Large deviations of affine processes

Matthew Varble

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Abstract

This is an abstract of the entire dissertation; summarize a history of large deviations and affine processes, then abstractly summarize our large deviations result.

Acknowledgment

This is where I acknowledge how I am useless without others.

Contents

Introduction

This is where I give the reader a little more history and detail regarding affine processes and large deviations, should they read this paper without already being well-versed in the subject.

Notation and conventions

Throughout, unless specifically referenced elsewhere, all notions of this text are formally defined and explored in [Kal02] or [JS03]. Most of our notation will coincide with these texts (as well as most other literature), except in regards to some particular conventions. Let us establish some of these here. A stochastic process X with a marginal-index-set I and state space $(\mathbb{X}, \mathscr{X})$ will be indifferently recognized as:

- a collection $X = (X_t)_{t \in I}$ of marginals $X_t : \Omega \to \mathbb{X}$,
- a map $X : \Omega \times I \to \mathbb{X}$,
- or its curried version $X : \Omega \to \mathbb{X}^I$.

With this convention, we find it appropriate to denote filtrations $\mathscr{F} = (\mathscr{F}_t)_{t \geq 0}$ of increasing σ-algebras $\mathscr{F}_t$. Seeing as $\mathscr{F}$ denotes the actual family of σ-algebras, we denote the joined algebra with an infinity subscript, $\mathscr{F}_\infty := \bigvee_{t \geq 0} \mathscr{F}_t$. The blackboard notation will generally correspond to a topological space, including those objects we typically introduce in analysis.

- The real $\mathbb{R}$, the complex $\mathbb{C} = \mathbb{R} \oplus \mathrm{i}\mathbb{R}$, the and nonnegative $\mathbb{R}_+ = [0, \infty)$ numbers with the usual Euclidean topologies.
- For real normed vector spaces $\mathbb{V}$, $\mathbb{W}$, the space $\mathbb{L}(\mathbb{V}, \mathbb{W})$ of real linear maps $\mathbb{V} \to \mathbb{W}$, equipped with operator norm.
$$|T| := \sup_{|v|=1} |Tv|$$
We also concisely denote $\mathbb{L}(\mathbb{V}) := \mathbb{L}(\mathbb{V}, \mathbb{V})$.
- For the a separable metric space $\mathbb{X}$ and an interval $I \subseteq \mathbb{R}_+$, the space $\mathbb{D}(I, \mathbb{X})$ of càdlàg functions, equipped with the Skorokhod J1 topology.
- For topological spaces $\mathbb{X}, \mathbb{Y}$, the space $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ of continuous functions, equipped with the supremum norm.

- For finite-dimensional normed vector spaces $\mathbb{V}, \mathbb{W}$ and open $\mathbb{U} \subseteq \mathbb{V}$, the subspace $\mathbb{C}^1(\mathbb{U}, \mathbb{W})$ of functions $f \in \mathbb{C}(\mathbb{U}, \mathbb{W})$ in which there is a derivative map $\mathrm{D}f \in \mathbb{C}(\mathbb{U}, \mathbb{L}(\mathbb{V}, \mathbb{W}))$.

$$\lim_{|v| \to 0} \frac{\big|f(u+v) - f(u) - \mathrm{D}f(u) \cdot v\big|}{|v|} = 0$$

For $f \in \mathbb{C}^1(\mathbb{U}, \mathbb{R})$, we denote $\nabla f \in \mathbb{C}(\mathbb{U}, \mathbb{V})$ the gradient,

$$\big\langle v, \nabla f(u) \big\rangle := \mathrm{D}f(u) \cdot v,$$

If there is some canonical ordered basis $(e_1, \ldots, e_{\dim \mathbb{V}})$ of $\mathbb{V}$, denote $\mathrm{D}_i f \in \mathbb{C}(\mathbb{U}, \mathbb{R})$ the i-th partial derivative.

$$\mathrm{D}_i f(u) := \mathrm{D}f(u) \cdot e_i, \quad i = 1, \ldots, d$$

- For finite-dimensional normed vector space $\mathbb{V}$ and open $\mathbb{U} \subseteq \mathbb{V}$, the subspace $\mathbb{C}^2(\mathbb{U}, \mathbb{R})$ of $f \in \mathbb{C}^1(\mathbb{U}, \mathbb{R})$ in which we also have $\nabla f \in \mathbb{C}^1(\mathbb{U}, \mathbb{V})$. In such a case, we denote $\mathrm{D}^2 f \in \mathbb{C}(\mathbb{U}, \mathbb{L}(\mathbb{V}))$ the Hessian.

$$\mathrm{D}^2 f(u) := \mathrm{D}\big(\nabla f(u)\big)$$

If there is some canonical ordered basis $(e_1, \ldots, e_{\dim \mathbb{V}})$ of $\mathbb{V}$, denote $\mathrm{D}_{ij} f \in \mathbb{C}(\mathbb{U}, \mathbb{R})$ the second-order ij-th partial derivative.

$$\mathrm{D}_{ij} f(u) := \big\langle e_i, \mathrm{D}^2 f(u) \cdot e_j \big\rangle, \quad i, j = 1, \ldots, d$$

For spaces $\mathbb{X}$ in which there is some canonical topology, we will denote the associated Borel algebra $\mathscr{B}(\mathbb{X})$. Particular examples of this convention are:

- the Borel algebra $\mathscr{B}(\mathbb{V})$ associated to the topology induced from a canonical inner-product $\langle \cdot, \cdot \rangle$ on a vector space $\mathbb{V}$.
- the Borel algebra $\mathscr{B}(\mathbb{X})$ associated to the relative topology of some subset $\mathbb{X}$ of a space $\mathbb{V}$ with itself some canonical topology.

In the case that we are dealing with a finite-dimensional real vectorspace $\mathbb{V}$ with inner-product $\langle \cdot, \cdot \rangle$, we assume some canonical orthonormal basis $e_1, \ldots, e_{\dim \mathbb{V}} \in \mathbb{V}$ and establish the associated isometric isomorphism $\mathbb{V} \equiv \mathbb{R}^d$.

$$v \in \mathbb{V} \quad \longleftrightarrow \quad \big(v^1, \ldots, v^{\dim \mathbb{V}}\big); \qquad v^i := \langle v, e_i \rangle, \quad i = 1, \ldots, \dim \mathbb{V}$$

Similarly identify any map $f : \mathbb{A} \to \mathbb{V}$ with component maps $f_1, \ldots, f_d : \mathbb{A} \to \mathbb{R}$.

$$f : \mathbb{A} \to \mathbb{V} \quad \longleftrightarrow \quad \big(f_1, \ldots, f_d\big) : \mathbb{A} \to \mathbb{R}^d; \qquad f_i(a) := \langle f(a), e_i \rangle$$

Extend the inner-product symmetrically to a bilinear form on $\mathbb{V} \oplus \mathrm{i}\mathbb{V}$,

$$\big\langle v_1 + \mathrm{i}w_1, v_2 + \mathrm{i}w_2 \big\rangle = \big(\langle v_1, v_2 \rangle - \langle w_1, w_2 \rangle\big) + \mathrm{i}\big(\langle v_1, w_2 \rangle + \langle w_1, v_2 \rangle\big),$$

and define the trace of an operator $T \in \mathbb{L}(\mathbb{V})$, as follows.

$$\mathrm{tr}(T) = \sum_{i=1}^{d} \langle e_i, T e_i \rangle$$

We adopt that $(\Omega, \Sigma, \mathrm{P})$ is an abstract probability space that—through the process of enlargement via Kolmogorov's extension theorem—we without loss of generality assume it is equipped with identifications of various quantities $X : \Omega \to \mathbb{X}$ into measurable spaces $(\mathbb{X}, \mathscr{X})$ associated with distributions μ on $(\mathbb{X}, \mathscr{X})$. We typically presume such maps X to be measurable without mention and will otherwise specify this fact explicitly by using the notation $X \in \Sigma/\mathscr{X}$. For each probability measure P on (Ω, Σ), we denote the P-distribution of such X by P_X or pushfoward notation, $X_\#\mathrm{P}$.

$$\mathrm{P}_X\Gamma \coloneqq (X_\#\mathrm{P})(\Gamma) \coloneqq \mathrm{P}(X \in \Gamma) \coloneqq \mathrm{P}\big(X^{-1}\Gamma\big), \quad \Gamma \in \mathscr{X}$$

For intuition, we will also denote integration against this distribution as follows.

$$\int_{\mathbb{X}} \mathrm{P}(X \in \mathrm{d}x) f(x) \coloneqq \int_{\mathbb{X}} \mathrm{P}_X(\mathrm{d}x) f(x) = \int_{\Omega} \mathrm{P}(\mathrm{d}\omega) f\big(X(\omega)\big) \eqqcolon \mathrm{E}_{\mathrm{P}} f(X)$$

Just as E_{P} denotes the expectation operator of the measure P, we will denote $\mathrm{E}_{\mathrm{P}}(\cdot|\mathscr{G})$ the conditional expectation operator of P associated with a filtration $\mathscr{G}$. Should we choose a target space $(\mathbb{Y}, \mathscr{Y})$ and a natural σ-algebra $Y^{-1}\mathscr{Y}$ from some quantity $Y \in \Sigma/\mathscr{Y}$, we denote $\mathrm{E}_{\mathrm{P}}(\cdot|Y = \cdot)$ the factoring of $\mathrm{E}_{\mathrm{P}}(\cdot|Y^{-1}\mathscr{Y})$ through Y.

$$\mathrm{E}_{\mathrm{P}}(X|Y = y) = \mathrm{E}_{\mathrm{P}}\big(X|Y^{-1}\mathscr{Y}\big)\Big|_{Y=y}$$

Also, any quantity $X : \Omega \to \mathbb{X}$ will be identified with the identity map on its codomain, so that we may abusingly use the convenient expectation notation.

$$\mathrm{E}_{\mathrm{P}_X} f(X) \coloneqq \mathrm{E}_{\mathrm{P}_X} f = \int_{\mathbb{X}} f(x) \mathrm{P}_X(\mathrm{d}x) = \int_{\Omega} f\big(X(\omega)\big) \mathrm{P}(\mathrm{d}\omega) = \mathrm{E}_{\mathrm{P}} f(X)$$

This will particularly be useful for when we discuss Markov processes and their associated identities.

Chapter I

Affine processes

Here I put a summary of chapter, along with a short history. It will include the following important notes.

