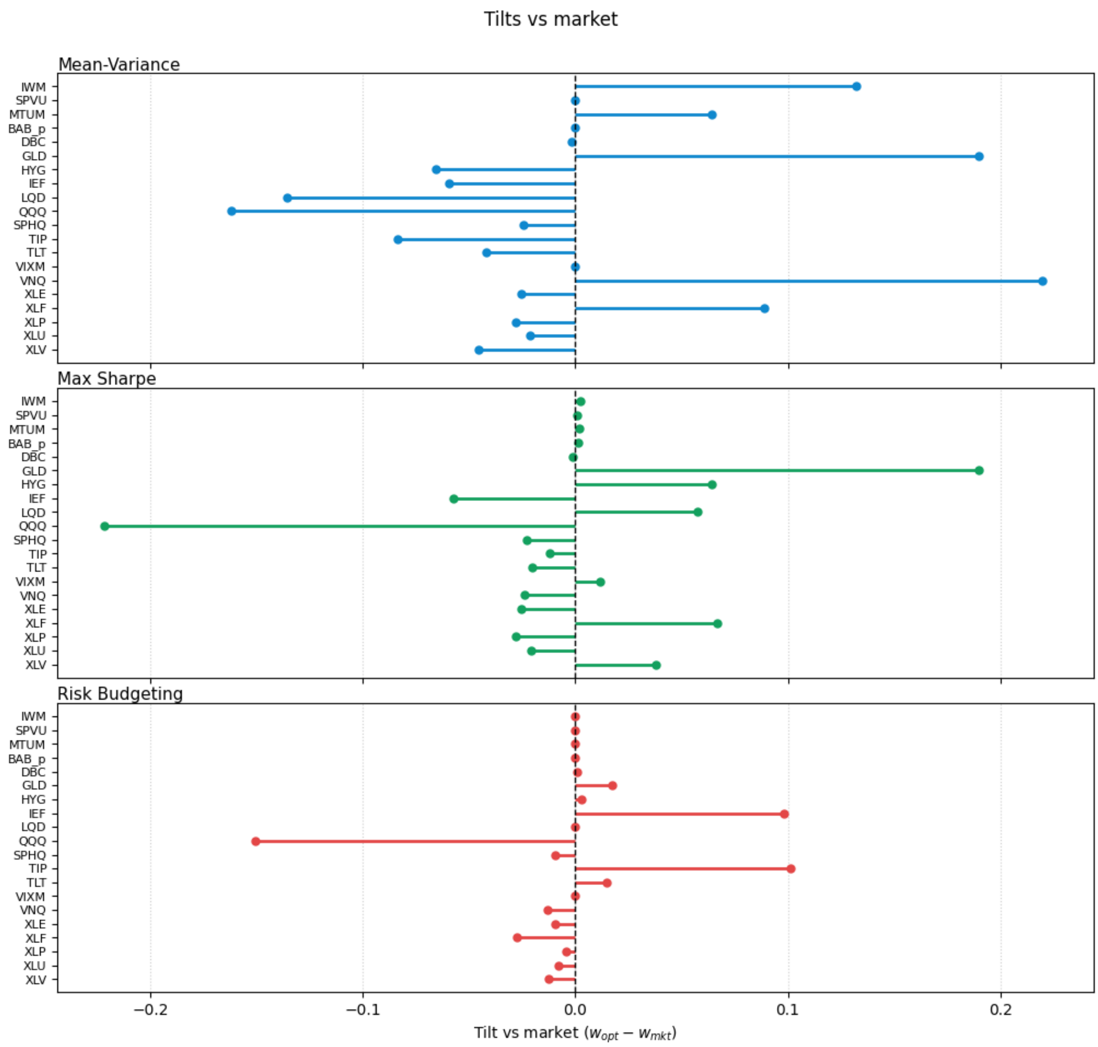


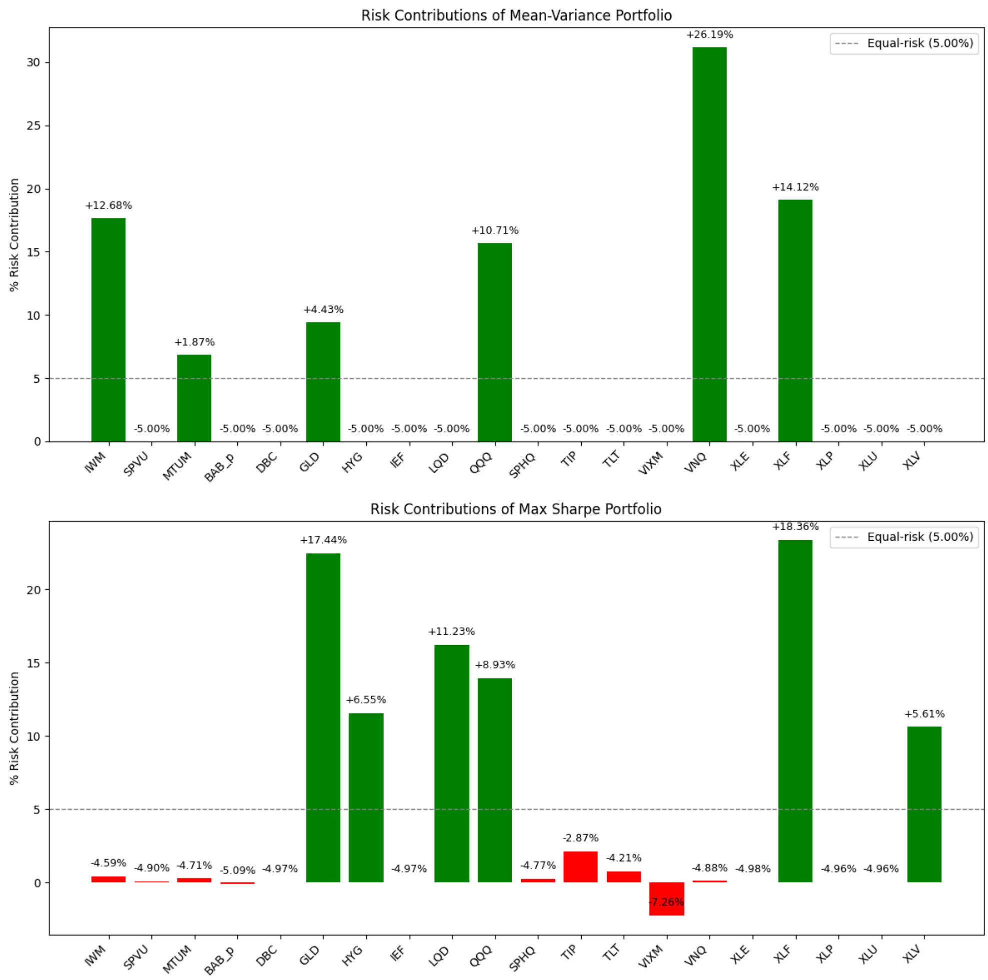
Figure 3.3: Optimal weight tilts with respect to market weights. Top: mean-variance; middle: max Sharpe ratio; bottom: risk budgeting.

3.2.3 Risk Contributions

Asset risk contributions are computed as

$$\mathcal{RC}_i = \frac{\mathbf{w}_i(\Sigma\mathbf{w})_i}{\mathbf{w}'\Sigma\mathbf{w}}$$

which measures the percentage share of total portfolio variance attributable to each asset. Results for all allocation methods are reported in Figure 3.4. As already discussed, the mean–variance portfolio is highly concentrated, allocating to only six assets, each of which contributes more than 5% (the equal-risk benchmark). The dominant driver of risk is `VNQ`, well above the equal-risk line. The maximum Sharpe portfolio exhibits a more balanced profile, with some allocations to low-risk or diversifying assets. `XLF` shows the largest excess risk contribution (+18.36% above equal risk), while `VIXM` records a negative contribution, consistent with its role as a hedge against equity risk. Finally, the risk-budgeting solution delivers the most even distribution of risk, with most assets lying within ±2% of the equal-risk line. A notable exception is `QQQ`, which stands out with a risk share more than 26% above the benchmark. It is worth noting that this is a direct consequence of the chosen budget: risk budgets $\boldsymbol{\beta}$ were set equal to the market portfolio weights, and `QQQ` has by far the largest prior market weight. Combined with its high volatility and strong correlation with other equity components, this specification leads the optimizer to allocate a disproportionate share of total Expected Shortfall risk to `QQQ`.



