Large deviations of affine processes

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Abstract

This is an abstract of the entire dissertation; summarize a history of large deviations and affine processes, then abstractly summarize our large deviations result.

Acknowledgment

This is where I acknowledge how I am useless without others.

Contents

Introduction

This is where I give the reader a little more history and detail regarding affine processes and large deviations, should they read this paper without already being well-versed in the subject.

Notation and conventions

I want this section to clear up notational similarities and differences with literature. Namely

- All the objects one needs for stochastic processes and their stochastic integration.
- All the spaces one often sees in real analysis.
- The space, functions, and parameters associated with a given affine process.

Chapter I

Affine processes

Here I put a summary of chapter, along with a short history. It will include the following important notes.

- Chapter addresses important fundamental results of affine processes.
- Chapter addresses consequences of results that are important for us, though not specified exactly in any of the literature.
- Chapter presents the information in an order of increasing complexity of concepts (versus the order in which it is typically proven).

I.1 Formulation

We start by specifying a state space on which our stochastic processes live. Let $\mathbb{V}$ be a finite-dimensional real vectorspace with inner-product $\langle\cdot,\cdot\rangle$. Equip $\mathbb{V}$ with the canonical topology and Borel algebra from $\langle\cdot,\cdot\rangle$. Denote the dimension $d \coloneqq \dim \mathbb{V}$ and establish the canonical isometric isomorphism $\mathbb{V} \equiv \mathbb{R}^d$ by specifying an orthonormal basis $e_1, \ldots, e_d \in \mathbb{V}$, so that we may identify components of vectors in $\mathbb{V}$.

$$v \in \mathbb{V} \quad \longleftrightarrow \quad v^i \coloneqq \langle v, e_i \rangle, \quad i = 1, \ldots, d$$

Similarly identify any map $f : \mathbb{A} \to \mathbb{V}$ with component maps $f_1, \ldots, f_d : \mathbb{A} \to \mathbb{R}$.

$$f : \mathbb{A} \to \mathbb{V} \quad \longleftrightarrow \quad (f_1, \ldots, f_d) : \mathbb{A} \to \mathbb{R}^d;\ f_i(a) = \langle f(a), e_i \rangle$$

Extend the inner-product symmetrically to a bilinear form on $\mathbb{V} \oplus \mathrm{i}\mathbb{V}$.

$$\big\langle v_1 + \mathrm{i}w_1, v_2 + \mathrm{i}w_2 \big\rangle = \big(\langle v_1, v_2\rangle - \langle w_1, w_2\rangle\big) + \mathrm{i}\big(\langle v_1, w_2\rangle + \langle w_1, v_2\rangle\big)$$

Fix a convex and closed $\mathbb{X} \subseteq \mathbb{V}$ satisfying $0 \in \mathbb{X}$ and $\operatorname{span} \mathbb{X} = \mathbb{V}$. Associate this space with the finite exponentials.

$$\mathcal{U}_{\mathbb{X}} \coloneqq \Big\{ u \in \mathbb{V} \oplus \mathrm{i}\mathbb{V} : \sup_{x \in \mathbb{X}} \exp \langle \Re(u), x \rangle < \infty \Big\}$$

We may now define the notion of an affine process on $\mathbb{X}$.

Definition I.1. *For a probability space* $(\Omega, \Sigma, \mathrm{P})$ *with filtration* $\mathscr{F} = (\mathscr{F}_t)_{t\geq 0}$*, an affine process* X *on* $\mathbb{X}$ *is a stochastically continuous, time-homogeneous* $(\mathrm{P}, \mathscr{F})$*-Markov process on* $\mathbb{X}$ *in which the bounded moments have the following log-affine dependence on the initial state.*

$$\begin{aligned} \mathrm{E}_{\mathrm{P}_x} \exp \langle u, X_t\rangle &= \exp \Psi(t,u,x) \\ \Psi(t,u,x) &= \psi_0(t,u) + \big\langle \psi(t,u), x\big\rangle, \end{aligned} \qquad t \geq 0,\ u \in \mathcal{U}_{\mathbb{X}} \tag{I.1}$$

Above, we are denoting $(\mathrm{P}_x)_{x\in\mathbb{X}}$ *the conditional distributions of* X *factored through the initial state (see Appendix* **??** *for further specification and notation).*

Remark I.2. *Our definitions of* $\mathbb{X}$ *and* Ψ *include the following conventions and motivations.*

(a) why we denote ψ_0, ψ *instead of KM* φ, ψ *or Cuchiero* Φ, Ψ

(b) how assumptions $0 \in \mathbb{X}$*,* $\operatorname{span} \mathbb{X} = \mathbb{V}$ *are nonrestrictive*

(c) how (I.1) decides the distribution of X *and how the distribution of the affine process decides* Ψ

(d) If we have a vectorspace $\mathbb{A}$ *and affine map* $\alpha : \mathbb{X} \to \mathbb{A}$ *determined by* $a_0, \ldots, a_d \in \mathbb{A}$ *via* $\alpha(x) = a_0 + \sum_{i=1}^d x^i a_i$*, then our linear assumptions* $0 \in \mathbb{X}$ *and* $\operatorname{span}\mathbb{X} = \mathbb{V}$ *uniquely determine* $a_0, \ldots, a_d \in \mathbb{A}$*. In particular, the map* Ψ *uniquely identifies its parts* $\psi_i : \mathbb{R}_+ \times \mathcal{U}_{\mathbb{X}} \to \mathbb{C}$ *for* $i = 0, \ldots, d$*.*

