Thinking of Commutative Algebra with Geometry.

By the ways commutative algebra is (usually) tought, one quickly arrives at the assumption that commutative algebra is a completely dry, tedious subject, relying heavily on complicated computations without deeper meaning. The aim of this short text is to show that this assumption is **wrong** (or at least not completely correct haha). As hinted on in the first exercise session, I want to explain how to convert complicated results in algebra into easy (or at least easier) to remember geometric pictures.

1 An Example.

The central objects in commutative algebra are polynomial rings over fields $A = k[x_1, \dots, x_n]$. By pluggin in coordinates, we turn these into geometric objects by considering the elements of A as functions $k^n \to k$. Given some element $f \in A$, we want to study the vanishing set

$$V(f) \coloneqq \{(x_1, \dots x_n) \in k^n \mid f(x_1, \dots, x_n) = 0\}.$$

More generally, given some ideal $I \subset A$, we define

$$V(I) = \{(x_1, \dots x_n) \in k^n \mid \forall f \in I : f(x_1, \dots, x_n) = 0\}.$$

This definition defines a map

$$V : \{\text{Ideals } I \text{ in } A\} \to \{\text{algebraic subsets } S \subset k^n\}$$

where we say that a set is *algebraic* if it lies in the image of V. Note that this operation is inclusion-reversing: If we have Ideals $I \subset J$, we have $V(J) \subset V(I)$ (almost) by definition. This opens the door to geometry-land, as we come from elements in (an abstract) ring, and obtain subsets of k^n, which we can think of as geometric objects. Instead of thinking of ideals of A, we want to think of their vanishing loci. Of course this comes with some losses (different ideals can have the same vanishing locus),[1] and we will have to do some work to understand the operator V, and even more work to extend this idea to the case where A is not of the form given above. But first, we want to have a look how this gives completely new perspective on weird results.

2 Converting a result into a picture

During the very first exercise session, we encountered the following statement.

Lemma. Let A be a commutative ring and $I \subset A$ be an ideal. There is a bijection

$$\{\text{Ideals } \overline{J} \subset A/I\} \leftrightarrow \{\text{Ideals } J \subset A \text{ such that } I \subset J\},$$

given by $\overline{J} \mapsto J + I$ (from left to right) and $J \mapsto J/I$ (from right to left).

The proof is completely formal and not very interesting. If one thinks of algebra as a hotchpotch of calculations, the statement too might seem quite random. But let us assume (with a bit of unnecessary loss of generality) that $A = \mathbb{R}[x, y]$ and $I = (f)$, where $f = x^2 + y^2 - 1$. Now

[1] Hilbert's Nullstellensatz fully describes how much information we lose

$V(I) = V(f) = S^1$ (the circle). If we are given an Ideal $J \subset A$ with $I \subset J$, we find that $V(J)$ is an algebraic subset of S^1. Hence, we want to think of the right hand side as algebraic subsets of S^1. Unfortunately, given some class $[f] \in A/I$, the mapping $[f] \mapsto V(f)$ is not well-defined, for example $V(0) \neq V(f)$ but $[0] = [f]$. However, it is well-defined once we intersect the image with $V(I)$. Indeed, two different representatives differ by functions that vanish on $V(I)$. This yields a new map