Large deviations of affine processes

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Abstract

This is an abstract of the entire dissertation; summarize a history of large deviations and affine processes, then abstractly summarize our large deviations result.

Acknowledgment

This is where I acknowledge how I am useless without others.

Contents

Introduction

This is where I give the reader a little more history and detail regarding affine processes and large deviations, should they read this paper without already being well-versed in the subject.

Notation and conventions

I want this section to clear up notational similarities and differences with literature. Namely

- All the objects one needs for stochastic processes and their stochastic integration.
- All the spaces one often sees in real analysis.
- The space, functions, and parameters associated with a given affine process.

Chapter I

Affine processes

Here I put a summary of chapter, along with a short history. It will include the following important notes.

- Chapter addresses important fundamental results of affine processes.
- Chapter addresses consequences of results that are important for us, though not specified exactly in any of the literature.
- Chapter presents the information in an order of increasing complexity of concepts (versus the order in which it is typically proven).

I.1 Formulation

We start by specifying a state space on which our stochastic processes live. Let $\mathbb{V}$ be a finite-dimensional real vectorspace with inner-product $\langle\cdot,\cdot\rangle$. Equip $\mathbb{V}$ with the canonical topology and Borel algebra from $\langle\cdot,\cdot\rangle$. Denote the dimension $d \coloneqq \dim\mathbb{V}$ and establish the canonical isometric isomorphism $\mathbb{V} \equiv \mathbb{R}^d$ by specifying an orthonormal basis $e_1, \ldots, e_d \in \mathbb{V}$, so that we may identify components of vectors in $\mathbb{V}$.

$$v \in \mathbb{V} \quad \longleftrightarrow \quad v^i \coloneqq \langle v, e_i\rangle, \quad i = 1, \ldots, d$$

Similarly identify any map $f : \mathbb{A} \to \mathbb{V}$ with component maps $f_1, \ldots, f_d : \mathbb{A} \to \mathbb{R}$.

$$f : \mathbb{A} \to \mathbb{V} \quad \longleftrightarrow \quad (f_1, \ldots, f_d) : \mathbb{A} \to \mathbb{R}^d;\ f_i(a) = \langle f(a), e_i\rangle$$

Extend the inner-product symmetrically to a bilinear form on $\mathbb{V} \oplus \mathrm{i}\mathbb{V}$.

$$\big\langle v_1 + \mathrm{i}w_1, v_2 + \mathrm{i}w_2\big\rangle = \big(\langle v_1, v_2\rangle - \langle w_1, w_2\rangle\big) + \mathrm{i}\big(\langle v_1, w_2\rangle + \langle w_1, v_2\rangle\big)$$

Fix a convex and closed $\mathbb{X} \subseteq \mathbb{V}$ satisfying $0 \in \mathbb{X}$ and $\operatorname{span}\mathbb{X} = \mathbb{V}$. Associate this space with the finite exponentials.

$$\mathcal{U}_{\mathbb{X}} \coloneqq \Big\{u \in \mathbb{V} \oplus \mathrm{i}\mathbb{V} : \sup_{x\in\mathbb{X}} \exp\langle\Re(u), x\rangle < \infty\Big\}$$

We may now define the notion of an affine process on $\mathbb{X}$.

Definition I.1. *For a probability space $(\Omega, \Sigma, \mathrm{P})$ with filtration $\mathscr{F} = (\mathscr{F}_t)_{t\geq 0}$, an affine process X on $\mathbb{X}$ is a stochastically continuous, time-homogeneous $(\mathrm{P}, \mathscr{F})$-Markov process on $\mathbb{X}$ in which the bounded moments have the following log-affine dependence on the initial state.*

$$\begin{aligned} \mathrm{E}_{\mathrm{P}_x} \exp \langle u, X_t \rangle &= \exp \Psi(t, u, x) \\ \Psi(t, u, x) &= \psi_0(t, u) + \big\langle \psi(t, u), x \big\rangle, \end{aligned} \qquad t \geq 0, \; u \in \mathcal{U}_{\mathbb{X}} \tag{I.1}$$

Above, we are denoting $(\mathrm{P}_x)_{x\in\mathbb{X}}$ the conditional distributions of X factored through the initial state (see Appendix **??** *for further specification and notation).*

Remark I.2. *Our definitions of $\mathbb{X}$ and Ψ include the following conventions and motivations.*

(a) why we denote ψ_0, ψ instead of KM φ, ψ or Cuchiero Φ, Ψ

(b) how assumptions $0 \in \mathbb{X}$, $\operatorname{span} \mathbb{X} = \mathbb{V}$ are nonrestrictive

(c) how (I.1) decides the distribution of X and how the distribution of the affine process decides Ψ

(d) If we have a vectorspace $\mathbb{A}$ and affine map $\alpha : \mathbb{X} \to \mathbb{A}$ determined by $a_0, \ldots, a_d \in \mathbb{A}$ via $\alpha(x) = a_0 + \sum_{i=1}^d x^i a_i$, then our linear assumptions $0 \in \mathbb{X}$ and $\operatorname{span} \mathbb{X} = \mathbb{V}$ uniquely determine $a_0, \ldots, a_d \in \mathbb{A}$. In particular, the map Ψ uniquely identifies its parts $\psi_i : \mathbb{R}_+ \times \mathcal{U}_{\mathbb{X}} \to \mathbb{C}$ for $i = 0, \ldots, d$.