(e) In [Cuc11, Theorem 1.2.7], it is shown that, without loss of generality on conditional distributions $(\mathrm{P}_x)_{x\in\mathbb{X}}$*, an affine process* X *can be chosen to have càdlàg paths. Thus, each distribution* P_x *may (and will) be recognized as a measure on the Borel algebra associated with the space* $\mathbb{D}([0,\infty), \mathbb{X})$ *of càdlàg functions equipped with the Skorokhod topology (see Appendix* **??***).*

While the tuple (ψ_0, ψ) in (I.1) is a simpler object than the distributions $(\mathrm{P}_x)_{x\in\mathbb{X}}$, selecting an *admissible* tuple—one (ψ_0, ψ) which actually can appear in (I.1)—is seemingly prohibitive. The following result is incredibly useful at demonstrating primitive affine objects which determine the time-dynamics of $\big(\psi_0(\cdot, u), \psi(\cdot, u)\big)$ for each $u \in \mathcal{D}_\Psi$. These implicitly depend on a *truncation function* χ, which we select as follows.

$$\chi : \mathbb{V} \to \mathbb{V}, \quad \chi(v) \coloneqq \begin{cases} v, & |v| \leq 1 \\ 0, & |v| > 1 \end{cases} \tag{I.2}$$

The importance of this function is for local integrability properties—the details of which we defer until Section I.4.

Theorem I.3. *Fix an affine process* X *on* $\mathbb{X}$*. There exist* $b_0^\chi, \ldots, b_d^\chi \in \mathbb{V}$*,* $a_0, \ldots, a_d \in \mathbb{L}(\mathbb{V})$*, and* $m_0, \ldots, m_d \in \mathbb{M}_s(\mathscr{B}(\mathbb{V}))$ *such that the following maps* $L_0, \ldots, L_d : \mathcal{U}_{\mathbb{X}} \to \mathbb{R}$*,*

$$L_i(u) \coloneqq \big\langle u, b_i^\chi(x)\big\rangle + \frac{1}{2}\big\langle u, a_i(x)u\big\rangle + \int_{\mathbb{V}} \Big(e^{\langle u,v\rangle} - 1 - \big\langle u, \chi(v)\big\rangle\Big) m_i(x, \mathrm{d}v),$$

determine the dynamics of each $\big(\psi_0(\cdot,u),\psi(\cdot,u)\big)$.

$$\begin{cases} \dot\psi_0(t,u) = L_0\big(\psi(t,u)\big) & t \geq 0 \\ \dot\psi(t,u) = L\big(\psi(t,u)\big) & t \geq 0 \\ \psi_0(0,u) = 0 \\ \psi(0,u) = u \end{cases}$$

Proof. This is simply a restatement of [Cuc11, Theorem 1.5.4].

Remark I.4. *By Remark I.2(d), Theorem I.3 is equivalent to the existence of maps*

$$\begin{aligned} \beta^\chi(x) &\coloneqq b_0^\chi + \sum_{i=1}^d x^i b_i^\chi, \\ \alpha(x) &\coloneqq a_0 + \sum_{i=1}^d x^i a_i, \\ \mu(x,\mathrm{d}v) &\coloneqq m_0(\mathrm{d}v) + \sum_{i=1}^d x^i m_i(\mathrm{d}v) \\ \Lambda(u,x) &= L_0(u) + \sum_{i=1}^d x^i L_i(u) \\ &\coloneqq \big\langle u, \beta^\chi(x)\big\rangle + \frac{1}{2}\big\langle u, \alpha(x)u\big\rangle + \int_{\mathbb{V}} \Big(e^{\langle u,v\rangle} - 1 - \big\langle u, \chi(v)\big\rangle\Big)\mu(x,\mathrm{d}v) \end{aligned} \tag{I.3}$$

which specify the dynamics of Ψ.

$$\forall x \in \mathbb{X}, \qquad \begin{cases} \dot\Psi(t,u,x) = \Lambda\big(\psi(t,u),x\big) & t \geq 0 \\ \Psi(0,u,x) = \langle u,x\rangle \end{cases} \tag{I.4}$$

Indeed, this is also because differentiability and our initial condition in (I.4) are linear.

Remark I.5. *In [Cuc11] there are immediate results on our functions* β^χ,α,μ *which are readily apparent in Section I.4, such as, for all* $x \in \mathbb{X}$*, the following are true.*

$$\begin{gathered} \alpha(x) \text{ is positive semidefinite} \\ \int_{\mathbb{V}} \big(1 \wedge |v|^2\big)\mu(x,\mathrm{d}v) < \infty \\ \mu(x,\{0\}) = 0 \end{gathered} \tag{I.5}$$

discuss how other papers try to clearly specify which other conditions on β^χ,α,μ *parameterize all admissible* (ψ_0,ψ)*, depending on the definition of* $\mathbb{X}$.

Henceforth, we fix X a càdlàg affine process with conditional distributions $(\mathrm{P}_x)_{x\in\mathbb{X}}$, induced filtration $\mathscr{F} = (\mathscr{F}_t)_{t\geq 0}$, and moment function Ψ as in Definition I.1. Also fix the parameters $(b_i^\chi, a_i, m_i, L_i)_{i=0}^d$ from Theorem I.3 and the associated functions $\beta^\chi,\alpha,\mu,\Lambda$ from Remark I.4.

