MATH 453 HW 10

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1. Sec 4.2 #1
   (a) Let F_n denote the number of such:
   When $n = 0$, there's 1 such subset, which is $\{\emptyset\}$, thus, $F_0 = 1$
   When $n = 1$, there are 2 such subsets, they are $\{\emptyset\}$ and $\{1\}$, thus, $F_1 = 2$
   When $n \geq 2$, conditioning on whether n is in the subsets:
   (1) If n is not in the subsets, then there are $n - 1$ numbers left that we can form the subsets from, we know that there are F_{n-1} such subsets
   (2) If n is in the subsets, since no two consecutive integers are allowed, the number $n - 1$ cannot be in the subsets, hence, we have $n - 2$ numbers left to form the subsets from, therefore, there are F_{n-2} ways to do so

   Hence, when $n \geq 2$, there are $F_n = F_{n-1} + F_{n-2}$ such subsets

   Thus, the number of subsets of $[n]$ that do not contain consecutive integers are:

$$F_n = \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ F_{n-1} + F_{n-2} & n \geq 2 \end{cases}$$

   (b) Let F_n denote the number of such binary numbers:
   When $n = 0$, there's 1 such binary, thus, $F_0 = 1$,
   When $n = 1$, there are 2 such binary numbers, they are 0 and 1, thus, $F_1 = 2$,
   When $n \geq 2$, conditioning on whether the binary number ends with 0:
   (1) If the binary number ends with 0, then the $(n-1)^{th}$ digit cannot be 0, so, there are F_{n-2} such binary numbers
   (2) If the binary number doesn't end with 0 (ends with 1), then there are F_{n-1} such binary numbers

   Hence, when $n \geq 2$, there are $F_n = F_{n-1} + F_{n-2}$ such subsets

   Thus, the number of $n -$ digit binary numbers that do not contain adjacent 0's is:

$$F_n = \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ F_{n-1} + F_{n-2} & n \geq 2 \end{cases}$$

   (c) Let F_n denote the number of ways to climb a flight in this fashion:
   When $n = 0$, there's 1 way to climb a flight, do nothing, thus, $F_0 = 1$
   When $n = 1$, there's 1 way to climb a flight, thus, $F_1 = 1$
   When $n \geq 2$, conditioning on how to reach the n^{th} stair:

(1) Reach the n^{th} stair directly from the $(n-2)^{th}$ stair, then there are F_{n-2} ways to climb the first $n-2$ stairs

(2) Reach the n^{th} stair from the $(n-1)^{th}$ stair, then there are F_{n-1} ways to climb the first $n-1$ stairs

Hence, when $n \geq 2$, there are $F_n = F_{n-1} + F_{n-2}$ ways to climb a flight

Thus, the number of ways to climb a flight of n stairs is:

$$F_n = \begin{cases} 1 & n = 0 \\ 1 & n = 1 \\ F_{n-1} + F_{n-2} & n \geq 2 \end{cases}$$

2. Sec 4.2 #2

Conditioning on the number of 2-tiles at the end of the tiling:

(1) When n is even, let F_{2i}, $0 \leq i \leq \frac{n}{2}$, denote the number of ways to tile the board:

When $i = 0$, meaning there's 0 2-tile at the end, which means it ends with a 1-tile, therefore, there are F_{2i-1} ways to tile the remaining $2i$ boards

When $i = 1$, meaning there's 1 2-tile at the end, which means the third last board is a 1-tile, there are F_{2i-3} ways to tile the remaining $2i-2$ boards

When $i = 2$, meaning there are 2 2-tiles at the end, which means the fifth last board is a 1-tile, there are F_{2i-5} ways to tile the remaining $2i-4$ boards

$\vdots$

When $i = \frac{n}{2}$, meaning there are $\frac{n}{2}$ 2-tiles, there are $F_0 = 1$ ways to tile the remaining 0 board

Therefore, in total, there are $1 + F_1 + F_3 + \cdots + F_{2i-1}$ ways to tile the board

(2) When n is odd, let F_{2i+1}, $0 \leq i \leq \frac{n-1}{2}$, denote the number of ways to tile the board:

When $i = 0$, meaning there's 0 2-tile at the end, since $2i+1$ is an odd number, at least 1 1-tile will be used, therefore, there are $2i$ boards left with F_{2i} ways to tile them

When $i = 1$, meaning there's 1 2-tile, similarly, there are F_{2i-2} ways to tile the remaining $2i-1$ boards

