Solutions to Sheet 1

Exercise 1

Determine the nilradical, the Jacobson radical and the units for each ring A below:

1. k a field and $A = k[T]$,
2. k a field and $A = k[\epsilon, T]/(\epsilon^2)$,
3. $n \geq 1$, k a field and $A = k[\![T_1, \dots, T_n]\!]$.

Solution.

1. *Nilradical.* If B is any commutative ring without zero divisors, then $B[T]$ doesn't have zero divisors. Indeed, if $f, g \in B[T]$ with $fg = 0$, we can look at the leading terms of f and g, obtaining $f = 0$ or $g = 0$. We now obtain $\mathrm{Nil}(A) = (0)$ as every element in the nilradical is a zero divisor.

   Units. Obviously, $k^\times \subset k[T]^\times$. We have the additive degree map $\deg : k[T]^\times \to \mathbb{N}_0$. If we have elements $f, g \in k[T]$ with $fg = 1$, then $0 = \deg(fg) = \deg(f) + \deg(g)$, thereby $\deg(f) = \deg(g) = 0$ and $f, g \in k^\times$. This shows that $k^\times \supset k[T]^\times$, and we have equality.

   Jacobson radical. Note that if B is any commutative ring and $f \in \mathrm{Jac}(B)$, then $1 + f \in B^\times$. Indeed, if we had $1 + f \notin B^\times$, we'd find some maximal ideal $\mathfrak{m}$ containing $1 + f$ (by Zorn's lemma). But now $f \in \mathfrak{m}$ (as $f \in \mathrm{Jac}(B)$) and $1 + f \in \mathfrak{m}$, hence $1 \in \mathfrak{m}$. This is a contradiction. Thereby we obtain that every $f \in \mathrm{Jac}(A)$ has degree 0, i.e., lies in k. As $A^\times \cap \mathrm{Jac}(A) = \emptyset$, we find $\mathrm{Jac}(A) = 0$. (As $\mathrm{Jac}(A) \supset \mathrm{Nil}(A)$, this is stronger than $\mathrm{Nil}(A) = 0$.)

2. *Nilradical and Jacobson radical.* We claim that if $I \subset \mathrm{Nil}(A)$, there is an equality $\mathrm{Nil}(A)/I = \mathrm{Nil}(A/I)$. Indeed, this can be seen directly by writing the nilradical as the intersection of prime ideals. The same statement is true for the Jacobson radical.

   We apply this statement with $I = (\varepsilon)$. As $\varepsilon^2 = 0$, we have $I \subset \mathfrak{p}$ for every prime ideal, hence $(\varepsilon) \subset \mathrm{Jac}(A)$. As $A/(\varepsilon) \cong k[T]$, we have $(0) = \mathrm{Nil}(A/(\varepsilon)) = \mathrm{Nil}(A)/(\varepsilon)$. This shows $\mathrm{Nil}(A) = (\varepsilon)$.

   The same proof, but with Jac in place of Nil (and maximal ideals instead of prime ideals) shows that $\mathrm{Jac}(A) = (\varepsilon)$.

   Units. There are probably smarter ways to do this, but let's try brute force. Suppose we have $f = f_1 + \varepsilon f_2$ and $g = g_1 + \varepsilon g_2$, where $f_i, g_i \in k[T]$, such that $fg = 1$. Now $1 = f_1 g_1 + \varepsilon(f_1 g_2 + f_2 g_1)$. It follows that $f_1 \in k^\times$, and we clam that this is also sufficient for $f \in A^\times$. Indeed, up to multiplication with a constant in $k^\times$, f is of the form $1 + \varepsilon f_2$, and now f admits an inverse $f^{-1} = 1 - \varepsilon f_2$.

3. *Units.* We first claim that every $f \in A$ with non-zero constant term is invertible. Indeed, after multiplying with a unit $c \in k^\times$ we may assume that $f = 1 + R$ with $R \in (T_1, \dots, T_n)$. Now, f admits the inverse $f^{-1} = \frac{1}{1-(1-f)} = \sum_{n=0}^{\infty} (1-f)^n \in k[\![T_1, \dots, T_n]\!]$.

   Jacobson radical. We first claim that A is a local ring, i.e., a ring with a unique maximal ideal. Indeed, we have seen that every element not lying in the ideal $\mathfrak{m} = (T_1, \dots, T_n)$ is invertible, hence $\mathfrak{m}$ is an ideal that contains all other ideals.

Nil radical. We want to show that A is reduced. More generally, we prove the following statement, from where the claim follows by induction.

If B is reduced, $B[\![T]\!]$ is reduced.

for the sake of contradiction, assume that $f \in B[\![T]\!]$ is a non-zero power series with $f^n = 0$. Write $f = a_d T^d + a_{d+1} T^{d+1} + \dots$ with $a_d \neq 0$. Now $f^n = 0$ implies $a_d^n = 0$, so $a_d = 0$ by reducedness of B. Hence $f = 0$.

Exercise 2

Prove the *Chinese remainder theorem*: Let A be a ring and $\mathfrak{a}, \mathfrak{b} \subset A$ two ideals such that $\mathfrak{a} + \mathfrak{b} = A$. Then the map

$$A/\mathfrak{a} \cap \mathfrak{b} \to A/\mathfrak{a} \times A/\mathfrak{b}, \quad r + \mathfrak{a} \cap \mathfrak{b} \mapsto (r + \mathfrak{a}, r + \mathfrak{b})$$

is an isomorphism. Moreover, show that $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a} \cdot \mathfrak{b}$, where $\mathfrak{a} \cdot \mathfrak{b}$ is the smalles ideal in A containing all products ab wth $a \in A$, $b \in B$. Show $a \cap b = ab$. Show that map has kernel $a \cap b$ and that homomorphism is surjective.

Solution. We first show that this map is well-defined, and indeed a homomorphism of rings. This is evident for the reduction-mod-$\mathfrak{a}$ and reduction-mod-$\mathfrak{b}$ maps $A \to A/\mathfrak{a}$ and $A/\mathfrak{b}$. By the universal property of the product of rings we obtain the map $A \to A/\mathfrak{a} \times A/\mathfrak{b}$. The kernel of this homomorphism is given by the elements in A which lie simultaneously in $\mathfrak{a}$ and $\mathfrak{b}$, hence we obtain an injective map

$$A/(\mathfrak{a} \cap \mathfrak{b}) \to A/\mathfrak{a} \times A/\mathfrak{b}.$$

To show surjectivity, it suffices to construct elements $a, b \in A$ such that $a \mapsto (0, 1)$ and $b \mapsto (1, 0)$. As $\mathfrak{a} + \mathfrak{b} = A$, there are elements $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ such that $a + b = 1$. These are the elements we are looking for! Indeed, as $a = 1 - b$ we find that a reduces to 1 mod $\mathfrak{b}$, and as $a \in \mathfrak{a}$ we find $(a + \mathfrak{a}, a + \mathfrak{b}) = (\mathfrak{a}, 1 + \mathfrak{b})$.

Remark. There is a more general version of the chinese remainder theorem which we will need in exercise 4. Namely, if $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ is a finite set of pairwise coprime ideals (meaning that for any choice $1 \leq i < j \leq n$ we have $\mathfrak{a}_i + \mathfrak{a}_j = A$), there is an isomorphism

$$A/(\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n) \cong A/\mathfrak{a}_1 \times \dots \times A/\mathfrak{a}_n.$$

To see this, one can either generalize the proof given above, or use induction after showing that the coprimality assumption implies that the ideals $(\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_{n-1})$ and $\mathfrak{a}_n$ are coprime.

We now show that $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a} \cdot \mathfrak{b}$. The inclusion $\mathfrak{a} \cap \mathfrak{b} \supset \mathfrak{a} \cdot \mathfrak{b}$ is obvious, as all products ab lie in both $\mathfrak{a}$ and $\mathfrak{b}$. To show the reverse inclusion, let $f \in \mathfrak{a} \cap \mathfrak{b}$. Again, let $a + b = 1$ with $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Then $fa + fb = f$, and the left hand side lies in $\mathfrak{a} \cdot \mathfrak{b}$ by definition.

Remark. Note that this statement is wrong if we drop the assumption that $\mathfrak{a} + \mathfrak{b} = 1$. Indeed, take for example $\mathfrak{a} = (4)$, $\mathfrak{b} = (6)$ as ideals of $\mathbb{Z}$. Then $\mathfrak{a}\mathfrak{b} = (24)$, while $\mathfrak{a} \cap \mathfrak{b} = (12)$. However, the assumption that $\mathfrak{a} + \mathfrak{b} = A$ is not necessary. In the case $A = k[X, Y]$, $\mathfrak{a} = (X)$ and $\mathfrak{b} = (Y)$ we still have $\mathfrak{a}\mathfrak{b} = (XY) = \mathfrak{a} \cap \mathfrak{b}$ even though $\mathfrak{a} + \mathfrak{b} = (X, Y) \neq A$.

Exercise 3

Recall that an element $e \in A$ in a ring A is called idempotent if $e^2 = e$.

1. Let A be a ring. Show that the map $e \mapsto (A_1 \coloneqq eA, A_2 \coloneqq (1-e)A)$ induces a bijection between the set $\mathrm{Idem}(A)$ of idempotents of A and the set of decompositions $A = A_1 \times A_2$ of rings.
2. Let $A = \mathbb{Z}/133\mathbb{Z}$. Determine $\mathrm{Idem}(A)$.

Solution.

1. The exercise does not make clear what it means by a decomposition. In the scope of this exercise, a decomposition of A is an isomorphism $\delta : A \to A_1 \times A_2$, where A_1 and A_2 are any two rings. We say that two decompositions $\delta_1 : A \to A_1 \times A_2$ and $\delta_2 : A \to B_1 \times B_2$ are isomorphic iff there are isomorphisms $\varphi_i : A_i \to B_i$, $i = 1, 2$ such that $(\varphi_1, \varphi_2) \circ \delta_1 = \delta_2$. We define the set D_A as the set of isomorphism classes of the set[1] of decompositions, and we'll show that the map specified in the exercise gives a bijection $\mathrm{Idem}(A) \to D_A$.

   First, note that $(1-e)^2 = (1-e)$ for any idempotent e.

   We have show that the map really is a map! That is, we show that for any idempotent element $e \in A$, there is an isomorphism $\delta_e : A \cong eA \times (1-e)A$, where eA and $(1-e)A$ carry the ring structure of A, but with identity given by e and $(1-e)$, respectively. Surjectivity is comes from the fact that $(ea, (1-e)b)$ has preimage $(ea + (1-e)b)$, and injectivity boils down to the calculation $\mathrm{Ker}(\delta_e) = (e) \cap (1-e) = (e) \cdot (1-e) = (0)$.

   Next, note that we also have a map $D_A \to \mathrm{Idem}(A)$ given by sending $\delta : A \to A_1 \times A_2$ to $e_\delta \coloneqq \delta^{-1}(1, 0)$. This map does not depend on the isomorphism class of δ as ring homomorphisms preserve the multiplicative unit. One quickly verifies that $\mathrm{Idem}(A) \to D_A \to \mathrm{Idem}(A)$ is the identity. The last thing to see is that $D_A \to \mathrm{Idem}(A) \to D_A$ is the identity as well, which is the same as showing that for a given decomposition $\delta : A \to A_1 \times A_2$, there is an isomorphism $\delta \cong \delta_{e_\delta}$. Such an isomorphism is the same as isomorphisms $\varphi_1 : e_\delta A \to A_1$, $\varphi_2 : (1 - e_\delta)A \to A_2$. As δ sends the ideal $(e) \subset A$ to the ideal generated by $(1, 0)$ in $A_1 \times A_2$, δ restricts to an isomorphism (of modules) $e_\delta A \to A_1 \times \{0\}$. This yields an isomorphism (of rings) $\varphi_1 : e_\delta A \to A_1$. Similarly for the second coordinate. Now (φ_1, φ_2) constitute an isomorphism $\delta \cong \delta_{e_\delta}$.

2. Note that $133 = 19 \times 7$, hence by the chinese remainder theorem $\mathbb{Z}/133 \cong \mathbb{Z}/19 \times \mathbb{Z}/7$. The right hand side is a product of fields, and it is clear that the only idempotents there are given by $(0, 0), (1, 0), (0, 1), (1, 1)$. As $1 = 19 \cdot 3 - 7 \cdot 8$, the isomorphism from the chinese remainder theorem is given by $(a, b) \mapsto 57b + 77a$, and we find that the non-trivial idempotents are given by 57 and 77.

Exercise 4

Let k be a field and let $k \to A$ be a ring homomorphism such that A is finite dimensional over k (i.e., regarded as a k-vector space, A has finite dimension).

1. Show that A is a field if A is an integral domain.
2. Deduce that each prime ideal in A is maximal.

[1] Actually I'm not sure if this really is a set, but whatever. The decompositions will certainly form a category (a groupoid), with morphisms the isomorphisms we described. The isomorphism classes do form a set as they all are represented by quotients of A.

3. Deduce that if A is reduced, then A is isomorphic to a finite product of finite field extensions l/k.

Solution.

1. Let $x \in A$ be nonzero. Let $\varphi : A \to A$ be the map obtained by multiplication with x, i.e., $\varphi(a) = xa$. Now φ is a morphism of k-vector spaces (as $\varphi(\lambda a+b) = \lambda\varphi(a)+\varphi(b)$ for $\lambda \in k$, $a, b \in A$.), and it is injective by the fact that A is an integral domain. Indeed, if $xa = 0$, we find $a = 0$ as there are no zero divisors and $x \neq 0$. But now φ is an injective morphism between k-vector spaces of the same dimension, hence an isomorphism. In particular, we find some element $x^{-1} \in A$ such that $1 = \varphi(x^{-1}) = xx^{-1}$. Hence every non-zero element of A has an inverse, and A is a field.

2. Let $\mathfrak{p} \in A$ be a prime ideal. We apply what we showed in part 1) to $A/\mathfrak{p}$. As $\mathfrak{p}$ is prime, $A/\mathfrak{p}$ is an integral domain. But also, the composition $k \to A \to A/\mathfrak{p}$ turns $A/\mathfrak{p}$ into a k-vector space with $\dim_k(A/\mathfrak{p}) \leq \dim_k(A)$ (surjective maps between vector spaces reduce dimension). In particular, $A/\mathfrak{p}$ is finite-dimensional over k. Now part 1) gives that $A/\mathfrak{p}$ is a field, and as an ideal is maximal if and only it's quotient ring is a field, we find that $\mathfrak{p}$ is maximal.

