Derivation of the Energy-Balance Equations

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TUNABLE COLLISIONS: HAMMER-STRING SIMULATION WITH TIME-VARIANT PARAMETERS

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1 Continuous-Domain Energy Balance: Equations (26) and (27)

We start with equation (21) in the paper:

$$d_t^2\mathbf{u} + \mathbf{R}d_t\mathbf{u} + \mathbf{K}\mathbf{u} = \mathbf{h}\bar{F}_{\mathrm{c}},$$

and pre-multiply with $(d_t\mathbf{u})^{\mathrm{T}}$, which yields

$$(d_t\mathbf{u})^{\mathrm{T}}d_t^2\mathbf{u} + (d_t\mathbf{u})^{\mathrm{T}}\mathbf{R}d_t\mathbf{u} + (d_t\mathbf{u})^{\mathrm{T}}\mathbf{K}\mathbf{u} = (\mathbf{h})^{\mathrm{T}}d_t\mathbf{u}\bar{F}_{\mathrm{c}}. \tag{1.1}$$

Now consider the following product-rule identities:

$$d_t\left[(d_t\mathbf{u})^{\mathrm{T}}d_t\mathbf{u}\right] = 2(d_t\mathbf{u})^{\mathrm{T}}d_t^2\mathbf{u}, \tag{1.2}$$

$$d_t\bar{u}_{\mathrm{s}} = d_t(\mathbf{h}^{\mathrm{T}}\mathbf{u}) = \mathbf{h}^{\mathrm{T}}d_t\mathbf{u} + (d_t\mathbf{h})^{\mathrm{T}}\mathbf{u} \tag{1.3}$$

$$d_t(\mathbf{u}^T\mathbf{K}\mathbf{u}) = 2(d_t\mathbf{u})^{\mathrm{T}}\mathbf{K}\mathbf{u} + \mathbf{u}^{\mathrm{T}}(d_t\mathbf{K})\mathbf{u}, \tag{1.4}$$

which we may use directly to substitute the relevant terms in (1.1):

$$d_t\underbrace{\left[(\tfrac{1}{2}d_t\mathbf{u})^{\mathrm{T}}d_t\mathbf{u} + \tfrac{1}{2}(\mathbf{u}^T\mathbf{K}\mathbf{u})\right]}_{\bar{H}_{\mathrm{str}}} = -(d_t\mathbf{u})^{\mathrm{T}}\mathbf{R}d_t\mathbf{u} + \tfrac{1}{2}\mathbf{u}^{\mathrm{T}}(d_t\mathbf{K})\mathbf{u} + \left[d_t\bar{u}_{\mathrm{s}} - (d_t\mathbf{h})^{\mathrm{T}}\mathbf{u}\right]\bar{F}_{\mathrm{c}}, \tag{1.5}$$

where $\bar{H}_{\mathrm{str}}$ denotes the scaled-form energy of the string. Now consider equation (10) of the paper:

$$d_t\{\bar{m}_{\mathrm{h}}d_t\bar{u}_{\mathrm{h}}\} = \bar{F}_{\mathrm{e}} - \bar{F}_{\mathrm{c}},$$

and multiply with $d_t\bar{u}_{\mathrm{h}}$, which gives

$$d_t\bar{u}_{\mathrm{h}} d_t\big\{\bar{m}_{\mathrm{h}} d_t\bar{u}_{\mathrm{h}}\big\} = d_t\bar{u}_{\mathrm{h}}\bar{F}_{\mathrm{e}} - d_t\bar{u}_{\mathrm{h}}\bar{F}_{\mathrm{c}}. \tag{1.6}$$