- Chapter addresses important fundamental results of affine processes.
- Chapter addresses consequences of mgf results that are important for us, though not specified exactly much in the literature

I.1 Formulation

We start by specifying our affine processes as in [KRM15]. That is to say, we fix a finite-dimensional real vectorspace $\mathbb{V}$ with inner-product $\langle \cdot, \cdot \rangle$ and select a convex, closed $\mathbb{X} \subseteq \mathbb{V}$ satisfying $0 \in \mathbb{X}$ and $\operatorname{span} \mathbb{X} = \mathbb{V}$. Associate this space with the finite exponentials.

$$\mathcal{U}_{\mathbb{X}} \coloneqq \left\{ u \in \mathbb{V} \oplus \mathrm{i}\mathbb{V} : \sup_{x \in \mathbb{X}} \exp \langle \Re(u), x \rangle < \infty \right\}$$

We may now define the notion of an affine process on $\mathbb{X}$.

Definition I.1. *For a probability space $(\Omega, \Sigma, \mathrm{P})$ with filtration $\mathscr{F} = (\mathscr{F}_t)_{t \geq 0}$, an affine process X on $\mathbb{X}$ is a stochastically continuous, time-homogeneous $(\mathrm{P}, \mathscr{F})$-Markov process on $\mathbb{X}$ in which the bounded moments have the following log-affine dependence on the initial state.*

$$\begin{aligned} \mathrm{E}_{\mathrm{P}_x} \exp \langle u, X_t \rangle &= \exp \Psi(t, u, x) \\ \Psi(t, u, x) &= \psi_0(t, u) + \big\langle \psi(t, u), x \big\rangle, \end{aligned} \qquad t \geq 0,\ u \in \mathcal{U}_{\mathbb{X}} \tag{I.1}$$

Above, we are denoting $(\mathrm{P}_x)_{x \in \mathbb{X}}$ the conditional P-distributions of X factored through the initial state $x \in \mathbb{X}$.

Remark I.2. *Our definitions of $\mathbb{X}$ and Ψ include the following conventions and motivations.*

(a) why we denote ψ_0, ψ instead of KM φ, ψ or Cuchiero Φ, Ψ

(b) *how assumptions* $0 \in \mathbb{X}$, $\operatorname{span}\mathbb{X} = \mathbb{V}$ *are nonrestrictive*

(c) *how (I.1) decides the distribution of* X *and how the distribution of the affine process decides* Ψ

(d) *If we have a vectorspace* $\mathbb{A}$ *and affine map* $\alpha : \mathbb{X} \to \mathbb{A}$ *determined by* $a_0, \dots, a_d \in \mathbb{A}$ *via* $\alpha(x) = a_0 + \sum_{i=1}^d x^i a_i$*, then our linear assumptions* $0 \in \mathbb{X}$ *and* $\operatorname{span}\mathbb{X} = \mathbb{V}$ *uniquely determine* $a_0, \dots, a_d \in \mathbb{A}$*. In particular, the map* Ψ *uniquely identifies its parts* $\psi_i : \mathbb{R}_+ \times \mathcal{U}_{\mathbb{X}} \to \mathbb{C}$ *for* $i = 0, \dots, d$*.*

In [Cuc11, Theorem 1.2.7], it is shown that, without loss of generality on conditional distributions $(\mathrm{P}_x)_{x\in\mathbb{X}}$, an affine process X can be chosen to have càdlàg paths. Thus, each distribution P_x may be recognized as a measure on the Borel algebra associated with the space $\mathbb{D}([0,\infty),\mathbb{X})$ of càdlàg functions equipped with the Skorokhod topology. We will impose this regularity, so that X is also a $(\mathrm{P}_x, \mathscr{F})$ jump-diffusion for each $x \in \mathbb{X}$. For relevant definitions and results pertaining to jump-diffusions, we refer the reader to Appendix A.

Theorem I.3. *An affine process* X *on* $\mathbb{X}$ *is a* $(\mathrm{P}_x, \mathscr{F})$ *jump-diffusion in which the differential* χ*-characteristics* $(\beta^\chi, \alpha, \mu)$ *are affine maps of the following form.*

$$\beta^\chi(x) \coloneqq b_0^\chi + \sum_{i=1}^d x^i b_i^\chi, \quad \alpha(x) \coloneqq a_0 + \sum_{i=1}^d x^i a_i, \quad \mu(x, \mathrm{d}v) \coloneqq m_0(\mathrm{d}v) + \sum_{i=1}^d x^i m_i(\mathrm{d}v)$$

The associated Lévy-Khintchine map Λ *then also affine,*

$$\begin{aligned}
\Lambda(u, x) &= \big\langle u, \beta^\chi(x)\big\rangle + \frac{1}{2}\big\langle u, \alpha(x)u\big\rangle + \int_{\mathbb{V}} \Big(e^{\langle u,v\rangle} - 1 - \big\langle u, \chi(v)\big\rangle\Big)\mu(x, \mathrm{d}v) \\
&= L_0(u) + \sum_{i=1}^d x^i L_i(u) \\
L_i(u) &\coloneqq \big\langle u, b_i^\chi(x)\big\rangle + \frac{1}{2}\big\langle u, a_i(x)u\big\rangle + \int_{\mathbb{V}} \Big(e^{\langle u,v\rangle} - 1 - \big\langle u, \chi(v)\big\rangle\Big)m_i(x, \mathrm{d}v),
\end{aligned}$$

and each $u \in \mathrm{i}\mathbb{V}$ *induces the following differential equation.*

$$\text{(I.2)} \qquad \begin{cases} \psi_0(t,u) = L_0\big(\psi(t,u)\big) & t \geq 0 \\ \psi(t,u) = L\big(\psi(t,u)\big) & t \geq 0 \\ \psi_0(0,u) = 0 \\ \psi(0,u) = u \end{cases}$$

Proof. This is simply a restatement of [Cuc11, Theorem 1.5.4].

Remark I.4. *By Remark I.2(d), the equation in (I.2) is equivalent to the following system of equations.*

$$\text{(I.3)} \qquad \forall\, x \in \mathbb{X}, \qquad \begin{cases} \dot{\Psi}(t,u,x) = \Lambda\big(\psi(t,u), x\big) & t \geq 0 \\ \Psi(0,u,x) = \langle u, x\rangle \end{cases}$$

Henceforth, we fix X a càdlàg affine process with conditional distributions $(\mathrm{P}_x)_{x\in\mathbb{X}}$ on $\mathbb{D}([0,\infty),\mathbb{X})$, induced filtration $\mathscr{F} = (\mathscr{F}_t)_{t\geq 0}$, and moment function Ψ as in Definition I.1. We will use the truncation function $\chi(v) = v1_{|v|\leq 1}$ and fix the differential χ-characteristics $(\beta^\chi, \alpha, \mu)$ and Lévy-Khintchine map Λ as in Theorem I.3.

I.2 Existence of real moments

This elaborates upon the extension of the transform formula in (I.1) and dynamics in (**??**) to real moments $u \in \mathbb{V}$. Clearly, should any extension exist for some $u \in \mathbb{V}$, the value $\Lambda(u, x) = \dot{\Psi}(0, u, x)$ should be well-defined. That said, we henceforth denote the following set.

$$\mathcal{D}_\Lambda \coloneqq \Big\{ u \in \mathbb{V} : \Lambda(u, x) \text{ is well-defined for all } x \in \mathbb{X} \Big\}$$

Quick inspection of $\Lambda(u, x)$ in (**??**) indicates that well-definition should depend solely on the integral term. The following results will perform the careful analysis to show that Λ has nice regularity on the finite moments of its associated jump kernel $\mu(\cdot, \mathrm{d}v)$.

I.3 Finite-dimensional distributions

Chapter II

Large deviations of affine processes

Chapter III

Large deviation rate functions

1. **Familiar large deviation principles.** Purpose of section is to familiarize with rate functions we already have and set the stage for how we operate with our theorem.
   (a) Cite Mogulskii's theorem.
   (b) Brownian motion of our regime: achieved with Mogulskii's theorem with Normal increments and exponential tightness (Schilder).
   (c) Poisson process of our regime: achieved with Mogulskii's theorem with Poisson increments and use exponential tightness.
   (d) Friedlin Wentzel (not so much our regime): use contraction mapping principles
   (e) Birth rate process: Can we similarly contraction map Poisson to get this?
   (f) Extension of Freidlin Wentzel to affine: KK

2. **Coupling states.** Indicate that when contraction mappings are not sufficient, we may *couple correlated states*, in the sense of looking at LDPs of joint processes.
   (a) Compound Poisson: Two *sources* of randomness; the arrivals and the jump sizes. Appeal to Duffy results for heuristical calculations.
   (b) Compound linear Hawkes: Similarly two *sources* of randomness. Duffy also gives us the calculations. Note on Zhu paper for *sidestep*; less general jumps, more general nonlinear relationship of arrivals.
   (c) The jumps of a general jump-diffusion do not have a well-posed notion of arrivals and jump sizes; we thus turn our focus to locally countable jump-diffusions, in which the three *sources* of randomness are the continuous local martingale, the arrival times, and the jump sizes. Note how this is discussed in next section.

3. **Locally countable affine processes.** Perform the necessary calculus and proceed to show our general formulation.
   (a) State result in numerous flavors, depending on which *base* quantities in which we choose to focus the large deviations.

$$\begin{aligned} X &= \beta(X) \cdot \ell + X^{\mathrm{c}} + \mathrm{id}_{\mathbb{V}} * \tilde{q}^X \\ N^X &= 1 * q^X \\ V^X &= \mathrm{id}_{\mathbb{V}} * q^X \end{aligned}$$

   i. **overdetermined flavor.** $(X, X^{\mathrm{c}}, N^X, V^X)$ produces an overdetermined system which requires another condition for $I(\xi, \omega, \eta, \gamma)$ to be finite.

$$\dot{\xi}(t) = \beta\big(\xi(t)\big) + \dot{\omega}(t) + \dot{\eta}(t)\dot{\gamma}(t)$$

   However, the rate function is very simple to understand.

   ii. **determine-continuous-noise flavor.** Normal term gets messy

   iii. **determine-arrivals flavor.** Poisson and jump-term-denominator gets messy

   iv. **determine-jumps flavor.** This is the one we have already presented; the jump-term-numerator gets messy.

(b) Discuss how the deviations of X from the dynamical system $X = \beta(X) \bullet \ell$ are imposed from *continuous deviations* X^{c} and *discontinuous deviations* $\mathrm{id}_{\mathbb{V}} * \tilde{q}^X$.