3. Let M be the set of maximal (or prime, they are the same by the above) ideals of A. We want to apply the chinese remainder theorem, but a priori we can't, because M might be infinite. We claim however that in our situation, M is finite. To show this, suppose that $(\mathfrak{m}_1, \mathfrak{m}_2, \dots)$ be an infinite sequence of elements in I. By the chinese remainder theorem, there is for any $N \in \mathbb{N}$ an isomorphism

$$A/(\mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_N) \cong A/\mathfrak{m}_1 \times \dots \times A/\mathfrak{m}_N.$$

The left-hand side has dimension $\leq \dim_k(A)$, as it is a quotient of A. Meanwhile, the right-hand side has dimension $\geq N$, as every quotient $A/\mathfrak{m}_i$ is a non-trivial k-vector space and thereby has dimension at least 1. If we choose $N > \dim_k(A)$, we arrive at a contradiction. Now $M = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$ is finite, and applying the chinese remainder theorem again yields the desired decomposition. All factors are field extensions of k of degree $\leq \dim_k(A)$, in particular finite.

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Solutions to Sheet 2

Exercise 1

Define $\zeta = \frac{-1+\sqrt{-3}}{2} \in \mathbb{C}$.

1. Show that ζ is a primitive third root of unity.
2. Show that the norm (for the field extension $\mathbb{Q}(\zeta)/\mathbb{Q}$ of an element $x + y\zeta \in \mathbb{Q}(\zeta)$, where $x, y \in \mathbb{Q}$, is given by $x^2 - xy + y^2$, and that this is non-negative for all $x, y \in \mathbb{Q}$.
3. Following the discussion of $\mathbb{Z}[i]$ from the lecture, show that a prime $p \neq 3$ is of the form $p = x^2 - xy + y^2$ for some $x, y \in \mathbb{Z}$ if and only if $p \equiv 1 \pmod 3$.

Solution.

1. We have
$$\zeta^3 = \left(\frac{1}{2}(-1+\sqrt{-3})\right)^3 = 1/8(-1+3\sqrt{-3}-9+3\sqrt{-3}) = 1.$$
As $\zeta \neq 1$ (and 3 has no non-trivial divisors), it is a primitive (third) root.

2. The norm is defined as the product of all galois-conjugates. The minimal polynomial of ζ is given by $f(x) = x^2 + x + 1 = (x - \zeta)(x - \overline{\zeta})$, so the only non-trivial element in the Galois-group $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ is given by the action defined via $\zeta \mapsto \overline{\zeta}$, which is the same as complex conjugation. We find
$$\mathrm{N}(x + \zeta y) = (x + \zeta y)(x + \overline{\zeta} y) = x^2 + (\zeta + \overline{\zeta})xy + \zeta\overline{\zeta}y^2.$$
The claim follows as $\zeta + \overline{\zeta} = -1$ and $\zeta\overline{\zeta} = 1$.

   It remains to show that the norm is always positive. The claim is trivial if x, y have different sign. If the sign is the same, we may wlog assume that both are positive. In that case, this is a special case of the AM-GM inequality. But for completeness, here is a calculation:
$$x^2 - xy + y^2 \geq x^2 - 2xy + y^2 = (x - y)^2 \geq 0$$

3. We want to show that there is an element $z = x + \zeta y \in \mathbb{Z}(\zeta)$ with $\mathrm{N}(z) = p$ if and only if $3 \mid p - 1$. We know from the lecture that $\mathbb{Z}[\zeta]$ is a principal ideal domain. First note that the "only if" part is trivial. Indeed, we have
$$x^2 - xy + y^2 \equiv \begin{cases} 1 \pmod 3, & \text{if } (x, y) = (1, 1), (0, 1), (1, 0) \\ 0 \pmod 3, & \text{if } (x, y) = (0, 0). \end{cases}$$
If $3 \mid x$ and $3 \mid y$ we find that $3 \mid \mathrm{N}(x + \zeta y)$, hence $\mathrm{N}(x + \zeta y)$ cannot be a prime. This shows that all primes of the form $x^2 - xy + y^2$ have residue 1 mod 3.

   To show the converse implication, let $p \in \mathbb{Z}$ be any prime. As $\mathbb{Z}[\zeta]$ is a PID, the prime elements $\pi \in \mathbb{Z}[\zeta]$ that divide p are in bijection with the maximal (equivalently, non-zero prime) ideals $\mathfrak{m} \subset \mathbb{Z}[\zeta]$ such that $\mathfrak{m} \cap \mathbb{Z} = (p)$. An easy computation shows (lecture 3) that these ideals are in bijection with the irreducible monic factors of $T^2 + T + 1$ in $\mathbb{F}_p[T]$. As $\mathbb{F}_p[T]$ has a non-trivial third root of unity if and only if $3 \mid p - 1$, we find that there are two prime ideals "above" (p) if $3 \mid p - 1$.

Hence, let π_1, π_2 be the two prime elements of $\mathbb{Z}[\zeta]$ that divide p and write $(p) = (\pi_1^{e_1})(\pi_2^{e_2})$. As in the lecture we find $\mathrm{N}(\pi_1) = \mathrm{N}(\pi_2) = p$, which implies $e_1 = e_2 = 1$. Now we have a primary decomposition $p = \pi_1\pi_2$, which implies that $\pi_1 = \overline{\pi}_2$, which gives the desired representation of p.

Exercise 2

1. Let A be a principal ideal domain that is not a field, and let $\mathfrak{m} \subset A$ be a maximal ideal. Prove that $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a one-dimensional vector space over $A/\mathfrak{m}$ for any $n \geq 0$.
2. Let $A = \mathbb{C}[x, y]$ and $\mathfrak{m} = (x, y)$. Compute $\dim_{A/\mathfrak{m}}(\mathfrak{m}^n/\mathfrak{m}^{n+1})$ for $n \geq 0$. Deduce that A is not a principal ideal domain.
3. Let $A = \mathbb{Z}[\sqrt{-3}]$. Show that A has a unique maximal ideal $\mathfrak{m}$ with $\mathfrak{m} \cap \mathbb{Z} = (2)$. Compute $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$. Deduce that A is not a principal ideal domain.

Solution.

1. Let $\pi \in A$ such that $(\pi) = \mathfrak{m}$. We have the map (of A-modules)

$$\varphi : A \to \mathfrak{m}^n/\mathfrak{m}^{n+1}, \quad a \mapsto a\pi^n.$$

   It is obviously surjective, and one quickly verifies that the kernel is given by (π). Hence we find $A/\mathfrak{m} \cong \mathfrak{m}^n/\mathfrak{m}^{n+1}$, and we are done.
2. We have $\mathfrak{m}^n = (x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n)$. These generators form a basis for $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ (they are generating and linearly independent over $\mathbb{C}$), hence the dimension is $n+1$. This contradicts what we showd for principal ideal domains once $n \geq 1$.
3. We first show that there is a unique maximal ideal of A with $\mathbb{Z} \cap \mathfrak{m} = (2)$. Indeed, those maximal ideals are in bijection with the maximal ideals of $\mathbb{F}_2[T]/(T^2 + 3)$. As $T^2 + 3$ factors in $\mathbb{F}_2[T]$ as $(T + 1)^2$, we find that $\mathfrak{m} = (2, \sqrt{-3} + 1)$ is the unique maximal ideal of $\mathbb{Z}[\sqrt{-3}]$ above (2).

   Now $\mathfrak{m}^2 = (4, 2\sqrt{-3} + 2, -2 + 2\sqrt{-3})$. Hence the elements 2 and $\sqrt{-3} + 1$ do not lie in $\mathfrak{m}^2$ as they have norm 4 (after choosing an embedding into $\mathbb{C}$), while all elements generating $\mathfrak{m}^2$ have norm 16. Hence there are at least 3 elements in $\mathfrak{m}/\mathfrak{m}^2$, thereby $\dim_{\mathbb{F}_2} \mathfrak{m}/\mathfrak{m}^2 \neq 1$.

Exercise 3

Let A be a unique factorization domain.

1. Show that for any prime element $\pi \in A$, the ideal $\mathfrak{p} = (\pi)$ is prime and only contains the prime ideals $\{0\}$ and $\mathfrak{p}$.
2. Conversely, let $0 \neq \mathfrak{p} \subset A$ be a prime ideal such that $\{0\}$ and $\mathfrak{p}$ are the only prime ideals of A that are contained in $\mathfrak{p}$. Show that $\mathfrak{p} = (\pi)$ for some prime element $\pi \in A$.
3. Assume that each non-zero prime ideal $\mathfrak{p} \subset A$ satisfies the assumption in 2). Show that A is a principal ideal domain.

Solution.

1. Let $0 \neq \mathfrak{q}$ be a prime contained in $\mathfrak{p}$. Take some nonzero element $q \in \mathfrak{q}$. Write $q = a\pi^n$, where $a \in A$ is an element not divisible by π. Now, as $\mathfrak{q}$ is prime, either $\pi^n \in \mathfrak{q}$ or $a \in \mathfrak{q}$. But we have $a \notin (\pi) \subset \mathfrak{q}$, hence $\pi^n \in \mathfrak{q}$. Induction shows that $\pi \in \mathfrak{q}$, which results in $\mathfrak{q} = \mathfrak{p}$.

2. Suppose $\pi \in \mathfrak{p}$ is a prime element contained in $\mathfrak{p}$. Then $(\pi) \subset \mathfrak{p}$, which by assumption shows $(\pi) = \mathfrak{p}$. We only need to show that there are prime ideals in any nonzero element $\mathfrak{p}$. For that sake, let $a \in \mathfrak{p}$. There is a finite decomposition $a = \prod_{i=1}^{n} p_i^{e_i}$, and we find that for some i, the prime element p_i lies in $\mathfrak{p}$.

3. Let $I \neq (0)$ be any ideal. Let $\pi_1, \dots, \pi_n$ be the finite set of primes such that $I \subset (\pi_i)$ (this is a finite set because any $f \in I$ has only a finite number of divisors), and let e_i be the maximal integer such that $I \subset (\pi_i^{e_i})$ holds. Write $\alpha = \pi_1^{e_1} \dots \pi_n^{e_n}$. We claim that $I = (\alpha)$. The inclusion "$I \subset (\alpha)$" is trivial.

   To show the other direction, it suffices to show that $\alpha \in I$. Suppose that $I = (g_i \mid i \in I)$. Write $g_i = h_i\alpha$ and inspect the ideal $I' = (h_i \mid i \in I)$. By construction there is no prime $\pi \in A$ such that $I' \subset (\pi)$, otherwise the factors e_i would not have been chosen maximal. But this shows that $I' = (1)$, i.e., $\alpha \in I$.

Exercise 4

1. Let A be any ring. Show that A contains minimal prime ideals.

2. Determine the minimal prime ideals of $\mathbb{Z}[x, y]/(xy)$.

Solution.

1. What does Zorn's Lemma say again? Ah. If in an ordered set we can show that any totally ordered chain has a minimal element, then there are minimal elements. As our ordered set we take the set of prime ideals, ordered by inclusion. To apply Zorn's lemma, let $\mathfrak{p}_1 \supset \mathfrak{p}_2 \supset \dots$ be a decreasing chain of prime ideals. We need to show that this chain has a minimal element, which is a prime ideal $\mathfrak{p}$ such that $\mathfrak{p}_i \supset \mathfrak{p}$. We set $\mathfrak{p} = \bigcap_{i \in \mathbb{N}} \mathfrak{p}_i$, and we have to show that this is a prime ideal. This is straight-forward. Assume that $ab \in \mathfrak{p}$. Assume $b \notin \mathfrak{p}$. Then, there is some i such that $b \notin \mathfrak{p}_i$, and hence $b \notin \mathfrak{p}_j$ for all $j \geq i$. But now, as all of the $\mathfrak{p}_i$ are prime, we find that $a \in \mathfrak{p}_i$ for all i. Hence $a \in \mathfrak{p}$, and we are done.

2. We use that minimal prime ideals of $\mathbb{Z}[x, y]/(xy)$ are exactly those prime ideals of $\mathbb{Z}[x, y]$ that are minimal among those containing (xy). Using that $\mathbb{Z}[x, y]$ is a UFD, we find that those prime ideals are given by (x) and (y).

Solutions to Sheet 3

Exercise 1

Let A be a PID. The arguments of $A = \mathbb{Z}$ from the lecture work verbatim to show that the prime ideals of $A[T]$ are

1. (0),
2. (f), $f \in A[T]$ irreducible,
3. (π, g) with $\pi \in A$ prime and $g \in A[T]$ a polynomial whose image in $(A/\pi)[T]$ is irreducible.

Show the following.

1. Assume that A has infinitely many prime ideals. Prove that the heights of the primes in (i), (ii) and (iii) are given by $0, 1, 2$ respectively. Show that each maximal ideal of $A[T]$ has height 2.
2. Let k be a field and set $A = k[\![u]\!]$. Show that $A[u^{-1}]$ is a field. Deduce that, in contrast to 1), the height 1 ideal $(uT - 1)$ is maximal.

Solution.

1. That (0) is of height zero is obvious. We showed on the last sheet that the only prime ideals contained in principal prime ideals (f) of UFDs are (0) and (f). As polynomial rings over UFDs are UFDs again, we are done with this case.

   The height of primes of the third form is at least 2. Indeed, we have inclusions $(0) \subset (\pi) \subset (\pi, g)$. We want to show that the height doesn't get larger than 2. The only thing that can go wrong is that there might be inclusions $(\pi, g) \subset (\pi', g')$.

   Assume we are given two prime ideals $\mathfrak{p} = (\pi, g) \subset (\pi', g') = \mathfrak{p}'$. By this inclusion we find $\mathfrak{p} \cap A = \mathfrak{p}' \cap A$, which shows $(\pi) = (\pi')$. But $A/(\pi)$ is a field, hence $A/(\pi)[T]$ is a PID and we find that the reductions of g and g' mod π are the same. This shows $(\pi, g) = (\pi, g')$, and we are done. (We have not used yet that there are infinitely many prime ideals).

