Solutions to Sheet 10

Exercise 1

Let k be a field and let $f : A \to B$ be a k-algebra homomorphism with B a finitely generated k-algebra. Let $\mathfrak{m} \subset B$ be a maximal ideal. Show that $f^{-1}(\mathfrak{m}) \subset A$ is a maximal idea.

Solution. Write $B = k[x_1, \ldots, x_n]/I$. If $\mathfrak{m} \subset B$ is maximal, then $B/\mathfrak{m} \cong K$, where K/k is a finite field extension by Hilbert's Nullstellensatz. We have the morphism

$$A/f^{-1}(\mathfrak{m}) \to B/\mathfrak{m} = K,$$

which is readily seen to be injective. Hence $A/f^{-1}(\mathfrak{m})$ is isomorphic to some sub-k-algebra of a finite field extension of k. But now it is a finite k-algebra, in particular a field itself. This shows that $f^{-1}(\mathfrak{m})$ is maximal.

Exercise 2

Let $n \geq 0$ and $Z \subset k^n$ be an algebraic subset. Show that $I(Z)$ is a prime ideal if and only if $Z = Z_1 \cap Z_2$ with Z_1, Z_2 algebraic implies $Z = Z_1$ or $Z = Z_2$.

Solution. A space sufficing the latter condition is called *irreducible*. I think all we know about $V(-)$ and $I(-)$ is

- Hilbert's Nullstellensatz: $I(V(J)) = \sqrt{J}$ and $V(I(Z)) = Z$.
- $I(-)$ and $V(-)$ are inclusion-reversing.
- $V(J_1 \cap J_2) = V(J_1 J_2) = V(J_1) \cup V(J_2)$ and $V(J_1 + J_2) = V(J_1) \cap V(J_2)$
- $I(Z_1 \cap Z_2) = I(Z_1) + I(Z_2)$ and $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$.
- The Zariski-Topology: This is the coarsest topology with sets of the form $V(I)$ closed.

If Z is irreducible and $f_1 f_2 \in I(Z)$, we have $V(f_1 f_2) \supset Z$ find $(V(f_1) \cap Z) \cup (V(f_2) \cap Z) = Z$, hence $V(f_1) \supset Z$ or $V(f_2) \supset Z$, which shows $f_1 \in I(Z)$ or $f_2 \in I(Z)$. Hence $I(Z)$ is prime.

On the contrary, if $I(Z)$ is prime and $Z = Z_1 \cup Z_2$, we find $I(Z) = I(Z_1 \cup Z_2) = I(Z_1)I(Z_2)$. Wlog, This implies $I(Z_1) = I(Z)$, hence $Z = V(I(Z)) = V(I(Z_1)) = Z_1$.

Exercise 3

A ring is called *Jacobson* if each prime ideal is the intersection of all maximal ideals containing it.

1. Show that a ring A is Jacobson if any only if for all primes $\mathfrak{p} \subset A$ and $a \notin \mathfrak{p}$ there exists a maximal ideal $\mathfrak{m} \subset A$ such that $a \notin \mathfrak{m}$ and $\mathfrak{p} \subset \mathfrak{m}$.

2. Let $f : A \to B$ be an injective, integral morphism and assume that B is Jacobson. Show that A is Jacobson. Deduce from the lecture that for each field k and $n \geq 0$ the ring $k[X_1, \dots, X_n]$ is Jacobson.

Solution.

1. There is not much to do. If A is Jacobson, then every prime ideal is the intersection containing it, hence for every $a \notin \mathfrak{p}$ there is some $\mathfrak{m} \supset \mathfrak{p}$ with $a \notin \mathfrak{m}$. The other direction is also readily verified.

2. First of all, note that if $\mathfrak{m} \subset B$ is maximal, $f^{-1}(\mathfrak{m}) \subset A$ is maximal as well. This follows directly from the going-up property of integral extension.

Also by going-up (or more generally, lying over) we find some $\mathfrak{q} \in \operatorname{Spec}(B)$ with $f^{-1}(\mathfrak{q}) = \mathfrak{p}$. As B is Jacobson we have $\mathfrak{q} = \bigcap_{\mathfrak{m} \supset \mathfrak{q}} \mathfrak{m}$, so that we obtain

$$\mathfrak{p} = f^{-1}(\mathfrak{q}) = \bigcap_{\mathfrak{m} \supset \mathfrak{q}} f^{-1}(\mathfrak{m}) = \bigcap_{f^{-1}(\mathfrak{m}) \supset \mathfrak{p}} f^{-1}(\mathfrak{m}).$$

Alternative proof. We can also use part 1. Let $\mathfrak{p} \in \operatorname{Spec}(A)$, $a \in A$ be any elements. By the lying-over property for integral extensions we find some prime $\mathfrak{q} \in \operatorname{Spec}(B)$ with $\mathfrak{q} \cap A = \mathfrak{p}$. Now there is some maximal ideal $\mathfrak{m} \in \operatorname{Spec}(B)$ with $\mathfrak{q} \subset \mathfrak{m}$ and $a \notin \mathfrak{m}$. But now let $\mathfrak{m}' = A \cap \mathfrak{m}$. This is a maximal ideal containing $\mathfrak{p}$, not containing a. We are done with part 1.