(c) Discuss how four quantities $X, X^{\mathrm{c}}, N^X, V^X$ heuristically relate in simple infinitesimal equality $(X, X^{\mathrm{c}}, N^X, V^X) \approx (\xi, \omega, \eta, \gamma)$.

$$\dot{\xi}(t) = \beta\big(\xi(t)\big) + \dot{\omega}(t) + \dot{\eta}(t) \cdot \dot{\gamma}(t)$$

(d) Each of the primitive deviations have a simple analogy, when we think of infinitesimals.

$\dot{\omega}(t)$	*normal deviations of covariance* $\alpha\big(\xi(t)\big)$
$\dot{\eta}(t)$	*Poisson deviations of rate* $\lambda\big(\xi(t)\big)$
$\dot{\gamma}(t)$	*jump deviations from* $\kappa\big(\xi(t), \mathrm{d}v\big)$
$\xi(t) = \beta\big(\xi(t)\big) + \dot{\omega}(t) + \dot{\eta}(t) \cdot \dot{\gamma}(t)$	*all combined deviations*

(e) Think of results from first section in this regard.

   i. birth is $\dot{\xi}(t) = \dot{\eta}(t)$, so we only need $\xi \approx X$.

   ii. diffusion is $\dot{\xi}(t) = \beta\big(\xi(t)\big) + \dot{\omega}(t)$, and so we only need $\xi \approx X$ and rate function includes $\xi(t) - \beta\big(\xi(t)\big)$ where $\dot{\omega}$ is.

   iii. compound Poisson is $\dot{\xi}(t) = \dot{\eta}(t) \cdot \dot{\gamma}(t)$, so we choose one of the following pairs $(\xi, \eta) \approx (X, N^X)$, $(\xi, \gamma) \approx (X, V^X)$, or $(\eta, \gamma) \approx (N^X, V^X)$.

   iv. compound linear Hawkes is $\dot{\xi}(t) = \beta\big(\xi(t)\big) + \dot{\eta}(t) \cdot \dot{\gamma}(t)$, so we can choose $(\xi, \eta) \approx (X, N^X)$ or $(\xi, \dot{\gamma}) \approx (X, V^X)$.

Appendix A

Jump-diffusions

TODO:

- Motivate why I chose to put this in the appendix. Big point: I want to resolve abstractions and rigor of [JS03] to the digestible notions of special jump-diffusions.
- Point to the various papers we use that do not consolidate a similar set of assumptions.

In order to discuss jump-diffusions on a finite-dimensional real vectorspace, one must have a decent understanding of semimartingales. A great text for a comprehensive study of this is [JS03], which we will refer to in our proofs. In terms of notational differences, we choose our probability space $(\Omega, \Sigma, \mathrm{P})$ and filtration $\mathscr{F} = (\mathscr{F}_t)_{t \geq 0}$, where $\mathscr{F}_\infty \subseteq \Sigma$ denotes the joined space. Furthermore, we do not explicitly write processes to take values in $\mathbb{R}^d$, but rather some vectorspace $\mathbb{V}$ with dimension $d := \dim \mathbb{V}$ and inner-product $\langle \cdot, \cdot \rangle$. Surely—due to our isometric isomorphism $\mathbb{V} \equiv \mathbb{R}^d$—any componentwise or linear notion, such as integration or differentiation may be taken as equivalent. Furthermore, we sometimes specify that a stochastic process X has a Borel state space $\mathbb{X} \subseteq \mathbb{V}$, as this is the case when studying affine processes. We find it important to highlight the following important notation of objects introduced in [JS03, Chapters I-II].

- Given $(\mathrm{P}, \mathscr{F})$ locally square-integrable martingales $M, N : \Omega \times \mathbb{R}_+ \to \mathbb{R}$, denote $\langle M, N \rangle$ the predictable quadratic covariation.
- Given $H, X : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ with H being $\mathscr{F}$ predictable and $(\mathrm{P}, \mathscr{F})$ locally bounded and X a $(\mathrm{P}, \mathscr{F})$ semimartingale, denote the stochastic integral process as follows.

$$H \bullet X_t = \int_0^t H_s \mathrm{d}X_s$$

  We may lift this concept componentwise and linearly. This allows us to choose the codomains of H, X to various combinations of $\mathbb{V}$ and $\mathbb{L}(\mathbb{V}, \mathbb{W})$ when evaluating $H \bullet X$, so long as such a combination allows for $H_t \cdot X_t$ to make sense.

- Denote $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ the identity map, so as to allow a concise notation for Lebesgue integration.

$$H \bullet \ell_t = \int_0^t H_s \mathrm{d}s$$

- Given a random measure $q : \Omega \times \mathscr{B}(\mathbb{R}_+ \times \mathbb{V}) \to [0, \infty]$, denote the stochstic integral process against some suitably integrable process $H : \Omega \times \mathbb{R}_+ \times \mathbb{V} \to \mathbb{R}$ as follows.

$$H * q_t = \int_{[0,t]\times\mathbb{V}} H_s(v) q(\mathrm{d}s, \mathrm{d}v)$$

  Denote its $(\mathrm{P}, \mathscr{F})$ predictable projection by $\hat{q}$ and the compensated measure $\tilde{q} = q - \hat{q}$. Also denote $H * \tilde{q}$ the compensated local martingale process for suitable $H \in G_{\mathrm{loc}}(q)$, as constructed in [JS03, Definition II.1.27]. Lift these integration notions to vector-valued H componentwise. Instead of choosing a canonical variable for integrating expressions in this form, we use the identity maps $\mathrm{id}_{\mathbb{V}}$ or ℓ.

$$f(\ell, \mathrm{id}_{\mathbb{V}}) * q_t = \int_{[0,t]\times\mathbb{V}} f(s, v) q(\mathrm{d}s, \mathrm{d}v)$$

- Given $(\mathrm{P}, \mathscr{F})$ semimartingales $X, Y : \Omega \times \mathbb{R}_+ \to \mathbb{R}$, denote $[X, Y]$ the quadratic covariation.

- Given a semimartingale X, denote X^{c} its continuous local martingale component and q^X its jump measure.

A.1 Formulation

As in [JS03, Definition III.2.18], a $(\mathrm{P}, \mathscr{F})$ jump-diffusion X on state space $(\mathbb{X}, \mathscr{B}(\mathbb{X}))$ is a $(\mathrm{P}, \mathscr{F})$ semimartingale in which the χ-characteristics $(B^\chi, A, \hat{q}^X)$ have the following decompositions,

$$B_t^\chi = \int_0^t \beta^\chi(X_s)\mathrm{d}s, \quad A_t = \int_0^t \alpha(X_s)\mathrm{d}s, \quad \hat{q}^X(\mathrm{d}s, \mathrm{d}v) = \mu(X_s, \mathrm{d}v)\mathrm{d}s, \tag{A.1}$$

where the functions have the following properties.

- $\beta^\chi : \mathbb{X} \to \mathbb{V}$ is Borel measurable, $\beta^\chi \in \mathscr{B}(\mathbb{X})/\mathscr{B}(\mathbb{V})$.

- $\alpha : \mathbb{X} \to \mathbb{L}(\mathbb{V})$ is Borel measurable, $\alpha \in \mathscr{B}(\mathbb{X})/\mathscr{B}(\mathbb{L}(\mathbb{V}))$, and $\alpha(x)$ is self-adjoint and nonnegative for each $x \in \mathbb{X}$.

- $\mu : \mathbb{X} \times \mathscr{B}(\mathbb{V}) \to [0, \infty]$ is a transition kernel from $\mathbb{X}$ to $\mathbb{V}$, and it satisfies the following properties for each $x \in \mathbb{X}$.

$$\mu\big(x, \{0\}\big) = 0, \qquad \int_{\mathbb{V}} \big(1 \wedge |v|^2\big)\mu(x, \mathrm{d}v) < \infty$$

In other words, our jump-diffusion X has the following canonical semimartingale representation (see [JS03, Theorem II.2.34] for definition).

$$\begin{aligned} X &= X_0 + \beta^\chi(X) \bullet \ell + X^{\mathrm{c}} + \chi * \tilde{q}^X + (\mathrm{id}_{\mathbb{V}} - \chi) * q^X \\ \langle X^{\mathrm{c},i}, X^{\mathrm{c},j}\rangle &= \alpha_{ij}(X) \bullet \ell \\ \hat{q}^X(\mathrm{d}s, \mathrm{d}v) &= \mu(X_s, \mathrm{d}v)\mathrm{d}s \end{aligned} \tag{A.2}$$

Remark A.1. *(a) Note that we differ slightly from the definition we reference by imposing a time-homogenity formulation. There is no loss of generality in doing so, because we may always extend the state to $\mathbb{R}_+ \times \mathbb{X}$ via $\hat{X}_t = (t, X_t)$.*

(b) Note that (A.1) can be written concisely by using the identity ℓ on $\mathbb{R}_+$.

$$B^\chi_t = \beta^\chi(X) \bullet \ell_t, \quad A_t = \alpha(X) \bullet \ell_t, \quad \hat{q}^X([0,t], \mathrm{d}v) = \mu(X, \mathrm{d}v) \bullet \ell_t$$

(c) If we have a jump-diffusion with χ-characteristics in (A.1), we call $(\beta^\chi, \alpha, \mu)$ the differential χ-characteristics*. We see from (A.2) that β^χ and $\beta^{\hat{\chi}}$ relate between different truncation functions $\chi, \hat{\chi}$ with the simple identity.*

$$\beta^{\hat{\chi}}(x) = \beta^\chi(x) + \int_{\mathbb{V}} \big(\hat{\chi}(v) - \chi(v)\big)\mu(x, \mathrm{d}v) \tag{A.3}$$

(d) The conditions on $\alpha(x)$ and $\mu(x, \mathrm{d}v)$ are immediate consequences of (A.1). For the most general setting, see the corresponding result for any *semimartingale, in [JS03, Proposition II.2.9].*

Example A.2. *Fix a probability space $(\Omega, \Sigma, \mathrm{P})$ and filtration $\mathscr{F} = (\mathscr{F})_{t\geq 0}$.*

Just as with $(\mathbb{R}^d, \mathscr{B}(\mathbb{V}))$, we say that W is a standard $(\mathrm{P}, \mathscr{F})$ Brownian motion on $(\mathbb{V}, \mathscr{B}(\mathbb{V}))$ if it is a continuous $(\mathrm{P}, \mathscr{F})$ martingale with predictable quadratic covariation as follows.

$$\langle W^i, W^j \rangle_t = \begin{cases} t & i = j \\ 0 & \textit{otherwise} \end{cases}$$

It is clear that W is a $(\mathrm{P}, \mathscr{F})$ jump-diffusion with differential χ-characteristics $(0, \alpha, 0)$, where $\alpha(x) = \mathrm{id}_{\mathbb{V}}$ for all $x \in \mathbb{X}$.