When $i = 2$, meaning there are 2 2-tiles, there are F_{2i-4} ways to tile the remaining $2i-3$ boards

$\vdots$

When $i = \frac{n-1}{2}$, meaning there are $\frac{n-1}{2}$ 2-tiles, there are F_0 ways to tile the remaining 1 board

Therefore, in total, there are $F_{2i} + F_{2i-2} + F_{2i-4} + \cdots + F_0$ ways to tile the board

To sum up, we use $2n+1$ and $2n$ to denote odd number of boards and even number of boards, thus, there are $F_{2n} = 1 + \sum_{k=0}^{n-1} F_{2k+1}$ or $F_{2n+1} = \sum_{k=0}^{n} F_{2k}$ ways to tile the entire board, depending on the number of boards

3. Sec 4.2 #10
   (a) 3-bracelet: $\{\{1\},\{2\},\{3\}\}$, $\{\{1,2\},\{3\}\}$, $\{\{1\},\{2,3\}\}$, $\{\{3,1\},2\}$
   4-bracelet: $\{\{1\},\{2\},\{3\},\{4\}\}$, $\{\{1,2\},\{3\},\{4\}\}$, $\{\{1\},\{2,3\},\{4\}\}$, $\{\{1\},\{2\},\{3,4\}\}$, $\{\{4,1\},\{2\},\{3\}\}$, $\{\{1,2\},\{3,4\}\}$, $\{\{4,1\},\{2,3\}\}$
   5-bracelet: $\{\{1\},\{2\},\{3\},\{4\},\{5\}\}$, $\{\{1,2\},\{3\},\{4\},\{5\}\}$, $\{\{1\},\{2,3\},\{4\},\{5\}\}$, $\{\{1\},\{2\},\{3,4\},\{5\}\}$, $\{\{1\},\{2\},\{3\},\{4,5\}\}$, $\{\{5,1\},\{2\},\{3\},\{4\}\}$, $\{\{1,2\},\{3,4\},\{5\}\}$, $\{\{1\},\{2,3\},\{4,5\}\}$, $\{\{5,1\},\{2,3\},\{4\}\}$, $\{\{5,1\},\{2\},\{3,4\}\}$, $\{\{1,2\},\{3\},\{4,5\}\}$
   (b) Conditioning on whether the first square is alone by itself (covered by a 1-tile), or covered by a 2-tile with another square (square 2 or square n):
      (1) If it is alone by itself, then there are $n-1$ squares left with β_{n-1} ways to tile them
      (2) If it is not alone by itself, then there are $n-2$ squares left with β_{n-2} ways to tile them

   Therefore, in total, there are $\beta_n = \beta_{n-1} + \beta_{n-2}$ ways to tile the n-bracelets

   We define $\beta_0 = 2$, because there are 2 different 0-bracelet, closed and open, we define $\beta_1 = 1$, because there's only 1 way to tile a 1-bracelet as it cannot be closed

   (c) Answer 1: Since $L_0 = 2$, $L_1 = 1$ and $F_0 = F_1 = 1$, $L_2 = L_0 + L_1 > F_2 = F_0 + F_1$, therefore, $L_3 = L_1 + L_2 > F_3 = F_1 + F_2 \dots L_n = L_{n-2} + L_{n-1} > F_n = F_{n-2} + F_{n-1}$ because one of L_{n-2} and L_{n-1} is bigger than both F_{n-2} and F_{n-1}, and the other is at least equal, only when $n = 1$, $L_1 = F_1$ thus, $L_n \geq F_n$ for all $n \geq 0$

   Answer 2: $Lucas\ number$ is the number of ways to tile a n-bracelet, which is circular, the head and tail can be connect, while $Fibonacci\ number$ is the number of ways to tile a n-board, whose head and tail cannot to connect, therefore, for all $n \geq 2$, the number of ways to tile a circular n-bracelet is more than the number of ways to tile a n-board. When $n = 0$, $L_0 = 2$ and $F_0 = 1$, when $n = 1$, $L_1 = F_1 = 1$, therefore, for all $n \geq 0$, $L_n \geq F_n$

4. Sec 4.2 #11
   How many ways are there to tile a n-bracelet?
   Answer 1: There are L_n ways
   Answer 2: Conditioning on whether the bracelet is closed:
   (1) The bracelet is open. Then the question is equivalent to tiling a n-board, which has F_n ways of doing so
   (2) The bracelet is closed. Then the first and the n^{th} bracelets are covered by a 2-tile, which leaves us $n-2$ bracelets to tile, and there are F_{n-2} ways of doing so