   We also have to show that every maximal ideal is of the form (π, g). To this end, we have to show that every ideal of the form (f) is contained in some ideal (π, g). But if we write such f as $f = a_d T^d + \cdots + a_0$ and choose some prime $\pi \in A$ that does not divide a_d, we find that the reduction of f mod π is monic, at least up to multiplication with some unit. Hence we can choose some irreducible factor $g \in (A/\pi)[T]$ of f and lift it to a function $\tilde{g} \in A[T]$. We find that $(\pi, \tilde{g})$ is prime and contains (f), as desired.

2. We have $A[T]/(uT - 1) = A[u^{-1}]$. Note that A is a local ring and in particular a principal ideal domain (but with only a single prime). We have seen on a prior sheet that every element $x \in A \setminus (u)$ is invertible, hence we done.

Exercise 2

Let k be an algebraically closed field and let

$$\varphi : k[x, y] \to k[u, v], \quad x \mapsto u, \ y \mapsto uv.$$

1. Use exercise 1 to show that the maximal ideals of $k[x, y]$ are precisely the ideals

$$\mathfrak{m}_{\lambda,\mu} := (x - \lambda, y - \mu), \quad \lambda, \mu \in k.$$

2. Show that φ induces an isomorphism $k[x, y][x^{-1}] \to k[u, v][u^{-1}]$.
3. For each $(\lambda, \mu) \in k^2$ calculate $\mathrm{Spec}(\varphi)^{-1}(\mathfrak{m}_{\lambda,\mu})$.

Solution.

1. By exercise 1, the maximal ideals are precisely the ideals of the form (π, g) where $\pi \in k[x]$ is prime and $g \in k[x, y]$ is an element with reduction mod π is irreducible. As k is algberically closed, we find that $(\pi) = (x - \lambda)$ for some $\lambda \in k$. Now $k[x]/(x - \lambda) \cong k$, the isomorphism is given by $x \mapsto \lambda$. Hence $g(\lambda, y) \in k[y]$ needs to be irreducible, i.e., of the form $y - \mu$.
2. We can give an isomorphism $k[u, v][1/v] \to k[x, y][1/x]$ via $u \mapsto x$ and $v \mapsto y/x$. Checking that this is an isomorphism is straight-forward.
3. Note that $\mathrm{Spec}(\pi)^{-1}(\mathfrak{m}_{\lambda,\mu})$ is equal to the set of prime ideals $\mathfrak{p} \subset k[u, v]$ for which $\varphi(\mathfrak{m}_{\lambda,\mu}) \subset \mathfrak{p}$. By the homomorphism theorem, we find

$$\{\mathfrak{p} \subset k[u, v] \text{ prime} \mid \varphi(\mathfrak{m}_{\lambda,\mu}) \subset \mathfrak{p}\} \xrightarrow{1:1} \mathrm{Spec}(k[u, v]/\varphi(\mathfrak{m}_{\lambda,\mu})).$$

We find $\varphi(\mathfrak{m}_{\lambda,\mu}) = (u - \lambda, uv - \mu)$, hence

$$k[u, v]/\varphi(\mathfrak{m}_{\lambda,\mu}) \cong k[u, v]/(u - \lambda, uv - \mu) \cong k[v]/(v\lambda - \mu).$$

But this can be calculated explicitely:

$$k[v]/(v\lambda - \mu) \cong \begin{cases} k, & \text{if } \lambda \neq 0 \\ k[v], & \text{if } \lambda = 0 \text{ and } \mu = 0 \\ 0, & \text{if } \lambda = 0 \text{ and } \mu \neq 0 \end{cases}$$

Exercise 3

Let A be a ring of Krull dimension $n := \dim A$. Show that

$$n + 1 \leq \dim A[T] \leq 2n + 1.$$

Solution. Let

$$0 = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_m$$

be a maximal chain of prime ideals in $A[T]$. Write $\mathfrak{p}_i = \mathfrak{q}_i \cap A$. We obtain an ascending chain of prime ideals

$$0 = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_m$$

in A, which by the assumption on the dimension of A contains at most $n+1$ different prime ideals. We will show that $\mathfrak{p}_i = \mathfrak{p}_{i+1}$ implies $\mathfrak{p}_{i+1} \neq \mathfrak{p}_{i+2}$, which shows $m \leq 2n+1$.

More generally, we'll show that for every prime ideal $\mathfrak{p} \subset A$ the set of prime ideals $\mathfrak{q} \subset A[T]$ with $\mathfrak{q} \cap A = \mathfrak{p}$ (the primes above $\mathfrak{p}$) can have chains of length at most two. Let $\mathfrak{p} \subset A$ be such a prime ideal. For $S = A \setminus \mathfrak{p} \subset A \subset A[T]$ we make use of the bijection

$$\{\text{prime ideals in } S^{-1}A\} \xrightarrow{1:1} \{\text{prime ideals in } A \text{ not intersecting } S\},$$

which shows that there is no difference between primes above $\mathfrak{p}$ in A and in $A_\mathfrak{p} = S^{-1}A$. Hence we may assume that $\mathfrak{p}$ is maximal in A. Now $A/\mathfrak{p}$ is a field, hence $A/\mathfrak{p}[T]$ is a PID, hence of dimension 1. The inclusion-preserving bijection

$$\{\mathfrak{q} \subset A/\mathfrak{p}[T]\} \xrightarrow{1:1} \{\mathfrak{q} \subset A[T] \mid A[T]\mathfrak{p} \subset \mathfrak{q}\}$$

solves the exercise.

Exercise 4

Let A be a ring and $S, T \subset A$ multiplicative subsets with $S \subset T$.

1. Let $\iota_S : A \to S^{-1}A$ be the natural ring homomorphism. Show that $\iota_S^{-1}((S^{-1}A)^\times)$ is the saturation $\overline{S}$ of S.
2. Show that there exists a unique ring homomorphism $\iota : S^{-1}A \to T^{-1}A$ such that $\iota \circ \iota_S = \iota_T$.
3. Deduce that ι is an isomorphism if and only if $\overline{S} = \overline{T}$.

Solution.

First a reminder: The saturation of a subset $S \subset A$ is given by the set

$$\overline{S} = \{s \in A \mid \exists a \in A : as \in S\}.$$

1. First remember that ι_S is given by $a \mapsto \frac{a}{1}$. Now let's try to work out what the units in $S^{-1}A$ are. Remember that $S^{-1}A$ has underlying set

$$(A \times S)/\sim_S, \quad \text{where} \quad (a,s) \sim_S (a',s') \text{ iff } as' = a's.$$

   In particular, we find that an element $(a,1)$ $(= \frac{a}{1} = \iota_S(a))$ lies in the units of $S^{-1}A$ if and only if there are $a' \in A$, $s' \in S$ with $(aa', s') \sim_S (1,1)$. This condition is equivalent to $(aa', 1) \sim_S (s', 1)$, which translates directly to what we had to show.

2. We define ι on representing objects using the inclusion $A \times S \to A \times T$. It is clear that this morphism behaves well under the equivalence relations $\sim_S$ and $\sim_T$ (here $\sim_T$ is defined the same way as for $\sim_S$), so we obtain a well-defined function

$$S^{-1}A \cong (A \times S)/\sim_S \to (A \times T)/\sim_T \cong T^{-1}A.$$

   One readily checks that this indeed gives a map of rings (with addition and multiplication defined accordingly). One also readily checks that $\iota \circ \iota_S = \iota_T$.

3. We show that the saturation of S is maximal among the subsets $S \subset S' \subset A$ with $S'^{-1}A \cong S^{-1}A$ (where the induced morphism is given by ι). We first note that there is no difference between localizing at S and localizing at $\overline{S}$. Indeed, given some $s \in \overline{S}$, there is some $a \in A$ with $as \in S$. But now, given any $b \in A$, the element $\frac{b}{s} \in \overline{S}^{-1}A$ lies in the same equivalence class as $\frac{ba}{sa} \in S^{-1}A$. (Alternatively, this follows directly from what we showed in part 1: We have $\overline{S}^{-1}A = \overline{S}^{-1}(S^{-1}A) = S^{-1}A$, where we used in the last equality that $\overline{S} \subset (S^{-1}A)^{\times}$). Next, any subset $S \subset T \subset A$ that is not contained in $\overline{S}$ has non-isomorphic localization. Indeed, assume $t \in T \setminus \overline{S}$. Then the equivalence class of $\frac{1}{t} \in T^{-1}A$ does not lie in the image of ι by construction. Finally, note that whenever $S \subset T \subset \overline{S}$, we have $\overline{T} = \overline{S}$.

   This solves the exercise in an instant. For the one direction, if $\overline{S} = \overline{T}$, we find

$$S^{-1}A \cong \overline{S}^{-1}A = \overline{T}^{-1}A \cong T^{-1}A.$$

   For the other direction, if $S^{-1}A \cong T^{-1}A$, the result above directly implies $\overline{S} = \overline{T}$.

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Solutions to Sheet 4

Exercise 1

Let A be a ring.

1. Assume that $f_n \in A[\![T]\!]$, $n \geq 0$, is a sequence of elements such that $f_n \in (T)^n$ for all $n \geq 0$. Show that there exists a unique element $f \in A[\![T]\!]$ such that $f - \sum_{k=0}^{n} f_k \in (T)^{n+1}$ for all $n \geq 0$.

2. Assume that A is noetherian. Show that $A[\![T]\!]$ is noetherian.

Solution.

1. We can just write down f. We need to find coefficients a_n such that $f = \sum_{n=0}^{\infty} a_n T^n$ satisfies $f - \sum f_k \in (T)^{n+1}$. Write $f_k = \sum_{j=0}^{k} a_{kj} T^j + (T)^k$. One quickly verifies that $a_n = \sum_{k=0}^{n} a_{kn}$ does the job.

2. Similar to the proof that the polynomial ring over a noetherian ring is noetherian, we let $I \subset A[\![T]\!]$ denote any ideal and denote by I' the ideal of A generated by the leading coefficients of functions in f, namely $I' := (a_d \mid f = a_d T^d + a_{d+1} T^{d+1} + \dots \in A[\![T]\!])$. As A is noetherian, there is a finite number of elements $f_1, \dots, f_n$ such that the leading (non-zero) coefficients of f_i gerate I'. Upon multiplying with powerst of T, we may assume that all f_i are of the form $f_i = a_{id} T^d + \dots$ with $a_{id} \neq 0$ for some some suitable d.

   Now we claim that any $g \in I \cap T^d$ also lies in $(f_1, \dots, f_n)$. Indeed, writing $g = b_d T^d + b_{d+1} T^{d+1} + \dots$, we find that $b_d \in I'$, so we can eliminate the term $b_d T^d$ from g without leaving $I \cap T^d$. But now $g' = g - b_d T^d \in I \cap (T^{d+1})$. Upon repeatedly eliminating leading coefficients, we find $g \in (f_1, \dots, f_n)$.

   To finish the argument, note that $A[\![T]\!]/(T^d) \cong A[T]/(T^d)$ is noetherian. Hence the image of I in this quotient is finitely generated, by $(g_1, \dots, g_m)$, say. Choose lifts $(\tilde{g}_1, \dots, \tilde{g}_m)$. Now, by construction, $I = (g_1, \dots, g_m, f_1, \dots, f_n)$.

Exercise 2

1. Let A be the ring of power series in $\mathbb{C}[\![z]\!]$ with a positive radius of convergence. Show that A is noetherian.

2. Show that the ring of holomorphic functions is not noetherian.

Solution.

1. One can quickly verify that all ideals of A are of the form (z^d). Indeed, every function that does not vanish at 0 does not have a root in some neighbourhood of 0 (by the identity theorem), hence admits a holomorphic inverse there. This shows that the units in A are given by $A \setminus (z)$. Now any non-unit is of the form $z^d u$ with u invertible and $d \geq 1$. The claim follows.

2. The hint commanded us to make use of the equation $\sin(2x) = 2\sin\cos(x)$. This shows that there is an infinite descending chain of ideals $(\sin(x)) \subset (\sin(x/2)) \subset (\sin(x/4)) \subset \dots$. It is clear that this chain does not get stationary, by looking at the real roots of those functions.

Exercise 3

Let $n \geq 1$. For an $n \times n$ matrix M over some ring A denote by $\chi_M(T) = \det(T \cdot \mathrm{Id} - M)$ its characteristic polynomial.

1. Let $A = \mathbb{Z}[a_{ij} \mid 1 \leq i, j \leq n]$ and $M := (a_{ij})_{ij} \in \mathrm{Mat}_n(A)$. Show that $\chi_M(M) = 0$.
2. Deduce a general form of the theorem of Cayley-Hamilton: Let A be a ring and let M be any $n \times n$ matrix over A. Then $\chi_M(M) = 0$.

Solution.

1. Since A is integral, we can pass to the field of fractions of A. Now the regular cayley hamilton applies. (Note that the calculation of the determinant does not depend on whether we are in the field of fractions or not).
2. There is a surjective map $\pi : \mathbb{Z}[a \in A] \to A$ given by $a \mapsto a$. By part 1 we find that $\chi_M(M) = 0$ in $\mathbb{Z}[a \in A]$. Now $0 = \pi(\chi_M(M)) = \chi_M(M)$. Done?

Exercise 4

Let A be a principal ideal theorem.

1. Let $a \in A \setminus \{0\}$ and let $\pi \in A$ be prime. Set $B := A/(a)$. For any $n \geq 0$ show that
$$\dim_{A/(\pi)} \pi^n B/\pi^{n+1}B = \begin{cases} 0, & \text{if } \nu_\pi(a) \leq n \\ 1, & \text{if } \nu_\pi(a) \geq n+1. \end{cases}$$
2. Assume that $M = A^r \oplus A/(a_1) \oplus \cdots \oplus A/(a_k)$, $N = A^s \oplus A/(b_1) \oplus \cdots \oplus A/(b_l)$ with $a_i, b_i \in A$ non zero and $a_1 \mid a_2 \mid \cdots \mid a_k$, $b_1 \mid \cdots \mid b_l$. Show that if $M \cong N$, then $r = s$, $k = l$ and $(a_i) = (b_i)$ for all i.

Solution.

1. First, note that there are isomorphism (of $A/(\pi)$ vector spaces)
$$\pi^n B/\pi^{n+1}B \cong \pi^n \left(A/(a)\right)/\pi^{n+1}\left(A/(a)\right) \cong \pi^n A/((\pi^{n+1}, a) \cap (\pi^n)).$$
We have seen before that for principal ideal domains, we have $\dim_{A/(\pi)}(\pi^n A)/(\pi^{n+1}A) = 1$. The right hand side of the equation above is the same as $\pi^n A/\pi^{n+1}A$ if $a \in (\pi^{n+1})$, otherwise we quotient out by some non-zero subspace. Hence we see that the quotient above vanishes if and only if $a \notin \pi^{n+1}$, that is, if and only if $\nu_\pi(a) \geq n+1$.