(e) In [Cuc11, Theorem 1.2.7], it is shown that, without loss of generality on conditional distributions $(\mathrm{P}_x)_{x\in\mathbb{X}}$, an affine process X can be chosen to have càdlàg paths. Thus, each distribution P_x may (and will) be recognized as a measure on the Borel algebra associated with the space $\mathbb{D}([0, \infty), \mathbb{X})$ of càdlàg functions equipped with the Skorokhod topology (see Appendix **??***).*

While the tuple (ψ_0, ψ) in (I.1) is a simpler object than the distributions $(\mathrm{P}_x)_{x\in\mathbb{X}}$, selecting an *admissible* tuple—one (ψ_0, ψ) which actually can appear in (I.1)—is seemingly prohibitive. The following result is incredibly useful at demonstrating primitive affine objects which determine the time-dynamics of $\big(\psi_0(\cdot, u), \psi(\cdot, u)\big)$ for each $u \in \mathcal{D}_\Psi$. These implicitly depend on a *truncation function* χ, which we select as follows.

$$\chi : \mathbb{V} \to \mathbb{V}, \quad \chi(v) \coloneqq \begin{cases} v, & |v| \leq 1 \\ 0, & |v| > 1 \end{cases} \tag{I.2}$$

The importance of this function is for local integrability properties—the details of which we defer until Section I.4.

Theorem I.3. *Fix an affine process X on $\mathbb{X}$. There exist $b_0^\chi, \ldots, b_d^\chi \in \mathbb{V}$, $a_0, \ldots, a_d \in \mathbb{L}(\mathbb{V})$, and $m_0, \ldots, m_d \in \mathbb{M}_s(\mathscr{B}(\mathbb{V}))$ such that the following maps $L_0, \ldots, L_d : \mathcal{U}_{\mathbb{X}} \to \mathbb{R}$,*

$$L_i(u) \coloneqq \big\langle u, b_i^\chi(x) \big\rangle + \frac{1}{2} \big\langle u, a_i(x) u \big\rangle + \int_{\mathbb{V}} \Big(e^{\langle u, v \rangle} - 1 - \big\langle u, \chi(v) \big\rangle \Big) m_i(x, \mathrm{d}v),$$

determine the dynamics of each $\big(\psi_0(\cdot,u),\psi(\cdot,u)\big)$.

$$\begin{cases} \dot\psi_0(t,u) = L_0\big(\psi(t,u)\big) & t\geq 0\\ \dot\psi(t,u) = L\big(\psi(t,u)\big) & t\geq 0\\ \psi_0(0,u)=0 \\ \psi(0,u)=u \end{cases}$$

Proof. This is simply a restatement of [Cuc11, Theorem 1.5.4].

Remark I.4. *By Remark I.2(d), Theorem I.3 is equivalent to the existence of maps*

$$\begin{aligned} \beta^\chi(x) &\coloneqq b_0^\chi + \sum_{i=1}^d x^i b_i^\chi, \\ \alpha(x) &\coloneqq a_0 + \sum_{i=1}^d x^i a_i, \\ \mu(x,\mathrm{d}v) &\coloneqq m_0(\mathrm{d}v) + \sum_{i=1}^d x^i m_i(\mathrm{d}v) \\ \Lambda(u,x) &= L_0(u) + \sum_{i=1}^d x^i L_i(u) \\ &\coloneqq \big\langle u,\beta^\chi(x)\big\rangle + \frac{1}{2}\big\langle u,\alpha(x)u\big\rangle + \int_{\mathbb{V}} \Big(e^{\langle u,v\rangle} - 1 - \big\langle u,\chi(v)\big\rangle\Big)\mu(x,\mathrm{d}v) \end{aligned} \tag{I.3}$$

which specify the dynamics of Ψ.

$$\forall x\in\mathbb{X},\qquad \begin{cases} \dot\Psi(t,u,x) = \Lambda\big(\psi(t,u),x\big) & t\geq 0\\ \Psi(0,u,x) = \langle u,x\rangle \end{cases} \tag{I.4}$$

Indeed, this is also because differentiability and our initial condition in (I.4) are linear.

Remark I.5. *In [Cuc11] there are immediate results on our functions* β^χ,α,μ *which are readily apparent in Section I.4, such as, for all* $x\in\mathbb{X}$*, the following are true.*

$$\begin{gathered} \alpha(x) \text{ is positive semidefinite} \\ \int_{\mathbb{V}} \big(1\wedge|v|^2\big)\mu(x,\mathrm{d}v) < \infty \\ \mu(x,\{0\}) = 0 \end{gathered} \tag{I.5}$$

discuss how other papers try to clearly specify which other conditions on β^χ,α,μ *parameterize all admissible* (ψ_0,ψ)*, depending on the definition of* $\mathbb{X}$.

Henceforth, we fix X a càdlàg affine process with conditional distributions $(\mathrm{P}_x)_{x\in\mathbb{X}}$, induced filtration $\mathscr{F}=(\mathscr{F}_t)_{t\geq 0}$, and moment function Ψ as in Definition I.1. Also fix the parameters $(b_i^\chi,a_i,m_i,L_i)_{i=0}^d$ from Theorem I.3 and the associated functions $\beta^\chi,\alpha,\mu,\Lambda$ from Remark I.4.

I.2 Existence of real moments

This elaborates upon the extension of the transform formula in (I.1) and dynamics in (I.4) to real moments $u \in \mathbb{V}$. Clearly, should any extension exist for some $u \in \mathbb{V}$, the value $\Lambda(u,x) = \dot{\Psi}(0,u,x)$ should be well-defined. That said, we henceforth denote the following set.

$$\mathcal{D}_\Lambda := \Big\{ u \in \mathbb{V} : \Lambda(u,x) \text{ is well-defined for all } x \in \mathbb{X} \Big\}$$

Quick inspection of $\Lambda(u,x)$ in (I.3) indicates that well-definition should depend solely on the integral term. The following results will perform the careful analysis to show that Λ has nice regularity on the finite moments of its associated jump kernel $\mu(\cdot, \mathrm{d}v)$.

