Solutions to Sheet 11

Exercise 1

Let k be a field and let A, B be two finitely generated k-algebras. Show

$$\dim(A \otimes_k B) = \dim(A) + \dim(B).$$

Solution. Remember noether normalization? It tells us that given any finitely generated k-algebra A of dimension n, there is some integral extension $k[x_1, \dots, x_n] \hookrightarrow A$. Similarly, B arises as an integral extension $k[y_1, \dots, y_m] \hookrightarrow B$. We now have an injection

$$k[x_1, \dots, x_n, y_1, \dots, y_m] = k[x_1, \dots, x_n] \otimes_k k[y_1, \dots, y_m] \hookrightarrow A \otimes_k k[y_1, \dots, y_m] \hookrightarrow A \otimes_k B.$$

Note that both maps are base changes of integral maps, thereby integral themself. To see this, look at the following diagram where every square is co-cartesian (i.e., in every square, the top-right is isomorphic to the tensor product along the corners)

$$\begin{tikzcd}
B \arrow[r] & k[x_1, \dots, x_n] \otimes_k B \arrow[r, "\therefore\text{integral}"'] & A \otimes_k B \\
k[y_1, \dots, y_m] \arrow[u, "\text{integral}"'] \arrow[r] & k[x_1, \dots, x_n, y_1, \dots, y_m] \arrow[u, "\therefore\text{integral}"'] \arrow[r, "\therefore\text{integral}"'] & A \otimes_k k[y_1, \dots, y_m] \arrow[u, "\therefore\text{integral}"'] \\
k \arrow[u] \arrow[r] & k[x_1, \dots, x_n] \arrow[u] \arrow[r, hook, "\text{integral}"'] & A \arrow[u]
\end{tikzcd}$$

Hence the map above is integral. As integral homomorphisms preserve dimension, we find $\dim(A \otimes_k B) = n + m = \dim(A) + \dim(B)$. Indeed, by going up we find that $\dim(A \otimes_k B) \geq n + m$. If the inequality was strict, we could apply Noether normalization again, eventually finding an integral extension of the form $k[x_1, \dots, x_{n+m}] \hookrightarrow k[x_1, \dots, x_{n+m+1}]$, which is absurd.

Exercise 2

Let k be a field, and consider the k-algebra morphism

$$\varphi : k[x, y]/(y^2 - x^3) \to k[t], \quad x \mapsto t^2, y \mapsto t^3.$$

Show that φ is finite, induces a bijection on Spec and is not an isomorphism.

Solution. This is not an isomorphism because t does not lie in the image.

To show that φ induces a bijection on spectra, note that it is an isomorphism if we invert x and t:

$$k[x^{\pm 1}, y]/(y^2 - x^3) = k[x^{\pm 1}, x^{3/2}] = k[x^{\pm \frac{1}{2}}] \cong k[t^{\pm 1}], \quad x^{\frac{1}{2}} \mapsto t.$$

In other words, restricting $\mathrm{Spec}(\varphi)$ to $\mathrm{Spec}(A) \setminus \{(x)\}$ yields an isomorphism to $\mathrm{Spec}(k[t]) \setminus \{(t)\}$. But one easily checks that the preimage of (t) is given by the ideal generated by (x), hence we have a bijection on spectra. (Geometrically, φ gives a parametrization of the cusp, given by $t = x/y$. In particular $t = 0$ implies $x = 0$. This is one standard example of normalization)

To show finiteness, note that $(1, t, t^2, \dots)$ generates $k[t]$ as an $k[x, y]/(y^2 - x^3)$-module. But $t^2 = x \cdot 1 \in k[x, y]/(y^2 - x^3) \cdot 1$, so $(1, t)$ is a generating tuple. Hence the map is finite.

Exercise 3

In this exercise we denote by $\operatorname{MinSpec}(A)$ the set of minimal prime ideals of a ring A.

1. Let $A_1, \dots, A_n$ be rings and let B be their product. Show that

$$\operatorname{MinSpec}(B) = \bigcup_{i=1}^{n} \operatorname{MinSpec}(A_i).$$

2. Let $f : A \to B$ be an injective and integral ring homomorphism. Show that the inclusion

$$\operatorname{MinSpec}(A) \subseteq \operatorname{Spec}(f)(\operatorname{MinSpec}(B))$$

   and give an example where the inclusion is strict.

