Math 453 HW11

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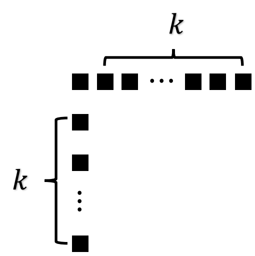
1. Sec 4.4 #2
   The idea is that $z_i - z_{i+1}$ equals the number of parts of size i in the conjugate, for $i = 1, 2, \dots, k$, and we define $z_{k+1} := 0$ therefore, the conjugate of this partition is $\sum_{i=0}^{k} i(z_i - z_{i+1})$

2. Sec 4.4 #3
   The statement is true. Let $n = 2k + 1$, where k is any integer, because n is odd. We can draw a self-conjugate Ferrers diagram like so:



   Therefore, there exists at least one partition that's self-conjugate

3. Sec 4.4 #4
   We know that
   $$P(n, 2) = P(n - 2, at\ most\ 2\ parts)$$
   Begin with Theorem 4.4.2 and we get
   $$P(m, at\ most\ 2\ parts) = P(m, largets\ part\ at\ most\ 2)$$
   Therefore, the OGF for the partitions of m with at most 2 parts is
   $$\sum_{m \geq 0} P(m, at\ most\ 2\ parts) x^m = \frac{1}{(1-x)(1-x^2)}$$
   The right-hand side can be written as
   $$\frac{1}{(1-x)(1-x^2)} = \frac{1}{(1-x)(1-x)(1+x)} = \frac{A}{(1-x)^2} + \frac{B}{1-x} + \frac{C}{1+x}$$
   Solve the equation and we get:
   $$\begin{cases} A = \frac{1}{2} \\ B = \frac{1}{4} \\ C = \frac{1}{4} \end{cases}$$
   Therefore, the decomposition makes right-hand side

$$\frac{1}{(1-x)(1-x^2)} = \frac{1/2}{(1-x)^2} + \frac{1/4}{1-x} + \frac{1/4}{1-(-x)}$$

Then, we seek the coefficient of x^m in

$$\frac{1}{2}\sum_{m\geq 0}\left(\binom{2}{m}\right)x^m + \frac{1}{4}\sum_{m\geq 0}\left(\binom{1}{m}\right)x^m + \frac{1}{4}\sum_{m\geq 0}x^{-m}$$

Therefore

$$P(m, at\ most\ 2\ parts) = \frac{1}{2}\left(\binom{2}{m}\right) + \frac{1}{4}\left(\binom{1}{m}\right) + \frac{(-1)^n}{4}$$

By simplifying the first 2 terms

$$\frac{1}{2}\left(\binom{2}{m}\right) + \frac{1}{4}\left(\binom{1}{m}\right) = \frac{2m+3}{4}$$

Therefore

$$P(n, at\ most\ 2\ parts) = \frac{2n+3+(-1)^n}{4}$$

Replace n with $n-2$ in this equation and we get

$$P(n-2, at\ most\ 2\ parts) = \frac{2(n-2)+3+(-1)^{n-2}}{4} = \frac{2n-1+(-1)^n}{4}$$

Since

$$P(n,2) = P(n-2, at\ most\ 2\ parts)$$

We conclude that

$$P(n,2) = \left\{\frac{2n-1+(-1)^n}{4}\right\}$$

4. Sec 5.2 #1

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 3 & 2 & 8 & 7 & 1 & 5 & 4 \end{pmatrix} = (1,6)(2,3)(4,8)(5,7)$$

Therefore, 4 cycles

5. Sec 5.2 #2

(1) $\binom{5}{1,4}\cdot 4! + \binom{5}{2,3}\cdot 2!\cdot 3! = 240$

(2) $\binom{5}{1,1,3}\cdot 3! + \binom{5}{1,2,2}\cdot 2!\cdot 2! = 60 + 400 = 460$

6. Sec 5.2 #3

(a) $\pi^{-1} = (1\ 5\ 3)(2)(4\ 6)$ and $\tau^{-1} = (1\ 2\ 3\ 4\ 5\ 6)$

(b) $\pi \circ \tau = (1\ 4\ 5\ 6)(2\ 3)$ and $\tau \circ \pi = (1\ 2)(3\ 4\ 5\ 6)$

