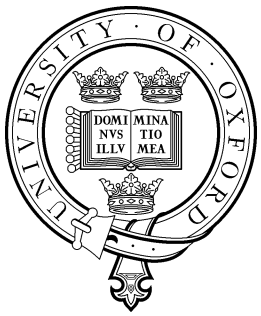
Network modelling of credit concentration risk



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Abstract

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Concentration risk is the risk arising from high exposure to single risk factors. Standard concentration measures rely mainly on the aggregation of the measures of each portfolio element at, for example the sectorial level (Lütkebohmert, 2008). They underestimate, however, direct economic links between the elements of the portfolio and therefore the underlying structure of the portfolio.

This thesis builds on previous work on concentration risk with a network model (Sindel, 2009), which used a varying parameter for progressively eliminating interdependencies between obligors, and analysed the exposure of the component with largest exposure to measure the portfolio's concentration risk. While this method is able to measure the properties of specific portfolios, the expected theoretical properties of the model have not been studied yet.

This thesis studies the theoretically expected distribution of the largest component in the sparsified correlation matrix by making a link between the varying parameter and the distribution function of the interobligor correlations, and interpreting these on the light of the random graph model. For this, it makes the assumptions that (1) the exposure is homogeneously distributed among the obligors and that (2) the correlations between obligors are statistically independent. Under these assumptions, we find that the distribution of the expected largest component is strongly dominated by the so-called giant component and therefore strongly dependent on the assumed distribution of the correlations.

Contents

Notation

Following notation will be used throughout this document:

k	the degree of a vertex
c	the mean degree of a graph
$\langle k \rangle$	the average degree of a graph
p_k	the probability that a randomly chosen vertex has degree k
S	the fraction of vertices that belong to the giant component of a graph
u	the fraction of vertices that do not belong to the giant component of a graph
ρ	the so-called ramping parameter, $\rho \in [0, 1]$

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Chapter 1

Introduction

Banking regulators (Basel II) require market participants to identify and monitor concentration risk in their credit portfolios. Concentration risk is the risk arising from high exposure to single risk factors. Standard concentration measures rely mainly on the aggregation of the measures of each portfolio element at, for example, the economic sector of activity (Lütkebohmert, 2008). While these models performs well at predefined aggregation levels, they usually underestimate direct obligor-specific economic links and, possibly, the latent dependency structure of the portfolio. The importance of structural measures has been confirmed by recent financial events, where large losses arising from worldwide contagion effects were registered (Kazi and Salloy, 2013).

Network models are especially apt at identifying structural properties (Newman, 2010), as they take not only model the elements of a system (the nodes or vertices) but also the relations between those elements (the edges). Network models are proving to be useful tools for providing early-warning signals of systemic risk (Squartini et al., 2013), measuring liquidity risk (Karas and Schoors, 2012), identifying sectors from time-series correlations (Onnela et al., 2004; Fenn et al., 2009; Fenn et al., 2011), as well as insights into finding diversified baskets of assets in the classical investment framework (Pozzi et al., 2013). Many other use cases are described in the recent book (Soramäki and Cook, 2016).

Previous work on concentration risk relied on a network intepretation of a portfolio (Sindel, 2009) by using a varying (ramping) parameter for progressively eliminating interdependencies between obligors, and analysed the relative exposure size of the component with largest exposure. Using a weight function, this relative exposure size is integrated over all parameter values in order to measure the portfolio's concentration risk.

While this method is able to measure properties of specific portfolios, little is hitherto know about the expected properties of the model for a given portfolios. The main goal of this thesis is to connect ramping parameter model with theoretical work behind random graphs. For this, we study the properties of the ramping parameter under the most widely theoretical graph model: the random graph model described by (Erdös and Rényi, 1959). Firstly, having the model inspected theoretically allows for a better understanding of its properties. Secondly, even though the theoretical models do not capture all the properties of real world graphs, it provides useful baselines for comparison with the results with real portfolios.

1.1 Credit concentration risk

Under Basel II (Basel, 2007), banks are required to compute the capital requirements for their credit portfolios using the ASFR (Asymptotic Single Risk Factor) (), and are allowed to use their own internal risk measurements - IRB approach. The ASRF model is based on the results from (Vasicek, 1991) depends on three main quantities:

PD the Probability of Default: the probability that a considered obligor will default within the next 12 months,

LGD Loss Given Default: the expected fractional loss in case of the obligor's default,

EAD Exposure At Default: the expected exposure an obligor has when the default event occurs as seen from today.

While we urge the reader to refer to (Sindel, 2009) for a more comprehensive description of the meaning of the quantities and how these are combined in the model, it is important to state that the ASRF model makes two important assumptions on the portfolios:

1. they are perfectly fine grained, thereby containing no idiosyncratic risk, i.e. no Concentration Risk

2. there is only one source of systematic risk.

In order to circumvent this limitation, the concentration risk, i.e. the probability of losses arising from an outstanding credit exposure to a given set of counterparties through some common risk factors, is accounted for by the so-called granularity adjustment (GA) (Lütkebohmert, 2008).

We outline the different strategies behind the existing methodologies for assessing the Concentration Risk

- concentration of absolute exposure, e.g. by means of the HHI (Herfindahl-Hischmann Index)
- a measurement of relative exposure concentrations, e.g. by using the Lorenz-Curve
- comparison with a given benchmark portfolio

As can be seen, the first two measures here rely solely on the exposure structure of the portfolio. The measure proposed by (Sindel, 2009) goes beyond this by relying on both the exposure- and the correlation-structure of the portfolio.

1.2 The ramping parameter model

The ramping parameter model was developed in (Sindel, 2009) and aims at assessing the concentration risk in real world loan portfolios. In contrast to other methods for studying concentration risk, the ramping parameter takes into account both, the exposure- and the dependency-structure of the portfolio. A particular strength of this methodology is that it can be used universally.

The ramping parameter monitors how the component structure - notably the component with largest exposure - of a graph representing the considered portfolio develops, as edges are removed increasingly according to their strength. We begin by defining some quantitites.

Definition 1.2.1. Portfolio A credit portfolio is a tuple (O, C), where O is a set of n exposures from n different obligors and C an $n \times n$ matrix. The C matrix represents the interdependency (or correlation) between each obligors i and j, and each element $c_{ij} \in [0, 1]$.

Remark. Throughout this work, the matrix C is symmetric, so that $\forall i, j c_{ij} = c_{ji}$, the values of $c_{ij} \in [0, 1]$ and $c_{ii} = 0$. Even though in the real world the dependency between obligors is not necessarily symmetric, e.g. in the case where obligor i is a subsidiary of obligor j, the symmetry assumption renders the model more analytically tractable.

The approach proposed by (Sindel, 2009) for quantifying the concentration risk of a loan portfolio works can be summarized as follows:

- The mutual dependency between the obligors in a portfolio is represented by a matrix C. This matrix will be referred to through the document as correlation or interdependency matrix. This matrix is symmetric, its values lie $c_{ij} \in [0, 1]$ [1].

- A so-called ramping parameter $\rho \in [0, 1]$ is used to define the effective correlation, or adjacency matrix $A(\rho)$ given the value of this parameter:

$$a_{ij}(\rho) = \begin{cases} 1 \text{ if } c_{ij} \geq \rho, \\ 0 \text{ otherwise.} \end{cases} \tag{1.1}$$

  At $\rho = 0$, the effective matrix will contain all connections in the original matrix. At $\rho = 1$, all obligors in the effective matrix A will be disconnected.

- The exposure-dependence is taken into account by computing the largest effective exposure for every value of ρ. In particular, the ratio $R(\rho)$ of the maximum of all exposures of all identified connected components C_k and the total exposure of the portfolio is computed. Concretely this means the following steps:

  i) compute the set of k connected components $C_1, \ldots, C_k$ of the graph G defined by the adjacency matrix $A(\rho)$ [2], which effectively is a partition of the set of obligors,

  ii) compute $R(\rho) = \max_{C_k} \left(\frac{\sum_{j \in C_k} \text{EAD}_j}{\sum_i \text{EAD}_i} \right)$, where C_k is a connected component

  $R(\rho)$ describes the risk of a portfolio associated with the default of its biggest clump for a given ρ

- The concentration risk CR is determined by considering the area under a curve weighted by $P(\rho)$, so that

$$CR = \int_0^1 R(\rho) P(\rho) d\rho \tag{1.2}$$

The original work experiments with several possible values for the weighted curve. In the context of this thesis, the consideration of a weighted curve is postponed for future work. Indeed, we see the results from this work as a possible contribution to determining a weighted curve that accounts for the expected distribution of $R(\rho)$.