I.2 Existence of real moments

This elaborates upon the extension of the transform formula in (I.1) and dynamics in (I.4) to real moments $u \in \mathbb{V}$. Clearly, should any extension exist for some $u \in \mathbb{V}$, the value $\Lambda(u,x) = \dot{\Psi}(0,u,x)$ should be well-defined. That said, we henceforth denote the following set.

$$\mathcal{D}_\Lambda \coloneqq \Big\{ u \in \mathbb{V} : \Lambda(u,x) \text{ is well-defined for all } x \in \mathbb{X} \Big\}$$

Quick inspection of $\Lambda(u,x)$ in (I.3) indicates that well-definition should depend solely on the integral term. The following results will perform the careful analysis to show that Λ has nice regularity on the finite moments of its associated jump kernel $\mu(\cdot, \mathrm{d}v)$.

Lemma I.6. *We have $u \in \mathcal{D}_\Lambda$ if and only if each $x \in \mathbb{X}$ is such that*

$$\int_{|v|>1} e^{\langle u,v\rangle} \mu(x,\mathrm{d}v) < \infty$$

Proof. To each $u, v \in \mathbb{V}$, Taylor's theorem gives us $\gamma_{u,v} \in [0,1]$ such that

$$e^{\langle u,v\rangle} = 1 + \langle u,v\rangle + \frac{1}{2} e^{\gamma_{u,v}\langle u,v\rangle} \langle u,v\rangle^2.$$

This allows us to see that, for each $x \in \mathbb{X}$, $\Lambda(u,x)$ and $\int_{|v|>1} e^{\langle u,v\rangle}\mu(x,\mathrm{d}v)$ differ by finite expressions.

$$\begin{aligned}
&\left|\Lambda(u,x) - \int_{|v|>1} e^{\langle u,v\rangle}\mu(x,\mathrm{d}v)\right| \\
&= \left|\langle u, \beta^{\chi}(x)\rangle + \frac{1}{2}\langle u, \alpha(x) u\rangle + \int_{|v|\le 1} \big(e^{\langle u,v\rangle} - 1 - \langle u,v\rangle\big)\mu(x,\mathrm{d}v) - \int_{|v|>1}\mu(x,\mathrm{d}v)\right| \\
&\le \left|\langle u, \beta^{\chi}(x)\rangle + \frac{1}{2}\langle u, \alpha(x) u\rangle\right| + \left|\int_{|v|\le 1} \frac{1}{2} e^{\gamma_{u,v}\langle u,v\rangle}\langle u,v\rangle^2 \mu(x,\mathrm{d}v)\right| + \int_{|v|>1}\mu(x,\mathrm{d}v) \\
&\le \left|\langle u, \beta^{\chi}(x)\rangle + \frac{1}{2}\langle u, \alpha(x) u\rangle\right| + \left(\frac{1}{2}e^{|u|} + 1\right)\int_{\mathbb{V}} \big(1 \wedge |v|^2\big)\mu(x,\mathrm{d}v)
\end{aligned}$$

Thus, one can be defined as a finite displacement of the other.

Lemma I.7. *$\mathcal{D}_\Lambda$ is convex.*

Proof. We use the characterization of $\mathcal{D}_\Lambda$ from Lemma I.6. Let $u, u' \in \mathcal{D}_\Lambda$, $\gamma \in (0,1)$, and use Hölder's inequality to see the following.

$$\begin{aligned}
&\int_{|v|>1} e^{\langle u' + \gamma(u-u'), v\rangle}\mu(x,\mathrm{d}v) \\
&= \int_{|v|>1} \left|(e^{\langle u,v\rangle})^{\gamma}\cdot(e^{\langle u',v\rangle})^{1-\gamma}\right|\mu(x,\mathrm{d}v) \\
&\le \left(\int_{|v|>1}\left|(e^{\langle u,v\rangle})^{\gamma}\right|^{\frac{1}{\gamma}}\mu(x,\mathrm{d}v)\right)^{\gamma}\left(\int_{|v|>1}\left|(e^{\langle u',v\rangle})^{1-\gamma}\right|^{\frac{1}{1-\gamma}}\mu(x,\mathrm{d}v)\right)^{1-\gamma}
\end{aligned}$$

$$= \left(\int_{|v|>1} e^{\langle u,v\rangle}\mu(x,\mathrm{d}v)\right)^{\gamma}\left(\int_{|v|>1} e^{\langle u',v\rangle}\mu(x,\mathrm{d}v)\right)^{1-\gamma}$$
$$< \infty$$

An arbitrary convex combination now satisfies $\gamma u + (1-\gamma)u' \in \mathcal{D}_\Lambda$.

Lemma I.8. *For each $x \in \mathbb{X}$, the map $\Lambda(\cdot, x)$ is continuously differentiable on $\mathcal{D}_\Lambda^\circ$, with derivative $D\Lambda(\cdot, x) : \mathcal{D}_\Lambda^\circ \to \mathbb{L}(\mathbb{V}, \mathbb{R})$ as follows.*

$$D\Lambda(u,x)w = \left\langle \beta^\chi(x) + \alpha(x)u + \int_{\mathbb{V}} \left(e^{\langle u,v\rangle}v - \chi(v)\right)\mu(x,\mathrm{d}v), w\right\rangle, \quad u \in \mathcal{D}_\Lambda^\circ \tag{I.6}$$