Exercise 4

Let A be a local ring and M a finitely presented, flat A-module. Show that M is free. *Hint:* Let $\mathfrak{m} \subset A$ be the maximal ideal. Use prev sheet to construct a short exact sequence $0 \to K \to A^n \to M \to 0$ with K finitely generated and $(A/\mathfrak{m})^n \to M/\mathfrak{m}M$ an isomorphism. Now use flatness of M and the snake lemma to check that $0 \to K/\mathfrak{m}K \to (A/\mathfrak{m})^n \to M/\mathfrak{m} \to 0$ is again short exact.

Solution. We follow the hint. Write $k = A/\mathfrak{m}$. Note that we can choose n as the k-dimension of $M/\mathfrak{m}$: The dimension is finite by finite-generatedness of M and right-exactness of tensoring with $A/\mathfrak{m} = k$. By Nakayama's Lemma, any choice of generators of $M/\mathfrak{m}$ lifts to generators of M. Hence we can construct a surjective morphism of A-modules $A^n \to M$ which is an isomorphism up to tensoring with k. Note that $\mathfrak{m}A \hookrightarrow A$, so after tensoring with M we find $\mathfrak{m}M \hookrightarrow M$. Also, tensoring the exact sequence

$$0 \to K \to A^n \to M \to 0$$

with $\mathfrak{m}$ yields the exact sequence

$$\mathfrak{m}K \to (A\mathfrak{m})^n \to \mathfrak{m}M \to 0.$$

All information up to now is encoded in the following diagram with exact rows.

$$\begin{tikzcd}
 & \mathfrak{m}K \arrow[r] \arrow[d] & (\mathfrak{m}A)^n \arrow[r] \arrow[d, hook] & \mathfrak{m}M \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d, two heads] & M \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & 0 \arrow[r] & (A/\mathfrak{m})^n \arrow[r] & M/\mathfrak{m}M \arrow[r] & 0
\end{tikzcd}$$

The snake lemma on the top two rows yields a short exact sequence

$$0 \to K/\mathfrak{m}K \to (A/\mathfrak{m})^n \to M/\mathfrak{m}M \to 0,$$

and we obtain $K/\mathfrak{m}K = 0$, i.e. $K = \mathfrak{m}K$. But K is finitely generated (as M is finitely presented), and this implies $K = 0$ by Nakayama.

There is a better way to think about the homological algebra here. We know already that tensoring is right-exact, but in general not left-exact. As it turns out, the failure of left-exactness can be captured by certain *higher derived* tensor products, also known as Tor-functors. The idea is simple, albeit unintuitive if you have never encountered cohomology groups: Given a short exact seqeunce of A-modules

$$0 \to M' \to M \to M'' \to 0$$

and another A-module N, there should be certain functors $\mathrm{Tor}^i_A(N, -)$ which capture the failure of left-exactness in that they fit into a long exact sequence

$$\begin{aligned} \dots \mathrm{Tor}^2(N, M'') \to \mathrm{Tor}^1(N, M') \to \mathrm{Tor}^1(N, M) \to \mathrm{Tor}^1(N, M'') \\ \to N \otimes_A M' \to N \otimes_A M \to N \otimes_A M'' \to 0. \end{aligned}$$

One can show that Tor^i_A is symmetric, i.e., $\mathrm{Tor}^i(M, N) = \mathrm{Tor}^i(N, M)$. Using Tor, one finds that M being flat is the same as $\mathrm{Tor}^i(M, N) = 0$ for all $i > 0$. This should make sense: If we have any exact sequence ending in N, then thensoring with M shouldn't make this not-exact, so $\mathrm{Tor}^1(M, N) = 0$. Knowing this, we see that any sequence ending in M is universally exact, i.e., still exact if we tensor it with any other A-module N. In particular, exactness of the sequence

$$0 \to K \to A^n \to M \to 0$$

implies exactness of the sequence

$$\mathrm{Tor}^1_A(M, A/\mathfrak{m}) = 0 \to K/\mathfrak{m}K \to (A/\mathfrak{m}A)^n \to M/\mathfrak{m}M \to 0.$$