[1] this thesis relaxes the interpretation and the domain of the values, but still follows the original definition

[2] A definition of these terms is provided in 2

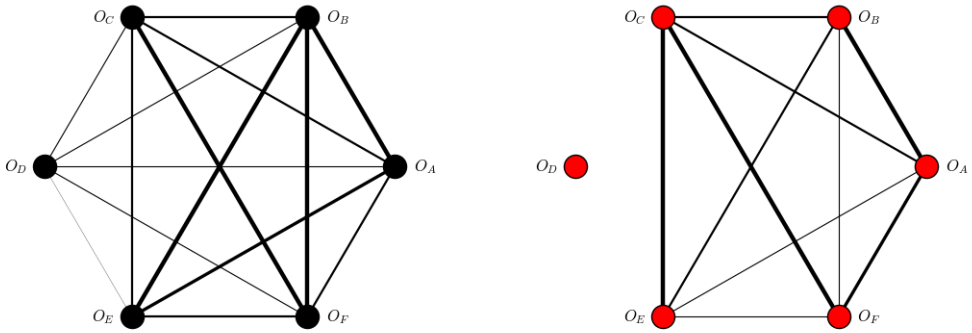


Figure 1.1: Stylized portfolio with six obligors (A, B, C, D, E, and F). The relative strength of the inter-obligor connections ρ_{ij} is illustrated by the line-width. In the left figure, the interdependency between customer D and the other customers is rather small. Additional to the individual inter-customer correlation each customer also has an individual obligation (symbolized by the differently coloured circles), indicated as $O_i,\ i \in \{A, B, C, D, E, F\}$. Both the obligation of the individual customer O_i and the inter-customer correlations ρ_{ij} are required to compute the Concentration Risk as suggested by our measure. For a particular value of the correlation strength obligor D becomes independent from all other obligors of the portfolio.

1.3 Random graph models

Random graph models are widely used to understand the properties of real world networks (Newman, 2002). They provide a means of generating and analysing networks with statistical properties that resemble those of real-world networks. The most popular random graph model is credited to (Erdös and Rényi, 1959). One reason for its popularity is the fact that it is very simple (and analytically tractable). Although its properties do not necessarily resemble those observed in social, financial or technical ecosystems, its dynamics already capture a very interesting phenomenon that is also relevant for this thesis: the emergence of a so-called giant component.

A connected component is a set of nodes, between which there is always a connecting path. The giant component is a connected component for which the fraction of nodes it contains remains constant as the network grows in size. It is also the largest component in the graph, and can be usually seen in many real world networks [3].

[3]For example, in the case of the internet network, the existence of a giant component is extremely important, as it guarantees that as much nodes (computers) as possible can communicate with each

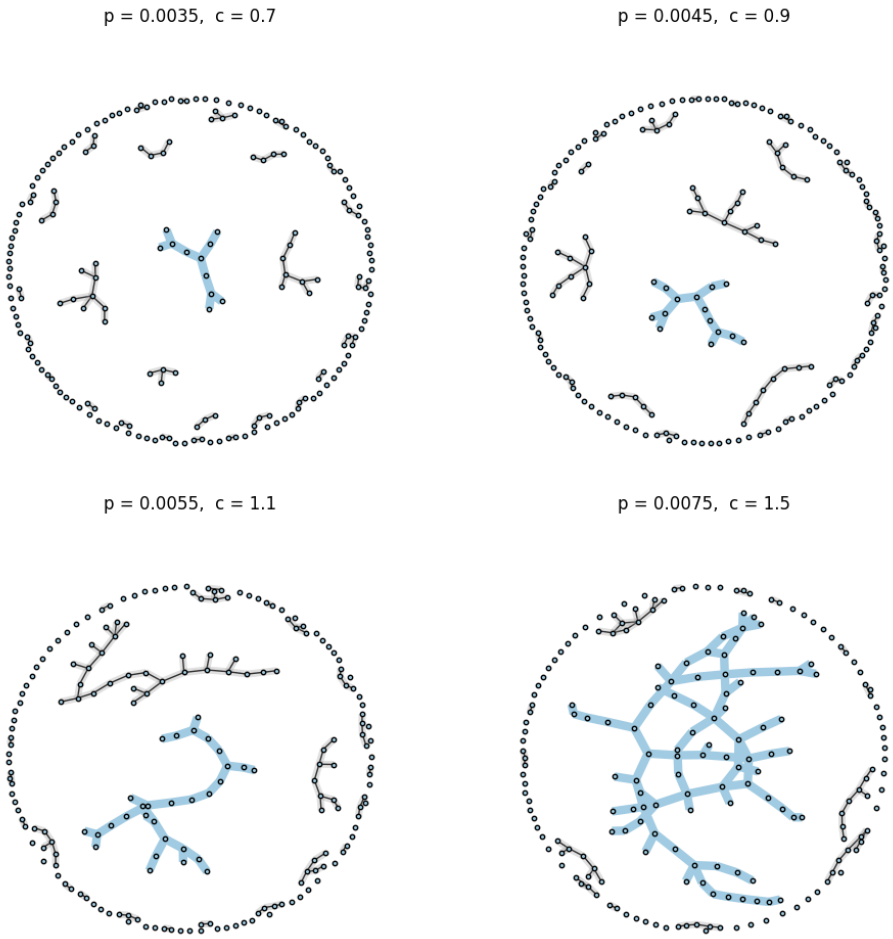


Figure 1.2: This figure illustrates the sudden appearance of a giant connected component (lower plots) in the random graph model. The graphs, each with $n = 200$ nodes are generated randomly, with a mean degree of c and an edge probability p. The random graph model predicts a giant component when $c \geq 1$

Not only its existence is of interest, but the emergence of the giant component is an important phenomenon. It emerges almost spontaneously as the edge density reaches a critical value, leading to a phase transition, as explained by (Newman, 2002). This can also be seen in Figure 1.2

One important property where the random graph model does not mimic the real world networks is the distribution of a node's degree. The degree is a property of the node in a graph that describes how many connections (or how important these connections are) the node has to the other nodes in the graph. Real-world networks

otgher

tend to have heavy tailed distributions, where as the random graph model has a binomial distribution / Poisson distribution. One of the these heavy tail distributions is the power law distribution, which is scale-free, as was and was shown by (Albert and Barabási, 2002) to lead to the so-called small world phenomenon (Watts and Strogatz, 1998).

1.4 Organisation of this document

This document is organized as follows. Chapter 2 describes the fundamental concepts of graph theory and of the random graph model. The random graph model provides a statistical description of the degree distribution and the component structure of random graphs, which may include both a giant component and a set of small components. The component structure is of high relevance for providing a connection to the ramping parameter model described in Section 1.2.

In chapter 3, we described expected properties of the ramping parameter in the case of random graphs. This connection, based on the expected component structure, provides useful information to better understand the behaviour of the ramping parameter model for the estimation of credit concentration risk in real-world portfolios.

We finalize in chapter 4 by providing a summary of the results of this thesis and a discussion of possible continuations of this work. Software code used throughout this thesis is provided in the appendix.

Chapter 2

Random graphs and networks

The networks studied in this document can be represented by a single graph. In general, the term *graph* is used for the mathematical object defined below and the term *network* for some model that relies on one (or more) graphs.

We begin by defining some terminology and proceed to described models of random graphs.

2.1 Terminology and essential results

Definition 2.1.1. A graph G is a tuple

$$G = (V, E), \tag{2.1}$$

where V is the set of vertices (or nodes), and E is the set of edges (or links). Each edge connects exactly one pair of vertices, and a vertex-pair can be connected by maximally one edge, i.e., multi-connections are not allowed. Let furthermore n denote the number of vertices $n = |V|$ and L the number of edges $L = |E|$.

In this thesis, the set V will correspond to the set of obligors in a portfolio, and E to the interdependencies between the obligors. In the case of a social network, for example, V would be the set of persons and E could mean that two persons (nodes) are acquainted (connected) with each other.

Definition 2.1.2. A weighted graph G is a graph where a number $x \in \mathbb{R}$ can be assigned to an element of E.

Definition 2.1.3. An unweighted graph G is a graph where no number is assigned to an element of E.

Definition 2.1.4. The adjacency matrix A of a graph $G = (V, E)$ is a matrix of size $n \times n$, such that each element a_{ij} of the matrix is given by

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge between } i \text{ and } j, \text{ i.e. } \{i, j\} \in E \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

If G is a weighted graph, then the

$$a_{ij} = \begin{cases} w_{ij} & \text{if there is an edge between } i \text{ and } j \text{ with weight } w_{ij} \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

Remark. Throughout this document we will be concerned with graphs without self-edges, so that $\forall i,\ a_{ii} = 0$.

Definition 2.1.5. An undirected graph G is a graph where its adjacency matrix A is symmetric, i.e. $\forall i, j,\ a_{ij} = a_{ji}$.