Similarly, we say that p is a standard $(\mathrm{P}, \mathscr{F})$ Poisson random measure on $\mathscr{B}(\mathbb{R}_+ \times \mathbb{V})$ if its $(\mathrm{P}, \mathscr{F})$ predictable projection is the Lebesgue measure $\hat{p}(\mathrm{d}s, \mathrm{d}v) = \mathrm{d}s \otimes \mathrm{d}v$ (identifying measures on $\mathscr{B}(\mathbb{R}^d)$ as those on $\mathscr{B}(\mathbb{V})$). By [JS03, Theorem II.4.8], this p is the same as a Poisson point process with Lebesgue intensity, which has infinitely many jumps on any nonempty interval of time. The accumulated jumps $\mathrm{id}_{\mathbb{V}} * p$ form a $(\mathrm{P}, \mathscr{F})$ jump-diffusion with parameters as follows.

$$\beta^\chi(x) = \int_{\mathbb{V}} \chi(v)\mathrm{d}v, \quad \alpha(x) = 0, \quad \mu(x, \mathrm{d}v) = \mathrm{d}v,$$

because we have the following decomposition.

$$\begin{aligned} \mathrm{id}_{\mathbb{V}} * p &= \chi * p + (\mathrm{id}_{\mathbb{V}} - \chi) * p \\ &= \chi * \hat{p} + \chi * \tilde{p} + (\mathrm{id}_{\mathbb{V}} - \chi) * p \\ &= \beta^\chi \bullet \ell + \chi * \tilde{p} + (\mathrm{id}_{\mathbb{V}} - \chi) * p \end{aligned}$$

Note that the infinite activity of p means that the last term cannot be compensated.

We will see at the end of this section that these two objects W and p are the fundamental building blocks of all jump-diffusions.

The following Lemma will be repeatedly used as a shortcut of Itô's formula and various identities that always apply with jump-diffusions.

Lemma A.3. *Let X be a jump-diffusion with differential χ-characteristics $(\beta^\chi, \alpha, \mu)$ and $f \in \mathbb{C}^2(\mathbb{V}, \mathbb{R})$. The composition $f(X)$ has the following semimartingale representation.*

$$\begin{aligned} f(X_t) = f(X_0) &+ \Big(\mathrm{D} f(X) \cdot \beta^\chi(X)\Big) \bullet \ell_t + \frac{1}{2} \operatorname{tr}\Big(\mathrm{D}^2 f(X) \circ \alpha(X)\Big) \bullet \ell_t + \mathrm{D} f(X_-) \bullet X^{\mathrm{c}} \\ &+ \Big(\mathrm{D} f(X_-) \cdot \chi\Big) * \tilde{q}_t^X + \Big(f(X_- + \mathrm{id}_{\mathbb{V}}) - f(X_-) - \mathrm{D} f(X_-) \cdot \chi\Big) * q_t^X \end{aligned}$$

Proof. Apply Itô's formula [JS03, Theorem I.4.57] and use the predictable covariation identity in (A.2) to get the following.

$$\begin{aligned} f(X_t) &= f(X_0) + \sum_{i=1}^d \mathrm{D}_i f(X_-) \bullet X_t^i + \frac{1}{2} \sum_{i,j=1}^d \mathrm{D}_{ij} f(X_-) \bullet \langle X^{\mathrm{c},i}, X^{\mathrm{c},j} \rangle_t \\ &\quad + \sum_{0<s\le t} \Big(f(X_s) - f(X_{s-}) - \sum_{i=1}^d \mathrm{D} f_i(X_{s-}) \Delta X_s \Big) \\ &= f(X_0) + \mathrm{D} f(X_-) \bullet X_t + \frac{1}{2} \sum_{i,j=1}^d \mathrm{D}_{ij} f(X_-) \bullet \big(\alpha_{ij}(X) \bullet \ell\big)_t \\ &\quad + \Big(f(X_- + \mathrm{id}_{\mathbb{V}}) - f(X_-) - \mathrm{D} f(X_-) \cdot \mathrm{id}_{\mathbb{V}} \Big) * q_t^X \end{aligned}$$

Using the iterated stochastic integral formula [JS03, Remark I.4.37], we may simplify the above equation to the following.

$$\begin{aligned} f(X_t) = f(X_0) &+ \mathrm{D} f(X_-) \bullet X_t + \frac{1}{2} \operatorname{tr}\Big(\mathrm{D}_{ij} f(X_-) \circ \alpha(X)\Big) \bullet \ell_t \\ &+ \Big(f(X_- + \mathrm{id}_{\mathbb{V}}) - f(X_-) - \mathrm{D} f(X_-) \cdot \mathrm{id}_{\mathbb{V}} \Big) * q_t^X \end{aligned}$$

Now substitute our representation of (A.2) and repeat the iterated stochastic integral to get the following.

$$\begin{aligned} f(X_t) &= f(X_0) + \mathrm{D} f(X_-) \bullet \big(X_0 + \beta^\chi(X) \bullet \ell + X^{\mathrm{c}} + \chi * \tilde{q}^X + (\mathrm{id}_{\mathbb{V}} - \chi) * q^X \big)_t \\ &\quad + \frac{1}{2} \operatorname{tr}\Big(\mathrm{D}^2 f(X_-) \circ \alpha(X)\Big) \bullet \ell_t + \Big(f(X_- + \mathrm{id}_{\mathbb{V}}) - f(X_-) - \mathrm{D} f(X_-) \cdot \mathrm{id}_{\mathbb{V}} \Big) * q_t^X \\ &= f(X_0) + \Big(\mathrm{D} f(X_-) \cdot \beta^\chi(X)\Big) \bullet \ell_t + \frac{1}{2} \operatorname{tr}\Big(\mathrm{D}^2 f(X_-) \circ \alpha(X)\Big) \bullet \ell_t + \mathrm{D} f(X_-) \bullet X^{\mathrm{c}} \\ &\quad + \mathrm{D} f(X_-) \bullet \big(\chi * \tilde{q}^X\big)_t + \Big(f(X_- + \mathrm{id}_{\mathbb{V}}) - f(X_-) - \mathrm{D} f(X_-) \cdot \chi \Big) * q_t^X \end{aligned}$$

Furthermore, since $X_- = X$ on all but a countable amount of jumps, we may rewrite the Lebesgue integrals.

$$\begin{aligned} f(X_t) = f(X_0) &+ \Big(\mathrm{D} f(X) \cdot \beta^\chi(X)\Big) \bullet \ell_t + \frac{1}{2} \operatorname{tr}\Big(\mathrm{D}^2 f(X) \circ \alpha(X)\Big) \bullet \ell_t + \mathrm{D} f(X_-) \bullet X^{\mathrm{c}} \\ &+ \mathrm{D} f(X_-) \bullet \big(\chi * \tilde{q}^X\big)_t + \Big(f(X_- + \mathrm{id}_{\mathbb{V}}) - f(X_-) - \mathrm{D} f(X_-) \cdot \chi \Big) * q_t^X \end{aligned} \tag{A.4}$$

For the remaining equality, we construct localizing sequence $(T_n)_{n\in\mathbb{N}}$ of $\mathscr{F}$ stopping times,

$$T_n(\omega) \coloneqq \inf\big\{ t > 0 : X_t(\omega) > n \big\} \wedge n, \quad \omega \in \Omega,\ n \in \mathbb{N}, \tag{A.5}$$

to see that $\mathrm{D}f(X_-)$ is $(\mathrm{P}, \mathscr{F})$ locally bounded.

$$\left|\mathrm{D}f(X_{s-}^{T_n})\right| \leq \sup_{|x|\leq n} \left|\mathrm{D}f(x)\right|$$

Thus, by [JS03, Proposition II.1.30], we may rewrite the following.

$$\mathrm{D}f(X_-) \bullet \left(\chi * \tilde{q}^X\right)_t = \left(\mathrm{D}f(X_-) \cdot \chi\right) * \tilde{q}_t^X,$$

which when substituted into (A.4) gives us our desired identity.

In the above lemma, the final term in the semimartingale decomposition of $f(X)$ is typically not able to be compensated into a local martingale. If we did have local integrability of the following quantity,

$$\left|f(X_- + \mathrm{id}_{\mathbb{V}}) - f(X_-) + \mathrm{D}f(X_-) \cdot \chi\right| * \hat{q}^X,$$

then by [JS03, Proposition II.1.28] we could rewrite $f(X)$ into a canonical special semimartingale decomposition.

$$\begin{aligned} f(X_t) &= f(X_0) + \mathcal{L}f(X) \bullet \ell_t + \mathrm{D}f(X_-) \bullet X^{\mathrm{c}} + \left(f(X_- + \mathrm{id}_{\mathbb{V}}) - f(X_-)\right) * \tilde{q}_t^X \\ \mathcal{L}f(x) &:= \mathrm{D}f(x) \cdot \beta^\chi(x) + \frac{1}{2} \operatorname{tr}\Big(\mathrm{D}^2 f(x) \circ \alpha(x)\Big) \\ &\qquad + \int_{\mathbb{V}} \Big(f(x+v) - f(x) - \mathrm{D}f(x) \cdot \chi(v)\Big) \mu(x, \mathrm{d}v) \end{aligned} \tag{A.6}$$

So long as f is bounded, we can guarantee this special semimartingale property.

Proposition A.4. *Let X and f as in Lemma A.3, and further impose f is bounded. Then the composition $f(X)$ is a special semimartingale with the decomposition as in (A.6).*

Proof. Seeing as f is bounded, [JS03, Lemma I.4.24] tells us that $f(X)$ is a special semimartingale. By [JS03, Proposition I.4.23], it is then the case that the following term is locally integrable.

$$\Big(f(X_- + \mathrm{id}_{\mathbb{V}}) - f(X_-) - \mathrm{D}f(X_-) \cdot \chi\Big) * q_t^X$$

By our discussion above, this suffices to conclude (A.6).

This operator $\mathcal{L}$ in (A.6) gives a nice closed from for suitable $f(X)$, and so we reserve it the term of *generator* associated with X. Note that we do not mark dependence on χ, as any other truncation function $\hat{\chi}$ will produce the same operator; see Remark A.1(c) and note that the displacement from β^χ and $\beta^{\hat{\chi}}$ would be the same as that in the integral term. One particular setting in which this result is useful is establishing a Lévy-Khintchine formula for jump-diffusions.

