Large deviations of affine processes

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Abstract

This is an abstract of the entire dissertation; summarize a history of large deviations and affine processes, then abstractly summarize our large deviations result.

Acknowledgment

This is where I acknowledge how I am useless without others.

Contents

Introduction

This is where I give the reader a little more history and detail regarding affine processes and large deviations, should they read this paper without already being well-versed in the subject.

Notation and conventions

I want this section to clear up notational similarities and differences with literature. Namely

- All the objects one needs for stochastic processes and their stochastic integration.
- All the spaces one often sees in real analysis.
- The space, functions, and parameters associated with a given affine process.

Chapter I

Affine processes

Here I put a summary of chapter, along with a short history. It will include the following important notes.

- Chapter addresses important fundamental results of affine processes.
- Chapter addresses consequences of results that are important for us, though not specified exactly in any of the literature.
- Chapter presents the information in an order of increasing complexity of concepts (versus the order in which it is typically proven).

I.1 Formulation

We start by specifying a state space on which our stochastic processes live. Let $\mathbb{V}$ be a finite-dimensional real vectorspace with inner-product $\langle\cdot,\cdot\rangle$. Equip $\mathbb{V}$ with the canonical topology and Borel algebra from $\langle\cdot,\cdot\rangle$. Denote the dimension $d \coloneqq \dim \mathbb{V}$ and establish the canonical isometric isomorphism $\mathbb{V} \equiv \mathbb{R}^d$ by specifying an orthonormal basis $e_1, \ldots, e_d \in \mathbb{V}$, so that we may identify components of vectors in $\mathbb{V}$.

$$v \in \mathbb{V} \quad \longleftrightarrow \quad v^i \coloneqq \langle v, e_i\rangle, \quad i = 1, \ldots, d \tag{I.1}$$

Similarly identify any map $f : \mathbb{A} \to \mathbb{V}$ with component functions $f_1, \ldots, f_d : \mathbb{A} \to \mathbb{R}$. Extend the inner-product to a complex bilinear form on $\mathbb{V} \oplus \mathrm{i}\mathbb{V}$, linearly and symmetrically.

$$\big\langle v_1 + \mathrm{i}w_1, v_2 + \mathrm{i}w_2\big\rangle = \big(\langle v_1, v_2\rangle - \langle w_1, w_2\rangle\big) + \mathrm{i}\big(\langle v_1, w_2\rangle + \langle w_1, v_2\rangle\big) \tag{I.2}$$

Fix a convex and closed $\mathbb{X} \subseteq \mathbb{V}$ satisfying $0 \in \mathbb{X}$ and $\operatorname{span} \mathbb{X} = \mathbb{V}$. Associate this space with the finite exponentials.

$$\mathcal{U}_{\mathbb{X}} \coloneqq \Big\{u \in \mathbb{V} \oplus \mathrm{i}\mathbb{V} : \sup_{x\in\mathbb{X}} \exp\langle \Re(u), x\rangle < \infty\Big\} \tag{I.3}$$

We may now define the notion of an affine process on $\mathbb{X}$.

Definition I.1. *For a probability space* $(\Omega, \Sigma, \mathrm{P})$ *with filtration* $\mathscr{F} = (\mathscr{F}_t)_{t\geq 0}$*, an affine process* X *on* $\mathbb{X}$ *is a stochastically continuous, time-homogeneous* $(\mathrm{P}, \mathscr{F})$*-Markov process on* $\mathbb{X}$ *in which the bounded complex moments have the following log-affine dependence on the initial state.*

$$\begin{aligned} \mathrm{E}_{\mathrm{P}_x} \exp\langle u, X_t\rangle &= \exp \Psi(t,u,x) \\ \Psi(t,u,x) &= \psi_0(t,u) + \big\langle \psi(t,u), x\big\rangle, \end{aligned} \qquad t \geq 0,\ u \in \mathcal{U}_{\mathbb{X}} \tag{I.4}$$

Above, we are denoting $(\mathrm{P}_x)_{x\in\mathbb{X}}$ *the conditional distributions of* X *factored through the initial state (see Appendix* **??** *for further specification and notation).* □

Remark I.2. *Our definitions of* $\mathbb{X}$ *and* Ψ *include the following conventions and motivations.*

(a) *why we denote* ψ_0, ψ *instead of KM* φ, ψ *or Cuchiero* Φ, Ψ

(b) *how assumptions* $0 \in \mathbb{X}$, $\operatorname{span}\mathbb{X} = \mathbb{V}$ *are nonrestrictive*

(c) *how (I.4) decides the distribution of* X *and how the distribution of the affine process decides* Ψ

