Solutions to Sheet 5

Exercise 1

Let A be a ring and let $\mathfrak{a}_1, \dots, \mathfrak{a}_n \subset A$ be ideals such that $\bigcap_{i=1}^n \mathfrak{a}_i = \{0\}$. Assume that each ring $A/\mathfrak{a}_i$ is noetherian. Show that A is noetherian.

Solution. Let $\pi_i : A \to A/\mathfrak{a}_i$ denote the projections. We have the map

$$\pi = (\pi_1, \dots, \pi_n) : A \to A/\mathfrak{a}_1 \times \cdots \times A/\mathfrak{a}_n.$$

As the $\mathfrak{a}_i$ have intersection $\{0\}$, π is injective. Hence A is isomorphic to the subring $\mathrm{Im}(\pi) \subset A/\mathfrak{a}_1 \times \cdots \times A/\mathfrak{a}_n$. This shows that A is isomorphic to the subring of a noetherian ring, thereby noetherian.

Exercise 2

Consider the matrix

$$S \coloneqq \begin{pmatrix} -36 & 14 & -24 \\ 18 & 6 & 12 \end{pmatrix}.$$

Determine its elementary divisors and the kernel/cokernel of the map $\mathbb{Z}^3 \xrightarrow{S} \mathbb{Z}^2$ (up to isomorphy).

Solution.

Exercise 3

Let A be a ring, let $\mathfrak{a} \subset A$ be an ideal and let M, N_i, $i \in I$, be A-modules for some set I.

1. Show that there exists a unique isomorphism

$$\Phi : \bigoplus_{i \in I} (N_i \otimes_A M) \to \left(\bigoplus_{i \in I} N_i \right) \otimes_A M$$

   such that $\Phi((\dots, 0, n_i \otimes m, 0 \dots)) = (\dots, 0, n_i, 0, \dots) \otimes m$ for all $n_i \in N_i$, $i \in I$, $m \in M$.

2. Show that there exists a unique isomorphism

$$\Psi : A/\mathfrak{a} \otimes_A M \to M/\mathfrak{a}M$$

   such that $\Psi((a + \mathfrak{a}) \otimes m) \mapsto am + \mathfrak{a}M$ for all $a \in A$, $m \in M$.

Solution.

1. By the unique property of the direct sum, defining Φ is the same as defining morphisms $\Phi_i : N_i \otimes_A M \to (\bigoplus_{i \in I} N_i) \otimes_A M$. Note that $N_i \otimes_A M$ is generated by elements of the

form $n_i \otimes m$ (with $n_i \in N_i$ and $m \in M$), and the exercise already specifies how Φ_i is defined on those elements, namely by

$$\Phi_i(n_i \otimes m) = (\dots, 0, n_i, 0, \dots) \otimes m \in \left(\bigoplus_{i\in I} N_i\right) \otimes_A M.$$

One could check now that this is well defined (remember that the elements $n_i \otimes m$ are only defined up to the relation $(an) \otimes m \sim n \otimes (am)$). But we use the universal property. The map is exactly the map that comes from the bilinear map $N_i \times M \to (\bigoplus_{i\in I} N_i) \otimes_A M$, $(n_i, m) \mapsto (\dots, 0, n_i, 0, \dots) \otimes m$. By construction, Ψ_i is a bijection on its image (really, we just put 0s everywhere else), and the images of Ψ_i have intersection $\{0\}$ and generate all of $(\bigoplus_{i\in I} N_i) \otimes M$. Hence Ψ is an isomorphism.

2. Again, we use the universal property. The mapping

$$A/\mathfrak{a} \times M \to M/\mathfrak{a}M, \quad (a + \mathfrak{a}, m) \mapsto am + \mathfrak{a}M.$$

is well-defined and bilinear, which is easy to check. This gives the desired map $\Psi : A/\mathfrak{a} \otimes_A M \to M/\mathfrak{a}M$. It is surjective as $\Psi(1 \otimes m) = m + \mathfrak{a}M$, and injective because if $\Psi((a + \mathfrak{a}) \otimes m) = 0 + \mathfrak{a}M$, we have $am \in \mathfrak{a}M$. Hence $am = a'm'$ for some $a' \in \mathfrak{a}, m' \in M$. In particular,

$$a \otimes m = 1 \otimes (am) = 1 \otimes (a'm') = a' \otimes m' = 0 \in A/\mathfrak{a} \otimes_A M.$$

This shows injectivity of Ψ, and we are done.

Exercise 4

Let A be a ring and let M, N be A-modules. A bilinear map $(-,-) : M \times M \to N$ is called symmetric if $(m_1, m_2) = (m_2, m_1)$ for all $m_1, m_2 \in M$. It is called alternating if $(m, m) = 0$ for all $m \in M$.

