Solutions to Sheet 12

Exercise 1

Let A be a ring and let G be a finite group acting on A by ring automorphisms. Let A^G be the ring of invariants of G in A.

1. Show that A is integral over A^G.
2. Assume that A is a domain with quotient field K. Show that $K^G = \mathrm{Quot}(A^G)$.

Solution.

1. Let $x \in A$ be any element. Then
$$P_x(T) = \prod_{g \in G} (T - g(x))$$
is a monic polynimial with coefficients invariant under G (by symmetry). As $P_x(x) = 0$, x is integral over A^G, and we are done.
2. G acts on K via $g(\frac{x}{y}) = \frac{g.x}{g.y}$. One readily verifies $\mathrm{Quot}(A^G) \subseteq K^G$. For the other inclusion, assume $\frac{x}{y} \in K^G$. Now we find for any $g \in G$
$$\frac{gx}{gy} = \frac{x \prod_{\mathrm{id} \neq h \in G} h(g(y))}{\prod_{h \in G} h(g(y))} = \frac{x \prod_{g \neq h \in G} h(y)}{\prod_{h \in G} h(y)} = \frac{x}{gy}.$$
This implies $x = gx$. After taking inverses, we also find $y = gy$, and we are done.

Exercise 2

1. Let A be a normal domain with quotient field K and let G be a finite group acting on A by ring automorphisms. Show that A^G is normal.
2. Let k be a field of characteristic $\neq 2$. Show that $k[x, y, z]/(z^2 - xy)$ is normal.

Solution.

1. This is just collecting what we did in exercise 1. A is algebraically closed in its quotient field K. Also, A is integral over A^G and K is integral over $K^G = \mathrm{Quot}(A^G)$. But now K is integral over A^G, in particular $K^G \subseteq K$ is integral over A^G.
2. We have $k[x, y, z]/(z^2 - xy) \cong k[x^2, xy, y^2] = k[x, y]^G$, where $G = \{\pm 1\}$ acts via
$$(-1).f(x, y) = f(-x, -y).$$
Now we are in the situation of part 1, and as $k[x, y]$ is normal, we are done.

Exercise 3

Let L/K be a finite Galois extension of number fields with Galois group G. Show that $\mathcal{O}_L$ is stable under action of G and that $\mathcal{O}_L^G = \mathcal{O}_K$.

Solution. To show that G is invariant under the action of G, let $x \in \mathcal{O}_L$ be an element with $f(x) = 0$, where $f \in \mathcal{O}_K[T]$ is monic and irreducible. Let $\sigma \in \mathrm{Gal}(L/K)$. Write f^σ for the polynomial that arises when applying sigma to the coefficients of f. Now $f^\sigma(\sigma x) = \sigma(f(x)) = 0$. (I just realized we have $f^\sigma = f$. Whatever.)

We now show the second statement. By Galois theory, we know that $L^G = K$. As $\mathcal{O}_L \subseteq L$ this shows $\mathcal{O}_L^G = \mathcal{O}_L \cap L^G$. By definition, $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ in $\mathcal{O}_L$. This directly shows $\mathcal{O}_L^G \supseteq \mathcal{O}_K$. The other direction follows because every element in $\mathcal{O}_L \cap K$ is integral over $\mathcal{O}_K$, which (by the definition of the integral closure) implies that $\mathcal{O}_L \cap K \subseteq \mathcal{O}_K$.

Exercise 4

Let k be a field and let $A \coloneqq k[x,y]/(y^2 - x^3 - x^2)$.

1. Show that A is a domain.
2. Show that $t = y/x \in \mathrm{Quot}(A)$ does not lie in A.
3. Show that t is integral over A.
4. Show that $\mathrm{Quot}(A) = k(t)$ and that $k[t] \subseteq \mathrm{Quot}(A)$ is the normalization of A.

Solution.

1. We have
$$k(x)[y]/(y^2 - x^2(x+1)) \cong k(x)[y]/((y/x)^2 - (x+1)),$$
and this is a quadratic field extension. In particular, $(y^2 - x^3 - x^2)$ is irreducible in $k(x)[y]$, hence also irreducible in $k[x,y]$. Alternatively, the Eisenstein criterion over $k[x]$ works.
2. Suppose $x/y \in A$. Now we have $(x,y) = (x)$. But in $k[x,y]/(y^2 - x^3 - x^2)$, we have $y \notin (x)$.
3. We have $t^2 = \frac{y^2}{x^2} = x + 1 \in A$.
4. What does this even mean?! The normalization is simply the integral closure of A in $\mathrm{Quot}(A)$. First, note $k(t) \subseteq \mathrm{Quot}(A)$ because $t \in \mathrm{Quot}(A)$. For the reverse statement, note that the calculation in part 1 shows that
$$\mathrm{Quot}(A) \subseteq k(x)[y]/(y^2 - x^3 - x^2) = k(x)[t]/(t^2 - x - 1) = k(t).$$
Let $N \subseteq \mathrm{Quot}(A)$ denote the normalization of A. We have $N \supseteq k[t]$ because t is integral over A, and $N \subseteq k[t]$ because $k[t]$ is integrally closed in $k(t) = \mathrm{Quot}(A)$.

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