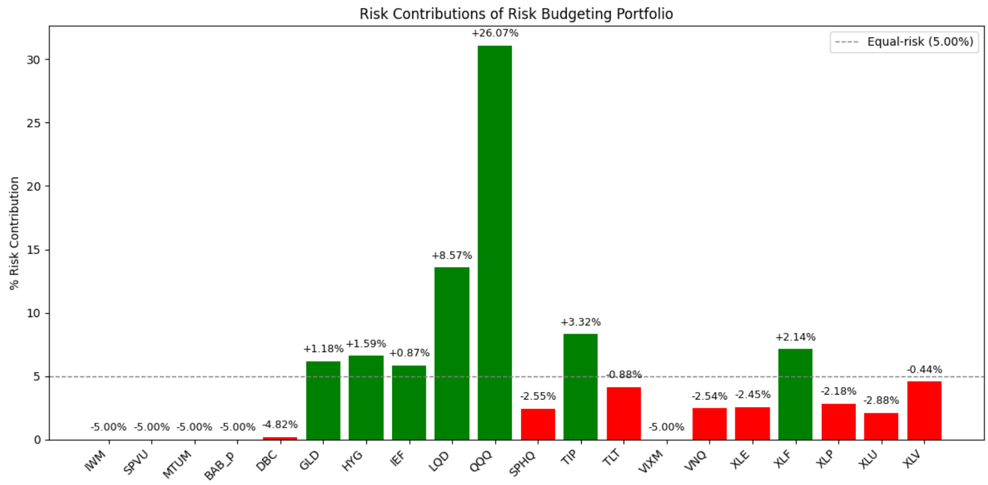


Figure 3.4: Percentage risk contributions under different allocation methods. The gray dashed line marks the equal-risk benchmark (5%). Bars are colored green when above and red when below this reference. Numbers on top of the bars indicate deviations from equal risk. Top: mean–variance; middle: maximum Sharpe ratio; bottom: risk budgeting.

3.2.4 Performance Statistics

To compare the efficiency of the different allocation schemes, ex-ante performance measures are reported. For each portfolio, the expected return (μ), variance, volatility, Sharpe ratio, and diversification ratio (DR) are computed. The diversification ratio is defined as

$$\mathrm{DR} = \frac{\mathbf{w}\boldsymbol{\sigma}}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}}$$

where $\boldsymbol{\sigma} = \mathrm{diag}(\boldsymbol{\Sigma})$ denotes the vector of individual asset volatilities. It measures the ratio between the weighted average of asset volatilities and the actual portfolio volatility. Higher values indicate greater diversification, since the portfolio variance is reduced relative to the average of the standalone volatilities. Table 3.3 summarizes the obtained results.

Portfolio	μ [% p.a.]	Var [% p.a.]	Vol (p.a.)	Sharpe (p.a.)	DR
$\mathbf{w}_{\mathrm{eq}}$	4.72	0.87	0.09	0.51	1.94
$\mathbf{w}_{\mathrm{mv}}$	8.16	2.37	0.15	0.53	1.30
$\mathbf{w}_{\mathrm{ms}}$	5.41	0.87	0.09	0.58	1.58
$\mathbf{w}_{\mathrm{rb}}$	4.39	0.67	0.08	0.54	1.58

Table 3.3: Annualized performance statistics of the optimized portfolios: mean return (μ), variance, volatility, Sharpe ratio, and diversification ratio (DR).

