Large deviations of affine processes

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Abstract

This is an abstract of the entire dissertation; summarize a history of large deviations and affine processes, then abstractly summarize our large deviations result.

Acknowledgment

This is where I acknowledge how I am useless without others.

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Introduction

This is where I give the reader a little more history and detail regarding affine processes and large deviations, should they read this paper without already being well-versed in the subject.

Notation and conventions

Throughout, unless specifically referenced elsewhere, all notions of this text are formally defined and explored in [Kal02] or [JS03]. Most of our notation will coincide with these texts (as well as most other literature), except in regards to some particular conventions. Let us establish some of these here. A stochastic process X with a marginal-index-set I and state space $(\mathbb{X}, \mathscr{X})$ will be indifferently recognized as:

- a collection $X = (X_t)_{t \in I}$ of marginals $X_t : \Omega \to \mathbb{X}$,
- a map $X : \Omega \times I \to \mathbb{X}$,
- or its curried version $X : \Omega \to \mathbb{X}^I$.

With this convention, we find it appropriate to denote filtrations $\mathscr{F} = (\mathscr{F}_t)_{t \geq 0}$ of increasing σ-algebras $\mathscr{F}_t$. Seeing as $\mathscr{F}$ denotes the actual family of σ-algebras, we denote the joined algebra with an infinity subscript, $\mathscr{F}_\infty \coloneqq \bigvee_{t \geq 0} \mathscr{F}_t$. The blackboard notation will generally correspond to a topological space, including those objects we typically introduce in analysis.

- The real $\mathbb{R}$, the complex $\mathbb{C} = \mathbb{R} \oplus \mathrm{i}\mathbb{R}$, the and nonnegative $\mathbb{R}_+ = [0, \infty)$ numbers with the usual Euclidean topologies.
- For real normed vector spaces $\mathbb{V}$, $\mathbb{W}$, the space $\mathbb{L}(\mathbb{V}, \mathbb{W})$ of real linear maps $\mathbb{V} \to \mathbb{W}$, equipped with operator norm.
$$|T| \coloneqq \sup_{|v|=1} |Tv|$$
We also concisely denote $\mathbb{L}(\mathbb{V}) \coloneqq \mathbb{L}(\mathbb{V}, \mathbb{V})$.
- For the a separable metric space $\mathbb{X}$ and an interval $I \subseteq \mathbb{R}_+$, the space $\mathbb{D}(I, \mathbb{X})$ of càdlàg functions, equipped with the Skorokhod J1 topology.
- For topological spaces $\mathbb{X}, \mathbb{Y}$, the space $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ of continuous functions, equipped with the supremum norm.

- For finite-dimensional normed vector spaces $\mathbb{V}, \mathbb{W}$ and open $\mathbb{U} \subseteq \mathbb{V}$, the subspace $\mathbb{C}^1(\mathbb{U}, \mathbb{W})$ of functions $f \in \mathbb{C}(\mathbb{U}, \mathbb{W})$ in which there is a derivative map $\mathrm{D}f \in \mathbb{C}(\mathbb{U}, \mathbb{L}(\mathbb{V}, \mathbb{W}))$.

$$\lim_{|v| \to 0} \frac{\big| f(u+v) - f(u) - \mathrm{D}f(u) \cdot v \big|}{|v|} = 0$$

  For $f \in \mathbb{C}^1(\mathbb{U}, \mathbb{R})$, we denote $\nabla f \in \mathbb{C}(\mathbb{U}, \mathbb{V})$ the gradient,

$$\big\langle v, \nabla f(u) \big\rangle \coloneqq \mathrm{D}f(u) \cdot v,$$

  If there is some canonical ordered basis $(e_1, \ldots, e_{\dim \mathbb{V}})$ of $\mathbb{V}$, denote $\mathrm{D}_i f \in \mathbb{C}(\mathbb{U}, \mathbb{R})$ the i-th partial derivative.

$$\mathrm{D}_i f(u) \coloneqq \mathrm{D}f(u) \cdot e_i, \quad i = 1, \ldots, d$$

- For finite-dimensional normed vector space $\mathbb{V}$ and open $\mathbb{U} \subseteq \mathbb{V}$, the subspace $\mathbb{C}^2(\mathbb{U}, \mathbb{R})$ of $f \in \mathbb{C}^1(\mathbb{U}, \mathbb{R})$ in which we also have $\nabla f \in \mathbb{C}^1(\mathbb{U}, \mathbb{V})$. In such a case, we denote $\mathrm{D}^2 f \in \mathbb{C}(\mathbb{U}, \mathbb{L}(\mathbb{V}))$ the Hessian.

$$\mathrm{D}^2 f(u) \coloneqq \mathrm{D}\big(\nabla f(u)\big)$$

  If there is some canonical ordered basis $(e_1, \ldots, e_{\dim \mathbb{V}})$ of $\mathbb{V}$, denote $\mathrm{D}_{ij} f \in \mathbb{C}(\mathbb{U}, \mathbb{R})$ the second-order ij-th partial derivative.

$$\mathrm{D}_{ij} f(u) \coloneqq \big\langle e_i, \mathrm{D}^2 f(u) \cdot e_j \big\rangle, \quad i, j = 1, \ldots, d$$

For spaces $\mathbb{X}$ in which there is some canonical topology, we will denote the associated Borel algebra $\mathscr{B}(\mathbb{X})$. Particular examples of this convention are:

- the Borel algebra $\mathscr{B}(\mathbb{V})$ associated to the topology induced from a canonical inner-product $\langle \cdot, \cdot \rangle$ on a vector space $\mathbb{V}$.
- the Borel algebra $\mathscr{B}(\mathbb{X})$ associated to the relative topology of some subset $\mathbb{X}$ of a space $\mathbb{V}$ with itself some canonical topology.

In the case that we are dealing with a finite-dimensional real vectorspace $\mathbb{V}$ with inner-product $\langle \cdot, \cdot \rangle$, we assume some canonical orthonormal basis $e_1, \ldots, e_{\dim \mathbb{V}} \in \mathbb{V}$ and establish the associated isometric isomorphism $\mathbb{V} \equiv \mathbb{R}^d$.

$$v \in \mathbb{V} \quad \longleftrightarrow \quad \big(v^1, \ldots, v^{\dim \mathbb{V}}\big); \qquad v^i \coloneqq \langle v, e_i \rangle, \quad i = 1, \ldots, \dim \mathbb{V}$$

Similarly identify any map $f : \mathbb{A} \to \mathbb{V}$ with component maps $f_1, \ldots, f_d : \mathbb{A} \to \mathbb{R}$.

$$f : \mathbb{A} \to \mathbb{V} \quad \longleftrightarrow \quad \big(f_1, \ldots, f_d\big) : \mathbb{A} \to \mathbb{R}^d; \qquad f_i(a) \coloneqq \langle f(a), e_i \rangle$$

Extend the inner-product symmetrically to a bilinear form on $\mathbb{V} \oplus \mathrm{i}\mathbb{V}$,

$$\big\langle v_1 + \mathrm{i}w_1, v_2 + \mathrm{i}w_2 \big\rangle = \big(\langle v_1, v_2 \rangle - \langle w_1, w_2 \rangle\big) + \mathrm{i}\big(\langle v_1, w_2 \rangle + \langle w_1, v_2 \rangle\big),$$

and define the trace of an operator $T \in \mathbb{L}(\mathbb{V})$, as follows.

$$\mathrm{tr}(T) = \sum_{i=1}^{d} \langle e_i, T e_i \rangle$$

We adopt that $(\Omega, \Sigma, \mathrm{P})$ is an abstract probability space that—through the process of enlargement via Kolmogorov's extension theorem—we without loss of generality assume it is equipped with identifications of various quantities $X : \Omega \to \mathbb{X}$ into measurable spaces $(\mathbb{X}, \mathscr{X})$ associated with distributions μ on $(\mathbb{X}, \mathscr{X})$. We typically presume such maps X to be measurable without mention and will otherwise specify this fact explicitly by using the notation $X \in \Sigma/\mathscr{X}$. For each probability measure P on (Ω, Σ), we denote the P-distribution of such X by P_X or pushfoward notation, $X_\#\mathrm{P}$.

$$\mathrm{P}_X\Gamma \coloneqq (X_\#\mathrm{P})(\Gamma) \coloneqq \mathrm{P}(X \in \Gamma) \coloneqq \mathrm{P}\big(X^{-1}\Gamma\big), \quad \Gamma \in \mathscr{X}$$

For intuition, we will also denote integration against this distribution as follows.

$$\int_{\mathbb{X}} \mathrm{P}(X \in \mathrm{d}x) f(x) \coloneqq \int_{\mathbb{X}} \mathrm{P}_X(\mathrm{d}x) f(x) = \int_{\Omega} \mathrm{P}(\mathrm{d}\omega) f\big(X(\omega)\big) \eqqcolon \mathrm{E}_\mathrm{P} f(X)$$

Just as E_P denotes the expectation operator of the measure P, we will denote $\mathrm{E}_\mathrm{P}(\cdot|\mathscr{G})$ the conditional expectation operator of P associated with a filtration $\mathscr{G}$. Should we choose a target space $(\mathbb{Y}, \mathscr{Y})$ and a natural σ-algebra $Y^{-1}\mathscr{Y}$ from some quantity $Y \in \Sigma/\mathscr{Y}$, we denote $\mathrm{E}_\mathrm{P}(\cdot|Y = \cdot)$ the factoring of $\mathrm{E}_\mathrm{P}(\cdot|Y^{-1}\mathscr{Y})$ through Y.

$$\mathrm{E}_\mathrm{P}(X|Y = y) = \mathrm{E}_\mathrm{P}\big(X|Y^{-1}\mathscr{Y}\big)\Big|_{Y=y}$$

Also, any quantity $X : \Omega \to \mathbb{X}$ will be identified with the identity map on its codomain, so that we may abusingly use the convenient expectation notation.

$$\mathrm{E}_{\mathrm{P}_X} f(X) \coloneqq \mathrm{E}_{\mathrm{P}_X} f = \int_{\mathbb{X}} f(x)\mathrm{P}_X(\mathrm{d}x) = \int_{\Omega} f\big(X(\omega)\big)\mathrm{P}(\mathrm{d}\omega) = \mathrm{E}_\mathrm{P} f(X)$$

This will particularly be useful for when we discuss Markov processes and their associated identities.