2. The exercise is confusing, because depending on how one thinks about modules over rings (especially if one thinks of them as generalized vector spaces) it might seem tautological. The main problem is that plain isomorphisms don't necessarily respect direct sums. Over non-commutative rings, there even are examples of modules S for which $S \oplus S \cong S$! Over PIDs however, everything seems to be well-behaved. We solve the exercise in two steps.
Step 1: $r = s$. Let $K = \operatorname{Frac}(A)$ denote the field of fractions of A. Then upon tensoring with K, the torsion part of M and N vanishes. Now by $M \cong N$, we find $K^r \cong K \otimes M \cong K \otimes N \cong K^s$. As every basis of a finite dimensional vector space has the same number of elements, this shows $r = s$. Appearently you have also already seen this in the lecture.
Step 2: The torsion part. We fix some prime element $\pi \in A$ and use part 1 of the exercise. The isomorphism $M \cong N$ yields an isomorphism $\pi^n M/\pi^{n+1}M \cong \pi^n N/\pi^{n+1}N$. Choosing $n = \nu_\pi(a_k)$, this shows that the π-adic valuations of a_k and b_l agree. Moreover, the number
$$d_n = \dim_{A/(\pi)} \pi^n M/\pi^{n+1}M = \dim_{A/(\pi)} \pi^n N/\pi^{n+1}N$$
is the number of elements among the a_i such that $\nu_\pi(a_i) = n$. Iterating over all prime numbers π, this shows that

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Solutions to Sheet 5

Exercise 1

Let A be a ring and let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n \subset A$ be ideals such that $\bigcap_{i=1}^n \mathfrak{a}_i = \{0\}$. Assume that each ring $A/\mathfrak{a}_i$ is noetherian. Show that A is noetherian.

Solution. Let $\pi_i : A \to A/\mathfrak{a}_i$ denote the projections. We have the map

$$\pi = (\pi_1, \ldots, \pi_n) : A \to A/\mathfrak{a}_1 \times \cdots \times A/\mathfrak{a}_n.$$

As the $\mathfrak{a}_i$ have intersection $\{0\}$, π is injective. Hence A is isomorphic to the subring $\operatorname{Im}(\pi) \subset A/\mathfrak{a}_1 \times \cdots \times A/\mathfrak{a}_n$. This shows that A is isomorphic to the subring of a noetherian ring, thereby noetherian.

Exercise 2

Consider the matrix

$$S \coloneqq \begin{pmatrix} -36 & 14 & -24 \\ 18 & 6 & 12 \end{pmatrix}.$$

Determine its elementary divisors and the kernel/cokernel of the map $\mathbb{Z}^3 \xrightarrow{S} \mathbb{Z}^2$ (up to isomorphy).

Solution. We want to find simpler representatives of the residue class of S in the double quotient $\mathrm{GL}_2(A) \backslash \mathrm{Mat}_{2\times 3}(A) / \mathrm{GL}_3(A)$. We add twice the lower row to the upper row (which is the same as multiplying by $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ from the left), which gives

$$S \sim \begin{pmatrix} 0 & 26 & 0 \\ 18 & 6 & 12 \end{pmatrix}.$$

Further transformations yield

$$\begin{pmatrix} 0 & 26 & 0 \\ 18 & 6 & 12 \end{pmatrix} \to \begin{pmatrix} 0 & 26 & 0 \\ 6 & 6 & 12 \end{pmatrix} \to \begin{pmatrix} 0 & 26 & 0 \\ 6 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 6 & 0 & 0 \\ 0 & 26 & 0 \end{pmatrix}.$$

This allows us to calculate kernel and cokernel of S. We find

$$\operatorname{Ker}(S) \cong \mathbb{Z}, \quad \operatorname{Coker}(S) \cong \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/26\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/78\mathbb{Z}.$$

This shows that the elementary divisors are given by 2 and 78.

Exercise 3

Let A be a ring, let $\mathfrak{a} \subset A$ be an ideal and let M, N_i, $i \in I$, be A-modules for some set I.

1. Show that there exists a unique isomorphism

$$\Phi : \bigoplus_{i\in I} (N_i \otimes_A M) \to \left(\bigoplus_{i \in I} N_i \right) \otimes_A M$$

such that $\Phi((\ldots, 0, n_i \otimes m, 0 \ldots)) = (\ldots, 0, n_i, 0, \ldots) \otimes m$ for all $n_i \in N_i$, $i \in I$, $m \in M$.

2. Show that there exists a unique isomorphism

$$\Psi : A/\mathfrak{a} \otimes_A M \to M/\mathfrak{a}M$$

such that $\Psi((a+\mathfrak{a}) \otimes m) \mapsto am + \mathfrak{a}M$ for all $a \in A$, $m \in M$.

Solution. This exercise looks like you'd have to do lots of calculations, but there is the following rule:

NEVER DO ANYTHING EXPLICITLY WHEN WORKING WITH TENSOR PRODUCTS.

1. We could try to solve this by somehow checking that the map is well-defined, working everything out element-wise, and in the end showing that the isomorphism we obtain is somehow unique. But this is messy, and probably confusing to anyone who wants to follow the argument. It is much cleaner to work with universal properties. Note that $\bigoplus_{i\in I}(N_i \otimes_A M)$ satisfies the following universal property:

   For any A-module P and any tuple of bilinear maps $(\varphi_i : N_i \times M \to P)_i$,
   there is a unique linear map $\Phi : \bigoplus_{i\in I}(N_i \otimes M) \to P$ such that $\Phi(n_i \otimes m) = \varphi_i(n_i, m)$.

   That $\bigoplus_{i\in I}(N_i \otimes_A M)$ satisfies this universal property is easy to see. The UP of the tensor product gives linear maps $N_i \otimes M \to P$ associated to φ_i, and we obtain φ by the UP of the direct sum. But note that $(\bigoplus_{i\in I} N_i) \otimes M$ satisfies the same UP. Indeed, one easily checks that a tuple of bilinear maps $(\varphi_i : N_i \times M \to P)_{i\in I}$ is the same data as a single bilinear map $(\varphi : (\bigoplus_{i\in I} N_i) \times M \to P)$. This automatically gives a unique isomorphism

   $$(\bigoplus_{i\in I} N_i) \otimes M \cong \bigoplus_{i\in I}(N_i \otimes M),$$

   which is of the desired form by construction.

2. I lied to you, this time we do things explicitely. The mapping

   $$A/\mathfrak{a} \times M \to M/\mathfrak{a}M, \quad (a+\mathfrak{a}, m) \mapsto am + \mathfrak{a}M.$$

   is well-defined and bilinear, which is easy to check. This gives the desired map $\Psi : A/\mathfrak{a} \otimes_A M \to M/\mathfrak{a}M$. It is surjective as $\Psi(1 \otimes m) = m + \mathfrak{a}M$, and injective because if $\Psi((a+\mathfrak{a}) \otimes m) = 0 + \mathfrak{a}M$, we have $am \in \mathfrak{a}M$. Hence $am = a'm'$ for some $a' \in \mathfrak{a}, m' \in M$. In particular,

   $$a \otimes m = 1 \otimes (am) = 1 \otimes (a'm') = a' \otimes m' = 0 \in A/\mathfrak{a} \otimes_A M.$$

   This shows injectivity of Ψ, and we are done.

   HAHAHA FOOLS! The proof above doesn't work! Namely, to show injectivity, it does not suffice to check that there are no nontrivial elements of the form $a \otimes m$ that get sent to zero. There might still be linear combinations of such elements which are getting sent to zero. But showing that $\sum a_i \otimes m_i \mapsto 0 \implies \sum a_i \otimes m_i = 0$ is really hard, there is no way to get a handle on the sum.

   So we try UPs again. We show that for any bilinear map $(-,-) : A/\mathfrak{a} \times M \to P$ there is a unique linear map $\varphi : M/\mathfrak{a}M \to P$ with $\varphi(am) = (a, m)$. This can be checked directly.

Exercise 4

Let A be a ring and let M, N be A-modules. A bilinear map $(-,-) : M \times M \to N$ is called symmetric if $(m_1, m_2) = (m_2, m_1)$ for all $m_1, m_2 \in M$. It is called alternating if $(m, m) = 0$ for all $m \in M$.

1. Show that there exists an A-module $\mathrm{Sym}^2_A(M)$ and a symmetric bilinear map $\iota : M \times M \to \mathrm{Sym}^2_A(M)$ with the following universal property: For every A-module N and for every symmetric bilinear map $(-,-) : M \times M \to N$ there exists a unique A-linear map $\Phi : \mathrm{Sym}^2_A(M) \to N$ usch that for all $m_1, m_2 \in M$
$$(m_1, m_2) = \Psi(\iota(m_1, m_2)).$$
Construct similarly an A-module $\Lambda^2_A(M)$ with a universal alternating bilinear map $\gamma : M \times M \to \Lambda^2_A(M)$.

2. Show that $\mathrm{Sym}^2_A(A^n)$ and $\Lambda^2_A(A^n)$ are free A-modules of ranks $\frac{n(n+1)}{2}$ and $\frac{n(n-1)}{2}$.

Solution.

1. Okay, the Sym-construction should be somehow similar to the construciton of $\otimes$, and ideally all proofs of properties simply follow from the universal property of the tensor product. In the construction of the tensor product, (m_1, m_2) corresponds to the image of $\varphi(m_1 \otimes m_2)$ for some suitable morphism φ. Imposing that $(m_1, m_2) = (m_2, m_1)$ corresponds to the statement that in Sym^2_A, any morphism should send $(m_1 \otimes m_2 - m_2 \otimes m_1)$ to zero. Building on this, we define $\mathrm{Sym}^2_A(M)$ as $(M \otimes_A M)/G$, where G is the A-module generated by elements of the form $(m_1 \otimes m_2 - m_2 \otimes m_1)$. We check that this works. With the notation of the exercise, we first obtain a morphism $\psi : M \otimes_A M \to N$ by the UP of the tensor product.
$$\begin{tikzcd}
M \times M \arrow[rr, "(m_1,m_2)\mapsto m_1\otimes m_2"] \arrow[dr, "(-,-)"'] & & M \otimes M \arrow[dl, dashed, "\psi"] \arrow[dr] & \\
& N & & \mathrm{Sym}^2_A(M) \cong (M \otimes_A M)/G \arrow[ll, dashed, "\Psi"]
\end{tikzcd}$$
By construction, we have $G \subset \operatorname{Ker}\psi$, so by the universal property of kernels, ψ extends uniquely to a morphism $\Psi : \mathrm{Sym}^2_A(M) \cong (M \otimes_A M)/G \to N$.
We define $\Lambda^2_A(M)$ similarly, this time we define G as submodule of $M \otimes_A M$ generated by elements of the form $(m \otimes m)$.

2. We'll again first focus on Sym^2_A. First of all, note that the set of bilinear maps $(-,-) : A^n \times A^n \to N$ with values in an A-module N is the same as the set of matrices $(a_{ij})_{i,j=1,\dots,n}$ with $a_{ij} \in N$. The argument essentially comes from linear algebra; we simply associate to $(-,-)$ the matrix $((e_i, e_j))_{ij}$. Now, note that the subset of symmetric bilinear forms corresponds to those matrices with $a_{ij} = a_{ji}$. The set of these matrices has a natural structure of a free A-module of rank $\frac{n(n+1)}{2}$. We need to show that this number is equal to the rank of Sym^2_A. But for any A-module N, we have established the isomorphisms
$$\begin{aligned} N^{\frac{n(n+1)}{2}} &\cong \{M = (a_{ij})_{ij} \mid a_{ij} \in N \text{ and } a_{ij} = a_{ji}\} \\ &\cong \mathrm{SymBiHom}(A^n, A^n; N) \cong \mathrm{Hom}_A(\mathrm{Sym}^2_A(A^2), N). \end{aligned}$$

Here, $\mathrm{SymBiHom}(A^n, A^n; N)$ denotes the space of symmetric bilinear maps $A^n \times A^n \to N$. The functor sending N to $N^{\frac{n(n+1)}{2}}$ is represented by $A^{\frac{n(n+1)}{2}}$. Hence, utilizing the Yoneda-lemma, we find that $A^{\frac{n(n+1)}{2}} \cong \mathrm{Sym}^2_A(A^n)$.

For $\Lambda^2_A(A^n)$, we do exactly the same. The only thing that changes is the set of matrices we look at, as this time we have isomorphisms

$$\{M = (a_{ij})_{ij} \mid a_{ij} \in N \text{ and } a_{ij} = -a_{ji} \text{ and } a_{ii} = 0\} \cong \mathrm{AltBiHom}_A(A^n, A^n, N).$$

The space of matrices is quickly seen to be isomorphic to $N^{\frac{n(n-1)}{2}}$.

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Solutions to Sheet 6

Exercise 1

Let A be a ring, $f \in A$ a non-zero divisor, $\mathfrak{a} = (f)$ and $\mathfrak{b} \subset A$ an ideal. Show that the natural map

$$\mathfrak{a} \otimes_A \mathfrak{b} \to \mathfrak{a} \cdot \mathfrak{b}, \quad a \otimes b \mapsto a \cdot b$$

is an isomorphism.

Solution. As $\mathfrak{a} = (f)$, we have an isomorphism $\varphi : A \xrightarrow{\sim} \mathfrak{a}$, given by $a \mapsto fa$. Also note that $\varphi|_{\mathfrak{b}} : \mathfrak{b} \to \mathfrak{ab}$ is an isomorphism. Now we have the diagram

$$\begin{tikzcd}
\mathfrak{a} \otimes_A \mathfrak{b} & & \mathfrak{ab} \\
A \otimes_A \mathfrak{b} \arrow[u, "\varphi\otimes\mathfrak{b}"] \arrow[rr, "\sim"] & & \mathfrak{b}, \arrow[u, "\varphi|_{\mathfrak{b}}"']
\end{tikzcd}$$

where all arrows are isos, yielding an isomorphism $\mathfrak{a} \otimes_A \mathfrak{b} \to \mathfrak{ab}$.

