Math 453 HW 12

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1. Sec 5.3 #1
   $I(f) = black, black, white, red$
   $R_1(f) = red, black, black, white$
   $R_2(f) = white, red, black, black$
   $R_3(f) = black, white, red, black$
   $F_1(f) = black, red, white, black$
   $F_2(f) = white, black, black, red$
   $F_{1,2}(f) = black, black, red, white$
   $F_{2,3}(f) = red, white, black, black$

2. Sec 5.3 #2
   Apply CFB theorem and we get

$$|\mathcal{O}| = \frac{1}{|G|} \sum_{\pi \in G} |\mathrm{fix}_G(\pi)|$$

And the symmetry group is

motion	product of disjoint cycles	$\left|\mathbf{fix}_{D_9}(\boldsymbol{\pi})\right|$
I	$(1)(2)(3)(4)(5)(6)(7)(8)(9)$	k^9
R_1	$(1\ 3\ 9\ 7)(2\ 6\ 8\ 4)(5)$	k^3
R_2	$(1\ 9)(2\ 8)(3\ 7)(4\ 6)(5)$	k^5
R_3	$(1\ 7\ 9\ 3)(2\ 4\ 8\ 6)(5)$	k^3
F_1	$(1)(2\ 4)(3\ 7)(5)(6\ 8)(9)$	k^6
F_2	$(1\ 3)(2)(4\ 6)(5)(7\ 9)(8)$	k^6
F_3	$(1\ 9)(2\ 6)(3)(4\ 8)(5)(7)$	k^6
F_4	$(1\ 7)(2\ 8)(3\ 9)(4)(5)(6)$	k^6

Therefore, there're $\frac{1}{8}(k^9 + 4k^6 + k^5 + 2k^3)$ ways

3. Sec 5.3 #3
   (a) Let a be the edge between ball 1 and 2, b be the edge between ball 2 and 3, etc.

motion	product of disjoint cycles	$\left|\mathbf{fix}_{D_4}(\boldsymbol{\pi})\right|$
I	$(a)(b)(c)(d)$	2^4
R_1	$(a\ b\ c\ d)$	2^1
R_2	$(a\ c)(b\ d)$	2^2
R_3	$(a\ d\ c\ b)$	2^1
F_1	$(a)(b\ d)(c)$	2^3
F_2	$(a\ c)(b)(d)$	2^3

$F_{1,2}$	$(a\ b)(c\ d)$	2^2
$F_{2,3}$	$(a\ d)(b\ c)$	2^2

Apply CFB theorem and we get $\frac{1}{|G|}\sum_{\pi\in G}|\text{fix}_G(\pi)| = \frac{1}{8}(2^4 + 2\cdot 2^1 + 3\cdot 2^2 + 2\cdot 2^3) = 6$

(b) Let a be the edge between ball 1 and 2, b be the edge between ball 2 and 3, etc.

motion	**product of disjoint cycles**	$\lvert\mathbf{fix}_{D_4}(\boldsymbol{\pi})\rvert$
I	$(1)(2)(3)(4)(a)(b)(c)(d)$	2^8
R_1	$(1\ 2\ 3\ 4)(a\ b\ c\ d)$	2^2
R_2	$(1\ 3)(2\ 4)(a\ c)(b\ d)$	2^4
R_3	$(1\ 4\ 2\ 3)(a\ d\ c\ b)$	2^2
F_1	$(1)(2\ 4)(3)(a)(b\ d)(c)$	2^6
F_2	$(1\ 4)(2\ 3)(a\ c)(b)(d)$	2^5
$F_{1,2}$	$(1\ 3)(2\ 4)(a\ b)(c\ d)$	2^4
$F_{2,3}$	$(1)(2\ 4)(3)(a\ d)(b\ c)$	2^5

Apply CFB theorem and we get $\frac{1}{|G|}\sum_{\pi\in G}|\text{fix}_G(\pi)| = \frac{1}{8}(2^8 + 2\cdot 2^2 + 2\cdot 2^4 + 2^5 + 2^6) = 83$

4. Sec 5.3 #4

(1) Suppose $\pi(f) = f$ is true. By definition 5.3.1, the action of G on C^A is defined by

$$\big(\pi(f)\big)(a) \coloneqq f\big(\pi^{-1}(a)\big)$$

Replace $\pi(f)$ with f in the above equation and we get

$$f(a) \coloneqq f\big(\pi^{-1}(a)\big)$$

This means that

$$a = \pi^{-1}(a)$$

is true for all $a \in A$, which means that the value of a is irrelevant, then f must be constant

(2) Suppose f is constant. Again, by definition 5.3.1, we get

$$\big(\pi(f)\big)(a) \coloneqq f\big(\pi^{-1}(a)\big)$$

Since the value passed to function f will not affect the output of f, we can replace $\pi^{-1}(a)$ on the right-hand side of the equation with anything we want, for example, a, then the equation becomes

$$\big(\pi(f)\big)(a) \coloneqq f(a)$$

Therefore, $\pi(f) = f$

Combine (1) and (2), hence, theorem 5.3.6 is proven

5. Sec 5.3 #6

The symmetry group has size 2, hence, there are $\frac{1}{2}(2^8 + 2^4) = 136$ ways