Chapter I

Affine processes

Here I put a summary of chapter, along with a short history. It will include the following important notes.

- Chapter addresses important fundamental results of affine processes.
- Chapter addresses consequences of mgf results that are important for us, though not specified exactly much in the literature

I.1 Formulation

We start by specifying our affine processes as in [KRM15]. That is to say, we fix a finite-dimensional real vectorspace $\mathbb{V}$ with inner-product $\langle \cdot, \cdot \rangle$ and select a convex, closed $\mathbb{X} \subseteq \mathbb{V}$ satisfying $0 \in \mathbb{X}$ and $\operatorname{span} \mathbb{X} = \mathbb{V}$. Associate this space with the finite exponentials.

$$\mathcal{U}_{\mathbb{X}} \coloneqq \left\{ u \in \mathbb{V} \oplus \mathrm{i}\mathbb{V} : \sup_{x \in \mathbb{X}} \exp \langle \Re(u), x \rangle < \infty \right\}$$

We may now define the notion of an affine process on $\mathbb{X}$.

Definition I.1. *For a probability space $(\Omega, \Sigma, \mathrm{P})$ with filtration $\mathscr{F} = (\mathscr{F}_t)_{t \geq 0}$, an affine process X on $\mathbb{X}$ is a stochastically continuous, time-homogeneous $(\mathrm{P}, \mathscr{F})$-Markov process on $\mathbb{X}$ in which the bounded moments have the following log-affine dependence on the initial state.*

$$\begin{aligned} \mathrm{E}_{\mathrm{P}_x} \exp \langle u, X_t \rangle &= \exp \Psi(t, u, x) \\ \Psi(t, u, x) &= \psi_0(t, u) + \big\langle \psi(t, u), x \big\rangle, \end{aligned} \qquad t \geq 0,\ u \in \mathcal{U}_{\mathbb{X}} \tag{I.1}$$

Above, we are denoting $(\mathrm{P}_x)_{x \in \mathbb{X}}$ the conditional P-distributions of X factored through the initial state $x \in \mathbb{X}$.

Remark I.2. *Our definitions of $\mathbb{X}$ and Ψ include the following conventions and motivations.*

(a) why we denote ψ_0, ψ instead of KM φ, ψ or Cuchiero Φ, Ψ

(b) *how assumptions* $0 \in \mathbb{X}$, $\operatorname{span} \mathbb{X} = \mathbb{V}$ *are nonrestrictive*

(c) *how* (I.1) *decides the distribution of* X *and how the distribution of the affine process decides* Ψ

(d) *If we have a vectorspace* $\mathbb{A}$ *and affine map* $\alpha : \mathbb{X} \to \mathbb{A}$ *determined by* $a_0, \dots, a_d \in \mathbb{A}$ *via* $\alpha(x) = a_0 + \sum_{i=1}^d x^i a_i$, *then our linear assumptions* $0 \in \mathbb{X}$ *and* $\operatorname{span} \mathbb{X} = \mathbb{V}$ *uniquely determine* $a_0, \dots, a_d \in \mathbb{A}$. *In particular, the map* Ψ *uniquely identifies its parts* $\psi_i : \mathbb{R}_+ \times \mathcal{U}_{\mathbb{X}} \to \mathbb{C}$ *for* $i = 0, \dots, d$.

In [Cuc11, Theorem 1.2.7], it is shown that, without loss of generality on conditional distributions $(\mathrm{P}_x)_{x \in \mathbb{X}}$, an affine process X can be chosen to have càdlàg paths. Thus, each distribution P_x may be recognized as a measure on the Borel algebra associated with the space $\mathbb{D}([0, \infty), \mathbb{X})$ of càdlàg functions equipped with the Skorokhod topology. We will impose this regularity, so that X is also a $(\mathrm{P}_x, \mathscr{F})$ jump-diffusion for each $x \in \mathbb{X}$. For relevant definitions and results pertaining to jump-diffusions, we refer the reader to Appendix A.

Theorem I.3. *An affine process* X *on* $\mathbb{X}$ *is a* $(\mathrm{P}_x, \mathscr{F})$ *jump-diffusion in which the differential* χ*-characteristics* $(\beta^\chi, \alpha, \mu)$ *are affine maps of the following form.*

$$\beta^\chi(x) \coloneqq b_0^\chi + \sum_{i=1}^d x^i b_i^\chi, \quad \alpha(x) \coloneqq a_0 + \sum_{i=1}^d x^i a_i, \quad \mu(x, \mathrm{d}v) \coloneqq m_0(\mathrm{d}v) + \sum_{i=1}^d x^i m_i(\mathrm{d}v)$$

The associated Lévy-Khintchine map Λ *then also affine,*

$$\begin{aligned} \Lambda(u, x) &= \big\langle u, \beta^\chi(x) \big\rangle + \frac{1}{2} \big\langle u, \alpha(x) u \big\rangle + \int_{\mathbb{V}} \Big(e^{\langle u, v \rangle} - 1 - \big\langle u, \chi(v) \big\rangle \Big) \mu(x, \mathrm{d}v) \\ &= L_0(u) + \sum_{i=1}^d x^i L_i(u) \\ L_i(u) &\coloneqq \big\langle u, b_i^\chi(x) \big\rangle + \frac{1}{2} \big\langle u, a_i(x) u \big\rangle + \int_{\mathbb{V}} \Big(e^{\langle u, v \rangle} - 1 - \big\langle u, \chi(v) \big\rangle \Big) m_i(x, \mathrm{d}v), \end{aligned}$$

and each $u \in \mathrm{i}\mathbb{V}$ *induces the following differential equation.*

$$\text{(I.2)} \qquad \begin{cases} \psi_0(t, u) = L_0\big(\psi(t, u)\big) & t \geq 0 \\ \psi(t, u) = L\big(\psi(t, u)\big) & t \geq 0 \\ \psi_0(0, u) = 0 \\ \psi(0, u) = u \end{cases}$$

Proof. This is simply a restatement of [Cuc11, Theorem 1.5.4].

Remark I.4. *By Remark I.2(d), the equation in (I.2) is equivalent to the following system of equations.*

$$\text{(I.3)} \qquad \forall\, x \in \mathbb{X}, \qquad \begin{cases} \dot{\Psi}(t, u, x) = \Lambda\big(\psi(t, u), x\big) & t \geq 0 \\ \Psi(0, u, x) = \langle u, x \rangle \end{cases}$$

Henceforth, we fix X a càdlàg affine process with conditional distributions $(\mathrm{P}_x)_{x \in \mathbb{X}}$ on $\mathbb{D}([0, \infty), \mathbb{X})$, induced filtration $\mathscr{F} = (\mathscr{F}_t)_{t \geq 0}$, and moment function Ψ as in Definition I.1. We will use the truncation function $\chi(v) = v 1_{|v| \leq 1}$ and fix the differential χ-characteristics $(\beta^\chi, \alpha, \mu)$ and Lévy-Khintchine map Λ as in Theorem I.3.

I.2 Existence of real moments

This elaborates upon the extension of the transform formula in (I.1) and dynamics in (**??**) to real moments $u \in \mathbb{V}$. Clearly, should any extension exist for some $u \in \mathbb{V}$, the value $\Lambda(u, x) = \dot{\Psi}(0, u, x)$ should be well-defined. That said, we henceforth denote the following set.

$$\mathcal{D}_\Lambda \coloneqq \Big\{ u \in \mathbb{V} : \Lambda(u, x) \text{ is well-defined for all } x \in \mathbb{X} \Big\}$$

Quick inspection of $\Lambda(u, x)$ in (**??**) indicates that well-definition should depend solely on the integral term. The following results will perform the careful analysis to show that Λ has nice regularity on the finite moments of its associated jump kernel $\mu(\cdot, \mathrm{d}v)$.

I.3 Finite-dimensional distributions

Chapter II

Large deviations of affine processes

Chapter III

Large deviation rate functions

Appendix A

Jump-diffusions

TODO:

- Motivate why I chose to put this in the appendix. Big point: I want to resolve abstractions and rigor of [JS03] to the digestible notions of special jump-diffusions.
- Point to the various papers we use that do not consolidate a similar set of assumptions.