Exercise 2

Let A be a ring, let I be a set and let $M, N_i, i \in I$ be A-modules.

1. Assume that M is finitely generated (resp. finitely presented). Show that the natural map
$$M \otimes_A \prod_{i \in I} N_i \to \prod_{i \in I} M \otimes_A N_i$$
is surjective (resp. bijective).

2. Take $A = \mathbb{Z}[X_0, X_1, \ldots]$, $J = (X_0, X_1, \ldots)$. Show that the natural map $A/J \otimes_A A[\![T]\!] \to A/J[\![T]\!]$ is not injective.

Solution.

1. First let's recall what finitely generated and finitely presented meant. An A-module M is finitely generated if there exists a surjective morphism of A-modules
$$A^{\oplus n} \to M.$$
Furthermore, we call M finitely presented if the kernel of this map is again finitely generated (that is, there is a finite number of relations among the images of the generators), which is to say that there is an exact sequence
$$A^m \to A^n \to M \to 0$$
for some integers $m, n \geq 0$.

Next, let's find out what the *natural map* is. We have for $i \in I$ the projections $\prod_{i\in I} N_i \to N_i$, which we can tensor with M to obtain maps $M \otimes_A \prod_{i\in I} N_i \to M \otimes_A N_i$. The collection of these maps gives the desired $M \otimes_A \prod_{i\in I} N_i \to \prod_{i\in I} M \otimes_A N_i$.

Note that if $M \cong A^{\oplus n}$, this natural map is an isomorphism, as we have

$$A^{\oplus n} \otimes_A \prod_i N_i \cong (A \otimes_A \prod_i N_i)^{\oplus n} \cong (\prod_i N_i)^{\oplus n} \cong \prod_i (A^{\oplus n} \otimes_A N_i).$$

Here we used the commutativity of finite direct sums and tensor products and that of finite direct sums and products (note that finite sums are isomorphic to finite products).

This puts us in the following situation, where we can use the 5-lemma.

$$\begin{tikzcd}
A^{\oplus m} \otimes_A \prod_{i\in I} N_i \arrow[r] \arrow[d, "\sim"] & A^{\oplus n} \otimes_A \prod_{i\in I} N_i \arrow[r] \arrow[d, "\sim"] & M \otimes_A \prod_{i\in I} N_i \arrow[r] \arrow[d, "\therefore \sim"] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
\prod_{i\in I}(A^{\oplus m} \otimes_A N_i) \arrow[r] & \prod_{i\in I}(A^{\oplus n} \otimes_A N_i) \arrow[r] & \prod_{i\in I}(M \otimes_A N_i) \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

It may not be clear why the top and bottom row are exact. Here we will also not give a complete answer. But the exactness of the top row follows directly by the fact that the tensor product is right exact (a good way to remember what right-exactness means is to remember that right-exact functors *turn cokernels into cokernels*), and exactness of the bottom row follows from the fact that products are exact (not in general for abelian categories but in the case of the category of R-modules). You'll have to find out what that last sentence means by yourself.

2. Note that $A/J \otimes_A A[\![T]\!] \cong A[\![T]\!]/JA[\![T]\!]$. Take the element $f := \sum_{i=1}^{\infty} x_i T^i \in A[\![T]\!]$. As all elements in $JA[\![T]\!]$ only have a finite number of x_i arise in the coefficients, we find $f \notin JA[\![T]\!]$, hence $f \neq 0$ in $A[\![T]\!]/JA[\![T]\!]$. But $f \mapsto 0$ under the natural map: All the coefficients x_i get sent to zero.

Exercise 3

Let k be a field, K/k an algebraic field extension, and $\overline{k}$ an alebraic closure of k.

1. If $V \to W$ is a k-linear injection of k-vector spaces, show that $V \otimes \overline{k} \to W \otimes \overline{k}$ is a $\overline{k}$-linear injection.
2. Show that K/k is separable if and only if the ring $K \otimes_k \overline{k}$ is reduced.

Solution.

1. All k-vector spaces are injective, hence every injective map $V \to W$ admits a section $W \to V$. Tensoring the section with $\overline{k}$ yields a section of $V \otimes_k \overline{k} \to W \otimes_k \overline{k}$.
2. We show that the following statements are equivalent:
   (a) K/k is separable.
   (b) For all $\alpha \in K$, $k[\alpha]/k$ is separable.
   (c) $k[\alpha] \otimes_k \overline{k}$ is reduced for all $\alpha \in K$.
   (d) $K \otimes_k \overline{k}$ is reduced.

(a) $\iff$ (b) is by definition. We show (b) $\iff$ (c). Let f be the minimal polynomial of some $\alpha \in K$. As K is algebraic over k, f decomposes in $\overline{k}$ as $f(x) = \prod_{i=1}^n (x - a_i)^{d_i}$ with $a_i \neq a_j$ whenever $i \neq j$. Now we find

$$k[\alpha] \otimes_k \overline{k} \cong (k[x]/f(x)) \otimes_k \overline{k} \cong \overline{k}[x]/f(x) \cong k[x]/(x-a_1)^{d_1} \times \cdots \times k[x]/(x-a_n)^{d_n},$$

which is reduced if and only if $d_1 = \cdots = d_n = 1$, which is the case if and only if $k[\alpha]$ is reduced over k.

For (d) $\implies$ (c), we use part 1. The arguments there show that $k[\alpha] \to K$ is injective, hence $k[\alpha] \otimes_k \overline{k}$ is isomorphic to a subring of $K \otimes_k \overline{k}$. But a subring of a reduced subring is reduced.

Lastly we show (a) $\implies$ (d). Let $\zeta = \sum_{i=1}^n \alpha_i \otimes b_i \in K \otimes_k \overline{k}$ be some element. Here, the α_i are elements of K, and as K is separated, we find that $k[\alpha_1, \ldots, \alpha_n]/k$ is a finite separated extension. But now, by the primitive element theorem, there is some $\alpha \in K$ with $k[\alpha] \cong k[\alpha_1, \ldots, \alpha_n]$, and $k[\alpha] \otimes_k \overline{k}$ is reduced by (c) $\iff$ (a).

Exercise 4

Let A be a ring and let I be an *invertible* A-module, i.e., there exists an A-module J such that $I \otimes_A J \cong A$. Let $\varphi : M \to N$ be a homomorphism of A-modules.

1. Show that φ is nonzero (resp. injective, resp. surjective) if and only if $\varphi \otimes_A I : M \otimes_A I \to N \otimes_A I$ is so.
2. Show that I is finitely generated.

Solution.

1. We have seen in the lecture that tensor products preserve surjectivity.

   To see that $\varphi = 0$ if and only if $\varphi \otimes_A I = 0$, just tensor with J.

   Lastly, suppose that $\varphi \otimes_A I$ is injective. Let $\psi : K \to M$ be the kernel of φ. We need to show that $\psi = 0$. We are in the following situation:

$$\begin{tikzcd} K \arrow[r, "\psi"] \arrow[rr, bend right, "0"'] & M \arrow[r, "\varphi"] & N \end{tikzcd} \qquad \begin{tikzcd} I \otimes K \arrow[r, "\psi\otimes I"] \arrow[rr, bend right, "0"'] & I \otimes M \arrow[r, "\varphi\otimes I"] & I \otimes N \end{tikzcd}$$

   We know that $\varphi \otimes I$ is injective, hence $\psi \otimes I$ has to be zero. But by preserving 0, this shows that $\psi = 0$, hence the kernel of φ vanishes. This shows that φ is injective. The same argument replaced with J shows that φ is injective if $\varphi \otimes I$ is.

2. We have an isomorphism $\varphi : I \otimes_A J \cong A$. Let's look at the preimage of 1 under φ. It is given by some finite sum $\varphi^{-1}(1) = \sum_{k=1}^n i_k \otimes j_k$. We claim that $i_1, \ldots, i_n$ generate I. Indeed, look at the morphism $\psi : A^n \to I$, $e_k \mapsto i_k$. Upon tensoring with J_k we obtain a morphism $\psi \otimes_A J : J^n \to A$, and $1 \in A$ lies in the image. Hence $\psi \otimes_A J$ is surjective. But this shows that ψ is surjective (by part 1).

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Solutions to Sheet 7

Exercise 1

Let $A \to B$ be a homomorphism of rings, let M be an A-module and let N be a B-module.

1. Show that the map
$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_B(B \otimes_A M, N), \quad \varphi \mapsto (b \otimes m \mapsto b\varphi(m))$$
is a well-defined isomorphism.
2. Show that the map
$$M \otimes_A N \to (M \otimes_A B) \otimes_B N, \quad m \otimes n \mapsto (m \otimes 1) \otimes n$$
is a well-defined isomorphism.
3. Deduce that $S^{-1}M_1 \otimes_A S^{-1}M_2 \cong S^{-1}M_1 \otimes_{S^{-1}A} S^{-1}M_2$ for two A-modules M_1, M_2 and a multiplicative subset $S \subset A$.

Solution.

1. A function $\varphi \in \mathrm{Hom}_B(B \otimes_A M, N)$ is uniquely determined by its values on elementary tensors. We have $\varphi(b \otimes m) = b\varphi(1 \otimes m)$, so in reality φ is uniquely determined by its values on $1 \otimes m$. But any such morphism gives rise to a A-linear map via $m \mapsto 1 \otimes m \mapsto \varphi(1 \otimes m)$, and conversely any $\psi \in \mathrm{Hom}_A(M, N)$ yields a unique morphism via $b \otimes m \mapsto b\psi(m) \in \mathrm{Hom}_B(B \otimes_A M, N)$. These constructions are quickly checked to be mutually inverse.

   Remark. This is a special case of the so called *Hom-Tensor adjunction.* It states that there is a natural isomorphism
$$\mathrm{Hom}_B(M \otimes_A L, N) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_B(L, N)).$$
   In more fany terms, this says that the functors $\mathrm{Hom}_B(L, -) : \mathsf{Mod}_B \to \mathsf{Mod}_A$ and $- \otimes_A L : \mathsf{Mod}_A \to \mathsf{Mod}_B$ is an adjoint pair, for any B-module L.

2. Again, universal properties. Of course, we'll want to show that this is an isomorphism of B-modules. We do this by using the universal property. What is a B-linear morphism $\varphi : (M \otimes_A B) \otimes_B N \to P$? The same as a B-bilinear map $\Phi : (M \otimes_A B) \times N \to P$. But as $\Phi(m \otimes b, n) = b\Phi(m \otimes 1, n)$, any such bilinear map is uniquely determined by its values on elements of the form $(m \otimes 1, n)$, hence it really is the same as a A-bilinear map $M \times N \to P$, given by $(m, n) \mapsto (m \otimes 1, n) \mapsto \Phi(m \otimes 1, n)$. This construction is quickly verified to be an isomorphism. But now $(M \otimes_A B) \otimes_B N$ satisfies the universal property of $M \otimes_A N$.

3. We apply the above with $S^{-1}M_1 = M$ and $S^{-1}M_2 = N$ and $B = S^{-1}A$. Note that $S^{-1}M_1 \otimes_A S^{-1}A \cong S^{-1}(S^{-1}M_1) \cong S^{-1}M_1$, which gives (following the above)
$$S^{-1}M_1 \otimes_A S^{-1}M_2 \cong (M \otimes_A S^{-1}A) \otimes_{S^{-1}A} S^{-1}M_2 \cong S^{-1}M_1 \otimes_{S^{-1}A} S^{-1}M_2.$$

Exercise 2

Let A be a ring. We define the *support* of an A-module M as $\operatorname{Supp}(M) := \{\mathfrak{p} \in \operatorname{Spec}(A) \mid M_\mathfrak{p} \neq 0\}$.

1. Assume M is finitely generated. Show that $\operatorname{Supp}(M) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid M \otimes_A k(\mathfrak{p}) \neq 0\}$, where $k(\mathfrak{p}) = \operatorname{Quot}(A/\mathfrak{p})$.
2. Assume M, N are finitely generated A-modules. Show $\operatorname{Supp}(M \otimes_A N) = \operatorname{Supp}(M) \cap \operatorname{Supp}(N)$.

Solution.

1. We will show that $M_\mathfrak{p} \neq 0$ if and only if $M \otimes_A k(\mathfrak{p}) \neq 0$. The map $A \to k(\mathfrak{p})$ factors through the map $A_\mathfrak{p} \to k(\mathfrak{p})$, and we find $M \otimes_A k(\mathfrak{p}) = M_\mathfrak{p} \otimes_{A_\mathfrak{p}} k(\mathfrak{p})$, this directly gives the implication $M \otimes_A k(\mathfrak{p}) \neq 0 \implies M_\mathfrak{p} \neq 0$.

   For the other direction, we use Nakayama's Lemma. It (or at least one version of it) states that if $N \neq 0$ is a finitely generated module over a local ring B with maximal ideal I, we have $IN \neq N$. In our situation, if we assume $M_\mathfrak{p} \neq 0$, Nakayama says

   $$M_\mathfrak{p} \otimes_{A_\mathfrak{p}} k(\mathfrak{p}) \cong M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \neq 0.$$

   Done.
2. We'll show that $(M \otimes_A N) \otimes k(\mathfrak{p}) \neq 0$ if and only if $M \otimes_A k(\mathfrak{p}) \neq 0$ and $N \otimes_A k(\mathfrak{p}) \neq 0$. Exercise 1.2 gives the isomorphism

   $$(M \otimes_A k(\mathfrak{p})) \otimes_{k(\mathfrak{p})} (N \otimes_A k(\mathfrak{p})) \cong M \otimes_A (N \otimes_A k(\mathfrak{p})) \cong (M \otimes_A N) \otimes_A k(\mathfrak{p}).$$

   From here we can directly check the desired equivalence.

Exercise 3

Let A be a ring, let $S \subset A$ be a multiplicative subset and let M, N be A-modules.

1. Assume that M is finitely presented A-module. Show that the map

   $$S^{-1} \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N), \quad \varphi/s \mapsto (m/t \mapsto \varphi(m)/st)$$

   is a well-defined isomorphism.
2. Construct a counterexample to the above if M is only assumed to be finitely generated.

Solution.

