Solutions to Sheet 6

Exercise 1

Let A be a ring, $f \in A$ a non-zero divisor, $\mathfrak{a} = (f)$ and $\mathfrak{b} \subset A$ an ideal. Show that the natural map

$$\mathfrak{a} \otimes_A \mathfrak{b} \to \mathfrak{a} \cdot \mathfrak{b}, \quad a \otimes b \mapsto a \cdot b$$

is an isomorphism.

Solution. As $\mathfrak{a} = (f)$, we have an isomorphism $\varphi : A \xrightarrow{\sim} \mathfrak{a}$, given by $a \mapsto fa$. Also note that $\varphi|_{\mathfrak{b}} : \mathfrak{b} \to \mathfrak{a}\mathfrak{b}$ is an isomorphism. Now we have the diagram

$$\begin{tikzcd}
\mathfrak{a} \otimes_A \mathfrak{b} & \mathfrak{a}\mathfrak{b} \\
A \otimes_A \mathfrak{b} \arrow[u, "\varphi\otimes\mathfrak{b}"] \arrow[r, "\sim"] & \mathfrak{b}, \arrow[u, "\varphi|_{\mathfrak{b}}"']
\end{tikzcd}$$

where all arrows are isos, yielding an isomorphism $\mathfrak{a} \otimes_A \mathfrak{b} \to \mathfrak{a}\mathfrak{b}$.

Exercise 2

Let A be a ring, let I be a set and let $M, N_i, i \in I$ be A-modules.

1. Assume that M is finitely generated (resp. finitely presented). Show that the natural map
$$M \otimes_A \prod_{i \in I} N_i \to \prod_{i \in I} M \otimes_A N_i$$
is surjective (resp. bijective).

2. Take $A = \mathbb{Z}[X_0, X_1, \dots]$, $J = (X_0, X_1, \dots)$. Show that the natural map $A/J \otimes_A A[\![T]\!] \to A/J[\![T]\!]$ is not injective.

Solution.

1. First let's recall what finitely generated and finitely presented meant. An A-module M is finitely generated if there exists a surjective morphism of A-modules
$$A^{\oplus n} \to M.$$
Furthermore, we call M finitely presented if the kernel of this map is again finitely generated (that is, there is a finite number of relations among the images of the generators), which is to say that there is an exact sequence
$$A^m \to A^n \to M \to 0$$
for some integers $m, n \geq 0$.

Next, let's find out what the *natural map* is. We have for $i \in I$ the projections $\prod_{i\in I} N_i \to N_i$, which we can tensor with M to obtain maps $M \otimes_A \prod_{i\in I} N_i \to M \otimes_A N_i$. The collection of these maps gives the desired $M \otimes_A \prod_{i\in I} N_i \to \prod_{i\in I} M \otimes_A N_i$.

Note that if $M \cong A^{\oplus n}$, this natural map is an isomorphism, as we have

$$A^{\oplus n} \otimes_A \prod_i N_i \cong (A \otimes_A \prod_i N_i)^{\oplus n} \cong (\prod_i N_i)^{\oplus n} \cong \prod_i (A^{\oplus n} \otimes_A N_i).$$

Here we used the commutativity of finite direct sums and tensor products and that of finite direct sums and products (note that finite sums are isomorphic to finite products).

This puts us in the following situation, where we can use the 5-lemma.

$$\begin{tikzcd}
A^{\oplus m} \otimes_A \prod_{i\in I} N_i \arrow[r] \arrow[d, "\sim"] & A^{\oplus n} \otimes_A \prod_{i\in I} N_i \arrow[r] \arrow[d, "\sim"] & M \otimes_A \prod_{i\in I} N_i \arrow[r] \arrow[d, "\therefore \sim"] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
\prod_{i\in I}(A^{\oplus m} \otimes_A N_i) \arrow[r] & \prod_{i\in I}(A^{\oplus n} \otimes_A N_i) \arrow[r] & \prod_{i\in I}(M \otimes_A N_i) \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

2. Note that $A/J \otimes_A A[\![T]\!] \cong A[\![T]\!]/JA[\![T]\!]$. Take the element $f := \sum_{i=1}^{\infty} x_i T^i \in A[\![T]\!]$. As all elements in $JA[\![T]\!]$ only have a finite number of x_i arise in the coefficients, we find $f \notin JA[\![T]\!]$, hence $f \neq 0$ in $A[\![T]\!]/JA[\![T]\!]$. But $f \mapsto 0$ under the natural map: All the coefficients x_i get sent to zero.

Exercise 3

Let k be a field, K/k an algebraic field extension, and $\overline{k}$ an alebraic closure of k.