The following product-rule identities hold:

$$d_t\big\{\bar{m}_{\mathrm{h}} d_t\bar{u}_{\mathrm{h}}\big\} = d_t\bar{m}_{\mathrm{h}} d_t\bar{u}_{\mathrm{h}} + \bar{m}_{\mathrm{h}} d_t^2\bar{u}_{\mathrm{h}} \tag{1.7}$$

$$d_t\left[(d_t u_{\mathrm{h}})^2\right] = 2\, d_t u_{\mathrm{h}}\, d_t^2 u_{\mathrm{h}}. \tag{1.8}$$

By substitution into (1.6) one obtains:

$$d_t\Big[\underbrace{\tfrac{1}{2}\bar{m}_{\mathrm{h}}(d_t u_{\mathrm{h}})^2}_{\bar{H}_{\mathrm{h}}}\Big] = -d_t\bar{m}_{\mathrm{h}}(d_t\bar{u}_{\mathrm{h}})^2 + d_t\bar{u}_{\mathrm{h}}\bar{F}_{\mathrm{e}} - d_t\bar{u}_{\mathrm{h}}\bar{F}_{\mathrm{c}}, \tag{1.9}$$

where $\bar{H}_{\mathrm{h}}$ denotes the scaled-form kinetic energy of the hammer. Now let's add equations (1.5) and (1.9) and obtain:

$$d_t\left[\bar{H}_{\mathrm{str}} + \bar{H}_{\mathrm{h}}\right] = -(d_t\mathbf{u})^{\mathrm{T}}\mathbf{R} d_t\mathbf{u} + \tfrac{1}{2}\mathbf{u}^{\mathrm{T}}(d_t\mathbf{K})\mathbf{u} + d_t\underbrace{[\bar{u}_{\mathrm{s}} - \bar{u}_{\mathrm{h}}]}_{\bar{y}}\bar{F}_{\mathrm{c}} - (d_t\mathbf{h})^{\mathrm{T}}\mathbf{u}\bar{F}_{\mathrm{c}} - d_t\bar{m}_{\mathrm{h}}(d_t\bar{u}_{\mathrm{h}})^2 + d_t\bar{u}_{\mathrm{h}}\bar{F}_{\mathrm{e}}. \tag{1.10}$$

From equation (16) in the paper we have

$$\bar{F}_{\mathrm{c}} = -\psi\partial_{\bar{y}}\psi = -\psi\frac{d_t\psi}{d_t\bar{y}} = -\frac{d_t(\frac{1}{2}\psi^2)}{d_t\bar{y}} + \psi\frac{(g_{\bar{\kappa}} d_t\bar{\kappa} + g_\alpha d_t\alpha)}{d_t\bar{y}}.$$

Substitution into (1.10) yields

$$\begin{aligned} d_t\overbrace{\left[\bar{H}_{\mathrm{str}} + \bar{H}_{\mathrm{h}} + \tfrac{1}{2}\psi^2\right]}^{\bar{H}} = &-(d_t\mathbf{u})^{\mathrm{T}}\mathbf{R} d_t\mathbf{u} + \tfrac{1}{2}\mathbf{u}^{\mathrm{T}}(d_t\mathbf{K})\mathbf{u} - (d_t\mathbf{h})^{\mathrm{T}}\mathbf{u}\bar{F}_{\mathrm{c}} + d_t\bar{u}_{\mathrm{h}}\bar{F}_{\mathrm{e}} \\ &- (d_t\bar{u}_{\mathrm{h}})^2 d_t\bar{m}_{\mathrm{h}} + \psi\left(g_{\bar{\kappa}} d_t\bar{\kappa} + g_\alpha d_t\alpha\right), \end{aligned} \tag{1.11}$$

which is the result expressed in equations (25) and (26) of the paper.

