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Dynamics of self-similar tilings

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Abstract. This paper investigates dynamical systems arising from the action by translations on the orbit closures of self-similar and self-affine tilings of $\mathbb{R}^d$. The main focus is on spectral properties of such systems which are shown to be uniquely ergodic. We establish criteria for weak mixing and pure discrete spectrum for wide classes of such systems. They are applied to a number of examples which include tilings with polygonal and fractal tile boundaries; systems with pure discrete, continuous and mixed spectrum.

0. *Introduction*

In this paper we study the properties of dynamical systems arising from self-similar tilings. A *self-similar tiling* of the plane has a finite set of tiles up to translation. Its main feature is the 'inflation-subdivision' property: there is a complex number λ with $|\lambda| > 1$, such that multiplication by λ maps every tile into a union of tiles. Self-similar tilings—in the sense we use—were introduced by Thurston [**Thur**], and studied in [**Ken1, Ken4, Prag**]. Closely related classes of tilings were considered in [**GS**, 10.1], (similarity tilings), [**LP**], [**Peyr**], and [**Rad2**] (substitution-tilings). Examples of self-similar tilings include many polygonal tilings [**GS**, Ch. 10], among them the Penrose tiling, and tilings with fractal boundary [**Dek2, Dek3, Rau, Bed1, IK, IO1, Ken1**].

If $\mathcal{T}$ is a plane tiling, the *tiling space* corresponding to $\mathcal{T}$ can be defined as the closure of the set of all its translations $\{\mathcal{T} - g, g \in \mathbb{R}^2\}$ in an appropriate topology. The group $\mathbb{R}^2$ continuously acts on the tiling space by translations, and we get a *tiling dynamical system*. Our goal is to study ergodic properties of such systems. Tiling dynamical systems in a different setting were considered by Rudolph [**Rud**]. The investigation of tiling dynamical systems as actions of various groups of rigid motions, including $\mathbb{R}^2$, was initiated by Radin [**RW, Rad1, Rad2, Rad3, BR**] and Robinson [**Ro1**]. There is an interesting aspect which we do not touch upon in this paper: connection with 'finite type' systems in two dimensions. For this development see [**Moz, Rad1**].

Methods of the theory of word substitutions play an important role in the papers of Radin [**Rad1, Rad2**]. We consider a smaller class of tilings but carry out the

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generalization of substitution dynamics much further. This involves symbolic dynamics, ergodic theory, algebraic number theory, and combinatorial geometry. Some results are obtained for self-affine tilings in $\mathbb{R}^d$; the case of $d = 1$ is also of interest.

Below we give a summary of our main results (for $d \leq 2$) and discuss briefly the contents of each section.

In §1 the setting of tiling dynamical systems is worked out (without the assumption of self-similarity). We also develop the important technical tool of 'cylinder sets'.

In §2 self-affine and self-similar tilings are defined and the *unique composition property* is introduced. The latter means, roughly speaking, that the process of inflation and subdivision can be reversed. The analogous condition for substitutions is known as *recognizability*. The unique composition property can be easily checked for the most important examples.

In §3 it is shown that the self-affine tiling dynamical system is uniquely ergodic. The proof relies on the known fact that in a self-affine tiling all patches occur with uniform frequencies [**LP, GH**].

In §4 we prove that the dynamical system arising from a self-affine tiling is never strongly mixing.

Next we study when there exist non-constant eigenfunctions (this is equivalent to being not weakly mixing). Necessary conditions are established in §4 and sufficient conditions are given in §5. Recall that an algebraic integer θ is a *real Pisot number* if it is greater than one and all its Galois conjugates are less than one in modulus, and a *complex Pisot number* if all the conjugates, except $\overline{\theta}$, have modulus less than one. We obtain the following results.

Consider a self-similar tiling of $\mathbb{R}^d$, $d \leq 2$, *with expansion constant* λ. *For* $d = 1$, *the tiling dynamical system is not weakly mixing if and only if* $|\lambda|$ *is a real Pisot number. For* $d = 2$, *assuming the unique composition property, the system is not weakly mixing if and only if* λ *is a complex Pisot number.*

In fact, the unique composition property is needed only for the implication (Pisot) $\Rightarrow$ (not weakly mixing). For $d = 1$ every non-periodic self-similar tiling has the unique composition property by a result of Mossé [**Mos**]. It is not known whether this is true for $d = 2$†.

In §6 we investigate when the spectrum of a tiling dynamical system is pure discrete. This is a subtle question, not completely resolved even for substitutions. We present an algorithm, the *method of overlaps*, which, in principle, gives an answer for any particular tiling. To formulate the main result, let $\mathcal{T}$ be a self-similar tiling of the complex plane $\mathbb{C}$, with non-real expansion constant λ. For $x \in \mathbb{C}$, denote by D_x the union of tiles T such that $T + x$ is also a tile of $\mathcal{T}$. We write $\mathrm{dens}(A)$ for the density of a set $A \subset \mathbb{C}$; it is easy to see that $\mathrm{dens}(D_x)$ always exists.

The tiling dynamical system has pure discrete spectrum if and only if λ *is complex Pisot and for some* $x \neq 0$,

$$\lim_{n\to\infty} \mathrm{dens}(D_{\lambda^n x}) = 1.$$

† See Note at the end of the paper.

Finally, in §7 the theory is applied to concrete examples, which include tilings with tiles having polygonal and 'fractal' boundary; systems with pure discrete, continuous, and mixed spectrum.

Connections with quasicrystals and diffraction spectrum served as a strong motivation for us. They will be discussed below, on a less rigorous level than the rest of the paper.

Non-periodic tilings are often used as models of aperiodic structures in physics. This is one of the main topics in the recent book by Senechal [**Sen**] which contains an extensive bibliography. If a tiling is given, the atomic arrangement can be formed by placing atoms at the vertices. Of course, this only makes sense for polygonal tilings. An alternative is to put atoms into specially chosen 'control points'; this can be done consistent with the hierarchical structure of a self-similar tiling (see §5), even for tiles with fractal boundary. (We note in passing that tilings with fractal boundary also occur as 'atomic surfaces' for one-dimensional self-similar structures [**LGJJ**].)

Given an atomic arrangement $\mathcal{V}$ in $\mathbb{R}^d$, consider the distribution $f(x) = \sum_{v\in\mathcal{V}} \delta_v$, where δ_v is Dirac's delta. The X-ray diffraction of $\mathcal{V}$ can be described using the Fourier transform of the autocorrelation γ:

$$\gamma := \lim_{L\to\infty} (2L)^{-d} \sum_{v,w\in\mathcal{V}\cap[-L,L]^d} \delta_{v-w}.$$

Hof [**Hof1**] proved (under some mild conditions) that $\hat{\gamma}$ is a positive measure and hence can be decomposed into a discrete and continuous part. The discrete part corresponds to 'sharp, bright spots' in the diffraction picture. The continuous part is sometimes called the 'diffuse spectrum'. One of the commonly used definitions of a quasicrystal is that of an atomic structure which has a discrete component in its diffraction spectrum. There are many works in which the diffraction spectrum of atomic arrangements arising from tilings is computed (in some cases numerically): [**BT, Bru2, Godr1, Godr2, GLu, GLa, Sen**] is just a small sample.

We will deal with a different kind of spectrum: the spectral measure of the unitary group arising from translations on the tiling space. There are works which relate it to diffraction spectrum [**Dw1, Ro3**]. One is led to the conclusion that the measure $\hat{\gamma}$ is a specific scalar part of the spectral measure. This allows us to give the following interpretation of our results.

(a) Pure discrete dynamical spectrum implies pure discrete diffraction spectrum. We give a necessary and sufficient condition for the former which applies to several examples (Example 7.1, 7.2, special cases of Example 7.10) considered in the physics literature [**BT, Godr1, GLu**]. The authors of [**BT, Godr1**] pointed out that their methods did not rule out a continuous component.

(b) Continuous dynamical spectrum implies that there is no discrete diffraction spectrum. In the literature on quasicrystals [**BT, Godr2, GLu, Sen**] the *Pisot condition* is given as a criterion for the presence of discrete diffraction spectrum, although not always on a rigorous mathematical basis. Our results in §4 imply that for a self-similar plane tiling to have a discrete spectral component it is necessary that the expansion constant is a *complex* Pisot number. This implies that the biggest eigenvalue of the 'inflation-subdivision' matrix is a real Pisot number. Example 4.7 shows that this 'real Pisot condition' is not sufficient.

(c) Results of §5 (construction of eigenfunctions) do not immediately imply anything for the diffraction spectrum, since the discrete component may be lost when taking the part of the spectral measure. However, this should not happen in a 'generic' situation, unless there is a special reason like symmetry of some kind.

As far as we are interested in eigenfunctions, the connection can go the other way: Hof [**Hof1**] pointed out that in some cases the discrete part of $\hat{\gamma}$ is given by $\sum_{y\in\mathcal{V}^*}|c_y|^2\delta_y$, where c_y is a continuous eigenfunction for the tiling dynamical system.

An interesting problem, which we do not address in this paper, is to study the nature of the continuous component of the spectrum, if it is present. The intuition is that it should usually be singular continuous, but the absolutely continuous part is possible, by analogy with the Rudin–Shapiro substitution (see [**Queff**]). The similar question for diffraction spectrum was studied numerically in [**Godr2, GLu**]. (Notice, however, that the dynamical spectrum cannot be pure absolutely continuous for tilings considered in this paper, since that would imply mixing.)

Notation. For the reader's convenience, we present a list of some symbols used in the paper, with references to the sections where they are defined.

$B_R(y) = \{g : \|g - y\| < R\}$	Vol = Lebesgue measure in $\mathbb{R}^d$
$\mathcal{T}$ = tiling, §1	$X_{\mathcal{T}}$ = tiling space, §1
$\mathcal{K}_{\mathcal{T}}$ = translation group, §1	Γ_g = translation on $X_{\mathcal{T}}$, §1
$X_{P,U}$ = cylinder set, Definition 1.3	$\eta(\mathcal{T})$, Lemma 1.5(i)
$L_P(A)$, A^{+r}, A^{-r}, §3.2	freq(P) = frequency of a patch, Theorem 3.3
$\Xi(\mathcal{T})$, Theorem 4.1	$c(T)$ = control point, §5
$\mathcal{O}_{y,T,S}$ = overlap, §6	$[\mathcal{O}_y]$ = equivalence class of overlaps, §6.

1. *Tilings and tiling dynamical systems*

A *tile* is a set $T \subset \mathbb{R}^d$ which is a closure of its interior. A *tiling* of $\mathbb{R}^d$ is a set $\mathcal{T}$ of tiles such that $\mathbb{R}^d = \cup\{T : T \in \mathcal{T}\}$; distinct tiles have non-intersecting interiors Int(T), and each compact set in $\mathbb{R}^d$ intersects a finite number of tiles in $\mathcal{T}$.

We will assume that the tiling has finitely many tile types up to translation. Each tile will be labeled with its 'tile type', an element of $\{1, 2, \ldots, m\}$. Any two tiles of the same type must be translations of each other.

Remarks. (a) It is often required that tiles be polytopes or at least homeomorphic to the ball in $\mathbb{R}^d$. We do not assume this; tiles can even be disconnected. Of course, in most examples the tiles will be topological balls.

(b) Two tiles, which are translations of each other, may not be of the same type. For instance, one can have a tiling of $\mathbb{R}^d$ by unit cubes with vertices at $\mathbb{Z}^d$ and an arbitrary choice of labels from $\{1, 2, \ldots, m\}$—essentially an element of the d-dimensional m-shift.

(c) Some authors, see [**Rad2, Ro1**], consider other transformation groups to identify tiles of the same type. In this paper we are concerned with the translation group only.

Suppose that $\mathcal{T}$ and $\mathcal{S}$ are tilings with the same set of tile types. Two tiles $T \in \mathcal{T}$ and $S \in \mathcal{S}$ are *equivalent*, $T \approx S$, if they have the same tile type. Then, in particular, $S = T + g$ for some $g \in \mathbb{R}^d$. A finite collection of tiles in $\mathcal{T}$ (with tile types marked) is called a $\mathcal{T}$*-patch*. We say that a $\mathcal{T}$-patch P^1 and an $\mathcal{S}$-patch P^2 are *equal*, $P^1 = P^2$, if they are composed of the same tiles having the same tile types. They are *equivalent* if there exists $g \in \mathbb{R}^d$ such that $P^1 + g = P^2$. We will sometimes write 'diameter of a patch', or 'a patch contained in A' which will refer to the union of tiles in the patch. If $U \subset \mathbb{R}^d$ then the $\mathcal{T}$*-patch of* U is the set of $\mathcal{T}$-tiles intersecting U.

Definition. A tiling is said to have a *finite number of local patterns* if for each $r > 0$ there are finitely many equivalence classes of $\mathcal{T}$-patches of diameter less than r.

The tiling $\mathcal{T}$ is said to have the *local isomorphism* property if for any $\mathcal{T}$-patch P there exists $R = R(P) > 0$ such that every open ball $B_R(y)$ contains a $\mathcal{T}$-patch equivalent to P.

The term 'local isomorphism' is used in [**GS, RW**]. Other authors use different terms: tilings with local isomorphism = recurrent tilings in [**BT**] = repetitive tilings in [**Sen**] = almost periodic tilings in [**Ro1**]. Tilings which have both local isomorphism and a finite number of local patterns are called quasi-homogeneous in [**Thur, Prag**] and quasi-periodic in [**Ken4**].

Tiling space and tiling dynamical system. Let $\mathcal{T}$ be a tiling of $\mathbb{R}^d$. We define the *tiling space* $X_{\mathcal{T}}$ as the set of all tilings $\mathcal{S}$ of $\mathbb{R}^d$ with the property that every $\mathcal{S}$-patch is equivalent to some $\mathcal{T}$-patch (assuming that $\mathcal{S}$ has the same set of tile types as $\mathcal{T}$.) The tiling space will be equipped with the following metric:

$$\rho(\mathcal{S}_1, \mathcal{S}_2) = \min\{1, \tilde{\rho}(\mathcal{S}_1, \mathcal{S}_2)\},$$

where

$$\begin{aligned}\tilde{\rho}(\mathcal{S}_1, \mathcal{S}_2) \quad = \quad & \inf\{\varepsilon : \text{there exists an } \mathcal{S}_1\text{-patch } P_1 \text{ and an } \mathcal{S}_2\text{-patch } P_2 \text{ such that} \\ & P_1 \supset B_{1/\varepsilon}(0),\, P_2 \supset B_{1/\varepsilon}(0), \text{ and } P_1 = P_2 + g \\ & \text{for some } g \in \mathbb{R}^d,\, \|g\| \le \varepsilon\}.\end{aligned}$$

Our definition of the metric is similar to the one used by Rudolph [**Rud**]. The axioms of metric are easily verified. Roughly speaking, two tilings are close if they have the same pattern in a large neighborhood of the origin, up to a small translation.

Some authors [**RW, Rad2, Ro1**] define the tiling space as the set of all tilings from a given set of prototiles, often satisfying some 'matching rules'. Then the metric can be defined using the Hausdorff distance between the unions of boundaries of tiles intersecting $B_{1/\varepsilon}(0)$. Our definition restricts the tiling space to the orbit closure under the translation action (see Lemma 1.2 below). In some cases (the Penrose tiling, for instance) it is known that matching rules 'force' any member of this larger space to be an element of the smaller space, up to rotation.

LEMMA 1.1. ([**RW**]) *Let* $\mathcal{T}$ *be a tiling. Then the tiling space* $(X_{\mathcal{T}}, \rho)$ *is compact.*

In fact, Radin and Wolff [**RW**] prove compactness of the larger space of all tilings with the given set of prototiles. In the case when there are finitely many local patterns, which will be our main concern, this result was mentioned by Rudolph [**Rud**]; it can be proved by a standard diagonalization argument.

For any $g \in \mathbb{R}^d$ consider the translation $\Gamma_g : \mathcal{S} \mapsto \mathcal{S} - g$. Clearly, Γ_g is a homeomorphism of $X_{\mathcal{T}}$ and, moreover, we get a jointly continuous action of the group $\mathbb{R}^d$. The resulting system $(X_{\mathcal{T}}, \Gamma_g)$ will be called the *tiling dynamical system*. The set $\{\mathcal{S} - g : g \in \mathbb{R}^d\}$ is the *orbit* of $\mathcal{S}$. Notice that $X_{\mathcal{T}}$ is the orbit closure of $\mathcal{T}$.

Recall that a dynamical system is *minimal* if every orbit is dense.

LEMMA 1.2. (essentially [**RW**]) *The tiling dynamical system* $(X_{\mathcal{T}}, \Gamma_g)$ *is minimal if and only if* $\mathcal{T}$ *has the local isomorphism property.*

Proof. This follows by a standard argument from topological dynamics (see [**Queff**, V. 12]). □

Let $\mathcal{K}_{\mathcal{T}} \subset \mathbb{R}^d$ denote the *translation group* of the tiling $\mathcal{T}$:

$$\mathcal{K}_{\mathcal{T}} = \{g \in \mathbb{R}^d : \mathcal{T} - g = \mathcal{T}\}.$$

It is clear that $\mathcal{K}_{\mathcal{T}}$ is isomorphic to $\mathbb{Z}^k$ for some $0 \le k \le d$. The tiling $\mathcal{T}$ is said to be *periodic* if $k = d$ (then $\mathcal{K}_{\mathcal{T}}$ is called a *lattice*), and *non-periodic* if $k = 0$.

The translation group is a subgroup of a larger group, called the *symmetry group* [**Ro1**] (the set of rigid motions of $\mathbb{R}^d$ preserving the tiling). Robinson [**Ro1**] introduced the notion of a *quasi-symmetry group* as the set of rigid motions which preserve the tiling space as a whole, not necessarily the individual tilings. The existence of symmetries and quasi-symmetries has important consequences for the tiling dynamical system, see [**Ro1, Rad1**].

Cylinder sets. In symbolic dynamics a cylinder set is determined by fixing certain terms of a sequence. By analogy we define a ‘cylinder set’ in the tiling space as the set of tilings with a given pattern in certain locations. Such sets will play an important technical role; in particular, they form a topology base for $X_{\mathcal{T}}$.

Definition 1.3. Let P be a patch of $\mathcal{T}$ or its translate and let $U \subset \mathbb{R}^d$ be a measurable set. Define the *cylinder set* $X_{P,U}$ as

$$X_{P,U} = \{\mathcal{S} \in X_{\mathcal{T}} : P - g \text{ is an } \mathcal{S}\text{-patch for some } g \in U\}.$$

The following properties are straightforward.

LEMMA 1.4. (Properties of cylinder sets)
(i) $U_1 \subset U_2 \Rightarrow X_{P,U_1} \subset X_{P,U_2}$;
(ii) *if* P_1 *is a subpatch of* P_2, *then* $X_{P_1,U} \supset X_{P_2,U}$;
(iii) $X_{P-h,U} = X_{P,U+h} = \Gamma_h(X_{P,U})$;
(iv) *if* $P_1 \cup P_2$ *is a patch, then* $X_{P_1 \cup P_2,U} \subset X_{P_1,U} \cap X_{P_2,U}$;
(v) $X_{P,U_1 \cup U_2} = X_{P,U_1} \cup X_{P,U_2}$.

In general, $U_1 \cap U_2 = \emptyset$ does not imply that $X_{P,U_1} \cap X_{P,U_2} = \emptyset$. This becomes true, however, in some important special cases.

LEMMA 1.5.