Proposition A.5. *Fix a jump-diffusion X with differential χ-characteristics $(\beta^\chi, \alpha, \mu)$. Then, for each $u \in \mathrm{i}\mathbb{V}$, the process $\exp\big(\langle u, X\rangle - \Lambda(u, X) \bullet \ell\big)$ is a complex-valued $(\mathrm{P}, \mathscr{F})$ local martingale, where $\Lambda : \mathrm{i}\mathbb{V} \times \mathbb{X} \to \mathbb{R}$ is the associaed* Lévy-Khintchine map.

$$\Lambda(u, x) = \big\langle u, \beta^\chi(x)\big\rangle + \frac{1}{2}\big\langle u, \alpha(x)\big\rangle + \int_{\mathbb{V}} \big(e^{\langle u, v\rangle} - 1 - \langle u, \chi(v)\rangle\big) \mu(x, \mathrm{d}v),$$

Proof. For a fixed $u \in \mathrm{i}\mathbb{V}$, note that the map f_u, defined by $f_u(v) = \exp\langle u, v\rangle$ is bounded. Thus, by Proposition A.4, we have

$$f_u(X_t) = f_u(X_0) + \mathcal{L}f_u(X) \bullet \ell_t + M_t,$$

where M is a $(\mathrm{P}, \mathscr{F})$ local martingale. Observe that the partial derivatives of f are as follows,

$$\mathrm{D}_i f_u(x) = f_u(x) u_i, \qquad \mathrm{D}_{ij} f_u(x) = f_u(x) u_i u_j, \tag{A.7}$$

so we have the following equation.

$$\begin{aligned}
\mathcal{L}f_u(x) &= \mathrm{D}f_u(x) \cdot \beta^\chi(x) + \frac{1}{2}\operatorname{tr}\Big(\mathrm{D}^2 f_u(x) \circ \alpha(x)\Big) \\
&\qquad + \int_{\mathbb{V}} \Big(f_u(x+v) - f_u(x) - \mathrm{D}f(x) \cdot \chi(v)\Big)\mu(x, \mathrm{d}v) \\
&= f_u(x)\big\langle u, \beta^\chi(x)\big\rangle + \frac{1}{2} f_u(x)\big\langle u, \alpha(x) u\big\rangle + f_u(x) \int_{\mathbb{V}} \Big(f_u(v) - 1 - \big\langle u, \chi(v)\big\rangle\Big)\mu(x, \mathrm{d}v) \\
&= f_u(x) \cdot \Lambda(u, x)
\end{aligned}$$

Denoting $A = f_u(X) = \exp\langle u, X\rangle$ and $B = \exp\big(-\Lambda(u, X) \bullet \ell\big)$, we now use the fact that B is $\mathscr{F}$ predictable and of finite-variation, so [JS03, Proposition I.4.49(b)] gives us the following.

$$\begin{aligned}
&\exp\Big(\langle u, X\rangle - \Lambda(u, X) \bullet \ell\Big) \\
&= A_t B_t \\
&= A_0 B_0 + A_- \bullet B_t + B \bullet A_t \\
&= \exp\langle u, X_0\rangle + A_- \bullet \Big(\big(-B \cdot \Lambda(u, X)\big) \bullet \ell\Big)_t + B \bullet \Big(f_u(X_0) + \mathcal{L}f_u(X) \bullet \ell + M\Big)_t \\
&= \exp\langle u, X_0\rangle - \Big(A \cdot B \cdot \Lambda(u, X)\Big) \bullet \ell_t + \Big(B \cdot f_u(X) \cdot \Lambda(u, X)\Big) \bullet \ell_t + B \bullet M_t \\
&= \exp\langle u, X_0\rangle + B \bullet M_t
\end{aligned}$$

This identity and [JS03, Remark I.4.34(b)] concludes the proof.

It turns out that each of the preceding results is sufficient in characterizing a semimartingale X as a jump-diffusion.

Theorem A.6. *The following statements are equivalent for a stochastic process X on state space $(\mathbb{X}, \mathscr{B}(\mathbb{X}))$.*

(a) X is a $(\mathrm{P}, \mathscr{F})$ jump-diffusion with differential χ-characteristics $(\beta^\chi, \alpha, \mu)$.

(b) For each bounded $f \in \mathbb{C}^2(\mathbb{V}, \mathbb{R})$, the process $f(X_t) - \mathcal{L}f(X_t) \bullet \ell_t$ is a $(\mathrm{P}, \mathscr{F})$ local martingale, where

$$\mathcal{L}f(x) \coloneqq \mathrm{D}f(x) \cdot \beta^\chi(x) + \frac{1}{2}\operatorname{tr}\Big(\mathrm{D}^2 f(x) \circ \alpha(x)\Big) + \int_{\mathbb{V}} \Big(f(x+v) - f(x) - \mathrm{D}f(x) \cdot \chi(v)\Big)\mu(x, \mathrm{d}v)$$

(c) For each $u \in \mathrm{i}\mathbb{V}$, the process $\exp\Big(\langle u, X\rangle - \Lambda(u, X) \bullet \ell\Big)$ is a $(\mathrm{P}, \mathscr{F})$ local martingale, where Λ is our Lévy-Khintchine map.

$$\Lambda(u, x) = \big\langle u, \beta^\chi(x)\big\rangle + \frac{1}{2}\big\langle u, \alpha(x)\big\rangle + \int_{\mathbb{V}} \big(e^{\langle u, v\rangle} - 1 - \langle u, \chi(v)\rangle\big)\mu(x, \mathrm{d}v),$$

(d) *Denoting* $(\mathrm{P}_x)_{x\in\mathbb{X}}$ *the* P*-conditional distributions of* X *factored through the initial state* X_0 *and selecting Borel functions* σ, c *to satisfy,*

$$\begin{aligned} \sigma &: \mathbb{X} \to \mathbb{L}(\mathbb{V}) & \sigma\sigma^*(x) &= \alpha(x) \\ c &: \mathbb{X}\times\mathbb{V} \to \mathbb{V} & \mu(x,\Gamma) &= \int 1_\Gamma\big(c(x,v)\big)\mathrm{d}v \end{aligned}$$

each P_x *is a solution to the equation associated with a standard Brownian motion* W *and Poisson random measure* p*, where* $\chi' = \mathrm{id}_\mathbb{V} - \chi$*.*

$$X_t = x + \beta^\chi(X)\bullet\ell_t + \sigma(X_-)\bullet W_t + \big(\chi\circ c(X_-,\mathrm{id}_\mathbb{V})\big) * \tilde{p}_t + \big(\chi'\circ c(X_-,\mathrm{id}_\mathbb{V})\big) * p_t$$

Proof. This is simply restating [JS03, Theorems II.2.42, II.2.49, and III.2.26] in terms of our identities from the previous propositions and lemmas. The choice of standard intensity $\mathrm{d}t\otimes\mathrm{d}v$ for the Poisson random measure is such that the jump factor $\mathrm{d}v$ satisfies the atomless and infinite properties in [JS03, Remark III.2.28(3)].

A.2 Special jump-diffusions

We now turn our focus to $(\mathrm{P},\mathscr{F})$ jump-diffusions which are additionally *special* in the sense of them having a semimartingale decomposition in which the finite-variation term is predictable. When looking at the canonical representation of a jump-diffusion X with χ-characteristics (β^χ,α,μ), it is clear how to make this predictable.

$$\begin{aligned} X_t &= X_0 + \beta^\chi(X)\bullet\ell_t + X^\mathrm{c}_t + \chi * \tilde{q}^X + (\mathrm{id}_\mathbb{V}-\chi) * q^X \\ &= X_0 + \beta^\chi(X)\bullet\ell_t + (\mathrm{id}_\mathbb{V}-\chi) * \hat{q}^X + X^\mathrm{c}_t + \mathrm{id}_\mathbb{V} * \tilde{q}^X \\ &= X_0 + \left(\beta^\chi(X) + \int_\mathbb{V}\big(v-\chi(v)\big)\mu(X,\mathrm{d}v)\right)\bullet\ell_t + X^\mathrm{c}_t + \mathrm{id}_\mathbb{V} * \tilde{q}^X \end{aligned} \tag{A.8}$$

In such a case, it is nice to define the function $\beta : \mathbb{X}\to\mathbb{V}$,

$$\beta(x) := \beta^\chi(x) + \int_\mathbb{V}\big(v-\chi(v)\big)\mu(x,\mathrm{d}v), \tag{A.9}$$

so that (A.8) may be simplified to a concise special semimartingale decomposition.

$$X_t = X_0 + \beta(X)\bullet\ell + X^\mathrm{c} + \mathrm{id}_\mathbb{V} * \tilde{q}^X_t$$

We call the triplet (β,α,μ) that results from (A.9) the *special differential characteristics* and its components β,α,μ the *drift*, *diffusion*, and *jump kernel*, respectively.

The calculus of (A.8) begs the question that $(\mathrm{id}_\mathbb{V}-\chi)*q^X$ can be compensated which is not generally the case—otherwise, the term *special* would be a misnomer! The next result specifies conditions on which we may perform the above calculus.

