

# Solutions to Sheet 13

## Exercise 1

Let  $K$  be a number field of degree  $d$ .

1. Show that there exists a constant  $C$ , depending only on  $K$ , with the following property. If  $I$  is a principal ideal of  $\mathcal{O}_K$ , then  $I = \alpha \mathcal{O}_K$  for some  $\alpha \in \mathcal{O}_K$  such that  $|\sigma(\alpha)| \leq C N(I)^{1/d}$  for every real or complex embedding of  $K$ .
2. Let  $K \subset \mathbb{R}$  be a quadratic number field. Let  $\eta \in \mathcal{O}_K^\times$  be such that  $\mathcal{O}_K^\times = \{\pm \eta^n; n \in \mathbb{Z}\}$ . Show that, in this case, one can take  $C = \max\{|\eta|, |\eta|^{-1}\}^{1/2}$ .
3. Do there exists  $x, y \in \mathbb{Z}$  with  $x^2 - 82y^2 = 2$ ?

## Solution.

1. Assume that  $I = (\alpha)$ . Then  $I = (u\alpha)$  for every unit  $u \in \mathcal{O}_K^\times$ . Look at the morphism

$$\mathcal{L} : \mathcal{O}_K \setminus \{0\} \rightarrow \mathbb{R}^{r+s}, \xi \mapsto (\log |\sigma_1(\xi)|, \dots, \log |\sigma_r(\xi)|, 2 \log |\sigma_{r+1}(\xi)|, \dots, 2 \log |\sigma_{r+s}(\xi)|).$$

We have seen in the lecture that the image of  $\mathcal{O}_K^\times$  under this map is a lattice in a linear subspace  $V$ . Here  $V$  is given by those vectors of  $\mathbb{R}^{r+s}$  whose coordinates sum up to 0. We are interested in the image of the set  $\alpha \mathcal{O}_K^\times$  under  $\mathcal{L}$ . This is contained in the affine-linear subspace  $\mathcal{L}(\alpha) + V$ .

The exercise now translates to: There is some constant  $C > 0$  depending only on  $K$  such that there is a point  $x = (x_1, \dots, x_r, 2x_{r+1}, \dots, 2x_{r+s}) \in \mathcal{L}(\alpha) + \mathcal{L}(\mathcal{O}_K^\times) \subset \mathcal{L}(\alpha) + V$  with

$$\max x_i \leq \frac{1}{d} \log N(I) + \log(C)$$

We define a norm  $\|-\|$  via

$$\|(x_1, \dots, x_n)^t\| = \max(|x_1|, \dots, |x_s|, \frac{1}{2} |x_{s+1}|, \dots, \frac{1}{2} |x_{s+r}|)$$

and a linear function  $S : \mathbb{R}^{r+s} \rightarrow \mathbb{R}$  by

$$S((x_1, \dots, x_{r+s})^t) = \sum_{i=1}^{r+s} x_i.$$

Let  $W$  be the vector space spanned by

$$w_0 = (1, \dots, 1, 2, \dots, 2)$$

with  $r$  ones and  $s$  twos. Write  $\mathcal{L}(\alpha) = w + v$  where  $v \in V$  and  $w \in W$ . We have

$$S(\mathcal{L}(\alpha)) = \log N(I),$$

hence we obtain (as  $w \in \mathbb{R}w_0$ ,  $S(v) = 0$  and  $N(I) \geq 1$ )

$$\|w\| = \frac{1}{d} |S(w)| = \frac{1}{d} |S(w + v)| = \frac{1}{d} |S(\mathcal{L}(\alpha))| = \frac{1}{d} \log N(I).$$

All we need to do is to find a point of  $\mathcal{L}(\alpha\mathcal{O}_K^\times)$  close to  $w$ . For  $\nu \in V$  define  $d(\nu) = \inf_{\gamma \in \mathcal{L}(\mathcal{O}_K^\times)} (\|\nu - \gamma\|)$ , and set  $C = \sup_{\nu \in V} d(\nu)$ . This is well-defined because  $\mathcal{L}(\mathcal{O}_K^\times)$  is a lattice in  $V$ . Now  $C$  only depends on  $K$ , and we find a point

$$\mathcal{L}(\alpha) + \mathcal{L}(\mathcal{O}_K^\times) \ni w + \nu_0 = \xi = (\xi_1, \dots, \xi_r, 2\xi_{r+1}, \dots, 2\xi_{r+s})$$

with  $\nu_0 \in V$  and  $\|\nu_0\| \leq C$ . In particular,

$$\max \xi_i \leq \|\xi\| \leq \|w\| + \|\nu_0\| \leq \frac{1}{d} \log N(I) + C.$$

This solves the exercise.