   Therefore, there are in total $F_n + F_{n-2}$ ways to tile a n-bracelet. Thus, $L_n = F_n + F_{n-2}$

5. Sec 4.3 #1
   $3x^4 = 3S(4,0)(x)_0 + 3S(4,1)(x)_1 + 3S(4,2)(x)_2 + 3S(4,3)(x)_3 + 3S(4,4)(x)_4$
   $= 3(x)_1 + 21(x)_2 + 18(x)_3 + 3(x)_4$
   $-x^3 = -(x)_1 - 3(x)_2 - (x)_3$
   $4x = 4(x)_1$
   Therefore, $3x^4 - x^3 + 4x + 10 = 6(x)_1 + 18(x)_2 + 17(x)_3 + 3(x)_4$

6. Sec 4.3 #2
   $3(x)_4 = 3x(x-1)(x-2)(x-3)$
   $-12(x)_3 = -12x(x-1)(x-2)$
   $4(x)_1 = 4x$
   Therefore, $3(x)_4 - 12(x)_3 + 4(x)_1 - 17 = 3x^4 - 30x^3 + 69x^2 - 38x - 17$

7. Sec 4.3 #6
   Since $(x)^{(n)} = x(x+1)(x+2)\dots(x+n-1)$, $(-1)^n(x)^{(n)} = (-1)^n x(x+1)(x+2)\dots(x+n-1)$,
   and $(-x)_n = (-x)(-x-1)(-x-2)\dots(-x-n+1) = (-1)^n(x+1)(x+2)\dots(x+n-1)$,
   Therefore, for any $n \geq 0$, $(-x)_n = (-1)^n(x)^{(n)}$

8. Sec 4.3 #7
   Since
   $$s(n,k) = (-1)^{n+k}c(n,k)$$
   Multiply both sides by x^k and we get:
   $$s(n,k)x^k = (-1)^{n+k}c(n,k)x^k$$
   then sum over all $k \geq 0$:
   $$\sum_{k\geq 0} s(n,k)x^k = \sum_{k\geq 0}(-1)^{n+k}c(n,k)x^k$$
   we know that
   $$\sum_{k\geq 0} s(n,k)x^k = (x)_n$$
   therefore
   $$\sum_{k\geq 0}(-1)^{n+k}c(n,k)x^k = (x)_n$$
   Replace x with $-x$ and we get:
   $$\sum_{k\geq 0}(-1)^{n+k}c(n,k)(-x)^k = (-x)_n$$
   The left-hand side equals:
   $$\sum_{k\geq 0}(-1)^{n+k}c(n,k)(-1)^k(x)^k = \sum_{k\geq 0}(-1)^{n+2k}c(n,k)(x)^k = \sum_{k\geq 0}(-1)^n c(n,k)(x)^k$$
   Because $2k$ is an even number, it wouldn't have effect on the sign of the equation

Therefore, we get:

$$\sum_{k\geq 0}(-1)^n c(n,k)(x)^k = (-x)_n$$

In the previous question, we proved that

$$(-x)_n = (-1)^n (x)^{(n)}$$

Therefore

$$\sum_{k\geq 0}(-1)^n c(n,k)(x)^k = (-1)^n (x)^{(n)}$$

Cancel the constant $(-1)^n$ on both sides and eventually, we get:

$$\sum_{k\geq 0} c(n,k)(x)^k = (x)^{(n)}$$

9. Sec 4.3 #8

Using the $Binomial\ Theorem$, we know that

$$(1+x)^n = \sum_{k\geq 0}\binom{n}{k}x^k$$

And $Theorem$ 4.3.4 tells us that

$$x^k = \sum_{j=0}^{k} S(k,j)(x)_j$$

Therefore, we get

$$(1+x)^n = \sum_{k\geq 0}\binom{n}{k}\left(\sum_{j=0}^{k} S(k,j)(x)_j\right) = \sum_{k\geq 0}\sum_{j=0}^{k}\binom{n}{k}S(k,j)(x)_j$$

Then, we switch the order of summation to get:

$$(1+x)^n = \sum_{j\geq 0}\sum_{k\geq j}\binom{n}{k}S(k,j)(x)_j = \sum_{j\geq 0}(x)_j\sum_{k\geq j}\binom{n}{k}S(k,j)$$

Therefore, the coefficient of $(x)_j$ is:

$$a_k = \sum_{k\geq j}\binom{n}{k}S(k,j)$$