In order to discuss jump-diffusions on a finite-dimensional real vectorspace, one must have a decent understanding of semimartingales. A great text for a comprehensive study of this is [JS03], which we will refer to in our proofs. In terms of notational differences, we choose our probability space $(\Omega, \Sigma, \mathrm{P})$ and filtration $\mathscr{F} = (\mathscr{F}_t)_{t \geq 0}$, where $\mathscr{F}_\infty \subseteq \Sigma$ denotes the joined space. Furthermore, we do not explicitly write processes to take values in $\mathbb{R}^d$, but rather some vectorspace $\mathbb{V}$ with dimension $d := \dim \mathbb{V}$ and inner-product $\langle \cdot, \cdot \rangle$. Surely—due to our isometric isomorphism $\mathbb{V} \equiv \mathbb{R}^d$—any componentwise or linear notion, such as integration or differentiation may be taken as equivalent. Furthermore, we sometimes specify that a stochastic process X has a Borel state space $\mathbb{X} \subseteq \mathbb{V}$, as this is the case when studying affine processes. We find it important to highlight the following important notation of objects introduced in [JS03, Chapters I-II].

- Given $(\mathrm{P}, \mathscr{F})$ locally square-integrable martingales $M, N : \Omega \times \mathbb{R}_+ \to \mathbb{R}$, denote $\langle M, N \rangle$ the predictable quadratic covariation.
- Given $H, X : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ with H being $\mathscr{F}$ predictable and $(\mathrm{P}, \mathscr{F})$ locally bounded and X a $(\mathrm{P}, \mathscr{F})$ semimartingale, denote the stochastic integral process as follows.

$$H \bullet X_t = \int_0^t H_s \mathrm{d}X_s$$

  We may lift this concept componentwise and linearly. This allows us to choose the codomains of H, X to various combinations of $\mathbb{V}$ and $\mathbb{L}(\mathbb{V}, \mathbb{W})$ when evaluating $H \bullet X$, so long as such a combination allows for $H_t \cdot X_t$ to make sense.

- Denote $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ the identity map, so as to allow a concise notation for Lebesgue integration.

$$H \bullet \ell_t = \int_0^t H_s \mathrm{d}s$$

- Given a random measure $q : \Omega \times \mathscr{B}(\mathbb{R}_+ \times \mathbb{V}) \to [0, \infty]$, denote the stochstic integral process against some suitably integrable optional process $H : \Omega \times \mathbb{R}_+ \times \mathbb{V} \to \mathbb{R}$ as follows.
$$H * q_t = \int_{[0,t]\times\mathbb{V}} H_s(v) q(\mathrm{d}s, \mathrm{d}v)$$
Denote its $(\mathrm{P}, \mathscr{F})$ predictable projection by $\hat{q}$ and the compensated measure $\tilde{q} = q - \hat{q}$. Lift these integration notions to vector-valued H componentwise. Instead of choosing a canonical variable for integrating expressions in this form, we use the identity maps $\mathrm{id}_{\mathbb{V}}$ or ℓ.
$$f(\ell, \mathrm{id}_{\mathbb{V}}) * q_t = \int_{[0,t]\times\mathbb{V}} f(s, v) q(\mathrm{d}s, \mathrm{d}v)$$
- Given a random measure q, we denote $G_{\mathrm{loc}}(q)$ the set of processes W which we can take a compensated stochastic integral, $W * \tilde{q}$, as in [JS03, Definition 1.27(a)].
- Given $(\mathrm{P}, \mathscr{F})$ semimartingales $X, Y : \Omega \times \mathbb{R}_+ \to \mathbb{R}$, denote $[X, Y]$ the quadratic covariation.
- Given a semimartingale X, denote X^{c} its continuous local martingale component and q^X its jump measure.

A.1 Formulation

As in [JS03, Definition III.2.18], a $(\mathrm{P}, \mathscr{F})$ jump-diffusion X on state space $(\mathbb{X}, \mathscr{B}(\mathbb{X}))$ is a $(\mathrm{P}, \mathscr{F})$ semimartingale in which the χ-characteristics $(B^\chi, A, \hat{q}^X)$ have the following decompositions,

$$B_t^\chi = \int_0^t \beta^\chi(X_s)\mathrm{d}s, \quad A_t = \int_0^t \alpha(X_s)\mathrm{d}s, \quad \hat{q}^X(\mathrm{d}s, \mathrm{d}v) = \mu(X_s, \mathrm{d}v)\mathrm{d}s, \tag{A.1}$$

where the functions have the following properties.

- $\beta^\chi : \mathbb{X} \to \mathbb{V}$ is Borel measurable, $\beta^\chi \in \mathscr{B}(\mathbb{X})/\mathscr{B}(\mathbb{V})$.
- $\alpha : \mathbb{X} \to \mathbb{L}(\mathbb{V})$ is Borel measurable, $\alpha \in \mathscr{B}(\mathbb{X})/\mathscr{B}(\mathbb{L}(\mathbb{V}))$, and $\alpha(x)$ is self-adjoint and nonnegative for each $x \in \mathbb{X}$.
- $\mu : \mathbb{X} \times \mathscr{B}(\mathbb{V}) \to [0, \infty]$ is a transition kernel from $\mathbb{X}$ to $\mathbb{V}$, and it satisfies the following properties for each $x \in \mathbb{X}$.
$$\mu\big(x, \{0\}\big) = 0, \qquad \int_{\mathbb{V}} \big(1 \wedge |v|^2\big)\mu(x, \mathrm{d}v) < \infty$$

Remark A.1. *(a) Note that we differ slightly from the definition we reference by imposing a time-homogenity formulation. There is no loss of generality in doing so, because we may always extend the state to $\mathbb{R}_+ \times \mathbb{X}$ via $\hat{X}_t = (t, X_t)$.*

(b) Note that (A.1) can be written concisely by using the identity λ on $\mathbb{R}_+$.
$$B_t^\chi = \beta^\chi(X) \bullet \ell_t, \quad A_t = \alpha(X) \bullet \ell_t, \quad \hat{q}^X([0, t], \mathrm{d}v) = \mu(X, \mathrm{d}v) \bullet \ell_t$$

(c) *If we have a jump-diffusion with χ-characteristics in (A.1), we call $(\beta^\chi, \alpha, \mu)$ the* differential χ-characteristics. *Just as with usual characteristics, there are simple expressions which relate β^χ and $\beta^{\hat\chi}$ between different truncation functions $\chi, \hat\chi$.*

$$\beta^{\hat\chi}(x) = \beta^\chi(x) + \int_{\mathbb{V}} \big(\hat\chi(v) - \chi(v)\big)\mu(x, \mathrm{d}v) \tag{A.2}$$

(d) *The conditions on $\alpha(x)$ and $\mu(x, \mathrm{d}v)$ are immediate consequences of (A.1). For the most general setting, see the corresponding result for* any *semimartingale, in [JS03, Proposition II.2.9].*

The following Lemma will be repeatedly used as a shortcut of Itô's formula and various identities that always apply with jump-diffusions.

Lemma A.2. *Let X be a jump-diffusion with differential χ-characteristics $(\beta^\chi, \alpha, \mu)$ and $f \in \mathbb{C}^2(\mathbb{V}, \mathbb{R})$. The composition $f(X)$ has the following semimartingale representation.*

$$\begin{aligned} f(X_t) = f(X_0) &+ \Big(\mathrm{D}f(X) \cdot \beta^\chi(X)\Big) \bullet \ell_t + \frac{1}{2} \operatorname{tr} \Big(\mathrm{D}^2 f(X) \circ \alpha(X)\Big) \bullet \ell_t + \mathrm{D}f(X_-) \bullet X^{\mathrm{c}} \\ &+ \Big(\mathrm{D}f(X_-) \cdot \chi\Big) * \tilde q^X_t + \Big(f(X_- + \mathrm{id}_{\mathbb{V}}) - f(X_-) - \mathrm{D}f(X_-) \cdot \chi\Big) * q^X_t \end{aligned}$$

Proof. Our jump-diffusion X has the following canonical semimartingale representation (see [JS03, Theorem II.2.34] for definition).

$$\begin{aligned} X &= X_0 + \beta^\chi(X) \bullet \ell + X^{\mathrm{c}} + \chi * \tilde q^X + (\mathrm{id}_{\mathbb{V}} - \chi) * q^X \\ \langle X^{\mathrm{c},i}, X^{\mathrm{c},j} \rangle &= \alpha_{ij}(X) \bullet \ell \\ \hat q^X(\mathrm{d}s, \mathrm{d}v) &= \mu(X_s, \mathrm{d}v)\mathrm{d}s \end{aligned} \tag{A.3}$$

Now apply Itô's formula [JS03, Theorem I.4.57] and use the predictable covariation identity in (A.3) to get the following.

$$\begin{aligned} f(X_t) &= f(X_0) + \sum_{i=1}^d \mathrm{D}_i f(X_-) \bullet X^i_t + \frac{1}{2} \sum_{i,j=1}^d \mathrm{D}_{ij} f(X_-) \bullet \langle X^{\mathrm{c},i}, X^{\mathrm{c},j} \rangle_t \\ &\quad + \sum_{0<s\le t} \Big(f(X_s) - f(X_{s-}) - \sum_{i=1}^d \mathrm{D}f_i(X_{s-})\Delta X_s\Big) \\ &= f(X_0) + \mathrm{D}f(X_-) \bullet X_t + \frac{1}{2} \sum_{i,j=1}^d \mathrm{D}_{ij} f(X_-) \bullet \big(\alpha_{ij}(X) \bullet \ell\big)_t \\ &\quad + \Big(f(X_- + \mathrm{id}_{\mathbb{V}}) - f(X_-) - \mathrm{D}f(X_-) \cdot \mathrm{id}_{\mathbb{V}}\Big) * q^X_t \end{aligned}$$