In terms of expected return, the mean–variance portfolio achieves the highest level (8.16%), but at the cost of substantially larger volatility, which reduces its efficiency relative to other schemes. The maximum Sharpe portfolio attains the best risk–return trade-off, with a Sharpe ratio of 0.58, while also improving diversification compared to the market benchmark. The risk-budgeting portfolio delivers the lowest volatility (0.08) and a Sharpe ratio comparable to that of maximum Sharpe, while achieving a high degree of diversification (DR = 1.58). Finally, the equally weighted portfolio shows moderate returns and volatility, but stands out for its very high diversification ratio (1.94). In general, mean–variance optimization pushes returns higher at the cost of concentration risk; maximum Sharpe achieves the most balanced trade-off between return and volatility, with moderate diversification; and risk budgeting attains low volatility and balanced risk contributions, positioning it between equal weights and maximum Sharpe in its overall profile.

3.2.5 Portfolio Exposures

To further investigate the sources of risk and return, the optimized portfolios are regressed on the Fama–French five-factor model augmented with momentum (MOM) and betting-against-beta (BAB). The estimated exposures are reported in Table 3.4. Alongside the factor loadings, the regression α (annualized) and the coefficient of determination (R^2) are shown. The results indicate that the mean–variance portfolio retains the strongest exposure to the market factor, while the maximum Sharpe and risk-budgeting portfolios substantially reduce market beta and display higher loadings on style factors such as CMA and BAB. The maximum Sharpe portfolio also delivers the highest alpha (1.44% p.a.), albeit with a lower R^2, reflecting greater deviation from systematic risk premia. The lower R^2 values for maximum Sharpe and risk-budgeting portfolios suggest that their performance is less tightly linked to standard factors, consistent with deliberate tilts away from market beta. Overall, the findings highlight a trade-off: the mean–variance portfolio is largely explained by systematic risk premia, whereas the maximum Sharpe allocation departs more strongly from them, producing a higher alpha but with lower explanatory power.

Portfolio	α [% p.a.]	MKT	SMB	HML	RMW	CMA	MOM	BAB	R^2
MV	0.84	0.783	0.126	0.092	-0.014	0.012	-0.030	0.138	0.889
MS	1.44	0.435	0.001	0.017	-0.016	0.065	-0.043	0.141	0.728
RB	0.04	0.384	0.013	-0.073	0.038	0.056	-0.029	0.111	0.724

Table 3.4: Factor exposures of optimized portfolios based on the Fama–French five-factor model augmented with momentum (MOM) and betting-against-beta (BAB). Reported are annualized α, factor loadings, and R^2.

Future Improvements

Several refinements could enhance the robustness and realism of the current framework:

- Improved covariance estimation. While this analysis used the Ledoit–Wolf shrinkage estimator, a natural extension would be to combine shrinkage with time-varying models such as EWMA or multivariate GARCH. This would allow the covariance matrix to adapt more flexibly to changing market regimes.

- Beta-scaled views. In the present implementation, factor views were imposed directly on ETF proxies. A more refined approach would normalize views by the target factor betas of each proxy, thereby approximating unit tilts to the underlying factors rather than their investable surrogates. In doing so, one should also account for cross-beta contamination, since proxies may load on multiple factors simultaneously. Adjusting for these cross-loadings would yield cleaner, more factor-specific views, although at the cost of introducing additional estimation noise.

- Machine learning for view formation. A further extension would be to automate the generation and updating of views through machine-learning models trained on macroeconomic or market indicators. This would allow the BL framework to adapt dynamically as new information arrives, rather than relying solely on manually specified priors.

- Stress testing and robustness checks. The optimized allocations could be evaluated under stressed scenarios (e.g., crisis periods) or alternative priors, to assess the sensitivity of results to model assumptions.

- Transaction costs and liquidity constraints. Introducing frictions such as bid–ask spreads, turnover penalties, or liquidity limits would make the optimization more realistic and highlight trade-offs between theoretical optimality and implementability.