(i) *Let $\mathcal{T}$ be any tiling (with a finite number of tile types) and $\eta = \eta(\mathcal{T}) > 0$ is such that every tile $T \in \mathcal{T}$ contains a ball of diameter η. Then for any $\mathcal{T}$-patch P,*

$$0 < \|y_1 - y_2\| < \eta \Rightarrow X_{P,\{y_1\}} \cap X_{P,\{y_2\}} = \emptyset.$$

(ii) *Suppose that $\mathcal{T}$ has the local isomorphism property. Then for any $r > 0$ there exists $R > 0$ such that if P is a $\mathcal{T}$-patch containing $B_R(0)$ and $U_1, U_2 \subset B_r(0)$, then*

$$(U_1 \cap (U_2 - g) = \emptyset, \forall g \in \mathcal{K}_{\mathcal{T}}) \Longrightarrow X_{P,U_1} \cap X_{P,U_2} = \emptyset.$$

Proof. (i) Observe that if P is an $\mathcal{S}$-patch for some $\mathcal{S} \in X_{\mathcal{T}}$, then $P - y$ cannot be an $\mathcal{S}$-patch for any $y < \eta$, since the tiles in $P - y$ will have interiors intersecting the boundaries of the corresponding tiles in P. The statement follows.

(ii) Suppose that the claim is false. Then there exist $r > 0$, $\mathcal{T}$-patches $P_n \supset B_n(0)$, sets $U_1^{(n)}, U_2^{(n)} \subset B_r(0)$ with $U_1^{(n)} \cap (U_2^{(n)} - g) = \emptyset$ for $g \in \mathcal{K}_{\mathcal{T}}$, and tilings $\mathcal{S}_n \in X_{P_n,U_1^{(n)}} \cap X_{P_n,U_2^{(n)}}$, for $n \geq 1$. By Definition 1.3, one can find $g_n^{(i)} \in U_i^{(n)}$ such that $P_n - g_n^{(i)}$ is an $\mathcal{S}_n$-patch, for $i = 1, 2$. But then the distance between $\mathcal{S}_n$ and $\mathcal{S}_n + g_n^{(1)} - g_n^{(2)}$ tends to zero (in the metric of $X_{\mathcal{T}}$). Using the compactness of $X_{\mathcal{T}}$ and passing to a subsequence, we can assume that $\mathcal{S}_n \to \mathcal{S} \in X_{\mathcal{T}}$, and $g_n^{(i)} \to g^{(i)}$ for $i = 1, 2$. Then $\mathcal{S} = \mathcal{S} + g^{(1)} - g^{(2)}$. If we show that $g^{(1)} - g^{(2)} \notin \mathcal{K}_{\mathcal{T}}$, this will be a contradiction, since by the local isomorphism property all tilings in $X_{\mathcal{T}}$ have the same translation group.

Observe that for any $g \in \mathcal{K}_{\mathcal{T}}$, we have $\mathcal{S}_n - g = \mathcal{S}_n$, so $P_n - g_n^{(1)} - g$ and $P_n - g_n^{(2)}$ are both $\mathcal{S}_n$-patches. Since $U_1^{(n)} \cap (U_2^{(n)} - g) = \emptyset$, by the part (i) of this lemma,

$$\inf\{\|g_n^{(1)} - g_n^{(2)} - g\| : g \in \mathcal{K}_{\mathcal{T}}\} \geq \eta.$$

Thus $g^{(1)} - g^{(2)} = \lim(g_n^{(1)} - g_n^{(2)}) \notin \mathcal{K}_{\mathcal{T}}$, and we are done. □

Our next goal is to represent $X_{\mathcal{T}}$ as a disjoint union of cylinder sets having arbitrarily small diameters.

LEMMA 1.6. *Let $\mathcal{T}$ be a tiling with the local isomorphism property and a finite number of local patterns. For any $\epsilon > 0$ there exists a finite family of disjoint cylinder sets X_{P_l,V_l} having diameter less than ϵ, such that*

$$X_{\mathcal{T}} = \bigcup_l X_{P_l,V_l}. \tag{1}$$

Proof. Consider $\mathcal{S}$-patches of the ball $B_{1/\epsilon}(0)$ for all $\mathcal{S} \in X_{\mathcal{T}}$. There are finitely many equivalence classes of such patches; choose their representatives $P_1, \ldots, P_k$. For each i consider the set U_i of vectors $y \in \mathbb{R}^d$ such that $P_i - y$ is a $B_{1/\epsilon}(0)$-patch (that is, every tile of $P_i - y$ intersects the ball). Then $X_{\mathcal{T}} = \cup_{i=1}^k X_{P_i,U_i}$ and this union is disjoint. All the sets U_i, $i = 1, \ldots, k$, lie in a ball of some fixed radius, say, the maximum diameter of a tile. Decompose the sets U_i into finite disjoint unions $U_i = \cup_\nu V_i^\nu$ so that $\mathrm{diam}(V_i^\nu) \leq \epsilon$. We have

$$X_{\mathcal{T}} = \bigcup_{i=1}^k \bigcup_\nu X_{P_i,V_i^{(\nu)}},$$

and by the definition of the metric on the tiling space, $\mathrm{diam}(X_{P_i,V_i^{(\nu)}}) \le \epsilon$. It remains to check that the sets $X_{P_i,V_i^{(\nu)}}$ for different ν are disjoint (for ϵ sufficiently small). This will follow from Lemma 1.5(ii), once it is verified that for all i,

$$U_i \cap (U_i - g) = \emptyset, \quad g \in \mathcal{K}_{\mathcal{T}}, g \neq 0.$$

To this end, suppose $y \in U_i \cap (U_i - g)$. Then $P_i - y$ is the $\mathcal{S}$-patch of $B_{1/\epsilon}(0)$ for some $\mathcal{S} \in X_{\mathcal{T}}$, and $P_i - y - g$ is the $\mathcal{S}'$-patch of $B_{1/\epsilon}(0)$ for some $\mathcal{S}' \in X_{\mathcal{T}}$. But $\mathcal{S} - g = \mathcal{S}$ for $g \in \mathcal{K}_{\mathcal{T}}$, so $P_i - y - g$ is an $\mathcal{S}$-patch as well. Every tile of both $P_i - y$ and $P_i - y - g$ intersects $B_{1/\epsilon}(0)$, which is only possible if $g = 0$. □

2. *Self-affine and self-similar tilings*

Definition. Let ϕ be an expansive linear mapping $\mathbb{R}^d \to \mathbb{R}^d$, that is, all its eigenvalues are greater than one in modulus. We will assume throughout the paper that ϕ is diagonalizable over $\mathbb{C}$. A tiling $\mathcal{T}$ will be called *ϕ-subdividing* if:
(a) for each $T \in \mathcal{T}$, ϕT is a union of tiles in $\mathcal{T}$;
(b) $T \approx T' \Longleftrightarrow \phi T \approx \phi T'$.
In other words, (a) means that ϕT forms a $\mathcal{T}$-patch for any $\mathcal{T}$-tile T, and (b) means that such patches for T, T' are equivalent if and only if T, T' are of the same type.

A tiling will be called *self-affine with expansion map ϕ* if it is ϕ-subdividing, has a finite number of local patterns and the local isomorphism property. If ϕ is a similarity the tiling will be called *self-similar*. For a self-similar tiling of $\mathbb{R}$ one can speak about the *expansion constant*. If $\mathcal{T}$ is a self-similar tiling of the plane $\mathbb{R}^2 \cong \mathbb{C}$, the linear map ϕ can be represented as the multiplication by $\lambda \in \mathbb{C}$ which will be called the (*complex*) *expansion constant*.

Let $\mathcal{T}$ be a ϕ-subdividing tiling. Then $\phi\mathcal{T} = \{\phi T : T \in \mathcal{T}\}$ is another ϕ-subdividing tiling with tiles composed of $\mathcal{T}$-tiles. They will be sometimes called *tiles of the second level*. It is obvious but important that any $\mathcal{T}$-patch will become a $\phi\mathcal{T}$-patch after some composition and possibly adding 'missing' tiles. Of course, one can go on to consider a whole hierarchy of tilings $\phi^2\mathcal{T}, \phi^3\mathcal{T}$ etc, each of which is ϕ-subdividing and has tiles composed of lower-level tiles (tiles of $\phi^n\mathcal{T}$ are tiles of the $(n+1)$st level).

Remarks. (a) Perhaps it would be more consistent to allow any affine map ϕ in the definition of a ϕ-subdividing tiling. However, assuming that ϕ is linear does not lead to loss of generality, as far as dynamics is concerned: if a tiling $\mathcal{T}$ has the ϕ_1-subdividing property, where $\phi_1 x = \phi x + g$, with ϕ linear, then the tiling $\mathcal{T} + (\phi - I)^{-1} g$ has the ϕ-subdividing property.

(b) The expansion map ϕ (or expansion constant λ) is not determined uniquely by the tiling. For instance, any ϕ-subdividing tiling is also ϕ^k-subdividing.

(c) What is known about the tiles of a self-similar tiling? It appears that their boundaries are either polygonal or 'fractal' (see examples below). Narbel [**Nar**] showed that the boundary of a two-dimensional self-similar tiling cannot have smooth non-linear pieces. Here we mention just one simple fact: if a tiling with expansion constant $\lambda \in \mathbb{C}$ has polygonal tiles, then $\lambda^k \in \mathbb{R}$ for some $k \in \mathbb{N}$ (and therefore, by taking the power, we can assume that the expansion constant is real). Indeed, since there are finitely many tile types up to translation, the sides of the tiles have finitely many directions. By the subdividing property, this set of directions is invariant under the multiplication by λ. The claim follows.

Examples and bibliographical notes. Here we mention tilings without going into their construction. §7 is devoted to a careful treatment of several examples.

(a) The simplest examples are periodic tilings of the plane: (1) by parallelograms; (2) by triangles. They are both self-similar with any expansion constant $k > 1, k \in \mathbb{N}$. However, the periodic tiling by hexagons ('honeycombs') is not self-affine.

(b) The Penrose tiling [**Pnr**] is probably the most famous example, and the one which attracted much attention to non-periodic tilings. The Penrose tiling comes in several variants; one of them (with 'thin' and 'thick' triangles as tiles) is self-similar with expansion constant $\frac{1}{2}(1+\sqrt{5})e^{3\pi i/5}$.

(c) There are many non-periodic self-similar tilings with polygonal tiles congruent to each other. Such tilings were considered by Grünbaum and Shephard [**GS**, 10.1] (who called them similarity tilings). Examples include tilings by triominoes and hexaminoes [**GS**, p. 529], the domino tiling and others [**Nar**].

(d) Lunnon and Pleasants [**LP**] studied 'tilings with inflation' which are much like our self-similar tilings but have polygonal tiles. Peyrière [**Peyr**] and Narbel [**Nar**] define self-similarity using the dual graph of the tiling. Definitions which we use are essentially due to Thurston [**Thur**] and Kenyon [**Ken1, Ken4**]. Kenyon [**Ken1**] and Praggastis [**Prag**] use the term 'self-similar' for self-affine tilings. Radin [**Rad2**] investigated 'substitution-tilings' which have much in common with our self-similar tilings but may have infinitely many tiles up to translation.

(e) On the other end of the spectrum from polygonal tilings are tilings by so-called 'fractiles', sets with nowhere differentiable boundary, often having Hausdorff dimension between 1 and 2. First examples of this kind were periodic (lattice) tilings with one tile type [**Giles, Gilb, Dek2**]. Recently they were classified by Bandt and Gelbrich [**BG**], see also [**Ken2, Vince, Ge1**] for related classes of tilings. Dekking [**Dek2, Dek3**] showed how to construct such tilings from free group endomorphisms. His method was applied to more general situations (not necessarily one tile type and non-periodic) by Ito and Kimura [**IK**], Ito and Ohtsuki [**IO1**] and Kenyon [**Ken4**]. A related construction uses the projection of a stepped surface [**Bed1, IO3, Ken1**]. A different (but also related) method, based on numeration systems with complex base, was employed in [**Gilb, Rau, Ken1, Petr, Prag**]. Particular examples of both kinds are given in §7.

(f) In the recent papers [**LW1, LW2**] Lagarias and Wang investigate *self-affine tiles* (such a tile is a compact set of positive Lebesgue measure whose affinely inflated copy is a union of translations of the tile, disjoint in measure). They proceed to characterize tilings of $\mathbb{R}^d$ by self-affine tiles (which are sometimes, but not always, self-affine tilings with one tile type, according to our definition). The paper [**LW1**] extends and clarifies the work of Kenyon [**Ken2**]. It also contains a bibliography on applications of self-affine tiles to wavelets.

(g) Self-affine tilings with nowhere differentiable boundary arise in connection with Markov partitions for hyperbolic toral automorphisms (by projecting the partition onto the unstable subspace). It is known that Markov partitions are non-smooth in dimension higher than two. The actual constructions [**Bed1, Bed2, Prag, IO2**] are based on one of the methods mentioned in (e).

Which λ (respectively ϕ) can be expansion constants (maps) of a self-similar (self-affine) tiling? It follows from [**Lind**] that for $d = 1$ they are characterized as real Perron numbers (a real Perron number λ is an algebraic integer > 0 whose Galois conjugates are all less than λ in modulus). This has a remarkable generalization.

THEOREM 2.1. (Thurston, Kenyon)

(i) *A self-similar tiling of the plane with expansion constant λ exists if and only if λ is a complex Perron number, that is, an algebraic integer whose Galois conjugates, except $\bar{\lambda}$, are less than $|\lambda|$ in modulus.*

(ii) *A self-affine tiling of $\mathbb{R}^d$ with diagonalizable expansion map ϕ exists if and only if the eigenvalues of ϕ are algebraic integers, and for each eigenvalue of multiplicity $k > 0$ its Galois conjugates which are larger or equal in modulus are also eigenvalues of ϕ and have multiplicity $\geq k$.*

Part (i) was announced by Thurston in his lecture notes [**Thur**] with a proof of necessity. Kenyon [**Ken1**] obtained part (ii) and has recently written a careful proof of sufficiency in part (i) [**Ken4**].

Definition. Let $\mathcal{T}$ be a ϕ-subdividing tiling and let T_j, $j = 1, \dots, m$, be the representatives of all tile types. The $m \times m$ matrix $M = M_{\mathcal{T}} = (M_{ij})$, where M_{ij} is the number of tiles of type i in ϕT_j, is called the *subdivision matrix* of the tiling $\mathcal{T}$. The decomposition

$$\phi T_j = \cup_{i=1}^{m} \cup_{k=1}^{M_{ij}} (T_i + g_{ik}^{j}) \tag{2}$$

will be called the *canonical subdivision* of ϕT_j.

LEMMA 2.2. (Praggastis [**Prag**, 1.4]) *A ϕ-subdividing tiling with a finite number of local patterns is self-affine (that is, has the local isomorphism property) if and only if its subdivision matrix is primitive ($M^k > 0$ for some $k > 0$).*

Let Vol denote the d-dimensional Lebesgue measure.

LEMMA 2.3. (Praggastis [**Prag**, 1.6]) *For any tile T of a self-affine tiling*, $\mathrm{Vol}(\partial T) = 0$.

By the Perron–Frobenius theory (see [**Snt**]), a primitive matrix M has an eigenvalue $\theta > 0$, called the *Perron eigenvalue*, which is greater in modulus than all other eigenvalues. The matrix M has strictly positive right and left eigenvectors corresponding to θ, called *Perron eigenvectors* and denoted by $\vec{r}$ and $\vec{l}$, such that

$$\lim_{n\to\infty} \theta^{-n} M^n = \vec{r} \cdot \vec{l}, \quad \vec{l} \cdot \vec{r} = 1.$$

Moreover, M has no other positive eigenvectors.

COROLLARY 2.4. *The Perron eigenvalue for the subdivision matrix of a self-affine tiling with expansion map ϕ is $|\det \phi|$. The vector $\vec{l} = (\mathrm{Vol}(T_j))_1^m$ is a left Perron eigenvector. Thus, for some $r_i > 0$, $i \leq m$,*

$$\lim_{n\to\infty} |\det \phi|^{-n} (M^n)_{ij} = r_i \, \mathrm{Vol}(T_j), \quad \sum_{i=1}^{m} r_i \, \mathrm{Vol}(T_i) = 1.$$

Proof. We only need to check that $\vec{l}M = |\det\phi|\vec{l}$, since $\vec{l}$ is strictly positive. By Lemma 2.3 and (2),

$$|\det\phi|\,\mathrm{Vol}(T_j) = \mathrm{Vol}(\phi T_j) = \sum_{i=1}^{m}\sum_{k=1}^{M_{ij}} \mathrm{Vol}(T_i + g_{ik}^{j}) = \sum_{i=1}^{m} M_{ij}\,\mathrm{Vol}(T_i),$$

which implies the desired statement. □

By Corollary 2.4, if λ is an expansion constant for a self-similar tiling of the plane, then $\det(\phi) = \lambda\bar{\lambda} = |\lambda|^2$ is a real Perron number. This is, however, weaker than λ being a complex Perron number, which characterizes such expansion constants.

As was already mentioned, the ϕ-subdividing property implies that any $\mathcal{T}$-patch will form a $\phi\mathcal{T}$-patch after some composition and adding missing tiles. If this can be done uniquely, at least deep inside the original patch, we will say that $\mathcal{T}$ has the *unique composition property*. The formal definition follows. (We remark that a closely related condition is considered in [**GS, Sen**].)

Definition 2.5. Let $[U]^{\mathcal{T}} = \{T \in \mathcal{T} : T \cap U \neq \emptyset\}$ stand for the $\mathcal{T}$-patch of a set U. A ϕ-subdividing tiling $\mathcal{T}$ is said to have the unique composition property if for any $r > 0$ there exists $R > 0$ such that for any $y_1, y_2, g \in \mathbb{R}^d$, then

$$[B_R(y_1)]^{\mathcal{T}} = [B_R(y_2)]^{\mathcal{T}} + g \Rightarrow [B_r(y_1)]^{\phi\mathcal{T}} = [B_r(y_2)]^{\phi\mathcal{T}} + g.$$

There is another way to state the unique composition property. Suppose that $\mathcal{T}$ is ϕ-subdividing. Then for any tile $T = T_j + g$, the (translated) formula (2) gives a specific 'subdivision rule' for ϕT. Thus for any tiling $\mathcal{S} \in X_{\mathcal{T}}$ we can consider the tiling $\phi\mathcal{S}$ which is then subdivided into tiles equivalent to $\mathcal{T}$-tiles. The resulting tiling will be an element of the tiling space $X_{\mathcal{T}}$, so we get the 'inflate and subdivide' map $\Psi : X_{\mathcal{T}} \to X_{\mathcal{T}}$. Clearly, Ψ is continuous and has the fixed point $\mathcal{T} : \Psi\mathcal{T} = \mathcal{T}$. Notice that Ψ and the action by translations are related as follows:

$$\Psi(\mathcal{S} - g) = \Psi(\mathcal{S}) - \phi g, \quad g \in \mathbb{R}^d.$$

LEMMA 2.6. *Let $\mathcal{T}$ be a ϕ-subdividing tiling. The following are equivalent:*
(i) *$\mathcal{T}$ has the unique composition property;*
(ii) *the map Ψ is invertible;*
(iii) *for any tiling $\mathcal{S} \in X_{\mathcal{T}}$ there is a unique tiling $\mathcal{S}' \in X_{\phi\mathcal{T}}$ such that each $\mathcal{S}'$-tile is a union of $\mathcal{S}$-tiles.*

Proof. Cearly, (ii)$\Leftrightarrow$ (iii), since (iii) just means that the subdivision map from $X_{\phi\mathcal{T}}$ to $X_{\mathcal{T}}$ is invertible. The equivalence of (i) and (iii) is an easy consequence of compactness. □

LEMMA 2.7. *A ϕ-subdividing tiling with the unique composition property is non-periodic.*

Proof. The proof is basically an argument of Grünbaum and Shephard [**GS**, 10.1.1]. Suppose that $\mathcal{T} - g = \mathcal{T}$ for some non-zero $g \in \mathbb{R}^d$. Then we have $\Psi^{-1}(\mathcal{T}) - \phi^{-1}g = \Psi^{-1}(\mathcal{T} - g) = \Psi^{-1}(\mathcal{T})$. Since $\Psi^{-1}(\mathcal{T}) = \mathcal{T}$, we get that $\mathcal{T} - \phi^{-1}g = \mathcal{T}$. Repeating this leads to a contradiction since a small non-zero translation of a $\mathcal{T}$-tile cannot be a $\mathcal{T}$-tile. □

Word substitutions. In most papers on 'inflation tilings', 'substitution tilings', or 'self-similar tilings', substitutions serve as a starting point. We will also use many methods and ideas from substitution dynamics, so it seems appropriate to review the relevant definitions. Our main source on substitutions was [**Queff**].