Using the iterated stochastic integral formula [JS03, Remark I.4.37], we may simplify the above equation to the following.

$$\begin{aligned} f(X_t) = f(X_0) &+ \mathrm{D}f(X_-) \bullet X_t + \frac{1}{2} \operatorname{tr} \Big(\mathrm{D}_{ij} f(X_-) \circ \alpha(X)\Big) \bullet \ell_t \\ &+ \Big(f(X_- + \mathrm{id}_{\mathbb{V}}) - f(X_-) - \mathrm{D}f(X_-) \cdot \mathrm{id}_{\mathbb{V}}\Big) * q^X_t \end{aligned}$$

Now substitute our representation of (A.3) and repeat the iterated stochastic integral to get the following.

$$\begin{aligned}
f(X_t) &= f(X_0) + \mathrm{D}f(X_-) \bullet \big(X_0 + \beta^{\chi}(X) \bullet \ell + X^{\mathrm{c}} + \chi * \tilde{q}^X + (\mathrm{id}_{\mathbb{V}} - \chi) * q^X\big)_t \\
&\quad + \frac{1}{2} \operatorname{tr}\Big(\mathrm{D}^2 f(X_-) \circ \alpha(X)\Big) \bullet \ell_t + \Big(f(X_- + \mathrm{id}_{\mathbb{V}}) - f(X_-) - \mathrm{D}f(X_-) \cdot \mathrm{id}_{\mathbb{V}}\Big) * q_t^X \\
&= f(X_0) + \Big(\mathrm{D}f(X_-) \cdot \beta^{\chi}(X)\Big) \bullet \ell_t + \frac{1}{2} \operatorname{tr}\Big(\mathrm{D}^2 f(X_-) \circ \alpha(X)\Big) \bullet \ell_t + \mathrm{D}f(X_-) \bullet X^{\mathrm{c}} \\
&\quad + \mathrm{D}f(X_-) \bullet \big(\chi * \tilde{q}^X\big)_t + \Big(f(X_- + \mathrm{id}_{\mathbb{V}}) - f(X_-) - \mathrm{D}f(X_-) \cdot \chi\Big) * q_t^X
\end{aligned}$$

Furthermore, since $X_- = X$ on all but a countable amount of jumps, we may rewrite the Lebesgue integrals.

$$\begin{aligned}
f(X_t) &= f(X_0) + \Big(\mathrm{D}f(X) \cdot \beta^{\chi}(X)\Big) \bullet \ell_t + \frac{1}{2} \operatorname{tr}\Big(\mathrm{D}^2 f(X) \circ \alpha(X)\Big) \bullet \ell_t + \mathrm{D}f(X_-) \bullet X^{\mathrm{c}} \\
&\quad + \mathrm{D}f(X_-) \bullet \big(\chi * \tilde{q}^X\big)_t + \Big(f(X_- + \mathrm{id}_{\mathbb{V}}) - f(X_-) - \mathrm{D}f(X_-) \cdot \chi\Big) * q_t^X
\end{aligned} \tag{A.4}$$

For the remaining equality, we construct localizing sequence $(T_n)_{n\in\mathbb{N}}$ of $\mathscr{F}$ stopping times,

$$T_n(\omega) \coloneqq \inf\big\{t > 0 : X_t(\omega) > n\big\} \wedge n, \quad \omega \in \Omega,\ n \in \mathbb{N}, \tag{A.5}$$

to see that $\mathrm{D}f(X_-)$ is $(\mathrm{P}, \mathscr{F})$ locally bounded.

$$\big|\mathrm{D}f(X_{s-}^{T_n})\big| \le \sup_{|x|\le n} \big|\mathrm{D}f(x)\big|$$

Thus, by [JS03, Proposition II.1.30], we may rewrite the following.

$$\mathrm{D}f(X_-) \bullet \big(\chi * \tilde{q}^X\big)_t = \big(\mathrm{D}f(X_-) \cdot \chi\big) * \tilde{q}_t^X,$$

which when substituted into (A.4) gives us our desired identity.

In the above lemma, the final term in the semimartingale decomposition of $f(X)$ is typically not able to be compensated into a local martingale. If we did have local integrability of the following quantity,

$$\Big|f(X_- + \mathrm{id}_{\mathbb{V}}) - f(X_-) + \mathrm{D}f(X_-) \cdot \chi\Big| * \hat{q}^X,$$

then by [JS03, Proposition II.1.28] we could rewrite $f(X)$ into a canonical special semimartingale decomposition.

$$\begin{aligned}
f(X_t) &= f(X_0) + \mathcal{L}f(X) \bullet \ell_t + \mathrm{D}f(X_-) \bullet X^{\mathrm{c}} + \big(f(X_- + \mathrm{id}_{\mathbb{V}}) - f(X_-)\big) * \tilde{q}_t^X \\
\mathcal{L}f(x) &\coloneqq \mathrm{D}f(x) \cdot \beta^{\chi}(x) + \frac{1}{2} \operatorname{tr}\Big(\mathrm{D}^2 f(x) \circ \alpha(x)\Big) \\
&\qquad + \int_{\mathbb{V}} \Big(f(x+v) - f(x) - \mathrm{D}f(x) \cdot \chi(v)\Big) \mu(x, \mathrm{d}v)
\end{aligned} \tag{A.6}$$

So long as f is bounded, we can guarantee this special semimartingale property.

Proposition A.3. *Let X and f as in Lemma A.2, and further impose f is bounded. Then the composition $f(X)$ is a special semimartingale with the decomposition as in (A.6).*

Proof. Seeing as f is bounded, [JS03, Lemma I.4.24] tells us that $f(X)$ is a special semimartingale. By [JS03, Proposition I.4.23], it is then the case that the following term is locally integrable.

$$\Big(f(X_- + \mathrm{id}_{\mathbb{V}}) - f(X_-) - \mathrm{D}f(X_-)\cdot\chi\Big) * q_t^X$$

By our discussion above, this suffices to conclude (A.6).

This operator $\mathcal{L}$ in (A.6) gives a nice closed from for suitable $f(X)$, and so we reserve it the term of *generator* associated with X. Note that we do not mark dependence on χ, as any other truncation function $\hat{\chi}$ will produce the same operator; see Remark A.1(c) and note that the displacement from β^χ and $\beta^{\hat{\chi}}$ would be the same as that in the integral term. One particular setting in which this result is useful is establishing a Lévy-Khintchine formula for jump-diffusions.

Proposition A.4. *Fix a jump-diffusion X with differential χ-characteristics $(\beta^\chi, \alpha, \mu)$. Then, for each $u \in \mathrm{i}\mathbb{V}$, the process $\exp\big(\langle u, X\rangle - \Lambda(u,X)\bullet\ell\big)$ is a complex-valued $(\mathrm{P},\mathscr{F})$ local martingale, where $\Lambda : \mathrm{i}\mathbb{V}\times\mathbb{X}\to\mathbb{R}$ is the associaed Lévy-Khintchine map.*

$$\Lambda(u,x) = \big\langle u, \beta^\chi(x)\big\rangle + \frac{1}{2}\big\langle u, \alpha(x)\big\rangle + \int_{\mathbb{V}} \big(e^{\langle u,v\rangle} - 1 - \langle u, \chi(v)\rangle\big)\mu(x,\mathrm{d}v),$$

Proof. For a fixed $u \in \mathrm{i}\mathbb{V}$, note that the map f_u, defined by $f_u(v) = \exp\langle u, v\rangle$ is bounded. Thus, by Proposition A.3, we have

$$f_u(X_t) = f_u(X_0) + \mathcal{L}f_u(X)\bullet\ell_t + M_t,$$

where M is a $(\mathrm{P},\mathscr{F})$ local martingale. Observe that the partial derivatives of f are as follows,

$$\mathrm{D}_i f_u(x) = f_u(x)u_i, \qquad \mathrm{D}_{ij} f_u(x) = f_u(x)u_iu_j, \tag{A.7}$$

so we have the following equation.

$$\begin{aligned}\mathcal{L}f_u(x) &= \mathrm{D}f_u(x)\cdot\beta^\chi(x) + \frac{1}{2}\operatorname{tr}\Big(\mathrm{D}^2 f_u(x)\circ\alpha(x)\Big)\\ &\qquad + \int_{\mathbb{V}}\Big(f_u(x+v) - f_u(x) - \mathrm{D}f(x)\cdot\chi(v)\Big)\mu(x,\mathrm{d}v)\\ &= f_u(x)\big\langle u, \beta^\chi(x)\big\rangle + \frac{1}{2}f_u(x)\big\langle u, \alpha(x)u\big\rangle + f_u(x)\int_{\mathbb{V}}\Big(f_u(v) - 1 - \big\langle u, \chi(v)\big\rangle\Big)\mu(x,\mathrm{d}v)\\ &= f_u(x)\cdot\Lambda(u,x)\end{aligned}$$

Denoting $A = f_u(X) = \exp\langle u, X\rangle$ and $B = \exp\big(-\Lambda(u,X)\bullet\ell\big)$, we now use the fact that B is $\mathscr{F}$ predictable and of finite-variation, so [JS03, Proposition I.4.49(b)] gives us the following.

$$\begin{aligned}&\exp\Big(\langle u, X\rangle - \Lambda(u,X)\bullet\ell\Big)\\ &= A_tB_t\\ &= A_0B_0 + A_-\bullet B_t + B\bullet A_t\end{aligned}$$

$$
\begin{aligned}
&= \exp\langle u, X_0\rangle + A_- \bullet \Big(\big(-B\cdot\Lambda(u,X)\big)\bullet\ell\Big)_t + B\bullet\Big(f_u(X_0)+\mathcal{L}f_u(X)\bullet\ell + M\Big)_t \\
&= \exp\langle u, X_0\rangle - \Big(A\cdot B\cdot\Lambda(u,X)\Big)\bullet\ell_t + \Big(B\cdot f_u(X)\cdot\Lambda(u,X)\Big)\bullet\ell_t + B\bullet M_t \\
&= \exp\langle u, X_0\rangle + B\bullet M_t
\end{aligned}
$$

This identity and [JS03, Remark I.4.34(b)] concludes the proof.