(d) *If we have a vectorspace* $\mathbb{A}$ *and affine map* $\alpha : \mathbb{X} \to \mathbb{A}$ *determined by* $a_0, \ldots, a_d \in \mathbb{A}$ *via* $\alpha(x) = a_0 + \sum_{i=1}^d x^i a_i$*, then our linear assumptions* $0 \in \mathbb{X}$ *and* $\operatorname{span}\mathbb{X} = \mathbb{V}$ *uniquely determine* $a_0, \ldots, a_d \in \mathbb{A}$*. In particular, the map* Ψ *uniquely identifies its parts* $\psi_i : \mathbb{R}_+ \times \mathcal{U}_{\mathbb{X}} \to \mathbb{C}$ *for* $i = 0, \ldots, d$.

(e) *In [Cuc11, Theorem 1.2.7], it is shown that, without loss of generality on conditional distributions* $(\mathrm{P}_x)_{x\in\mathbb{X}}$*, an affine process* X *can be chosen to have càdlàg paths. Thus, each distribution* P_x *may (and will) be recognized as a measure on the Borel algebra associated with the space* $\mathbb{D}([0,\infty), \mathbb{X})$ *of càdlàg functions equipped with the Skorokhod topology (see Appendix* **??***).* □

While the tuple (ψ_0, ψ) in (I.4) is a simpler object than the distributions $(\mathrm{P}_x)_{x\in\mathbb{X}}$, selecting an *admissible* tuple—one (ψ_0, ψ) which actually can appear in (I.4)—is seemingly prohibitive. The following result is incredibly useful at demonstrating *parameters* β^χ, α, μ of an affine process, that easily specify admissible pairs (ψ_0, ψ).

Theorem I.3. *Fix an affine process* X *on* $\mathbb{X}$*. There exists affine functions* β^χ, α, μ *of the form,*

$$\beta^\chi(x) := b_0^\chi + \sum_{i=1}^d x^i b_i^\chi, \qquad b_0^\chi, \ldots, b_d^\chi \in \mathbb{V} \tag{I.5}$$

$$\alpha(x) := a_0 + \sum_{i=1}^d x^i a_i, \qquad a_0, \ldots, a_d \in \mathbb{L}(\mathbb{V}) \tag{I.6}$$

$$\mu(x, \mathrm{d}v) := m_0(\mathrm{d}v) + \sum_{i=1}^d m_i(\mathrm{d}v), \qquad m_0, \ldots, m_d \in \mathbb{M}_1(\mathscr{B}(\mathbb{V})) \tag{I.7}$$

which induce Ψ *as follows. For each* $u \in \mathcal{U}_{\mathbb{X}}$*, we have affine function* $\Lambda(u, \cdot) : \mathbb{X} \to \mathbb{R}$*, defined by*

$$\Lambda(u,x) := \big\langle u, \beta^\chi(x)\big\rangle + \frac{1}{2}\big\langle u, \alpha(x)u\big\rangle + \int_{\mathbb{V}} \big(e^{\langle u,v\rangle} - 1 - \langle u, \chi(v)\rangle\big)\mu(x, \mathrm{d}v) \tag{I.8}$$

$$\chi(v) \coloneqq v 1_{|v|\le 1} \tag{I.9}$$

$$\forall x \in \mathbb{X} \qquad \begin{cases} \dot{\Psi}(t,u,x) = \Lambda\big(\psi(t,u),x\big) & t \ge 0 \\ \Psi(0,u,x) = \langle u, x\rangle \end{cases} \tag{I.10}$$

(I.11)

I.2 Existence of real moments

Lemma I.4. *The expression $\Lambda(u,x)$ is well-defined for all $x \in \mathbb{X}$ if and only if $u \in \mathcal{D}_\Lambda$.* □

Proof. To each $u, v \in \mathbb{V}$, Taylor's theorem gives us $\gamma_{u,v} \in [0,1]$ such that

$$e^{\langle u,v\rangle} = 1 + \langle u,v\rangle + \frac{1}{2} e^{\gamma_{u,v}\langle u,v\rangle}\langle u,v\rangle^2. \tag{I.12}$$

This allows us to see that, for each $x \in \mathbb{X}$, $\Lambda(u,x)$ and $\int_{|v|>1} e^{\langle u,v\rangle}\mu(x,\mathrm{d}v)$ differ by finite expressions.