Definition 2.1.6. A connected component (or just component) of an undirected graph $G = (V, E)$ is a subgraph $G' = (V', E')$ where:

(i) V' and E' are subsets of V, E respectively

(ii) any two vertices $a, binV'$ are connected to each other by a sequence of edges $[e_1, \ldots, e_k]$ such that $e_1, \ldots, e_k \in E'$

(iii) given a vertex $a \in V'$, for all vertices b such that $a, b \in V'$, $b \in E'$.

Definition 2.1.7. The degree k_i of a node i in an undirected graph G is the sum of the elements of the i^{th} row (column) of its adjacency matrix A, i.e. $k_i = \sum_{j=1}^{j=n} a_{ij}$.

Remark. In case G is unweighted, then the degree of node i is equal to the number of edges that connect to that node. In case G is weighted, then it corresponds to the sum of the weights of the edges that connect to that node.

Definition 2.1.8. The degree matrix D is a graph G is an $n \times n$ matrix where its diagonal contains the degree of the corresponding vertices, i.e. where

$$d_{ij} = \begin{cases} k_i & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.1.9. The Laplacian [1] matrix L of a graph G is given by $L = D - A$, where D is the degree matrix and A the adjacency matrix, as defined above.

[1] Please note that there may be similar alternative definitions of the Laplacian matrix of a graph, but its properties equivalent w.r.t. the requirements for this thesis.

Remark. The graph Laplacian turns up in a large variety of places in network modelling, including diffusion processes, random walks, graph partitioning and, the most important for the purpose of this thesis, network connectivity.

There is, in fact, a connection between the eigenvectors of the Laplacian matrix of a graph and its connected components, which will allow us to easily compute the component structure of particular instances of graphs. In fact, a graph G with multiple connected components C_k will have a Laplacian matrix L that is a block diagonal matrix. After reordering of the vertices, each block in this matrix will be the corresponding Laplacian matrix for each component C_k.

Theorem 2.1.1. *Consider a graph G, with a Laplacian matrix L. Then,*

(i) all eigenvalues of L $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1} \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ are ≥ 0

(ii) $\lambda_j = 0 \iff$ the corresponding eigenvector defines a connected component.

Proof. The proof is left here undone, as it can be found in most books on networks, e.g. (Newman, 2010). □

Remark. In order to find the connected components of a graph, one can compute the null-eigenvalues and eigenvectors of its Laplacian matrix.

Example: We consider the following example of a graph which has following (rows

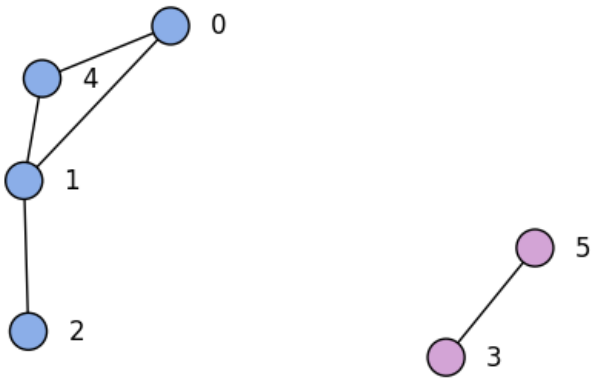


Figure 2.1: Example graph with 2 components and 6 nodes.

and columns 4,5 permutated) adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and Laplacian matrix

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

The block- structure of the Laplacian matrix is well visible and easy to correspond to the community structure in the plotted graph.

2.2 Random graphs

A random graph is a graph that is generated by randomly sampling from a collection of graphs, i.e. a random variable defined in a probability space with a probability distribution.

Definition 2.2.1. The Erdös-Rényi random graph $G_{n,p}$ is the random graph containing n nodes obtained by connecting pairs of vertices with an edge with probability p. Each edge exists independently with the same probability.

Under this model, the probability of any simple graph G, i.e. without multiple edges or self-loops, with n nodes and m edges is given by

$$P(G) = p^m(1-p)^{\binom{n}{2}-m}$$

This model was studied by the Hungarian mathematicians Paul Erdös (1913-1996) and Alfréd Rényi (1921-1970), and has lead to a large corpus of research on random graphs. As with other models studied in this thesis, statements made about the model are statements made about the collection of graphs, rather than any specific instance of a graph.

The random graph model is especially interesting because several of its statistical properties can be derived analytically. The properties of the networks it generates, however, differ substantially from those observed in real world networks (Albert and Barabási, 2002).

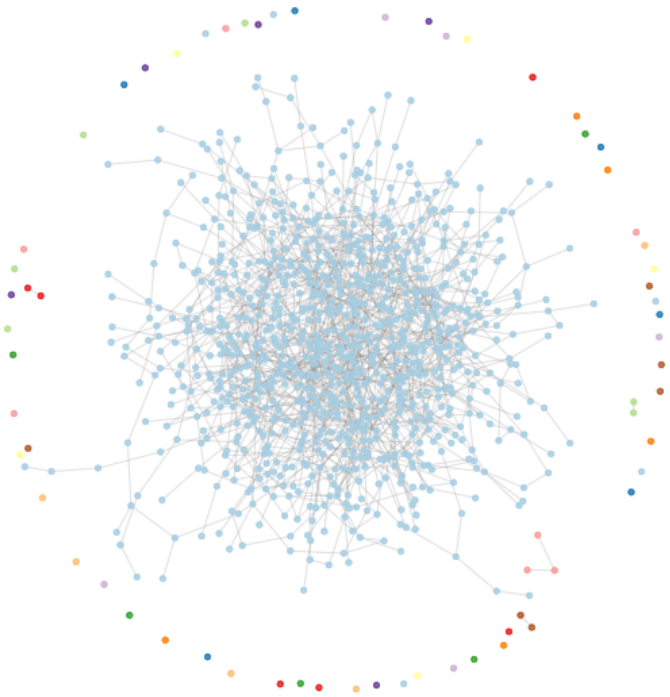


Figure 2.2: Plot of a random graph with $n = 1000$ nodes, edge probability $p = 0.003$ and $c = 3$. Nodes belonging to the same connected component are coloured identically. We can see a dominating large component in the graph through the light blue nodes in the centre. The existence of this dominating component is described in Section 3.1.1.

2.2.1 Mean degree and degree distribution

In the random graph model, the existence of any edge is independent from each other and solely determined by the same probability value p. As such, the probability of a graph with m edges drawn from the $G(n, p)$ model is given by a binomial distribution choosing m edges out of a universe of $\binom{n}{2}$ edges

$$P(m) = \binom{\binom{n}{2}}{m} p^m (1-p)^{\binom{n}{2}-m} \tag{2.4}$$

Therefore, the expected number of edges is given by

$$\langle m \rangle = \sum_{m=0}^{\binom{n}{2}} m P(m) = \binom{n}{2} p \tag{2.5}$$

Using this result, we can derive the expected mean node degree c in a graph

$$c = \sum_{m=0}^{\binom{n}{2}} \frac{2m}{n} P(m) = \frac{2}{n} \binom{n}{2} p = (n-1)p, \tag{2.6}$$

More generally, we can not only derive the mean, but also the entire distribution of the node degree. In fact, any node in the graph is connected with independent probability p to any of the remaining $n-1$ nodes in the graph. Hence, the probability of having a particular degree k, i.e. being connected to exactly k other nodes, is $p^k(1-p)^{n-1-k}$. Given that there are $\binom{n-1}{k}$ possible sets of k vertices, the probability distribution of the node degree is given by

$$p_k = \binom{n-1}{k} p^k (1-p)^{n-1-k} \tag{2.7}$$

We sometimes wish to consider not only small but also large networks. When n is assumed to be large, some useful approximations can be made, which render the model more analytically tractable.

When $n \to \infty$, we can rewrite

$$ln[(1-p)^{n-1-k}] = (n-1-k)\ln\left(1 - \frac{c}{n-1}\right) \tag{2.8}$$

$$\simeq (n-1-k)\frac{c}{n-1} \tag{2.9}$$

$$\simeq -c, \tag{2.10}$$

and therefore

$$(1-p)^{n-1-k} \simeq e^{-c} \tag{2.11}$$

by using the Taylor expansion of $\ln\left(1 - \frac{c}{n-1}\right)$ and approximating to the first term. These approximations hold exactly when $n \to \infty$, as does the following approximation when

$$\binom{n-1}{k} = \frac{(n-1)!}{(n-1-k)!k!} \simeq \frac{(n-1)^k}{k!} \tag{2.12}$$

By applying both approximations under the assumption that $n \to \infty$, we can rewrite equation 2.7 as follows

$$p_k = \frac{(n-1)^k}{k!} p^k e^{-c} = e^{-c}\frac{c^k}{k!} \tag{2.13}$$

which is the familiar Poisson distribution.