Definition. Let $\mathcal{A}$ be a finite alphabet, $\mathcal{A}^* = \cup_{n\geq 1}\mathcal{A}^n$ (the set of finite words), and let $\mathcal{A}^{\mathbb{N}}$ be the set of infinite words with letters in $\mathcal{A}$. A *substitution* is a map $\zeta : \mathcal{A} \to \mathcal{A}^*$. We will assume that ζ is injective. The substitution is extended to $\mathcal{A}^*$ and $\mathcal{A}^{\mathbb{N}}$ by concatenation. The length of a word w is denoted by $|w|$.

Assume that $|\zeta^n(\alpha)| \to \infty$, as $n \to \infty$, for $\alpha \in \mathcal{A}$. Then one can find $k \geq 1$ and $u = u_0u_1u_2\ldots \in \mathcal{A}^{\mathbb{N}}$ such that $\zeta^k(u) = u$ (see [**Queff**, V.1]). For the purposes of dynamics, one can replace ζ with ζ^k, so without loss of generality it can be assumed that $\zeta(u) = u$. It is convenient to set $\mathcal{A} = \{1, 2, \ldots, m\}$, where $1 = u_0$; then $\zeta(1) = 1w, |w| \geq 1,$ and $u = 1w\zeta(w)\zeta^2(w)\ldots.$

The space $\mathcal{A}^{\mathbb{N}}$ is compact in the product topology. The *substitution space* X_ζ is the orbit closure of u under the shift map σ:

$$X_\zeta = \mathrm{Clos}\{\sigma^n u : n \geq 0\}.$$

The pair (X_ζ, σ) forms the *substitution dynamical system*.

Alternatively, one can consider a two-sided substitution dynamical system (X'_ζ, σ) where X'_ζ can be defined as the set of all sequences $x \in \mathcal{A}^{\mathbb{Z}}$ such that every block of x occurs in u.

The *substitution matrix* is a $m \times m$ matrix with the entries $M_\zeta(i, j)$ equal to the number of letters i in $\zeta(j)$. The substitution dynamical system is minimal if and only if the matrix M_ζ is primitive.

There are obvious parallels between substitution and subdividing tiling dynamical systems: the ϕ-subdividing tiling plays the role of the fixed point u, the action of Ψ corresponds to the substitution action ζ while the action by translations corresponds to the shift. Further, the subdivision matrix $M_{\mathcal{T}}$ is the analog of M_ζ, in both cases primitivity is equivalent to minimality.

One-dimensional tilings. Suppose that we have a tiling of $\mathbb{R}$ by connected tiles. Thus the tiles are intervals; they are distinguished by their labels—'tile types'—and possibly, but not necessarily, their lengths. We will label the tile types by elements of $\mathcal{A} = \{1, 2, \ldots, m\}$ and let s_i be their lengths. Such a tiling of $\mathbb{R}$ can be identified with a pair (x, t) where $x \in \mathcal{A}^{\mathbb{Z}}$ and $t \in [0, s_{x(0)})$ (t is the distance from 0 to the left endpoint of the tile which covers 0). Consider the set of tilings arising from the substitution space X'_ζ (the two-sided version). On this set there is a natural $\mathbb{R}$-action by translations. It is easy to see that the resulting tiling dynamical system is a flow under the function $f(x) = s_{x_0}$, built over the substitution dynamical system [**BR**]. Much of the theory of substitutions can be carried over to this setting.

We are concerned with the special case, when the tiling space arises from a ϕ-subdividing tiling. This is equivalent to the choice of $(s_i)_{i=1}^m$ as a left Perron eigenvector for the substitution matrix M_ζ. The map ϕ is the multiplication by some $\lambda, |\lambda| > 1$. The tiling dynamical system remains unchanged if we replace λ by λ^2, so we can assume that the expansion constant is positive.

LEMMA 2.8. *A non-periodic self-similar tiling of $\mathbb{R}$ with connected tiles and positive expansion constant has the unique composition property.*

Proof. By a theorem of Mossé [**Mos**], the fixed point u of a non-periodic primitive substitution ζ is bilaterally recognizable which is precisely the substitution version of the unique composition property. In view of the connection indicated above, the result follows. □

It should be mentioned that tilings of the line with disconnected tiles have also been considered [**Ken2, LW3**].

3. *Unique ergodicity*

Let $(X_{\mathcal{T}}, \Gamma_g)$ be the tiling dynamical system arising from a tiling $\mathcal{T}$ of $\mathbb{R}^d$ with finitely many local patterns and the local isomorphism property. From the theory of group actions on a compact space we know that there is an invariant measure, that is, a Borel probability measure on $X_{\mathcal{T}}$, such that $\mu(E) = \mu(E - g)$, $g \in \mathbb{R}^d$. If such a measure is unique, the system is said to be *uniquely ergodic*.

THEOREM 3.1. *Let $\mathcal{T}$ be a self-affine tiling of $\mathbb{R}^d$. Then the tiling dynamical system $(X_{\mathcal{T}}, \Gamma_g)$ is uniquely ergodic.*

Remark. This result is basically known, although not exactly in our setting. Unique ergodicity for substitution dynamical systems was proved by Michel [**Mi1**] (see also [**Queff**]). Radin [**Rad2**] proves unique ergodicity for 'substitution-tilings'. Notice, however, that Radin considers a larger tiling space which is rotation-invariant. Thus he needs an additional assumption that relative rotations of congruent tiles are irrational, the property which cannot hold when there are finitely many tiles up to translation.

Notation 3.2. Let P be a patch and let $A \subset \mathbb{R}^d$. Denote by $L_P(A)$ the number of $\mathcal{T}$-patches contained in A and equivalent to P. (Of course, this depends on the tiling $\mathcal{T}$ which will always be fixed.) If we specify a tile in P then distinct $\mathcal{T}$-patches equivalent to P must have distinct specified tiles (notice that the whole patches may overlap). This implies a useful estimate

$$L_P(A) \le (1/V_{\min}) \operatorname{Vol}(A), \tag{3}$$

where $V_{\min}$ is the minimal volume of a $\mathcal{T}$-tile. We will also be using the following notation:

$$A^{+r} = \{x : \operatorname{dist}(x, A) \le r\}, \quad A^{-r} = \{x \in A : \operatorname{dist}(x, \partial A) \ge r\}.$$

A sequence of sets $A_n \subset \mathbb{R}^d$ is a *Van Hove* sequence if for any $r > 0$,

$$\lim_{n\to\infty} \operatorname{Vol}((\partial A_n)^{+r}) / \operatorname{Vol}(A_n) = 0$$

(this term is used in statistical mechanics, see [**Rue, GH**]).

THEOREM 3.3. *Let $\mathcal{T}$ be a tiling with a finite set of local patterns and the local isomorphism property. The tiling dynamical system $(X_{\mathcal{T}}, \Gamma_g)$ is uniquely ergodic if there exist uniform frequencies of $\mathcal{T}$-patches in the sense of Van Hove, that is, for any $\mathcal{T}$-patch P there is a number* $\mathrm{freq}(P) > 0$ *such that for any Van Hove sequence* $\{A_n\}$,

$$\lim_{n\to\infty} L_P(A_n)/\operatorname{Vol}(A_n) = \mathrm{freq}(P).$$

Notice that if the frequency $\mathrm{freq}(P)$ exists, it has to be positive by local isomorphism.

THEOREM 3.4. *Let $\mathcal{T}$ be a self-affine tiling. Then the frequencies of $\mathcal{T}$-patches exist in the sense of Van Hove.*

Clearly, Theorem 3.3 and Theorem 3.4 imply Theorem 3.1. We omit the proof of Theorem 3.4: it was given by Geerse and Hof [**GH**] for self-similar tilings whose tiles are polytopes, but the argument readily extends to our situation, due to Lemma 2.3 of Praggastis. The existence of uniform frequencies of individual tiles in the Penrose tiling was established by Grünbaum and Shephard [**GS**, 10.5.5]. Uniform frequencies of patches in cubes were shown to exist by Lunnon and Pleasants [**LP**] (again for polytopes); Peyrière [**Peyr**] has a similar result for the Penrose tiling. All the proofs rely on the Perron–Frobenius theory.

Proof of Theorem 3.3. The tiling dynamical system $(X_{\mathcal{T}}, \Gamma_g)$ is uniquely ergodic if and only if for any function f continuous on $X_{\mathcal{T}}$ and any Van Hove sequence $\{A_n\}$,

$$(I_{A_n} f)(\mathcal{S}) := \frac{1}{\operatorname{Vol}(A_n)} \int_{A_n} f(\mathcal{S} - g)\, dg \to \text{constant}, \quad n \to \infty, \tag{4}$$

uniformly in $\mathcal{S} \in X_{\mathcal{T}}$, with the constant depending on f but not on A_n. This can be proved similar to the case of a $\mathbb{Z}$-action (see [**Furst**, §3.2]). Since the orbit $\{\mathcal{T} - h, h \in \mathbb{R}^d\}$ is dense in $X_{\mathcal{T}}$, an equivalent statement is that (4) holds for $\mathcal{S} = \mathcal{T} - h$ uniformly in h. Next, notice that $(I_{A_n} f)(\mathcal{T} - h) = (I_{A_n+h} f)(\mathcal{T})$ and $\{A_n + h\}$ is a Van Hove sequence for any $h \in \mathbb{R}^d$. Thus, unique ergodicity is equivalent to the existence of the limit $\lim_{n\to\infty}(I_{A_n} f)(\mathcal{T})$, independent of the Van Hove sequence $\{A_n\}$.

Using the decomposition (1) of the tiling space, we can approximate a continuous function on $X_{\mathcal{T}}$ by a step-function in the sup norm. Thus, f in (4) can be assumed a characteristic function of a cylinder set $X_{P,U}$ with U sufficiently small.

Suppose that $\operatorname{diam}(U) < \eta$, where $\eta = \eta(\mathcal{T}) > 0$ is such that every tile $T \in \mathcal{T}$ contains a ball of diameter η. By the definition of the cylinder set (Definition 1.3),

$$J_n(f) := \int_{A_n} f(\mathcal{T} - g)\, dg = \operatorname{Vol}\{g \in A_n : P - k + g \text{ is a } \mathcal{T} - \text{patch for some } k \in U\}.$$

Let $P_\nu = P + g_\nu$, $\nu \geq 1$, be the list of all $\mathcal{T}$-patches equivalent to P. Then

$$J_n(f) = \operatorname{Vol}\{g \in A_n : \exists \nu \geq 1, \exists k \in U, -k + g = g_\nu\} = \operatorname{Vol}\left[\cup_{\nu\geq 1}(A_n \cap (U + g_\nu))\right].$$

By Lemma 1.5(i), the distance between distinct g_ν is at least $\eta > \operatorname{diam}(U)$, so the sets $U + g_\nu$ are disjoint. Let

$$r = \max\{\|k\| : k \in U\} + \max\{\|y\| : y \in P\}.$$

Then for any patch $P_\nu \subset A_n^{-r}$ we have $U + g_\nu \subset A_n$, and for any patch $P_\nu \subset \mathbb{R}^d \setminus A_n^{+r}$ we have $U + g_\nu \cap A_n = \emptyset$. It follows that

$$\mathrm{Vol}(U) L_P(A_n^{-r}) \geq J_n(f) \geq \mathrm{Vol}(U) L_P(A_n^{+r}).$$

Observe that for a Van Hove sequence $\mathrm{Vol}(A_n) \sim \mathrm{Vol}(A_n^{+r}) \sim \mathrm{Vol}(A_n^{-r})$ and (3) implies $L_P(A_n^{-r}) \sim L_P(A_n) \sim L_P(A_n^{+r})$, as $n \to \infty$. Since $(I_{A_n} f)(\mathcal{T}) = J_n(f)/\mathrm{Vol}(A_n)$, Theorem 3.3 follows. This also concludes the proof of Theorem 3.1. □

In the context of Theorem 3.1 let μ denote the unique translation-invariant measure on the tiling space $X_{\mathcal{T}}$. The constant in (4) must be the integral of f with respect to μ. Thus we get the following result.

COROLLARY 3.5. *Let $\mathcal{T}$ be a self-affine tiling with expansion map ϕ and let $\eta = \eta(\mathcal{T}) > 0$ be such that every $\mathcal{T}$-tile contains a ball of diameter η. Then for any $\mathcal{T}$-patch P and any $U \subset \mathbb{R}^d$ with* $\mathrm{diam}(U) < \eta$*, we have*

$$\mu(X_{P,U}) = \mathrm{Vol}(U) \lim_{n\to\infty} \frac{L_P(\phi^n T)}{\mathrm{Vol}(\phi^n T)} = \mathrm{Vol}(U) \lim_{\mathrm{Vol}(Q)\to\infty} \frac{L_P(Q)}{\mathrm{Vol}(Q)} = \mathrm{Vol}(U)\,\mathrm{freq}(P).$$

Here T is any $\mathcal{T}$-tile and Q is a d-dimensional cube. Every cylinder set has positive measure.

Proof. To prove the first equality, observe that $\phi^n T$ is a Van Hove sequence, since ϕ is expansive and $\mathrm{Vol}(\partial T) = 0$. Certainly, a sequence of cubes with volumes tending to infinity is Van Hove as well, which yields the second equality. □

4. *Strong and weak mixing*

THEOREM 4.1. *If $\mathcal{T}$ is a self-affine tiling, then the $\mathbb{R}^d$-action $(X_{\mathcal{T}}, \mu, \Gamma_g)$ is not mixing.*

Remark. This is analogous to the theorem of Dekking and Keane [**DK**] from substitution dynamics. It is an open problem to find a mixing tiling dynamical system. Radin conjectured that the pinwheel tiling [**Rad3**] gives rise to such a system. The pinwheel tiling has many common features with our self-similar tilings, but there is a crucial distinction: the set of tiles is finite up to translations and rotations but not up to translations only.

Proof. Consider the set of translation vectors between $\mathcal{T}$-tiles of the same type:

$$\Xi(\mathcal{T}) = \{x \in \mathbb{R}^d : \exists T, T' \in \mathcal{T}, T \approx T', T' = T + x\}. \tag{5}$$

Let $X_{P,U}$ be a cylinder set with $\mathrm{diam}(U) < \eta$, where $\eta = \eta(\mathcal{T})$ is such that every $\mathcal{T}$-tile contains a ball of diameter η.

LEMMA 4.2. *For $x \in \Xi(\mathcal{T})$ there exists $\delta = \delta(x)$ independent of P and U, such that for all $n > n(P)$,*

$$\mu[X_{P,U} \cap \Gamma_{-\phi^n x}(X_{P,U})] > \delta\mu(X_{P,U}).$$

Notice that the theorem immediately follows from Lemma 4.2. Indeed, if the tiling dynamical system were mixing, we would have

$$\mu[X_{P,U} \cap \Gamma_{-\phi^n x}(X_{P,U})] \to [\mu(X_{P,U})]^2, \quad n \to \infty.$$

Choosing U small enough, one can ensure that $0 < \mu(X_{P,U}) < \delta$ (see Corollary 3.5) which leads to a contradiction.

Proof of Lemma 4.2. By Lemma 1.4, we have for n such that $P \cap (P + \phi^n x) = \emptyset$:

$$X_{P,U} \cap \Gamma_{-\phi^n x}(X_{P,U}) = X_{P,U} \cap X_{P+\phi^n x,U} \supset X_{P\cup(P+\phi^n x),U}.$$

Fix any tile, say, T_k of type k. By Corollary 3.5,

$$\mu(X_{P\cup(P+\phi^n x),U}) = \mathrm{Vol}(U) \lim_{N\to\infty} \frac{L_{P\cup(P+\phi^n x)}(\phi^N T_k)}{\mathrm{Vol}(\phi^N T_k)}. \tag{6}$$

Recall that $L_{P\cup(P+\phi^n x)}(\phi^N T_k)$ is the number of patches in $\phi^N T_k$ equivalent to $P \cup (P + \phi^n x)$. We are going to estimate it from below.

Since $x \in \Xi(\mathcal{T})$, there exist two $\mathcal{T}$-tiles T_j and T_j' of (some) type j such that $T_j' = T_j + x$. One can find a $\mathcal{T}$-tile of (some) type i, which will be denoted T_i, and $k_0 > 0$, so that $\phi^{k_0} T_i$ contains both T_j and $T_j' = T_j + x$. Then $\phi^{n+k_0} T_i$ contains $\phi^n T_j$ and $\phi^n T_j + \phi^n x$. This means that for any $\mathcal{T}$-patch P_ν in $\phi^n T_j$ equivalent to P, the patch $P_\nu \cup (P_\nu + \phi^n x)$ is in $\phi^{n+k_0} T_i$. Thus, we get at least $L_P(\phi^n T_j)$ patches in $\phi^{n+k_0} T_i$ equivalent to $P \cup (P + \phi^n x)$ (see Figure 4.1).

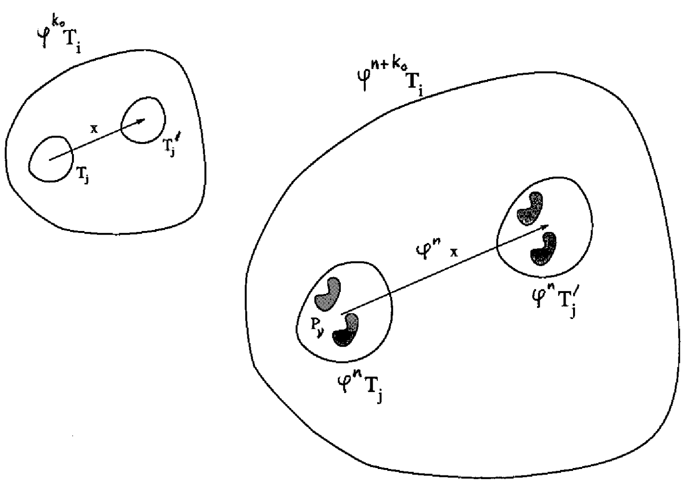

FIGURE 4.1. $L_{P\cup(P+\phi^n x)}(\phi^{n+k_0} T_i) \geq L_P(\phi^n T_j)$.

