Solutions to Sheet 2

Exercise 1

Determine the nilradical, the Jacobson radical and the units for each ring A below:

1. k a field and $A = k[T]$,
2. k a field and $A = k[\epsilon, T]/(\epsilon^2)$,
3. $n \geq 1$, k a field and $A = k[\![T_1, \dots, T_n]\!]$.

Solution.

1. *Nilradical.* If B is any commutative ring without zero divisors, then $B[T]$ doesn't have zero divisors. Indeed, if $f, g \in B[T]$ with $fg = 0$, we can look at the leading terms of f and g, obtaining $f = 0$ or $g = 0$. We now obtain $\mathrm{Nil}(A) = (0)$ as every element in the nilradical is a zero divisor.
   Units. Obviously, $k^\times \subset k[T]^\times$. We have the additive degree map $\deg : k[T]^\times \to \mathbb{N}_0$. If we have elements $f, g \in k[T]$ with $fg = 1$, then $0 = \deg(fg) = \deg(f) + \deg(g)$, thereby $\deg(f) = \deg(g) = 0$ and $f, g \in k^\times$. This shows that $k^\times \supset k[T]^\times$, and we have equality.
   Jacobson radical. Note that if B is any commutative ring and $f \in \mathrm{Jac}(B)$, then $1+f \in B^\times$. Indeed, if we had $1 + f \notin B^\times$, we'd find some maximal ideal $\mathfrak{m}$ containing $1 + f$ (by Zorn's lemma). But now $f \in \mathfrak{m}$ (as $f \in \mathrm{Jac}(B)$) and $1 + f \in \mathfrak{m}$, hence $1 \in \mathfrak{m}$. This is a contradiction. Thereby we obtain that every $f \in \mathrm{Jac}(A)$ has degree 0. As $A^\times \cap \mathrm{Jac}(A) = \emptyset$, we find $\mathrm{Jac}(A) = 0$. As $\mathrm{Jac}(A) \supset \mathrm{Nil}(A)$, this is stronger than $\mathrm{Nil}(A) = 0$.

2. *Nilradical.* We claim that if $I \subset \mathrm{Nil}(A)$, there is an equality $\mathrm{Nil}(A)/I = \mathrm{Nil}(A/I)$. Indeed, we have Indeed, this can be seen Same works with Jacobson.

3. For A is inverse limit (i.e., can be described by comp systems). Hence Units are power series whose first term is invertible. For units: Claim: If $I \subset \mathrm{Nil}(B)$ then $B^\times = \pi^{-1}((B/I)^\times)$. Take $I = (T_1, \dots, T_n)$.

Exercise 2

Prove the *Chinese remainder theorem*: Let A be a ring and $\mathfrak{a}, \mathfrak{b} \subset A$ two ideals such that $\mathfrak{a} + \mathfrak{b} = A$. Then the map

$$A/\mathfrak{a} \cap \mathfrak{b} \to A/\mathfrak{a} \times A/\mathfrak{b}, \quad r + \mathfrak{a} \cap \mathfrak{b} \mapsto (r + \mathfrak{a}, r + \mathfrak{b})$$

is an isomorphism. Moreover, show that $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a} \cdot \mathfrak{b}$, where $\mathfrak{a} \cdot \mathfrak{b}$ is the smalles ideal in A containing all products ab wth $a \in A$, $b \in B$. Show $a \cap b = ab$. Show that map has kernel $a \cap b$ and that homomorphism is surjective.

Solution. We first show that this map is well-defined, and indeed a homomorphism of rings. This is evident for the reduction-mod-$\mathfrak{a}$ and reduction-mod-$\mathfrak{b}$ maps $A \to A/\mathfrak{a}$ and $A/\mathfrak{b}$. By the universal property of the product of rings we obtain the map $A \to A/\mathfrak{a} \times A/\mathfrak{b}$. The kernel of this homomorphism is given by the elements in A which lie simultaneously in $\mathfrak{a}$ and $\mathfrak{b}$, hence we obtain an injective map

$$A/(\mathfrak{a} \cap \mathfrak{b}) \to A/\mathfrak{a} \times A/\mathfrak{b}.$$

To show surjectivity, it suffices to construct elements $a, b \in A$ such that $a \mapsto (0,1)$ and $a \mapsto (1,0)$. As $\mathfrak{a} + \mathfrak{b} = A$, there are elements $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ such that $a + b = 1$. These are the elements we are looking for! Indeed, as $a = 1 - b$ we find that a reduces to 1 mod $\mathfrak{b}$, and as $a \in \mathfrak{a}$ we find $(a + \mathfrak{a}, a + \mathfrak{b}) = (\mathfrak{a}, 1 + \mathfrak{b})$.

We now show that $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a} \cdot \mathfrak{b}$. The inclusion $\mathfrak{a} \cap \mathfrak{b} \supset \mathfrak{a} \cdot \mathfrak{b}$ is obvious, as all products ab lie in both $\mathfrak{a}$ and $\mathfrak{b}$. To show the reverse inclusion, let $f \in \mathfrak{a} \cap \mathfrak{b}$. Again, let $a + b = 1$ with $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Then $fa + fb = f$, and the left hand side lies in $\mathfrak{a} \cdot \mathfrak{b}$ by definition.

Exercise 3

1. okay. How to make sense of "decompositions": $\{(A_1, A_2, \alpha : A \cong A_1 \times A_2)\}$. Weddeburn's theorem.

2. $133 = 7 \cdot 19$. We have $1 = 3 \times 19 - 8 \cdot 7$. Hence the idempotents are $(57, 77, 0, 1)$.

Exercise 4

1. Integral domain => mult by $x \in A \setminus \{0\}$ injective => Mult surjective => Field

2. Apply 1) to $A/\mathfrak{p}$.

3. Use $0 = \mathrm{Nil}(A) = \cap_i \mathfrak{m}_i$. Generalize CRT to get $A/\bigcap_i \mathfrak{m}_i \cong \prod_i A/\mathfrak{m}_i$. By A finite dimensional, there are only finitely many factors.