

# Solutions to Sheet 14

## Exercise 1

Let  $m \in \mathbb{N}$  and let  $\chi$  be a Dirichlet character modulo  $m$ .

1. Show that  $\log |1 - w| = -\operatorname{Re} \sum_{k=1}^{\infty} \frac{w^k}{k}$  for all  $w \in \mathbb{C}$  with  $|w| < 1$ .
2. Show that  $3 + 4 \cos \theta + \cos 2\theta \geq 0$  for all  $\theta \in \mathbb{R}$ .
3. Show that  $|1 - w|^3 |1 - wu|^4 |1 - wu^2| \leq 1$  for all  $w \in [0, 1)$  and all  $u \in \mathbb{C}$  with  $|u| = 1$ .
4. Show that  $|\zeta(\sigma)^3| |L(\sigma + it; \chi)|^4 |L(\sigma + 2it; \chi^2)| \geq 1$  for all  $\sigma \in (1, \infty)$  and  $t \in \mathbb{R}$ .
5. Deduce that  $L(1 + it; \chi) \neq 0$  for all  $t \in \mathbb{R} \setminus \{0\}$ .

**Solution.** This exercise reviews standard methods to prove zero-free regions of  $L$ -functions, and most introductory texts to analytic number theory should cover the content of this exercise (see for example [Jörg Brüdern, *Einführung in die analytische Zahlentheorie*, p. 101f]). Hence I will only sketch the solution.

1. The power series is that of the standard branch of the logarithm. The claim now follows by its properties.
2. This is a trick found by Hadamard and de la Vallée Poussin. One quickly checks that

$$3 + 4 \cos \alpha + \cos 2(\alpha) = 2(1 + \cos \alpha)^2 \geq 0.$$

3. Okay clearly the previous two exercises want us to take logarithm. We write  $u = e^{i\theta}$  and find

$$\begin{aligned} \log(|1 - w|^3 |1 - wu|^4 |1 - wu^2|) &= 3 \log(|1 - w|) + 4 \log |1 - wu| + \log(|1 - wu^2|) \\ &= - \sum_k \frac{w^k}{k} (3 + 4 \cos(k\theta) + \cos(2k\theta)) \leq 0. \end{aligned}$$

In the last inequality we used 2.

4. Developing this in a Euler product, the terms that occur are exactly of the form from exercise 3, but (multiplicatively) inverted. The claim follows.
5. Suppose that  $L(1 + it; \chi) = 0$  for some  $t \neq 0$ . Then also

$$\lim_{\sigma \rightarrow 1^+} |\zeta(\sigma)^3| |L(\sigma + it; \chi)|^4 |L(\sigma + 2it; \chi^2)| = 0,$$

as all the functions are analytic the degree 3-pole of  $\zeta(\sigma)$  at  $\sigma = 1$  gets eaten by  $L(\sigma + it; \chi)^4$ , which is a degree-4-zero at this point. This contradicts part 4. (Here we used that all the functions have (meromorphic) continuations to the whole plane.)

## Exercise 2

Let  $\chi$  be a non-trivial Dirichlet character modulo 8. Show that

$$L(1; \chi) > 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} = \frac{64}{105}.$$

**Solution.** The idea is that every Dirichlet character mod 8 is real as every element in  $(\mathbb{Z}/2\mathbb{Z})$  has order at most 2. Furthermore, we have  $\chi(3)\chi(5) = \chi(7)$ . Hence the smallest possible value that the series

$$L(1; \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} = \sum_{k \in 8\mathbb{N}} \left( \frac{\chi(k+1)}{k+1} + \frac{\chi(k+3)}{k+3} + \frac{\chi(k+5)}{k+5} + \frac{\chi(k+7)}{k+7} \right)$$

can ever take is if  $\chi(3) = \chi(5) = (-1)$  and  $\chi(7) = 1$ . In this case the above becomes

$$L(1; \chi) = \sum_{k \in 8\mathbb{N}} \left( \frac{1}{k+1} - \frac{1}{k+3} - \frac{1}{k+5} + \frac{1}{k+7} \right)$$

One quickly checks that for each  $k \in \mathbb{N}$  (as  $x \mapsto \frac{1}{x}$  is a convex function),

$$\frac{1}{k+1} - \frac{1}{k+3} - \frac{1}{k+5} + \frac{1}{k+7} > 0,$$

and the claim follows after truncating the series above at  $k = 1$ .