Now decompose $\phi^N T_k$ (for $N > n + k_0$) into tiles of the tiling $\phi^{n+k_0}\mathcal{T}$. Observe that the number of $\phi^{n+k_0}\mathcal{T}$-tiles contained in $\phi^N T_k$ and equivalent to $\phi^{n+k_0} T_i$, equals $L_{T_i}(\phi^{N-n-k_0} T_k)$. Thereby we obtain

$$L_{P\cup(P+\phi^n x)}(\phi^N T_k) \geq L_{T_i}(\phi^{N-n-k_0} T_k) L_P(\phi^n T_j).$$

By the definition of the subdivision matrix M, $L_{T_i}(\phi^{N-n-k_0}T_k) = (M^{N-n-k_0})_{ik}$. It follows from (6) and Corollary 2.4 that

$$\begin{aligned}\mu(X_{P\cup(P+\phi^n x),U}) &\geq \mathrm{Vol}(U)\lim_{N\to\infty}\frac{(M^{N-n-k_0})_{ik}L_P(\phi^n T_j)}{|\det\phi|^N\,\mathrm{Vol}(T_k)}\\ &= \mathrm{Vol}(U)|\det\phi|^{-n-k_0}r_i L_P(\phi^n T_j)\end{aligned}$$

(here (r_i) is a right Perron eigenvector for M). On the other hand, using Corollary 3.5 again, we obtain

$$\mu(X_{P,U}) = \mathrm{Vol}(U)\lim_{n\to\infty}\frac{L_P(\phi^n T_j)}{|\det\phi|^n\,\mathrm{Vol}(T_j)}.$$

It follows that

$$\liminf_{n\to\infty}\frac{\mu(X_{P\cup(P+\phi^n x),U})}{\mu(X_{P,U})} \geq r_i\,\mathrm{Vol}(T_j)|\det\phi|^{-k_0}.$$

This implies the lemma with $\delta = \frac{1}{2}r_i\,\mathrm{Vol}(T_j)|\det\phi|^{-k_0}$ which is independent of P. □

The same Lemma 4.2 which was used to show that the tiling dynamical system is not mixing, can be applied to obtain sufficient conditions for weak mixing. We remark that examples of weakly mixing tiling dynamical systems (not self-similar) were recently constructed by Berend and Radin [**BR**], using the results of Mozes [**Moz**].

The $\mathbb{R}^d$-action $(X_{\mathcal{T}}, \Gamma_g)$ gives rise to a group of unitary operators $\{U_g\}$ on $L^2(X_{\mathcal{T}}, \mu)$:

$$U_g f(\mathcal{S}) = f(\mathcal{S} - g).$$

A vector $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ is said to be an eigenvalue for the $\mathbb{R}^d$-action if there exists an eigenfunction $f \in L^2(X_{\mathcal{T}}, \mu)$, that is, $f \not\equiv 0$ and

$$U_g f(\mathcal{S}) = e^{2\pi i\langle g,\alpha\rangle} f(\mathcal{S}), \quad g \in \mathbb{R}^d.$$

Here and below we use the notation $\langle u, v\rangle = \sum_{i=1}^d u_i\overline{v_i}$, for $u, v \in \mathbb{C}^d$. Recall that a dynamical system is weakly mixing if and only if it has no non-constant eigenfunctions.

THEOREM 4.3. *Let $\mathcal{T}$ be a self-affine tiling of $\mathbb{R}^d$ with expansion map ϕ. If $\alpha \in \mathbb{R}^d$ is an eigenvalue for $(X_{\mathcal{T}}, \mu, \Gamma_g)$, then for any $x \in \Xi(\mathcal{T})$ we have* $\lim_{n\to\infty} e^{2\pi i\langle\phi^n x,\alpha\rangle} = 1$.

This is analogous to a theorem due to Host [**Host**]. The converse is also true, provided $\mathcal{T}$ has the unique composition property, as will be shown in the next section.

Proof of Theorem 4.3. Suppose that $f(\mathcal{S})$ is a measurable eigenfunction, corresponding to the eigenvalue $\alpha \in \mathbb{R}^d$. By ergodicity, $|f| =$ constant a.e., so one can assume $|f| = 1$ a.e. Let us approximate f by a step-function in the L^1-norm. Fix $\epsilon > 0$. It is well known that on a compact space every Borel measure is regular, so continuous functions are dense in $L^1(X_{\mathcal{T}}, \mu)$. Let $\tilde{f}$ be a function continuous on $X_{\mathcal{T}}$, such that $\|f - \tilde{f}\|_1 < \epsilon/2$. By Lemma 1.6, there is a decomposition $X_{\mathcal{T}} = \cup_l X_{P_l,V_l}$ into disjoint cylinder sets of arbitrarily small diameter. Then $\tilde{f}$ can be approximated by a linear combination of their characteristic functions χ_l, that is, for some c_l,

$$\left\|\tilde{f} - \sum_l c_l\chi_l\right\|_\infty < \epsilon/2.$$

Clearly, $g = \sum_l c_l \chi_l$ satisfies $\|f - g\|_1 < \epsilon$. Let

$$A_{n,\epsilon} = \bigcup_l Y_l, \quad \text{where } Y_l = X_{P_l,V_l} \cap X_{P_l+\phi^n x, V_l}.$$

This union is also disjoint. If ϵ is sufficiently small, Lemma 4.2 implies for $n > n_1(\epsilon)$ that

$$\mu(A_{n,\epsilon}) \geq \delta \sum_l \mu(X_{P_l,V_l}) = \delta\mu(X_{\mathcal{T}}) = \delta > 0,$$

where δ is independent of ϵ. Now we can write

$$\begin{aligned} I &:= \int_{A_{n,\epsilon}} |f(\mathcal{S} - \phi^n x) - f(\mathcal{S})|\, d\mu = \int_{A_{n,\epsilon}} |e^{2\pi i\langle \phi^n x,\alpha\rangle} f(\mathcal{S}) - f(\mathcal{S})|\, d\mu \\ &= \mu(A_{n,\epsilon})|e^{2\pi i\langle \phi^n x,\alpha\rangle} - 1| \geq \delta |e^{2\pi i\langle \phi^n x,\alpha\rangle} - 1|. \end{aligned}$$

On the other hand,

$$\begin{aligned} I &\leq \int_{A_{n,\epsilon}} |f(\mathcal{S} - \phi^n x) - g(\mathcal{S} - \phi^n x)|\, d\mu + \int_{A_{n,\epsilon}} |g(\mathcal{S} - \phi^n x) - g(\mathcal{S})|\, d\mu \\ &\quad + \int_{A_{n,\epsilon}} |g(\mathcal{S}) - f(\mathcal{S})|\, d\mu. \end{aligned}$$

The first and the third integrals do not each exceed $\|f - g\|_1 < \epsilon$. The second integral is equal to zero:

$$\int_{A_{n,\epsilon}} |g(\mathcal{S} - \phi^n x) - g(\mathcal{S})|\, d\mu = \sum_l \int_{Y_l} |g(\mathcal{S} - \phi^n x) - g(\mathcal{S})|\, d\mu = 0.$$

Indeed, for $\mathcal{S} \in Y_l$ we have $\mathcal{S} \in X_{P_l,V_l}$, so $g(\mathcal{S}) = c_l$, and $\mathcal{S} \in X_{P_l+\phi^n x,V_l}$ so $\mathcal{S} - \phi^n x \in X_{P_l,V_l}$, hence $g(\mathcal{S} - \phi^n x) = c_l$. We conclude that $I \leq 2\epsilon$, whereby

$$|e^{2\pi i\langle \phi^n x,\alpha\rangle} - 1| \leq 2\epsilon/\delta.$$

Since δ does not depend on $\epsilon \to 0$, the theorem is proved. □

Next we deduce some necessary conditions for the existence of non-trivial eigenvalues. Since ϕ is diagonalizable, there is a basis for $\mathbb{C}^d$ consisting of eigenvectors. Let $E_1, \ldots, E_r$ be the eigenspaces corresponding to distinct eigenvalues $\theta_1, \ldots, \theta_r$, and let P_i denote the projection onto E_i parallel to E_j, $j \neq i$. Then $x = \sum_{i=1}^r P_i x$. All the eigenvalues θ_i have modulus greater than one (ϕ is expansive) and are algebraic integers (this is the easy part of the Thurston–Kenyon Theorem 2.1). We have

$$\langle \phi^n x, \alpha\rangle = \sum_{i=1}^r \langle P_i x, \alpha\rangle \theta_i^n = \sum_{i=1}^r \langle x, P_i^* \alpha\rangle \theta_i^n, \tag{7}$$

where P_i^* is the operator adjoint to P_i (transpose complex-conjugate).

Definition. A set of algebraic integers $\Theta = (\theta_1, \ldots, \theta_r)$ is a *Pisot family* if for all $i \leq r$, every Galois conjugate γ of θ_i with $|\gamma| \geq 1$ is in Θ. For $r = 1$ this reduces to $|\theta_1|$ being a real Pisot number, and for $r = 2$, to θ_1 being a complex Pisot number.

THEOREM 4.4. *Let $\mathcal{T}$ be a self-affine tiling with ϕ, P_i, θ_i as above. If the tiling dynamical system $(X_{\mathcal{T}}, \mu, \Gamma_g)$ has an eigenvalue $\alpha \in \mathbb{R}^d \setminus \{0\}$, then the set $\Theta = \{\theta_i : P_i^*\alpha \neq 0\}$ is a Pisot family.*

For $d \leq 2$ we get the following result.

COROLLARY 4.5. *Let $\mathcal{T}$ be a self-similar tiling of $\mathbb{R}^d$, $d \leq 2$, with expansion constant λ. If the tiling dynamical system $(X_{\mathcal{T}}, \mu, \Gamma_g)$ has a non-zero eigenvalue, then $|\lambda|$ is a real Pisot number, and λ itself is a complex Pisot number (for $d = 2$).*

Indeed, the implication Theorem 4.4 $\Rightarrow$ Corollary 4.5 is obvious for $d = 1$. If $d = 2$ and the tiling is self-similar, the eigenvalues are λ and $\bar{\lambda}$, or $\lambda \in \mathbb{R}$ is an eigenvalue of multiplicity two. Since λ is complex Pisot if and only if $\bar{\lambda}$ is complex Pisot, and the modulus of a complex Pisot number is real Pisot, the claim follows.

Proof of Theorem 4.4. Let $\alpha \in \mathbb{R}^d, \alpha \neq 0$, be an eigenvalue. By Theorem 4.3, $\exp[2\pi i \langle \phi^n x, \alpha\rangle] \to 1$, for any $x \in \Xi(\mathcal{T})$. First consider the case $d = 1$. Let $\lambda \in \mathbb{R}$ be the expansion constant for $\mathcal{T}$. We know that λ is algebraic and $\lambda^n \beta \to 0 (\mathrm{mod} 1)$, for some $\beta \in \mathbb{R}, \beta \neq 0$. By the Pisot Theorem (see [**Sa**, p. 4]), $|\lambda|$ is a real Pisot number. This is, in fact, a direct proof of Corollary 4.5 for $d = 1$.

For $d > 1$ we need the following generalization of the Pisot Theorem.

LEMMA 4.6. (Mauduit) *Let $\theta_1, \ldots, \theta_r$ be distinct algebraic integers with $|\theta_j| \geq 1$, and $(b_1, \ldots, b_r) \in (\mathbb{R} \setminus \{0\})^r$ be such that* $\exp[2\pi i \sum_{j=1}^r b_j \theta_j^n] \to 1$, *as $n \to \infty$. Then:*
(i) *$\{\theta_1, \ldots, \theta_r\}$ is a Pisot family;*
(ii) *for some $C > 0$, $\rho \in (0, 1)$, $|\exp[2\pi i \sum_{j=1}^r b_j \theta_j^n] - 1| < C\rho^n$;*
(iii) *$b_j \in \mathbb{Q}(\theta_j)$ for $j = 1, \ldots, r$.*

This is a special case of [**Mau**, L.2]; (ii) is not stated explicitly there but easily follows from the proof. Only (i) is needed now; parts (ii) and (iii) will be used in the next section.

Let us conclude the proof of Theorem 4.4. By Lemma 4.6(i), the set $\Theta_1(x) = \{\theta_i : \langle x, P_i^*\alpha\rangle \neq 0\}$ is a Pisot family. By the local isomorphism property, the directions of vectors $x \in \Xi(\mathcal{T})$ (translations between tiles of the same type) are dense in the unit sphere in $\mathbb{R}^d$. Since $\mathbb{R}^d$ spans $\mathbb{C}^d$ as a complex space, one can find $x \in \Xi(\mathcal{T})$ with $\langle x, P_i^*\alpha\rangle \neq 0$ for i such that $P_i^*\alpha \neq 0$. Then $\Theta = \Theta_1(x)$ is a Pisot family. □

Example 4.7. The number $\lambda = -1 + i\sqrt{2 + \sqrt{8}}$ is a complex Perron number but not a complex Pisot number. Thus, by Theorem 2.1 there exists a self-similar tiling of the plane with expansion constant λ, while by Corollary 4.5 the corresponding dynamical system is weakly mixing. At the same time, $\lambda\bar{\lambda} = 3 + \sqrt{8}$ is a real Pisot number. This shows that the presence of a discrete spectral component cannot be inferred from the fact that the subdivision matrix has the Pisot property, as sometimes suggested in the physics literature.

To justify our claim, observe that λ is a root of a polynomial $f(x) = p(q(x))$, where $p(u) = u^2 + 6u + 1$ and $q(x) = x^2 + 2x$. Then $f(x) = x^4 + 4x^3 + 10x^2 + 12x + 1$;

its roots are found from $p(u) = 0$, $u_{1,2} = 3 \pm \sqrt{8}$, $q(\theta_j) = u_{1,2}$. Thus the conjugates of $\lambda = \theta_1$ are $\overline{\lambda} = \theta_2$, $\theta_3 = -1 - \sqrt{\sqrt{8} - 2} \approx -1.91$, and $\theta_4 = -1 + \sqrt{\sqrt{8} + 2} \approx -0.0898$. Since $|\lambda| > 2$, we see that λ is complex Perron, but not complex Pisot.

5. *Eigenfunctions*

In the previous section, necessary conditions for the existence of measurable eigenfunctions were given. Here we present a construction of eigenfunctions, which in many cases leads to a complete characterization of (not) weakly mixing tiling dynamical systems.

THEOREM 5.1.

(i) *Let $\mathcal{T}$ be a self-affine tiling of $\mathbb{R}^d$ with expansion map ϕ, having the unique composition property. Then $\alpha \in \mathbb{R}^d$ is an eigenvalue of the measure-preserving system $(X_{\mathcal{T}}, \mu, \Gamma_g)$ if and only if*

$$\lim_{n\to\infty} e^{2\pi i \langle \phi^n x, \alpha\rangle} = 1 \quad \textit{for all } x \in \Xi(\mathcal{T}). \tag{8}$$

Moreover, if (8) holds, the eigenfunction can be chosen continuous.

(ii) *Let $\mathcal{T}$ be a self-similar tiling of $\mathbb{R}$ with connected tiles and expansion constant λ. The tiling dynamical system is not weakly mixing if and only if $|\lambda|$ is a real Pisot number. If λ is real Pisot and $\mathcal{T}$ is non-periodic, there exists $a \in \mathbb{R}$, $a \neq 0$, such that the set of eigenvalues contains $a\mathbb{Z}[\lambda^{-1}]$.*

(iii) *Let $\mathcal{T}$ be a self-similar tiling of $\mathbb{R}^2 \cong \mathbb{C}$ with expansion constant $\lambda \in \mathbb{C}$, having the unique composition property. Then the tiling dynamical system is not weakly mixing if and only if λ is a complex Pisot number. Moreover, if λ is a non-real Pisot number, there exists $a \in \mathbb{C}, a \neq 0$, such that the set of eigenvalues contains $\{(\alpha_1, \alpha_2) : \alpha_1 + i\alpha_2 \in a\mathbb{Z}[\overline{\lambda}^{-1}]\}$.*

Remarks. (a) Part (i) of the theorem is analogous to a characterization of eigenvalues for substitution dynamical systems due to Host [**Host**], under the assumption of recognizability. In both cases the description is indirect and does not immediately indicate whether non-zero eigenvalues exist.

(b) It is interesting to compare the simple criterion for one-dimensional self-similar tilings (part (ii)) with the rather complicated situation for substitutions.

Let ζ be a primitive non-periodic substitution on m letters, with the substitution matrix M_ζ. One has to deal separately with rational and irrational eigenvalues. In general, the existence of rational eigenvalues is not determined by M_ζ, but if $\det(M_\zeta) = \pm 1$, no non-trivial rational eigenvalues exist. As for irrational eigenvalues, there are two cases.

Case 1: the characteristic polynomial $g_\zeta(t)$ of M_ζ is irreducible over $\mathbb{Q}$. Existence of irrational eigenvalues is determined, although in a fairly complicated manner, by $g_\zeta(t)$ [**Liv2, Sol1**]. If the polynomial $g_\zeta(t)$ is Pisot, there are such eigenvalues [**Host**]. If m is prime, the Pisot condition is both necessary and sufficient for the presence of irrational eigenvalues [**Sol1**]. If m is not prime, there are non-Pisot examples with such eigenvalues [**Sol1**]. (It should be mentioned that the paper [**Sol1**] deals with stationary adic transformations rather than substitution dynamical systems; Livshits [**Liv2**] proved that they are isomorphic.)

Case 2: the characteristic polynomial of M_ζ is reducible. The situation gets even more complicated; in any case, having a Pisot number as the largest eigenvalue of M_ζ does not guarantee that non-trivial eigenvalues exist [**Liv3**].

Note added in proof. Recently Ferenczi, Mauduit and Nogueira gave a complete algebraic characterization of eigenvalues in [**FMN**].

Theorem 5.1 deals with non-periodic tilings, except for the case $d = 1$. The following result has to do with periodic tilings and is well known; we provide it for completeness. Notice that 'sub-periodic' cases (when the translation group $\mathcal{K}_\mathcal{T}$ has positive rank less than d) are not covered.

LEMMA 5.2. *Let $\mathcal{T}$ be a periodic tiling of $\mathbb{R}^d$ with translation group (lattice) $\mathcal{K}_\mathcal{T}$. Then the set of eigenvalues for the tiling dynamical system $(X_\mathcal{T}, \mu, \Gamma_g)$ is the dual lattice*

$$\mathcal{K}_\mathcal{T}^* = \{\alpha \in \mathbb{R}^d : \langle g, \alpha\rangle \in \mathbb{Z}, \forall g \in \mathcal{K}_\mathcal{T}\}$$

Proof. Since the tiling is periodic, the quotient space $\mathbb{R}^d/\mathcal{K}_\mathcal{T}$ is compact. It follows that the orbit of $\mathcal{T}$ is closed, so $X_\mathcal{T}$ can be identified with this quotient space (topologically, the d-dimensional torus). Let $\alpha \in \mathcal{K}_\mathcal{T}^*$, and set

$$f_\alpha(\mathcal{S}) = e^{2\pi i\langle y,\alpha\rangle}, \quad \text{for } \mathcal{S} = \mathcal{T} - y.$$

This is well defined because α is from the dual lattice. Clearly, f_α is a continuous eigenfunction corresponding to α. It is also not hard to see that these eigenfunctions form a basis for $L^2(X_\mathcal{T}, \mu)$. In fact, this tiling dynamical system is topologically conjugate to the Kronecker action on the d-dimensional torus (see [**Furst**]). □

For the proof of Theorem 5.1 we will need the notion of control points [**Thur, Ken1, Prag**]. These are special reference points for tiles with the property that ϕ maps control points into control points.

Definition. For each $\mathcal{T}$-tile T fix a tile γT in the patch ϕT; choose γT with the same relative position for all tiles of the same type. This defines a map $\gamma : \mathcal{T} \to \mathcal{T}$ called the *tile map*. Then define the *control point* for a tile $T \in \mathcal{T}$ by

$$c(T) = \bigcap_{n=0}^{\infty} \phi^{-n}(\gamma^n T).$$

It is clear that control points have the following properties:
(a) $T' = T + c(T') - c(T)$, for any tiles T, T' of the same type;
(b) $\phi(c(T)) = c(\gamma T)$, for $T \in \mathcal{T}$.
Control points are also fixed for tiles of any tiling $\mathcal{S} \in X_\mathcal{T}$: they have the same relative position as in $\mathcal{T}$-tiles. At the same time control points are determined for second, third, etc level tiles by $c(\phi T) = \phi c(T)$.