- Improved BAB proxy. The analysis relied on a simple ETF-based construction of the betting-against-beta factor. Using a more accurate proxy, or replicating the original long–short methodology, would provide cleaner exposure to the intended risk premium and reduce measurement error in the factor regressions.

- Updated market weights. The market weights were computed using the 2024 Global Market Portfolio as reference. Employing more recent estimates would improve the accuracy of the implied equilibrium returns.

Conclusions

The analysis was conducted on a universe of U.S. exchange-traded funds complemented by factor proxies. The relevant factors were identified through regression analysis and proxied with ETFs to ensure investability. All data were resampled to weekly frequency, and the Ledoit–Wolf shrinkage estimator was applied to the covariance matrix to obtain stable inputs.

The Black–Litterman model was then introduced by combining equilibrium returns with subjective views, with the confidence levels of the latter calibrated using Idzorek's method. This produced posterior expected returns from which optimal unconstrained allocations were derived for three investor profiles. As expected, the Kelly investor produced a highly imbalanced portfolio with extreme leverage and short selling, while the Average and Trustee investors generated more moderate allocations. Already at this stage, the influence of views was visible, with posterior weights deviating systematically from the market benchmark.

Constraints were subsequently imposed, and three allocation methods were implemented: mean–variance optimization, maximum Sharpe ratio optimization, and Expected Shortfall risk budgeting. All three solvers performed well computationally and satisfied feasibility requirements. The mean–variance solution was highly concentrated, the maximum Sharpe allocation incorporated the views most effectively, and the risk budgeting portfolio emphasized diversification and a balanced distribution of risk.

Further analyses included weight tilts, variance-based risk contributions, ex-ante performance measures, and factor exposures. The tilt analysis provided a clear picture of how optimized allocations deviated from the market benchmark, with patterns that were consistent with both diversification and the imposed views. Risk contributions confirmed the concentration tendency of mean–variance, the more balanced but still input-sensitive structure of maximum Sharpe, and the even distribution of risk achieved by risk budgeting. Performance statistics highlighted the trade-offs across methods: mean–variance delivered the highest expected return but also the highest volatility, maximum Sharpe achieved the best risk–return efficiency, and risk budgeting combined lower risk with strong diversification. Finally, factor regressions showed that maximum Sharpe generated the highest alpha with reduced market exposure, whereas risk budgeting delivered moderate, broad-based exposures consistent with its risk-spreading objective.

Overall, the project demonstrated how the Black–Litterman model, combined with constrained optimization, generates allocations that diversify away from market concentration, integrate subjective views, and achieve varying balances of return, risk, and diversification depending on the chosen optimization criterion.

Appendix A

Fama-French Factors

SM	Small Value	**BV**	Big Value
SN	Small Neutral	**BN**	Big Neutral
SG	Small Growth	**BG**	Big Growth
SR	Small Robust	**BR**	Big Robust
SW	Small Weak	**BW**	Big Weak
SC	Small Conservative	**BC**	Big Conservative
SA	Small Aggressive	**BA**	Big Aggressive
SV	Small Value	**BV**	Big Value

Table A.1: Notation for factors definitions

$$\mathrm{SMB_{B/M}} = \tfrac{1}{3}\big(\mathrm{SV} + \mathrm{SN} + \mathrm{SG}\big) - \tfrac{1}{3}\big(\mathrm{BV} + \mathrm{BN} + \mathrm{BG}\big)$$

$$\mathrm{SMB_{OP}} = \tfrac{1}{3}\big(\mathrm{SR} + \mathrm{SN} + \mathrm{SW}\big) - \tfrac{1}{3}\big(\mathrm{BR} + \mathrm{BN} + \mathrm{BW}\big)$$

$$\mathrm{SMB_{INV}} = \tfrac{1}{3}\big(\mathrm{SC} + \mathrm{SN} + \mathrm{SA}\big) - \tfrac{1}{3}\big(\mathrm{BC} + \mathrm{BN} + \mathrm{BA}\big)$$

$$\mathrm{SMB} = \tfrac{1}{3}\big(\mathrm{SMB_{B/M}} + \mathrm{SMB_{OP}} + \mathrm{SMB_{INV}}\big)$$