$$\begin{aligned}
&\left|\Lambda(u,x) - \int_{|v|>1} e^{\langle u,v\rangle}\mu(x,\mathrm{d}v)\right| \\
&= \left|\langle u, \beta^\chi(x)\rangle + \frac{1}{2}\langle u, \alpha(x)u\rangle + \int_{|v|\le 1}\big(e^{\langle u,v\rangle} - 1 - \langle u,v\rangle\big)\mu(x,\mathrm{d}v) - \int_{|v|>1}\mu(x,\mathrm{d}v)\right| && \text{(I.13)}\\
&\le \left|\langle u, \beta^\chi(x)\rangle + \frac{1}{2}\langle u, \alpha(x)u\rangle\right| + \left|\int_{|v|\le 1}\frac{1}{2}e^{\gamma_{u,v}\langle u,v\rangle}\langle u,v\rangle^2\mu(x,\mathrm{d}v)\right| + \int_{|v|>1}\mu(x,\mathrm{d}v) && \text{(I.14)}\\
&\le \left|\langle u, \beta^\chi(x)\rangle + \frac{1}{2}\langle u, \alpha(x)u\rangle\right| + \left(\frac{1}{2}e^{|u|} + 1\right)\int_{\mathbb{V}}\big(1 \wedge |v|^2\big)\mu(x,\mathrm{d}v) && \text{(I.15)}
\end{aligned}$$

Thus, one can be defined as a finite displacement of the other. □

Remark I.5. *Seeing as $0 \in \mathbb{X}$, we may define the following map $L_0 : \mathcal{D}_\Lambda \to \mathbb{R}$.*

$$L_0(u) \coloneqq \Lambda(u,0) = \langle u, b_0^\chi\rangle + \frac{1}{2}\langle u, a_0 u\rangle + \int_{\mathbb{V}}\big(e^{\langle u,v\rangle} - 1 - \langle u, \chi(v)\rangle\big)m_0(\mathrm{d}v) \tag{I.16}$$

Fix $i = 1,\ldots,d$. Because $\operatorname{span}\mathbb{X} = \mathbb{V}$*, we may produce a linear combination of elements of $\mathbb{X}$ to produce our standard basis vector e_i, say $e_i =: \sum_{\ell=1}^m \gamma_\ell x_\ell$. From here, we may define $L_i : \mathcal{D}_\Lambda \to \mathbb{R}$ as follows.*

$$\begin{aligned}
L_i(u) &\coloneqq \sum_{\ell=1}^m \gamma_\ell\big(\Lambda(u,x_\ell) - \Lambda(u,0)\big) && \text{(I.17)}\\
&= \langle u, b_i^\chi\rangle + \frac{1}{2}\langle u, a_i u\rangle + \int_{\mathbb{V}}\big(e^{\langle u,v\rangle} - 1 - \langle u, \chi(v)\rangle\big)m_i(\mathrm{d}v)
\end{aligned}$$

In other words, the affine structure of our maps β, α, μ and the linear assumptions on $\mathbb{X}$ allow us to extract component maps $L_0 : \mathcal{D}_\Lambda \to \mathbb{R}$, $L : \mathcal{D}_\Lambda \to \mathbb{V}$ which build Λ.

$$\Lambda(u,x) = L_0(u) + \big\langle L(u), x\big\rangle = L_0(u) + \sum_{i=1}^d x^i L_i(u) \tag{I.18}$$

□

Lemma I.6. *$\mathcal{D}_\Lambda$ is convex.* □

Proof. Let $u, u' \in \mathcal{D}_\Lambda$, $\gamma \in (0,1)$, and use Hölder's inequality to see the following.

$$\int_{|v|>1} e^{\langle u'+\gamma(u-u'),v\rangle}\mu(x,\mathrm{d}v)$$

$$= \int_{|v|>1} \left|(e^{\langle u,v\rangle})^\gamma \cdot (e^{\langle u',v\rangle})^{1-\gamma}\right|\mu(x,\mathrm{d}v) \tag{I.19}$$

$$\le \left(\int_{|v|>1} \left|(e^{\langle u,v\rangle})^\gamma\right|^{\frac{1}{\gamma}}\mu(x,\mathrm{d}v)\right)^\gamma \left(\int_{|v|>1} \left|(e^{\langle u',v\rangle})^{1-\gamma}\right|^{\frac{1}{1-\gamma}}\mu(x,\mathrm{d}v)\right)^{1-\gamma} \tag{I.20}$$

$$= \left(\int_{|v|>1} e^{\langle u,v\rangle}\mu(x,\mathrm{d}v)\right)^\gamma \left(\int_{|v|>1} e^{\langle u',v\rangle}\mu(x,\mathrm{d}v)\right)^{1-\gamma} \tag{I.21}$$

$$< \infty \tag{I.22}$$

An arbitrary convex combination now satisfies $\gamma u + (1-\gamma)u' \in \mathcal{D}_\Lambda$. □