2.2.2 Giant component

Although very simple, the random graph model possesses one very interesting property: the sudden appearance of the so-called giant component by varying the mean degree c. A giant component is a component whose size is proportional to the size of the network n, and its sudden appearance is called a *phase transition*.

Consider the random graph $G_{n,p}$ with two extreme parametrisations:

$p = 0$, there are no edges in the graph, and therefore the largest component is of size 1,

$p = 1$ every vertex is connected with every other vertex, and so the largest connected component has size n.

Under $p = 1$, the size of the largest component grows with the network. This is called a giant component, and its size can be computed exactly in the limit of a large network size.

Definition 2.2.2. Let u denote the average fraction of vertices not in the giant component. Let S be the reciprocal of u, $S = 1 - u$, i.e. the average relative size of the giant component.

By definition the giant component exists if and only if $u < 1$. Suppose that, under the $G(n, p)$, vertex i does not belong to the giant component. Consider another vertex j. Either

i) i and j are not connected with probability $1 - p$, or

ii) there is an edge between i and j and j also does not belong to the giant component, which has a probability of pu.

As such, the probability that there is an edge (i, j) is $1 - p + pu$. If we now consider every node in the graph, then the probability that node i is not in the giant component is u, and depends on it only being connected to nodes not being in the giant component. Since every edge in the graph exists independently, then we have

$$u = (1 - p + pu)^{n-1} = \left[1 - \frac{c}{n-1}(1 - u)\right]^{n-1} \tag{2.14}$$

If we take the logarithm of both sides,

$$\ln u = \ln \left[1 - \frac{c}{n-1}(1 - u)\right]^{n-1} = (n - 1) \ln \left(1 - \frac{c}{n-1}(1 - u)\right) \tag{2.15}$$

When n is large, $\frac{c}{n-1}$ is very small, so that we following approximation holds

$$\ln \left(1 - \frac{c}{n-1}(1 - u)\right) \approx -\frac{c}{n-1}(1 - u) \tag{2.16}$$

and therefore,

$$\ln u \approx -c(1 - u) \tag{2.17}$$

By taking the exponential of both sides, we can write it as

$$u \approx e^{-c(1-u)} \tag{2.18}$$

Alternatively, one can consider the reciprocal of u, i.e. the fraction S of nodes that do belong to the giant component.

$$S = 1 - e^{-cS} \tag{2.19}$$

This equation has a trivial solution at $S = 0$, which is the only solution if $c \leq 1$, i.e. there is no giant component if $c \leq 1$. For $c > 1$, there is not only the trivial solution but also another solution at $S > 0$. (Newman, 2010) shows that the non-trivial solution best describes the giant component.
Although very simple, equation 2.19 has no closed-form solution. It can be expressed through the W-function [2]

$$S = 1 + \frac{W(-ce^{-c})}{c} \tag{2.20}$$

, for which there are standardized numerical software module, e.g. (SymPy Development Team, 2016).

2.3 Generating functions

In order to determine other properties of the random graph model, it is helpful to consider generating functions, much like (Newman et al., 2000) does.

We first consider the function $G_0(x)$, the probability distribution of vertex degrees k, defined as

$$G_0(x) = \sum_{k=0}^{\infty} p_k x^k \tag{2.21}$$

where p_k is the probability that a given vertex of the graph has degree k. The generating function has the interesting property that it encodes entire probability distribution. In particular, it is possible to recover the probability p_k by taking the k^{th} derivative of the G_0

$$p_k = \frac{1}{k!} \frac{d^k G_0}{dx^k}\Big|_{x=0}. \tag{2.22}$$

The derivatives of this function can be used to derive other statistical quantities of the distribution. For example, one can obtain the average degree c of the nodes in a graph

$$c = \langle k \rangle = \sum_k k p_k = G_0'(1) \tag{2.23}$$

[2]The Lambert-W function is a set of functions representing the inverse relation of the function $f(z) = ze^z$ where z is a complex number, i.e. $z = f^{-1}(ze^z) = W(ze^z)$

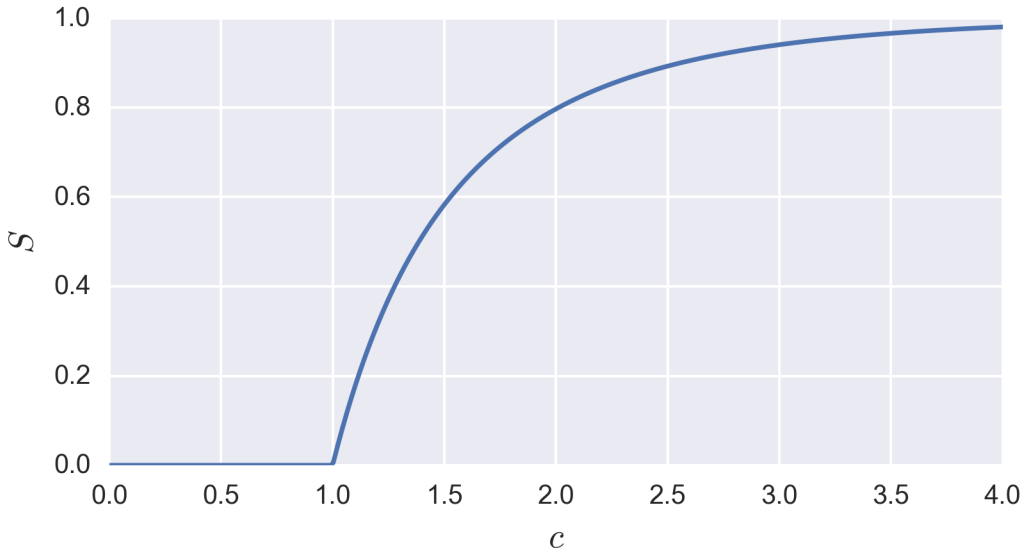


Figure 2.3: The fraction of the graph occupied by the giant component, S, as a function of the average expected degree c, and computed according to equation 2.20. The phase transition at $c = 1$ is well visible. For $c < 1$, no giant component is present, for $c > 1$ it quickly occupies a dominating fraction of the network.

Another important generating function is that of the probability of a randomly chosen vertex to belong to a component of a given size. This generating function $h(z)$ is given by

$$h(z) = \sum_{s=1}^{\infty} \pi_s z^s, \tag{2.24}$$

where π_s is the probability that a randomly chosen vertex belongs to a small component of size s. This function will be instrumental in being able to describe the component structure of the graph, as can be seen in Section 2.3.1.

2.3.1 Component sizes

Consider the probability that $P(s|k)$ that node a with degree k is part of a small component of size s. We can express this probability based on the probability that all k connected nodes belong to a sub-component with with sizes $s_1, \ldots, s_k$, given by $\Pi_{j=1}^{k} \pi_{s_j}$ such that $\sum s_j = 1$:

$$P(s|k) = \sum_{s_1=1}^{\infty} \ldots \sum_{s_k=1}^{\infty} \left[\Pi_{j=1}^{k} \pi_{s_j}\right] \delta(s-1, \sum_j s_j), \tag{2.25}$$

and δ is the Kronecker delta.

As seen above, π_s is the probability that a randomly chosen vertex belongs to a component of size s. π_s can be expressed through $P(s|k)$ as

$$\pi_s = \sum_{k=0}^{\infty} p_k P(s|k) = \sum_{k=0}^{\infty} p_k \sum_{s_1=1}^{\infty} \cdots \sum_{s_k=1}^{\infty} \left[\Pi_{j=1}^{k} \pi_{s_j}\right] \delta(s-1, \sum_j s_j) \tag{2.26}$$

$$= e^{-c} \sum_{k=0}^{\infty} \frac{c^k}{k!} \sum_{s_1=1}^{\infty} \cdots \sum_{s_k=1}^{\infty} \left[\Pi_{j=1}^{k} \pi_{s_j}\right] \delta(s-1, \sum_j s_j) \tag{2.27}$$

Substituting in the generating function given by equation 2.24, we get

$$h(z) = \sum_{s=1}^{\infty} z^s e^{-c} \sum_{k=0}^{\infty} \frac{c^k}{k!} \sum_{s_1=1}^{\infty} \cdots \sum_{s_k=1}^{\infty} \left[\Pi_{j=1}^{k} \pi_{s_j}\right] \delta(s-1, \sum_j s_j) \tag{2.28}$$

$$= e^{-c} \sum_{k=0}^{\infty} \frac{c^k}{k!} \sum_{s_1=1}^{\infty} \cdots \sum_{s_k=1}^{\infty} \left[\Pi_{j=1}^{k} \pi_{s_j}\right] z^{1+\sum_j s_j} \tag{2.29}$$

$$= z e^{-c} \sum_{k=0}^{\infty} \frac{c^k}{k!} \sum_{s_1=1}^{\infty} \cdots \sum_{s_k=1}^{\infty} \left[\Pi_{j=1}^{k} \pi_{s_j} z^{s_j}\right] \tag{2.30}$$

$$= z e^{-c} \sum_{k=0}^{\infty} \frac{c^k}{k!} \left[\sum_{s=1}^{\infty} \pi_s z^s\right]^k \tag{2.31}$$

$$= z e^{-c} \sum_{k=0}^{\infty} \frac{c^k}{k!} \left[h(z)\right]^k \tag{2.32}$$

$$= z \exp\left(c\left(h(z)-1\right)\right) \tag{2.33}$$

and obtain a surprisingly simple form, which does not have a closed form solution.