Solution. A module M over a product of rings $A_1, \dots, A_n$ is the same as modules M_i over each of the rings A_i. Indeed, set $M_i = e_i M$ with $e_i \in A_1 \times \cdots \times A_n$ the i-th standard entry. Now e_j annihilates e_i for $i \neq j$ and one can check that $M \cong e_1 M \times \cdots \times e_n M$.

For part 1, this yields that there is an inclusion preserving bijection $\operatorname{Spec}(B) = \bigcup_{i=1}^{n} \operatorname{Spec}(A_i)$. Indeed, any ideal is of the form $I = I_1 \times \cdots \times I_n$, and for this to be prime we need $I_i = \mathfrak{p} \in \operatorname{Spec}(A_i)$ for some $1 \leq i \leq n$ and $I_j = A_j$ for all $j \neq i$. One easily checks that all those ideals are prime. And if, say, $I_1 \subsetneq A_1$ and $I_2 \subsetneq A_2$ are proper ideals, then $(1, 0) \notin I_1 \times I_2$ and $(0, 1) \notin I_1 \times I_2$, but $(1, 0) \cdot (0, 1) = (0, 0) \in I_1 \times I_2$, so $I_1 \times I_2$ has no chance to be prime.

For part 2, by lying over we have that $\operatorname{Spec}(f)$ is surjective. So given any prime $\mathfrak{p} \in \operatorname{Spec}(A)$ we find some $\mathfrak{q} \in \operatorname{Spec}(B)$ with $f^{-1}(\mathfrak{q}) = \mathfrak{p}$. But now there is some minimal prime $\mathfrak{q}' \subseteq \mathfrak{q}$, and we find $f^{-1}(\mathfrak{q}') \subseteq f^{-1}(\mathfrak{q}) = \mathfrak{p}$. But by minimality of $\mathfrak{p}$ this implies $f^{-1}(\mathfrak{q}') = \mathfrak{p}$. Hence every minimal prime of A arises as the preimage of some minimal prime of B. This is what we had to show.

Exercise 4

Let k be an algebraically closed field and let $Z \subset k^4$ be the vanishing locus of the ideal $(xz, yz, xw, yw) \subset k[x, y, z, w]$. Determine the irreducible components of Z and their intersections.

Solution. Consider the projections $k[x, y, z, w] \to k[x, y]$ and $k[x, y, z, w] \to k[z, w]$. These yield a homomorphism $k[x, y, z, w] \to k[x, y] \times k[z, w]$. The kernel is given by the intersection of the kerenels of the two individual maps, which is $(z, w) \cap (x, y) = (xz, yz, xw, yw)$. This yields an injective homomorphism

$$A := k[x, y, z, w]/(xz, yz, xw, yw) \to k[x, y] \times k[z, w].$$

One easily sees that this is finite. Indeed, the right hand side is generated by $(1, 0)$ and $(0, 1)$ as A-modules. We are now in a position to apply the results of exercise 3. The set of minimal primes of $k[x, y]$ is the singleton $\{(0)\}$. By 3.1 we find

$$\operatorname{MinSpec}(k[x, y] \times k[z, w] = \{(1, 0), (0, 1)\}.$$

We have $f^{-1}((1, 0)) = (x, y)$ and $f^{-1}((0, 1)) = (z, w)$. Hence 3.2 gives

$$\operatorname{MinSpec}(k[x, y, z, w]/(xz, yz, xw, yw)) \subseteq \{(x, y), (z, w)\}.$$

But there is at least one minimal prime and symmetry forces equality.

Now, as irreducible components are in bijection with minimal primes, $\mathrm{Spec}(A)$ has two irreducible components, given by $V(x, y)$ and $V(z, w)$. Their intersection is given by $V(x, y, z, w) = \{(0, 0, 0, 0)\}$, the set containing only the origin.

Max von Consbruch, email: s6mavonc@uni-bonn.de. Date: July 3, 2023