2 Discrete-Domain Energy Balance: Equations (58) and (59)

Let's start with equation (38) of the paper:

$$\delta_t^2\mathbf{u}^n + \widehat{\mathbf{R}}^n\delta_{t\cdot}\mathbf{u}^n + \mu_t^2\widehat{\mathbf{K}}^n\mu_t^2\mathbf{u}^n = \bar{F}_{\mathrm{c}}^n\mu_{t\cdot}\mathbf{h}^n,$$

and pre-multiply with $(\delta_{t\cdot}\mathbf{u}^n)^{\mathrm{T}}$, which yields:

$$(\delta_{t\cdot}\mathbf{u}^n)^{\mathrm{T}}\delta_t^2\mathbf{u}^n + (\delta_{t\cdot}\mathbf{u}^n)^{\mathrm{T}}\widehat{\mathbf{R}}^n\delta_{t\cdot}\mathbf{u}^n + (\delta_{t\cdot}\mathbf{u}^n)^{\mathrm{T}}\mu_t^2\widehat{\mathbf{K}}^n\mu_t^2\mathbf{u}^n = (\delta_{t\cdot}\mathbf{u}^n)^{\mathrm{T}}\mu_{t\cdot}\mathbf{h}^n\bar{F}_{\mathrm{c}}^n. \tag{2.1}$$

To make further useful manipulations, we must first consider the product identities articulated in equation (36) of the paper:

$$\delta_t\left\{(\delta_t u^n)^2\right\} = 2\delta_{t\cdot}u^n\delta_t^2 u^n,$$

$$\delta_t \left\{ (\mu_t u^n)^2 \right\} = 2\delta_{t\cdot} u^n \mu_t^2 u^n.$$

From these we may derive that

$$\begin{aligned}
\delta_{t\cdot} u^n \, \delta_t^2 u^n + \mu_t^2 k^n \delta_{t\cdot} u^n \mu_t^2 u^n &= \delta_t \overbrace{\left\{ \tfrac{1}{2} (\delta_t u^n)^2 \right\}}^{T^n} + \mu_t^2 k^n \delta_t \left\{ \tfrac{1}{2} (\mu_t u^n)^2 \right\} \\
&= \delta_t T^n + \frac{1}{2\Delta_t} \left[\mu_t^2 k^n (\mu_t u^{n+\frac{1}{2}})^2 - \mu_t^2 k^n (\mu_t u^{n-\frac{1}{2}})^2 \right] \\
&= \delta_t T^n + \frac{1}{2\Delta_t} \left[\frac{\mu_t k^{n+\frac{1}{2}} + \mu_t k^{n-\frac{1}{2}}}{2} (\mu_t u^{n+\frac{1}{2}})^2 - \frac{\mu_t k^{n+\frac{1}{2}} + \mu_t k^{n-\frac{1}{2}}}{2} (\mu_t u^{n-\frac{1}{2}})^2 \right] \\
&= \delta_t T^n + \frac{1}{\Delta_t} \left[\underbrace{\tfrac{1}{2} \mu_t k^{n+\frac{1}{2}} (\mu_t u^{n+\frac{1}{2}})^2}_{V^{n+\frac{1}{2}}} - \underbrace{\tfrac{1}{2} \mu_t k^{n-\frac{1}{2}} (\mu_t u^{n-\frac{1}{2}})^2}_{V^{n-\frac{1}{2}}} \right] \\
&\quad - \frac{1}{4\Delta_t} \left[(\mu_t k^{n+\frac{1}{2}} - \mu_t k^{n-\frac{1}{2}}) (\mu_t u^{n+\frac{1}{2}})^2 - (\mu_t k^{n-\frac{1}{2}} - \mu_t k^{n+\frac{1}{2}}) (\mu_t u^{n-\frac{1}{2}})^2 \right] \\
&= \delta_t \left\{ T^n + V^n \right\} - \tfrac{1}{2} \underbrace{\left[\frac{\mu_t k^{n+\frac{1}{2}} - \mu_t k^{n-\frac{1}{2}}}{\Delta_t} \right]}_{\delta_{t\cdot} k^n} \underbrace{\left[\frac{(\mu_t u^{n+\frac{1}{2}})^2 + (\mu_t u^{n-\frac{1}{2}})^2}{2} \right]}_{\mu_t \left\{ (\mu_t u^n)^2 \right\}} \\
&= \delta_t \left\{ T^n + V^n \right\} - \left[\tfrac{1}{4} \mu_t u^{n+\frac{1}{2}} \, \delta_{t\cdot} k^n \mu_t u^{n+\frac{1}{2}} + \tfrac{1}{4} \mu_t u^{n-\frac{1}{2}} \, \delta_{t\cdot} k^n \mu_t u^{n-\frac{1}{2}} \right]. \qquad (2.2)
\end{aligned}$$

