Solutions to Sheet 7

Exercise 1

Let $A \to B$ be a homomorphism of rings, let M be an A-module and let N be a B-module.

1. Show that the map
$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_B(B \otimes_A M, N), \quad \varphi \mapsto (b \otimes m \mapsto b\varphi(m))$$
is a well-defined isomorphism.
2. Show that the map
$$M \otimes_A N \to (M \otimes_A B) \otimes_B N, \quad m \otimes n \mapsto (m \otimes 1) \otimes n$$
is a well-defined isomorphism.
3. Deduce that $S^{-1}M_1 \otimes_A S^{-1}M_2 \cong S^{-1}M_1 \otimes_{S^{-1}A} S^{-1}M_2$ for two A-modules M_1, M_2 and a multiplicative subset $S \subset A$.

Solution.

1. A function $\varphi \in \mathrm{Hom}_B(B \otimes_A M, N)$ is uniquely determined by its values on elementary tensors. We have $\varphi(b \otimes m) = b\varphi(1 \otimes m)$, so in reality φ is uniquely determined by its values on $1 \otimes m$. But any such morphism gives rise to a A-linear map via $m \mapsto 1 \otimes m \mapsto \varphi(1 \otimes m)$, and conversely any $\psi \in \mathrm{Hom}_A(M, N)$ yields a unique morphism via $b \otimes m \mapsto b\psi(m) \in \mathrm{Hom}_B(B \otimes_A M, N)$. These constructions are quickly checked to be mutually inverse.

   Remark. This is a special case of the so called *Hom-Tensor adjunction*. It states that there is a natural isomorphism
$$\mathrm{Hom}_B(M \otimes_A L, N) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_B(L, N)).$$
   In more fany terms, this says that the functors $\mathrm{Hom}_B(L, -) : \mathsf{Mod}_B \to \mathsf{Mod}_A$ and $- \otimes_A L : \mathsf{Mod}_A \to \mathsf{Mod}_B$ is an adjoint pair, for any B-module L.

2. Again, universal properties. Of course, we'll want to show that this is an isomorphism of B-modules. We do this by using the universal property. What is a B-linear morphism $\varphi : (M \otimes_A B) \otimes_B N \to P$? The same as a B-bilinear map $\Phi : (M \otimes_A B) \times N \to P$. But as $\Phi(m \otimes b, n) = b\Phi(m \otimes 1, n)$, any such bilinear map is uniquely determined by its values on elements of the form $(m \otimes 1, n)$, hence it really is the same as a A-bilinear map $M \times N \to P$, given by $(m, n) \mapsto (m \otimes 1, n) \mapsto \Phi(m \otimes 1, n)$. This construction is quickly verified to be an isomorphism. But now $(M \otimes_A B) \otimes_B N$ satisfies the universal property of $M \otimes_A N$.

3. We apply the above with $S^{-1}M_1 = M$ and $S^{-1}M_2 = N$ and $B = S^{-1}A$. Note that $S^{-1}M_1 \otimes_A S^{-1}A \cong S^{-1}(S^{-1}M_1) \cong S^{-1}M_1$, which gives (following the above)
$$S^{-1}M_1 \otimes_A S^{-1}M_2 \cong (M \otimes_A S^{-1}A) \otimes_{S^{-1}A} S^{-1}M_2 \cong S^{-1}M_1 \otimes_{S^{-1}A} S^{-1}M_2.$$

Exercise 2

Let A be a ring. We define the *support* of an A-module M as $\mathrm{Supp}(M) := \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_\mathfrak{p} \neq 0\}$.

1. Assume M is finitely generated. Show that $\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M \otimes_A k(\mathfrak{p}) \neq 0\}$, where $k(\mathfrak{p}) = \mathrm{Quot}(A/\mathfrak{p})$.
2. Assume M, N are finitely generated A-modules. Show $\mathrm{Supp}(M \otimes_A N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$.

Solution.

1. We will show that $M_\mathfrak{p} \neq 0$ if and only if $M \otimes_A k(\mathfrak{p}) \neq 0$. The map $A \to k(\mathfrak{p})$ factors through the map $A_\mathfrak{p} \to k(\mathfrak{p})$, and we find $M \otimes_A k(\mathfrak{p}) = M_\mathfrak{p} \otimes_{A_\mathfrak{p}} k(\mathfrak{p})$, this directly gives the implication $M \otimes_A k(\mathfrak{p}) \neq 0 \implies M_\mathfrak{p} \neq 0$.