1. If $V \to W$ is a k-linear injection of k-vector spaces, show that $V \otimes \overline{k} \to W \otimes \overline{k}$ is a $\overline{k}$-linear injection.
2. Show that K/k is separable if and only if the ring $K \otimes_k \overline{k}$ is reduced.

Solution.

1. All k-vector spaces are injective, hence every injective map $V \to W$ admits a section $W \to V$. Tensoring the section with $\overline{k}$ yields a section of $V \otimes_k \overline{k} \to W \otimes_k \overline{k}$.
2. We show that the following statements are equivalent:
   (a) K/k is separable.
   (b) For all $\alpha \in K$, $k[\alpha]/k$ is separable.
   (c) $k[\alpha] \otimes_k \overline{k}$ is reduced for all $\alpha \in K$.
   (d) $K \otimes_k \overline{k}$ is reduced.

   (a) $\iff$ (b) is by definition. We show (b) $\iff$ (c). Let f be the minimal polynomial of some $\alpha \in K$. As K is algebraic over k, f decomposes in $\overline{k}$ as $f(x) = \prod_{i=1}^{n}(x - a_i)^{d_i}$ with $a_i \neq a_j$ whenever $i \neq j$. Now we find

$$k[\alpha] \otimes_k \overline{k} \cong (k[x]/f(x)) \otimes_k \overline{k} \cong \overline{k}[x]/f(x) \cong k[x]/(x-a_1)^{d_1} \times \cdots \times k[x]/(x-a_n)^{d_n},$$

which is reduced if and only if $d_1 = \cdots = d_n = 1$, which is the case if and only if $k[\alpha]$ is reduced over k.

For (d) $\implies$ (c), we use part 1. The arguments there show that $k[\alpha] \to K$ is injective, hence $k[\alpha] \otimes_k \overline{k}$ is isomorphic to a subring of $K \otimes_k \overline{k}$. But a subring of a reduced subring is reduced.

Lastly we show (a) $\implies$ (d). Let $\zeta = \sum_{i=1}^n \alpha_i \otimes b_i \in K \otimes_k \overline{k}$ be some element. Here, the α_i are elements of K, and as K is separated, we find that $k[\alpha_1, \ldots, \alpha_n]/k$ is a finite separated extension. But now, by the primitive element theorem, there is some $\alpha \in K$ with $k[\alpha] \cong k[\alpha_1, \ldots, \alpha_n]$, and $k[\alpha] \otimes_k \overline{k}$ is reduced by (c) $\iff$ (a).

Exercise 4

Let A be a ring and let I be an *invertible* A-module, i.e., there exists an A-module J such that $I \otimes_A J \cong A$. Let $\varphi : M \to N$ be a homomorphism of A-modules.

1. Show that φ is nonzero (resp. injective, resp. surjective) if and only if $\varphi \otimes_A I : M \otimes_A I \to N \otimes_A I$ is so.
2. Show that I is finitely generated.

Solution.

1. We have seen in the lecture that tensor products preserve surjectivity.

   To see that $\varphi = 0$ if and only if $\varphi \otimes_A I = 0$, just tensor with J.

   Lastly, suppose that $\varphi \otimes_A I$ is injective. Let $\psi : K \to M$ be the kernel of φ. We need to show that $\psi = 0$. We are in the following situation:

$$\begin{tikzcd} K \arrow[r, "\psi"] \arrow[rr, bend right, "0"'] & M \arrow[r, hook, "\varphi"] & N \end{tikzcd} \qquad \begin{tikzcd} I \otimes K \arrow[r, "\psi \otimes I"] \arrow[rr, bend right, "0"'] & I \otimes M \arrow[r, hook, "\varphi \otimes I"] & I \otimes N \end{tikzcd}$$

   We know that $\varphi \otimes I$ is injective, hence $\psi \otimes I$ has to be zero. But by preserving 0, this shows that $\psi = 0$, hence the kernel of φ vanishes. This shows that φ is injective. The same argument replaced with J shows that φ is injective if $\varphi \otimes I$ is.

2. We have an isomorphism $\varphi : I \otimes_A J \cong A$. Let's look at the preimage of 1 under φ. It is given by some finite sum $\varphi^{-1}(1) = \sum_{k=1}^n i_k \otimes j_k$. We claim that $i_1, \ldots, i_n$ generate I. Indeed, look at the morphism $\psi : A^n \to I$, $e_k \mapsto i_k$. Upon tensoring with J_k we obtain a morphism $\psi \otimes_A J : J^n \to A$, and $1 \in A$ lies in the image. Hence $\psi \otimes_A J$ is surjective. But this shows that ψ is surjective (by part 1).