Lemma I.6. *We have $u \in \mathcal{D}_\Lambda$ if and only if each $x \in \mathbb{X}$ is such that*

$$\int_{|v|>1} e^{\langle u,v\rangle} \mu(x, \mathrm{d}v) < \infty$$

Proof. To each $u, v \in \mathbb{V}$, Taylor's theorem gives us $\gamma_{u,v} \in [0,1]$ such that

$$e^{\langle u,v\rangle} = 1 + \langle u,v\rangle + \frac{1}{2} e^{\gamma_{u,v}\langle u,v\rangle} \langle u,v\rangle^2.$$

This allows us to see that, for each $x \in \mathbb{X}$, $\Lambda(u,x)$ and $\int_{|v|>1} e^{\langle u,v\rangle}\mu(x,\mathrm{d}v)$ differ by finite expressions.

$$\begin{aligned}
&\left| \Lambda(u,x) - \int_{|v|>1} e^{\langle u,v\rangle}\mu(x,\mathrm{d}v) \right| \\
&= \left| \langle u, \beta^{\mathcal{X}}(x)\rangle + \frac{1}{2}\langle u, \alpha(x) u\rangle + \int_{|v|\le 1} \big(e^{\langle u,v\rangle} - 1 - \langle u,v\rangle\big)\mu(x,\mathrm{d}v) - \int_{|v|>1} \mu(x,\mathrm{d}v) \right| \\
&\le \left| \langle u, \beta^{\mathcal{X}}(x)\rangle + \frac{1}{2}\langle u, \alpha(x) u\rangle \right| + \left| \int_{|v|\le 1} \frac{1}{2} e^{\gamma_{u,v}\langle u,v\rangle}\langle u,v\rangle^2 \mu(x,\mathrm{d}v) \right| + \int_{|v|>1}\mu(x,\mathrm{d}v) \\
&\le \left| \langle u, \beta^{\mathcal{X}}(x)\rangle + \frac{1}{2}\langle u, \alpha(x) u\rangle \right| + \left(\frac{1}{2} e^{|u|} + 1 \right) \int_{\mathbb{V}} \big(1 \wedge |v|^2\big)\mu(x,\mathrm{d}v)
\end{aligned}$$

Thus, one can be defined as a finite displacement of the other.

Lemma I.7. *$\mathcal{D}_\Lambda$ is convex.*

Proof. We use the characterization of $\mathcal{D}_\Lambda$ from Lemma I.6. Let $u, u' \in \mathcal{D}_\Lambda$, $\gamma \in (0,1)$, and use Hölder's inequality to see the following.

$$\begin{aligned}
&\int_{|v|>1} e^{\langle u' + \gamma(u-u'), v\rangle} \mu(x,\mathrm{d}v) \\
&\quad = \int_{|v|>1} \big| (e^{\langle u,v\rangle})^{\gamma} \cdot (e^{\langle u',v\rangle})^{1-\gamma} \big| \mu(x,\mathrm{d}v) \\
&\quad \le \left(\int_{|v|>1} \big| (e^{\langle u,v\rangle})^{\gamma} \big|^{\frac{1}{\gamma}} \mu(x,\mathrm{d}v) \right)^{\gamma} \left(\int_{|v|>1} \big| (e^{\langle u',v\rangle})^{1-\gamma} \big|^{\frac{1}{1-\gamma}} \mu(x,\mathrm{d}v) \right)^{1-\gamma}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_{|v|>1} e^{\langle u,v\rangle}\mu(x,\mathrm{d}v)\right)^{\gamma}\left(\int_{|v|>1} e^{\langle u',v\rangle}\mu(x,\mathrm{d}v)\right)^{1-\gamma} \\
&< \infty
\end{aligned}
$$

An arbitrary convex combination now satisfies $\gamma u + (1-\gamma)u' \in \mathcal{D}_\Lambda$.

Lemma I.8. *For each $x \in \mathbb{X}$, the map $\Lambda(\cdot, x)$ is continuously differentiable on $\mathcal{D}_\Lambda^\circ$, with derivative $D\Lambda(\cdot, x) : \mathcal{D}_\Lambda^\circ \to \mathbb{L}(\mathbb{V}, \mathbb{R})$ as follows.*

$$
D\Lambda(u,x)w = \left\langle \beta^{\chi}(x) + \alpha(x)u + \int_{\mathbb{V}} \left(e^{\langle u,v\rangle}v - \chi(v)\right)\mu(x,\mathrm{d}v), w\right\rangle, \quad u \in \mathcal{D}_\Lambda^\circ \tag{I.6}
$$

Proof. Fix $x \in \mathbb{X}$, $u \in \mathcal{D}_\Lambda^\circ$. Let $\epsilon > 0$ such that $B(u,\epsilon) \subseteq \mathcal{D}_\Lambda$. For all $0 < \delta < \epsilon$ and $i = 1, \ldots, d$, we now have the following identity

$$
\begin{aligned}
\frac{\Lambda(u+\delta e_i, x) - \Lambda(u,x)}{\delta} &= \langle e_i, \beta^{\chi}(x)\rangle + \langle e_i, \alpha(x)u\rangle + \frac{1}{2}\langle \delta e_i, \alpha(x)u\rangle \\
&\quad + \int_{|v|\le 1} \frac{1}{\delta}\left(e^{\langle u+\delta e_i,v\rangle} - e^{\langle u,v\rangle} - \langle \delta e_i, v\rangle\right)\mu(x,\mathrm{d}v) \\
&\quad + \int_{|v|>1} \frac{1}{\delta}\left(e^{\langle u+\delta e_i,v\rangle} - e^{\langle u,v\rangle}\right)\mu(x,\mathrm{d}v)
\end{aligned} \tag{I.7}
$$

Evaluating the limit of (I.7) as $\delta \to 0$ is now a matter of exchanging the limit with integration; we will do this by using the dominated convergence theorem.