Proof. Fix $x \in \mathbb{X}$, $u \in \mathcal{D}_\Lambda^\circ$. Let $\epsilon > 0$ such that $B(u,\epsilon) \subseteq \mathcal{D}_\Lambda$. For all $0 < \delta < \epsilon$ and $i = 1, \ldots, d$, we now have the following identity

$$\begin{aligned}\frac{\Lambda(u+\delta e_i, x) - \Lambda(u,x)}{\delta} &= \langle e_i, \beta^\chi(x)\rangle + \langle e_i, \alpha(x)u\rangle + \frac{1}{2}\langle \delta e_i, \alpha(x)u\rangle \\ &+ \int_{|v|\le 1} \frac{1}{\delta}\Big(e^{\langle u+\delta e_i, v\rangle} - e^{\langle u,v\rangle} - \langle \delta e_i, v\rangle\Big)\mu(x,\mathrm{d}v) \\ &+ \int_{|v|>1} \frac{1}{\delta}\Big(e^{\langle u+\delta e_i, v\rangle} - e^{\langle u,v\rangle}\Big)\mu(x,\mathrm{d}v)\end{aligned} \tag{I.7}$$

Evaluating the limit of (I.7) as $\delta \to 0$ is now a matter of exchanging the limit with integration; we will do this by using the dominated convergence theorem.

For the first integral, Taylor's theorem provides us $\gamma_0, \gamma_1 \in [0,1]$ such that the following hold.

$$\begin{aligned} e^{\langle u+\delta e_i, v\rangle} &= 1 + \langle u+\delta e_i, v\rangle + \frac{1}{2}\langle u+\delta e_i, v\rangle^2 e^{\gamma_0\langle u+\delta e_i, v\rangle} \\ e^{\langle u,v\rangle} &= 1 + \langle u, v\rangle + \frac{1}{2}\langle u,v\rangle^2 e^{\gamma_1\langle u,v\rangle}\end{aligned}$$

This shows us that, for all $0 < \delta < \epsilon$ and $|v| \le 1$,

$$\begin{aligned}\left|\frac{1}{\delta}\Big(e^{\langle u+\delta e_i, v\rangle} - e^{\langle u,v\rangle} - \langle \delta e_i, v\rangle\Big)\right| &= \left|\frac{1}{2}\langle u+\delta e_i, v\rangle^2 e^{\gamma_0\langle u+\delta e_i, v\rangle} + \frac{1}{2}\langle u,v\rangle^2 e^{\gamma_1\langle u,v\rangle}\right| \\ &\le \Big(\big(|u|+\epsilon\big)^2 e^{|u|+\epsilon}\Big)|v|^2.\end{aligned}$$

This dominating function is integrable (recall the finiteness in (I.5) of Remark I.5),

$$\int_{|v|\le 1} \Big(\big(|u|+\epsilon\big)^2 e^{|u|+\epsilon}\Big)|v|^2\mu(x,\mathrm{d}v) \le \Big(\big(|u|+\epsilon\big)^2 e^{|u|+\epsilon}\Big)\int_{\mathbb{V}}(1\wedge |v|^2)\mu(x,\mathrm{d}v) < \infty,$$

so we may apply the dominated convergence theorem.

$$\begin{aligned}&\lim_{\delta\to 0}\int_{|v|\le 1}\frac{1}{\delta}\Big(e^{\langle u+\delta e_i, v\rangle} - e^{\langle u,v\rangle} - \langle \delta e_i, v\rangle\Big)\mu(x,\mathrm{d}v) \\ &\quad = \int_{|v|\le 1}\lim_{\delta\to 0}\frac{1}{\delta}\Big(e^{\langle u+\delta e_i, v\rangle} - e^{\langle u,v\rangle} - \langle \delta e_i, v\rangle\Big)\mu(x,\mathrm{d}v)\end{aligned}$$

$$= \int_{|v|\le 1} \left(e^{\langle u,v\rangle} v_i - v_i\right)\mu(x,\mathrm{d}v) \tag{I.8}$$

For the second integral, we again use Taylor's theorem to establish for each $0 < \delta < \epsilon/2$, some $\gamma_\delta \in [0,\delta]$ such that

$$e^{\langle u+\delta e_i, v\rangle} = e^{\langle u,v\rangle} + \langle \delta e_i, v\rangle e^{\langle u+\gamma_\delta e_i, v\rangle}$$

This way, we have the following dominating function.

$$\left|\frac{1}{\delta}\left(e^{\langle u+\delta e_i,v\rangle} - e^{\langle u,v\rangle}\right)\right| \le \left|\langle e_i, v\rangle e^{\langle u+\gamma_\delta e_i,v\rangle}\right| \le |v_i| e^{\langle u,v\rangle + \epsilon|v_i|/2}$$

The claim is that this dominating function is integrable. To see this, first note that because we have the following limit,

$$\lim_{|v|\to\infty} \frac{|v_i|e^{\langle u,v\rangle+\epsilon|v_i|/2}}{e^{\langle u,v\rangle+2\epsilon|v_i|/3}} = \lim_{|v|\to\infty} \frac{|v_i|}{e^{\epsilon|v_i|/6}} = 0$$

There exists $M > 0$ such that for all $|v| > M$,

$$|v_i|e^{\langle u,v\rangle+\epsilon|v_i|/2} < e^{\langle u,v\rangle+2\epsilon|v_i|/3}.$$

We now see that

$$\begin{aligned}
&\int_{|v|>1} |v_i|e^{\langle u,v\rangle+\epsilon|v_i|/2}\mu(x,\mathrm{d}v)\\
&\quad= \int_{1<|v|\le M} |v_i|e^{\langle u,v\rangle+\epsilon|v_i|/2}\mu(x,\mathrm{d}v) + \int_{|v|>M} |v_i|e^{\langle u,v\rangle+\epsilon|v_i|/2}\mu(x,\mathrm{d}v)\\
&\quad\le \int_{1<|v|\le M} Me^{(|u|+\epsilon/2)M}\mu(x,\mathrm{d}v) + \int_{|v|>M} e^{\langle u,v\rangle+2\epsilon|v_i|/3}\mu(x,\mathrm{d}v)\\
&\quad\le Me^{(|u|+\epsilon/2)M}\int_{\mathbb{V}}(1\wedge|v|^2)\mu(x,\mathrm{d}v) + \sum_{\ell=0}^{1}\int_{|v|>1} e^{\langle u+2\epsilon e_i/3, v\rangle}\mu(x,\mathrm{d}v)\\
&\quad< \infty.
\end{aligned}$$

We again use the dominated convergence theorem to deduce the following.

