Solutions to Sheet 2

Exercise 1

Define $\zeta = \frac{-1+\sqrt{-3}}{2} \in \mathbb{C}$.

1. Show that ζ is a primitive third root of unity.
2. Show that the norm (for the field extension $\mathbb{Q}(\zeta)/\mathbb{Q}$ of an element $x + y\zeta \in \mathbb{Q}(\zeta)$, where $x, y \in \mathbb{Q}$, is given by $x^2 - xy + y^2$, and that this is non-negative for all $x, y \in \mathbb{Q}$.
3. Following the discussion of $\mathbb{Z}[i]$ from the lecture, show that a prime $p \neq 3$ is of the form $p = x^2 - xy + y^2$ for some $x, y \in \mathbb{Z}$ if and only if $p \equiv 1 \pmod 3$.

Solution.

1. We have
$$\zeta^3 = \left(\frac{1}{2}(-1+\sqrt{-3})\right)^3 = 1/8(-1+3\sqrt{-3}-9+3\sqrt{-3}) = 1.$$
As $\zeta \neq 1$ (and 3 has no non-trivial divisors), it is a primitive (third) root.
2. The norm is defined as the product of all galois-conjugates. The minimal polynomial of ζ is given by $f(x) = x^2 + x + 1 = (x - \zeta)(x - \overline{\zeta})$, so the only non-trivial element in the Galois-group $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ is given by the action defined via $\zeta \mapsto \overline{\zeta}$, which is the same as complex conjugation. We find
$$\mathrm{N}(x + \zeta y) = (x + \zeta y)(x + \overline{\zeta} y) = x^2 + (\zeta + \overline{\zeta})xy + \zeta\overline{\zeta}y^2.$$
The claim follows as $\zeta\overline{\zeta}$ and $\zeta + \overline{\zeta}$ are given by the constant and the negative of the second-to-highest coefficient of the minimal polynomial of ζ (which are both given by 1).

Exercise 2

1. Let A be a principal ideal domain that is not a field, and let $\mathfrak{m} \subset A$ be a maximal ideal. Prove that $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a one-dimensional vector space over $A/\mathfrak{m}$ for any $n \geq 0$.
2. Let $A = \mathbb{C}[x, y]$ and $\mathfrak{m} = (x, y)$. Compute $\dim_{A/\mathfrak{m}}(\mathfrak{m}^n/\mathfrak{m}^{n+1})$ for $n \geq 0$. Deduce that A is not a principal ideal domain.
3. Let $A = \mathbb{Z}[\sqrt{-3}]$. Show that A has a unique maximal ideal $\mathfrak{m}$ with $\mathfrak{m} \cap \mathbb{Z} = (2)$. Compute $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$. Deduce that A is not a principal ideal domain.

Solution.

1. Let $\pi \in A$ such that $(\pi) = \mathfrak{m}$. We have the map (of A-modules)
$$\varphi : A \to \mathfrak{m}^n/\mathfrak{m}^{n+1}, \quad a \mapsto a\pi^n$$
. It is obviously surjective, and one quickly verifies that the kernel is given by (π). Hence we find $A/\mathfrak{m} \cong \mathfrak{m}^n/\mathfrak{m}^{n+1}$, and we are done.
2. We have $\mathfrak{m}^n = (x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n)$. These generators form a basis for $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ (they are generating and linearly independent over $\mathbb{C}$), hence the dimension is $n+1$. This contradicts what we showd for principal ideal domains once $n \geq 1$.

Exercise 3

Let A be a unique factorization domain.

1. Show that for any prime element $\pi \in A$, the ideal $\mathfrak{p} = (\pi)$ is prime and only contains the prime ideals $\{0\}$ and $\mathfrak{p}$.
2. Conversely, let $0 \neq \mathfrak{p} \subset A$ be a prime ideal such that $\{0\}$ and $\mathfrak{p}$ are the only prime ideals of A that are contained in $\mathfrak{p}$. Show that $\mathfrak{p} = (\pi)$ for some prime element $\pi \in A$.
3. Assume that each non-zero prime ideal $\mathfrak{p} \subset A$ satisfies the assumption in 2). Show that A is a principal ideal domain.

Exercise 4

1. Let A be any ring. Show that A contains minimal prime ideals.
2. Determine the minimal prime ideals of $\mathbb{Z}[x, y]/(xy)$.

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