For the first integral, Taylor's theorem provides us $\gamma_0, \gamma_1 \in [0,1]$ such that the following hold.

$$
\begin{aligned}
e^{\langle u+\delta e_i,v\rangle} &= 1 + \langle u+\delta e_i, v\rangle + \frac{1}{2}\langle u+\delta e_i, v\rangle^2 e^{\gamma_0\langle u+\delta e_i,v\rangle} \\
e^{\langle u,v\rangle} &= 1 + \langle u,v\rangle + \frac{1}{2}\langle u,v\rangle^2 e^{\gamma_1\langle u,v\rangle}
\end{aligned}
$$

This shows us that, for all $0 < \delta < \epsilon$ and $|v| \le 1$,

$$
\begin{aligned}
\left|\frac{1}{\delta}\left(e^{\langle u+\delta e_i,v\rangle} - e^{\langle u,v\rangle} - \langle \delta e_i, v\rangle\right)\right| &= \left|\frac{1}{2}\langle u+\delta e_i, v\rangle^2 e^{\gamma_0\langle u+\delta e_i,v\rangle} + \frac{1}{2}\langle u,v\rangle^2 e^{\gamma_1\langle u,v\rangle}\right| \\
&\le \left(\left(|u|+\epsilon\right)^2 e^{|u|+\epsilon}\right)|v|^2.
\end{aligned}
$$

This dominating function is integrable (recall the finiteness in (I.5) of Remark I.5),

$$
\int_{|v|\le 1} \left(\left(|u|+\epsilon\right)^2 e^{|u|+\epsilon}\right)|v|^2\mu(x,\mathrm{d}v) \le \left(\left(|u|+\epsilon\right)^2 e^{|u|+\epsilon}\right)\int_{\mathbb{V}} (1 \wedge |v|^2)\mu(x,\mathrm{d}v) < \infty,
$$

so we may apply the dominated convergence theorem.

$$
\begin{aligned}
&\lim_{\delta\to 0}\int_{|v|\le 1} \frac{1}{\delta}\left(e^{\langle u+\delta e_i,v\rangle} - e^{\langle u,v\rangle} - \langle \delta e_i, v\rangle\right)\mu(x,\mathrm{d}v) \\
&\quad = \int_{|v|\le 1} \lim_{\delta\to 0}\frac{1}{\delta}\left(e^{\langle u+\delta e_i,v\rangle} - e^{\langle u,v\rangle} - \langle \delta e_i, v\rangle\right)\mu(x,\mathrm{d}v)
\end{aligned}
$$

$$= \int_{|v|\le 1} \left(e^{\langle u,v\rangle} v_i - v_i\right)\mu(x,\mathrm{d}v) \tag{I.8}$$

For the second integral, we again use Taylor's theorem to establish for each $0 < \delta < \epsilon/2$, some $\gamma_\delta \in [0,\delta]$ such that

$$e^{\langle u+\delta e_i, v\rangle} = e^{\langle u,v\rangle} + \langle \delta e_i, v\rangle e^{\langle u+\gamma_\delta e_i, v\rangle}$$

This way, we have the following dominating function.

$$\left|\frac{1}{\delta}\left(e^{\langle u+\delta e_i,v\rangle} - e^{\langle u,v\rangle}\right)\right| \le \left|\langle e_i, v\rangle e^{\langle u+\gamma_\delta e_i,v\rangle}\right| \le |v_i| e^{\langle u,v\rangle + \epsilon|v_i|/2}$$

The claim is that this dominating function is integrable. To see this, first note that because we have the following limit,

$$\lim_{|v|\to\infty} \frac{|v_i|e^{\langle u,v\rangle+\epsilon|v_i|/2}}{e^{\langle u,v\rangle+2\epsilon|v_i|/3}} = \lim_{|v|\to\infty} \frac{|v_i|}{e^{\epsilon|v_i|/6}} = 0$$

There exists $M > 0$ such that for all $|v| > M$,

$$|v_i|e^{\langle u,v\rangle+\epsilon|v_i|/2} < e^{\langle u,v\rangle+2\epsilon|v_i|/3}.$$

We now see that

$$\begin{aligned}
&\int_{|v|>1} |v_i|e^{\langle u,v\rangle+\epsilon|v_i|/2}\mu(x,\mathrm{d}v)\\
&= \int_{1<|v|\le M} |v_i|e^{\langle u,v\rangle+\epsilon|v_i|/2}\mu(x,\mathrm{d}v) + \int_{|v|>M} |v_i|e^{\langle u,v\rangle+\epsilon|v_i|/2}\mu(x,\mathrm{d}v)\\
&\le \int_{1<|v|\le M} Me^{(|u|+\epsilon/2)M}\mu(x,\mathrm{d}v) + \int_{|v|>M} e^{\langle u,v\rangle+2\epsilon|v_i|/3}\mu(x,\mathrm{d}v)\\
&\le Me^{(|u|+\epsilon/2)M}\int_{\mathbb{V}}(1\wedge|v|^2)\mu(x,\mathrm{d}v) + \sum_{\ell=0}^{1}\int_{|v|>1} e^{\langle u+2\epsilon e_i/3, v\rangle}\mu(x,\mathrm{d}v)\\
&< \infty.
\end{aligned}$$

We again use the dominated convergence theorem to deduce the following.