Lemma A.7. *Let* X *be a* $(\mathrm{P},\mathscr{F})$ *jump-diffusion with differential* χ*-characteristics* (β^χ,α,μ)*, such that* μ *satisfies the following condition.*

$$x\mapsto \int_\mathbb{V}\big|v-\chi(v)\big|\mu(x,\mathrm{d}v) \textit{ is bounded on compact subsets}$$

Then, X *is special with drift* β *as in (A.9).*

Proof. By choosing a $\mathscr{F}$ localizing sequence $(T_n)_{n\in\mathbb{N}}$ as in (A.5), our hypothsesis gives us the following integrability.

$$\mathrm{E_P}\big|\mathrm{id}_{\mathbb{V}} - \chi\big| * \hat{q}^X_{T_n} = \mathrm{E_P}\int_0^{T_n}\int_{\mathbb{V}}\big|v - \chi(v)\big|\mu(X_t, \mathrm{d}v)\mathrm{d}t \le n \cdot \sup_{|x|\le n}\int_{\mathbb{V}}\big|v-\chi(v)\big|\mu(x,\mathrm{d}v) < \infty$$

Now, [JS03, Proposition II.1.28] allows us to compensate as we did in (A.8)

Seeing as $(\mathrm{id}_{\mathbb{V}} - \chi) * q^X$ may be compensated for special jump-diffusions X, all of the characterizing objects of Theorem A.6 may be rewritten in terms of our drift β—effectively, χ becomes the identity.

$$\mathcal{L}f(x) \coloneqq \mathrm{D}f(x)\cdot\beta(x) + \frac{1}{2}\,\mathrm{tr}\left(\mathrm{D}^2 f(x)\circ\alpha(x)\right) + \int_{\mathbb{V}}\Big(f(x+v) - f(x) - \mathrm{D}f(x)\cdot v\Big)\mu(x,\mathrm{d}v)$$

$$\Lambda(u,x) = \big\langle u, \beta(x)\big\rangle + \frac{1}{2}\big\langle u,\alpha(x)\big\rangle + \int_{\mathbb{V}}\big(e^{\langle u,v\rangle} - 1 - \langle u,v\rangle\big)\mu(x,\mathrm{d}v),$$

$$X_t = x + \beta(X)\bullet\ell_t + \sigma(X_-)\bullet W_t + c(X_-,\mathrm{id}_{\mathbb{V}}) * \tilde{p}_t$$

A.3 Locally countable jump-diffusions

We see that a $(\mathrm{P},\mathscr{F})$ jump-diffusion X is special if the accumulated large jumps $(\mathrm{id}_{\mathbb{V}}-\chi)*q^X$ may be compensated. To this end, being special is a condition on the jumps *away* from the origin. We now turn our focus to jump-diffusions X in which the jumps *near* the origin behave nicely. For any jump-diffusion X, we may count the jumps with the jump process N^X.

(A.10)
$$N^X_t \coloneqq \sum_{0<s\le t} 1_{\Delta X_s \neq 0} = 1 * q^X_t$$

For many jump diffusions, it may be the case that we P-almost-surely have $N^X_t = \infty$ for all $t > 0$. We way that X has $(\mathrm{P},\mathscr{F})$ *locally countable*, so long as N^X is $(\mathrm{P},\mathscr{F})$ locally integrable. Below, we state how to verify this using the differential characteristics.

Lemma A.8. *Fix a $(\mathrm{P},\mathscr{F})$ jump-diffusion X with differential χ-characteristics (β^χ,α,μ) satisfying*

$$x \mapsto \mu\big(x,\mathbb{V}\big) \text{ is bounded on compact sets,}$$

then X is locally countable. Moreover, we may define $\lambda : \mathbb{X}\to\mathbb{R}_+$ and probability kernel $\kappa : \mathbb{X}\times\mathscr{B}(\mathbb{V})\to[0,1]$ by the following factoring.

$$\lambda(x) \coloneqq \mu(x,\mathbb{V}), \quad \mu(x,\mathrm{d}v) \eqqcolon \lambda(x)\kappa(x,\mathrm{d}v)$$

Also, N has $(\mathrm{P},\mathscr{F})$ intensity $\lambda(X)$.

Proof. Select the sequence $(T_n)_{n\in\mathbb{N}}$ as in (A.5). Note now that, since the constant function 1 is predictable,

$$\mathrm{E_P}N^X_{T_n} = \mathrm{E_P}1 * q^X_{T_n} = \mathrm{E_P}1*\hat{q}^X_{T_n} = \mathrm{E_P}\int_0^{T_n}\mu(X_t,\mathbb{V})\mathrm{d}t \le n\cdot\sup_{|x|\le n}\mu(x,\mathbb{V}) < \infty$$

This means that N^X is locally integrable, making X locally countable. Moreover, by [JS03, Theorem II.1.8],

$$N^X - \int_0^t \lambda(X_s)\mathrm{d}s = 1 * q^X - \int_0^t \int_{\mathbb{V}} \mu(X_s, \mathrm{d}v)\mathrm{d}s = 1 * q^X - 1 * \hat{q}^X$$

is a $(\mathrm{P}, \mathscr{F})$ local martingale, which finishes the proof.

Remark A.9. *(a) Such objects λ, κ always exist with our assumption of the Lemma. Seeing as μ is a transition kernel from $(\mathbb{X}, \mathscr{B}(\mathbb{X}))$ to $(\mathbb{V}, \mathscr{B}(\mathbb{V}))$, we have our desired measurability.*

$$\lambda := \mu(\cdot, \mathbb{V}) \in \mathscr{B}(\mathbb{X})/\mathscr{B}(\mathbb{R})$$

Constructing κ should be obvious algebra, so long as we have no zero measures; otherwise, we may define

$$\kappa(x, \Gamma) := \delta_0(\Gamma) \cdot 1_{\lambda^{-1}\{0\}}(x) + \frac{\mu(x, \Gamma)}{\lambda(x)} 1_{\mathbb{X}-\lambda^{-1}\{0\}}(x),$$

where δ_0 is the degenerate measure at $0 \in \mathbb{V}$. This ensures that any $\kappa(\cdot, \Gamma) \in \mathscr{B}(\mathbb{X})/\mathscr{B}(\mathbb{R})$ and any $\kappa(x, \cdot)$ a probability measure on $\mathscr{B}(\mathbb{V})$; also, when $\mu(x, \cdot)$ is the zero measure,

$$\mu(x, \mathrm{d}v) = 0 = \lambda(x) \cdot \delta_0(\mathrm{d}v) = \lambda(x)\kappa(x, \mathrm{d}v),$$

and otherwise,

$$\mu(x, \mathrm{d}v) = \mu(x, \mathbb{V}) \frac{\mu(x, \mathrm{d}v)}{\mu(x, \mathbb{V})} = \lambda(x)\kappa(x, \mathrm{d}v).$$

(b) As far as we know, there is no widely accepted source which explores jump-diffusions to the extent of declaring a notion like locally countable*, as we have. This means that there is likely some clash of terminology, should such a concept already exist.*

A.4 Real moments of jump-diffusions

We now turn our focus to the real moments of $(\mathrm{P}, \mathscr{F})$ jump-diffusions and the extension of our Lévy-Khintchine map Λ to real moments.

$$\Lambda(u, x) = \langle u, \beta^\chi(x) \rangle + \frac{1}{2}\langle u, \alpha(x) \rangle + \int_{\mathbb{V}} \left(e^{\langle u, v \rangle} - 1 - \langle u, \chi(v) \rangle\right)\mu(x, \mathrm{d}v), \quad u \in \mathbb{V},\ x \in \mathbb{X}$$

The above expression may be infinite, as the final term includes an unbounded integral over a possibly infinite measure. That said, we find it imperative to denote the following sets of finiteness.

$$\mathcal{D}_\Lambda(x) := \Big\{u \in \mathbb{V} : \Lambda(u, x) < \infty\Big\}, \qquad \mathcal{D}_\Lambda := \bigcap_{x \in \mathbb{X}} \mathcal{D}_\Lambda(x) \tag{A.11}$$

The following results will explore the nature of the maps $\Lambda(\cdot, x) : \mathcal{D}_\Lambda(x) \to \mathbb{R}$ for fixed differentiable χ-characteristics $(\beta^\chi, \alpha, \mu)$, where our truncation function χ is defined by $\chi(v) = v 1_{|v| \leq 1}$. Note that there is no loss of generality in selecting this truncation function, since they all evaluate Λ identically.

Lemma A.10. *For any $x \in \mathbb{X}$, we have $u \in \mathcal{D}_\Lambda(x)$ if and only if $\int_{|v|>1} e^{\langle u,v\rangle}\mu(x,\mathrm{d}v) < \infty$.*

Proof. To each $u, v \in \mathbb{V}$, Taylor's theorem gives us $\gamma_{u,v} \in [0,1]$ such that

$$e^{\langle u,v\rangle} = 1 + \langle u,v\rangle + \frac{1}{2}e^{\gamma_{u,v}\langle u,v\rangle}\langle u,v\rangle^2.$$

This allows us to see that, for each $x \in \mathbb{X}$, $\Lambda(u,x)$ and $\int_{|v|>1} e^{\langle u,v\rangle}\mu(x,\mathrm{d}v)$ differ by finite expressions.

$$\begin{aligned}
&\left|\Lambda(u,x) - \int_{|v|>1} e^{\langle u,v\rangle}\mu(x,\mathrm{d}v)\right| \\
&= \left|\langle u, \beta^\chi(x)\rangle + \frac{1}{2}\langle u, \alpha(x)u\rangle + \int_{|v|\le 1}\big(e^{\langle u,v\rangle} - 1 - \langle u,v\rangle\big)\mu(x,\mathrm{d}v) - \int_{|v|>1}\mu(x,\mathrm{d}v)\right| \\
&\le \left|\langle u, \beta^\chi(x)\rangle + \frac{1}{2}\langle u, \alpha(x)u\rangle\right| + \left|\int_{|v|\le 1}\frac{1}{2}e^{\gamma_{u,v}\langle u,v\rangle}\langle u,v\rangle^2\mu(x,\mathrm{d}v)\right| + \int_{|v|>1}\mu(x,\mathrm{d}v) \\
&\le \left|\langle u, \beta^\chi(x)\rangle + \frac{1}{2}\langle u, \alpha(x)u\rangle\right| + \left(\frac{1}{2}e^{|u|} + 1\right)\int_{\mathbb{V}}\big(1 \wedge |v|^2\big)\mu(x,\mathrm{d}v)
\end{aligned}$$

Thus, one can be defined as a finite displacement of the other.

Lemma A.11. *For each $x \in \mathbb{X}$, $\mathcal{D}_\Lambda(x)$ is convex.*

Proof. We use our characterization of $\mathcal{D}_\Lambda(x)$ from Lemma A.10. Let $u, u' \in \mathcal{D}_\Lambda(x)$, $\gamma \in (0,1)$, and use Hölder's inequality to see the following.

$$\begin{aligned}
&\int_{|v|>1} e^{\langle u' + \gamma(u-u'), v\rangle}\mu(x,\mathrm{d}v) \\
&\quad = \int_{|v|>1}\left|(e^{\langle u,v\rangle})^\gamma \cdot (e^{\langle u',v\rangle})^{1-\gamma}\right|\mu(x,\mathrm{d}v) \\
&\quad \le \left(\int_{|v|>1}\left|(e^{\langle u,v\rangle})^\gamma\right|^{\frac{1}{\gamma}}\mu(x,\mathrm{d}v)\right)^\gamma \left(\int_{|v|>1}\left|(e^{\langle u',v\rangle})^{1-\gamma}\right|^{\frac{1}{1-\gamma}}\mu(x,\mathrm{d}v)\right)^{1-\gamma} \\
&\quad = \left(\int_{|v|>1} e^{\langle u,v\rangle}\mu(x,\mathrm{d}v)\right)^\gamma \left(\int_{|v|>1} e^{\langle u',v\rangle}\mu(x,\mathrm{d}v)\right)^{1-\gamma} \\
&\quad < \infty
\end{aligned}$$

An arbitrary convex combination now satisfies $\gamma u + (1-\gamma)u' \in \mathcal{D}_\Lambda(x)$.