It turns out that each of the preceding results is sufficient in characterizing a semimartingale X as a jump-diffusion.

Theorem A.5. *The following statements are equivalent for a stochastic process X on state space $(\mathbb{X}, \mathscr{B}(\mathbb{X}))$.*

1. *X is a $(\mathrm{P}, \mathscr{F})$ jump-diffusion with differential χ-characteristics $(\beta^\chi, \alpha, \mu)$.*
2. *For each bounded $f \in \mathbb{C}^2(\mathbb{V}, \mathbb{R})$, the process $f(X_t) - \mathcal{L}f(X_t)\bullet\ell_t$ is a $(\mathrm{P}, \mathscr{F})$ local martingale, where*

$$\mathcal{L}f(x) \coloneqq \mathrm{D}f(x)\cdot\beta^\chi(x) + \frac{1}{2}\operatorname{tr}\Big(\mathrm{D}^2 f(x)\circ\alpha(x)\Big) + \int_{\mathbb{V}}\Big(f(x+v) - f(x) - \mathrm{D}f(x)\cdot\chi(v)\Big)\mu(x, \mathrm{d}v)$$

3. *For each $u \in \mathrm{i}\mathbb{V}$, the process $\exp\Big(\langle u, X\rangle - \Lambda(u,X)\bullet\ell\Big)$ is a $(\mathrm{P}, \mathscr{F})$ local martingale, where Λ is our Lévy-Khintchine map.*

$$\Lambda(u,x) = \big\langle u, \beta^\chi(x)\big\rangle + \frac{1}{2}\big\langle u, \alpha(x)\big\rangle + \int_{\mathbb{V}}\big(e^{\langle u,v\rangle} - 1 - \langle u, \chi(v)\rangle\big)\mu(x, \mathrm{d}v),$$

4. *Denoting $(\mathrm{P}_x)_{x\in\mathbb{X}}$ the P-conditional distributions of X factored through the initial state X_0 and selecting Borel functions σ, c to satisfy,*

$$
\begin{aligned}
\sigma &: \mathbb{X} \to \mathbb{L}(\mathbb{V}) \quad && \sigma\sigma^*(x) = \alpha(x) \\
c &: \mathbb{X}\times\mathbb{V} \to \mathbb{V} \quad && \mu(x, \Gamma) = \int 1_\Gamma\big(c(x,v)\big)\mathrm{d}v
\end{aligned}
$$

each P_x is a solution to the equation associated with a standard Brownian motion W and Poisson random measure p, where $\chi' = \mathrm{id}_{\mathbb{V}} - \chi$.

$$X_t = x + \beta^\chi(X)\bullet\ell_t + \sigma(X_-)\bullet W_t + \big(\chi\circ c(X_-, \mathrm{id}_{\mathbb{V}})\big) * \tilde{p}_t + \big(\chi'\circ c(X_-, \mathrm{id}_{\mathbb{V}})\big) * p_t$$

Proof. This is simply restating [JS03, Theorems II.2.42, II.2.49, and III.2.26] in terms of our identities from the previous propositions and lemmas. The choice of standard intensity $\mathrm{d}t\otimes\mathrm{d}v$ for the Poisson random measure is such that the jump factor $\mathrm{d}v$ satisfies the atomless and infinite properties in [JS03, Remark III.2.28(3)].

A.2 Special jump-diffusions

We now turn our focus to $(\mathrm{P}, \mathscr{F})$ jump-diffusions which are additionally *special* in the sense of them having a semimartingale decomposition in which the finite-variation term is predictable. When looking at the canonical representation of a jump-diffusion X with χ-characteristics $(\beta^\chi, \alpha, \mu)$, it is clear how to make this predictable.

$$\begin{aligned} X_t &= X_0 + \beta^\chi(X) \bullet \ell_t + X_t^{\mathrm{c}} + \chi * \tilde{q}^X + (\mathrm{id}_{\mathbb{V}} - \chi) * q^X \\ &= X_0 + \beta^\chi(X) \bullet \ell_t + (\mathrm{id}_{\mathbb{V}} - \chi) * \hat{q}^X + X_t^{\mathrm{c}} + \mathrm{id}_{\mathbb{V}} * \tilde{q}^X \\ &= X_0 + \left(\beta^\chi(X) + \int_{\mathbb{V}} \big(v - \chi(v)\big)\mu(X, \mathrm{d}v)\right) \bullet \ell_t + X_t^{\mathrm{c}} + \mathrm{id}_{\mathbb{V}} * \tilde{q}^X \end{aligned} \tag{A.8}$$

In such a case, it is nice to define the function $\beta : \mathbb{X} \to \mathbb{V}$,

$$\beta(x) := \beta^\chi(x) + \int_{\mathbb{V}} \big(v - \chi(v)\big)\mu(x, \mathrm{d}v), \tag{A.9}$$

so that (A.8) may be simplified to a concise special semimartingale decomposition.

$$X_t = X_0 + \beta(X) \bullet \ell + X^{\mathrm{c}} + \mathrm{id}_{\mathbb{V}} * \tilde{q}_t^X$$

We call the triplet (β, α, μ) that results from (A.9) the *special differential characteristics* and its components β, α, μ the *drift*, *diffusion*, and *jump kernel*, respectively.

The calculus of (A.8) begs the question that $(\mathrm{id}_{\mathbb{V}} - \chi) * q^X$ can be compensated which is not generally the case—otherwise, the term *special* would be a misnomer! The next result specifies conditions on which we may perform the above calculus.

Lemma A.6. *Let X be a $(\mathrm{P}, \mathscr{F})$ jump-diffusion with differential χ-characteristics $(\beta^\chi, \alpha, \mu)$, such that μ satisfies the following condition.*

$$x \mapsto \int_{\mathbb{V}} \big|v - \chi(v)\big|\mu(x, \mathrm{d}v) \text{ is bounded on compact subsets}$$

Then, X is special with drift β as in (A.9).

Proof. By choosing a $\mathscr{F}$ localizing sequence $(T_n)_{n\in\mathbb{N}}$ as in (A.5), our hypothsesis gives us the following integrability.

$$\mathrm{E}_{\mathrm{P}}\big|\mathrm{id}_{\mathbb{V}} - \chi\big| * \hat{q}_{T_n}^X = \mathrm{E}_{\mathrm{P}} \int_0^{T_n} \int_{\mathbb{V}} \big|v - \chi(v)\big|\mu(X_t, \mathrm{d}v)\mathrm{d}t \le n \cdot \sup_{|x|\le n} \int_{\mathbb{V}} \big|v - \chi(v)\big|\mu(x, \mathrm{d}v) < \infty$$

Now, [JS03, Proposition II.1.28] allows us to compensate as we did in (A.8)

Seeing as $(\mathrm{id}_{\mathbb{V}} - \chi) * q^X$ may be compensated for special jump-diffusions X, all of the characterizing objects of Theorem A.5 may be rewritten in terms of our drift β—effectively, χ becomes the identity.

$$\begin{gathered} \mathcal{L}f(x) := \mathrm{D}f(x) \cdot \beta(x) + \frac{1}{2}\operatorname{tr}\left(\mathrm{D}^2 f(x) \circ \alpha(x)\right) + \int_{\mathbb{V}} \Big(f(x+v) - f(x) - \mathrm{D}f(x) \cdot v\Big)\mu(x, \mathrm{d}v) \\ \Lambda(u, x) = \big\langle u, \beta(x)\big\rangle + \frac{1}{2}\big\langle u, \alpha(x)\big\rangle + \int_{\mathbb{V}} \big(\mathrm{e}^{\langle u, v\rangle} - 1 - \langle u, v\rangle\big)\mu(x, \mathrm{d}v), \\ X_t = x + \beta(X) \bullet \ell_t + \sigma(X_-) \bullet W_t + c(X_-, \mathrm{id}_{\mathbb{V}}) * \tilde{p}_t \end{gathered}$$

A.3 Locally countable jump-diffusions

We see that a $(\mathrm{P}, \mathscr{F})$ jump-diffusion X is special if the accumulated large jumps $(\mathrm{id}_{\mathbb{V}} - \chi) * q^X$ may be compensated. To this end, being special is a condition on the jumps *away* from the origin. We now turn our focus to jump-diffusions X in which the jumps *near* the origin behave nicely. For any jump-diffusion X, we may count the jumps with the jump process N^X.

$$N_t^X \coloneqq \sum_{0<s\leq t} 1_{\Delta X_s \neq 0} = 1 * q_t^X \tag{A.10}$$

For many jump diffusions, it may be the case that we P-almost-surely have $N_t^X = \infty$ for all $t > 0$. We way that X has $(\mathrm{P}, \mathscr{F})$ *locally countable*, so long as N^X is $(\mathrm{P}, \mathscr{F})$ locally integrable. Below, we state how to verify this using the differential characteristics.