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{|v|>1} \frac{1}{\delta}\left(e^{\langle u+\delta e_i,v\rangle} - e^{\langle u,v\rangle}\right)\mu(x,\mathrm{d}v)\\
&= \int_{|v|>1} \lim_{\delta\to 0}\frac{1}{\delta}\left(e^{\langle u+\delta e_i,v\rangle} - e^{\langle u,v\rangle}\right)\mu(x,\mathrm{d}v)\\
&= \int_{|v|>1} e^{\langle u,v\rangle} v_i\mu(x,\mathrm{d}v) \qquad \text{(I.9)}
\end{aligned}$$

Combining equations (I.7), (I.8), and (I.9) now yields our desired identity.

$$\frac{\partial}{\partial u_i}\Lambda(u,x) = \left\langle e_i, \beta^\chi(x) + \alpha(x)u + \int_{\mathbb{V}}\left(e^{\langle u,v\rangle}v - \chi(v)\right)\mu(x,\mathrm{d}v)\right\rangle$$

Continuity of $\frac{\partial}{\partial u_i}\Lambda(u,x)$ for $u \in \mathcal{D}_\Lambda^\circ$ involves very similar dominated convergence theorem arguments as above. From here, it is clear that Λ is continuously differentiable with the form in (I.6).

Lemma I.9. *If $0 \in \mathcal{D}_\Lambda^\circ$, then for each $x \in \mathbb{X}$, we have the following integrability.*

$$\int_{\mathbb{V}} |v - \chi(v)| \mu(x, \mathrm{d}v) < \infty$$

This allows us to define the following function $\beta : \mathbb{X} \to \mathbb{V}$ and rewrite Λ.

$$\beta(x) := \beta^\chi(x) + \int_{\mathbb{V}} \big(v - \chi(v)\big) \mu(x, \mathrm{d}v)$$

$$\Lambda(u, x) = \langle u, \beta(x) \rangle + \frac{1}{2} \langle u, \alpha(x) u \rangle + \int_{\mathbb{V}} \big(e^{\langle u,v \rangle} - 1 - \langle u, v \rangle\big) \mu(x, \mathrm{d}v) \tag{I.10}$$

$$D\Lambda(u, x) w = \Big\langle w, \beta(x) + \alpha(x) u + \int_{\mathbb{V}} v \big(e^{\langle u,v \rangle} - 1\big) \mu(x, \mathrm{d}v) \Big\rangle$$

Proof. If $0 \in \mathcal{D}_\Lambda^\circ$, then there exists some $\delta > 0$ such that each $v \in \mathbb{V}$ with $|v| \leq \delta$ also satisfies $v \in \mathcal{D}_\Lambda$. Observe the following implication of this fact, for each $x \in \mathbb{X}$.

$$\begin{aligned}
\int_{\mathbb{V}} \big|v - \chi(v)\big| \mu(x, \mathrm{d}v) &= \int_{|v|>1} |v| \mu(x, \mathrm{d}v) \\
&\leq \int_{|v|>1} \frac{\sqrt{d}}{\delta} \exp\Big(\frac{\delta |v|}{\sqrt{d}}\Big) \mu(x, \mathrm{d}v) \\
&\leq \frac{\sqrt{d}}{\delta} \int_{|v|>1} \exp\Big(\max_{i=1}^{d} \max_{\ell=0}^{1} \big\langle (-1)^\ell \delta e^i, v \big\rangle\Big) \mu(x, \mathrm{d}v) \\
&\leq \frac{\sqrt{d}}{\delta} \sum_{i=1}^{d} \sum_{\ell=0}^{1} \int_{|v|>1} \exp \big\langle (-1)^\ell \delta e^i, v \big\rangle \mu(x, \mathrm{d}v) \\
&< \infty
\end{aligned}$$

From here, the rest of the proof is simple algebra.

Now that we have elaborated upon the relevant results of Λ as a scalar field, we proceed to show the existence of a solutions to the dynamical system it imposes. These tighly coincide with the moments of our affine process.

Definition I.10. *For each $\tau \geq 0$ and $u \in \mathcal{D}_\Lambda$, we say a function $Q^u : [0, \tau] \times \mathbb{X} \to \mathbb{R}$ satisfies* $\mathrm{system}(\Lambda, \tau, u)$ *if the following hold.*

$$\forall\, t \in [0, \tau],\ x \in \mathbb{X}, \quad Q^u(t, x) = q_0^u(t) + \langle q^u(t), x \rangle,$$

$$\forall\, x \in \mathbb{X}, \quad \begin{cases} \dot{Q}^u(t, x) = \Lambda\big(q^u(t), x\big), & t \in [0, \tau] \\ Q^u(0, x) = \langle u, x \rangle \end{cases}$$

Now define the following sets.

$$\mathcal{D}_\Psi(\tau) := \Big\{ u \in \mathcal{D}_\Lambda : \textit{there exists a solution to } \mathrm{system}(\Lambda, \tau, u) \Big\}$$

$$\mathcal{D}_\Psi := \bigcup_{\tau \geq 0} \Big(\{\tau\} \times \mathcal{D}_\Psi(\tau) \Big)$$

Theorem I.11. *(a) There exists a map $\Psi : \mathcal{D}_\Psi \times \mathbb{X} \to \mathbb{R}$ of the form*

$$\Psi(t,u,x) = \psi_0(t,u) + \big\langle \psi(t,u), x \big\rangle$$

such that, for each $(\tau,u) \in \mathcal{D}_\Psi$, $\Psi(\cdot,u,\cdot)$ is a solution to system(Λ,τ,u) *dominated by all other such solutions. Moreover, this map satisfies the following for each $(\tau,u) \in \mathcal{D}_\Psi$ and $x \in \mathbb{X}$.*

$$\mathrm{E}_{\mathrm{P}_x} \exp \langle u, X_t \rangle = \exp \Psi(t,u,x), \quad t \in [0,\tau] \tag{I.11}$$

(b) If $\tau \geq 0$, $u \in \mathbb{V}$, and $x \in \mathbb{X}^\circ$ are such that $\mathrm{E}_{\mathrm{P}_x} \exp \langle u, X_\tau \rangle < \infty$, then $(\tau,u) \in \mathcal{D}_\Psi$.