## Exercise 3

Let  $m \in \mathbb{N}$ , set  $\zeta_m = e^{2\pi i/m}$  and let  $K \subset \mathbb{Q}(\zeta_m)$  be a number field of degree  $d$ . Set  $G = \text{Gal}(K/\mathbb{Q})$  and identify  $\widehat{G}$  with a subgroup of the group of Dirichlet characters modulo  $m$  as in the lecture.

1. Show that for each  $\chi \in \widehat{G}$ , there exists a unique  $f = f_\chi \in \mathbb{N}$  (called the *conductor* of  $\chi$ ) such that  $f \mid m$  and  $\chi$  is the composition of the canonical homomorphism  $(\mathbb{Z}/m\mathbb{Z})^\times \rightarrow (\mathbb{Z}/f\mathbb{Z})^\times$  with a primitive Dirichlet character  $\chi^{\text{prim}}$  modulo  $f$ .
2. Let  $p$  be a prime number such that  $m = m'p^e$  for some  $m', e \in \mathbb{N}$  such that  $p \nmid m'$ . Set  $\zeta_{m'} = e^{2\pi i/m'}$  and  $L = K \cap \mathbb{Q}(\zeta_{m'})$ . Show that

$$\prod_{\mathfrak{p} \mid p\mathcal{O}_K} (1 - N(\mathfrak{p})^{-s}) = \prod_{\mathfrak{q} \mid p\mathcal{O}_L} (1 - N(\mathfrak{q})^{-s}) = \prod_{\chi \in \widehat{G}} (1 - \chi^{\text{prim}}(p)p^{-s})$$

3. Show that  $\zeta_K(s) = \prod_{\chi \in \widehat{G}} L(s; \chi^{\text{prim}})$  for all  $s \in \mathbb{C} \setminus \{1\}$  with  $\text{Re } s > 1 - 1/d$ .

**Solution.**

1. We let  $f > 1$  be the minimal integer with the desired property.
2. We are in the following situation.

$$\begin{array}{ccc} K & \longrightarrow & \mathbb{Q}(\zeta_m) = \mathbb{Q}(\zeta_{p^e}) \cdot \mathbb{Q}(\zeta_{m'}) \\ \downarrow & & \uparrow \\ L & \longrightarrow & \mathbb{Q}(\zeta_{m'}) \\ \downarrow & & \uparrow \\ \mathbb{Q} & \longrightarrow & \mathbb{Q} \end{array}$$

We know that the extension  $\mathbb{Q} \hookrightarrow \mathbb{Q}(\zeta'_m)$  is unramified above  $p$  as  $p$  does not divide the discriminant of  $\mathbb{Q}(\zeta'_m)$ . Hence  $p$  is also unramified in  $L$  (by multiplicativity of ramification indices along extensions of fields), and we find that

$$p\mathcal{O}_K = \mathfrak{q}_1 \dots \mathfrak{q}_r$$

for  $r$  pairwise distinct primes  $\mathfrak{q}_i \subset \mathcal{O}_L$ . Let  $f_i$  be the residue degree of  $\mathfrak{q}_i$  over  $p$ , then on the side of the Euler factors we find

$$\prod_{\mathfrak{q}|p\mathcal{O}_L} (1 - N(\mathfrak{q})^{-s}) = \prod_{i=1}^r (1 - p^{-f_i s}).$$

As  $\mathbb{Q}(\zeta_m) \cong \mathbb{Q}(\zeta'_m)[T]/(T^{p^e} - 1)$  we find that above  $p$ , the extension  $\mathbb{Q}(\zeta'_m) \hookrightarrow \mathbb{Q}(\zeta_m)$  is totally ramified, hence the extension  $L \hookrightarrow K$  is totally ramified as well (by multiplicativity of residue degrees along extensions). Thereby we can write  $\mathfrak{q}_i \mathcal{O}_K = \mathfrak{p}_i^{e_i}$  for prime ideals  $\mathfrak{p}_i \subset \mathcal{O}_K$ . But the Euler factors *forget* about the numbers  $e_i$ . Hence we also find

$$\prod_{\mathfrak{p}|p\mathcal{O}_K} (1 - N(\mathfrak{p})^{-s}) = \prod_{i=1}^r (1 - p^{-f_i s}).$$