Lemma A.7. *Fix a $(\mathrm{P}, \mathscr{F})$ jump-diffusion X with differential χ-characteristics $(\beta^\chi, \alpha, \mu)$ satisfying*

$$x \mapsto \mu\big(x, \mathbb{V}\big) \text{ is bounded on compact sets,}$$

then X is locally countable. Moreover, we may define $\lambda : \mathbb{X} \to \mathbb{R}_+$ and probability kernel $\kappa : \mathbb{X} \times \mathscr{B}(\mathbb{V}) \to [0, 1]$ by the following factoring.

$$\lambda(x) \coloneqq \mu(x, \mathbb{V}), \quad \mu(x, \mathrm{d}v) = \lambda(x)\kappa(x, \mathrm{d}v)$$

Also, N has $(\mathrm{P}, \mathscr{F})$ intensity $\lambda(X)$.

Proof. Select the sequence $(T_n)_{n\in\mathbb{N}}$ as in (A.5). Note now that, since the constant function 1 is predictable,

$$\mathrm{E_P} N_{T_n}^X = \mathrm{E_P} 1 * q_{T_n}^X = \mathrm{E_P} 1 * \hat{q}_{T_n}^X = \mathrm{E_P} \int_0^{T_n} \mu(X_t, \mathbb{V})\mathrm{d}t \leq n \cdot \sup_{|x|\leq n} \mu(x, \mathbb{V}) < \infty$$

This means that N^X is locally integrable, making X locally countable. Moreover, by [JS03, Theorem II.1.8],

$$N^X - \int_0^t \lambda(X_s)\mathrm{d}s = 1 * q^X - \int_0^t \int_{\mathbb{V}} \mu(X_s, \mathrm{d}v)\mathrm{d}s = 1 * q^X - 1 * \hat{q}^X$$

is a $(\mathrm{P}, \mathscr{F})$ local martingale, which finishes the proof.

A.4 Real moments of jump-diffusions

We now turn our focus to the real moments of $(\mathrm{P}, \mathscr{F})$ jump-diffusions and the extension of our Lévy-Khintchine map Λ to real moments.

$$\Lambda(u, x) = \big\langle u, \beta^\chi(x)\big\rangle + \frac{1}{2}\big\langle u, \alpha(x)\big\rangle + \int_{\mathbb{V}} \big(e^{\langle u,v\rangle} - 1 - \langle u, \chi(v)\rangle\big)\mu(x, \mathrm{d}v), \quad u \in \mathbb{V},\ x \in \mathbb{X}$$

The above expression may be infinite, as the final term includes an unbounded integral over a possibly infinite measure. That said, we find it imperative to denote the following sets of finiteness.

$$\mathcal{D}_\Lambda(x) \coloneqq \Big\{u \in \mathbb{V} : \Lambda(u, x) < \infty\Big\}, \qquad \mathcal{D}_\Lambda \coloneqq \bigcap_{x\in\mathbb{X}} \mathcal{D}_\Lambda(x) \tag{A.11}$$

The following results will explore the nature of the maps $\Lambda(\cdot,x):\mathcal{D}_\Lambda(x)\to\mathbb{R}$ for fixed differentiable χ-characteristics (β^χ,α,μ), where our truncation function χ is defined by $\chi(v)=v1_{|v|\leq 1}$. Note that there is no loss of generality in selecting this truncation function, since they all evaluate Λ identically.

Lemma A.8. *For any $x\in\mathbb{X}$, we have $u\in\mathcal{D}_\Lambda(x)$ if and only if $\int_{|v|>1}e^{\langle u,v\rangle}\mu(x,\mathrm{d}v)<\infty$.*

Proof. To each $u,v\in\mathbb{V}$, Taylor's theorem gives us $\gamma_{u,v}\in[0,1]$ such that

$$e^{\langle u,v\rangle}=1+\langle u,v\rangle+\frac{1}{2}e^{\gamma_{u,v}\langle u,v\rangle}\langle u,v\rangle^2.$$

This allows us to see that, for each $x\in\mathbb{X}$, $\Lambda(u,x)$ and $\int_{|v|>1}e^{\langle u,v\rangle}\mu(x,\mathrm{d}v)$ differ by finite expressions.

$$\begin{aligned}
&\left|\Lambda(u,x)-\int_{|v|>1}e^{\langle u,v\rangle}\mu(x,\mathrm{d}v)\right|\\
&=\left|\langle u,\beta^\chi(x)\rangle+\frac{1}{2}\langle u,\alpha(x)u\rangle+\int_{|v|\leq 1}\big(e^{\langle u,v\rangle}-1-\langle u,v\rangle\big)\mu(x,\mathrm{d}v)-\int_{|v|>1}\mu(x,\mathrm{d}v)\right|\\
&\leq\left|\langle u,\beta^\chi(x)\rangle+\frac{1}{2}\langle u,\alpha(x)u\rangle\right|+\left|\int_{|v|\leq 1}\frac{1}{2}e^{\gamma_{u,v}\langle u,v\rangle}\langle u,v\rangle^2\mu(x,\mathrm{d}v)\right|+\int_{|v|>1}\mu(x,\mathrm{d}v)\\
&\leq\left|\langle u,\beta^\chi(x)\rangle+\frac{1}{2}\langle u,\alpha(x)u\rangle\right|+\left(\frac{1}{2}e^{|u|}+1\right)\int_{\mathbb{V}}\big(1\wedge|v|^2\big)\mu(x,\mathrm{d}v)
\end{aligned}$$

Thus, one can be defined as a finite displacement of the other.

Lemma A.9. *For each $x\in\mathbb{X}$, $\mathcal{D}_\Lambda(x)$ is convex.*

Proof. We use our characterization of $\mathcal{D}_\Lambda(x)$ from Lemma A.8. Let $u,u'\in\mathcal{D}_\Lambda(x)$, $\gamma\in(0,1)$, and use Hölder's inequality to see the following.

$$\begin{aligned}
&\int_{|v|>1}e^{\langle u'+\gamma(u-u'),v\rangle}\mu(x,\mathrm{d}v)\\
&\quad=\int_{|v|>1}\left|(e^{\langle u,v\rangle})^\gamma\cdot(e^{\langle u',v\rangle})^{1-\gamma}\right|\mu(x,\mathrm{d}v)\\
&\quad\leq\left(\int_{|v|>1}\left|(e^{\langle u,v\rangle})^\gamma\right|^{\frac{1}{\gamma}}\mu(x,\mathrm{d}v)\right)^\gamma\left(\int_{|v|>1}\left|(e^{\langle u',v\rangle})^{1-\gamma}\right|^{\frac{1}{1-\gamma}}\mu(x,\mathrm{d}v)\right)^{1-\gamma}\\
&\quad=\left(\int_{|v|>1}e^{\langle u,v\rangle}\mu(x,\mathrm{d}v)\right)^\gamma\left(\int_{|v|>1}e^{\langle u',v\rangle}\mu(x,\mathrm{d}v)\right)^{1-\gamma}\\
&\quad<\infty
\end{aligned}$$

An arbitrary convex combination now satisfies $\gamma u+(1-\gamma)u'\in\mathcal{D}_\Lambda(x)$.

Lemma A.10. *For each $x\in\mathbb{X}$, the map $\Lambda(\cdot,x)$ is continuously differentiable on $\mathcal{D}_\Lambda(x)^\circ$, with derivative $\mathrm{D}\Lambda(\cdot,x):\mathcal{D}_\Lambda(x)^\circ\to\mathbb{L}(\mathbb{V},\mathbb{R})$ as follows.*

$$\mathrm{D}\Lambda(u,x)w=\left\langle\beta^\chi(x)+\alpha(x)u+\int_{\mathbb{V}}\big(e^{\langle u,v\rangle}v-\chi(v)\big)\mu(x,\mathrm{d}v),w\right\rangle,\quad u\in\mathcal{D}_\Lambda(x)^\circ \tag{A.12}$$

Proof. Fix $x \in \mathbb{X}$, $u \in \mathcal{D}_\Lambda(x)^\circ$. Let $\epsilon > 0$ such that $B(u,\epsilon) \subseteq \mathcal{D}_\Lambda(x)$. For all $0 < \delta < \epsilon$ and $i = 1, \ldots, d$, we now have the following identity

$$\begin{aligned} \frac{\Lambda(u+\delta e_i, x) - \Lambda(u,x)}{\delta} &= \langle e_i, \beta^\chi(x)\rangle + \langle e_i, \alpha(x)u\rangle + \frac{1}{2}\langle \delta e_i, \alpha(x)u\rangle \\ &\quad + \int_{|v|\le 1} \frac{1}{\delta}\Big(e^{\langle u+\delta e_i, v\rangle} - e^{\langle u,v\rangle} - \langle \delta e_i, v\rangle\Big)\mu(x,\mathrm{d}v) \\ &\quad + \int_{|v|>1} \frac{1}{\delta}\Big(e^{\langle u+\delta e_i, v\rangle} - e^{\langle u,v\rangle}\Big)\mu(x,\mathrm{d}v) \end{aligned} \tag{A.13}$$

Evaluating the limit of (A.13) as $\delta \to 0$ is now a matter of exchanging the limit with integration; we will do this by using the dominated convergence theorem.

