Solutions to Sheet 2

Exercise 1

Define $\zeta = \frac{-1+\sqrt{-3}}{2} \in \mathbb{C}$.

1. Show that ζ is a primitive third root of unity.
2. Show that the norm (for the field extension $\mathbb{Q}(\zeta)/\mathbb{Q}$ of an element $x + y\zeta \in \mathbb{Q}(\zeta)$, where $x, y \in \mathbb{Q}$, is given by $x^2 - xy + y^2$, and that this is non-negative for all $x, y \in \mathbb{Q}$.
3. Following the discussion of $\mathbb{Z}[i]$ from the lecture, show that a prime $p \neq 3$ is of the form $p = x^2 - xy + y^2$ for some $x, y \in \mathbb{Z}$ if and only if $p \equiv 1 \pmod 3$.

Exercise 2

1. Let A be a principal ideal domain that is not a field, and let $\mathfrak{m} \subset A$ be a maximal ideal. Prove that $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a one-dimensional vector space over $A/\mathfrak{m}$ for any $n \geq 0$.
2. Let $A = \mathbb{C}[x, y]$ and $\mathfrak{m} = (x, y)$. Compute $\dim_{A/\mathfrak{m}}(\mathfrak{m}^n/\mathfrak{m}^{n+1})$ for $n \geq 0$. Deduce that A is not a principal ideal domain.
3. Let $A = \mathbb{Z}[\sqrt{-3}]$. Show that A has a unique maximal ideal $\mathfrak{m}$ with $\mathfrak{m} \cap \mathbb{Z} = (2)$. Compute $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$. Deduce that A is not a principal ideal domain.

Exercise 3

Let A be a unique factorization domain.

1. Show that for any prime element $\pi \in A$, the ideal $\mathfrak{p} = (\pi)$ is prime and only contains the prime ideals $\{0\}$ and $\mathfrak{p}$.
2. Conversely, let $0 \neq \mathfrak{p} \subset A$ be a prime ideal such that $\{0\}$ and $\mathfrak{p}$ are the only prime ideals of A that are contained in $\mathfrak{p}$. Show that $\mathfrak{p} = (\pi)$ for some prime element $\pi \in A$.
3. Assume that each non-zero prime ideal $\mathfrak{p} \subset A$ satisfies the assumption in 2). Show that A is a principal ideal domain.

Exercise 4

1. Let A be any ring. Show that A contains minimal prime ideals.
2. Determine the minimal prime ideals of $\mathbb{Z}[x, y]/(xy)$.

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