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{|v|>1}\frac{1}{\delta}\left(e^{\langle u+\delta e_i,v\rangle} - e^{\langle u,v\rangle}\right)\mu(x,\mathrm{d}v)\\
&\quad= \int_{|v|>1}\lim_{\delta\to 0}\frac{1}{\delta}\left(e^{\langle u+\delta e_i,v\rangle} - e^{\langle u,v\rangle}\right)\mu(x,\mathrm{d}v)\\
&\quad= \int_{|v|>1} e^{\langle u,v\rangle} v_i\mu(x,\mathrm{d}v) \qquad \text{(I.9)}
\end{aligned}$$

Combining equations (I.7), (I.8), and (I.9) now yields our desired identity.

$$\frac{\partial}{\partial u_i}\Lambda(u,x) = \left\langle e_i, \beta^\chi(x) + \alpha(x)u + \int_{\mathbb{V}}\left(e^{\langle u,v\rangle}v - \chi(v)\right)\mu(x,\mathrm{d}v)\right\rangle$$

Continuity of $\frac{\partial}{\partial u_i}\Lambda(u,x)$ for $u \in \mathcal{D}_\Lambda^\circ$ involves very similar dominated convergence theorem arguments as above. From here, it is clear that Λ is continuously differentiable with the form in (I.6).

Lemma I.9. *If $0 \in \mathcal{D}_\Lambda^\circ$, then for each $x \in \mathbb{X}$, we have the following integrability.*

$$\int_{\mathbb{V}} |v - \chi(v)| \mu(x, \mathrm{d}v) < \infty$$

This allows us to define the following function $\beta : \mathbb{X} \to \mathbb{V}$ and rewrite Λ.

$$\beta(x) \coloneqq \beta^\chi(x) + \int_{\mathbb{V}} \big(v - \chi(v)\big) \mu(x, \mathrm{d}v)$$

$$\Lambda(u, x) = \langle u, \beta(x)\rangle + \frac{1}{2}\langle u, \alpha(x) u\rangle + \int_{\mathbb{V}} \big(e^{\langle u, v\rangle} - 1 - \langle u, v\rangle\big) \mu(x, \mathrm{d}v) \tag{I.10}$$

$$D\Lambda(u, x) w = \Big\langle w, \beta(x) + \alpha(x) u + \int_{\mathbb{V}} v \big(e^{\langle u, v\rangle} - 1\big) \mu(x, \mathrm{d}v)\Big\rangle$$

Proof. If $0 \in \mathcal{D}_\Lambda^\circ$, then there exists some $\delta > 0$ such that each $v \in \mathbb{V}$ with $|v| \le \delta$ also satisfies $v \in \mathcal{D}_\Lambda$. Observe the following implication of this fact, for each $x \in \mathbb{X}$.

$$\begin{aligned}
\int_{\mathbb{V}} \big|v - \chi(v)\big| \mu(x, \mathrm{d}v) &= \int_{|v|>1} |v| \mu(x, \mathrm{d}v) \\
&\le \int_{|v|>1} \frac{\sqrt{d}}{\delta} \exp\Big(\frac{\delta |v|}{\sqrt{d}}\Big) \mu(x, \mathrm{d}v) \\
&\le \frac{\sqrt{d}}{\delta} \int_{|v|>1} \exp\Big(\max_{i=1}^{d} \max_{\ell=0}^{1} \big\langle (-1)^\ell \delta e^i, v\big\rangle\Big) \mu(x, \mathrm{d}v) \\
&\le \frac{\sqrt{d}}{\delta} \sum_{i=1}^{d} \sum_{\ell=0}^{1} \int_{|v|>1} \exp \big\langle (-1)^\ell \delta e^i, v\big\rangle \mu(x, \mathrm{d}v) \\
&< \infty
\end{aligned}$$

From here, the rest of the proof is simple algebra.

Now that we have elaborated upon the relevant results of Λ as a scalar field, we proceed to show the existence of a solutions to the dynamical system it imposes. These tighly coincide with the moments of our affine process.

Definition I.10. *For each $\tau \ge 0$ and $u \in \mathcal{D}_\Lambda$, we say a function $Q^u : [0, \tau] \times \mathbb{X} \to \mathbb{R}$ satisfies* $\mathrm{system}(\Lambda, \tau, u)$ *if the following hold.*

$$\forall\, t \in [0, \tau],\ x \in \mathbb{X}, \quad Q^u(t, x) = q_0^u(t) + \langle q^u(t), x\rangle,$$

$$\forall\, x \in \mathbb{X}, \quad \begin{cases} \dot{Q}^u(t, x) = \Lambda\big(q^u(t), x\big), & t \in [0, \tau] \\ Q^u(0, x) = \langle u, x\rangle \end{cases}$$

Now define the following sets.