Lemma A.12. *For each $x \in \mathbb{X}$, the map $\Lambda(\cdot, x)$ is continuously differentiable on $\mathcal{D}_\Lambda(x)^\circ$, with derivative $\mathrm{D}\Lambda(\cdot,x) : \mathcal{D}_\Lambda(x)^\circ \to \mathbb{L}(\mathbb{V},\mathbb{R})$ as follows.*

$$\mathrm{D}\Lambda(u,x)w = \left\langle \beta^\chi(x) + \alpha(x)u + \int_{\mathbb{V}}\big(e^{\langle u,v\rangle}v - \chi(v)\big)\mu(x,\mathrm{d}v), w\right\rangle, \quad u \in \mathcal{D}_\Lambda(x)^\circ \tag{A.12}$$

Proof. Fix $x \in \mathbb{X}$, $u \in \mathcal{D}_\Lambda(x)^\circ$. Let $\epsilon > 0$ such that $B(u,\epsilon) \subseteq \mathcal{D}_\Lambda(x)$. For all $0 < \delta < \epsilon$ and $i = 1, \dots, d$, we now have the following identity

$$\begin{aligned}\frac{\Lambda(u+\delta e_i, x) - \Lambda(u,x)}{\delta} &= \langle e_i, \beta^\chi(x)\rangle + \langle e_i, \alpha(x)u\rangle + \frac{1}{2}\langle \delta e_i, \alpha(x)u\rangle \\ &\quad + \int_{|v|\le 1} \frac{1}{\delta}\Big(e^{\langle u+\delta e_i, v\rangle} - e^{\langle u,v\rangle} - \langle \delta e_i, v\rangle\Big)\mu(x,\mathrm{d}v) \\ &\quad + \int_{|v|>1} \frac{1}{\delta}\Big(e^{\langle u+\delta e_i, v\rangle} - e^{\langle u,v\rangle}\Big)\mu(x,\mathrm{d}v)\end{aligned} \tag{A.13}$$

Evaluating the limit of (A.13) as $\delta \to 0$ is now a matter of exchanging the limit with integration; we will do this by using the dominated convergence theorem.

For the first integral, Taylor's theorem provides us $\gamma_0, \gamma_1 \in [0,1]$ such that the following hold.

$$\begin{aligned} e^{\langle u+\delta e_i, v\rangle} &= 1 + \langle u+\delta e_i, v\rangle + \frac{1}{2}\langle u+\delta e_i, v\rangle^2 e^{\gamma_0\langle u+\delta e_i, v\rangle} \\ e^{\langle u,v\rangle} &= 1 + \langle u,v\rangle + \frac{1}{2}\langle u,v\rangle^2 e^{\gamma_1\langle u,v\rangle}\end{aligned}$$

This shows us that, for all $0 < \delta < \epsilon$ and $|v| \le 1$,

$$\begin{aligned}\left|\frac{1}{\delta}\Big(e^{\langle u+\delta e_i, v\rangle} - e^{\langle u,v\rangle} - \langle \delta e_i, v\rangle\Big)\right| &= \left|\frac{1}{2}\langle u+\delta e_i, v\rangle^2 e^{\gamma_0\langle u+\delta e_i, v\rangle} + \frac{1}{2}\langle u,v\rangle^2 e^{\gamma_1\langle u,v\rangle}\right| \\ &\le \Big(\big(|u|+\epsilon\big)^2 e^{|u|+\epsilon}\Big)|v|^2.\end{aligned}$$

This dominating function is integrable,

$$\int_{|v|\le 1}\Big(\big(|u|+\epsilon\big)^2 e^{|u|+\epsilon}\Big)|v|^2\mu(x,\mathrm{d}v) \le \Big(\big(|u|+\epsilon\big)^2 e^{|u|+\epsilon}\Big)\int_{\mathbb{V}}(1\wedge|v|^2)\mu(x,\mathrm{d}v) < \infty,$$

so we may apply the dominated convergence theorem.

$$\begin{aligned} &\lim_{\delta\to 0}\int_{|v|\le 1}\frac{1}{\delta}\Big(e^{\langle u+\delta e_i, v\rangle} - e^{\langle u,v\rangle} - \langle \delta e_i, v\rangle\Big)\mu(x,\mathrm{d}v) \\ &\quad = \int_{|v|\le 1}\lim_{\delta\to 0}\frac{1}{\delta}\Big(e^{\langle u+\delta e_i, v\rangle} - e^{\langle u,v\rangle} - \langle \delta e_i, v\rangle\Big)\mu(x,\mathrm{d}v) \\ &\quad = \int_{|v|\le 1}\Big(e^{\langle u,v\rangle}v_i - v_i\Big)\mu(x,\mathrm{d}v) \qquad \text{(A.14)}\end{aligned}$$

For the second integral, we again use Taylor's theorem to establish for each $0 < \delta < \epsilon/2$, some $\gamma_\delta \in [0,\delta]$ such that

$$e^{\langle u+\delta e_i, v\rangle} = e^{\langle u,v\rangle} + \langle \delta e_i, v\rangle e^{\langle u+\gamma_\delta e_i, v\rangle}$$

This way, we have the following dominating function.

$$\left|\frac{1}{\delta}\Big(e^{\langle u+\delta e_i, v\rangle} - e^{\langle u,v\rangle}\Big)\right| \le \left|\langle e_i, v\rangle e^{\langle u+\gamma_\delta e_i, v\rangle}\right| \le |v_i| e^{\langle u,v\rangle + \epsilon|v_i|/2}$$

The claim is that this dominating function is integrable. To see this, first note that because we have the following limit,

$$\lim_{|v|\to\infty} \frac{|v_i| e^{\langle u,v\rangle + \epsilon|v_i|/2}}{e^{\langle u,v\rangle + 2\epsilon|v_i|/3}} = \lim_{|v|\to\infty} \frac{|v_i|}{e^{\epsilon|v_i|/6}} = 0$$

There exists $M > 0$ such that for all $|v| > M$,

$$|v_i| e^{\langle u,v\rangle + \epsilon|v_i|/2} < e^{\langle u,v\rangle + 2\epsilon|v_i|/3}.$$

We now see that

$$\begin{aligned}
&\int_{|v|>1} |v_i| e^{\langle u,v\rangle + \epsilon|v_i|/2} \mu(x, \mathrm{d}v) \\
&= \int_{1<|v|\le M} |v_i| e^{\langle u,v\rangle + \epsilon|v_i|/2} \mu(x, \mathrm{d}v) + \int_{|v|>M} |v_i| e^{\langle u,v\rangle + \epsilon|v_i|/2} \mu(x, \mathrm{d}v) \\
&\le \int_{1<|v|\le M} M e^{(|u|+\epsilon/2)M} \mu(x, \mathrm{d}v) + \int_{|v|>M} e^{\langle u,v\rangle + 2\epsilon|v_i|/3} \mu(x, \mathrm{d}v) \\
&\le M e^{(|u|+\epsilon/2)M} \int_{\mathbb{V}} (1 \wedge |v|^2) \mu(x, \mathrm{d}v) + \sum_{\ell=0}^{1} \int_{|v|>1} e^{\langle u + 2\epsilon e_i/3, v\rangle} \mu(x, \mathrm{d}v) \\
&< \infty.
\end{aligned}$$

We again use the dominated convergence theorem to deduce the following.

$$\begin{aligned}
&\lim_{\delta\to 0} \int_{|v|>1} \frac{1}{\delta} \Big(e^{\langle u+\delta e_i, v\rangle} - e^{\langle u,v\rangle} \Big) \mu(x, \mathrm{d}v) \\
&= \int_{|v|>1} \lim_{\delta\to 0} \frac{1}{\delta} \Big(e^{\langle u+\delta e_i, v\rangle} - e^{\langle u,v\rangle} \Big) \mu(x, \mathrm{d}v) \\
&= \int_{|v|>1} e^{\langle u,v\rangle} v_i \mu(x, \mathrm{d}v) \qquad \text{(A.15)}
\end{aligned}$$

Combining equations (A.13), (A.14), and (A.15) now yields our desired identity.

$$\mathrm{D}_i \Lambda(u, x) = \Big\langle e_i, \beta^{\chi}(x) + \alpha(x) u + \int_{\mathbb{V}} \Big(e^{\langle u,v\rangle} v - \chi(v) \Big) \mu(x, \mathrm{d}v) \Big\rangle$$

Continuity of $\mathrm{D}_i \Lambda(u, x)$ for $u \in \mathcal{D}_\Lambda(x)^\circ$ involves very similar dominated convergence theorem arguments as above. From here, it is clear that Λ is continuously differentiable with the form in (A.12).

As we have seen in Lemmas A.7 and A.8, if we have local boundedness of certain integrals of a jump kernel μ, we can leverage these to $(\mathrm{P}, \mathscr{F})$ local conditions of the associated jump-diffusion X. Throughout the remainder of this section, we impose the following uniform-boundedness principle for the kernel μ.

$$\begin{aligned}
&\int_{\mathbb{V}} f(v) \mu(x, \mathrm{d}v) < \infty \text{ for all } x \in \mathbb{X} \\
&\implies \quad x \mapsto \int_{\mathbb{V}} f(v) \mu(x, \mathrm{d}v) \text{ bounded on compact sets} \qquad \text{(A.16)}
\end{aligned}$$

With this assumption, we get some nice results on finite exponential moments of X.