1. Show that there exists an A-module $\mathrm{Sym}^2_A(M)$ and a symmetric bilinear map $\iota : M \times M \to \mathrm{Sym}^2_A(M)$ with the following universal property: For every A-module N and for every symmetric bilinear map $(-,-) : M \times M \to N$ there exists a unique A-linear map $\Phi : \mathrm{Sym}^2_A(M) \to N$ usch that for all $m_1, m_2 \in M$

$$(m_1, m_2) = \Psi(\iota(m_1, m_2)).$$

Construct similarly an A-module $\Lambda^2_A(M)$ with a universal alternating bilinear map $\gamma : M \times M \to \Lambda^2_A(M)$.

2. Show that $\mathrm{Sym}^2_A(A^n)$ and $\Lambda^2_A(A^n)$ are free A-modules of ranks $\frac{n(n+1)}{2}$ and $\frac{n(n-1)}{2}$.

Solution.

1. Okay, the Sym-construction should be somehow similar to the construciton of $\otimes$, and ideally all proofs of properties simply follow from the universal property of the tensor product. In the construction of the tensor product, (m_1, m_2) corresponds to the image of $\varphi(m_1 \otimes m_2)$ for some suitable morphism φ. Imposing that $(m_1, m_2) = (m_2, m_1)$ corresponds to the statement that in Sym^2_A, any morphism should send $(m_1 \otimes m_2 - m_2 \otimes m_1)$

to zero. Building on this, we define $\mathrm{Sym}^2_A(M)$ as $(M \otimes_A M)/G$, where G is the A-module generated by elements of the form $(m_1 \otimes m_2 - m_2 \otimes m_1)$. We check that this works. With the notation of the exercise, we first obtain a morphism $\psi : M \otimes_A M \to N$ by the UP of the tensor product.

$$\begin{tikzcd}
M \times M \arrow[r, "(m_1,m_2)\mapsto m_1\otimes m_2"] \arrow[dr, "{(-,-)}"'] & M \otimes_A M \arrow[d, dashed, "\psi"'] \arrow[dr, "\mathrm{proj}"] & \\
& N & \mathrm{Sym}^2_A(M) \arrow[l, dashed, "\Psi"]
\end{tikzcd}$$

By construction, we have $G \subset \operatorname{Ker}\psi$, so by the universal property of kernels, ψ extends uniquely to a morphism $\Psi : \mathrm{Sym}^2_A(M) \cong (M \otimes_A M)/G \to N$.

We define $\Lambda^2_A(M)$ similarly, this time we define G as submodule of $M \otimes_A M$ generated by elements of the form $(m \otimes m)$.

2. We'll again first focus on Sym^2_A. First of all, note that the set of bilinear maps $(-,-) : A^n \times A^n \to N$ with values in an A-module N is the same as the set of matrices $(a_{ij})_{i,j=1,\dots,n}$ with $a_{ij} \in N$. The argument essentially comes from linear algebra; we simply associate to $(-,-)$ the matrix $((e_i, e_j))_{ij}$. Now, note that the subset of symmetric bilinear forms corresponds to those matrices with $a_{ij} = a_{ji}$. The set of these matrices has a natural structure of a free A-module of rank $\frac{n(n+1)}{2}$. We need to show that this number is equal to the rank of Sym^2_A. But for any A-module N, we have established the isomorphisms

$$\begin{aligned} N^{\frac{n(n+1)}{2}} &\cong \{M = (a_{ij})_{ij} \mid a_{ij} \in N \text{ and } a_{ij} = a_{ji}\} \\ &\cong \mathrm{SymBiHom}(A^n, A^n; N) \cong \mathrm{Hom}_A(\mathrm{Sym}^2_A(A^2), N). \end{aligned}$$

The functor sending N to $N^{\frac{n(n+1)}{2}}$ is represented by $A^{\frac{n(n+1)}{2}}$. Hence, utilizing the Yoneda-lemma, we find that $A^{\frac{n(n+1)}{2}} \cong \mathrm{Sym}^2_A(A^n)$.

For $\Lambda^2_A(A^n)$, we do exactly the same. The only thing that changes is the set of matrices we look at, as this time we have isomorphisms

$$\{M = (a_{ij})_{ij} \mid a_{ij} \in N \text{ and } a_{ij} = -a_{ji} \text{ and } a_{ii} = 0\} \cong \mathrm{AltBiHom}_A(A^n, A^n, N).$$

The space of matrices is quickly seen to be isomorphic to $N^{\frac{n(n-1)}{2}}$.

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