Please note how, if we replace k^n with $\widehat{K}^n_{i,i}$, the second term in the penultimate line is a consistent approximation to the components of the dot product $\frac{1}{2}\mathbf{u}^{\mathrm{T}}(d_t\mathbf{K})\mathbf{u}$ appearing on the RHS of (1.11). In vector form, with $\widehat{\mathbf{K}}$ being a diagonal matrix, we can write this as

$$\begin{aligned}
(\delta_{t\cdot}\mathbf{u}^n)^{\mathrm{T}} \delta_t^2 \mathbf{u}^n + (\delta_{t\cdot}\mathbf{u}^n)^{\mathrm{T}} \mu_t^2 \widehat{\mathbf{K}}^n \mu_t^2 \mathbf{u}^n &= \delta_t \left\{ \tfrac{1}{2} (\delta_t \mathbf{u}^n)^{\mathrm{T}} \delta_t \mathbf{u}^n + \tfrac{1}{2} (\mu_t \mathbf{u}^n)^{\mathrm{T}} \mu_t \widehat{\mathbf{K}}^n \mu_t \mathbf{u}^n \right\} \\
&\quad - \left[\tfrac{1}{4} (\mu_t \mathbf{u}^{n+\frac{1}{2}})^{\mathrm{T}} \delta_{t\cdot} \widehat{\mathbf{K}}^n \mu_t \mathbf{u}^{n+\frac{1}{2}} + \tfrac{1}{4} (\mu_t \mathbf{u}^{n-\frac{1}{2}})^{\mathrm{T}} \delta_{t\cdot} \widehat{\mathbf{K}}^n \mu_t \mathbf{u}^{n-\frac{1}{2}} \right]. \qquad (2.3)
\end{aligned}$$

Similar to the identities in equation (36) of the paper, we may derive a discrete product rule of the form:

$$\delta_{t\cdot} \left\{ u^n q^n \right\} = \mu_{t\cdot} u^n \, \delta_{t\cdot} q^n + \delta_{t\cdot} q^n \, \delta_{t\cdot} u^n. \qquad (2.4)$$

This allows writing

$$\begin{aligned}
(\delta_{t\cdot}\mathbf{u}^n)^{\mathrm{T}} \mu_{t\cdot} \mathbf{h}^n &= (\mu_{t\cdot} \mathbf{h}^n)^{\mathrm{T}} \delta_{t\cdot} \mathbf{u}^n \\
&= \delta_{t\cdot} \underbrace{\left\{ (\mathbf{h}^n)^{\mathrm{T}} \mathbf{u}^n \right\}}_{\bar{u}_{\mathrm{s}}^n} - \delta_{t\cdot} (\mathbf{h}^n)^{\mathrm{T}} \mu_{t\cdot} \mathbf{u}^n. \qquad (2.5)
\end{aligned}$$