1. First, note that we always (without hypothesis on M) obtain such a map. This follows (for example) from exercise 1.1 with $B = S^{-1}A$. We obtain the isomorphism

   $$\operatorname{Hom}_A(M, S^{-1}N) \xrightarrow{\sim} \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N).$$

Also, the natural map $N \to S^{-1}N$ yields a map

$$\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(M, S^{-1}N).$$

Finally, as multiplication with any $s \in S$ gives an isomorphism on the right hand side, we obtain a morphism

$$S^{-1}\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(M, S^{-1}N) \xrightarrow{\sim} \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N).$$

by the universal property of localization on modules. One readily checks that this morphism is the one provided by the exercise.

Now we have to show that this is an isomorphism if M is finitely presented. As usual, we write M as part of a short exact sequence

$$0 \to A^m \to A^n \to M \to 0.$$

Now we use that $\operatorname{Hom}_A(-, N)$ is right-exact. Hence applying $\operatorname{Hom}_A(-, N)$ yields an exact sequence

$$0 \to 0 \to \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(A^n, N) \cong N^n \to \operatorname{Hom}_A(A^m, N) \cong A^m.$$

Localizing at S is exact, so we obtain

$$0 \to 0 \to S^{-1}\operatorname{Hom}_A(M, N) \to (S^{-1}N)^n \to (S^{-1}N)^m.$$

Similarly, we can localize at S first and then apply $\operatorname{Hom}_{S^{-1}A}(S^{-1}(-), S^{-1}N)$, which yields the exact sequence

$$0 \to 0 \to \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \to (S^{-1}N)^n \to (S^{-1}N)^m.$$

Now we can use the 5-lemma again!

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & S^{-1}\operatorname{Hom}_A(M, N) \arrow[r] \arrow[d, "\therefore \text{iso}"] & (S^{-1}A)^n \arrow[r] \arrow[d, equal] & (S^{-1}A)^m \arrow[d, equal] \\
0 \arrow[r] & 0 \arrow[r] & \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \arrow[r] & (S^{-1}A)^n \arrow[r] & (S^{-1}A)^m
\end{tikzcd}$$

2. A standard example seems to be the following. Let $A = k[x, y_1, y_2, \dots]$ be the polynomial ring in variables indexed by $\mathbb{N}$. Let $M = A/(y_1, y_2, \dots)$, $N = A/(xy_1, x^2y_2, \dots)$ and $S = \{1, x, x^2, \dots\}$. Now let's compare both sides of the morphism. Note that M is generated by 1, so that any A-linear morphism $\varphi : M \to N$ is uniquely determined by the value of $\varphi(1) \in N$. Now we have $0 = y_1\varphi(1) = y_2\varphi(1) = \dots$, which shows that any lift $\tilde{\varphi(1)} \in R$ is infintely divisible by x, hence $\varphi(1) = 0$. On the left hand side, we find that $S^{-1}M \cong S^{-1}N \cong k[x^{\pm 1}]$, so there are many $S^{-1}A$-linear morphisms $S^{-1}M \to S^{-1}N$.

Exercise 4

Let A be a principal domain and let $f \in A \setminus \{0\}$ be a non-unit. Show that the $A[T]$-module $(f, T) \subset A[T]$ is not flat.

Solution. Consider the map given by multiplication with f, which we will denote as $\varphi : A \to A$. It is injective. Note that $A \cong A[T]/(T)$. We want to show that $(f, T) \otimes_{A[T]} A$ is not injective, showing that (f, T) is not flat. We have an isomorphism (of $A[T]$-modules)

$$(f, T) \otimes_{A[T]} A \cong (f, T)/T(f, T),$$

and $(f, T) \otimes \varphi$ corresponds to the endomorphism given by multiplication with f under this identification. Now, $T \neq 0$ in $(f, T)/T(f, T)$, but $fT = \varphi(T) = 0$.

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Solutions to Sheet 8

Exercise 1

Let A be a ring and $\mathfrak{a} \subset A$ an ideal. Show that $A/\mathfrak{a}$ is finitely presented if and only if $\mathfrak{a}$ is a finitely generated ideal.

Solution. Remember that an A-algebra B is of finite presentation iff there is an isomorphism $A[X_1, \ldots, X_n]/(f_1, \ldots, f_r) \cong B$ with $f_i \in A[X_1, \ldots, X_n]$. If $\mathfrak{a}$ is finitely generated, clearly $A/\mathfrak{a}$ is of finite presentation. Now suppose that $A/\mathfrak{a} \cong A[X_1, \ldots, X_n]/(f_1, \ldots, f_r)$. We have the following diagram with the horizontals being short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & (f_1, \ldots, f_r) \arrow[r] \arrow[d, "\alpha"'] & A[X_1, \ldots, X_n] \arrow[r] \arrow[d, two heads, "\beta"'] & A[X_1, \ldots, X_n]/(f_1, \ldots, f_r) \arrow[r] \arrow[d, "\gamma"', "\cong"] & 0 \arrow[d, equal] \\
0 \arrow[r] & \mathfrak{a} \arrow[r] & A \arrow[r] & A/\mathfrak{a} \arrow[r] & 0
\end{tikzcd}$$

Here, the map β exists because every map is also a morphism of A-algebras, and in particular send 1 to 1. Now $\alpha(f_i)$ is defined by the image of f_i in A, which lies in $\mathfrak{a}$ as $\alpha(f_i) = 0$ after projection to $A/\mathfrak{a}$ (by commutativity of the diagram). We need to show that α is surjective. There are many ways to see this, for example we can use functoriality of kernels and the fact that β splits, or we can use the snake lemma, or simply do a diagram chase.

Exercise 2

Let k be a field. Show that the ring extensions $k[X+Y] \to k[X,Y]/(XY)$ and $k[X^2-1] \to k[X]$ are integral.

Solution.

1. Let $f(T) = T^2 - T(X+Y)$. Then $f(X) = X^2 - X(X+Y) = -XY = 0$ in $k[X,Y]/(XY)$.
2. Let $f(T) = T^2 - 1 - (X^2 - 1)$. Then $f(X) = 0$.

In both cases, the extension is generated by elements for which we found monic polynomials that have those elements as roots, hence they are generated by algebraic elements, hence algebraic.

Exercise 3

Let $\varphi : A \to B$ be a finite morphism of rings, i.e., $A \to B$ is a ring homomorphism which makes B a finite A-module. Show that the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ has finite fibers.

Solution. First, we find out what the fiber above a prime $\mathfrak{p} \in \mathrm{Spec}(A)$ is. Writing down definitions, we find that it's given by it is given by

$$\{\mathfrak{q} \in \mathrm{Spec}(B) \mid \varphi^{-1}(\mathfrak{q}) = \mathfrak{p}\} = \{\mathfrak{q} \in \mathrm{Spec}(B) \mid \varphi(\mathfrak{p}) \subset \mathfrak{q} \subset \varphi(\mathfrak{p}') \ \forall \mathfrak{p}' \supset \mathfrak{p}\}$$

By the homomorphism theorems, this is given by $\mathrm{Spec}(B \otimes_A \kappa(\mathfrak{p}))$. But as B is a finite A-module, there is a surjection (of A-modules) $A^n \to B$, which turns into a surjection (of $\kappa(\mathfrak{p})$-vector spaces) $\kappa(\mathfrak{p})^n \to B \otimes_A \kappa(\mathfrak{p})$. Hence $B \otimes_A \kappa(\mathfrak{p}) =: B_{\kappa(\mathfrak{p})}$ is a finite $\kappa(\mathfrak{p})$-algebra (in the sense that it is finitely generated as an A-module). We now use ideas from Sheet 1, exercise 4. First, note that every prime in $B_{\kappa(\mathfrak{p})}$ is maximal, because every finite integral extension of a field is a field. Now given any set $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_N\}$ of prime (hence maximal) ideals we have an isomorphism

$$B_{\kappa(\mathfrak{p})}/(\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_N) \xrightarrow{\sim} B_{\kappa(\mathfrak{p})}/\mathfrak{m}_1 \times \cdots \times B_{\kappa(\mathfrak{p})}/\mathfrak{m}_N.$$

The object on the left has $\kappa(\mathfrak{p})$-dimension $\leq \dim_{\kappa(\mathfrak{p})} B_{\kappa(\mathfrak{p})}$, and the object on the right has $\kappa(\mathfrak{p})$-dimension $\geq N$. In particular, there aren't more that $\dim_{\kappa(\mathfrak{p})} B_{\kappa(\mathfrak{p})}$ prime ideals in $B_{\kappa(\mathfrak{p})}$.

Exercise 4

Prove the 5-lemma.

Solution. I don't want to prove the 5-lemma. I feel like the proof is a bit involved and you don't get much insight from proving something so elementary. However, I also don't want to discourage you from reading up the proof if you feel like it! There are many proofs of this statement in various levels of complicatedness and generality. If you are just interested in how to prove the 5-lemma for modules over rings (as in the exercise), you can simply do a diagram chase. This has been done (for example) on Wikipedia.[1] There is a proof making extensive use of the snake lemma[2], and this even generalizes to arbitrary abelian categories. The most elegant proof I know of uses spectral sequences. It is stated as exercise 1.7.C in Ravi Vakil's book.[3] ((I love this book))

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[1] `https://en.wikipedia.org/wiki/Five_lemma`
[2] https://math.stackexchange.com/...
[3] `https://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf`

Solutions to Sheet 9

Exercise 1

Assume that $d \in \mathbb{Z}$ is not a square. Determine all $x, y, z \in \mathbb{Z}$ with $\gcd(x, y, z) = 1$ and $x^2 - dy^2 = z^2$.

Solution. We do the same as in the lecture. First note that

$$L = \{(x, y, z) \mid x^2 - dy^2 = z^2\} \cong \{(x, y) \in \mathbb{Q}^2 \mid x^2 - dy^2 = 1\} = L'.$$

Just as in the lecture we try to simultaneously solve the equations

$$\begin{aligned} x^2 - dy^2 &= 1 \\ qx + q &= y \end{aligned}$$

for $q \in \mathbb{Q}$. Some calculations later we arrive at the unique non-trivial solution $(x, y) = (\frac{1+dq^2}{1-dq^2}, \frac{2q}{1+dq^2})$. Writing $q = \frac{u}{v}$ with $(u, v) = 1$, we find that all solutions are of the form

$$(x, y, z) = \begin{cases} (v^2 + du^2, 2uv, v^2 - du^2), & \text{if } v^2 + du^2 \text{ odd} \\ (\frac{v^2+du^2}{2}, uv, \frac{v^2-du^2}{2}), & \text{if } v^2 + du^2 \text{ even.} \end{cases}$$

Exercise 2

Let k be an algebraically closed field and let $f(x) \in k[x]$ be a polynomial. Determine the set $\operatorname{Spec}(k[x, y]/(y^2 - f(x)))$ and the cardinality of all fibers of the map

$$\operatorname{Spec}(k[x, y]/(y^2 - f(x))) \to \operatorname{Spec}(k[x])$$

that is induced by the k algebra homomorphism $k[x] \to k[x, y]/(y^2 - f(x))$, $x \mapsto x$.

Solution. We have seen that the prime ideals of $k[x, y]$ are those of the form $(x - a, y - b)$ for $a, b \in k$. The prime ideals of $k[x, y]/(y^2 - f(x))$ are now those which contain $y^2 - f(x)$.

In the following, we assume $\operatorname{char} k \neq 2$. There are two types of prime ideals in $k[x]$. Those of the form $(x - a)$ for $a \in k$ and the zero-ideal. Let $\pi : \operatorname{Spec}(k[x, y]/(y^2 - f(x))) \to \operatorname{Spec}(k[x])$ denote the morphism on spectra induced by the inclusion. We calculate the fibers. On the *special* fibers we find

$$\pi^{-1}((x - a)) = \operatorname{Spec}(k[x, y]/(y^2 - f(x)) \otimes_{k[x],\ x \mapsto a} k).$$

We can calculate the tensor product explicitely. We find

$$k[x, y]/(y^2 - f(x)) \otimes_{k[x]} k = k[x, y]/(y^2 - f(x), x - a) = k[y]/(y^2 - f(a)).$$

And here we have

$$k[y]/(y^2 - f(a)) = \begin{cases} k^2, & \text{if } f(a) \neq 0 \\ k[y]/(y^2), & \text{if } f(a) = 0. \end{cases}$$

Hence the fibers either are given by two distinct "degree 1"-primes or by a single "degree 2"-prime.

At the *generic* fiber we have

$$\pi^{-1}((0)) = \operatorname{Spec}(k[x,y]/(y^2 - f(x)) \otimes_{k[x],\ x\mapsto x} k(x)).$$

Here the algebra calculates to

$$k[x,y]/(y^2 - f(x)) \otimes_{k[x],\ x\mapsto x} k(x) = k(x)[y]/(y^2 - f(x)).$$

Now

$$k(x)[y]/(y^2 - f(x)) \cong \begin{cases} k(x)[y]/y^2, & \text{if } f(x) = 0 \\ k(x)^2, & \text{if } f(x) = g(x)^2 \neq 0 \\ k(x)[\sqrt{f(x)}], & \text{otherwise.} \end{cases}$$

In the first case we have one prime ideal, in the second there are two, in the third there is one again. Note that in all cases, we are somehow "degree 2". In all three cases, the algebras lying over the primes are $k(x)$-algebras of dimension 2.

Remarks. Two remarks on calculations like this.

1. When calculating fibers as above, there is a neat formula to calculate tensor products, which I call *Torsten's magic potion formula.*[1] It is given by the following:

$$\begin{aligned} &k[y_1, \dots, y_m]/I \otimes_{\varphi, k[x_1,\dots,x_n], \psi} k[z_1, \dots, z_l]/J \\ &\qquad \cong k[y_1, \dots, y_m, z_1, \dots, z_l]/(I, J, \varphi(x_1) - \psi(x_1), \dots, \varphi(x_n) - \psi(x_n)). \end{aligned}$$

2. Let $f : A \to B$ and $\mathfrak{p} \in \operatorname{Spec}(A)$. In the last exercise session we discussed how using the homomorphism theorems, $\operatorname{Spec}(f)^{-1}(\mathfrak{p}) = \operatorname{Spec}(B \otimes_A k(\mathfrak{p}))$ because the prime ideals in $B \otimes_A k(\mathfrak{p})$ are identify with those prime ideals "above and below" $\mathfrak{p}$. Here, I'd like to discuss a perhaps less tedious way of arriving at this formula.