2. Let  $\eta$  be as in the question and assume  $|\eta| < 1$ . Let  $\sigma_1, \sigma_2 : K \hookrightarrow \mathbb{R}$  be the two real embeddings of  $K$  (i.e.,  $\sigma_1$  is the identity on  $K \subset \mathbb{R}$  and  $\sigma_2$  is "conjugation"). Suppose  $I = (\alpha)$ . Then we can pick  $n$  such that

$$\sigma_1(\eta^n \alpha) = |\eta|^n |\sigma_1(\alpha)| \in (|\eta|^{1/2} N(I)^{1/2}, |\eta|^{-1/2} N(I)^{1/2}).$$

Now

$$|\sigma_2(\eta^n \alpha)| = |\eta|^{-n} |\sigma_2(\alpha)| = (|\eta|^{-n} \sigma_1(\alpha)^{-1}) N(I) \leq \frac{1}{|\eta|^{1/2}} N(I)^{1/2}.$$

3. Let  $K = \mathbb{Q}(\sqrt{82})$ . A unit as in 2 is given by  $\eta = 9 + \sqrt{82}$ . This can be checked similarly to sheet 11, exercise 4. Also, note that (by Dedekind-Kummer)  $2\mathcal{O}_K = \mathfrak{p}^2$  for some prime ideal  $\mathfrak{p}$ . Now  $N(\mathfrak{p}) = 2$ , and a solution to  $x^2 - 82y^2 = 2$  would result in  $\mathfrak{p}$  being principal. By part one and two, it suffices to show that there is no solution with

$$\max |x \pm \sqrt{82}y| \leq N(2\mathcal{O}_K)^{1/2} \sqrt{9 + \sqrt{82}} < 7.$$

But there are no such solutions as  $\sqrt{82} > 7$ .

## Exercise 2

Show that

$$\zeta_{\mathbb{Q}(\sqrt{6})} \zeta_{\mathbb{Q}(\sqrt{7})} \zeta_{\mathbb{Q}(\sqrt{42})} = \zeta_{\mathbb{Q}(\sqrt{6}, \sqrt{7})} \zeta_{\mathbb{Q}}^2.$$

**Solution.** Here we will only sketch a solution. The details are tedious. Recall that from the lecture (Example 3.12, say) we have for quadratic fields  $K = \mathbb{Q}(\sqrt{m})$  with discriminand  $\Delta_K$

$$p\mathcal{O}_K = \begin{cases} \text{prime ideal,} & \text{if } \left(\frac{\Delta_K}{p}\right) = -1 \\ \mathfrak{p}_1 \mathfrak{p}_2, & \text{(totally split) if } \left(\frac{\Delta_K}{p}\right) = 1 \\ \mathfrak{p}^2, & \text{(totally ramified) if } \left(\frac{\Delta_K}{p}\right) = 0. \end{cases}$$

Also, note that for  $K$  as above we have

$$\begin{aligned} \zeta_K(s) &= \prod_{\mathfrak{p}} \left(1 - \frac{1}{N(\mathfrak{p}^s)}\right)^{-1} = \prod_{p \in \mathbb{Z}} \prod_{\text{prime } \mathfrak{p} | p\mathcal{O}_K} \left(1 - \frac{1}{N(\mathfrak{p}^s)}\right)^{-1} \\ &= \prod_{\left(\frac{\Delta_K}{p}\right) = -1} \left(1 - \frac{1}{N(\mathfrak{p})^{2s}}\right)^{-1} \prod_{\left(\frac{\Delta_K}{p}\right) = 1} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-2} \prod_{\left(\frac{\Delta_K}{p}\right) = 0} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1} \end{aligned}$$

This yields some expansion of  $\zeta_{\mathbb{Q}(\sqrt{6})}\zeta_{\mathbb{Q}(\sqrt{7})}\zeta_{\mathbb{Q}(\sqrt{42})}$  in terms of factors indexed by prime numbers.

Write  $L = \mathbb{Q}(\sqrt{6}, \sqrt{7})$ . We want to do something similar as above for the function  $\zeta_{\mathbb{Q}(\sqrt{6}, \sqrt{7})}$ . Recall that if  $p\mathcal{O}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g}$  we have  $e_1 = \cdots = e_g = e$ ,  $f(\mathfrak{p}_1) = \cdots = f(\mathfrak{p}_g) = f$  and  $efg = 4$ . Also, as  $\text{Gal}(L/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$  is not cyclic, we cannot have inert primes, i.e., we never have  $f = 4$ . Indeed, if there was an inert prime  $\mathfrak{p} \mid p$ , we'd find  $\text{Gal}(L/\mathbb{Q}) = D(\mathfrak{p}|p) \cong \text{Gal}(\kappa(\mathfrak{p})/\mathbb{F}_p)$ , and the latter (being the Galois group of an extension of finite fields) is cyclic. Hence, the only possible splitting behaviours of a prime  $p \in \mathbb{Z}$  are:

$$\begin{aligned} g = 4 &\implies \prod_{\mathfrak{p}|p} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1} = (1 - p^{-s})^{-4} \\ f = 2, g = 2 &\implies \prod_{\mathfrak{p}|p} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1} = (1 - p^{-2s})^{-2} \\ e = 2, f = 2, g = 1 &\implies \prod_{\mathfrak{p}|p} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1} = (1 - p^{-2s})^{-1} \\ e = 2, f = 1, g = 2 &\implies \prod_{\mathfrak{p}|p} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1} = (1 - p^{-s})^{-2} \\ e = 4, f = 1, g = 1 &\implies \prod_{\mathfrak{p}|p} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1} = (1 - p^{-s})^{-1}. \end{aligned}$$