$$\mathcal{D}_\Psi(\tau) \coloneqq \Big\{u \in \mathcal{D}_\Lambda : \textit{there exists a solution to } \mathrm{system}(\Lambda, \tau, u)\Big\}$$

$$\mathcal{D}_\Psi \coloneqq \bigcup_{\tau \ge 0} \Big(\{\tau\} \times \mathcal{D}_\Psi(\tau)\Big)$$

Theorem I.11. *(a) There exists a map $\Psi : \mathcal{D}_\Psi \times \mathbb{X} \to \mathbb{R}$ of the form*

$$\Psi(t,u,x) = \psi_0(t,u) + \big\langle \psi(t,u), x \big\rangle$$

such that, for each $(\tau,u) \in \mathcal{D}_\Psi$, $\Psi(\cdot,u,\cdot)$ is a solution to system(Λ,τ,u) *dominated by all other such solutions. Moreover, this map satisfies the following for each $(\tau,u) \in \mathcal{D}_\Psi$ and $x \in \mathbb{X}$.*

$$\mathrm{E}_{\mathrm{P}_x} \exp \langle u, X_t \rangle = \exp \Psi(t,u,x), \quad t \in [0,\tau] \tag{I.11}$$

(b) If $\tau \geq 0$, $u \in \mathbb{V}$, and $x \in \mathbb{X}^\circ$ are such that $\mathrm{E}_{\mathrm{P}_x} \exp \langle u, X_\tau \rangle < \infty$, then $(\tau,u) \in \mathcal{D}_\Psi$.

Proof. With Remark I.4, this is the same as [KRM15, Theorem 2.14].

Now that we have two characterizations for the space $\mathcal{D}_\Psi$, we seek to understand properties of it and the associated moment map $\Psi : \mathcal{D}_\Psi \times \mathbb{X} \to \mathbb{R}$.

Proposition I.12. *(a) For each $\tau > 0$, $\mathcal{D}_\Psi(\tau)$ is open in $\mathcal{D}_\Lambda^\circ$,*

(b) For each $\tau > 0$ and $u \in \mathcal{D}_\Psi(\tau) \cap \mathcal{D}_\Lambda^\circ$, $\Psi(\cdot,u,\cdot)$ from Theorem I.11 is the unique solution to system(Λ,τ,u).

(c) Ψ is continuously differentiable on $\mathcal{D}_\Psi^\circ \times \mathbb{X}$.

Proof. Fix $\tau > 0$ and $u \in \mathcal{D}_\Psi(\tau) \cap \mathcal{D}_\Lambda^\circ$. Because $u \in \mathcal{D}_\Psi(\tau)$, $\Psi(\cdot,u,\cdot)$ exists on $[0,\tau] \times \mathbb{X}$ as a solution to system(Λ,τ,u). As mentioned in Remark I.4, the function $\psi(\cdot,u)$ associated with $\Psi(\cdot,u,\cdot)$ is a solution to the following equation,

$$\begin{cases} \dot\psi(t,u) = f\big(t,\psi(t,u)\big) & t \in [0,\tau] \\ \psi(0,u) = u \end{cases} \tag{I.12}$$

where $f : \mathbb{R} \times \mathcal{D}_\Lambda^\circ \to \mathbb{V}$ is defined by $f(t,u) \coloneqq L(u)$. Seeing as f is continuously differentiable on $\mathbb{R} \times \mathcal{D}_\Lambda^\circ$ by Lemma I.8, we may use [Wal98, III.13 Theorem X] to ensure some $\epsilon > 0$ such that the band

$$S_\epsilon \coloneqq \Big\{ (t,v) \in [0,\tau] \times \mathbb{V} : |v - \psi(t,u)| \leq \epsilon \Big\}$$

is contained in $\mathbb{R} \times \mathcal{D}_\Lambda^\circ$ and provides us to each $(t_0,v) \in S_\epsilon$ a unique solution $q(\cdot,t_0,v)$ to the following initial value problem,

$$\begin{cases} \dot q(t,t_0,v) = f\big(t,q(t,t_0,v)\big) & t \in [t_0,\tau] \\ q(t_0,t_0,v) = v \end{cases}$$

which is continuously differentiable with derivatives $\partial_{t_0} q(t,t_0,v) \in \mathbb{V}$ and $Dq(t,t_0,v) \in \mathbb{L}(\mathbb{V})$ satisfying the following equations.

$$\partial_{t_0} q(t,t_0,v) = -f(t_0,u) + \int_{t_0}^t Df\big(s,q(s,t_0,v)\big)\partial_{t_0} q(s,t_0,v)\mathrm{d}s$$

$$Dq(t,t_0,v) = \mathrm{id}_\mathbb{V} + \int_{t_0}^t Df\big(s,q(s,t_0,v)\big) Dq(s,t_0,v)\mathrm{d}s$$

In particular, for each $v \in B(u, \epsilon)$, we have $|v - \psi(0, u)| = |v - u| < \epsilon$, and so $(0, v) \in S_\epsilon$; this allows us to disregard the middle coordinate and have $q : [0, \tau] \times B(u, \epsilon) \to \mathbb{V}$ such that $q(\cdot, v)$ is the unique solution to

$$\begin{cases} \dot{q}(t, v) = L\big(q(t, v)\big), \quad t \in [0, \tau] \\ q(0, v) = v \end{cases}$$

and the derivative in the second coordinate $Dq(t, v) \in \mathbb{L}(\mathbb{V})$ satisfies the following equation.

$$Dq(t, v) = \mathrm{id}_{\mathbb{V}} + \int_0^t DL\big(q(s, v)\big) Dq(s, v) \mathrm{d}s$$