   We will need another description of $\operatorname{Spec}(R)$, which is

$$\operatorname{Spec}(R) = \{f : R \to K\}/\sim,$$

   where K are arbitrary fields and $(f_1 : R \to K_1) \sim (f_2 : R \to K_2)$ if and only if there is some field K' with morphisms $K_1 \to K'$, $K_2 \to K'$ such that $f_1 = f_2$ after applying those morphisms. The bijections are given by sending $\mathfrak{p} \in \operatorname{Spec}(R)$ to the morphism $R \to k(\mathfrak{p})$ (in one direction), and by sending f to $\operatorname{Ker}(f)$ (in the opposite direction). With this description, a morphism of rings induces a morphism on spectra by precomposition. Remember the universal property of the tensor product of rings:

$$\begin{tikzcd} T & & \\ & A \otimes_R B \arrow[ul, dashed, "\exists!"'] & B \arrow[l] \arrow[ull, bend right] \\ & A \arrow[u] \arrow[uul, bend left] & R \arrow[l] \arrow[u] \end{tikzcd}$$

   That is, given two R-algebras A and B and T a morphism $A \otimes_R B \to T$ is the same as R-algebra morphisms $A \to T$, $B \to T$ such that everything commutes with the structure morphisms from R.

[1] I do not know who Torsten is, or whether it's *Torsten* or *Thorsten*.

Now back to the fiber. We find

$$\mathrm{Spec}(f)^{-1}(A \to k(\mathfrak{p})) = \{[g : B \to K] \mid g \circ f \sim (A \to k(\mathfrak{p}))\}$$

and the set on the right is exactly given by the set of morphisms g such that there are commutative squares

$$\begin{tikzcd} B \arrow[r, "g"'] & K \\ A \arrow[u, "f"] \arrow[r] & k(\mathfrak{p}) \arrow[u] \end{tikzcd}$$

up to equivalence, which is the same as $\mathrm{Spec}(B \otimes_A k(\mathfrak{p}))$ by the universal property of the tensor product.

Exercise 3

Let $m, n \geq 1$ and let $\zeta_m = \mathrm{e}^{2\pi\mathrm{i}/m} \in \mathbb{C}$ be a primitive m-th root of unity. Set $G := \langle \zeta_m \rangle \subset \mathbb{C}^\times$. We let G act on $A := \mathbb{C}[T_1, \dots, T_n]$ via $(g, f(T_1, \dots, T_n)) \mapsto g \cdot f := f(gT_1, \dots, gT_n)$.

1. Determine the ring of invariante $A^G := \{f \in A \mid g \cdot f = f \text{ for all } g \in G\}$.
2. Set $m = n = 2$. Find a presentation $A^G \cong \mathbb{C}[X_1, \dots, X_k]/(h_1, \dots, h_l)$.

Solution.

1. We simply write down what happens. Let $f = \sum_{\mathbf{i}=(i_1,\dots,i_n)\in\mathbb{N}^n} a_{\mathbf{i}} T^{\mathbf{i}} \in \mathbb{C}[T_1, \dots, T_N]$. Now applying ζ_m gives
$$\zeta_m f = \sum_{k=0}^{\infty} \zeta_m^k \sum_{|\mathbf{i}|=k} a_{\mathbf{i}} T^{\mathbf{i}},$$
where $|\mathbf{i}| = \sum_{j=1}^n i_j$. Now it is easy to see that $\zeta_m f = f$ if and only if the only $a_{\mathbf{i}} = 0$ whenever $m \nmid |\mathbf{i}|$.
2. By the above, we find that $A^G = \mathbb{C}[T_1^2, T_1T_2, T_2^2]$. This is also given by $\mathbb{C}[X, Y, Z]/(Y^2 - XZ) =: B$. To see that $A^G \cong B$, look at $\mathbb{C}[X, Y, Z] \to \mathbb{C}[T_1, T_2]$, $X \mapsto T_1^2, Y \mapsto T_1T_2, Z \mapsto T_2^2$. The kernel of this morphism contains $(Y^2 - XZ)$. Also, the image, A^G, has Krull-dimension at least 2, as we have the chain of prime ideals $0 \subset (T_1^2, T_1T_2) \subset (T_1^2, T_1T_2, T_2^2)$. By Krull's PID theorem, the dimension of $\mathbb{C}[X, Y, Z]/(Y^2 - XZ)$ is two. Hence the kernel is generated by $Y^2 - XZ$, as any other generator would decrease dimension even more.

Exercise 4

Let A be a ring and M be a finitely generated A-module. Let $n \geq 1$ and let $f : A^n \to M$ be a surjection. Show that $K := \mathrm{Ker}(f)$ is finitely generated.

Solution. As M is finitely generated, there is a short exact sequence $0 \to Q \to A^m \to M \to 0$ with Q finitely generated. Our situation is now the following.

$$\begin{tikzcd} 0 \arrow[r] & Q \arrow[r] \arrow[d, dashed, "\exists\beta?"] & A^m \arrow[r, "g"] \arrow[d, dashed, "\exists\alpha?"] & M \arrow[r] \arrow[d, dashed, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & K \arrow[r] & A^n \arrow[r, "f"] & M \arrow[r] & 0 \end{tikzcd}$$

We want to construct morphisms α and β making the diagram above commute, in the hope of being able to apply the snake lemma then. First, we construct α. It suffices to find values for $\alpha(e_i)$. We simply choose any $\alpha(e_i) \in f^{-1}(g(e_i))$. Now by the universal property of kernels, we also get β. We want to show that K is finitely generated. The snake lemma gives a short exact sequence

$$0 \to \operatorname{Coker} \beta \to \operatorname{Coker} \alpha \to 0.$$

Hence, $\operatorname{Coker} \beta \cong \operatorname{Coker} \alpha \cong A^n / \operatorname{Im}(\alpha)$ is finitely generated. We also have the short exact sequence

$$0 \to \operatorname{Im}(\beta) \to K \to \operatorname{Coker}(\beta) \to 0.$$

As $\operatorname{Im}(\beta)$ is finitely generated, we obtain that K is finitely generated. Indeed, let $(f_1, \dots, f_n)$ be generators of $\operatorname{Im}(\beta)$ and $(g_1, \dots, g_m)$ be lifts of generators of $\operatorname{Coker}(\beta) = K / \operatorname{Im}(\beta)$. Now we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & A^n \arrow[r] \arrow[d, dashed, "e_i \mapsto f_i"] & A^{m+n} \arrow[r] \arrow[d] & A^m \arrow[r] \arrow[d, dashed, "e_j \mapsto g_j"] & 0 \\
0 \arrow[r] & \operatorname{Im}(\beta) \arrow[r] & K \arrow[r] & \operatorname{Coker}(\beta) \arrow[r] & 0
\end{tikzcd}$$

from where we can use the snake lemma agin.

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Solutions to Sheet 10

Exercise 1

Let k be a field and let $f : A \to B$ be a k-algebra homomorphism with B a finitely generated k-algebra. Let $\mathfrak{m} \subset B$ be a maximal ideal. Show that $f^{-1}(\mathfrak{m}) \subset A$ is a maximal idea.

Solution. Write $B = k[x_1, \ldots, x_n]/I$. If $\mathfrak{m} \subset B$ is maximal, then $B/\mathfrak{m} \cong K$, where K/k is a finite field extension by Hilbert's Nullstellensatz. We have the morphism

$$A/f^{-1}(\mathfrak{m}) \to B/\mathfrak{m} = K,$$

which is readily seen to be injective. Hence $A/f^{-1}(\mathfrak{m})$ is isomorphic to some sub-k-algebra of a finite field extension of k. But now it is a finite k-algebra, in particular a field itself. This shows that $f^{-1}(\mathfrak{m})$ is maximal.

Exercise 2

Let $n \geq 0$ and $Z \subset k^n$ be an algebraic subset. Show that $I(Z)$ is a prime ideal if and only if $Z = Z_1 \cap Z_2$ with Z_1, Z_2 algebraic implies $Z = Z_1$ or $Z = Z_2$.

Solution. A space sufficing the latter condition is called *irreducible*. I think all we know about $V(-)$ and $I(-)$ is

- Hilbert's Nullstellensatz: $I(V(J)) = \sqrt{J}$ and $V(I(Z)) = Z$.
- $I(-)$ and $V(-)$ are inclusion-reversing.
- $V(J_1 \cap J_2) = V(J_1 J_2) = V(J_1) \cup V(J_2)$ and $V(J_1 + J_2) = V(J_1) \cap V(J_2)$
- $I(Z_1 \cap Z_2) = I(Z_1) + I(Z_2)$ and $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$.
- The Zariski-Topology: This is the coarsest topology with sets of the form $V(I)$ closed.

If Z is irreducible and $f_1 f_2 \in I(Z)$, we have $V(f_1 f_2) \supset Z$ find $(V(f_1) \cap Z) \cup (V(f_2) \cap Z) = Z$, hence $V(f_1) \supset Z$ or $V(f_2) \supset Z$, which shows $f_1 \in I(Z)$ or $f_2 \in I(Z)$. Hence $I(Z)$ is prime.

On the contrary, if $I(Z)$ is prime and $Z = Z_1 \cup Z_2$, we find $I(Z) = I(Z_1 \cup Z_2) = I(Z_1)I(Z_2)$. Wlog, This implies $I(Z_1) = I(Z)$, hence $Z = V(I(Z)) = V(I(Z_1)) = Z_1$.

Exercise 3

A ring is called *Jacobson* if each prime ideal is the intersection of all maximal ideals containing it.

1. Show that a ring A is Jacobson if any only if for all primes $\mathfrak{p} \subset A$ and $a \notin \mathfrak{p}$ there exists a maximal ideal $\mathfrak{m} \subset A$ such that $a \notin \mathfrak{m}$ and $\mathfrak{p} \subset \mathfrak{m}$.

2. Let $f : A \to B$ be an injective, integral morphism and assume that B is Jacobson. Show that A is Jacobson. Deduce from the lecture that for each field k and $n \geq 0$ the ring $k[X_1, \dots, X_n]$ is Jacobson.

Solution.

1. There is not much to do. If A is Jacobson, then every prime ideal is the intersection containing it, hence for every $a \notin \mathfrak{p}$ there is some $\mathfrak{m} \supset \mathfrak{p}$ with $a \notin \mathfrak{m}$. The other direction is also readily verified.

2. First of all, note that if $\mathfrak{m} \subset B$ is maximal, $f^{-1}(\mathfrak{m}) \subset A$ is maximal as well. This follows directly from the going-up property of integral extension.

Also by going-up (or more generally, lying over) we find some $\mathfrak{q} \in \operatorname{Spec}(B)$ with $f^{-1}(\mathfrak{q}) = \mathfrak{p}$. As B is Jacobson we have $\mathfrak{q} = \bigcap_{\mathfrak{m} \supset \mathfrak{q}} \mathfrak{m}$, so that we obtain

$$\mathfrak{p} = f^{-1}(\mathfrak{q}) = \bigcap_{\mathfrak{m} \supset \mathfrak{q}} f^{-1}(\mathfrak{m}) = \bigcap_{f^{-1}(\mathfrak{m}) \supset \mathfrak{p}} f^{-1}(\mathfrak{m}).$$

Alternative proof. We can also use part 1. Let $\mathfrak{p} \in \operatorname{Spec}(A)$, $a \in A$ be any elements. By the lying-over property for integral extensions we find some prime $\mathfrak{q} \in \operatorname{Spec}(B)$ with $\mathfrak{q} \cap A = \mathfrak{p}$. Now there is some maximal ideal $\mathfrak{m} \in \operatorname{Spec}(B)$ with $\mathfrak{q} \subset \mathfrak{m}$ and $a \notin \mathfrak{m}$. But now let $\mathfrak{m}' = A \cap \mathfrak{m}$. This is a maximal ideal containing $\mathfrak{p}$, not containing a. We are done with part 1.

Exercise 4

Let A be a local ring and M a finitely presented, flat A-module. Show that M is free. *Hint:* Let $\mathfrak{m} \subset A$ be the maximal ideal. Use prev sheet to construct a short exact sequence $0 \to K \to A^n \to M \to 0$ with K finitely generated and $(A/\mathfrak{m})^n \to M/\mathfrak{m}M$ an isomorphism. Now use flatness of M and the snake lemma to check that $0 \to K/\mathfrak{m}K \to (A/\mathfrak{m})^n \to M/\mathfrak{m} \to 0$ is again short exact.

Solution. We follow the hint. Write $k = A/\mathfrak{m}$. Note that we can choose n as the k-dimension of $M/\mathfrak{m}$: The dimension is finite by finite-generatedness of M and right-exactness of tensoring with $A/\mathfrak{m} = k$. By Nakayama's Lemma, any choice of generators of $M/\mathfrak{m}$ lifts to generators of M. Hence we can construct a surjective morphism of A-modules $A^n \to M$ which is an isomorphism up to tensoring with k. Note that $\mathfrak{m}A \hookrightarrow A$, so after tensoring with M we find $\mathfrak{m} \otimes_A M \hookrightarrow M$. Also, tensoring the exact sequence

$$0 \to K \to A^n \to M \to 0$$

with $\mathfrak{m}$ yields the exact sequence

$$\mathfrak{m} \otimes_A K \to \mathfrak{m}^n \to \mathfrak{m} \otimes_A M \to 0.$$

All information up to now is encoded in the following diagram with exact rows.

$$\begin{tikzcd}
 & \mathfrak{m} \otimes K \arrow[r] \arrow[d] & \mathfrak{m}^n \arrow[r] \arrow[d, hook] & \mathfrak{m} \otimes_A M \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d, two heads] & M \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & 0 \arrow[r] & (A/\mathfrak{m})^n \arrow[r] & M/\mathfrak{m}M \arrow[r] & 0
\end{tikzcd}$$

The snake lemma on the top two rows yields a short exact sequence

$$0 \to K/\mathfrak{m}K \to (A/\mathfrak{m})^n \to M/\mathfrak{m}M \to 0,$$

and we obtain $K/\mathfrak{m}K = 0$, i.e. $K = \mathfrak{m}K$. But K is finitely generated (as M is finitely presented), and this implies $K = 0$ by Nakayama.