Using Eq. 2.33 we can describe the average size of the component a randomly chosen vertex belongs to by

$$\langle s \rangle = \frac{\sum_s s\pi_s}{\sum_s \pi_s} = \frac{h'(1)}{1-S} \tag{2.34}$$

Since

$$\begin{aligned} h'(z) &= \exp\left(c(h(z)-1)\right) + czh'(z)\exp\left(c(h(z)-1)\right) \\ &= \frac{h(z)}{z} + ch(z)h'(z), \end{aligned}$$

By further simplifying and observing that, as stated before, $h(1) = \sum_s pi_s = 1 - S$, we arrive at

$$h'(1) = \frac{h(1)}{1-ch(1)} = \frac{1-S}{1-c+cS} \tag{2.35}$$

and that considering Eq. 2.34, we can write

$$\langle s \rangle = \frac{1}{1-c+cS} \tag{2.36}$$

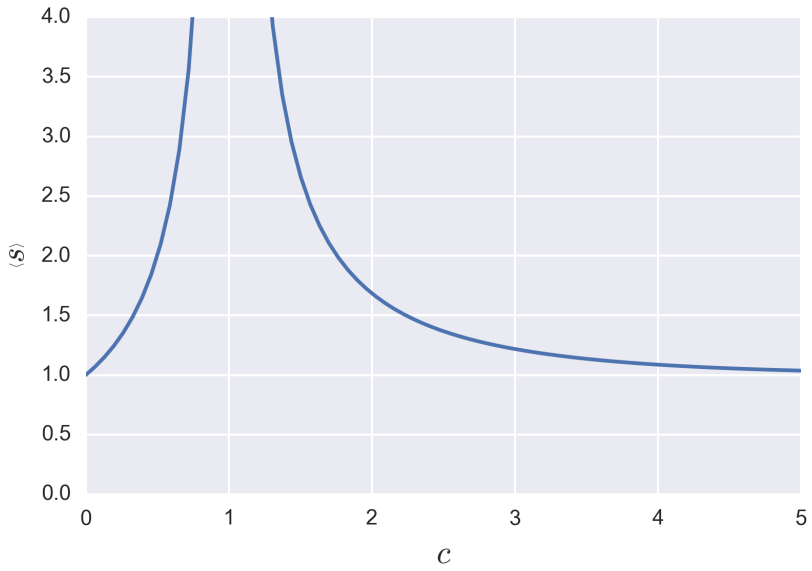


Figure 2.4: Mean size of small component to which a randomly chose vertex would belong to as a function of the mean degree c.

This equation will diverge for $c = 1$, since for this value of c, $S = 0$. This is clearly visible in figure 2.4.

Of large interest to this thesis is π_s, the probability that a randomly chose node belongs to a component with size s. It turns out that, although they cannot be directly extracted by derivation of Eq. 2.33, because the solution is not known. We turn to (Newman, 2010), who shows that making use of the *Lagrange inversion formula*, a series expansion of $h(z)$ can be derived and used to produce the identity

$$\pi_s = \frac{e^{-sc}(sc)^{s-1}}{s!} \tag{2.37}$$

2.4 Discussion

In this chapter we have looked at several properties of the random graph model. The random graph model is a relatively simplistic model that nonetheless exhibits interesting properties, such as the existence of a giant component after a certain average connectivity.

Other models of random graphs exist, such as the configuration model, which exhibit more realistic properties, but which are less prone to analytic inspection. In this thesis, we focus on the random graph model and its connection with the ramping parameter. This connection will be analysed in the next chapter.

Chapter 3

Ramping parameter under the random graph model

Previous work (Sindel, 2009) provided a connection between the clustered structure of a graph and an interpretation of concentration risk. The methodology presented by the authors considered the effects of removing edges with weight under a (varying) threshold, on the community structure of a fully connected obligor-correlation matrix. In particular, by computing the variation of the component structure of the graph, the authors have shown how to analyse concrete credit portfolios and identify sets of strongly connect obligors.

This thesis builds upon the aforementioned model and aims at describing the expected behaviour of the ramping parameter model for theoretical graph generation models under a few simplifying assumptions. In this chapter, we focus on the random graph model, a well understood theoretical graph model, described in more detail in Section 2.2.

In particular, we attempt to describe the expected behaviour of the curve designed by (Sindel, 2009), assuming that we have large, idealized, portfolios generated under simplifying assumptions, and according to random graph model that generates them. This curve could be hypothetically be later used to derived a more neutral weight-curve (see Equation 1.2) and more accurately compute the concentration risk.

3.1 Ramping parameter under random graph model

In this Section, we make use of the properties of the random graph model in order to make predictions about the ramping parameter and the concentration risk. Contrasting to the original work on the ramping parameter (Sindel, 2009), rather than computing the concentration risk of a particular portfolio, we wish to describe the

behaviour of the family of portfolios drawn from a given distribution. In order to do this, certain assumptions about the portfolios must be made. In particular, we assume:

- the portfolios to be large, i.e. we assume that $n \to \infty$, where n is the number of obligors
- the exposure of each individual obligor is $\frac{1}{n}$, i.e. the exposure is uniformly distributed amongst all obligors
- the elements of the dependency matrix to be independent from each other

Remark. The dependency to large n allows the results for the random graph model to hold exactly. In practice, however, even large portfolios will be finite and there will be some error. This error is here not analysed.

Definition 3.1.1. Let $\rho \in \mathbb{R}$, and the function f_ρ so that

$$f_\rho : C \to Aa_{ij} = \begin{cases} 1 & \text{if } c_{ij} > \rho \\ 0 & \text{if } c_{ij} \leq \rho \\ 0 & \text{if } i = j \end{cases} \tag{3.1}$$

Let g the pdf and G the cdf of each element c_{ij} of C.

By defining $p = 1 - G(\rho)$, we can write

$$P(a_{ij} = 0) = P(c_{ij} \leq \rho) = G(\rho) = 1 - p$$

and conversely

$$P(a_{ij} = 1) = P(c_{ij} > \rho) = 1 - G(\rho) = p$$

In a first step we define the space of correlation matrices

Definition 3.1.2. The space of correlation matrices $\mathcal{C}_{n\times n} = (C, \mathcal{F}, g_C)$ consists of all matrices where

- $C \in \mathbb{R}^{n\times n}$, such that $c_{ij} = c_{ji}$, $c_{ii} = 0$,
- $\mathcal{F}$ is the Borel $\sigma-$algebra on C,
- and all elements c_{ij} are i.i.d. random variables with a probability distribution g.

We want to identify matrices in $\mathcal{C}$ with adjacency matrices using a construction like the function 3.1 defined above. As the image of f depends only on $G(\rho)$ this map is not injective; more precisely two matrices C_1 and C_2 are mapped to the the same matrix A if for each pair $i \neq j$ of indices the equation $G(\rho) = 1 - p$ is satisfied. We use this relation to define equivalence classes:

Definition 3.1.3. Two matrices C_1 and C_2 are equivalent if the following condition is satisfied:

$$C_1 \sim C_2 \quad \Leftrightarrow \quad \forall i, j \;\; f_\rho(c_{ij}^1) = f_\rho(c_{ij}^2)\,. \tag{3.2}$$

Then the space of equivalence classes is denoted by $\widetilde{C}_{n\times n}$.

Theorem 3.1.1. *Equivalence to random graph model. Consider the following probability spaces:*

1. *the space of equivalence classes of correlation matrices* $\widetilde{C}_{n\times n}$,
2. *the space of unweighted adjacency matrices* $\mathcal{A}_{n\times n} = \left(A, 2^A, P_A\right)$,
3. *the space of random graphs* $G_{n,p}$

where,

$A \in \{0,1\}^{n\times n}$ *such that* $\forall i, j\, a_{ij} = a_{ji}$ *and* $a_{ii} = 0$,

$$P_A = \begin{cases} 1 & \text{with probability } p, \\ 0 & \text{with probability } 1-p \end{cases},$$

P_C *is some probability distribution,*

an adjacency matrix is a (0,1)-matrix with zeros on its diagonal.