But one easily checks that  $\Delta_L$  has only prime divisors 2, 3, 7, so that all other primes are unramified on  $\mathcal{O}_L$ . Using this and that

$$\left(\frac{42}{p}\right) = \left(\frac{6}{p}\right) \left(\frac{7}{p}\right),$$

we can show that the Euler factors at each prime  $p$  coincide for both functions.

### Exercise 3

Let  $K$  be a number field and let

$$\log : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$$

denote the principal branch of the complex logarithm. The variable  $\mathfrak{p}$  always runs over the non-zero primes of  $\mathcal{O}_K$  in the following.

1. Show that

$$\lim_{s \rightarrow 1^+} \frac{1}{\log(s-1)} \sum_{\mathfrak{p}} \log(1 - N(\mathfrak{p})^{-s}) = 1.$$

2. Show that

$$\sum_{\mathfrak{p}; f(\mathfrak{p}) > 1} \frac{1}{N(\mathfrak{p})} + \sum_{\mathfrak{p}} \sum_{n=2}^{\infty} \frac{1}{n N(\mathfrak{p})^n} < \infty.$$

3. Using 1 and 2, deduce that there exists infinitely many prime ideals  $\mathfrak{p}$  of  $\mathcal{O}_K$  with  $f(\mathfrak{p}) = 1$ .

**Solution.** The following is a bit unprecise as I sometimes forgot to insert absolute-value brackets. But it works out if we simply assume  $s \in \mathbb{R}$  everywhere.

1. We know that the Dedekind-zeta function  $\zeta_K(s)$  is holomorphic in  $\operatorname{Re} s > 1$ , has a pole of order 1 at  $s = 1$  with residue  $\kappa > 0$  and has an Euler product

$$\zeta_K(s) = \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-s})^{-1}.$$

Hence we find

$$\lim_{s \rightarrow 1^+} \left( (s-1) \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-s})^{-1} \right) = \kappa,$$

and the claim follows after taking logarithms.

2. Note that there are at most  $[K : \mathbb{Q}]$  prime ideals of  $\mathcal{O}_K$  above each prime number  $p \in \mathbb{Z}$ . If  $\mathfrak{p}$  lies above  $p$  and  $f(\mathfrak{p}) > 1$ , we have (by definition)  $N(\mathfrak{p}) \geq p^2$ . Hence we obtain that

$$\sum_{\mathfrak{p}, f(\mathfrak{p}) > 1} N(\mathfrak{p})^{-s} \leq [K : \mathbb{Q}] \sum_p p^{-2s},$$

which is (absolutely) convergent for  $\operatorname{Re} s > \frac{1}{2}$ . Similarly, we find that

$$\frac{1}{[K : \mathbb{Q}]} \sum_{\mathfrak{p}} \sum_{n=2}^{\infty} \frac{1}{n N(\mathfrak{p})^{sn}} \leq \sum_p \sum_{n=2}^{\infty} \frac{1}{n p^{sn}} < \sum_p p^{-2s} \sum_{n=0}^{\infty} p^{-sn} < \left( \sum_{n=0}^{\infty} 2^{-sn} \right) \sum_p p^{-2s}.$$

This is absolutely convergent for  $\operatorname{Re} s > 0$ .

3. Recall that  $\log(1+t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} t^n$  for  $|t| < 1$ . We plug this into the logarithm of the euler product to obtain for  $\operatorname{Re} s > 1$

$$\log \zeta_K(s) = - \sum_{\mathfrak{p}} \log(1 - N(\mathfrak{p})^{-s}) = \sum_{\mathfrak{p}} \sum_{n=1}^{\infty} \frac{1}{n} N(\mathfrak{p})^{-ns}.$$

Splitting off the  $n = 1$  terms, this yields

$$\log \zeta_K(s) = \sum_{\mathfrak{p}, f(\mathfrak{p})=1} N(\mathfrak{p})^{-s} + \sum_{\mathfrak{p}, f(\mathfrak{p})>1} N(\mathfrak{p})^{-s} + \sum_{\mathfrak{p}} \sum_{n=2}^{\infty} \frac{1}{n N(\mathfrak{p})^{sn}}.$$

The left hand side of this equation diverges for  $s \rightarrow 1^+$ , but the last two terms of the RHS remain finite by part 2. The claim follows.