MATH 453 HW 09

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1. Sec 4.1 #1
   (a) $\binom{16}{10,4,2} = \binom{16}{10} \times \binom{6}{4} \times \binom{2}{2} = 120120$
   (b) Suppose there's 0 loss, then we have in total 12 games with 10 wins and 2 ties. There are $\binom{12}{10,2} = 66$ ways to finish these 12 games in this fashion. Then we insert the 4 losses into these 12 games such that none of these 4 losses are adjacent. There are 13 spaces, therefore, there are $\binom{13}{4} = 715$ ways to insert them. Thus, there are $66 \times 715 = 47190$ ways in part (a) that do not have consecutive losses
   (c) Think of those 6 wins as a single win, because they must be adjacent. Then the question becomes: How many ways are there to arrange the list
   $$W^6WWWWLLLLTT, \text{ where } W^6 \text{ denotes 6 consecutive wins}$$
   such that W^6 is not next to any of the W's. Let's arrange W^6 and $LLLLTT$ first, there are $\binom{7}{1,4,2} = 105$ ways to arrange them. Once this is done, we have $8 - 2 = 6$ spaces to insert the other 4 wins into the list. This is equivalent to distributing 4 identical objects to 6 distinct recipients, which has $\left(\binom{6}{4}\right) = 126$ ways. Thus, in total, there are $105 \times 126 = 13230$ ways

2. Sec 4.1 #2
   (a) Every letter except for I is unique. First, we arrange $DVSBLTY$, which has $7! = 5040$ ways. After this, we insert the 5 I's into the list. There are 8 empty spaces to insert them into. This is equivalent to distributing 5 identical objects to 8 distinct recipients, which has $\left(\binom{8}{5}\right) = 792$ ways. Thus, in total, there are $5040 \times 792 = 3991680$ ways
   (b) First, there are 5040 ways to arrange $DVSBLTY$. Then we have 8 spaces to insert I's into the list, since no 2 I's is adjacent, there are $\binom{8}{5} = 56$ ways to do so. Thus, in total, there are $5040 \times 56 = 282240$ such lists

3. Sec 4.1 #3
   First, there are $\binom{120}{105}$ ways to select students to house in this dorm. Then, housing these 105 students in these different dorms without yet signing them to rooms is equivalent to distributing 105 distinct objects to 42 identical recipients, each of whom

receives 2 objects, and 7 other identical recipients, each of whom receives 3 objects. There are:

$$\binom{105}{\underbrace{2,2,\dots,2}_{42},\underbrace{3,3,\dots,3}_{7}} \Big/ (42! \times 7!)$$

ways to do so. Therefore, in total, there are

$$\binom{120}{105}\binom{105}{\underbrace{2,2,\dots,2}_{42},\underbrace{3,3,\dots,3}_{7}} \Big/ (42! \times 7!)$$

$$= \binom{120}{105}\frac{105!}{(2!)^{42}\cdot(42!)\cdot(3!)^{7}\cdot(7!)}$$

ways to do so

4. Sec 4.1 #4
   First, since students are assigned their own rooms, in question 3, the 42 identical recipients and 7 other identical recipients become distinct. Therefore, there are $\binom{120}{105}\frac{105!}{(2!)^{42}\cdot(3!)^{7}}$ ways to assign rooms to the 105 selected students. For the remaining 15 students, there are $\binom{15}{5,5,5} = \frac{15!}{(5!)^3}$ ways to arrange their rooms. Therefore, in total, there are $\binom{120}{105}\frac{105!}{(2!)^{42}\cdot(3!)^{7}}\cdot\frac{15!}{(5!)^3}$ to assign dorm rooms

5. Sec 4.1 #5
   Since its nest is at location $(12, 9, 10)$, there are in total $12 + 9 + 10 = 31$ steps it must take. This question is equivalent to distributing 31 distinct object to 3 distinct recipients such that the first receives 12 objects, the second receives 9 and the third receives 10. Therefore, there are $\binom{31}{12,9,10} = \frac{31!}{12!\cdot 9!\cdot 10!}$ ways for the mouse to travel to its nest

6. Sec 4.1 #12
   Suppose we want to find out the number of $k -$ composition of n, where $1 \le k \le n$. This is equivalent to inserting $k - 1$ bars into a list of n identical objects. For example, when $n = 5$ and $k = 3$, one way to do so is ■|■■|■■. When there are n objects, we have $n - 1$ empty spaces to insert these $k - 1$ bars, therefore, there are $\binom{n-1}{k-1}$ ways to place these $k - 1$ bars. To sum over all possible k, there are in total $\sum_{k=1}^{n}\binom{n-1}{k-1}$ ways. Binomial coefficient tells us that $\sum_{k=1}^{n}\binom{n-1}{k-1} = 2^{n-1}$, therefore, there are 2^{n-1} compositions of n

7. Sec 4.1 #13

Coefficient of x^n in $\sqrt{1-8x} = \binom{\frac{1}{2}}{n}(-8)^n = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(-\frac{2n-1}{2}\right)(-8)^n}{n!} = \frac{(1)(3)(5)\ldots(2n-1)(-1)^n(-8)^n}{2^n n!} = \frac{(1)(3)(5)\ldots(2n-1)4^n}{n!} = \frac{(1)(3)(5)\ldots(2n-1)2^n\cdot 2^n}{n!}\cdot\frac{n!}{n!} = \frac{(2n)!}{(k!)^2}\cdot 2^n = 2^n\binom{2n}{n}$

8. Sec 4.1 #15

   To find out the a_n, we divide these n elements to 2 parts, the first k elements and the next $n-k$ elements. Where $1 \le k \le n-1$, because either part must have at least 1 element. Given that there are a_k ways to compute the product of the first k elements and there are a_{n-k} ways to compute the product of the next $n-k$ elements, we know that $a_n = a_k a_{n-k}$ when k is known. Then we sum over all possible k and get the recurrence $a_n = \sum_{k=1}^{n-1} a_k a_{n-k}$ for all $n \ge 2$ with $a_1 = a_2 = 1$, $a_3 = 2$ and $a_4 = 5$