MATH 453 HW 07

Name: Changyu Wu

CWID: A20337986

1. Sec 3.3 #1

$$\sum_{k\geq 0} a_k x^k = \frac{1}{(1-x^2)(1-x^3)(1-x^6)(1-x^7)(1-x^8)}$$

2. Sec 3.3 #2

(a) We need to find the coefficient of x^{14} in $(x+x^2)^{10}$
Since $(x+x^2)^{10} = x^{10}(1+x)^{10}$, the concise OGF is $x^{10}(1+x)^{10}$, therefore, it is equivalent to finding the coefficient of x^4 in $(1+x)^{10}$

(b) The coefficient of x^k in $\frac{1}{(1-x)(1-x^6)(1-x^{12})(1-x^{24})(1-x^{30})}$

(c) The coefficient of x^{75} in $\frac{1}{(1-x^3)(1-x^5)(1-x^{10})(1-x^{12})}$

(d) Let z_i be the number of candies of type i, where $1 \leq i \leq 5$, then we must have $z_1 + z_2 + z_3 + z_4 + z_5 = 24$ and $2 \leq z_i \leq 16$, thus, we want the coefficient of x^{24} in $(x^2 + x^3 + x^4 + \cdots + x^{16})^5$, since $x^2 + x^3 + x^4 + \cdots + x^{16} = x^2(1 + x + x^2 + \cdots + x^{14}) = x^2 \cdot \frac{1-x^{15}}{1-x}$, the concise OGF is $x^{10} \cdot \left(\frac{1-x^{15}}{1-x}\right)^5$, therefore, it is equivalent to finding the coefficient of x^{14} in $\left(\frac{1-x^{15}}{1-x}\right)^5$

(e) We need to find the coefficient of x^{15} in $(1+x+x^2+\cdots+x^8)^3$
Since $(1+x+x^2+\cdots+x^8) = \frac{1-x^9}{1-x}$, the concise OGF is $\left(\frac{1-x^9}{1-x}\right)^3$, therefore, we need to find the coefficient of x^{15} in $\left(\frac{1-x^9}{1-x}\right)^3$

(f) The coefficient of x^{100} in $\frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}$

3. Sec 3.3 #3

(a) x^{60} in $\frac{1}{(1-x)^{23}} = \left(\!\binom{23}{60}\!\right) \approx 5.0826 \times 10^{19}$

(b) $\frac{1+x+x^4}{(1-x^5)} = \frac{1}{(1-x^5)} + \frac{x}{(1-x^5)} + \frac{x^4}{(1-x^5)}$, therefore, the coefficient of x^k is $\left(\!\binom{5}{k}\!\right) + \left(\!\binom{5}{k-1}\!\right) + \left(\!\binom{5}{k-4}\!\right)$

(c) $\left(\!\binom{8}{3-1}\!\right) = 36$

(d) $(x^9+x^{10}+x^{11}+\cdots)^3 = x^{27}(1+x+x^2+\cdots)^3$, therefore, the coefficient of x^{50} in $(x^9+x^{10}+x^{11}+\cdots)^3$ is equal to the coefficient of $x^{50-27}=x^{23}$ in $(1+x+x^2+\cdots)^3$, which is $\left(\!\binom{3}{23}\!\right)$

(e) $\frac{1+x}{(1-2x)^5} = \frac{1}{(1-2x)^5} + \frac{x}{(1-2x)^5}$, substitute $2x$ with y and we get $\frac{1}{(1-y)^5} + \frac{1}{2}\cdot\frac{y}{(1-y)^5}$, where the coefficient of $y^{k-1} = \left(\!\binom{5}{k-1}\!\right) + \frac{1}{2}\cdot\left(\!\binom{5}{k-2}\!\right)$, then replace y with $2x$ and we get $(2x)^{k-1} = 2^{k-1}\cdot x^{k-1} = \left(\!\binom{5}{k-1}\!\right) + \frac{1}{2}\cdot\left(\!\binom{5}{k-2}\!\right)$, thus, the coefficient of x^{k-1} is $2^{1-k}\cdot\left(\!\binom{5}{k-1}\!\right) + 2^{-k}\cdot\left(\!\binom{5}{k-2}\!\right)$

4. Sec 3.3 #4

   We want to find the coefficient of x^k in $(1+x^2+x^4+x^5)^{20}$, since $(1+x+x^2+\cdots+x^5 = \frac{1-x^6}{1-x}$, $1+x^2+x^4+x^5 = \frac{1-x^6}{1-x} - x - x^3 = \frac{1-x^6-x+x^2-x^3+x^4}{1-x}$, therefore, the concise OGF is $\left(\frac{1-x^6-x+x^2-x^3+x^4}{1-x}\right)^{20}$, thus, it is equivalent to finding the coefficient of x^k in $\left(\frac{1-x^6-x+x^2-x^3+x^4}{1-x}\right)^{20}$