From here, we may define $Q : [0, \tau] \times B(u, \epsilon) \times \mathbb{X} \to \mathbb{R}$ as follows.

$$\begin{aligned} Q(t, v, x) &\coloneqq q_0(t, v) + \big\langle q(t, v), x \big\rangle \\ q_0(t, v) &\coloneqq \int_0^t L_0\big(q(s, v)\big) \mathrm{d}s \\ L_0(v) &\coloneqq \Lambda(v, 0) \end{aligned}$$

Because the image of $q(\cdot, v)$ on $[0, \tau]$ remains in $\mathcal{D}_\Lambda^\circ$, on which L is continuously differentiable, q_0 is continuously differentiable with derivatives $\dot{q}_0$ and Dq_0 satisfying the following.

$$\begin{aligned} \dot{q}_0(t, v) &= L_0\big(q(s, v)\big) \\ Dq_0(t, v) &= \int_0^t DL_0\big(q(s, v)\big) Dq(s, v) \mathrm{d}s \end{aligned}$$

By linearity, $Q(\cdot, v, \cdot)$ is a solution to system(Λ, τ, v) and so $v \in \mathcal{D}_\Psi(\tau)$. We now have $B(u, \epsilon) \subseteq \mathcal{D}_\Psi(\tau)$, concluding part (a). Meanwhile, any solution Q^u to system(Λ, τ, u) is such that the associated q^u solves (I.12) and so $q^u = q(\cdot, u)$. From here, it is thus the case that $Q^u = Q(\cdot, u, \cdot)$. This means Ψ from Theorem I.11 is such that $\Psi(\cdot, u, \cdot)$ is the unique solution to system(Λ, u, τ), concluding part (b). Lastly, for each $x \in \mathbb{X}$, linearity also shows us that $\Psi(\cdot, \cdot, x)$ is continuously differentiable in a neighborhood of (t, u), with derivative in the second coordinate $D\Psi(\cdot, \cdot, x)$ satisfying the following.

$$\begin{aligned} D\Psi(t, u, x) &= D\psi_0(t, u) + D\psi(t, u) \cdot x \\ &= Dq_0(t, u) + Dq(t, u) \cdot x \\ &= \int_0^t DL_0\big(q(s, u)\big) Dq(s, u) \mathrm{d}s + \left(\mathrm{id}_{\mathbb{V}} + \int_0^t DL\big(q(s, u)\big) Dq(s, u) \mathrm{d}s \right) \cdot x \\ &= x + \int_0^t \left(DL_0\big(q(s, u)\big) Dq(s, u) + \sum_{i=1}^d x_i DL_i\big(q(s, u)\big) Dq(s, u) \right) \mathrm{d}s \\ &= x + \int_0^t D\left(L_0 + \sum_{i=1}^d x_i L_i \right) \big(q(s, u)\big) Dq(s, u) \mathrm{d}s \\ &= x + \int_0^t D\Lambda\big(q(s, u), x\big) Dq(s, u) \mathrm{d}s \\ &= x + \int_0^t D\Lambda\big(\psi(s, u), x\big) D\psi(s, u) \mathrm{d}s \end{aligned}$$

This concludes part (c).

Corollary I.13. *Suppose $0 \in \mathcal{D}_\Lambda^\circ$. For each $\tau > 0$, there exists $\gamma > 0$ with $B(0,\gamma) \subseteq \mathcal{D}_\Psi(\tau)$.*

Proof. Suppose $0 \in \mathcal{D}_\Lambda^\circ$ and fix $\tau > 0$. Theorem I.11(b) tells us $0 \in \mathcal{D}_\Psi(\tau)$. Thus, $0 \in \mathcal{D}_\Psi(\tau) \cap \mathcal{D}_\Lambda^\circ$ and so we may use Proposition I.12(a) to ensure $0 \in \mathcal{D}_\Psi(\tau)^\circ$. This means there is some $\gamma > 0$ such that $B(0,\gamma) \subseteq \mathcal{D}_\Psi(\tau)$.

Proposition I.14. *For each compact set $K \subseteq \mathcal{D}_\Lambda^\circ$, there exists $\delta > 0$ such that $K \subseteq \mathcal{D}_\Psi(\delta)$. Moreover, $\Psi(\cdot, u, \cdot)$ from Theorem I.11 is the unique solution to* system(Λ, δ, u) *for each $u \in K$.*

Proof. Firstly, we recognize that by virtue of $K \subseteq \mathcal{D}_\Lambda^\circ$ being compact, we have some $\epsilon > 0$ such that the associated open set

$$K^\epsilon := \left\{ u \in \mathbb{V} : \inf_{u' \in K} |u - u'| < \epsilon \right\}$$

has closure $\overline{K^\epsilon}$ contained in $\mathcal{D}_\Lambda^\circ$. Note in particular that this provides us with a buffer of radius ϵ around each point in $\mathcal{D}_\Lambda^\circ$.

$$\overline{B}(u,\epsilon) := \left\{ u' \in \mathbb{V} : |u' - u| \le \epsilon \right\}$$
$$\bigcup_{u \in \mathcal{D}_\Lambda} \overline{B}(u,\epsilon) \subseteq \overline{K^\epsilon} \subseteq \mathcal{D}_\Lambda^\circ$$

With these sets established, we mitigate the task of finding a solution Q^u to system(Λ, δ, u) to that of finding a solution q^u to the related equation.

$$\text{(I.13)} \qquad \begin{cases} \dot{q}^u(t) = L\big(q^u(t)\big) & t \in [0,\delta] \\ q^u(0) = u \end{cases}$$