The probability spaces are equivalent.

Proof. (2) $\Leftrightarrow$ (3): Given that (i) a graph is defined unequivocally by its adjacency matrix, that (ii) all elements of a_{ij} are by definition independent and that (iii) P_A follows the same probability distribution as in $G_{n,p}$, both probability spaces are equivalent.
(1) $\Leftrightarrow$ (2): The equivalence of $\widetilde{\mathcal{C}}_{n\times n}$ and $\mathcal{A}_n$ is a direct consequence of the definition 3.1.3 of the equivalence classes
□

By making an assumptions on the distribution of the portfolio interdependency or correlation matrix C, theorem 3.1.1 on the previous page allows us to apply the results of the random graph model to the study of the ramping parameter. The original work of (Sindel, 2009) assumes the c_{ij} to non-negative correlation values in the interval $[0, 1]$, and relies on the empirical distribution of the portfolio values.

In practice the matrix C can be generated by multiplying a set of (independent or correlated) risk factors with the dependencies of each obligor to these risk factors. Alternatively, it could also be generated either by expert knowledge (in which case one would expect the matrix to be rather sparse), by a pure correlation of stock prices of the obligors, or even by a hybrid approach.

In light of this, no general assumptions can be made over the distribution of the values of C and we will be studying the behaviour of the ramping parameter under the following distributions: (1) the uniform distribution, and a skewed distribution, (2) the truncated exponential distribution.

3.1.1 Giant component

As seen in Section 3.1.1, the giant component of the $G_{n,m}$ exists when $c \geq 1$, c being the mean degree of G. This means that a giant component is expected to exist when each vertex is in average connected to at least one other vertex. As we have seen, the relative size S of the giant component is given by:

$$S = 1 + \frac{W(-ce^{-c})}{c}, \tag{3.3}$$

where W is the Lambert-W function. Although there is no closed-form solution for this equation, it is possible to solve it numerically (e.g. by using the software package (SymPy Development Team, 2016)).

The largest connected component of the effective correlation matrix is essential to computing the ramping parameter. Whenever it exists, the giant component is therefore relevant.

We note that, under theorem 3.1.1, the mean degree c is connected to the distribution of the values of the correlation matrix by its c.d.f.. In fact, $c = p(n-1)$, which as we defined in 3.1.1 can be rewritten as $c = (1 - G(\rho))(n-1)$, where G is the c.d.f. of the elements of the correlation matrix.

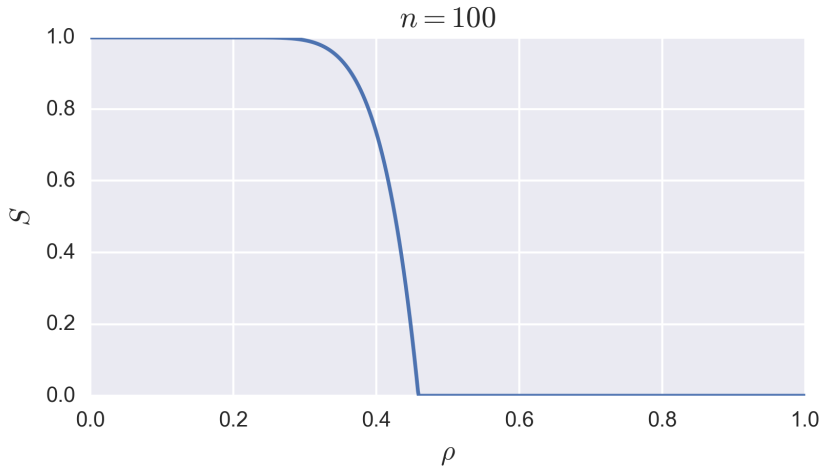


Figure 3.1: Relative size S of giant component as a function of the ramping parameter ρ given an uniform distribution of the correlation weights.

3.1.1.1 Uniform distribution of obligor correlations

By assuming that the $c_{ij} \sim U(0,1)$, the c.d.f. is trivially $G(\rho) = \rho$, and therefore $c = (1-\rho)(n-1)$. As we will see, the giant component therefore appears at a very early stage in the evolution of the ramping parameter, and so the community structure will be largely dominated by it, which can be seen in Figure 3.1.

3.1.1.2 Truncated exponential distribution

As we have seen in Section 3.1.1.1, the giant component clearly dominates the community structure of the graph for the most values of the ramping parameter. In a real world portfolio, it is likely that most correlations will be near zero, and that the latent structure of the inter-obligor matrix is revealed by a small set of strong interconnections.

In order to represent this, we assume that the c_{ij} are distributed according to the truncated exponential distribution. The distribution is similar to the exponential distribution, but it is restricted to the domain $[0,1]$. Like the exponential distribution, it is governed by one parameter: α. Its p.d.f. is as follows

$$g(x) = -\frac{\alpha e^{-\alpha x}}{1 - e^{-\alpha}} \tag{3.4}$$

and the c.d.f. given by

$$G(x) = \frac{1 - e^{-\alpha x}}{1 - e^{-\alpha}}. \tag{3.5}$$

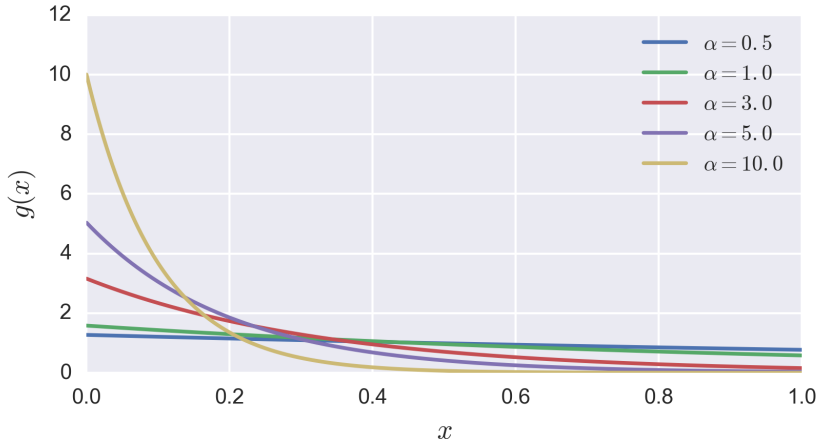


Figure 3.2: Probability density function of truncated exponential distribution with different values of the parameter α.

The behaviour of the p.d.f. can be seen in figure 3.2. Larger values of the parameter α lead to more probability being concentrated with lower values of the r.v.. The fact that much more probability density is concentrated on lower correlation values leads to an earlier and smoother disaggregation of the giant component. The point at which the giant component disappears is also strongly dependent on the value of the parameter α, which can be observed in figure 3.3 on the following page. This is already of great interest to the study of the ramping parameter, since it indicates a strong sensitivity to the distribution of the correlation values, independently of whether additional structure being available (which goes beyond of this chapter).

3.1.2 Small components' sizes

From the previous chapter, we know that in the random graph model, the probability distribution of π_s, i.e. the probability that a randomly chosen node in the graph belongs to a component of size s. We also know that this distribution complements the giant component

$$\sum_{s=1}^{\infty} \pi_s = 1 - S \tag{3.6}$$

and that it is given by

$$\pi_s = \frac{1}{s!}\left[\frac{d^{s-1}}{dh^{s-1}} e^{sc(h-1)}\right] = \frac{e^{-sc}(sc)^{s-1}}{s!}, \tag{3.7}$$

What we are interested is, however, the actual number of components with size s. This can be easily derived by considering that n_s will be related to π_s in that s nodes

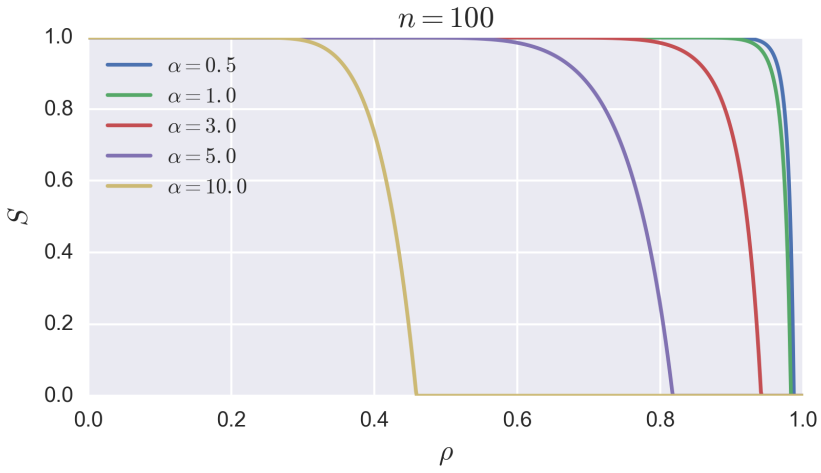


Figure 3.3: Relative size S of giant component as a function of the ramping parameter ρ given truncated exponential distributions of the correlation weights with different parameters α.

will belong to a component of size s. Therefore,

$$n_s = \frac{\pi_s}{s} = \frac{e^{-sc}(sc)^{s-1}}{s\,s!}. \tag{3.8}$$

As hinted above, the situation is clearly dependent on whether the giant component exists in the graph. In order to illustrate this, we show a histogram of the distribution of n_s in figure 3.4.