Proof of Theorem 5.1. (i) The necessity of condition (8) for the absence of weak mixing is contained in Theorem 4.3, so it remains to prove sufficiency.

By Lemma 2.2 the subdivision matrix $M = M_\mathcal{T}$ is primitive, so $M^k > 0$ for some $k > 0$. Without loss of generality, we can replace ϕ with ϕ^k. Indeed, this does not change the tiling dynamical system, condition (8) will still hold, and the unique composition property for ϕ carries over to ϕ^k.

Thus, we assume $M > 0$. This means that for every $\mathcal{T}$-tile T, the patch ϕT contains tiles of all types. This allows us to define control points for $\mathcal{T}$-tiles in a specific way. Choose the tile map so that γT has the same tile type for all $T \in \mathcal{T}$. Then $c(\gamma T) - c(\gamma S) \in \Xi(\mathcal{T})$ for any $T, S \in \mathcal{T}$ and by property (b) of control points,

$$\phi(c(T) - c(S)) \in \Xi(\mathcal{T}), \quad T, S \in \mathcal{T}. \tag{9}$$

Now we are ready to construct eigenfunctions. Let $\alpha \in \mathbb{R}^d$ satisfy (8) and $\mathcal{S} \in X_{\mathcal{T}}$. By the unique composition property and Lemma 2.6(ii), there exists a unique tiling $\mathcal{S}^{(1)} \in X_{\phi\mathcal{T}}$ whose tiles are unions of $\mathcal{S}$-tiles. Repeating this yields a unique sequence of tilings $\mathcal{S}^{(n)} \in X_{\phi^n\mathcal{T}}$, $\mathcal{S}^{(0)} = \mathcal{S}$, such that each $\mathcal{S}^{(j)}$-tile is a union of $\mathcal{S}^{(i)}$-tiles for $i < j$. Fix a sequence of tiles $T^{(n)} \in \mathcal{S}^{(n)}$ such that $0 \in T^{(n)}$ and $T^{(n+1)} \supset T^{(n)}$. Set

$$f_\alpha(\mathcal{S}) = \lim_{n\to\infty} \exp[-2\pi i \langle c(T^{(n)}), \alpha\rangle]. \tag{10}$$

We need to prove: (a) that the limit in (10) exists and does not depend on the choice of $T^{(0)}$; (b) that f_α is a continuous eigenfunction.

The tile $T^{(n+1)}$ is composed of $\mathcal{S}^{(n)}$-tiles. By the definition of control points, $c(T^{(n+1)})$ is the control point of one of these tiles, say $S^{(n)}$. Since $\mathcal{S}^{(n)} \in X_{\phi^n\mathcal{T}}$, the patch $S^{(n)} \cup T^{(n)} \subset T^{(n+1)}$ is equivalent to $\phi^n(S \cup T)$ for some $\mathcal{T}$-tiles S and T lying in the same $\phi\mathcal{T}$-tile. We have

$$c(T^{(n+1)}) - c(T^{(n)}) = c(S^{(n)}) - c(T^{(n)}) = \phi^n(c(S) - c(T)).$$

As S and T lie in the same $\phi\mathcal{T}$-tile, $|c(T) - c(S)| \le K$, where K is a uniform constant (the maximum diameter of a $\phi\mathcal{T}$-tile). Since $\mathcal{T}$ has a finite number of local patterns, the set of vectors $\{c(T) - c(S) : S, T \in \mathcal{T}, |c(T) - c(S)| \le K\}$ is finite. From (8) and (9) we have

$$\exp[-2\pi i \langle \phi^n(c(T) - c(S)), \alpha\rangle] \to 1.$$

Now writing the decomposition (7) for $x = c(T) - c(S)$ and applying Lemma 4.6(ii) yields, for some $C > 0$ and $\rho \in (0, 1)$, independent of $\mathcal{S}$,

$$|\exp[-2\pi i \langle c(T^{(n+1)}) - c(T^{(n)}), \alpha\rangle] - 1| < C\rho^n, \quad n \ge 0.$$

It follows that the limit in (10) exists.

Ambiguity in (10) might arise if the origin 0 were on the boundary of an $\mathcal{S}$-tile. Then we could make a different choice $\tilde{T}^{(0)}$ and get another sequence $\tilde{T}^{(n)}$. However, $\tilde{T}^{(n)} \cap T^{(n)}$ contains the origin for all n, so (notice that here we use the unique composition property!) $\tilde{T}^{(n)}$ and $T^{(n)}$ either coincide, or are adjacent. In the latter case

$$c(\tilde{T}^{(n)}) - c(T^{(n)}) = \phi^n(c(S) - c(T))$$

for some adjacent $\mathcal{T}$-tiles S and T. As above, Lemma 4.6(ii) implies

$$|\exp[-2\pi i \langle c(\tilde{T}^{(n)}) - c(T^{(n)}), \alpha\rangle] - 1| < C\rho^n,$$

which means that the limit in (10) will be the same.

By the definition of the unique composition property, to determine $T^{(n)}$ in the right-hand side of (10), only the knowledge of a sufficiently large $\mathcal{S}$-patch around the origin is needed. This means that the function f_α is continuous.

It remains to show that $f_\alpha(\mathcal{S} - g) = e^{2\pi i \langle g, \alpha\rangle} f(\mathcal{S})$. Let $\hat{T}^{(n)}$ be the sequence of tiles corresponding to $\mathcal{S} - g$ in the same way as $T^{(n)}$ correspond to $\mathcal{S}$, so that

$$f_\alpha(\mathcal{S} - g) = \lim_{n\to\infty} \exp[-2\pi i \langle c(\hat{T}^{(n)}), \alpha\rangle].$$

Observe that either $\hat{T}^{(n)} = T^{(n)} - g$ or $\hat{T}^{(n)}$ is a tile adjacent to $T^{(n)} - g$, for n large enough. In the first case the claim is obvious, since $c(T^{(n)} - g) = c(T^{(n)}) - g$. In the second case the error in the nth term is estimated by $C\rho^n$, exactly as above, so it vanishes in the limit. □

Next we deduce (ii) from (i). If $\mathcal{T}$ is a periodic tiling of the line, the expansion constant λ must be an integer, so $|\lambda|$ is a real Pisot number. The tiling dynamical system is not weakly mixing by Lemma 5.2.

Now suppose that $\mathcal{T}$ is a non-periodic self-similar tiling of $\mathbb{R}$ with expansion constant λ and interval tiles of lengths $s_1, \ldots, s_m$. If the tiling dynamical system has a non-trivial eigenvalue, $|\lambda|$ is a Pisot number by Corollary 4.5. Conversely, suppose that $|\lambda|$ is Pisot. Passing from λ to λ^2 we can assume that $\lambda > 0$, so it remains to verify the last statement of Theorem 5.1(ii). By Corollary 2.4, $(s_1, \ldots, s_m)$ is a left eigenvector of the subdivision matrix, corresponding to its Perron eigenvalue λ. It follows that $s_i/s_1 \in \mathbb{Q}(\lambda)$, so for some polynomial $q \in \mathbb{Z}[t]$, $q(\lambda) \neq 0$, we have $s_i \in (s_1/q(\lambda))\mathbb{Z}[\lambda]$. Let $a = q(\lambda)/s_1$.

In order to apply part (i), we need to check that

$$\exp[2\pi i \lambda^n \alpha x] \to 1, \quad \text{for } x \in \Xi(\mathcal{T}) \text{ and } \alpha \in a\mathbb{Z}[\lambda^{-1}].$$

Recall that $x \in \Xi(\mathcal{T})$ is a translation vector between two equivalent tiles, so x is an integral linear combination of s_i. Thus, $x\alpha \in \mathbb{Z}[\lambda]\mathbb{Z}[\lambda^{-1}] = \mathbb{Z}[\lambda^{-1}]$. It is well known (see [**Sa**]) that if λ is a real Pisot number, then

$$e^{2\pi i \lambda^n \theta} \to 1, \quad \text{for } \theta \in \mathbb{Z}[\lambda^{-1}],$$

so condition (8) is verified. Finally, the unique composition property for $\mathcal{T}$ holds by Lemma 2.8. Applying part (i), already proved, we get the desired statement. □

For the proof of part (iii) we will need a result due to Kenyon.

THEOREM 5.3. (Kenyon) *Let $\mathcal{T}$ be a self-similar tiling of the plane with expansion constant λ. If $\lambda \notin \mathbb{R}$, then $\Xi(\mathcal{T}) \subset b\mathbb{Z}[\lambda]$ for some $b \in \mathbb{C}$. If $\lambda \in \mathbb{R}$, then $\Xi(\mathcal{T}) \subset g_1\mathbb{Z}[\lambda] + g_2\mathbb{Z}[\lambda]$ for some basis $\{g_1, g_2\}$ of $\mathbb{R}^2$.*

This result is taken from [**Ken4**]; its proof is similar to the proof of [**Ken3**, Theorem 9]. We should point out that in all known examples of self-similar tilings the property stated in Theorem 5.3 is obvious from the construction. Actually, Kenyon states the result with $\mathbb{Q}(\lambda)$ instead of $\mathbb{Z}[\lambda]$. However, the free Abelian group generated by $\Xi(\mathcal{T})$ is finitely generated, since the tiling has finitely many local patterns, and the desired statement follows.

Proof of Theorem 5.1(iii). Let $\mathcal{T}$ be a self-similar tiling of the plane $\mathbb{R}^2 \cong \mathbb{C}$ with expansion constant λ. If the (measurable) tiling dynamical system has a non-trivial eigenvalue, then λ is a complex Pisot number by Corollary 4.5.

Conversely, suppose that λ is a non-real Pisot number. By Theorem 5.3, $\Xi(\mathcal{T}) \subset b\mathbb{Z}[\lambda]$. Let $\alpha = \alpha_1 + i\alpha_2 = 2\overline{b}^{-1}h(\overline{\lambda}^{-1})$ for some $h \in \mathbb{Z}[t]$. Notice that for $x = x_1 + ix_2 \in \mathbb{C}$, we have $\langle (x_1, x_2), (\alpha_1, \alpha_2)\rangle = \mathrm{Re}(x\overline{\alpha})$. Fix $k \geq \deg h$, so that $\lambda^k h(\lambda^{-1}) = g(\lambda)$, $g \in \mathbb{Z}[t]$. For $x \in \Xi(\mathcal{T})$ we have $x = bp(\lambda)$, for some $p \in \mathbb{Z}[t]$. Then

$$\mathrm{Re}(\lambda^n x\overline{\alpha}) = 2\,\mathrm{Re}[\lambda^{n-k}g(\lambda)p(\lambda)].$$

Since λ is an algebraic integer, $\sum_j \lambda_j^{n-k} g(\lambda_j)p(\lambda_j) \in \mathbb{Z}$, where the sum is taken over $\lambda_1 = \lambda$, $\lambda_2 = \overline{\lambda}$, and their Galois conjugates λ_j, $j \geq 3$. Since λ is complex Pisot, $|\lambda_j| < 1$ for $j \geq 3$, so

$$\exp[2\pi i\,\mathrm{Re}(\lambda^n x\overline{\alpha})] = \exp\left[2\pi i \sum_{j=1}^{2} \lambda_j^{n-k} g(\lambda_j)p(\lambda_j)\right] \to 1, \quad n \to \infty.$$

Now the claim (with $a = 2\overline{b}^{-1}$) follows from part (i).

Finally, suppose that λ is a real Pisot number. We have from Theorem 5.3 that $\Xi(\mathcal{T}) \subset g_1\mathbb{Z}[\lambda] + g_2\mathbb{Z}[\lambda]$. Let $\{g_1^*, g_2^*\}$ be the dual basis: $\langle g_i, g_j^*\rangle = \delta_{ij}$. One can easily see that condition (8) holds for $\alpha \in g_1^*\mathbb{Z}[\lambda^{-1}] + g_2^*\mathbb{Z}[\lambda^{-1}]$. This concludes the proof of (iii). $\square$

6. *Pure discrete spectrum*

A measure-preserving $\mathbb{R}^d$-action on a Lebesgue space is said to have pure discrete spectrum if the set of measurable eigenfunctions is total (complete) in $L^2(X, \mu)$ or, equivalently, if the projection spectral measure for the group of unitary operators $\{(U_g f)(x) = f(gx), g \in \mathbb{R}^d\}$ is pure discrete.

Deciding if the spectrum is pure discrete is difficult, even for a substitution dynamical system (the $\mathbb{Z}$-action which underlies the one-dimensional self-similar tiling dynamical system). It is conjectured that if the substitution matrix has irreducible characteristic polynomial and the largest eigenvalue is a real Pisot number, then the spectrum is pure discrete. To our knowledge, this has not yet been proved, although in many special cases the conjecture has been verified [**Rau, Mi2, Queff, Sol2**].

In this section we give a sufficient condition for a self-affine tiling dynamical system to have pure discrete spectrum, and then a necessary and sufficient condition in the case of a self-similar tiling in dimension $d \leq 2$. The proof is based on a concrete combinatorial-geometric algorithm ('method of overlaps'). The next section contains several examples of how the algorithm works. Another ingredient of the proof is an abstract operator theoretic result analogous to [**Sol2**].

Let $\mathcal{T}$ be a tiling of $\mathbb{R}^d$ and $x \in \mathbb{R}^d$. Introduce the following notation:

$$D_x = \bigcup\{T \in \mathcal{T} : T + x \in \mathcal{T}\}.$$

Let us point out that writing $T + x \in \mathcal{T}$ means $T + x = T' \approx T$, $T' \in \mathcal{T}$. Of course, the set D_x is non-empty if and only if x is in $\Xi(\mathcal{T})$, the set of translation vectors between

$\mathcal{T}$-tiles of the same type. The symbol $\mathrm{dens}(A)$ will stand for the density of a set $A \subset \mathbb{R}^d$: $\mathrm{dens}(A) = \lim \mathrm{Vol}(A \cap Q)/\mathrm{Vol}(Q)$, for cubes Q as $\mathrm{Vol}(Q) \to \infty$, if the limit exists. Observe that by Theorem 3.3 the density $\mathrm{dens}(D_x)$ exists and can be written as

$$\mathrm{dens}(D_x) = \sum_{i=1}^{m} \mathrm{freq}[T_i \cup (T_i + x)] \, \mathrm{Vol}(T_i),$$

where T_i are representatives of all tile types.

THEOREM 6.1. *Let $\mathcal{T}$ be a self-affine tiling of $\mathbb{R}^d$ with expansion map ϕ. If there exists a basis $\mathcal{B}$ for $\mathbb{R}^d$ such that for all $x \in \mathcal{B}$,*

$$\sum_{n=0}^{\infty} (1 - \mathrm{dens}(D_{\phi^n x})) < \infty, \tag{11}$$

then the tiling dynamical system $(X_{\mathcal{T}}, \mu, \Gamma_g)$ has pure discrete spectrum.

Remark. If condition (11) holds for x, it holds for ϕx as well. Thus, it is enough to check (11) for a single x provided the set $\{x, \phi x, \phi^2 x, \ldots\}$ spans $\mathbb{R}^d$. This is the case, for instance, when $\mathcal{T}$ is a self-similar tiling of the plane with non-real expansion constant.

THEOREM 6.2. *Let $\mathcal{T}$ be a self-similar tiling of $\mathbb{R}^d$, $d \le 2$, with expansion constant λ. The tiling dynamical system $(X_{\mathcal{T}}, \mu, \Gamma_g)$ has pure discrete spectrum if and only if λ is Pisot (real or complex) and*

$$\lim_{n\to\infty} \mathrm{dens}(D_{\lambda^n x}) = 1, \quad x \in \Xi(\mathcal{T}).$$

In the next section it is shown (Example 7.4) that a tiling dynamical system in $\mathbb{R}^2$ can have mixed (not pure discrete) spectrum when $\lambda = 2i$, a Pisot number. One-dimensional examples of this kind are well known; they arise from substitutions of constant length [**Queff**].

The following lemma is the operator theoretic result which we use to prove that the spectrum is pure discrete.

LEMMA 6.3. (Compare with [**Sol2**]) *Let $\mathcal{G} = \{U_g : g \in \mathbb{R}^d\}$ be a group of unitary operators on a Hilbert space H, and let ϕ be an expansive diagonalizable linear map on $\mathbb{R}^d$. Suppose that for a dense subset $\mathcal{F} \subset H$ and for some basis $\mathcal{B}$ of $\mathbb{R}^d$ we have*

$$\sum_{n} \|U_{\phi^n x} f - f\|^2 < \infty, \quad x \in \mathcal{B}, \, f \in \mathcal{F}. \tag{12}$$

Then $\mathcal{G}$ has pure discrete spectrum.

Proof. By the spectral theorem there exists a projection spectral measure $E(\cdot)$ on $\mathbb{R}^d$ such that

$$U_g = \int_{\mathbb{R}^d} e^{2\pi i \langle g,t\rangle} \, dE(t).$$

Let $\mu_f = (E(\cdot) f, f)$ be the spectral measure of $f \in H$, a positive Borel measure on $\mathbb{R}^d$ with the norm $\|f\|^2$. Then

$$\|U_{\phi^n x} f - f\|^2 = \int_{\mathbb{R}^d} |e^{2\pi i \langle \phi^n x, t\rangle} - 1|^2 \, d\mu_f(t).$$

Therefore, (12) implies that for any $\epsilon > 0$,

$$\sum_n \mu_f\{t : |e^{2\pi i\langle\phi^n x,t\rangle} - 1| > \epsilon\} < \infty, \quad \text{for } x \in \mathcal{B}.$$

Let $C_x = \{\alpha : e^{2\pi i\langle\phi^n x,\alpha\rangle} \to 1\}$. By the Borel–Cantelli Lemma, $\mu_f(\mathbb{R}^d \setminus C_x) = 0$, for $x \in \mathcal{B}$. Since this is true for f from a dense set in H, it follows that $E(\mathbb{R}^d \setminus C_x) = 0$ for $x \in \mathcal{B}$. Thus, if the set $C = \cap_{x\in\mathcal{B}} C_x$ is at most countable, the spectral measure $E = \sum_{\alpha\in C} E(\{\alpha\})$ is pure discrete.