We now show the equality

$$\prod_{\mathfrak{q}|p\mathcal{O}_L} (1 - N(\mathfrak{q})^{-s}) = \prod_{\chi \in \widehat{G}} (1 - \chi^{\text{prim}}(p)p^{-s}).$$

We make use of Theorem 6.13 in the lecture and write  $H = \text{Gal}(L/\mathbb{Q})$ . Applied to our situation, the Theorem states that

$$\zeta_L(s) = \prod_{\mathfrak{q}|m'\mathcal{O}_L} \frac{1}{1 - N(\mathfrak{q})^{-s}} \prod_{\chi \in \widehat{H}} L(s; \chi).$$

Hence (as  $p \nmid m'$ ) the Euler factor of  $\zeta_L(s)$  at  $p$  is given by

$$\prod_{\mathfrak{q}|p\mathcal{O}_L} \frac{1}{1 - N(\mathfrak{q})^{-s}} = \prod_{\chi \in \widehat{H}} \frac{1}{1 - \chi(p)p^{-s}}.$$

Now the argument gets a little wild<sup>1</sup>. First, using standard arguments in Galois theory, one can show that the diagram of abelian groups

$$\begin{array}{ccc} \text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) & \longrightarrow & G \\ \downarrow & & \downarrow \\ \text{Gal}(\mathbb{Q}(\zeta_{m'})/\mathbb{Q}) & \longrightarrow & H \end{array}$$

is a pushout diagram in the category of (finitely generated) abelian groups. Now we use that  $\text{Hom}_{\mathbf{Ab}}(-, \mathbb{C}^\times)$  is an exact functor (or equivalently,  $\mathbb{Q}/\mathbb{Z}$  is an injective object in  $\mathbf{Ab}$ ), to obtain that the dual diagram

$$\begin{array}{ccc} \widehat{H} & \longrightarrow & \widehat{(\mathbb{Z}/m'\mathbb{Z})}^\times \\ \downarrow & & \downarrow \\ \widehat{G} & \longrightarrow & \widehat{(\mathbb{Z}/m\mathbb{Z})}^\times \end{array}$$

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<sup>1</sup>I was too lazy to formulate out what the category-theoretic words mean. But all of the following can be made explicit, and I encourage you to do so if you haven't encountered the words used here before.

is a pullback-diagram, which is to say, we have  $\widehat{H} = \widehat{G} \cap (\widehat{\mathbb{Z}/m'\mathbb{Z}})^\times$  inside  $(\widehat{\mathbb{Z}/m\mathbb{Z}})^\times$  (where we identify the multiplicative residue groups with the respective cyclotomic Galois-groups). Now we are almost done. It is relatively straight-forward to check that whenever  $f \in \widehat{G}$ , we have

$$\chi^{\text{prim}}(p) = 0 \iff p \mid f_\chi \iff \chi \notin (\widehat{\mathbb{Z}/m'\mathbb{Z}})^\times \subset (\widehat{\mathbb{Z}/m\mathbb{Z}})^\times \iff \chi \notin \widehat{H}.$$

In particular, as we have  $\chi(p) = \chi^{\text{prim}}(p)$  whenever  $p \nmid f_\chi$ , we find

$$\prod_{\chi \in \widehat{H}} (1 - \chi(p)p^{-s}) = \prod_{\chi \in \widehat{G}} (1 - \chi^{\text{prim}}(p)p^{-s}).$$

3. This follows by comparing the factors of the respective Euler products.

#### Exercise 4

Let  $p$  be an odd prime number and let  $\zeta \in \mathbb{C}$  be a root of unity of order  $p$ . Show that  $1 + \zeta \in \mathcal{O}_{\mathbb{Q}(\zeta)}^\times$ .

**Solution.** Note that  $1 + \zeta = \frac{1 - \zeta^2}{1 - \zeta}$ . If we pick  $n \in \mathbb{N}$  such that  $2n \equiv 1 \pmod{p}$ , we have

$$\frac{1 - \zeta^2}{1 - \zeta} = \frac{1 - \zeta^2}{1 - \zeta^{2n}} = \left( \frac{1 - \zeta^{2n}}{1 - \zeta^2} \right)^{-1}.$$

And this is in  $\mathcal{O}_K$ , as

$$\frac{1 - \zeta^{2n}}{1 - \zeta^2} = 1 + \zeta^2 + \zeta^4 + \dots + \zeta^{2(n-1)}.$$