

Solutions to Sheet 10

Exercise 1

Let k be a field and let $f : A \rightarrow B$ be a k -algebra homomorphism with B a finitely generated k -algebra. Let $\mathfrak{m} \subset B$ be a maximal ideal. Show that $f^{-1}(\mathfrak{m}) \subset A$ is a maximal ideal.

Solution. Write $B = k[x_1, \dots, x_n]/I$. If $\mathfrak{m} \subset B$ is maximal, then $B/\mathfrak{m} \cong K$, where K/k is a finite field extension by Hilbert's Nullstellensatz. We have the morphism

$$A/f^{-1}(\mathfrak{m}) \rightarrow B/\mathfrak{m} = K,$$

which is necessarily injective, hence $A/f^{-1}(\mathfrak{m})$ is isomorphic to some sub- k -algebra of a finite field extension of k . But now it is a finite k -algebra, in particular a field itself. This shows that $f^{-1}(\mathfrak{m})$ is maximal.

Exercise 2

Let $n \geq 0$ and $Z \subset k^n$ be an algebraic subset. Show that $I(Z)$ is a prime ideal if and only if $Z = Z_1 \cap Z_2$ with Z_1, Z_2 algebraic implies $Z = Z_1$ or $Z = Z_2$.

Solution. A space satisfying the latter condition is called *irreducible*. I think all we know about $V(-)$ and $I(-)$ is

- Hilbert's Nullstellensatz: $I(V(J)) = \sqrt{J}$ and $V(I(Z)) = Z$.
- $I(-)$ and $V(-)$ are inclusion-reversing.
- $V(J_1 \cap J_2) = V(J_1 J_2) = V(J_1) \cup V(J_2)$ and $V(J_1 + J_2) = V(J_1) \cap V(J_2)$
- $I(Z_1 \cap Z_2) = I(Z_1) + I(Z_2)$ and $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$.
- The Zariski-Topology: This is the coarsest topology with sets of the form $V(I)$ closed.

If Z is irreducible and $f_1 f_2 \in I(Z)$, we have $V(f_1 f_2) \supset Z$ find $(V(f_1) \cap Z) \cup (V(f_2) \cap Z) = Z$, hence $V(f_1) \supset Z$ or $V(f_2) \supset Z$, which shows $f_1 \in I(Z)$ or $f_2 \in I(Z)$. Hence $I(Z)$ is prime.

On the contrary, if $I(Z)$ is prime and $Z = Z_1 \cup Z_2$, we find $I(Z) = I(Z_1 \cup Z_2) = I(Z_1)I(Z_2)$. Wlog, This implies $I(Z_1) = I(Z)$, hence $Z = V(I(Z)) = V(I(Z_1)) = Z_1$.

Exercise 3

A ring is called *Jacobson* if each prime ideal is the intersection of all maximal ideals containing it.

1. Show that a ring A is Jacobson if and only if for all primes $\mathfrak{p} \subset A$ and $a \notin \mathfrak{p}$ there exists a maximal ideal $\mathfrak{m} \subset A$ such that $a \notin \mathfrak{m}$ and $\mathfrak{p} \subset \mathfrak{m}$.

2. Let $f : A \rightarrow B$ be an injective, integral morphism and assume that B is Jacobson. Show that A is Jacobson. Deduce from the lecture that for each field k and $n \geq 0$ the ring $k[X_1, \dots, X_n]$ is Jacobson.

Solution.

1. There is not much to do. If A is Jacobson, then every prime ideal is the intersection containing it, hence for every $a \notin \mathfrak{p}$ there is some $\mathfrak{m} \supset \mathfrak{p}$ with $a \notin \mathfrak{m}$. The other direction is also readily verified.

2. First of all, note that if $\mathfrak{m} \subset B$ is maximal, $f^{-1}(\mathfrak{m}) \subset A$ is maximal as well. This follows directly from the going-up property of integral extension.