Let $\theta_i, i \le r$, be the distinct eigenvalues of ϕ and P_i the projections onto the eigenspaces in $\mathbb{C}^d$, as in §4. From (7) we have $\langle\phi^n x, \alpha\rangle = \sum_{i=1}^r \langle x, P_i^*\alpha\rangle\theta_i^n$. Lemma 4.6(iii) can be applied to obtain

$$\alpha \in C_x \Rightarrow \langle x, P_i^*\alpha\rangle \in \mathbb{Q}(\theta_i), \quad \text{for } i \le d.$$

Since $\mathcal{B}$ is a basis, the set $\{\langle x, P_i^*\alpha\rangle\}_{x\in\mathcal{B}}$ determines $P_i^*\alpha$ uniquely. It follows that there are at most countable many possibilities for $P_i^*\alpha$ if $\alpha \in \cap_{x\in\mathcal{B}} C_x$. Finally, $\alpha = \sum_{i=1}^r P_i^*\alpha$, so the set $\cap_{x\in\mathcal{B}} C_x$ is at most countable. The proof is complete. □

Proof of Theorem 6.1. Recall that linear combinations of characteristic functions of cylinder sets, with U arbitrarily small, are dense in $L^2(X_{\mathcal{T}}, \mu)$. Thus, in view of Lemma 6.3, it remains to show that (11) implies $\sum_n \|U_{\phi^n x} f - f\|^2 < \infty$, where f is the characteristic function of a cylinder set $X_{P,U}$ with U sufficiently small. We have

$$\begin{aligned} \|U_{\phi^n x} f - f\|^2 &= \int_{X_{\mathcal{T}}} |f(\mathcal{S} - \phi^n x) - f(\mathcal{S})|^2 \, d\mu(\mathcal{S}) \\ &= \mu(X_{P,U} \triangle X_{P+\phi^n x,U}) = 2[\mu(X_{P,U}) - \mu(X_{P,U} \cap X_{P+\phi^n x,U})]. \end{aligned} \tag{13}$$

The last equality uses the fact that μ is translation-invariant. By Lemma 1.4(vi), $X_{P,U} \cap X_{P+\phi^n x,U} \supset X_{P\cup P+\phi^n x,U}$, so

$$\|U_{\phi^n x} f - f\|^2 \le 2[\mu(X_{P,U}) - \mu(X_{P\cup P+\phi^n x,U})].$$

One can assume that $\operatorname{diam}(U) < \eta = \eta(\mathcal{T})$, so that Corollary 3.5 yields

$$\|U_{\phi^n x} f - f\|^2 \le 2\,\mathrm{Vol}(U) \lim_{\mathrm{Vol}(Q)\to\infty} \frac{L_P(Q) - L_{P\cup P+\phi^n x}(Q)}{\mathrm{Vol}(Q)}. \tag{14}$$

Say that a $\mathcal{T}$-tile T is 'bad' if $T + \phi^n x$ is not a $\mathcal{T}$-tile equivalent to T. We note that the set $\mathrm{Clos}(\mathbb{R}^d \setminus D_{\phi^n x})$ is exactly the union of bad tiles. The number of bad tiles in Q is at most $c_1 \mathrm{Vol}(Q \setminus D_{\phi^n x})$, where $c_1 = 1/V_{\min}$, and $V_{\min}$ is the minimal volume of a $\mathcal{T}$-tile. Say that a $\mathcal{T}$-patch P_ν equivalent to P is a 'bad patch', if $P_\nu + \phi^n x$ is not a $\mathcal{T}$-patch equivalent to P. A bad patch must have a bad tile. A bad tile cannot belong to more than K bad patches, where K is the number of tiles in P. Therefore, the number of bad patches in Q is at most $c_1 K \,\mathrm{Vol}(Q \setminus D_{\phi^n x})$.

Now observe that $L_P(Q) - L_{P\cup P+\phi^n x}(Q)$ is not greater than the number of bad patches in Q plus the number of patches P_ν equivalent to P such that $P_\nu \subset Q$, $P_\nu + \phi^n x \not\subset Q$. It follows that for $\delta = \|\phi^n x\| + \operatorname{diam}(P)$ we have

$$L_P(Q) - L_{P\cup P+\phi^n x}(Q) \le c_1 K \,\mathrm{Vol}(Q \setminus D_{\phi^n x}) + L_P((\partial Q)^{+\delta})$$

(here we use the notation introduced in §3.2). By (3),

$$L_P((\partial Q)^{+\delta})/\operatorname{Vol}(Q) \le (1/V_{\min})\operatorname{Vol}((\partial Q)^{+\delta})/\operatorname{Vol}(Q) \to 0, \quad \text{as } \operatorname{Vol}(Q) \to \infty.$$

Thus (14) implies

$$\|U_{\phi^n x} f - f\|^2 \le 2c_1 K \cdot \operatorname{Vol}(U) \liminf_{\operatorname{Vol}(Q)\to\infty} \frac{\operatorname{Vol}(Q \setminus D_{\phi^n x})}{\operatorname{Vol}(Q)} = 2c_1 K \cdot \operatorname{Vol}(U)(1 - \operatorname{dens}(D_{\phi^n x})).$$

We see that (12) holds and Lemma 6.3 can be applied to complete the proof. □

Before the proof of Theorem 6.2, we need to develop the 'overlap algorithm'.

Definition. Let $\mathcal{T}$ be a self-affine tiling of $\mathbb{R}^d$ with expansion map ϕ. Let $T, S \in \mathcal{T}$, $y \in \mathbb{R}^d$. We call the pair $\mathcal{O}_{y,T,S} = ((T + y), S)$ an *overlap* if $\operatorname{Int}((T + y) \cap S) \neq \emptyset$. Say that two overlaps are *equivalent* and write $\mathcal{O}_{y,T,S} \approx \mathcal{O}_{y',T',S'}$ if $T + y = T' + y' + g$, $S = S' + g$ for some $g \in \mathbb{R}^d$ and, moreover, $T \approx T'$, $S \approx S'$. For any $y \in \mathbb{R}^d$, there are finitely many equivalence classes of overlaps $\mathcal{O}_{y,T,S}$, because the tiling has a finite number of local patterns. Denote by $[\mathcal{O}_{y,T,S}]$ the equivalence class of an overlap. Further, let $\mathcal{O}_y$ stand for the set of all overlaps $\mathcal{O}_{y,T,S}$ with $T, S \in \mathcal{T}$ and $[\mathcal{O}_y]$ for the corresponding set of equivalence classes. An overlap $\mathcal{O}_{y,T,S}$ is a *coincidence* if $T + y = S$ and $T \approx S$. Sometimes we will write 'union of overlaps' meaning the union of sets $(T + y) \cap S$; this should not cause any confusion.

Now fix $x \in \Xi(\mathcal{T})$. We are going to construct a directed graph $\mathcal{G}_{\mathcal{O}}(\mathcal{T}, x)$ called the *subdivision graph for overlaps*. Its vertices are elements of $\cup_{n\ge 0}[\mathcal{O}_{\phi^n x}]$. We have

$$\phi[(T + \phi^n x) \cap S] = \bigcup_{T' \subset \phi T,\, S' \subset \phi S} ((T' + \phi^{n+1} x) \cap S'), \tag{15}$$

applying the canonical subdivision. An edge leads from $[\mathcal{O}_{\phi^n x,T,S}]$ to $[\mathcal{O}_{\phi^{n+1}x,T',S'}]$ for every overlap in the right-hand side of (15). This defines a directed graph which, in general, may have infinitely many vertices. Notice that an overlap may belong to both $[\mathcal{O}_{\phi^n x}]$ and $[\mathcal{O}_{\phi^{n+1} x}]$, so we may have loops.

Another useful remark is that the graph $\mathcal{G}_{\mathcal{O}}(\mathcal{T}, x)$ has a subgraph whose vertices are coincidences. All edges from overlaps-coincidences lead to coincidences. This subgraph is isomorphic to the subdivision graph for tiles (which is defined in the obvious manner from (2)).

For the rest of the section we will assume (unless stated otherwise) that $\mathcal{T}$ is a self-similar tiling of the complex plane with non-real expansion constant λ. Straightforward modifications have to be made to deal with the other two cases covered by Theorem 6.2 ($d = 1$ and $d = 2$, the expansion constant real).

PROPOSITION 6.4. *Suppose that λ is a non-real Pisot number, $x \in \Xi(\mathcal{T})$. Then the set of equivalence classes of overlaps $\cup_{n\ge 0}[\mathcal{O}_{\lambda^n x}]$ is finite, so that the graph $\mathcal{G}_{\mathcal{O}}(\mathcal{T}, x)$ is finite.*

Proof. Choose $k > 0$ so that $\lambda^k T$ contains tiles of all types for any $T \in \mathcal{T}$. We can replace λ by λ^k without loss of generality (having finitely many vertices in $\cup_{n\ge 0}[\mathcal{O}_{\lambda^{kn} x}]$ will imply that the whole graph $\mathcal{G}_{\mathcal{O}}(\mathcal{T}, x)$ is finite). Then we define control points $c(T)$ for $\mathcal{T}$-tiles as in §5, so that $\lambda(c(T) - c(S)) \in \Xi(\mathcal{T})$.

Let $\mathcal{O} = \mathcal{O}_{\lambda^n x,T,S}$ be an overlap; its equivalence class is determined by the types of T and S and the vector

$$\nu(\mathcal{O}) := c(T + \lambda^n x) - c(S) = c(T) - c(S) + \lambda^n x.$$

For $\mathcal{O}$ to be an overlap we must have $\mathrm{Int}[(T + \lambda^n x) \cap S] \neq \emptyset$, so

$$|\nu(\mathcal{O})| < 2 \max\{\mathrm{diam}(T) : T \in \mathcal{T}\}.$$

Recall that by Theorem 5.3, $\Xi(\mathcal{T}) \subset b\mathbb{Z}[\lambda]$ for some $b \in \mathbb{C}$.

LEMMA 6.5. *There exists $K = K(\mathcal{T}) > 0$ such that for any $T, S \in \mathcal{T}$ there is a polynomial $p_{T,S} \in \mathbb{Z}[x]$ with coefficients not greater than K in modulus, for which $c(T) - c(S) = b\lambda^{-1} p_{T,S}(\lambda)$.*

Proof of Lemma 6.5. The lemma follows from results of Praggastis [**Prag**] who expanded $c(T) - c(S)$ in base λ using a finite set of digits. We indicate a direct proof for completeness.

Consider the sequence of tiles $T = T^{(0)} \subset T^{(1)} \subset T^{(2)} \subset \cdots$ such that $T^{(k)}$ is a $\lambda^k \mathcal{T}$-tile (tile of the $(k+1)$st level). Let $S = S^{(0)} \subset S^{(1)} \subset S^{(2)} \subset \cdots$ be the analogous sequence for the tile S. Then for all n large enough, either $T^{(n)} = S^{(n)}$ or $T^{(n)}$ and $S^{(n)}$ are adjacent. Fix such an n. We have

$$c(T) - c(S) = \sum_{i=0}^{n-1} [c(T^{(i)}) - c(T^{(i+1)})] + c(T^{(n)}) - c(S^{(n)}) + \sum_{i=0}^{n-1} [c(S^{(i+1)} - c(S^{(i)})].$$

Now $c(T^{(n)}) - c(S^{(n)})$ is either zero or $\lambda^n (c(T') - c(S'))$ for some adjacent $\mathcal{T}$-tiles T', S'. Further, $c(T^{(i)}) - c(T^{(i+1)}) = \lambda^i (c(T_i'') - c(T_i'''))$ for some $\mathcal{T}$-tiles T_i'', T_i''' lying in the $\lambda\mathcal{T}$-tile $\lambda^{-i} T^{(i+1)}$. A similar formula holds for $c(S^{(i+1)}) - c(S^{(i)})$. Adding them all, we obtain the claim with $K = 2|\lambda| \max\{\mathrm{diam}(T) : T \in \mathcal{T}\}$. □

Continue with the proof of Proposition 6.4. We have $\nu(\mathcal{O}) = c(T) - c(S) + \lambda^n x$, where x is a fixed element of $\Xi(\mathcal{T}) \subset b\mathbb{Z}[\lambda]$. Combining this with Lemma 6.5 we see that $\nu(\mathcal{O}) = b\lambda^{-1} q_{T,S}(\lambda)$ for some polynomial $q_{T,S} \in \mathbb{Z}[x]$ having coefficients bounded by a constant independent of T, S, n. It remains to use the following lemma.

LEMMA 6.6. *Let λ be a complex Pisot number, $R \geq 0$. Then the set $X_R := \{x = p(\lambda) : p \in \mathbb{Z}[x]$ with coefficients $|p_j| \leq R\}$ is discrete.*

Proof of Lemma 6.6. This kind of result is rather standard; see [**Gar**, 1.51] and [**Ken3**, L.12]. We indicate the proof for completeness. Clearly, $X_R - X_R \subset X_{2R}$, so it is enough to show that 0 is an isolated point of X_{2R}. To this end, notice that $\Pi_{i=1}^m p(\lambda_i) \in \mathbb{Z}$, where $\lambda_1 = \lambda$ and $\lambda_2 = \bar{\lambda}, \lambda_3, \ldots, \lambda_m$ are the conjugates of λ. Thus, either $p(\lambda) = 0$ or $\Pi_{i=1}^m |p(\lambda_i)| \geq 1$, so

$$|p(\lambda)|^2 = |p(\lambda) p(\bar{\lambda})| \geq \frac{1}{\Pi_{i=3}^m |p(\lambda_i)|} \geq \left(\frac{1-\rho}{\max |p_j|} \right)^{m-2},$$

where $\rho = \max\{|\lambda_i|,\ i \geq 3\} < 1$. □

Since $|\nu(\mathcal{O})|$ is bounded by a constant independent of n, Lemma 6.6 implies that there are finitely many possibilities for the vector $\nu(\mathcal{O})$. The number of types of T and S is also finite, so there are finitely many classes of overlaps $\mathcal{O}_{\lambda^n x,T,S}$. This proves Proposition 6.4. □

Overlap Algorithm. We start building the subdivision graph for overlaps $\mathcal{G}_{\mathcal{O}}(\mathcal{T}, x)$ inductively. First one has to identify $[\mathcal{O}_x]$ (see Example 7.1 in the next section for a way this could be done). Then inflation and subdivision (15) are applied. Edges from $[\mathcal{O}_x]$ are drawn and new vertices are added, if necessary. Then the process is repeated with the new vertices-overlaps. It stops when $[\mathcal{O}_{\lambda^{n+1}x}] \subset [\mathcal{O}_x] \cup \cdots \cup [\mathcal{O}_{\lambda^n x}]$; Proposition 6.4 guarantees that this will happen if λ is non-real Pisot.

PROPOSITION 6.7. *Suppose that $\mathcal{T}$ is a self-similar tiling of the plane with expansion constant λ a non-real Pisot number, and $x \in X_{\mathcal{T}}$. The following are equivalent:*
(i) *the tiling dynamical system $(X_{\mathcal{T}}, \mu, \Gamma_g)$ has pure discrete spectrum;*
(ii) *from any vertex of the graph $\mathcal{G}_{\mathcal{O}}(\mathcal{T}, x)$ there is a path leading to a coincidence;*
(iii) $\mathrm{dens}(D_{\lambda^n x}) \to 1$, *as $n \to \infty$.*

Remark. In some sense the overlap algorithm is analogous to the *balanced block algorithm* developed for substitutions [**Mi2, Liv1**], [**Queff**, V.5]. However, it is not known whether the latter always terminates in the Pisot case; for the overlap algorithm this follows from Proposition 6.4. We should also mention that the idea of using coincidences goes back to the work of Dekking [**Dek1**] on substitutions of constant length.

Proof. We have for any $x \in \mathbb{C}$, $n \geq 0$,

$$\mathbb{C} = \bigcup_{T,S\in\mathcal{T}} ((T + \lambda^n x) \cap S). \tag{16}$$

Notice that the set $D_{\lambda^n x}$ is exactly the union of coincidences among the overlaps in (16). It is immediate from the definition that $\lambda D_{\lambda^n x} \subset D_{\lambda^{n+1}x}$.

First we prove (ii) ⇒ (i). Suppose that from any vertex of $\mathcal{G}_{\mathcal{O}}(\mathcal{T}, x)$ there is a path leading to a coincidence. Then we can find $l > 0$ such that for any overlap $\mathcal{O}_{\lambda^n x,T,S}$, the subdivision of the set $\lambda^l[(T + \lambda^n x) \cap S]$ contains a coincidence. We have

$$\mathrm{Vol}[\lambda^l((T + \lambda^n x) \cap S)] \leq |\lambda|^{2l}\,\mathrm{Vol}(S) \leq |\lambda|^{2l} V_{\max}.$$

The area of the overlap-coincidence is at least $V_{\min}$. It is not hard to see that this implies

$$\mathrm{dens}(\mathbb{C} \setminus D_{\lambda^{n+l}x}) \leq \left(1 - \frac{V_{\min}}{V_{\max}|\lambda|^{2l}}\right) \mathrm{dens}(\mathbb{C} \setminus D_{\lambda^n x}), \quad n \geq 0.$$

It follows that $1 - \mathrm{dens}(D_{\lambda^n x}) = \mathrm{dens}(\mathbb{C} \setminus D_{\lambda^n x})$ converges to zero geometrically. Theorem 6.1 applies and we conclude that the spectrum is pure discrete.

Next we prove (iii) ⇒ (ii). Suppose that for some overlap $\mathcal{O}_{\lambda^n x,T,S}$ there is no path leading from it to a coincidence. Then the decomposition of $\lambda^l[(T + \lambda^n x) \cap S]$ into the union of overlaps contains no coincidences for all $l > 0$, that is,

$$\lambda^l[(T + \lambda^n x) \cap S] \subset \mathbb{C} \setminus D_{\lambda^{n+l}x}, \quad l > 0.$$

Let Φ be the union of overlaps equivalent to $\mathcal{O}_{\lambda^n x,T,S}$ in $[\mathcal{O}_{\lambda^n x}]$. It follows from the existence of uniform frequencies (Theorem 3.4) that $\text{dens}(\Phi) > 0$. We have $\lambda^l \Phi \subset \mathbb{C} \setminus D_{\lambda^{n+l}x}$, $l > 0$, therefore,

$$\text{dens}(\mathbb{C} \setminus D_{\lambda^{n+l}x}) \geq \text{dens}(\lambda^l \Phi) = \text{dens}(\Phi) > 0,$$

which means that (iii) does not hold.

It remains to deduce (iii) from (i). It will be more convenient to get started with an arbitrary self-affine tiling $\mathcal{T}$ of $\mathbb{R}^d$ with expansion map ϕ. By Theorem 4.3, for every eigenvalue $\alpha \in \mathbb{R}^d$ we have $e^{2\pi i \langle \phi^n x, \alpha \rangle} \to 1$. This implies

$$(U_{\phi^n x} - I) f_\alpha \to 0, \tag{17}$$

in the norm of $L^2(X_{\mathcal{T}}, \mu)$, for the corresponding eigenfunction f_α. Pure discrete spectrum means that the linear span of eigenfunctions is dense in L^2. Since $\|U_{\phi^n x} - I\| \leq 2$, (17) holds for any L^2-function. Let T be a $\mathcal{T}$-tile and $\text{diam}(U) < \eta = \eta(\mathcal{T})$. Choosing f as the characteristic function of the cylinder set $X_{T,U}$, we see from (13) and (17) that

$$\mu(X_{T,U} \cap X_{T+\phi^n x, U}) \to \mu(X_{T,U}). \tag{18}$$

In general, the set in the left-hand side of (18) may not be a cylinder set.