Lemma I.7. *For each $x \in \mathbb{X}$, the map $\Lambda(\cdot,x)$ is continuously differentiable on $\mathcal{D}_\Lambda^\circ$, with derivative $D\Lambda(\cdot,x) : \mathcal{D}_\Lambda^\circ \to \mathbb{L}(\mathbb{V},\mathbb{R})$ as follows.*

$$D\Lambda(u,x)w = \left\langle \beta^\chi(x) + \alpha(x)u + \int_{\mathbb{V}} \left(e^{\langle u,v\rangle}v - \chi(v)\right)\mu(x,\mathrm{d}v), w\right\rangle, \quad u \in \mathcal{D}_\Lambda^\circ \tag{I.23}$$ □

Proof. Fix $x \in \mathbb{X}$, $u \in \mathcal{D}_\Lambda^\circ$. Let $\epsilon > 0$ such that $B(u,\epsilon) \subseteq \mathcal{D}_\Lambda$. For all $0 < \delta < \epsilon$ and $i = 1, \ldots, d$, we now have the following identity

$$\begin{aligned}\frac{\Lambda(u+\delta e^i,x) - \Lambda(u,x)}{\delta} &= \langle e^i, \beta^\chi(x)\rangle + \langle e^i, \alpha(x)u\rangle + \frac{1}{2}\langle \delta e^i, \alpha(x)u\rangle \\ &+ \int_{|v|\le 1} \frac{1}{\delta}\left(e^{\langle u+\delta e^i,v\rangle} - e^{\langle u,v\rangle} - \langle \delta e^i, v\rangle\right)\mu(x,\mathrm{d}v) \\ &+ \int_{|v|>1} \frac{1}{\delta}\left(e^{\langle u+\delta e^i,v\rangle} - e^{\langle u,v\rangle}\right)\mu(x,\mathrm{d}v)\end{aligned} \tag{I.24}$$

Evaluating the limit $\delta \to 0$ is now a matter of exchanging the limit with integration; we will do this by using the dominated convergence theorem.

For the first integral, Taylor's theorem provides us $\gamma_0, \gamma_1 \in [0,1]$ such that the following hold.

$$e^{\langle u+\delta e^i,v\rangle} = 1 + \langle u+\delta e^i, v\rangle + \frac{1}{2}\langle u+\delta e^i, v\rangle^2 e^{\gamma_0\langle u+\delta e^i,v\rangle} \tag{I.25}$$

$$e^{\langle u,v\rangle} = 1 + \langle u,v\rangle + \frac{1}{2}\langle u,v\rangle^2 e^{\gamma_1\langle u,v\rangle} \tag{I.26}$$

This shows us that, for all $0 < \delta < \epsilon$ and $|v| \le 1$

$$\left|\frac{1}{\delta}\left(e^{\langle u+\delta e^i,v\rangle} - e^{\langle u,v\rangle} - \langle \delta e^i, v\rangle\right)\right| = \left|\frac{1}{2}\langle u+\delta e^i, v\rangle^2 e^{\gamma_0\langle u+\delta e^i,v\rangle} + \frac{1}{2}\langle u,v\rangle^2 e^{\gamma_1\langle u,v\rangle}\right| \tag{I.27}$$

$$\le \left(\left(|u|+\epsilon\right)^2 e^{|u|+\epsilon}\right)|v|^2. \tag{I.28}$$

This dominating function is integrable,

$$\int_{|v|\leq 1}\Big(\big(|u|+\epsilon\big)^2 e^{|u|+\epsilon}\Big)|v|^2\mu(x,\mathrm{d}v) \leq \Big(\big(|u|+\epsilon\big)^2 e^{|u|+\epsilon}\Big)\int_{\mathbb{V}}(1\wedge |v|^2)\mu(x,\mathrm{d}v) < \infty, \tag{I.29}$$

so we may apply the dominated convergence theorem.