5. Sec 3.3 #6

   There are $(1+x)^5$ ways to distribute to these 5 adults and $(1+x+x^2+x^3+\cdots)^3$ ways to distribute to the 3 children, since we have 15 candies, we want to find the coefficient of x^{15} in $(1+x)^5(1+x+x^2+x^3+\cdots)^3$. Since $(1+x+x^2+x^3+\cdots)^3 = \frac{1}{(1-x)^3}$, we then need to find the coefficient of x^{15} in $\frac{(1+x)^5}{(1-x)^3}$, which is $\sum_{k=0}^{5}\binom{5}{k}\left(\!\binom{3}{15-k}\!\right) = 16472$

6. Sec 3.4 # 1

   We need to find the coefficient of x^{12} in $(x+x^2+x^3+x^4)^6$, since $x+x^2+x^3+x^4 = x(1+x+x^2+x^3) = x\cdot\frac{1-x^4}{1-x}$, the concise OGF is $x^6\left(\frac{1-x^4}{1-x}\right)^6$, therefore, it is equivalent to finding the coefficient of x^6 in $\left(\frac{1-x^4}{1-x}\right)^6$. $\left[\!\left[\left(\frac{1-x^4}{1-x}\right)^6\right]\!\right]_{x^6} = \left[\!\left[(1-x^4)^6\cdot\frac{1}{(1-x)^6}\right]\!\right]_{x^6} = \left[\!\left[\left(\sum_{j\geq 0}\binom{6}{j}(-x^4)^j\right)\left(\sum_{j\geq 0}\left(\!\binom{6}{j}\!\right)x^j\right)\right]\!\right]_{x^6} =$

$\left[\!\left[\left(\sum_{j\geq 0}\binom{6}{j}(-1)^j x^{4j}\right)\left(\sum_{j\geq 0}\left(\!\binom{6}{j}\!\right)x^j\right)\right]\!\right]_{x^6} = \left(\binom{6}{0}-\binom{6}{1}x^4+\binom{6}{2}x^8-\binom{6}{3}x^{12}+\cdots\right)\left(\sum_{k\geq 0}\left(\!\binom{6}{k}\!\right)x^k\right)$

Terms in each sum that contribute to the x^6 term:

term in first sum	term in second sum	resulting term in product
$\binom{6}{0}$	$\left(\!\binom{6}{6}\!\right)x^6$	$\binom{6}{0}\left(\!\binom{6}{6}\!\right)x^6$
$-\binom{6}{1}x^4$	$\left(\!\binom{6}{2}\!\right)x^2$	$-\binom{6}{1}\left(\!\binom{6}{2}\!\right)x^6$

So, the coefficient of x^6 is:

$\binom{6}{0}\left(\!\binom{6}{6}\!\right)-\binom{6}{1}\left(\!\binom{6}{2}\!\right)=336$

7. Sec 3.4 #2

Begin by equating the coefficient of x^k on the left-hand and right-hand sides. On the left,

$$\left[\!\left[\frac{1}{(1-x)^{m+n}}\right]\!\right]_{x^k}=\left(\!\binom{m+n}{k}\!\right)$$

On the right we know that

$$\frac{1}{(1-x)^m}\cdot\frac{1}{(1-x)^n}=\left(\sum_{j\geq 0}\left(\!\binom{m}{j}\!\right)x^j\right)\left(\sum_{j\geq 0}\left(\!\binom{n}{j}\!\right)x^j\right)$$

So, by the convolution formula

$$\left[\!\left[\frac{1}{(1-x)^m}\cdot\frac{1}{(1-x)^n}\right]\!\right]_{x^k}=\left(\sum_{j=0}^{k}\left(\!\binom{m}{j}\!\right)\right)\cdot\left(\sum_{j=0}^{k}\left(\!\binom{n}{j}\!\right)\right)$$

This proves the identity

$$\left(\!\binom{m+n}{k}\!\right)=\left(\sum_{j=0}^{k}\left(\!\binom{m}{j}\!\right)\right)\cdot\left(\sum_{j=0}^{k}\left(\!\binom{n}{j}\!\right)\right)$$

8. Sec 3.4 #3

Let $f(x)=(1+x)^n=a_0+a_1x+a_2x^2+\cdots+a_nx^n$

And $g(x)=\frac{1}{(1-x)^m}=b_0+b_1x+b_2x^2+\cdots+b_mx^m$,

therefore, by convolution formula, the coefficient of x^k is $\sum_{j=0}^{k}\binom{n}{j}\left(\!\binom{m}{k-j}\!\right)$

9. Sec 3.4 #4

How many ways are there to distribute k identical pieces of candy to n adults and m children such that each adult receives at most 1 piece of candy while each child can receive any number?

10. Sec 3.4 #6

Substitute x^j with y and we get $\frac{1}{(1-y)^n}$, where the coefficient of $y^m = \left(\!\binom{n}{m}\!\right)$, m is any integer at the interval $[0, n]$, then replace y with x^j and we get $x^{j \cdot m} = \left(\!\binom{n}{m}\!\right)$, therefore, for any number $k \in [0, nj]$ and $k = j \cdot m$, we get $m = \frac{k}{j}$, then replace m with $\frac{k}{j}$ and we get $x^k = \left(\!\binom{n}{\frac{k}{j}}\!\right)$