CLAIM. *Under the conditions of Proposition 6.7, assuming that U is sufficiently small, $X_{T,U} \cap X_{T+\lambda^n x, U} = X_{T \cup T + \lambda^n x, U}$, a cylinder set.*

Proof of the Claim. We only need to prove the inclusion '$\subset$' since the opposite inclusion always holds by Lemma 1.4(iv). By Proposition 6.4 the set $\cup_{n \geq 0}[\mathcal{O}_{\lambda^n x}]$ is finite. Therefore, one can choose $\delta > 0$ independent of n, so that if $T, T' \in \mathcal{T}$ are such that $|c(T + \lambda^n x) - c(T')| < \delta$, then $T + \lambda^n x = T'$. Suppose that $\text{diam}(U) < \delta$ and let $\mathcal{S} \in X_{T,U} \cap X_{T+\lambda^n x, U}$. Then T is an $(\mathcal{S} - k)$-tile and $T + \lambda^n x$ is an $(\mathcal{S} - k')$-tile for some $k, k' \in U$. It follows that $T' = T + \lambda^n x + k' - k \approx T$ is an $(\mathcal{S} - k)$-tile. Clearly, $|c(T + \lambda^n x) - c(T')| = |k' - k| < \delta$ and T, T' are both $(\mathcal{S} - k)$-tiles. Since locally $(\mathcal{S} - k)$ and $\mathcal{T}$ look the same (by the definition of the tiling space), we can conclude that $T + \lambda^n x = T'$, $k = k'$. This means by definition that $\mathcal{S} \in X_{T \cup T + \lambda^n x, U}$. The claim is verified. $\square$

Combining the claim with (18) and Corollary 3.5, we obtain that

$$\text{freq}[T \cup (T + \lambda^n x)] \to \text{freq}(T), \quad n \to \infty,$$

where $\text{freq}(P)$ is the uniform frequency of a patch P. Hence

$$\text{dens}(D_{\lambda^n x}) = \sum_{i=1}^{m} \text{freq}[T_i \cup (T_i + \lambda^n x)] \,\text{Vol}(T_i) \to \sum_{i=1}^{m} \text{freq}(T_i)\, \text{Vol}(T_i) = 1.$$

This concludes the proof of Proposition 6.7. In order to deduce Theorem 6.2 (in the case of non-real λ) it remains to observe that if the spectrum is pure discrete, λ must be complex Pisot by Corollary 4.5. $\square$

7. *Examples*

7.1. *Self-similar tilings of the plane with polygonal tiles.*

Example 7.1. A non-periodic tiling by triominoes [**GS**, p. 523]. The tiles are L-shaped triominoes, with sides parallel to the coordinate axes. In Figure 7.1(a) the subdivision rule is indicated. To define the self-similar tiling $\mathcal{T}$, consider the affine map ψ which maps the shaded tile onto the patch consisting of four tiles. The fixed point of ψ^{-1} (shown in Figure 7.1(a)) is chosen as the origin. Then ψ becomes linear; in fact, it is just the multiplication by $\lambda = 2i$ in the complex plane. Next we subdivide each tile according to the same rule and apply ϕ. We get a larger patch which contains the original patch as a subpatch (see Figure 7.1(b)). Repeating the procedure we obtain a tiling $\mathcal{T}$ of the whole plane. Clearly, two triominoes in this tiling can touch each other in finitely many ways, so there are finitely many local patterns. All the tiles are congruent, but since we use the group of translations there are four tile types (indicated in Figure 7.1(b)).

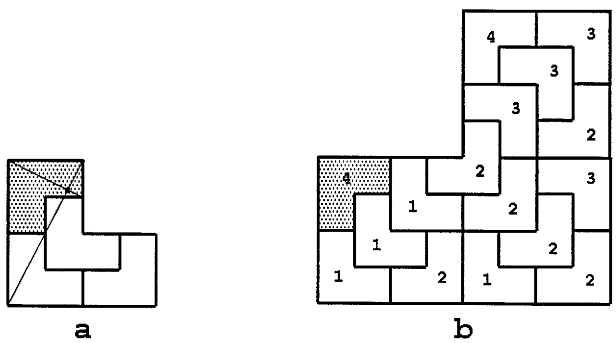

FIGURE 7.1. Triomino tiling.

The subdivision matrix is

$$M = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}, \quad M^2 > 0,$$

so $\mathcal{T}$ is self-similar by Lemma 2.2.

Next observe that $\mathcal{T}$ has the unique composition property. Indeed, for two $\mathcal{T}$-tiles T and S, the set $(\phi T + x) \cap (\phi S + y)$ cannot be a $\mathcal{T}$-patch, unless $\phi T + x = \phi S + y$. Therefore, tiles of the second level are composed uniquely. By Lemma 2.7, the tiling $\mathcal{T}$ is non-periodic.

Let us determine the discrete component of the spectrum for the tiling dynamical system. Recall that $\Xi(\mathcal{T})$ is the set of translation vectors between pairs of tiles of the same type. We see from Figure 7.1(b) that $\Xi(\mathcal{T}) \supset \{1+i, 2\}$. On the other hand, it is obvious that $\Xi(\mathcal{T}) \subset \mathbb{Z} \oplus i\mathbb{Z}$. Theorem 5.1 implies that the group of eigenvalues is $\mathbb{Z}_2 \oplus i\mathbb{Z}_2$, where $\mathbb{Z}_2 = \{p/2^m,\, p \in \mathbb{Z}, m \geq 1\}$ is the group of 2-adic rationals.

Finally, we apply Proposition 6.7 to show that the tiling dynamical system has pure discrete spectrum. Let $x = 1+i \in \Xi(\mathcal{T})$. The first step is to find all overlaps $(S+x, S')$, where S and S' are $\mathcal{T}$-tiles.

One can easily check that for $S + x \cap S'$ to have non-empty interior, it is necessary that $S \cap S' \neq \emptyset$. Call a pair (S, S') of $\mathcal{T}$-tiles with $S \neq S'$, $S \cap S' \neq \emptyset$ *adjacent*. We start by making an exhaustive list of adjacent pairs. Applying repeatedly the canonical subdivision, find all adjacent pairs in $\phi T_i, \phi^2 T_i, \phi^3 T_i$, etc (here $T_i, i \leq 4$, are representatives of tile types). Let $\mathcal{A}_k$ be the set of equivalence classes of adjacent pairs which occur in one of $\phi^k T_i$ (equivalence is defined modulo translations, as for overlaps). Since the tiling has a finite number of local patterns, $\mathcal{A}_k = \mathcal{A}_{k-1}$ for some k. We claim that $\mathcal{A}_k$ is the exhaustive list. Indeed, it is enough to show that $\mathcal{A}_{k+1} = \mathcal{A}_k$, since by induction $\mathcal{A}_n = \mathcal{A}_k, n \geq k$, and $\mathcal{A}_n$, for n large, contains all adjacent pairs by local isomorphism.

Let (S, S') be an adjacent pair in $\phi^{k+1} T_i$. We have $S \subset \phi T, S' \subset \phi T'$ for some $T, T' \in \mathcal{T}$. Moreover, $T \cap T' \neq \emptyset$, since $S \cap S' \neq \emptyset$. If $T = T'$ then $(S, S') \in \mathcal{A}_1$. Otherwise (T, T') is an adjacent pair in $\phi^k T_i$. By assumption, $\mathcal{A}_k = \mathcal{A}_{k-1}$, so there is an adjacent pair in $\phi^{k-1} T_j$, for some j, equivalent to (T, T'). Then (S, S') has an equivalent adjacent pair in $\phi^k T_j$, and the claim is verified.

In Figure 7.2(a) we present the list of adjacent pairs (S, S'); the complete list is obtained by applying reflections and rotations. The corresponding overlaps are shown in Figure 7.2(b) (notice that not all adjacent pairs lead to overlaps). Applying inflation and subdivision, we get the subdivision graph for overlaps (Figure 7.2(c)). One new overlap appears in the process.

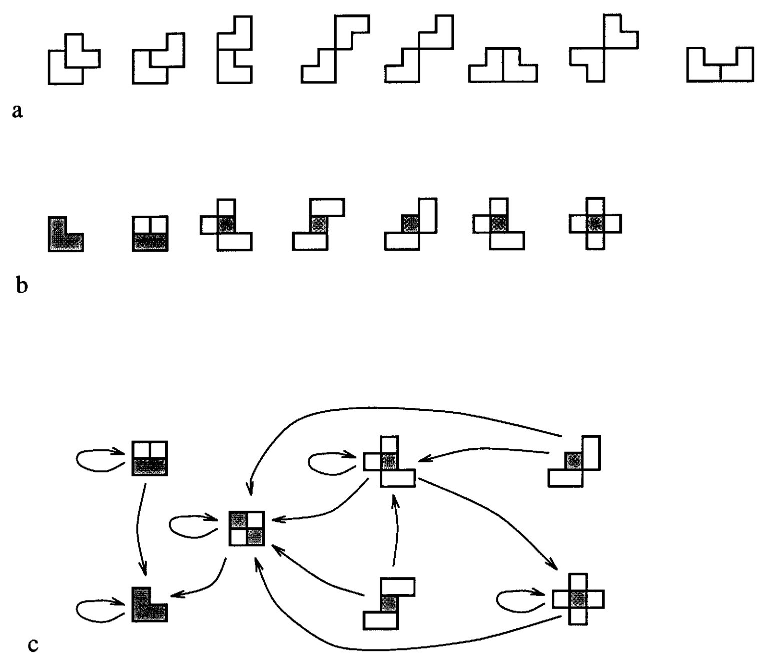

FIGURE 7.2. Triomino tiling: (a) adjacent pairs; (b) overlaps; (c) subdivision graph for overlaps.

Clearly, from every overlap there is a path leading to a coincidence, so by Proposition 6.7 the tiling dynamical system has pure discrete spectrum. (To be precise, the graph in Figure 7.2(c) is obtained from the actual subdivision graph by identifying congruent overlaps; this is permissible in our case since the subdivision rules agree with isometries of tiles.)

Example 7.2 Tiling by hexaminoes ('sphinxes') [**GS, Godr1**]. The subdivision rule is shown in Figure 7.3. This tiling can be constructed and analyzed similarly to the triomino example.

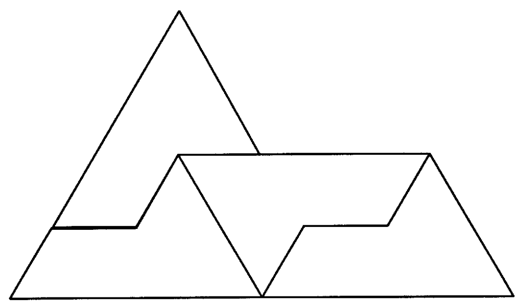

FIGURE 7.3. 'Sphinx' tiling.

It has 12 tiles (up to translation); all are congruent. It is self-similar with (the smallest in modulus) expansion constant $\lambda = 2e^{2\pi i/3}$. The unique composition property and hence non-periodicity is easily checked. The corresponding dynamical system has pure discrete spectrum but the verification is rather lengthy. If we identify the overlaps obtained by reflections and rotations, we get 24 equivalence classes of overlaps including one coincidence.

It is not hard to construct other non-periodic monohedral tilings (that is, tilings whose tiles are all congruent to each other). Several such examples can be found in [**Nar**]. In principle, all of them can be analyzed using Proposition 6.7.

Example 7.3. Domino tiling [**Nar**]. The tiles are dominoes composed of two unit squares; there are just two tile types. The subdivision rule and a patch of the tiling are shown in Figures 7.4(a) and (b). The expansion constant is $\lambda = 2i$; the subdivision matrix is

$$M = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

The tiling $\mathcal{T}$ of the plane $\mathbb{R}^2$ is generated by repeatedly applying inflation and subdivision, as in Example 7.1.

It is easy to see that $\mathcal{T}$ has the unique composition property. Indeed, consider a large patch of $\mathcal{T}$ and let P in Figure 7.4(c) be its subpatch. We claim that P itself is the only way (compatible with the tiling $\phi\mathcal{T}$) its tiles can fit into tiles of the second level. The only other possibility is shown in Figure 7.4(c), but clearly it cannot be extended to get tiles of the third level. Thus, $\mathcal{T}$ is non-periodic by Lemma 2.7.

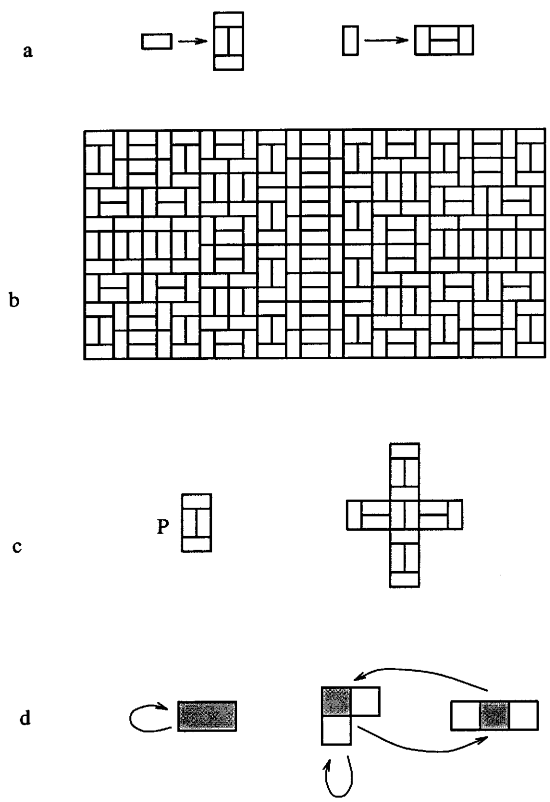

FIGURE 7.4. Domino tiling: (a) subdivision rule; (b) patch of the tiling; (c) unique composition; (d) subdivision graph for overlaps.

Let us determine the discrete spectral component. We have $\{1, i\} \subset \Xi(\mathcal{T}) \subset \mathbb{Z} \oplus i\mathbb{Z}$. Thus, by Theorem 5.1 the group of eigenvalues equals $\mathbb{Z}_2 \oplus i\mathbb{Z}_2$ (the same as in Example 7.1).

Next we apply Proposition 6.7 to find out whether the spectrum is pure discrete. Having just two tile types makes the computations easy. Let $x = 1 \in \Xi(\mathcal{T})$. It is quite obvious that there are just three equivalence classes of overlaps $\mathcal{O}_{x,T,S}$ up to translations *and* rotations. The subdivision graph of overlaps is shown in Figure 7.4(d). Choosing another element of $\Xi(\mathcal{T})$, say $x = i$, gives rise to an isomorphic subdivision graph. Since the non-coincidence overlaps do not lead to coincidences, the spectrum is not pure discrete.

We conclude that the tiling dynamical system has both discrete and continuous spectral components. In this respect it is similar to the dynamical system generated by Morse substitution $0 \to 01,\ 1 \to 10$, see [**Queff**].

Example 7.4. The Penrose tiling is certainly the most famous example. It has been extensively studied: see [**Pnr, Bru1, Bru3, GS, Rad1, Ro2, Sen**]. There are different versions: with ‘kites and darts’, rhombi of two kinds, and ‘thin’ and ‘thick’ triangles used as tiles. Only the last one can be made into a genuine self-similar tiling, although all variants are closely related. The tile types and subdivision rules can be found in [**Bru3**] or in [**Thur**]. The expansion constant can be chosen $\lambda = \frac{1}{2}(1+\sqrt{5})e^{3\pi i/5}$, a complex Pisot number. It has been noted [**GS**, p. 534] that the Penrose tiling has the unique composition property. Thus, by Theorem 5.1, the corresponding dynamical system has a large group of eigenvalues. Probably Proposition 6.7 can be applied to show that it has pure discrete spectrum, but the amount of work required is rather large.

As it turns out, another definition of the Penrose tiling leads to a precise description of the tiling dynamical system. Using the *grid method* developed by de Bruijn [**Bru1**], Robinson [**Ro2**] has shown that the Penrose tiling dynamical system is an almost 1:1 extension of the Kronecker $\mathbb{R}^2$ action on the four-dimensional torus (that is, action by group translations). It has measure-theoretic pure discrete spectrum with $\mathbb{Z}[\lambda]$ as the group of eigenvalues.

It is a rare occasion that a self-similar tiling can also be defined by the grid or projection method. So, for instance, Robinson's techniques do not apply to Examples 7.1–7.3.

7.2. *Self-affine tilings with nowhere differentiable boundary.* The literature on such tilings was discussed at the beginning of §2. Here we study two examples from a family constructed by Kenyon [**Ken1, Ken4**] and a class of tilings which comes from beta-expansions, following [**Prag**].

The definitions below are taken from [**Ken4**]. Let a, b, c be vectors in $\mathbb{R}^2$ which point in different directions. Denote by $F(a, b, c)$ the set of polygonal paths starting at the origin, whose edges are all equal (as vectors) to one of $\pm a, \pm b, \pm c$. There is a natural group structure on $F(a, b, c)$: to multiply two paths, one has to translate the second path so that it starts at the end of the first path, and erase any 'backtracking'. It is easy to see that this group is isomorphic to the free group on three generators which will be denoted by the same letters a, b, c. Now suppose there is a group endomorphism θ on $F(a, b, c)$. It is defined by arbitrarily assigning the words $\theta(a), \theta(b), \theta(c)$. In some cases it leads to a plane tiling.

Example 7.5. Let

$$\theta(a) = b, \quad \theta(b) = c, \quad \theta(c) = a^{-1}b^{-1}.$$

Take $a = 1, b = \lambda, c = \lambda^2 \in \mathbb{C}$, where $\lambda \approx 0.341164 + 1.16154i$ is the complex root of $\lambda^3+\lambda+1 = 0$. We start with three *basic parallelograms* $[a, b] = aba^{-1}b^{-1}$, $[b, c], [a, c]$. Notice that

$$\theta([a, b]) = [b, c], \quad \theta([b, c]) = (a^{-1}[a, c]a)(a^{-1}b^{-1}[b, c]ba), \quad \theta([a, c]) = a^{-1}[a, b]a. \tag{19}$$

This implies that the images of basic parallelograms (or rather, their interiors) can be tiled with translates of basic parallelograms (see Figure 7.5). Then we repeat the procedure and consider polygonal paths $A_n = \theta^n([a, b])$, $B_n = \theta^n([b, c])$, $C_n = \theta^n([a, c])$, $n \geq 1$.

LEMMA 7.6. *A_n is a simple closed polygonal path. Its interior can be tiled by translates of basic parallelograms. The same is true for B_n and C_n.*

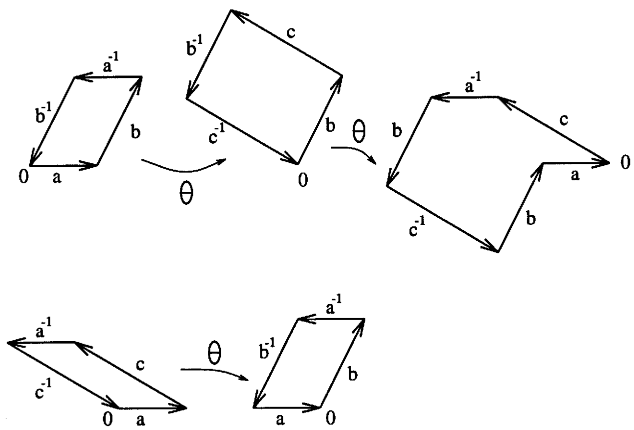

FIGURE 7.5. Endomorphism θ acting on basic parallelograms.

Proof outline. Although this is not new, neither Kenyon [**Ken4**], nor Dekking [**Dek2, Dek3**] give complete details. Notice that the example under consideration does not satisfy the condition of 'short range cancellations' [**Dek2, Dek3**]. A careful treatment of similar examples is due to Ito and Ohtsuki [**IO1**].

We will prove by induction a stronger statement: the interior of A_n can be tiled by translates of basic parallelograms *and* $\mathbb{R}^2$ can be tiled periodically by translates of A_n, so that the resulting parallelogram tiling is edge-to-edge. This is certainly true for A_1 which is a parallelogram.