Using (2.3) and (2.5), we may now rewrite (2.1) as

$$\begin{aligned}
\delta_t \overbrace{\left\{ \tfrac{1}{2} (\delta_t \mathbf{u}^n)^{\mathrm{T}} (\delta_t \mathbf{u}^n) + \tfrac{1}{2} (\mu_t \mathbf{u}^n)^{\mathrm{T}} \mu t \widehat{\mathbf{K}}^n \mu_t \mathbf{u}^n \right\}}^{\bar{H}^n_{\mathrm{str}}} &= \tfrac{1}{4} (\mu_t \mathbf{u}^{n+\frac{1}{2}})^{\mathrm{T}} \delta_{t\cdot} \widehat{\mathbf{K}}^n \mu_t \mathbf{u}^{n+\frac{1}{2}} + \tfrac{1}{4} (\mu_t \mathbf{u}^{n-\frac{1}{2}})^{\mathrm{T}} \delta_{t\cdot} \widehat{\mathbf{K}}^n \mu_t \mathbf{u}^{n-\frac{1}{2}} \\
&\quad - (\delta_{t\cdot} \mathbf{u}^n)^{\mathrm{T}} \widehat{\mathbf{R}}^n \delta_{t\cdot} \mathbf{u}^n + \delta_{t\cdot} \bar{u}_{\mathrm{s}}^n F_{\mathrm{c}}^n - \delta_{t\cdot} (\mathbf{h}^n)^{\mathrm{T}} \mu_{t\cdot} \mathbf{u}^n F_{\mathrm{c}}^n, \qquad (2.6)
\end{aligned}$$

where $\bar{H}^n_{\mathrm{str}}$ denotes the numerical string energy. Now let's recall equation (50) of the paper:

$$\delta_t\big\{\mu_t\bar{m}^n_{\mathrm{h}}\delta_t\bar{u}^n_{\mathrm{h}}\big\} = \bar{F}^n_{\mathrm{e}} - \bar{F}^n_{\mathrm{c}}$$

and pre-multiply with $\delta_{t\cdot}\bar{u}^n_{\mathrm{h}}$:

$$\delta_{t\cdot}\bar{u}^n_{\mathrm{h}}\delta_t\big\{\mu_t\bar{m}^n_{\mathrm{h}}\delta_t\bar{u}^n_{\mathrm{h}}\big\} = \delta_{t\cdot}\bar{u}^n_{\mathrm{h}}\bar{F}^n_{\mathrm{e}} - \delta_{t\cdot}u^n_{\mathrm{h}}\bar{F}^n_{\mathrm{c}}. \tag{2.7}$$

Using the identity in equation (37) of the paper:

$$\delta_t\left\{\mu_t q^n\delta_t u^n\right\}\delta_t u^n = \tfrac{1}{2}\delta_t\left\{\mu_t q^n(\delta_t u^n)^2\right\} + \delta_{t\cdot}q^n\delta_{t+}u^n\delta_{t-}u^n,$$

we may substitute the LHS of (2.7), and write this as

$$\delta_t\underbrace{\left\{\tfrac{1}{2}\mu_t\bar{m}^n_{\mathrm{h}}(\delta_t\bar{u}^n_{\mathrm{h}})^2\right\}}_{\bar{H}^n_{\mathrm{h}}} = \delta_{t\cdot}\bar{u}^n_{\mathrm{h}}\bar{F}^n_{\mathrm{e}} - \delta_{t\cdot}\bar{u}^n_{\mathrm{h}}\bar{F}^n_{\mathrm{c}} - \delta_{t\cdot}\bar{m}^n_{\mathrm{h}}\delta_{t+}\bar{u}^n_{\mathrm{h}}\delta_{t-}\bar{u}^n_{\mathrm{h}}, \tag{2.8}$$

where $\bar{H}^n_{\mathrm{h}}$ denotes the numerical hammer energy. Adding equations (2.6) and (2.8) yields:

$$\begin{aligned}\delta_t\left\{\bar{H}^n_{\mathrm{str}} + \bar{H}^n_{\mathrm{h}}\right\} &= \tfrac{1}{4}(\mu_t\mathbf{u}^{n+\frac{1}{2}})^{\mathrm{T}}\delta_{t\cdot}\widehat{\mathbf{K}}^n\mu_t\mathbf{u}^{n+\frac{1}{2}} + \tfrac{1}{4}(\mu_t\mathbf{u}^{n-\frac{1}{2}})^{\mathrm{T}}\delta_{t\cdot}\widehat{\mathbf{K}}^n\mu_t\mathbf{u}^{n-\frac{1}{2}}\\ &\quad - (\delta_{t\cdot}\mathbf{u}^n)^{\mathrm{T}}\widehat{\mathbf{R}}^n\delta_{t\cdot}\mathbf{u}^n - \delta_{t\cdot}(\mathbf{h}^n)^{\mathrm{T}}\mu_{t\cdot}u^nF^n_{\mathrm{c}} + \delta_{t\cdot}\bar{u}^n_{\mathrm{h}}\bar{F}^n_{\mathrm{e}}\\ &\quad + \delta_{t\cdot}\underbrace{[\bar{u}^n_{\mathrm{s}} - \bar{u}^n_{\mathrm{h}}]}_{\bar{y}^n}\bar{F}^n_{\mathrm{c}} - \delta_{t\cdot}\bar{m}^n_{\mathrm{h}}\delta_{t+}\bar{u}^n_{\mathrm{h}}\delta_{t-}\bar{u}^n_{\mathrm{h}}.\end{aligned} \tag{2.9}$$

From equation (54) of the paper, we may write:

$$\begin{aligned}\delta_{t\cdot}\bar{y}^n\bar{F}^n_{\mathrm{c}} &= -\mu_t\psi^n\cdot(\delta_t\psi^n) + \mu_t\psi^n\left(g^n_{\bar{\kappa}}\delta_{t\cdot}\bar{\kappa}^n + g^n_{\alpha}\delta_{t\cdot}\alpha^n\right)\\ &= -\delta_t\left\{\tfrac{1}{2}(\psi^n)^2\right\} + \mu_t\psi^n\left(g^n_{\bar{\kappa}}\delta_{t\cdot}\bar{\kappa}^n + g^n_{\alpha}\delta_{t\cdot}\alpha^n\right).\end{aligned} \tag{2.10}$$

Substitution into (2.9) yields

$$\begin{aligned}\delta_t\overbrace{\left\{\bar{H}^n_{\mathrm{str}} + \bar{H}^n_{\mathrm{h}} + \tfrac{1}{2}(\psi^n)^2\right\}}^{\bar{H}^n} &= \tfrac{1}{4}(\mu_t\mathbf{u}^{n+\frac{1}{2}})^{\mathrm{T}}\delta_{t\cdot}\widehat{\mathbf{K}}^n\mu_t\mathbf{u}^{n+\frac{1}{2}} + \tfrac{1}{4}(\mu_t\mathbf{u}^{n-\frac{1}{2}})^{\mathrm{T}}\delta_{t\cdot}\widehat{\mathbf{K}}^n\mu_t\mathbf{u}^{n-\frac{1}{2}}\\ &\quad - (\delta_{t\cdot}\mathbf{u}^n)^{\mathrm{T}}\widehat{\mathbf{R}}^n\delta_{t\cdot}\mathbf{u}^n - \delta_{t\cdot}(\mathbf{h}^n)^{\mathrm{T}}\mu_{t\cdot}u^nF^n_{\mathrm{c}} + \delta_{t\cdot}\bar{u}^n_{\mathrm{h}}\bar{F}^n_{\mathrm{e}}\\ &\quad - \delta_{t\cdot}\bar{m}^n_{\mathrm{h}}\delta_{t+}\bar{u}^n_{\mathrm{h}}\delta_{t-}\bar{u}^n_{\mathrm{h}} + \mu_t\psi^n\left(g^n_{\bar{\kappa}}\delta_{t\cdot}\bar{\kappa}^n + g^n_{\alpha}\delta_{t\cdot}\alpha^n\right),\end{aligned} \tag{2.11}$$

which is the result expressed in equations (58) and (59) of the paper.