For a fixed $u \in \mathcal{D}_\Lambda^\circ$, the existence of some $\delta_u > 0$ and solution q^u to (I.13) may be obtained from the usual fixed-point method (see [Wal98, II.6 Theorem III]). Indeed, Remark I.2(d) and Lemma I.8 provide us a Lipschitz property for L on $\overline{B}(u,\epsilon)$,

$$\big|L(v) - L(w)\big| \le |v - w| C_{u,\epsilon}, \qquad v, w \in \overline{B}(u,\epsilon)$$
$$C_{u,\epsilon} := \sup_{u' \in \overline{B}(u,\epsilon)} \Big| DL\big(u', x\big) \Big|$$

and so a Banach space $(\mathbb{B}_u, \|\cdot\|_{\mathbb{B}_u})$ defined by

$$\delta_u := 1 \wedge \frac{\epsilon}{\sup_{u' \in \overline{B}(u,\epsilon)} |L(u')|}$$
$$\mathbb{B}_u := \mathbb{C}([0,\delta_u], \mathbb{V})$$
$$\|f\|_{\mathbb{B}_u} := \sup_{t \in [0,\delta_u]} |f(t)| e^{-2C_{u,\epsilon} t}$$

is partially equipped with a map $T : \mathbb{C}([0,\delta_u], K) \to \mathbb{C}([0,\delta_u], \overline{K^\epsilon})$ defined by

$$Tf(t) := u + \int_0^t L\big(f(s)\big) \mathrm{d}s,$$

satisfying a contraction property,

$$\|Tf - Tg\|_{\mathbb{B}_u} \le \frac{1}{2}\|f - g\|_{\mathbb{B}_u},$$

which induces a unique solution $q^u \in \mathbb{C}([0,\delta_u], \overline{K^\epsilon})$ to the associated fixed-point equation, $Tq^u = q^u$. This solution q^u is thus a unique solution to (I.13).

From here, we define the following positive δ,

$$\delta := \inf_{u\in K} \delta_u \ge 1 \wedge \inf_{u\in K} \frac{\epsilon}{\sup_{u'\in\overline{B}(u,\epsilon)} |L(u')|} \ge 1 \wedge \frac{\epsilon}{\sup_{u'\in\overline{K}^\epsilon} |L(u')|} > 0$$

so that each $u \in K$ has a unique solution q^u to (I.13). This induces the following map $Q^u : [0,\delta] \times \mathbb{X} \to \mathbb{R}$ for each $u \in K$.

$$Q^u(t,x) := q_0^u(t) + \langle q^u(t), x\rangle$$
$$q_0^u(t) := \int_0^t L_0\big(q^u(s)\big)\mathrm{d}s$$

By linearity, Q^u is a solution to system(Λ,δ,u) for each $u \in K$, and so $K \subseteq \mathcal{D}_\Psi(\delta)$. For each $u \in K \subseteq \mathcal{D}_\Psi(\delta)$, a solution $\tilde{Q}^u$ to system(Λ,δ,u) is such that the associated $\tilde{q}^u$ solves (I.13) and so $\tilde{q}^u = q^u$. From here, it is thus the case that $\tilde{Q}^u = Q^u$. This means Ψ from Theorem I.11 is such that $\Psi(\cdot,u,\cdot)$ is the unique solution to system(Λ,u,δ) for all $u \in K$.

Proposition I.15. *For any compact subset $K \subseteq \mathcal{D}_\Psi^\circ$, there exists $C_K > 0$ such that the following holds for all $(t,u) \in K$.*

$$\big|\Psi(t,u,x) - \Psi(0,u,x)\big| \le C_K \cdot t \cdot \big(1 + |x|\big)$$

Proof (Proposition I.15). Let $K \subseteq \mathcal{D}_\Psi^\circ$ be compact. By Remark I.4 and Proposition I.12(c), we have that the functions ψ_i for $i = 0,\dots,d$ are continuously differentiable on $\mathcal{D}_\Psi^\circ$. Thus, we may define the following positive numbers.

$$C_{K,i} := \sup_{(t,u)\in K} |\dot{\psi}_i(t,u)|, \quad i = 0,\dots,d$$
$$C_K := \max\Big\{C_{K,0}, C_{K,1}\sqrt{d},\dots,C_{K,d}\sqrt{d}\Big\} < \infty$$

Using the fundamental theorem of calculus and that $\Psi(\cdot,u,\cdot)$ solves system(Λ,τ,u), we produce the following bound for all $(t,u) \in K$.

$$\begin{aligned}
\big|\Psi(t,u,x) - \Psi(0,u,x)\big| &= \Big|\psi_0(t,u) + \big\langle \psi(t,u) - u, x\big\rangle\Big| \\
&\le \big|\psi_0(t,u)\big| + \big|\psi(t,u) - u\big| \cdot |x| \\
&= \left|\int_0^t \dot{\psi}_0(s,u)\mathrm{d}s\right| + \left|\int_0^t \dot{\psi}_i(s,u)\mathrm{d}s\right| \cdot |x| \\
&\le C_{K,0} \cdot t + \Big(\sum_{i=1}^d C_{K,i}^2\Big)^{1/2} \cdot t \cdot |x| \\
&\le C_K \cdot t \cdot \big(1 + |x|\big)
\end{aligned}$$

I.3 Finite-dimensional distributions

I.4 Jump diffusions

Chapter II

Large deviations of affine processes

Appendix A

Jump-Diffusions

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