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{|v|\leq 1}\frac{1}{\delta}\Big(e^{\langle u+\delta e^i,v\rangle} - e^{\langle u,v\rangle} - \langle \delta e^i, v\rangle\Big)\mu(x,\mathrm{d}v) \\
&= \int_{|v|\leq 1}\lim_{\delta\to 0}\frac{1}{\delta}\Big(e^{\langle u+\delta e^i,v\rangle} - e^{\langle u,v\rangle} - \langle \delta e^i, v\rangle\Big)\mu(x,\mathrm{d}v) && \text{(I.30)}\\
&= \int_{|v|\leq 1}\Big(e^{\langle u,v\rangle}v_i - v_i\Big)\mu(x,\mathrm{d}v) && \text{(I.31)}
\end{aligned}$$

For the second integral, we again use Taylor's theorem to establish for each $0<\delta<\epsilon/2$, some $\gamma_\delta\in[0,\delta]$ such that

$$e^{\langle u+\delta e^i,v\rangle} = e^{\langle u,v\rangle} + \langle \delta e^i, v\rangle e^{\langle u+\gamma_\delta e^i,v\rangle} \tag{I.32}$$

This way, we have the following dominating function.

$$\left|\frac{1}{\delta}\Big(e^{\langle u+\delta e^i,v\rangle} - e^{\langle u,v\rangle}\Big)\right| \leq \left|\langle e^i, v\rangle e^{\langle u+\gamma_\delta e^i,v\rangle}\right| \leq |v_i| e^{\langle u,v\rangle + \epsilon|v_i|/2} \tag{I.33}$$

The claim is that this function is integrable. To see this, first note that because we have the following limit,

$$\lim_{|v|\to\infty}\frac{|v_i|e^{\langle u,v\rangle+\epsilon|v_i|/2}}{e^{\langle u,v\rangle+2\epsilon|v_i|/3}} = \lim_{|v|\to\infty}\frac{|v_i|}{e^{\epsilon|v_i|/6}} = 0 \tag{I.34}$$

There exists $M>0$ such that for all $|v|>M$,

$$|v_i|e^{\langle u,v\rangle+\epsilon|v_i|/2} < e^{\langle u,v\rangle+2\epsilon|v_i|/3} \tag{I.35}$$

We now see that

$$\begin{aligned}
&\int_{|v|>1}|v_i|e^{\langle u,v\rangle+\epsilon|v_i|/2}\mu(x,\mathrm{d}v) && \text{(I.36)}\\
&= \int_{1<|v|\leq M}|v_i|e^{\langle u,v\rangle+\epsilon|v_i|/2}\mu(x,\mathrm{d}v) + \int_{|v|>M}|v_i|e^{\langle u,v\rangle+\epsilon|v_i|/2}\mu(x,\mathrm{d}v) && \text{(I.37)}\\
&\leq \int_{1<|v|\leq M} Me^{(|u|+\epsilon/2)M}\mu(x,\mathrm{d}v) + \int_{|v|>M} e^{\langle u,v\rangle+2\epsilon|v_i|/3}\mu(x,\mathrm{d}v) && \text{(I.38)}\\
&\leq Me^{(|u|+\epsilon/2)M}\int_{\mathbb{V}}(1\wedge|v|^2)\mu(x,\mathrm{d}v) + \sum_{\ell=0}^{1}\int_{|v|>1} e^{\langle u+2\epsilon e^i/3,v\rangle}\mu(x,\mathrm{d}v) && \text{(I.39)}\\
&< \infty. && \text{(I.40)}
\end{aligned}$$

We again use the dominated convergence theorem to deduce the following.

$$\lim_{\delta\to 0}\int_{|v|>1}\frac{1}{\delta}\Big(e^{\langle u+\delta e^i,v\rangle} - e^{\langle u,v\rangle}\Big)\mu(x,\mathrm{d}v)$$

$$= \int_{|v|>1} \lim_{\delta \to 0} \frac{1}{\delta} \Big(e^{\langle u+\delta e^i, v\rangle} - e^{\langle u,v\rangle} \Big) \mu(x, \mathrm{d}v) \tag{I.41}$$

$$= \int_{|v|>1} e^{\langle u,v\rangle} v_i \mu(x, \mathrm{d}v) \tag{I.42}$$

Combining equations (I.25), (I.31), and (I.42) now yields our desired identity.

$$\frac{\partial}{\partial u_i}\Lambda(u,x) = \Big\langle e^i, \beta^\chi(x) + \alpha(x)u + \int_{\mathbb{V}} \Big(e^{\langle u,v\rangle} v - \chi(v) \Big) \mu(x, \mathrm{d}v) \Big\rangle \tag{I.43}$$

Continuity of $\frac{\partial}{\partial u_i}\Lambda(u,x)$ for $u \in \mathcal{D}_\Lambda^\circ$ involves very similar dominated convergence theorem arguments as above. $\square$

I.3 Jump diffusions

Chapter II

Large deviations of affine processes

Appendix A

Jump-Diffusions

Bibliography

[Cuc11] Christina Cuchiero. Affine and polynomial processes, 2011.