For the first integral, Taylor's theorem provides us $\gamma_0, \gamma_1 \in [0,1]$ such that the following hold.

$$\begin{aligned} e^{\langle u+\delta e_i, v\rangle} &= 1 + \langle u+\delta e_i, v\rangle + \frac{1}{2}\langle u+\delta e_i, v\rangle^2 e^{\gamma_0\langle u+\delta e_i, v\rangle} \\ e^{\langle u,v\rangle} &= 1 + \langle u,v\rangle + \frac{1}{2}\langle u,v\rangle^2 e^{\gamma_1\langle u,v\rangle} \end{aligned}$$

This shows us that, for all $0 < \delta < \epsilon$ and $|v| \le 1$,

$$\begin{aligned} \left|\frac{1}{\delta}\Big(e^{\langle u+\delta e_i, v\rangle} - e^{\langle u,v\rangle} - \langle \delta e_i, v\rangle\Big)\right| &= \left|\frac{1}{2}\langle u+\delta e_i, v\rangle^2 e^{\gamma_0\langle u+\delta e_i, v\rangle} + \frac{1}{2}\langle u,v\rangle^2 e^{\gamma_1\langle u,v\rangle}\right| \\ &\le \Big(\big(|u|+\epsilon\big)^2 e^{|u|+\epsilon}\Big)|v|^2. \end{aligned}$$

This dominating function is integrable,

$$\int_{|v|\le 1} \Big(\big(|u|+\epsilon\big)^2 e^{|u|+\epsilon}\Big)|v|^2\mu(x,\mathrm{d}v) \le \Big(\big(|u|+\epsilon\big)^2 e^{|u|+\epsilon}\Big)\int_{\mathbb{V}} (1 \wedge |v|^2)\mu(x,\mathrm{d}v) < \infty,$$

so we may apply the dominated convergence theorem.

$$\begin{aligned} &\lim_{\delta\to 0}\int_{|v|\le 1} \frac{1}{\delta}\Big(e^{\langle u+\delta e_i, v\rangle} - e^{\langle u,v\rangle} - \langle \delta e_i, v\rangle\Big)\mu(x,\mathrm{d}v) \\ &\quad = \int_{|v|\le 1} \lim_{\delta\to 0}\frac{1}{\delta}\Big(e^{\langle u+\delta e_i, v\rangle} - e^{\langle u,v\rangle} - \langle \delta e_i, v\rangle\Big)\mu(x,\mathrm{d}v) \\ &\quad = \int_{|v|\le 1} \Big(e^{\langle u,v\rangle}v_i - v_i\Big)\mu(x,\mathrm{d}v) \end{aligned} \tag{A.14}$$

For the second integral, we again use Taylor's theorem to establish for each $0 < \delta < \epsilon/2$, some $\gamma_\delta \in [0,\delta]$ such that

$$e^{\langle u+\delta e_i, v\rangle} = e^{\langle u,v\rangle} + \langle \delta e_i, v\rangle e^{\langle u+\gamma_\delta e_i, v\rangle}$$

This way, we have the following dominating function.

$$\left|\frac{1}{\delta}\Big(e^{\langle u+\delta e_i, v\rangle} - e^{\langle u,v\rangle}\Big)\right| \le \left|\langle e_i, v\rangle e^{\langle u+\gamma_\delta e_i, v\rangle}\right| \le |v_i| e^{\langle u,v\rangle + \epsilon|v_i|/2}$$

The claim is that this dominating function is integrable. To see this, first note that because we have the following limit,

$$\lim_{|v|\to\infty} \frac{|v_i| e^{\langle u,v\rangle+\epsilon|v_i|/2}}{e^{\langle u,v\rangle+2\epsilon|v_i|/3}} = \lim_{|v|\to\infty} \frac{|v_i|}{e^{\epsilon|v_i|/6}} = 0$$

There exists $M > 0$ such that for all $|v| > M$,

$$|v_i| e^{\langle u,v\rangle+\epsilon|v_i|/2} < e^{\langle u,v\rangle+2\epsilon|v_i|/3}.$$

We now see that

$$\begin{aligned}
&\int_{|v|>1} |v_i| e^{\langle u,v\rangle+\epsilon|v_i|/2}\mu(x,\mathrm{d}v) \\
&\quad= \int_{1<|v|\le M} |v_i| e^{\langle u,v\rangle+\epsilon|v_i|/2}\mu(x,\mathrm{d}v) + \int_{|v|>M} |v_i| e^{\langle u,v\rangle+\epsilon|v_i|/2}\mu(x,\mathrm{d}v) \\
&\quad\le \int_{1<|v|\le M} M e^{(|u|+\epsilon/2)M}\mu(x,\mathrm{d}v) + \int_{|v|>M} e^{\langle u,v\rangle+2\epsilon|v_i|/3}\mu(x,\mathrm{d}v) \\
&\quad\le M e^{(|u|+\epsilon/2)M} \int_{\mathbb{V}} (1\wedge|v|^2)\mu(x,\mathrm{d}v) + \sum_{\ell=0}^{1} \int_{|v|>1} e^{\langle u+2\epsilon e_i/3, v\rangle}\mu(x,\mathrm{d}v) \\
&\quad< \infty.
\end{aligned}$$

We again use the dominated convergence theorem to deduce the following.

$$\begin{aligned}
&\lim_{\delta\to0}\int_{|v|>1} \frac{1}{\delta}\Big(e^{\langle u+\delta e_i,v\rangle} - e^{\langle u,v\rangle}\Big)\mu(x,\mathrm{d}v) \\
&\quad= \int_{|v|>1} \lim_{\delta\to0}\frac{1}{\delta}\Big(e^{\langle u+\delta e_i,v\rangle} - e^{\langle u,v\rangle}\Big)\mu(x,\mathrm{d}v) \\
&\quad= \int_{|v|>1} e^{\langle u,v\rangle} v_i \mu(x,\mathrm{d}v)
\end{aligned} \tag{A.15}$$

Combining equations (A.13), (A.14), and (A.15) now yields our desired identity.

$$\mathrm{D}_i\Lambda(u,x) = \Big\langle e_i, \beta^{\chi}(x) + \alpha(x)u + \int_{\mathbb{V}} \Big(e^{\langle u,v\rangle} v - \chi(v)\Big)\mu(x,\mathrm{d}v)\Big\rangle$$

Continuity of $\mathrm{D}_i\Lambda(u,x)$ for $u \in \mathcal{D}_\Lambda(x)^\circ$ involves very similar dominated convergence theorem arguments as above. From here, it is clear that Λ is continuously differentiable with the form in (A.12).

As we have seen in Lemmas A.6 and A.7, if we have local boundedness of certain integrals of a jump kernel μ, we can leverage these to $(\mathrm{P},\mathscr{F})$ local conditions of the associated jump-diffusion X. Throughout the remainder of this section, we impose the following uniform-boundedness principle for the kernel μ.

$$\begin{aligned}
&\int_{\mathbb{V}} f(v)\mu(x,\mathrm{d}v) < \infty \text{ for all } x \in \mathbb{X} \\
&\implies \quad x \mapsto \int_{\mathbb{V}} f(v)\mu(x,\mathrm{d}v) \text{ bounded on compact sets}
\end{aligned} \tag{A.16}$$

With this assumption, we get some nice results on finite exponential moments of X.

Proposition A.11. *Fix a* $(\mathrm{P}, \mathscr{F})$ *jump-diffusion* X *with differential* χ*-characteristics* $(\beta^\chi, \alpha, \mu)$. *Suppose we have the regularity condition (A.16) above. If* $0 \in \mathcal{D}_\Lambda^\circ$, *then* X *is special.*

Proof. If $0 \in \mathcal{D}_\Lambda^\circ$, then there exists some $\delta > 0$ such that $\overline{B}(0,\delta) \subseteq \mathcal{D}_\Lambda$. Observe the following implication of this fact, for each $x \in \mathbb{X}$.

$$\begin{aligned}
\int_{\mathbb{V}} \big|v - \chi(v)\big| \mu(x, \mathrm{d}v) &= \int_{|v|>1} |v| \mu(x, \mathrm{d}v) \\
&\le \int_{|v|>1} \frac{\sqrt{d}}{\delta} \exp\Big(\frac{\delta |v|}{\sqrt{d}}\Big) \mu(x, \mathrm{d}v) \\
&\le \frac{\sqrt{d}}{\delta} \int_{|v|>1} \exp\Big(\max_{i=1}^{d} \max_{\ell=0}^{1} \big\langle (-1)^\ell \delta e^i, v \big\rangle\Big) \mu(x, \mathrm{d}v) \\
&\le \frac{\sqrt{d}}{\delta} \sum_{i=1}^{d} \sum_{\ell=0}^{1} \int_{|v|>1} \exp \big\langle (-1)^\ell \delta e^i, v \big\rangle \mu(x, \mathrm{d}v) \\
&< \infty
\end{aligned}$$

Our regularity condition (A.16) now allows us to apply Lemma A.6 to conclude X is special.