Theorem 3.1.2. *Size of the largest component. Let n_s be the distribution of the non-giant component sizes under the random graph model, k be the number of components in the graph, C_i the size of component i. Then $Y = max\{C_1, \ldots, C_k\}$, the size of the largest non-giant component in the network, is a r.v. with probability distribution given by $F_Y(y) = (\sum_{s=1}^{y} n_s)^k$.*

Proof. Since every edge in the network exists independently, the C_i variables are i.i.d. with p.d.f. given by equation 3.8. We know that

$$P(Y \leq y) = P(C_1 \leq y, \ldots, C_k \leq y) = \Pi_{i=1}^{k} P(C_i \leq y)$$

By definition, $P(C_i \leq x) = \sum_{s=1}^{y} n_s$, and so $F_Y(y) = (\sum_{s=1}^{y} n_s)^k$. □

Remark. Theorem 3.1.2 is only of limited help due to the fact that the number of components k in the graph is not known a priori in the random graph model. Its distribution is, however, known and can be used to compute a numerical estimate of the distribution above.

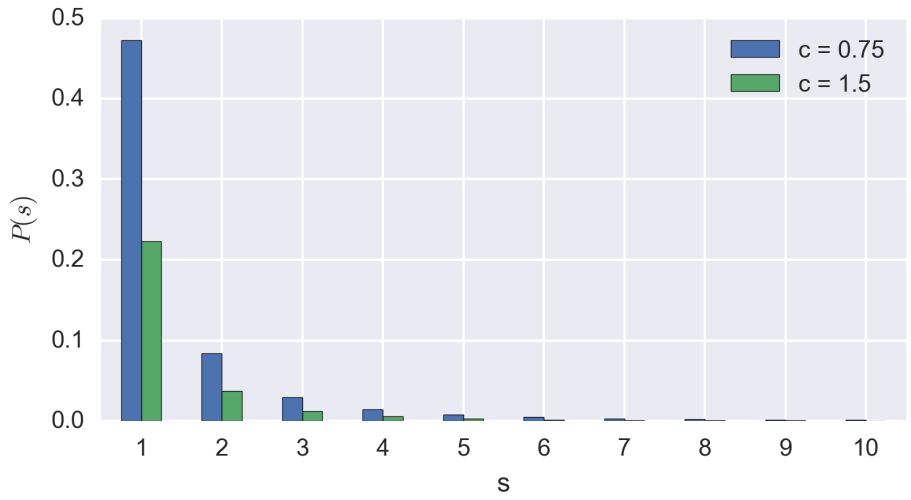


Figure 3.4: Histogram of n_s for graphs with two different mean degree values. $c = 0.75$ means that there is no giant component while $c = 1.5$ guarantees the presence of the giant component. As can be seen, the probabilities of the component sizes is significantly lower in the case where the giant component exists, the reason for this being the fact that a fraction of the nodes belongs to the giant component and not to a small component.

If we consider $c > 1$, then the size of the largest component is trivially given by S. In the case of $c \leq 1$, then we must consider the distribution Y given by Theorem 3.1.2. We consider the expected number of components to be given by

$$k = \sum_{s=1}^{\infty} n_s \tag{3.9}$$

which can be seen in Figure 3.6.

And finally, the expected maximal component size is simply given by the definition of the expected value.

$$E[Y] = \sum_{s} sP(Y = s) \tag{3.10}$$

and can be seen in Figure 3.5 for all values of the parameter ρ in different truncated exponential probabilities.

Theorem 3.1.3. *Expected ramping parameter value. Let Y be the size of the largest small component in a random graph, and S the relative size of the giant component. Let Pf be a uniformly exposed portfolio with correlation matrix C whose elements are independently distributed according to the cdf G. Then, the expected ramping parameter value for Pf and ρ by*

$$\max\left\{\frac{E[Y]}{n}, S\right\}$$

is given by c.

Proof. The largest component is either the the giant component (if it not exists, $S = 0$) or the largest small component, with expected value given by Equation 3.10. Because the exposures are uniformly distributed across obligors, $EAD_j = 1/n$, the value of

$$R(\rho) = \max_{C_k} \left(\frac{\sum_{j \in C_k} \mathrm{EAD}_j}{\sum_i \mathrm{EAD}_i} \right)$$

reduces trivially to

$$R(\rho) = \max_{C_k} \left(\frac{\sum_{j \in C_k} 1/n}{\sum_i 1/n} \right) = \max_{C_k} \frac{\sum_{j \in C_k} 1}{n}$$

and therefore

$$E[R(\rho)] = \max \left\{ \frac{E[Y]}{n}, S \right\}$$

□

An example of the curve defined by this value can be seen in Figure 3.5 above. The usefulness of this result is discussed in the following Section.

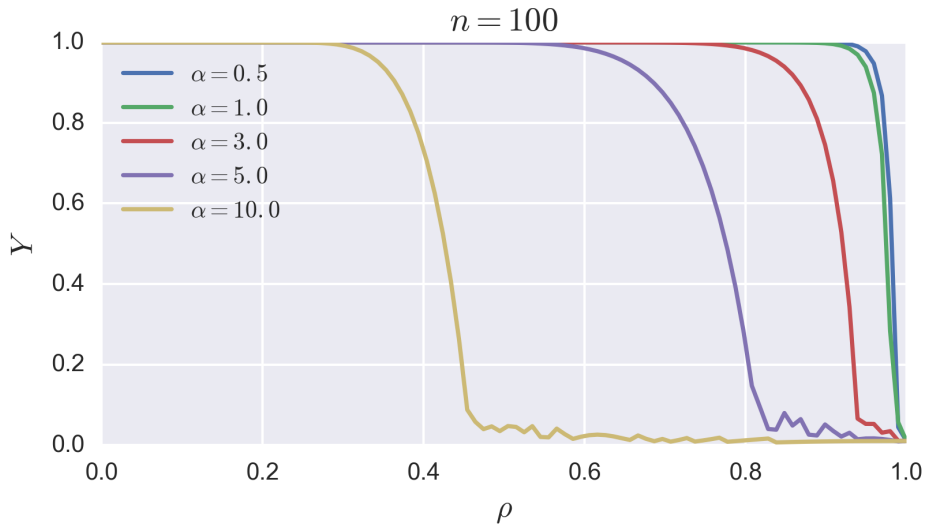


Figure 3.5: Expected relative size of the largest component. One can see the result of some numerical instability in the region near the phase transition. The reason behind this instability is still unknown.

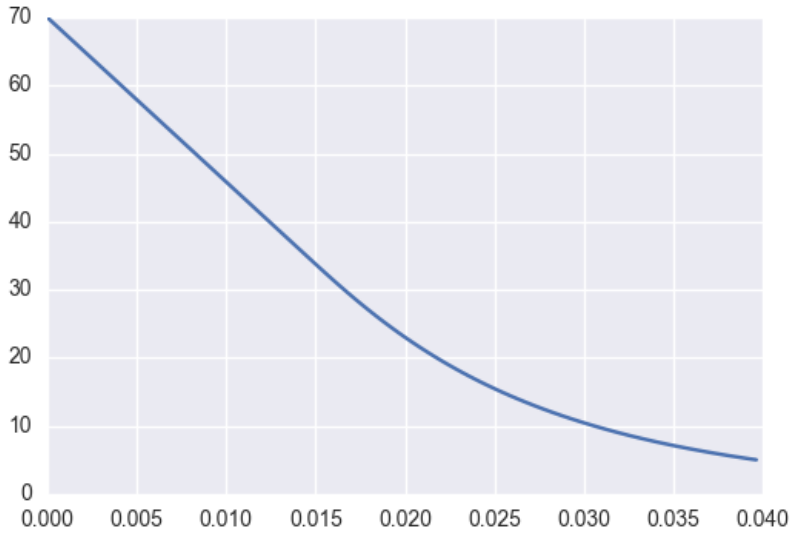


Figure 3.6: Expected number of components as a function of the probability p of the $G(n, p)$ model.