(c) $\pi^{-1} \circ (\tau \circ \pi^2) = (1\ 6\ 3\ 4)(2\ 5)$

(d) $\pi^{-2} = (1\ 3\ 5)(2)(4)(6)$ and $\tau^{-3} = (1\ 4)(2\ 5)(3\ 6)$

7. Sec 5.2 #4

(1) $(R,\cdot)$ is not a group because not every element in has an inverse. For example, let $e = 1$, when $a = 0$, there does not exist an x that could make $a \cdot x = e$ or $x \cdot a = e$

(2) $(R,\cdot)$ is a group if 0 is not an element

1. Closure: for any $a, b \in G$, we have $a \cdot b \in G$
2. Associativity: For each $a, b, c \in G$, we have $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
3. Existence of an identity: There exists an element $e \in G$, which is 1, such that for each $a \in G$, we have $a \cdot e = a$ and $e \cdot a = a$
4. Existence of inverse: Let $e = 1 \in G$, for each $a \in G$, there exists $x \in G$ such that $a \cdot x = e$, where $x = \frac{1}{a}$, and $x \cdot a = e$

Therefore, without 0 in the set, $(R,\cdot)$ is a group

8. Sec 5.2 #9

motion	**product of disjoint cycles**
I	$(1)(2)(3)(4)(5)(6)(7)(8)(9)$
R_1	$(1\ 3\ 9\ 7)(2\ 6\ 8\ 4)(5)$
R_2	$(1\ 9)(2\ 8)(3\ 7)(4\ 6)(5)$
R_3	$(1\ 7\ 9\ 3)(2\ 4\ 8\ 6)(5)$
F_1	$(1)(2\ 4)(3\ 7)(5)(6\ 8)(9)$
F_2	$(1\ 3)(2)(4\ 6)(5)(7\ 9)(8)$
F_3	$(1\ 9)(2\ 6)(3)(4\ 8)(5)(7)$
F_4	$(1\ 7)(2\ 8)(3\ 9)(4)(5)(6)$

9. Sec 5.2 #12

(a) $\pi^2 = \pi \circ \pi = (1)(2)(3)(4)(5)$

$\pi^3 = \pi \circ \pi \circ \pi = (1\ 3\ 4)(2\ 5)$

(b) Symmetries of a pentagon:

motion	**two-line form**	**product of disjoint cycles**
I	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$	$(1)(2)(3)(4)(5)$
R_1	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$	$(1\ 2\ 3\ 4\ 5)$
R_2	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$	$(1\ 3\ 5\ 2\ 4)$
R_3	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$	$(1\ 4\ 2\ 5\ 3)$
R_4	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$	$(1\ 5\ 4\ 3\ 2)$
F_1	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}$	$(1)(2\ 5)(5\ 3)$
F_2	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$	$(1\ 3)(2)(4\ 5)$

F_3	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$	$(1\ 5)(2\ 4)(3)$
F_4	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}$	$(1\ 2)(3\ 5)(4)$
F_5	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 & 5 \end{pmatrix}$	$(1\ 4)(2\ 3)(5)$

Therefore, the group table for the symmetries of a pentagon is

$\circ$	I	R_1	R_2	R_3	R_4	F_1	F_2	F_3	F_4	F_5
I	I	R_1	R_2	R_3	R_4	F_1	F_2	F_3	F_4	F_5
R_1	R_1	R_2	R_3	R_4	I	F_4	F_5	F_1	F_2	F_3
R_2	R_2	R_3	R_4	I	R_1	F_2	F_3	F_4	F_5	F_1
R_3	R_3	R_4	I	R_1	R_2	F_5	F_1	F_2	F_3	F_4
R_4	R_4	I	R_1	R_2	R_3	F_3	F_4	F_5	F_1	F_2
F_1	F_1	F_3	F_5	F_2	F_4	I	R_3	R_1	R_4	R_2
F_2	F_2	F_4	F_1	F_3	F_5	R_2	I	R_3	R_1	R_4
F_3	F_3	F_5	F_2	F_4	F_1	R_4	R_2	I	R_3	R_1
F_4	F_4	F_1	F_3	F_5	F_2	R_1	R_4	R_2	I	R_3
F_5	F_5	F_2	F_4	F_1	F_3	R_3	R_1	R_4	R_2	I