Proposition A.12. *Fix a* $(\mathrm{P}, \mathscr{F})$ *jump-diffusion* X *with differential* χ*-characteristics* $(\beta^\chi, \alpha, \mu)$. *Suppose we have the regularity condition (A.16) above. If* $u \in \mathcal{D}_\Lambda$, *then* $\exp\langle u, X\rangle$ *is special, and* $\exp\big(\langle u, X\rangle - \Lambda(u, X) \bullet \ell\big)$ *is a* $(\mathrm{P}, \mathscr{F})$ *local martingale.*

Proof. Using Lemma A.2 for the function $f_u(v) = \exp\langle u, v\rangle$ and its derivative identities as in (A.7), we get the following.

$$\begin{aligned}
\exp\langle u, X_t\rangle =\ & \exp\langle u, X_0\rangle + \exp\langle u, X_t\rangle \Big(\langle u, \beta^\chi(X)\rangle + \frac{1}{2}\langle u, \alpha(X) u\rangle\Big) \bullet \ell_t \\
& + \mathrm{D} f_u(X_-) \bullet X^{\mathrm{c}} + \Big(\exp\langle u, X_-\rangle \langle u, \chi\rangle\Big) * \tilde{q}_t^X \\
& + \exp\langle u, X_-\rangle \cdot \Big(\exp\langle u, \mathrm{id}_{\mathbb{V}}\rangle - 1 - \langle u, \chi\rangle\Big) * q^X
\end{aligned} \tag{A.17}$$

Note that localizing our final term on the sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times in (A.5), we get the following.

$$\begin{aligned}
& \mathrm{E}_{\mathrm{P}} \Big| \exp\langle u, X_-\rangle \Big(\exp\langle u, \mathrm{id}_{\mathbb{V}}\rangle - 1 - \langle u, \chi\rangle\Big)\Big| * \hat{q}^X_{T_n} \\
&= \mathrm{E}_{\mathrm{P}} \int_0^{T_n} \int_{\mathbb{V}} \Big| \exp\langle u, X_s\rangle \Big(\exp\langle u, v\rangle - 1 - \langle u, \chi(v)\rangle\Big)\Big| \mu(X_s, \mathrm{d}v) \mathrm{d}s \\
&\le n \cdot \sup_{|x|\le n} \Big(e^{\langle u, x\rangle} \int_{\mathbb{V}} \big| e^{\langle u, v\rangle} - 1 - \langle u, \chi(v)\rangle\big| \mu(x, \mathrm{d}v)\Big)
\end{aligned}$$

Seeing as $u \in \mathcal{D}_\Lambda$, the integral in the above quantity is finite, and so (A.16) gives us finiteness of the supremum. Using [JS03, Proposition II.1.28] now allows us to compensate the jump term in (A.17).

$$\exp\langle u, X_t\rangle = \exp\langle u, X_0\rangle + \Big(\exp\langle u, X_t\rangle \cdot \Lambda(u, X)\Big) \bullet \ell_t + \mathrm{D} f_u(X_-) \bullet X^{\mathrm{c}} + \Big(\exp\langle u, X_-\rangle \langle u, \chi\rangle\Big) * \tilde{q}_t^X$$

This is a representation of $\exp\langle u, X\rangle$ as an initial term, predictable term of finite variation, and a local martingale. Thus, it is a special semimartingale. From here, we may perform the product rule on $\exp\big(\langle u, X\rangle - \Lambda(u, X)\bullet\ell\big)$ as we did in Proposition A.4 to show that the process is a local martingale.

Theorem A.13. *Fix a $(\mathrm{P}, \mathscr{F})$ jump-diffusion X with differential χ-characteristics $(\beta^\chi, \alpha, \mu)$. Suppose we have the regularity condition (A.16) above and that $0 \in \mathcal{D}_\Lambda^\circ$. For each $h \in \mathbb{D}([0,\infty), \mathbb{V})$ of finite-variation with image contained in $\mathcal{D}_\Lambda$, the process $\exp(h\bullet X)$ is special and*

$$\exp\Big(h\bullet X - \Lambda(h, X)\bullet\ell\Big)$$

is a $(\mathrm{P}, \mathscr{F})$ local martingale.

Proof. We first note that Proposition A.11 allows us to conclude X is special. Perform Itô's formula [JS03, Theoerem I.4.57] in addition to its jump-diffusion variant in Lemma A.2 and various stochastic integral identities [JS03, Remarks I.4.36, I.4.37, Theorem I.4.40(d), Proposition II.1.30(b)].

$$\begin{aligned}
&\exp\big(h\bullet X_t\big)\\
&= \exp\big(h\bullet X_-\big)\bullet\big(h\bullet X\big)_t + \frac{1}{2}\exp\big(h\bullet X_-\big)\bullet\big\langle (h\bullet X)^{\mathrm{c}}, (h\bullet X)^{\mathrm{c}}\big\rangle_t\\
&\quad + \sum_{0<s\le t}\Big(\exp\big(h\bullet X_{s-} + \Delta\big(h\bullet X\big)_s\big) - \exp\big(h\bullet X_{s-}\big) - \exp\big(h\bullet X_{s-}\big)\Delta\big(h\bullet X\big)_s\Big)\\
&= \Big(\exp\big(h\bullet X_-\big)\cdot h\Big)\bullet X_t + \frac{1}{2}\exp\big(h\bullet X\big)\big\langle h, \alpha(X)h\big\rangle\bullet\ell_t\\
&\quad + \exp\big(h\bullet X_-\big)\Big(e^{\langle h, \mathrm{id}_{\mathbb{V}}\rangle} - 1 - \langle h, \mathrm{id}_{\mathbb{V}}\rangle\big) * q_t^X
\end{aligned}$$

$$\begin{aligned}
&= \Big(\exp\big(h\bullet X\big)\cdot\langle h, \beta\rangle + \frac{1}{2}\exp\big(h\bullet X\big)\big\langle h, \alpha(X)h\big\rangle\Big)\bullet\ell_t + \Big(\exp\big(h\bullet X_-\big)\cdot h\Big)\bullet X_t^{\mathrm{c}}\\
&\quad + \exp\big(h\bullet X_-\big)\langle h, \mathrm{id}_{\mathbb{V}}\rangle * \tilde{q}_t^X\\
&\quad + \exp\big(h\bullet X_-\big)\Big(e^{\langle h, \mathrm{id}_{\mathbb{V}}\rangle} - 1 - \langle h, \mathrm{id}_{\mathbb{V}}\rangle\Big) * q_t^X
\end{aligned} \tag{A.18}$$

Now, choosing our $(\mathrm{P}, \mathscr{F})$ localizing sequence $(T_n)_{n\in\mathbb{N}}$ as in A.5, we have the following bound.

$$\begin{aligned}
&\mathrm{E}_{\mathrm{P}}\Big|\exp\big(h\bullet X_-\big)\Big(e^{\langle h, \mathrm{id}_{\mathbb{V}}\rangle} - 1 - \langle h, \mathrm{id}_{\mathbb{V}}\rangle\Big) * \hat{q}_{T_n}^X\Big|\\
&= \mathrm{E}_{\mathrm{P}}\int_0^{T_n}\int_{\mathbb{V}}\Big|\exp\big(h\bullet X_s\big)\Big(e^{\langle h(s), v\rangle} - 1 - \langle h(s), v\rangle\Big)\Big|\mu(X_s, \mathrm{d}v)\mathrm{d}s\\
&\le n\cdot\sup_{|x|\le n}\sup_{s\in[0,n]} e^{|x|\cdot|h(s)|}\int_{\mathbb{V}}\big|e^{\langle h(s), v\rangle} - 1 - \langle h(s), v\rangle\big|\mu(x, \mathrm{d}v)
\end{aligned}$$

Seeing as $\Lambda(\cdot, x)$ is continuously differentiable, it is uniformly bounded on $\mathcal{D}_\Lambda^\circ$. This, along with assumption (A.16) allow us to conclude that the preceding expression is finite. Thus, we may compensate the final jump integral in (A.18).

$$\begin{aligned}
\exp\big(h\bullet X_t\big) &= \Big(\exp\big(h\bullet X\big)\cdot\Lambda(h, X)\Big)\bullet\ell_t + \Big(\exp\big(h\bullet X_-\big)\cdot h\Big)\bullet X_t^{\mathrm{c}}\\
&\quad + \exp\big(h\bullet X_-\big)\Big(e^{\langle h, \mathrm{id}_{\mathbb{V}}\rangle} - 1\Big) * q_t^X
\end{aligned}$$

The decomposition of $\exp(h \bullet X)$ into a predictable finite-variation process and a local martingale implies that it is special. Now, we write M as the local martingale term above, $A = \exp(h \bullet X)$, and $B = \exp(-\Lambda(h, X) \bullet \ell)$. We now recognize that B is predictable and finite-variation and use [JS03, Proposition I.4.49(b)] to conclude our proof.

$$\begin{aligned}
\exp\big(h \bullet X_t - \Lambda(h, X) \bullet \ell_t\big) &= A_t B_t \\
&= A_- \bullet B_t + B \bullet A_t \\
&= \big(A \cdot B \cdot -\Lambda(h, X)\big) \bullet \ell_t + B \bullet \Big(\big(\exp(h \bullet X) \cdot \Lambda(h, X)\big) \bullet \ell + M\Big)_t \\
&= \big(A \cdot B \cdot -\Lambda(h, X)\big) \bullet \ell_t + \big(B \cdot A \cdot \Lambda(h, X)\big) \bullet \ell_t + B \bullet M_t \\
&= B \bullet M_t
\end{aligned}$$

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