Proof. With Remark I.4, this is the same as [KRM15, Theorem 2.14].

Now that we have two characterizations for the space $\mathcal{D}_\Psi$, we seek to understand properties of it and the associated moment map $\Psi : \mathcal{D}_\Psi \times \mathbb{X} \to \mathbb{R}$.

Proposition I.12. *(a) For each $\tau > 0$, $\mathcal{D}_\Psi(\tau)$ is open in $\mathcal{D}_\Lambda^\circ$,*

(b) For each $\tau > 0$ and $u \in \mathcal{D}_\Psi(\tau) \cap \mathcal{D}_\Lambda^\circ$, $\Psi(\cdot,u,\cdot)$ from Theorem I.11 is the unique solution to system(Λ,τ,u).

(c) Ψ is continuously differentiable on $\mathcal{D}_\Psi^\circ \times \mathbb{X}$.

Proof. Fix $\tau > 0$ and $u \in \mathcal{D}_\Psi(\tau) \cap \mathcal{D}_\Lambda^\circ$. Because $u \in \mathcal{D}_\Psi(\tau)$, $\Psi(\cdot,u,\cdot)$ exists on $[0,\tau] \times \mathbb{X}$ as a solution to system(Λ,τ,u). As mentioned in Remark I.4, the function $\psi(\cdot,u)$ associated with $\Psi(\cdot,u,\cdot)$ is a solution to the following equation,

$$\begin{cases} \dot{\psi}(t,u) = f\big(t, \psi(t,u)\big) & t \in [0,\tau] \\ \psi(0,u) = u \end{cases} \tag{I.12}$$

where $f : \mathbb{R} \times \mathcal{D}_\Lambda^\circ \to \mathbb{V}$ is defined by $f(t,u) \coloneqq L(u)$. Seeing as f is continuously differentiable on $\mathbb{R} \times \mathcal{D}_\Lambda^\circ$ by Lemma I.8, we may use [Wal98, III.13 Theorem X] to ensure some $\epsilon > 0$ such that the band

$$S_\epsilon \coloneqq \Big\{ (t,v) \in [0,\tau] \times \mathbb{V} : |v - \psi(t,u)| \leq \epsilon \Big\}$$

is contained in $\mathbb{R} \times \mathcal{D}_\Lambda^\circ$ and provides us to each $(t_0,v) \in S_\epsilon$ a unique solution $q(\cdot,t_0,v)$ to the following initial value problem,

$$\begin{cases} \dot{q}(t,t_0,v) = f\big(t, q(t,t_0,v)\big) & t \in [t_0,\tau] \\ q(t_0,t_0,v) = v \end{cases}$$

which is continuously differentiable with derivatives $\partial_{t_0} q(t,t_0,v) \in \mathbb{V}$ and $Dq(t,t_0,v) \in \mathbb{L}(\mathbb{V})$ satisfying the following equations.

$$\partial_{t_0} q(t,t_0,v) = -f(t_0,u) + \int_{t_0}^t Df\big(s, q(s,t_0,v)\big) \partial_{t_0} q(s,t_0,v) \mathrm{d}s$$

$$Dq(t,t_0,v) = \mathrm{id}_\mathbb{V} + \int_{t_0}^t Df\big(s, q(s,t_0,v)\big) Dq(s,t_0,v) \mathrm{d}s$$

In particular, for each $v \in B(u,\epsilon)$, we have $|v-\psi(0,u)| = |v-u| < \epsilon$, and so $(0,v) \in S_\epsilon$; this allows us to disregard the middle coordinate and have $q : [0,\tau] \times B(u,\epsilon) \to \mathbb{V}$ such that $q(\cdot,v)$ is the unique solution to

$$\begin{cases} \dot{q}(t,v) = L\big(q(t,v)\big), & t \in [0,\tau] \\ q(0,v) = v \end{cases}$$

and the derivative in the second coordinate $Dq(t,v) \in \mathbb{L}(\mathbb{V})$ satisfies the following equation.

$$Dq(t,v) = \mathrm{id}_{\mathbb{V}} + \int_0^t DL\big(q(s,v)\big)Dq(s,v)\mathrm{d}s$$

From here, we may define $Q : [0,\tau] \times B(u,\epsilon) \times \mathbb{X} \to \mathbb{R}$ as follows.

$$\begin{aligned} Q(t,v,x) &\coloneqq q_0(t,v) + \big\langle q(t,v), x\big\rangle \\ q_0(t,v) &\coloneqq \int_0^t L_0\big(q(s,v)\big)\mathrm{d}s \\ L_0(v) &\coloneqq \Lambda(v,0) \end{aligned}$$

Because the image of $q(\cdot,v)$ on $[0,\tau]$ remains in $\mathcal{D}_\Lambda^\circ$, on which L is continuously differentiable, q_0 is continuously differentiable with derivatives $\dot{q}_0$ and Dq_0 satisfying the following.

$$\begin{aligned} \dot{q}_0(t,v) &= L_0\big(q(s,v)\big) \\ Dq_0(t,v) &= \int_0^t DL_0\big(q(s,v)\big)Dq(s,v)\mathrm{d}s \end{aligned}$$