   For the other direction, we use Nakayama's Lemma. It (or at least one version of it) states that if $N \neq 0$ is a finitely generated module over a local ring B with maximal ideal I, we have $IN \neq N$. In our situation, if we assume $M_\mathfrak{p} \neq 0$, Nakayama says

   $$M_\mathfrak{p} \otimes_{A_\mathfrak{p}} k(\mathfrak{p}) \cong M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \neq 0.$$

   Done.

2. We'll show that $(M \otimes_A N) \otimes k(\mathfrak{p}) \neq 0$ if and only if $M \otimes_A k(\mathfrak{p}) \neq 0$ and $N \otimes_A k(\mathfrak{p}) \neq 0$. Exercise 1.2 gives the isomorphism

   $$(M \otimes_A k(\mathfrak{p})) \otimes_{k(\mathfrak{p})} (N \otimes_A k(\mathfrak{p})) \cong M \otimes_A (N \otimes_A k(\mathfrak{p})) \cong (M \otimes_A N) \otimes_A k(\mathfrak{p}).$$

   From here we can directly check the desired equivalence.

Exercise 3

Let A be a ring, let $S \subset A$ be a multiplicative subset and let M, N be A-modules.

1. Assume that M is finitely presented A-module. Show that the map

   $$S^{-1}\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N), \quad \varphi/s \mapsto (m/t \mapsto \varphi(m)/st)$$

   is a well-defined isomorphism.

2. Construct a counterexample to the above if M is only assumed to be finitely generated.

Solution.

1. First, note that we always (without hypothesis on M) obtain such a map. This direclty follows (for example) from exercise 1.1 with $B = S^{-1}A$.

   Now we have to show that this is an isomorphism if M is finitely presented. As usual, we write M as part of a short exact sequence

   $$0 \to A^m \to A^n \to M \to 0.$$

Now we use that $\mathrm{Hom}_A(-, N)$ is right-exact. Hence applying $\mathrm{Hom}_A(-, N)$ yields an exact sequence

$$0 \to 0 \to \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(A^n, N) \cong N^n \to \mathrm{Hom}_A(A^m, N) \cong A^m.$$

Localizing at S is exact, so we obtain

$$0 \to 0 \to S^{-1}\mathrm{Hom}_A(M, N) \to (S^{-1}N)^n \to (S^{-1}N)^m.$$

Similarly, we can localize at S first and then apply $\mathrm{Hom}_{S^{-1}A}(S^{-1}(-), S^{-1}N)$, which yields the exact sequence

$$0 \to 0 \to \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \to (S^{-1}N)^n \to (S^{-1}N)^m.$$

Now we can use the 5-lemma again!

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & S^{-1}\mathrm{Hom}_A(M, N) \arrow[r] \arrow[d, "\therefore \text{iso}"] & (S^{-1}A)^n \arrow[r] \arrow[d, equal] & (S^{-1}A)^m \arrow[d, equal] \\
0 \arrow[r] & 0 \arrow[r] & \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \arrow[r] & (S^{-1}A)^n \arrow[r] & (S^{-1}A)^m
\end{tikzcd}$$

2. A standard example seems to be the following. Let $A = k[x, y_1, y_2, \dots]$ be the polynomial ring in variables indexed by $\mathbb{N}$. Let $M = A/(y_1, y_2, \dots)$, $N = A/(xy_1, x^2y_2, \dots)$ and $S = \{1, x, x^2, \dots\}$. Now let's compare both sides of the morphism. Note that M is generated by 1, so that any A-linear morphism $\varphi : M \to N$ is uniquely determined by the value of $\varphi(1) \in N$. Now we have $0 = y_1\varphi(1) = y_2\varphi(1) = \dots$, which shows that any lift $\widetilde{\varphi(1)} \in R$ is infintely divisible by x, hence $\varphi(1) = 0$. On the left hand side, we find that $S^{-1}M \cong S^{-1}N \cong k[x^{\pm 1}]$, so there are many $S^{-1}A$-linear morphisms $S^{-1}M \to S^{-1}N$.

Exercise 4

Let A be a principal domain and let $f \in A \setminus \{0\}$ be a non-unit. Show that the $A[T]$-module $(f, T) \subset A[T]$ is not flat.

Solution. Consider the map given by multiplication with f, which we will denote as $\varphi : A \to A$. It is injective. Note that $A \cong A[T]/(T)$. We want to show that $(f, T) \otimes_{A[T]} A$ is not injective, showing that (f, T) is not flat. We have an isomorphism (of $A[T]$-modules)

$$(f, T) \otimes_{A[T]} A \cong (f, T)/T(f, T),$$

and $(f, T) \otimes \varphi$ corresponds to the endomorphism given by multiplication with f under this identification. Now, $T \neq 0$ in $(f, T)/T(f, T)$, but $fT = \varphi(T) = 0$.

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