Chapter 4

Summary and discussion

This thesis explores properties the construction process of the ramping parameter model (Sindel, 2009) using theoretical results from the random graph model. To the best of our knowledge, this has not been addressed in the literature.

Usually, credit concentration risk measures take only one of the above aspects into consideration (Lütkebohmert, 2008). By contrast, the ramping parameter model is a measure of concentration risk in credit portfolios that takes both (i) the exposure to the obligors and (ii) the interdependencies between obligors, which renders it especially interesting.

The ramping parameter model works by considering the correlation matrix of obligors as a graph. For each value of a ramping parameter $\rho \in [0, 1]$, edges are removed from that graph if their weight (associated correlation or interdependency value) does not exceed the value of the parameter. The community structure of the reduced graph is then taken into consideration. Namely, the relative exposure-weight of the connected component with largest exposure $CR(\rho)$ is tracked for each value of the parameter. In the original work, the curve is weighted and integrated to provide a final concentration value. An interesting discussion on which weight curve should be used is also part of the original work. Unfortunately, the choice of weight curve lacks a theoretical backing from random graph theory, motivating this thesis.

The random graph model is a simple model of network formation that generates networks with different statistical properties than real-world networks, but from which analytic solutions can be derived. For studying the ramping parameter model, this is an interesting model to explore, since it has very relaxed assumptions:

- every edge exists independently from each other
- the probability of each edge is constant across the entire graph

and, as already mentioned, largely analytically tractable.

This thesis studies value of the ramping parameter model expected by the random graph model $G_{n,p}$, by making the following assumptions:

1. the value of EAD_i is equal for all obligors

2. the correlations between obligors are statistically independent

As show in chapter 3, under the random graph model, we show that it is possibly to calculate the expected size of the largest connected component, and, thanks to assumption (1), the expected value of the ramping parameter for each value ρ. In particular, this result takes into account the overall distribution of the correlation values of the original correlation matrix.

Not only were the theoretical results described, but also the Python code for computing the statistical quantities is provided.

4.1 Outlook and future work

Time is both the biggest friend and enemy of such a work. While it ensures the convergence to a possibly useful result, it leaves many questions unanswered.

For us, the main unanswered question is to which extent the theoretical curve computed in the end of section 3.1.2 can be used for determining the weight curve (see Equation 1.2 on page 4). Our hypothesis is that it could serve as some sort of null model and therefore lead to a weight curve and measure of concentration risk that takes the insights from the random graph models into account. It would also be important to observe the curve under the empirical distribution of correlations from a real portfolio.

Another question arises as to whether other, more expressive models of random graphs, can be used. One such model is the configuration model, which can express the properties of graphs with more realistic degree distributions, such as the popular power law distribution. In the literature research leading to this thesis, we came across theoretical results that describe the effect of edge removal from graphs with power-law and power-law like degree distributions (DuBois et al., 2012). These could be used in order to derive a theoretical distribution of the ramping parameter curve under the assumptions that certain obligors are intrinsically more densely connected than others. This is definitely more in line with what is observed in real world portfolios.

Lastly, a further possible extension of this work deals would allow for asymmetric interdependencies between obligors, i.e. the graphs under study would be directed

graphs. The treatment of directed graphs is, however, more challenging than the that of undirected graphs. In particular, the definition of node degree is not unequivocal (Newman, 2010), like in the case of undirected graphs.

Appendix A

A.1 Python code

main_code

December 16, 2016

In [1]:
```
import math
import networkx as nx
import pandas as pd
import numpy as np
from sympy import LambertW


## Random graph model
@np.vectorize
def giant_component_fraction(mean_degree):
        return 1 + LambertW(-mean_degree * np.exp(-mean_degree))\
           / mean_degree if mean_degree >= 1. else 0.

fct = np.vectorize(math.factorial)

def dist_node_component(s, c):
        return np.exp( - s * c) * c * (s * c) ** (s-2) / fct(s-1)

def dist_comp_sizes(s, c):
        return 1 / s * dist_node_component(s, c)

def avg_comp_size(c):
        S = giant_component_fraction(c)
        return 1 / (1 - c + c * S)

def largest_comp_fraction(c=0.4, n=100):
    s = np.linspace(1,20,20)

    S = np.asscalar(giant_component_fraction(c))
    dist_component_sizes = pd.Series(dist_comp_sizes(s, c), index=s)
    k = int(np.ceil( dist_component_sizes.sum() * n))
    x_s = (dist_component_sizes * n / k)
    cdf_s = x_s.cumsum()
    y = (cdf_s ** k)
    yy = y.diff()
    yy[1] = y[1]
    exp_yy = np.sum(yy.index.values * yy) / n
    return max(exp_yy, S)#, yy, k, dist_component_sizes, S
```

In [2]:
```
from bokeh.palettes import inferno
from matplotlib import pyplot as plt
```

In [3]:
```
from scipy.stats import expon
```

In [4]:
```
g = lambda x, α: expon.pdf(x, scale=1/α) / (1-np.exp(-α))
G = lambda x, α: expon.cdf(x, scale=1/α) / (1-np.exp(-α))
```

In [5]:
```
ρ = np.linspace(0, 1., 100)
n = 100
```

In [6]:
```
α = 5
n = 100
c = (1-G(ρ, α)) * (n-1)

f_largest = np.vectorize( lambda c: largest_comp_fraction(c=c, n=n))
l = f_largest(c)
```

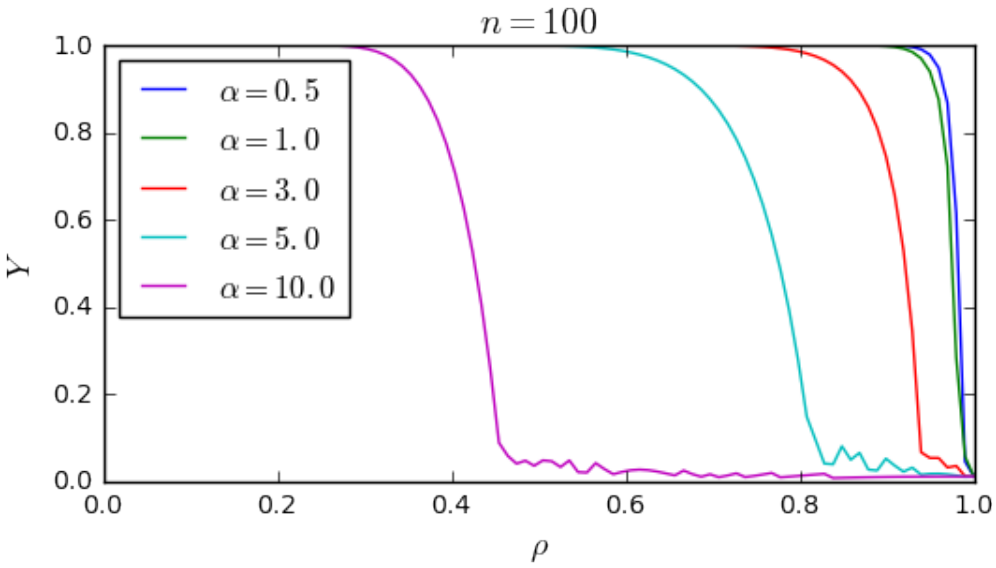
```
/usr/local/anaconda3/lib/python3.5/site-packages/ipykernel/__main__.py:16: RuntimeW
/usr/local/anaconda3/lib/python3.5/site-packages/ipykernel/__main__.py:16: RuntimeW
```

In [7]:
```
%matplotlib inline
plt.figure(figsize=(6,3))
plt.title("$n = %d$" % n, fontsize=14)
plt.hold(True)
plt.xlabel("$\\rho$", fontsize=13)
plt.ylabel("$Y$", fontsize=13)

alphas = [0.5, 1.0, 3., 5., 10.]
colors = inferno(len(alphas))
lines = []
vals = {}
for i, α in enumerate(alphas):
    c = (1-G(ρ, α)) * (n-1)
    l = f_largest(c)
    vals["%f" % α] = l
    ll, = plt.plot(ρ, l, label="$\\alpha = %.1f$" % α)
    lines.append(ll)

lgd = plt.legend(handles = lines, loc='best')
# plt.savefig("../figures/3_largest_component.png", bbox_extra_artists=(lgd
```

```
/usr/local/anaconda3/lib/python3.5/site-packages/ipykernel/__main__.py:16: RuntimeW
/usr/local/anaconda3/lib/python3.5/site-packages/ipykernel/__main__.py:16: RuntimeW
```


$n = 100$
$\alpha = 0.5$
$\alpha = 1.0$
$\alpha = 3.0$
$\alpha = 5.0$
$\alpha = 10.0$
Y
ρ
0.0
0.2
0.4
0.6
0.8
1.0

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