By linearity, $Q(\cdot,v,\cdot)$ is a solution to system(Λ,τ,v) and so $v \in \mathcal{D}_\Psi(\tau)$. We now have $B(u,\epsilon) \subseteq \mathcal{D}_\Psi(\tau)$, concluding part (a). Meanwhile, any solution Q^u to system(Λ,τ,u) is such that the associated q^u solves (I.12) and so $q^u = q(\cdot,u)$. From here, it is thus the case that $Q^u = Q(\cdot,u,\cdot)$. This means Ψ from Theorem I.11 is such that $\Psi(\cdot,u,\cdot)$ is the unique solution to system(Λ,u,τ), concluding part (b). Lastly, for each $x \in \mathbb{X}$, linearity also shows us that $\Psi(\cdot,\cdot,x)$ is continuously differentiable in a neighborhood of (t,u), with derivative in the second coordinate $D\Psi(\cdot,\cdot,x)$ satisfying the following.

$$\begin{aligned} D\Psi(t,u,x) &= D\psi_0(t,u) + D\psi(t,u)\cdot x \\ &= Dq_0(t,u) + Dq(t,u)\cdot x \\ &= \int_0^t DL_0\big(q(s,u)\big)Dq(s,u)\mathrm{d}s + \left(\mathrm{id}_{\mathbb{V}} + \int_0^t DL\big(q(s,u)\big)Dq(s,u)\mathrm{d}s\right)\cdot x \\ &= x + \int_0^t \Big(DL_0\big(q(s,u)\big)Dq(s,u) + \sum_{i=1}^d x_i DL_i\big(q(s,u)\big)Dq(s,u)\Big)\mathrm{d}s \\ &= x + \int_0^t D\Big(L_0 + \sum_{i=1}^d x_i L_i\Big)\big(q(s,u)\big)Dq(s,u)\mathrm{d}s \\ &= x + \int_0^t D\Lambda\big(q(s,u),x\big)Dq(s,u)\mathrm{d}s \\ &= x + \int_0^t D\Lambda\big(\psi(s,u),x\big)D\psi(s,u)\mathrm{d}s \end{aligned}$$

This concludes part (c).

Corollary I.13. *Suppose $0 \in \mathcal{D}_\Lambda^\circ$. For each $\tau > 0$, there exists $\gamma > 0$ with $B(0,\gamma) \subseteq \mathcal{D}_\Psi(\tau)$.*

Proof. Suppose $0 \in \mathcal{D}_\Lambda^\circ$ and fix $\tau > 0$. Theorem I.11(b) tells us $0 \in \mathcal{D}_\Psi(\tau)$. Thus, $0 \in \mathcal{D}_\Psi(\tau) \cap \mathcal{D}_\Lambda^\circ$ and so we may use Proposition I.12(a) to ensure $0 \in \mathcal{D}_\Psi(\tau)^\circ$. This means there is some $\gamma > 0$ such that $B(0,\gamma) \subseteq \mathcal{D}_\Psi(\tau)$.

Proposition I.14. *For each compact set $K \subseteq \mathcal{D}_\Lambda^\circ$, there exists $\delta > 0$ such that $K \subseteq \mathcal{D}_\Psi(\delta)$. Moreover, $\Psi(\cdot, u, \cdot)$ from Theorem I.11 is the unique solution to* system(Λ, δ, u) *for each $u \in K$.*

Proof. Firstly, we recognize that by virtue of $K \subseteq \mathcal{D}_\Lambda^\circ$ being compact, we have some $\epsilon > 0$ such that the associated open set

$$K^\epsilon := \Big\{ u \in \mathbb{V} : \inf_{u' \in K} |u - u'| < \epsilon \Big\}$$

has closure $\overline{K^\epsilon}$ contained in $\mathcal{D}_\Lambda^\circ$. Note in particular that this provides us with a buffer of radius ϵ around each point in $\mathcal{D}_\Lambda^\circ$.

$$\overline{B}(u,\epsilon) := \Big\{ u' \in \mathbb{V} : |u' - u| \le \epsilon \Big\}$$
$$\bigcup_{u \in \mathcal{D}_\Lambda} \overline{B}(u,\epsilon) \subseteq \overline{K^\epsilon} \subseteq \mathcal{D}_\Lambda^\circ$$

With these sets established, we mitigate the task of finding a solution Q^u to system(Λ, δ, u) to that of finding a solution q^u to the related equation.

$$\begin{cases} \dot{q}^u(t) = L\big(q^u(t)\big) & t \in [0,\delta] \\ q^u(0) = u \end{cases} \tag{I.13}$$

For a fixed $u \in \mathcal{D}_\Lambda^\circ$, the existence of some $\delta_u > 0$ and solution q^u to (I.13) may be obtained from the usual fixed-point method (see [Wal98, II.6 Theorem III]). Indeed, Remark I.2(d) and Lemma I.8 provide us a Lipschitz property for L on $\overline{B}(u,\epsilon)$,

$$\big|L(v) - L(w)\big| \le |v - w| C_{u,\epsilon}, \qquad v, w \in \overline{B}(u,\epsilon)$$
$$C_{u,\epsilon} := \sup_{u' \in \overline{B}(u,\epsilon)} \Big| DL\big(u', x\big) \Big|$$

and so a Banach space $(\mathbb{B}_u, \|\cdot\|_{\mathbb{B}_u})$ defined by

$$\delta_u := 1 \wedge \frac{\epsilon}{\sup_{u' \in \overline{B}(u,\epsilon)} |L(u')|}$$
$$\mathbb{B}_u := \mathbb{C}([0,\delta_u], \mathbb{V})$$
$$\|f\|_{\mathbb{B}_u} := \sup_{t \in [0,\delta_u]} |f(t)| e^{-2C_{u,\epsilon} t}$$

is partially equipped with a map $T : \mathbb{C}([0,\delta_u], K) \to \mathbb{C}([0,\delta_u], \overline{K^\epsilon})$ defined by

$$Tf(t) := u + \int_0^t L\big(f(s)\big) \mathrm{d}s,$$

satisfying a contraction property,

$$\|Tf - Tg\|_{\mathbb{B}_u} \leq \frac{1}{2}\|f - g\|_{\mathbb{B}_u},$$

which induces a unique solution $q^u \in \mathbb{C}([0, \delta_u], \overline{K^\epsilon})$ to the associated fixed-point equation, $Tq^u = q^u$. This solution q^u is thus a unique solution to (I.13).