Proposition A.13. *Fix a* $(\mathrm{P}, \mathscr{F})$ *jump-diffusion* X *with differential* χ*-characteristics* $(\beta^{\chi}, \alpha, \mu)$. *Suppose we have the regularity condition (A.16) above. If* $0 \in \mathcal{D}_{\Lambda}^{\circ}$, *then* X *is special.*

Proof. If $0 \in \mathcal{D}_{\Lambda}^{\circ}$, then there exists some $\delta > 0$ such that $\overline{B}(0, \delta) \subseteq \mathcal{D}_{\Lambda}$. Observe the following implication of this fact, for each $x \in \mathbb{X}$.

$$\begin{aligned} \int_{\mathbb{V}} \big|v - \chi(v)\big| \mu(x, \mathrm{d}v) &= \int_{|v|>1} |v| \mu(x, \mathrm{d}v) \\ &\le \int_{|v|>1} \frac{\sqrt{d}}{\delta} \exp\Big(\frac{\delta |v|}{\sqrt{d}}\Big) \mu(x, \mathrm{d}v) \\ &\le \frac{\sqrt{d}}{\delta} \int_{|v|>1} \exp\Big(\max_{i=1}^{d} \max_{\ell=0}^{1} \big\langle (-1)^{\ell} \delta e^{i}, v\big\rangle\Big) \mu(x, \mathrm{d}v) \\ &\le \frac{\sqrt{d}}{\delta} \sum_{i=1}^{d} \sum_{\ell=0}^{1} \int_{|v|>1} \exp\big\langle (-1)^{\ell} \delta e^{i}, v\big\rangle \mu(x, \mathrm{d}v) \\ &< \infty \end{aligned}$$

Our regularity condition (A.16) now allows us to apply Lemma A.7 to conclude X is special.

Proposition A.14. *Fix a* $(\mathrm{P}, \mathscr{F})$ *jump-diffusion* X *with differential* χ*-characteristics* $(\beta^{\chi}, \alpha, \mu)$. *Suppose we have the regularity condition (A.16) above. If* $u \in \mathcal{D}_{\Lambda}$, *then* $\exp\langle u, X\rangle$ *is special, and* $\exp\big(\langle u, X\rangle - \Lambda(u, X) \bullet \ell\big)$ *is a* $(\mathrm{P}, \mathscr{F})$ *local martingale.*

Proof. Using Lemma A.3 for the function $f_u(v) = \exp\langle u, v\rangle$ and its derivative identities as in (A.7), we get the following.

$$\begin{aligned} \exp\langle u, X_t\rangle = {}& \exp\langle u, X_0\rangle + \exp\langle u, X_t\rangle \Big(\langle u, \beta^{\chi}(X)\rangle + \frac{1}{2}\langle u, \alpha(X) u\rangle\Big) \bullet \ell_t \\ &+ \mathrm{D} f_u(X_-) \bullet X^{\mathrm{c}} + \Big(\exp\langle u, X_-\rangle \langle u, \chi\rangle\Big) * \tilde{q}_t^X \\ &+ \exp\langle u, X_-\rangle \cdot \Big(\exp\langle u, \mathrm{id}_{\mathbb{V}}\rangle - 1 - \langle u, \chi\rangle\Big) * q^X \end{aligned} \tag{A.17}$$

Note that localizing our final term on the sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times in (A.5), we get the following.

$$\begin{aligned} &\mathrm{E}_{\mathrm{P}} \Big| \exp\langle u, X_-\rangle \Big(\exp\langle u, \mathrm{id}_{\mathbb{V}}\rangle - 1 - \langle u, \chi\rangle\Big)\Big| * \hat{q}_{T_n}^X \\ &= \mathrm{E}_{\mathrm{P}} \int_0^{T_n} \int_{\mathbb{V}} \Big| \exp\langle u, X_s\rangle \Big(\exp\langle u, v\rangle - 1 - \langle u, \chi(v)\rangle\Big)\Big| \mu(X_s, \mathrm{d}v) \mathrm{d}s \\ &\le n \cdot \sup_{|x|\le n} \Big(e^{\langle u, x\rangle} \int_{\mathbb{V}} \big|e^{\langle u, v\rangle} - 1 - \langle u, \chi(v)\rangle\big| \mu(x, \mathrm{d}v)\Big) \end{aligned}$$

Seeing as $u \in \mathcal{D}_{\Lambda}$, the integral in the above quantity is finite, and so (A.16) gives us finiteness of the supremum. Using [JS03, Proposition II.1.28] now allows us to compensate the jump term in (A.17).

$$\exp\langle u, X_t\rangle = \exp\langle u, X_0\rangle + \Big(\exp\langle u, X_t\rangle \cdot \Lambda(u, X)\Big) \bullet \ell_t + \mathrm{D} f_u(X_-) \bullet X^{\mathrm{c}} + \Big(\exp\langle u, X_-\rangle \langle u, \chi\rangle\Big) * \tilde{q}_t^X$$

This is a representation of $\exp\langle u, X\rangle$ as an initial term, predictable term of finite variation, and a local martingale. Thus, it is a special semimartingale. From here, we may perform the product rule on $\exp\big(\langle u, X\rangle - \Lambda(u, X)\bullet\ell\big)$ as we did in Proposition A.5 to show that the process is a local martingale.

Theorem A.15. *Fix a $(\mathrm{P}, \mathscr{F})$ jump-diffusion X with differential χ-characteristics $(\beta^\chi, \alpha, \mu)$. Suppose we have the regularity condition (A.16) above and that $0 \in \mathcal{D}_\Lambda^\circ$. For each $h \in \mathbb{D}([0,\infty), \mathbb{V})$ of finite-variation with image contained in $\mathcal{D}_\Lambda$, the process $\exp(h\bullet X)$ is special and*

$$\exp\Big(h\bullet X - \Lambda(h, X)\bullet\ell\Big)$$

is a $(\mathrm{P}, \mathscr{F})$ local martingale.

Proof. We first note that Proposition A.13 allows us to conclude X is special. Perform Itô's formula [JS03, Theoerem I.4.57] in addition to its jump-diffusion variant in Lemma A.3 and various stochastic integral identities [JS03, Remarks I.4.36, I.4.37, Theorem I.4.40(d), Proposition II.1.30(b)].

$$\begin{aligned}
&\exp\big(h\bullet X_t\big)\\
&= \exp\big(h\bullet X_-\big)\bullet\big(h\bullet X\big)_t + \frac{1}{2}\exp\big(h\bullet X_-\big)\bullet\big\langle(h\bullet X)^{\mathrm{c}}, (h\bullet X)^{\mathrm{c}}\big\rangle_t\\
&\quad + \sum_{0<s\le t}\Big(\exp\big(h\bullet X_{s-} + \Delta\big(h\bullet X\big)_s\big) - \exp\big(h\bullet X_{s-}\big) - \exp\big(h\bullet X_{s-}\big)\Delta\big(h\bullet X\big)_s\Big)\\
&= \Big(\exp\big(h\bullet X_-\big)\cdot h\Big)\bullet X_t + \frac{1}{2}\exp\big(h\bullet X\big)\big\langle h, \alpha(X)h\big\rangle\bullet\ell_t\\
&\quad + \exp\big(h\bullet X_-\big)\Big(e^{\langle h, \mathrm{id}_{\mathbb{V}}\rangle} - 1 - \langle h, \mathrm{id}_{\mathbb{V}}\rangle\big) * q_t^X
\end{aligned}$$

$$\begin{aligned}
&= \Big(\exp\big(h\bullet X\big)\cdot\langle h, \beta\rangle + \frac{1}{2}\exp\big(h\bullet X\big)\big\langle h, \alpha(X)h\big\rangle\Big)\bullet\ell_t + \Big(\exp\big(h\bullet X_-\big)\cdot h\Big)\bullet X_t^{\mathrm{c}}\\
&\quad + \exp\big(h\bullet X_-\big)\langle h, \mathrm{id}_{\mathbb{V}}\rangle * \tilde{q}_t^X\\
&\quad + \exp\big(h\bullet X_-\big)\Big(e^{\langle h, \mathrm{id}_{\mathbb{V}}\rangle} - 1 - \langle h, \mathrm{id}_{\mathbb{V}}\rangle\Big) * q_t^X
\end{aligned}\tag{A.18}$$

Now, choosing our $(\mathrm{P}, \mathscr{F})$ localizing sequence $(T_n)_{n\in\mathbb{N}}$ as in A.5, we have the following bound.

$$\begin{aligned}
&\mathrm{E}_{\mathrm{P}}\Big|\exp\big(h\bullet X_-\big)\Big(e^{\langle h, \mathrm{id}_{\mathbb{V}}\rangle} - 1 - \langle h, \mathrm{id}_{\mathbb{V}}\rangle\Big) * \hat{q}_{T_n}^X\Big|\\
&= \mathrm{E}_{\mathrm{P}}\int_0^{T_n}\int_{\mathbb{V}}\Big|\exp\big(h\bullet X_s\big)\Big(e^{\langle h(s), v\rangle} - 1 - \langle h(s), v\rangle\Big)\Big|\mu(X_s, \mathrm{d}v)\mathrm{d}s\\
&\le n\cdot\sup_{|x|\le n}\sup_{s\in[0,n]} e^{|x|\cdot|h(s)|}\int_{\mathbb{V}}\big|e^{\langle h(s), v\rangle} - 1 - \langle h(s), v\rangle\big|\mu(x, \mathrm{d}v)
\end{aligned}$$

Seeing as $\Lambda(\cdot, x)$ is continuously differentiable, it is uniformly bounded on $\mathcal{D}_\Lambda^\circ$. This, along with assumption (A.16) allow us to conclude that the preceding expression is finite. Thus, we may compensate the final jump integral in (A.18).

$$\begin{aligned}
\exp\big(h\bullet X_t\big) &= \Big(\exp\big(h\bullet X\big)\cdot\Lambda(h, X)\Big)\bullet\ell_t + \Big(\exp\big(h\bullet X_-\big)\cdot h\Big)\bullet X_t^{\mathrm{c}}\\
&\quad + \exp\big(h\bullet X_-\big)\Big(e^{\langle h, \mathrm{id}_{\mathbb{V}}\rangle} - 1\Big) * q_t^X
\end{aligned}$$

The decomposition of $\exp(h \bullet X)$ into a predictable finite-variation process and a local martingale implies that it is special. Now, we write M as the local martingale term above, $A = \exp(h \bullet X)$, and $B = \exp(-\Lambda(h, X) \bullet \ell)$. We now recognize that B is predictable and finite-variation and use [JS03, Proposition I.4.49(b)] to conclude our proof.

$$\begin{aligned}
\exp\big(h \bullet X_t - \Lambda(h, X) \bullet \ell_t\big) &= A_t B_t \\
&= A_- \bullet B_t + B \bullet A_t \\
&= \big(A \cdot B \cdot -\Lambda(h, X)\big) \bullet \ell_t + B \bullet \Big(\big(\exp(h \bullet X) \cdot \Lambda(h, X)\big) \bullet \ell + M\Big)_t \\
&= \big(A \cdot B \cdot -\Lambda(h, X)\big) \bullet \ell_t + \big(B \cdot A \cdot \Lambda(h, X)\big) \bullet \ell_t + B \bullet M_t \\
&= B \bullet M_t
\end{aligned}$$

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