There is a better way to think about the homological algebra here. We know already that tensoring is right-exact, but in general not left-exact. As it turns out, the failure of left-exactness can be captured by certain *higher derived* tensor products, also known as Tor-functors. The idea is simple, albeit unintuitive if you have never encountered cohomology groups: Given a short exact seqeunce of A-modules

$$0 \to M' \to M \to M'' \to 0$$

and another A-module N, there are certain functors $\mathrm{Tor}_i^A(N, -)$ which capture the failure of left-exactness in that they fit into a long exact sequence

$$\begin{aligned} \ldots \mathrm{Tor}_2(N, M'') \to \mathrm{Tor}_1(N, M') \to \mathrm{Tor}_1(N, M) \to \mathrm{Tor}_1(N, M'') \\ \to N \otimes_A M' \to N \otimes_A M \to N \otimes_A M'' \to 0. \end{aligned}$$

One can show that Tor_i^A is symmetric, i.e., $\mathrm{Tor}_i(M, N) = \mathrm{Tor}_i(N, M)$. Using Tor, one finds that M being flat is the same as $\mathrm{Tor}_i(M, N) = 0$ for all $i > 0$. This should make sense: If we have any exact sequence ending in N, then thensoring with M shouldn't make this not-exact, so $\mathrm{Tor}_1(M, N) = 0$. Knowing this, we see that any sequence ending in M is universally exact, i.e., still exact if we tensor it with any other A-module N. In particular, exactness of the sequence

$$0 \to K \to A^n \to M \to 0$$

implies exactness of the sequence

$$\mathrm{Tor}_1^A(M, A/\mathfrak{m}) = 0 \to K/\mathfrak{m}K \to (A/\mathfrak{m}A)^n \to M/\mathfrak{m}M \to 0.$$

Solutions to Sheet 11

Exercise 1

Let k be a field and let A, B be two finitely generated k-algebras. Show

$$\dim(A \otimes_k B) = \dim(A) + \dim(B).$$

Solution. Remember noether normalization? It tells us that given any finitely generated k-algebra A of dimension n, there is some integral extension $k[x_1, \ldots, x_n] \hookrightarrow A$. Similarly, B arises as an integral extension $k[y_1, \ldots, y_m] \hookrightarrow B$. We now have an injection

$$k[x_1, \ldots, x_n, y_1, \ldots, y_m] = k[x_1, \ldots, x_n] \otimes_k k[y_1, \ldots, y_m] \hookrightarrow A \otimes_k k[y_1, \ldots, y_m] \hookrightarrow A \otimes_k B.$$

Note that both maps are base changes of integral maps, thereby integral themself. To see this, look at the following diagram where every square is co-cartesian (i.e., in every square, the top-right is isomorphic to the tensor product along the corners)

$$\begin{tikzcd}
A \arrow[r] & A \otimes_k k[y_1, \ldots, y_m] \arrow[r, "\therefore\text{integral}"'] & A \otimes_k B \\
k[x_1, \ldots, x_n] \arrow[u, hook, "\text{integral}"'] \arrow[r] & k[x_1, \ldots, x_n, y_1, \ldots, y_m] \arrow[u, "\therefore\text{integral}"'] \arrow[r, "\therefore\text{integral}"'] & k[x_1, \ldots, x_n] \otimes_k B \arrow[u, "\therefore\text{integral}"'] \\
k \arrow[u] \arrow[r] & k[y_1, \ldots, y_m] \arrow[u] \arrow[r, hook, "\text{integral}"'] & B \arrow[u]
\end{tikzcd}$$

Hence the map above is integral. As integral homomorphisms preserve dimension, we find $\dim(A \otimes_k B) = n + m = \dim(A) + \dim(B)$. Indeed, by going up we find that $\dim(A \otimes_k B) \geq n + m$. If the inequality was strict, we could apply Noether normalization again, eventually finding an integral extension of the form $k[x_1, \ldots, x_{n+m}] \hookrightarrow k[x_1, \ldots, x_{n+m+1}]$, which is absurd.

Exercise 2

Let k be a field, and consider the k-algebra morphism

$$\varphi : k[x, y]/(y^2 - x^3) \to k[t], \quad x \mapsto t^2, y \mapsto t^3.$$

Show that φ is finite, induces a bijection on Spec and is not an isomorphism.

Solution. This is not an isomorphism because t does not lie in the image.

To show that φ induces a bijection on spectra, note that it is an isomorphism if we invert x and t:

$$k[x^{\pm 1}, y]/(y^2 - x^3) = k[x^{\pm 1}, x^{3/2}] = k[x^{\pm \frac{1}{2}}] \cong k[t^{\pm 1}], \quad x^{\frac{1}{2}} \mapsto t.$$

In other words, restricting $\mathrm{Spec}(\varphi)$ to $\mathrm{Spec}(A) \backslash \{(x)\}$ yields an isomorphism to $\mathrm{Spec}(k[t]) \backslash \{(t)\}$. But one easily checks that the preimage of (t) is given by the ideal generated by (x), hence we have a bijection on spectra. (Geometrically, φ gives a parametrization of the cusp, given by $t = x/y$. In particular $t = 0$ implies $x = 0$. This is one standard example of normalization)

To show finiteness, note that $(1, t, t^2, \ldots)$ generates $k[t]$ as an $k[x, y]/(y^2 - x^3)$-module. But $t^2 = x \cdot 1 \in k[x, y]/(y^2 - x^3) \cdot 1$, so $(1, t)$ is a generating tuple. Hence the map is finite.

Exercise 3

In this exercise we denote by $\mathrm{MinSpec}(A)$ the set of minimal prime ideals of a ring A.

1. Let $A_1, \dots, A_n$ be rings and let B be their product. Show that

$$\mathrm{MinSpec}(B) = \bigcup_{i=1}^{n} \mathrm{MinSpec}(A_i).$$

2. Let $f : A \to B$ be an injective and integral ring homomorphism. Show that the inclusion

$$\mathrm{MinSpec}(A) \subseteq \mathrm{Spec}(f)(\mathrm{MinSpec}(B))$$

and give an example where the inclusion is strict.

Solution. A module M over a product of rings $A_1, \dots, A_n$ is the same as modules M_i over each of the rings A_i. Indeed, set $M_i = e_i M$ with $e_i \in A_1 \times \cdots \times A_n$ the i-th standard entry. Now e_j annihilates e_i for $i \neq j$ and one can check that $M \cong e_1 M \times \cdots \times e_n M$.

For part 1, this yields that there is an inclusion preserving bijection $\mathrm{Spec}(B) = \bigcup_{i=1}^{n} \mathrm{Spec}(A_i)$. Indeed, any ideal is of the form $I = I_1 \times \cdots \times I_n$, and for this to be prime we need $I_i = \mathfrak{p} \in \mathrm{Spec}(A_i)$ for some $1 \leq i \leq n$ and $I_j = A_j$ for all $j \neq i$. One easily checks that all those ideals are prime. And if, say, $I_1 \subsetneq A_1$ and $I_2 \subsetneq A_2$ are proper ideals, then $(1,0) \notin I_1 \times I_2$ and $(0,1) \notin I_1 \times I_2$, but $(1,0) \cdot (0,1) = (0,0) \in I_1 \times I_2$, so $I_1 \times I_2$ has no chance to be prime.

For part 2, by lying over we have that $\mathrm{Spec}(f)$ is surjective. So given any prime $\mathfrak{p} \in \mathrm{Spec}(A)$ we find some $\mathfrak{q} \in \mathrm{Spec}(B)$ with $f^{-1}(\mathfrak{q}) = \mathfrak{p}$. But now there is some minimal prime $\mathfrak{q}' \subseteq \mathfrak{q}$, and we find $f^{-1}(\mathfrak{q}') \subseteq f^{-1}(\mathfrak{q}) = \mathfrak{p}$. But by minimality of $\mathfrak{p}$ this implies $f^{-1}(\mathfrak{q}') = \mathfrak{p}$. Hence every minimal prime of A arises as the preimage of some minimal prime of B. This is what we had to show.

Exercise 4

Let k be an algebraically closed field and let $Z \subset k^4$ be the vanishing locus of the ideal $(xz, yz, xw, yw) \subset k[x, y, z, w]$. Determine the irreducible components of Z and their intersections.

Solution. Consider the projections $k[x, y, z, w] \to k[x, y]$ and $k[x, y, z, w] \to k[z, w]$. These yield a homomorphism $k[x, y, z, w] \to k[x, y] \times k[z, w]$. The kernel is given by the intersection of the kerenels of the two individual maps, which is $(z, w) \cap (x, y) = (xz, yz, xw, yw)$. This yields an injective homomorphism

$$A := k[x, y, z, w]/(xz, yz, xw, yw) \to k[x, y] \times k[z, w].$$

One easily sees that this is finite. Indeed, the right hand side is generated by $(1, 0)$ and $(0, 1)$ as A-modules. We are now in a position to apply the results of exercise 3. The set of minimal primes of $k[x, y]$ is the singleton $\{(0)\}$. By 3.1 we find

$$\mathrm{MinSpec}(k[x, y] \times k[z, w] = \{(1, 0), (0, 1)\}.$$

We have $f^{-1}((1, 0)) = (x, y)$ and $f^{-1}((0, 1)) = (z, w)$. Hence 3.2 gives

$$\mathrm{MinSpec}(k[x, y, z, w]/(xz, yz, xw, yw)) \subseteq \{(x, y), (z, w)\}.$$

But there is at least one minimal prime and symmetry forces equality.

Now, as irreducible components are in bijection with minimal primes, $\mathrm{Spec}(A)$ has two irreducible components, given by $V(x, y)$ and $V(z, w)$. Their intersection is given by $V(x, y, z, w) = \{(0, 0, 0, 0)\}$, the set containing only the origin.

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Solutions to Sheet 12

Exercise 1

Let A be a ring and let G be a finite group acting on A by ring automorphisms. Let A^G be the ring of invariants of G in A.

1. Show that A is integral over A^G.
2. Assume that A is a domain with quotient field K. Show that $K^G = \mathrm{Quot}(A^G)$.

Solution.

1. Let $x \in A$ be any element. Then
$$P_x(T) = \prod_{g \in G} (T - g(x))$$
is a monic polynimial with coefficients invariant under G (by symmetry). As $P_x(x) = 0$, x is integral over A^G, and we are done.
2. G acts on K via $g(\frac{x}{y}) = \frac{g.x}{g.y}$. One readily verifies $\mathrm{Quot}(A^G) \subseteq K^G$. For the other inclusion, assume $\frac{x}{y} \in K^G$. Now we can write
$$\frac{x}{y} = \frac{x \prod_{\mathrm{id} \neq h \in G} h(g(y))}{\prod_{h \in G} h(g(y))} = \frac{x \prod_{g \neq h \in G} h(y)}{\prod_{h \in G} h(y)} = \frac{x}{gy}.$$
This implies $x = gx$. After taking inverses, we also find $y = gy$, and we are done.

Exercise 2

1. Let A be a normal domain with quotient field K and let G be a finite group acting on A by ring automorphisms. Show that A^G is normal.
2. Let k be a field of characteristic $\neq 2$. Show that $k[x, y, z]/(z^2 - xy)$ is normal.

Solution.

1. This is just collecting what we did in exercise 1. A is algebraically closed in its quotient field K. Also, A is integral over A^G and K is integral over $K^G = \mathrm{Quot}(A^G)$. But now K is integral over A^G, in particular $K^G \subseteq K$ is integral over A^G.
2. We have $k[x, y, z]/(z^2 - xy) \cong k[x^2, xy, y^2] = k[x, y]^G$, where $G = \{\pm 1\}$ acts via
$$(-1).f(x, y) = f(-x, -y).$$
Now we are in the situation of part 1, and as $k[x, y]$ is normal, we are done.

Exercise 3

Let L/K be a finite Galois extension of number fields with Galois group G. Show that $\mathcal{O}_L$ is stable under action of G and that $\mathcal{O}_L^G = \mathcal{O}_K$.

Solution. To show that G is invariant under the action of G, let $x \in \mathcal{O}_L$ be an element with $f(x) = 0$, where $f \in \mathcal{O}_K[T]$ is monic and irreducible. Let $\sigma \in \mathrm{Gal}(L/K)$. Write f^σ for the polynomial that arises when applying sigma to the coefficients of f. Now $f^\sigma(\sigma x) = \sigma(f(x)) = 0$. (I just realized we have $f^\sigma = f$. Whatever.)

We now show the second statement. By Galois theory, we know that $L^G = K$. As $\mathcal{O}_L \subseteq L$ this shows $\mathcal{O}_L^G = \mathcal{O}_L \cap L^G$. By definition, $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ in $\mathcal{O}_L$. This directly shows $\mathcal{O}_L^G \supseteq \mathcal{O}_K$. The other direction follows because every element in $\mathcal{O}_L \cap K$ is integral over $\mathcal{O}_K$, which (by the definition of the integral closure) implies that $\mathcal{O}_L \cap K \subseteq \mathcal{O}_K$.

Exercise 4

Let k be a field and let $A \coloneqq k[x,y]/(y^2 - x^3 - x^2)$.

1. Show that A is a domain.
2. Show that $t = y/x \in \mathrm{Quot}(A)$ does not lie in A.
3. Show that t is integral over A.
4. Show that $\mathrm{Quot}(A) = k(t)$ and that $k[t] \subseteq \mathrm{Quot}(A)$ is the normalization of A.

Solution.

1. We have
$$k(x)[y]/(y^2 - x^2(x+1)) \cong k(x)[y]/((y/x)^2 - (x+1)),$$
and this is a quadratic field extension. In particular, $(y^2 - x^3 - x^2)$ is irreducible in $k(x)[y]$, hence also irreducible in $k[x,y]$. Alternatively, the Eisenstein criterion over $k[x]$ works.
2. Suppose $x/y \in A$. Now we have $(x,y) = (x)$. But in $k[x,y]/(y^2 - x^3 - x^2)$, we have $y \notin (x)$.
3. We have $t^2 = \frac{y^2}{x^2} = x + 1 \in A$.
4. What does this even mean?! The normalization is simply the integral closure of A in $\mathrm{Quot}(A)$. First, note $k(t) \subseteq \mathrm{Quot}(A)$ because $t \in \mathrm{Quot}(A)$. For the reverse statement, note that the calculation in part 1 shows that
$$\mathrm{Quot}(A) \subseteq k(x)[y]/(y^2 - x^3 - x^2) = k(x)[t]/(t^2 - x - 1) = k(t).$$
Let $N \subseteq \mathrm{Quot}(A)$ denote the normalization of A. We have $N \supseteq k[t]$ because t is integral over A, and $N \subseteq k[t]$ because $k[t]$ is integrally closed in $k(t) = \mathrm{Quot}(A)$.

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