Also by going-up (or more generally, lying over) we find some $\mathfrak{q} \in \text{Spec}(B)$ with $f^{-1}(\mathfrak{q}) = \mathfrak{p}$. As B is Jacobson we have $\mathfrak{q} = \bigcap_{\mathfrak{m} \supset \mathfrak{q}} \mathfrak{m}$, so that we obtain

$$\mathfrak{p} = f^{-1}(\mathfrak{q}) = \bigcap_{\mathfrak{m} \supset \mathfrak{q}} f^{-1}(\mathfrak{m}) = \bigcap_{f^{-1}(\mathfrak{m}) \supset \mathfrak{p}} f^{-1}(\mathfrak{m}).$$

Exercise 4

Let A be a local ring and M a finitely presented, flat A -module. Show that M is free. *Hint:* Let $\mathfrak{m} \subset A$ be the maximal ideal. Use prev sheet to construct a short exact sequence $0 \rightarrow K \rightarrow A^n \rightarrow M \rightarrow 0$ with K finitely generated and $(A/\mathfrak{m})^n \rightarrow M/\mathfrak{m}M$ an isomorphism. Now use flatness of M and the snake lemma to check that $0 \rightarrow K/\mathfrak{m}K \rightarrow (A/\mathfrak{m})^n \rightarrow M/\mathfrak{m}M \rightarrow 0$ is again short exact.

Solution. We follow the hint. Write $k = A/\mathfrak{m}$. Note that we can choose n as the k -dimension of M/\mathfrak{m} : The dimension is finite by finite-generatedness of M and right-exactness of tensoring with $A/\mathfrak{m} = k$. By Nakayama's Lemma, any choice of generators of M/\mathfrak{m} lifts to generators of M . Hence we can construct a surjective morphism of A -modules $A^n \rightarrow M$ which is an isomorphism up to tensoring with k . Note that $\mathfrak{m}A \hookrightarrow A$, so after tensoring with M we find $\mathfrak{m}M \hookrightarrow M$. Also, tensoring the exact sequence

$$0 \rightarrow K \rightarrow A^n \rightarrow M \rightarrow 0$$

with \mathfrak{m} yields the exact sequence

$$\mathfrak{m}K \rightarrow (\mathfrak{m}A)^n \rightarrow \mathfrak{m}M \rightarrow 0.$$

All information up to now is encoded in the following diagram with exact rows.

$$\begin{array}{ccccccc} \mathfrak{m}K & \longrightarrow & (\mathfrak{m}A)^n & \longrightarrow & \mathfrak{m}M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & A^n & \longrightarrow & M \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & (A/\mathfrak{m})^n & \longrightarrow & M/\mathfrak{m}M \longrightarrow 0 \end{array}$$

The snake lemma on the top two rows yields a short exact sequence

$$0 \rightarrow K/\mathfrak{m}K \rightarrow (A/\mathfrak{m})^n \rightarrow M/\mathfrak{m}M \rightarrow 0,$$

and we obtain $K/\mathfrak{m}K = 0$, i.e. $K = \mathfrak{m}K$. But K is finitely generated (as M is finitely presented), and this implies $K = 0$ by Nakayama.

There is a better way to think about the homological algebra here. We know already that tensoring is right-exact, but in general not left-exact. As it turns out, the failure of left-exactness can be captured by certain *higher derived* tensor products, also known as Tor-functors. The idea is simple, albeit unintuitive if you have never encountered cohomology groups: Given a short exact sequence of A -modules

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

and another A -module N , there should be certain functors $\mathrm{Tor}_A^i(N, -)$ which capture the failure of left-exactness in that they fit into a long exact sequence

$$\begin{aligned} \dots \mathrm{Tor}_A^2(N, M'') \rightarrow \mathrm{Tor}_A^1(N, M') \rightarrow \mathrm{Tor}_A^1(N, M) \rightarrow \mathrm{Tor}_A^1(N, M'') \\ \rightarrow N \otimes_A M' \rightarrow N \otimes_A M \rightarrow N \otimes_A M'' \rightarrow 0. \end{aligned}$$

One can show that Tor_A^i is symmetric, i.e., $\mathrm{Tor}_A^i(M, N) = \mathrm{Tor}_A^i(N, M)$. Using Tor, one finds that M being flat is the same as $\mathrm{Tor}_A^i(M, N) = 0$ for all $i > 0$. This should make sense: If we have any exact sequence ending in N , then tensoring with M shouldn't make this not-exact, so $\mathrm{Tor}_A^1(M, N) = 0$. Knowing this, we see that any sequence ending in M is universally exact, i.e., still exact if we tensor it with any other A -module N . In particular, exactness of the sequence

$$0 \rightarrow K \rightarrow A^n \rightarrow M \rightarrow 0$$

implies exactness of the sequence

$$\mathrm{Tor}_A^1(M, A/\mathfrak{m}) = 0 \rightarrow K/\mathfrak{m}K \rightarrow (A/\mathfrak{m}A)^n \rightarrow M/\mathfrak{m}M \rightarrow 0.$$