Solutions to Sheet 5

Exercise 1

Let A be a ring and let $\mathfrak{a}_1, \dots, \mathfrak{a}_n \subset A$ be ideals such that $\bigcap_{i=1}^n \mathfrak{a}_i = \{0\}$. Assume that each ring $A/\mathfrak{a}_i$ is noetherian. Show that A is noetherian.

Solution. When does this situation arise? One example are rings of the form $A/\bigcap_i \mathfrak{a}_i$.

Exercise 2

Consider the matrix

$$S := \begin{pmatrix} -36 & 14 & -24 \\ 18 & 6 & 12 \end{pmatrix}.$$

Determine its elementary divisors and the kernel/cokernel of the map $\mathbb{Z}^3 \xrightarrow{S} \mathbb{Z}^2$ (up to isomorphy).

Exercise 3

Let A be a ring, let $\mathfrak{a} \subset A$ be an ideal and let M, N_i, $i \in I$, be A-modules for some set I.

1. Show that there exists a unique isomorphism

$$\Phi : \bigoplus_{i\in I}(N_i \otimes_A M) \to \left(\bigoplus_{i\in I} N_i\right) \otimes_A M$$

   such that $\Phi((\dots, 0, n_i \otimes m, 0 \dots)) = (\dots, 0, n_i, 0, \dots) \otimes m$ for all $n_i \in N_i$, $i \in I$, $m \in M$.

2. Show that there exists a unique isomorphism

$$\Psi : A/\mathfrak{a} \otimes_A M \to M/\mathfrak{a}M$$

   such that $\Psi((a + \mathfrak{a}) \otimes m) \mapsto am + \mathfrak{a}M$ for all $a \in A$, $m \in M$.

Solution.

1. By the unique property of the direct sum, defining Φ is the same as defining morphisms $\Phi_i : N_i \otimes_A M \to (\bigoplus_{i\in I} N_i) \otimes_A M$. Note that $N_i \otimes_A M$ is generated by elements of the form $n_i \otimes m$ (with $n_i \in N_i$ and $m \in M$), and the exercise already specifies how Φ_i is defined on those elements, namely by

$$\Phi_i(n_i \otimes m) = (\dots, 0, n_i, 0, \dots) \otimes m \in \left(\bigoplus_{i\in I} N_i\right) \otimes_A M.$$

   One could check now that this is well defined (remember that the elements $n_i \otimes m$ are only defined up to the relation $(an) \otimes m \sim n \otimes (am)$). But we use the universal property. The map is exactly the map that comes from the bilinear map $N_i \times M \to (\bigoplus_{i\in I} N_i) \otimes_A M$, $(n_i, m) \mapsto (\dots, 0, n_i, 0, \dots) \otimes m$. By construction, Ψ_i is a bijection on its image (really, we just put 0s everywhere else), and the images of Ψ_i have intersection $\{0\}$ and generate all of $(\bigoplus_{i\in I} N_i) \otimes M$. Hence Ψ is an isomorphism.

2. Again, we use the universal property. The mapping

$$A/\mathfrak{a} \times M \to M/\mathfrak{a}M, \quad (a+\mathfrak{a}, m) \mapsto am + \mathfrak{a}M.$$

is well-defined and bilinear, which is easy to check. This gives the desired map $\Psi : A/\mathfrak{a} \otimes_A M \to M/\mathfrak{a}M$. It is surjective as $\Psi(1 \otimes m) = m + \mathfrak{a}M$, and injective because if $\Psi((a+\mathfrak{a}) \otimes m) = 0 + \mathfrak{a}M$, we have $am \in \mathfrak{a}M$. Hence $am = a'm'$ for some $a' \in \mathfrak{a}, m' \in M$. In particular,

$$a \otimes m = 1 \otimes (am) = 1 \otimes (a'm') = a' \otimes m' = 0 \in A/\mathfrak{a} \otimes_A M.$$

This shows injectivity of Ψ, and we are done.

Exercise 4

Let A be a ring and let M, N be A-modules. A bilinear map $(-,-) : M \times M \to N$ is called symmetric if $(m_1, m_2) = (m_2, m_1)$ for all $m_1, m_2 \in M$. It is called alternating if $(m, m) = 0$ for all $m \in M$.

1. Show that there exists an A-module $\mathrm{Sym}^2_A(M)$ and a symmetric bilinear map $\iota : M \times M \to \mathrm{Sym}^2_A(M)$ with the following universal property: For every A-module N and for every symmetric bilinear map $(-,-) : M \times M \to N$ there exists a unique A-linear map $\Phi : \mathrm{Sym}^2_A(M) \to N$ usch that for all $m_1, m_2 \in M$

$$(m_1, m_2) = \Psi(\iota(m_1, m_2)).$$

   Construct similarly an A-module $\Lambda^2_A(M)$ with a universal alternating bilinear map $\gamma : M \times M \to \Lambda^2_A(M)$.

2. Show that $\mathrm{Sym}^2_A(A^n)$ and $\Lambda^2_A(A^n)$ are free A-modules of ranks $\frac{n(n+1)}{2}$ and $\frac{n(n-1)}{2}$.

Solution.

1. Okay, the Sym-construction should be somehow similar to the construciton of $\otimes$, and ideally all proofs of properties simply follow from the universal property of the tensor product. In the construction of the tensor product, (m_1, m_2) corresponds to the image of $\varphi(m_1 \otimes m_2)$ for some suitable morphism φ. Imposing that $(m_1, m_2) = (m_2, m_1)$ corresponds to the statement that in Sym^2_A, any morphism should send $(m_1 \otimes m_2 - m_2 \otimes m_1)$ to zero. Building on this, we define $\mathrm{Sym}^2_A(M)$ as $(M \otimes_A M)/G$, where G is the A-module generated by elements of the form $(m_1 \otimes m_2 - m_2 \otimes m_1)$. We check that this works. With the notation of the exercise, we first obtain a morphism $\psi : M \otimes_A M \to N$ by the UP of the tensor product.

$$\begin{tikzcd}
M \times M \arrow[r, "(m_1,m_2)\mapsto m_1\otimes m_2"] \arrow[dr, "(-,-)"'] & M \otimes_A M \arrow[d, dashed, "\psi"] \arrow[dr, "\mathrm{proj}"] & \\
& N & \mathrm{Sym}^2_A(M) \arrow[l, dashed, "\Psi"]
\end{tikzcd}$$

   By construction, we have $G \subset \operatorname{Ker} \psi$, so by the universal property of kernels, ψ extends uniquely to a morphism $\Psi : \mathrm{Sym}^2_A(M) \cong (M \otimes_A M)/G \to N$.

   We define $\Lambda^2_A(M)$ similarly, this time we define G as submodule of $M \otimes_A M$ generated by elements of the form $(m \otimes m)$.

2. Note that similarly to vector spaces, $A^n \otimes_A A^n$ is the free module generated over the basis $e_i \otimes e_j$. In the case of Sym_A^2, we need to quotient out the submodule generated by elements of the form $m_1 \otimes m_2 - m_2 \otimes m_1$.

Max von Consbruch, email: s6mavonc@uni-bonn.de. Date: May 15, 2023