Suppose that the claim is true for A_n. Apply θ to all parallelograms of the corresponding tiling and then subdivide their images according to (19). The union of images of tiles comprising A_n will form the interior of A_{n+1}. There are two potential problems. One is that applying θ may involve cancellations. However, one can verify easily (by a separate straightforward induction) that no cancellations occur in $\theta(A_n) = A_{n+1}$ and $\theta(B_n) = B_{n+1}$. Cancellations do occur in $\theta(C_0) = \theta([a,c]) = \theta(aca^{-1}c^{-1}) = ba^{-1}b^{-1}b^{-1}ba = ba^{-1}b^{-1}a$, but not in $\theta(C_n) = C_{n+1}$ for $n \geq 1$. Another potential problem is that two parallelograms of the tiling might become overlapping after θ is applied. This is ruled out by direct inspection for adjacent pairs of parallelograms (since the tilings are edge-to-edge there are few possibilities to consider). Then it follows, by an elementary topological argument, that no two parallelograms can become overlapping. (If one is troubled by the cancellation in $\theta([a,c])$ one can go one step further and consider the tilings by translates of A_1, B_1, C_1. Applying θ to those never involves cancellations.) □

Observe that the tilings of $B_n, n \geq 1$, agree with each other and $\cup_{n=0}^{4} B_n$ covers a neighborhood of the origin (see Figure 7.6). The tilings of $\cup_{n=5k}^{5k+4} B_n$ cover increasing patches of the plane which tend to a non-periodic parallelogram tiling of $\mathbb{R}^2$. This tiling will be denoted $\mathcal{T}_{\mathcal{P}}$. It is interesting that $\mathcal{T}_{\mathcal{P}}$ looks like a 'stepped surface'; this can be made precise [**IO3, Ken1**].

The endomorphism θ can be applied to all tiles of $\mathcal{T}_{\mathcal{P}}$ to get a tiling $\theta(\mathcal{T}_{\mathcal{P}})$ by translates

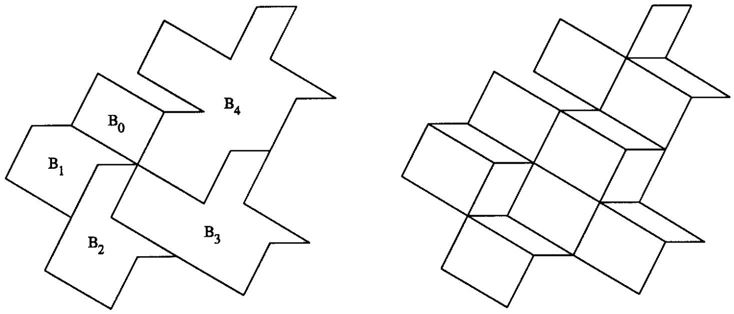

FIGURE 7.6. Constructing the parallelogram tiling $\mathcal{T}_{\mathcal{P}}$.

of A_1, B_1, C_1, and this can be repeated. It turns out that applying θ and rescaling leads to a self-similar tiling.

LEMMA 7.7. *The set $\lambda^{-n}A_n$ converges in Hausdorff metric to a simple closed curve, whose interior is a tile homeomorphic to the disk. The tilings $\lambda^{-n}\theta^n(\mathcal{T}_{\mathcal{P}})$ converge (in the Hausdorff metric) to a self-similar tiling with expansion constant λ, whose tiles are translates of T_a, the interior of $\lim_{n\to\infty}\lambda^{-n}A_n$, T_b, the interior of $\lim_{n\to\infty}\lambda^{-n}B_n$, and T_c, the interior of $\lim_{n\to\infty}\lambda^{-n}C_n$.*

Proof. The argument is known; it is outlined for completeness. Recall that $a = 1, b = \lambda, c = \lambda^2 \in \mathbb{C}$. We have $\theta(a) = \lambda a$ and $\theta(b) = \lambda b$. Further, $\theta(c) = a^{-1}b^{-1}$ and $\lambda c = \lambda^3 = -a - b$ (as vectors or complex numbers) by the choice of λ. It follows that $\theta(c)$ is a two-edge path with the same endpoints as λc, so the path $\lambda^{-1}\theta(c)$ has the same endpoints as c.

This implies the following description of the paths $\lambda^{-n}A_n$. They have edges equal (as vectors) to $\pm\lambda^{-n}, \pm\lambda^{-n+1}, \pm\lambda^{-n+2}$. The transition from $\lambda^{-n}A_n$ to $\lambda^{-n-1}A_{n+1}$ can be described as follows: the edges equal to $\pm\lambda^{-n}, \pm\lambda^{-n+1}$ remain unchanged, the edges λ^{-n+2} are replaced by pairs $-\lambda^{-n}, -\lambda^{-n+1}$, and the edges $-\lambda^{-n+2}$ are replaced by pairs $\lambda^{-n+1}, \lambda^{n}$ (in this order). All vertices of $\lambda^{-n}A_n$ remain vertices of $\lambda^{-n-1}A_{n+1}$ and one new vertex is created for each edge equal to $\pm\lambda^{-n+2}$.

It is not hard to deduce from this that there are parametrizations $\gamma_n : [0, 1] \to \lambda^{-n}A_n$ such that $\|\gamma_n - \gamma_{n+1}\|_\infty <$ constant λ^{-n}. This proves the convergence of $\lambda^{-n}A_n$ to a closed curve. Additional care is needed to show that the limiting curve is simple; this can be shown using the arguments from the proof of Lemma 7.6.

Similarly, one can deal with the other two tiles. We obtain a well-defined tiling $\mathcal{T} = \lim_{n\to\infty}\lambda^{-n}\theta^n\mathcal{T}_{\mathcal{P}}$. Now let us see how the tiles subdivide after multiplication by λ. It follows from the definition of T_a, T_b, T_c and (19) that $\lambda T_a = T_b$, $\lambda T_b = (T_b - g_a - g_b) \cup (T_c - g_a)$, $\lambda T_c = T_a - g_a$, where g_a is the vector from the origin to the end of the path $\lim_{n\to\infty}\lambda^{-n}\theta^n(a)$, and g_b is the vector from the origin to the end of the path $\lim_{n\to\infty}\lambda^{-n}\theta^n(b)$. These vectors are easy to find. However, by the same argument as above, the endpoints of $\lambda^{-n}\theta^n(a)$ are the same for all n. So we have $g_a = 1, g_b = \lambda$, and therefore

$$\lambda T_a = T_b, \quad \lambda T_b = (T_b - 1 - \lambda) \cup (T_c - 1), \quad \lambda T_c = T_a - 1. \tag{20}$$

This implies that the tiling $\mathcal{T}$ is λ-subdividing, with the subdivision matrix

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Finally, $\mathcal{T}$ has finitely many local patterns since they all arise from the local patterns of $\mathcal{T}_{\mathcal{P}}$ by the same renormalization procedure. Thus, $\mathcal{T}$ is self-similar. A patch of the tiling is shown in Figure 7.7 (in fact, this is the image of $\lambda^{-20}\theta^{20}$ applied to the tiling in Figure 7.6). □

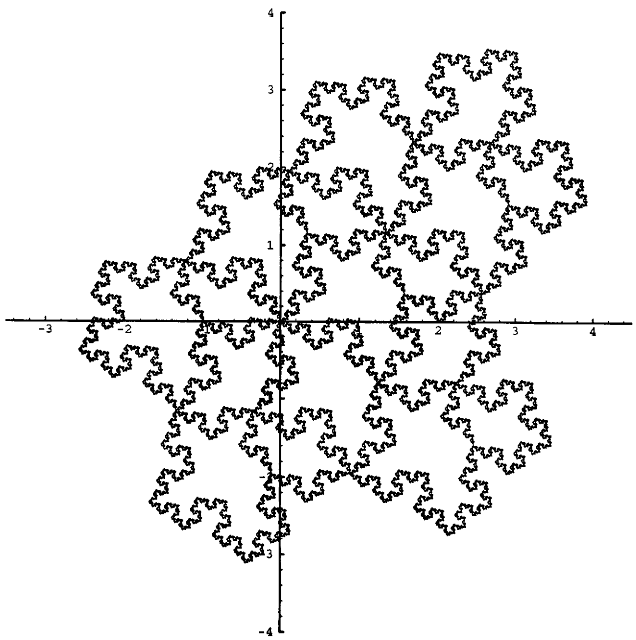

FIGURE 7.7. Patch of a self-similar tiling (Pisot case).

Observe that $\mathcal{T}$ has the unique composition property. Indeed, consider the 'second-level' tiling $\lambda\mathcal{T}$. Its patches of type λT_b are the only ones which contain tiles of type T_c in their subdivision, and $\mathcal{T}$-tiles of type T_a form second-level tiles of type λT_c. The remaining (after composing tiles of the type λT_b) tiles of type T_b have to be the second-level tiles of type λT_a. Thus, the composition procedure is always unique.

The expansion constant λ is a complex Pisot number (its conjugates are $\bar{\lambda}$ and ≈ -0.682328). By Theorem 5.1 the tiling dynamical system has a large discrete spectral component, with the group of eigenvalues containing $\mathbb{Z}[\lambda]$. It is probably possible to show that the spectrum is pure discrete with the help of Proposition 6.7, but we do not pursue it here.

Example 7.8. A tiling with continuous spectrum. This example is analogous to Example 7.5. Let

$$\theta(a) = b, \quad \theta(b) = c, \quad \theta(c) = a^{-3}b,$$

and λ is the non-real root of the equation $\lambda^3 + \lambda + 3 = 0$. (This is another special case of Kenyon's construction [**Ken4**].) The self-similar tiling with expansion constant λ is constructed exactly as above. The main difference is that now λ is not a complex Pisot number, so by Corollary 4.5, the tiling dynamical system is weakly mixing (has continuous spectrum). A patch of the tiling is shown in Figure 7.8.

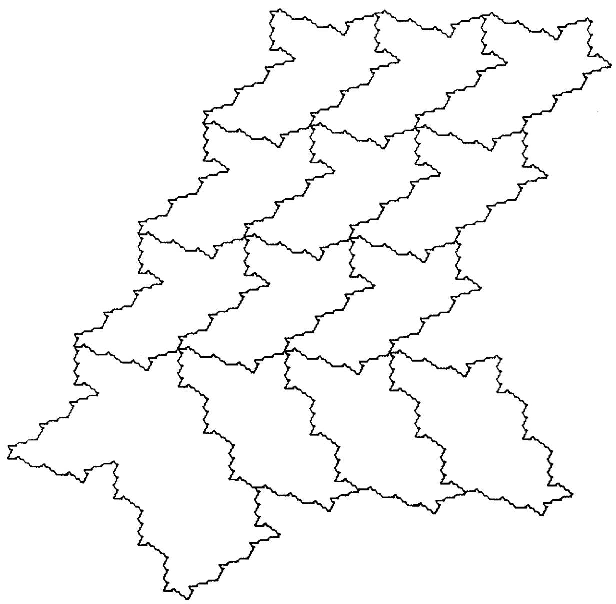

FIGURE 7.8. Patch of a self-similar tiling (non-Pisot case).

Example 7.9. Dual Pisot tilings. The following construction is hinted upon in [**Thur**] and analyzed in detail in [**Prag**]. Actually, we will restrict ourselves to a special case when the spectral properties of the tiling dynamical system can be completely determined.

Let $p(x) = x^d - k_1 x^{d-1} - k_2 x^{d-2} - \cdots - k_d \in \mathbb{Z}[x]$, where $k_1 \geq k_2 \geq \cdots \geq k_d = 1$. Then $p(x)$ has a unique positive zero $\beta > 1$, and all other zeros have moduli less than one [**Bra**]. Thus β is a Pisot number. Let

$$M = \begin{bmatrix} k_1 & k_2 & \dots & k_{d-1} & k_d \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix};$$

its eigenvalues are the zeros of $p(x)$. Consider the M-invariant decomposition of $\mathbb{R}^d$ into the sum $H_u + H_s$, where H_u, the unstable subspace, is the eigenspace corresponding to β, and H_s, the stable subspace, corresponds to other eigenvalues. Let π_u be the projection onto H_u parallel to H_s, and π_s the projection onto H_s parallel to H_u. Further, set $e_0 = [1, 0, \ldots, 0]^t \in \mathbb{R}^d$, $e_u = \pi_u e_0, e_s = \pi_s e_0$.

We start with a self-similar tiling $\mathcal{S}$ of the half-line $H_u^+ = \mathbb{R}_+ e_u$, with expansion constant β. It arises from the 'integral part' of the β-expansion. We will not go into the theory of β-expansions; the reader is referred to [**Re, Par, Blan, FS**] for definitions and needed results. It is clear that in base β the (lexicographically greatest) expansion of one is $1. = .k_1k_2\ldots k_d$. Thus, by a theorem of Parry [**Par**] β-expansions can be characterized as walks on a labeled graph $\mathcal{G}$. The graph is easy to describe. It has d vertices $V_1, \ldots, V_d$, with one edge labeled k_j leading from each V_j to V_{j+1} for $j \le d-1$, and k_j edges labeled $0, \ldots, k_j - 1$, leading from each of V_j to V_1. The incidence matrix of the graph is M^t (it disregards the labels).

Let $\mathcal{Z}_j$ be the set of finite sequences which correspond to walks in $\mathcal{G}$ ending at the vertex V_j and set $\mathcal{Z} = \cup_{j=1}^d \mathcal{Z}_j$. The tiles of $\mathcal{S}$ are line segments and control points can be defined as their left endpoints. Thus, the set of control points $c(\mathcal{S})$ completely determines the tiling. Define

$$c(\mathcal{S}) = \left\{ \sum_{i=-N}^{0} a_i \beta^{-i} e_u : \{a_i\}_{-N}^0 \in \mathcal{Z}, N \ge 0 \right\}.$$

There will be d tile types; tiles of type $j \le d$ have length $\sum_{i=j}^d k_i \beta^{-i} \|e_u\|$ and control points $\sum_{i=-N}^0 a_i \beta^{-i} e_u$ with $\{a_i\}_{-N}^0 \in \mathcal{Z}_j$. It is not difficult to show that $\mathcal{S}$ is a self-similar tiling with expansion constant β and subdivision matrix M (see [**Thur, Prag**]).

Next we build the 'dual' tiling of the $(d-1)$-dimensional space H_s with expansion map $\phi := (M|H_s)^{-1}$. Notice that ϕ is expansive because β is Pisot, and diagonalizable since $p(x)$ is irreducible. Let

$$\Omega_j = \operatorname{Clos} \left\{ \sum_{i=-N}^{0} a_i \phi^i e_s : \{a_i\}_{-N}^0 \in \mathcal{Z}_j, N \ge 0 \right\}, \quad 1 \le j \le d.$$

The sets Ω_j are bounded since ϕ is expansive. It follows from [**Prag**] that $\Omega_j = \operatorname{Clos}(\operatorname{Int}\Omega_j)$ and the origin lies in $\operatorname{Int}\Omega_j$. (In fact, in the special case under consideration this can be verified directly using [**FS**].) The sets Ω_j, $j \le d$, will be the representatives of tile types. Inspection of the graph $\mathcal{G}$ leads to the formulas

$$\phi\Omega_1 = \bigcup_{j=1}^{d} \left[\bigcup_{l=0}^{k_j-1} (\Omega_j + l\phi e_s) \right]; \quad \phi\Omega_j = \Omega_{j-1} + k_j \phi e_s, \quad j = 2, \ldots, d. \tag{21}$$

Further, one can show, as in [**Rau**], that interiors of the tiles in the representation of $\phi\Omega_1$ in (21) do not overlap, using the fact that $\operatorname{Vol}_{d-1}(\phi\Omega) = |\det\phi| \operatorname{Vol}_{d-1}(\Omega)$ and $|\det\phi| = \beta$ (recall that $\det M = k_d = 1$). To get a tiling of H_s, consider the sets

$$T_{j,a} = \Omega_j + \sum_{i=1}^{M} a_i \phi^i e_s, \quad \{a_i\}_1^M \in \mathcal{A}_j,$$

where $\mathcal{A}_j$ is the set of finite sequences corresponding to walks in the graph $\mathcal{G}$ which *start* at V_j. One can show that they form a self-affine tiling $\mathcal{T}$ of H_s with expansion map

ϕ and subdivision matrix M^t. (Applying an isomorphism between H_s and $\mathbb{R}^{d-1}$ makes this into a tiling of $\mathbb{R}^{d-1}$.)

Let us analyze the corresponding tiling dynamical system. The unique composition property is immediate since the tiles $\Omega_2, \ldots, \Omega_d$ uniquely determine the tile of the second level in which they lie. It can be deduced from Theorem 5.1 that the tiling dynamical system has a group of eigenvalues $\mathbb{Z}[\phi]e_s$.

PROPOSITION 7.10. *The tiling dynamical system arising from $\mathcal{T}$ has pure discrete spectrum.*

Proof. We are going to use Theorem 6.1. Let us check that $\sum(1 - \mathrm{dens}(D_{\phi^n e_s})) < \infty$. (This is sufficient since the vectors $e_0, Me_0, \ldots, M^{d-1}e_0$ span $\mathbb{R}^d$ and hence $e_s, \phi^{-1}e_s, \ldots, \phi^{-d+1}e_s$ span H_s.) It is clear from the definition of the tiling $\mathcal{T}$ that $T_{j,a}$ with $a = \{a_i\}_1^M \in \mathcal{A}_j$ lies in $D_{\phi^n e_s}$ if and only if for some $\{b_i\}_1^N \in \mathcal{A}_j$,

$$\sum_{i=1}^{M} a_i \phi^i e_s + \phi^n e_s = \sum_{i=1}^{N} b_i \phi^i e_s. \tag{22}$$

It follows from [**FS, Sol2**] that there exists $K = K(\beta) \in \mathbb{N}$ such that if the block $00\ldots0$ of length K occurs in $\{a_j\}_1^n$, then

$$\sum_{i=1}^{M} a_i \beta^{-i} + \beta^{-n} = \sum_{i=1}^{N} b_i \beta^{-i}$$

for some $\{b_i\}_1^N \in \mathcal{A}_j$. Since ϕ is diagonalizable with eigenvalues which are algebraic conjugates of β^{-1}, (22) will follow.

Let $L_{M,j}$ be the number of tiles $T_{j,a}$ with $a = \{a_j\}_1^M \in \mathcal{A}_j$. The number of tiles among them for which the block $00\ldots0$ of length K does not occur in $\{a_j\}_1^n$ is bounded above by $C\rho^n L_{M,j}$ for some $\rho \in (0,1)$. This implies, with a little bit of work, that

$$1 - \mathrm{dens}(D_{\phi^n e_s}) = \mathrm{dens}(\mathbb{C} \setminus D_{\phi^n e_s}) < C'\rho^n.$$

The series converges, so the spectrum is pure discrete. □

In the case $d = 3$, when the conjugates of β are $\lambda, \bar{\lambda}$, the constructed tiling is isomorphic to a self-similar tiling of the plane with expansion constant λ. It can be defined directly using expansions in base λ. Rauzy [**Rau**] investigated the case $k_1 = k_2 = k_3 = 1$. Other examples of this kind can be found in [**IK**] where the same tilings are constructed using the method of free group endomorphisms.

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Note added in November 1996. Since the submission of this paper, some progress has occurred. We generalized the method of [**Mos**] to show that any non-periodic self-affine tiling in any dimension has the unique composition property [**Sol3**]. Thus, the statement of Theorem 5.1 can be simplified. Conditions for pure discrete spectrum were extended to non-self-affine tilings in [**Sol4**]. Gähler and Klitzing [**GK**] determined the discrete part of the diffraction spectrum of self-similar tilings, assuming that the expansion constant is real. They have a condition similar to (8) and conclude that the expansion constant must be Pisot if the discrete part is non-trivial. Hof [**Hof2**] has shown that tilings obtained by the projection method (with a Riemann-integrable 'window') give rise to dynamical systems with pure discrete spectrum. Translational dynamics of tilings in a general setting was investigated by Dworkin [**Dw2**]. Gelbrich [**Ge2**] studied Galois dual tilings of which Example 7.9 is a special case. The theory of iterated function systems has been fruitfully applied to self-affine tilings by Bandt and co-authors in a series of papers, see [**B**] and references therein.

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