$$\mathrm{HML} = \tfrac{1}{2}\big(\mathrm{SV} + \mathrm{BV}\big) - \tfrac{1}{2}\big(\mathrm{SG} + \mathrm{BG}\big)$$

$$\mathrm{RMW} = \tfrac{1}{2}\big(\mathrm{SR} + \mathrm{BR}\big) - \tfrac{1}{2}\big(\mathrm{SW} + \mathrm{BW}\big)$$

$$\mathrm{CMA} = \tfrac{1}{2}\big(\mathrm{SC} + \mathrm{BC}\big) - \tfrac{1}{2}\big(\mathrm{SA} + \mathrm{BA}\big)$$

Appendix B

Computation of the Diagonal Elements of Ω According to Idzorek

Idzorek [16] introduces a practical method to compute the uncertainty matrix $\mathbf{\Omega}$ from user-specified *confidence levels* in each view. The approach links the confidence level directly to the *portfolio tilt* induced by the view. The notation is the same reported in Table 2.1.

For each view $k = 1, \ldots, K$:

1. Construct the 100% posterior return: Treat the k-th view as being known with complete certainty (i.e. $\mathbf{\Omega} \equiv \mathbf{0}$). This is done by using modified version of Equation 2.6:

$$\boldsymbol{\mu}_{k,100\%} = \boldsymbol{\pi} + \tau \Sigma \boldsymbol{p}'_k \big(\boldsymbol{p}_k \, \tau \Sigma \boldsymbol{p}'_k\big)^{-1} (q_k - \boldsymbol{p}_k \boldsymbol{\pi}),$$

   here $\boldsymbol{p}_k$ represent the k-th row of the pick matrix, and q_k is the k-th element of the view vector. Since the view is treated as 100% certain, there is no uncertainty term. Note that here τ cancels out.

2. Calculate the $\mathbf{w}_{k,100\%}$ weights: Use unconstrained optimization formula:

$$\mathbf{w}_{k,100\%} = \frac{1}{2\lambda_{\text{mkt}}} \mathbf{\Sigma}^{-1} \boldsymbol{\mu}_{k,100\%}$$

3. Compute maximum departures from marked capitalization weights under 100% confidence assumption:

$$D_{k,100\%} = \mathbf{w}_{k,\,100\%} - \mathbf{w}_{\text{mkt}}$$

4. Determine the partial-confidence tilt for the given confidence level C_k:

$$\text{Tilt}_k = D_{k,100\%} \cdot C_k$$

5. Estimate the target weight vector $\mathrm{w}_{k,\%}$ based on the tilt:

$$\mathbf{w}_{k,\,\%} = \mathbf{w}_{\text{mkt}} + \text{Tilt}_k$$

6. Find the value of ω_k that, subject to $\omega_k > 0$, solves the following optimization problem:
$$\min \sum (\mathbf{w}_{k,\%} - \mathbf{w}_k)^2$$
where
$$\mathbf{w}_k = [2\lambda_{\text{mkt}}\boldsymbol{\Sigma}]^{-1} \left[\boldsymbol{\Sigma}^{-1} + \frac{1}{\omega_k}\boldsymbol{p}_k'\boldsymbol{p}_k\right]^{-1} \left[\boldsymbol{\Sigma}^{-1}\boldsymbol{\pi} + \frac{Q_k}{\omega_k}\boldsymbol{p}_k'\right]$$
This is the k-th diagonal element of the uncertainty matrix (i.e. $\tilde{\boldsymbol{\Omega}}_{kk} = \omega_k$)

7. Repeat Steps 1-6 for all the views to build the complete $\tilde{\boldsymbol{\Omega}}$ matrix.

8. Assemble the final uncertainty matrix:
$$\boldsymbol{\Omega} = \tau\,\tilde{\boldsymbol{\Omega}}$$

Bibliography

[1] Federal Reserve Bank of St. Louis. *3-Month Treasury Bill Secondary Market Rate, Discount Basis*. URL: `https://fred.stlouisfed.org/series/TB3MS`.