From here, we define the following positive δ,

$$\delta := \inf_{u \in K} \delta_u \geq 1 \wedge \inf_{u \in K} \frac{\epsilon}{\sup_{u' \in \overline{B}(u,\epsilon)} |L(u')|} \geq 1 \wedge \frac{\epsilon}{\sup_{u' \in \overline{K}^\epsilon} |L(u')|} > 0$$

so that each $u \in K$ has a unique solution q^u to (I.13). This induces the following map $Q^u : [0, \delta] \times \mathbb{X} \to \mathbb{R}$ for each $u \in K$.

$$Q^u(t, x) := q_0^u(t) + \langle q^u(t), x \rangle$$
$$q_0^u(t) := \int_0^t L_0\big(q^u(s)\big)\mathrm{d}s$$

By linearity, Q^u is a solution to system(Λ, δ, u) for each $u \in K$, and so $K \subseteq \mathcal{D}_\Psi(\delta)$. For each $u \in K \subseteq \mathcal{D}_\Psi(\delta)$, a solution $\tilde{Q}^u$ to system(Λ, δ, u) is such that the associated $\tilde{q}^u$ solves (I.13) and so $\tilde{q}^u = q^u$. From here, it is thus the case that $\tilde{Q}^u = Q^u$. This means Ψ from Theorem I.11 is such that $\Psi(\cdot, u, \cdot)$ is the unique solution to system(Λ, u, δ) for all $u \in K$.

Proposition I.15. *For any compact subset $K \subseteq \mathcal{D}_\Psi^\circ$, there exists $C_K > 0$ such that the following holds for all $(t, u) \in K$.*

$$\big|\Psi(t, u, x) - \Psi(0, u, x)\big| \leq C_K \cdot t \cdot \big(1 + |x|\big)$$

Proof (Proposition I.15). Let $K \subseteq \mathcal{D}_\Psi^\circ$ be compact. By Remark I.4 and Proposition I.12(c), we have that the functions ψ_i for $i = 0, \ldots, d$ are continuously differentiable on $\mathcal{D}_\Psi^\circ$. Thus, we may define the following positive numbers.

$$C_{K,i} := \sup_{(t,u) \in K} |\dot{\psi}_i(t, u)|, \quad i = 0, \ldots, d$$
$$C_K := \max\Big\{C_{K,0}, C_{K,1}\sqrt{d}, \ldots, C_{K,d}\sqrt{d}\Big\} < \infty$$

Using the fundamental theorem of calculus and that $\Psi(\cdot, u, \cdot)$ solves system(Λ, τ, u), we produce the following bound for all $(t, u) \in K$.

$$\begin{aligned}
\big|\Psi(t, u, x) - \Psi(0, u, x)\big| &= \Big|\psi_0(t, u) + \big\langle \psi(t, u) - u, x \big\rangle\Big| \\
&\leq \big|\psi_0(t, u)\big| + \big|\psi(t, u) - u\big| \cdot |x| \\
&= \left|\int_0^t \dot{\psi}_0(s, u)\mathrm{d}s\right| + \left|\int_0^t \dot{\psi}_i(s, u)\mathrm{d}s\right| \cdot |x| \\
&\leq C_{K,0} \cdot t + \Big(\sum_{i=1}^d C_{K,i}^2\Big)^{1/2} \cdot t \cdot |x| \\
&\leq C_K \cdot t \cdot \big(1 + |x|\big)
\end{aligned}$$

I.3 Finite-dimensional distributions

I.4 Affine jump diffusions

Chapter II

Large deviations of affine processes

Chapter III

Large deviation rate functions

Appendix A

Jump diffusions

This is where I motivate my choice of including this Appendix. Big point: I want to resolve the abstractions and rigor of [JS03] to the digestible notions and calculus of diffusions.

A.1 Markov processes

1. Cite [Kal02] for the notion of a conditional distribution
2. Establish the convention that each quantity X on a space $\mathbb{X}$ will be identitified with the identity map on this space, allowing the nice abuse of notation.

$$\mathrm{E}_{\mu}\big(f(X)|Y=y\big) = \int_{\mathbb{X}} f(x)\mu_{X|Y}(\mathrm{d}x|y)$$

3. State the definition of a Markov process.
4. State the definition of a time-homogeneous Markov process.
5. State the definition of transition kernel
6. State how transition kernels decide distribution on (weak) path space
7. State the definition of stochastic continuity of a Markov process.

A.2 Semimartingales

1. Resolve $X_t : \Omega \to \mathbb{V}$ with $X : \Omega \times \mathbb{R}_+ \to \mathbb{V}$.
2. State the notion of optional, predictable.
3. State the notion of finite variation.
4. State the notion of a martingale and its predictable compensator.
5. State the notion of a semimartingale.
6. State Itô's formula.

7. State the notion of stochastic integration against a semimartingale.
8. State the notion of a random measure.
9. State the notion of characteristics.

A.3 Jump diffusions

1. State the definition of a jump diffusion in terms of its characteristics. Establish the notion of drift, diffusion, jump.
2. Restate Itô in terms of drift, diffusion, jump.
3. Establish the concept of the generator in terms of the drift, diffusion, jump.
4. Establish how the generator on the exponential induces the Λ map in terms of drift, diffusion, jump.
5. Establish a Girsanov theorems associated with Λ.

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