[2] Eugene F. Fama and Kenneth R. French. "Common risk factors in the returns on stocks and bonds". In: *Journal of financial economics* 33.1 (1993), pp. 3–56.

[3] Eugene F. Fama and Kenneth R. French. "A five-factor asset pricing model". In: *Journal of financial economics* 116.1 (2015), pp. 1–22.

[4] Kenneth R. French. *Description of Fama/French Factors*. URL: `https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`.

[5] AQR Capital Management, LLC. *Betting Against Beta: Equity Factors, Daily*. `https://www.aqr.com/Insights/Datasets/Betting-Against-Beta-Equity-Factors-Daily`. Dataset last updated June 30, 2025. 2025.

[6] Andrea Frazzini and Lasse Heje Pedersen. "Betting against beta". In: *Journal of financial economics* 111.1 (2014), pp. 1–25.

[7] Jorge Mina, Jerry Yi Xiao, et al. "Return to RiskMetrics: the evolution of a standard". In: *RiskMetrics Group* 1.1 (2001), p. 11.

[8] Olivier Ledoit and Michael Wolf. "A well-conditioned estimator for large-dimensional covariance matrices". In: *Journal of Multivariate Analysis* 88.2 (2004), pp. 365–411.

[9] Harry Markowitz. "Portfolio Selection". In: *The Journal of Finance* 7.1 (1952), pp. 77–91. ISSN: 0022-1082. DOI: `10.1111/j.1540-6261.1952.tb01525.x`. URL: `https://doi.org/10.1111/j.1540-6261.1952.tb01525.x`.

[10] Richard O Michaud. "The Markowitz optimization enigma: Is 'optimized'optimal?" In: *Financial analysts journal* 45.1 (1989), pp. 31–42.

[11] Victor DeMiguel, Lorenzo Garlappi, and Raman Uppal. "Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy?" In: *The review of Financial studies* 22.5 (2009), pp. 1915–1953.

[12] Fischer Black and Robert Litterman. "Asset allocation: combining investor views with market equilibrium". In: *Goldman Sachs Fixed Income Research* 115.1 (1990), pp. 7–18.

[13] Attilio Meucci. "The black-litterman approach: Original model and extensions". In: *Shorter version in, The Encyclopedia of Quantitative Finance, Wiley* (2010).

[14] Dodard, Frederic and Le, Amy and State Street Global Advisors. *Global Market Portfolio 2024: Rise of the Middle East*. State Street Global Advisors Insights. Oct. 2024.

[15] *Market Capitalization: What It Means & How To Calculate It.* Yahoo Finance. Accessed July 2025. 2023. URL: https://finance.yahoo.com/news/market-capitalization-calculate-164302483.html.

[16] Thomas Idzorek. "A step-by-step guide to the Black-Litterman model: Incorporating user-specified confidence levels". In: *Forecasting expected returns in the financial markets.* Elsevier, 2007, pp. 17–38.

[17] *Wall Street Mojo.* URL: https://www.wallstreetmojo.com/capital-market-line/.

[18] P. Palomar Daniel. *PORTFOLIO OPTIMIZATION: Theory and Application.* CAMBRIDGE University Press, 2025.

[19] Sbastien Maillard, Thierry Roncalli, and Jerome Teiletche. *On the properties of equally-weighted risk contributions portfolios. SSRN.* 2008.

[20] Thierry Roncalli. *Introduction to risk parity and budgeting.* CRC press, 2013.

[21] Kris Boudt, Peter Carl, and Brian G Peterson. "Asset allocation with conditional value-at-risk budgets". In: *Journal of Risk* 15.3 (2012), pp. 39–68.

[22] Basel Committee on Banking Supervision. *Minimum capital requirements for market risk.* Tech. rep. Final standard for the Fundamental Review of the Trading Book (FRTB). Bank for International Settlements, Jan. 2019. URL: https://www.bis.org/publ/bcbs457.htm.