Exercise 1

1. The squarefull non-squares up to onehundred are $8, 27, 32, 72$.

2. It suffices to show that any squarefull prime power can be written uniquely as $p^k = a^2b^3$ with b square-free. But this is the same as writing $k = 2a + 3b$ with $0 \leq b \leq 1$, and this is possible in a unique way once $k \geq 2$.

3. Using the above and that b is square-free iff $\mu^2(b) = 1$, we may write

$$\sum_{n \text{ squarefull}} n^{-s} = \sum_{a,b} \frac{\mu^2(b)}{a^{2s}b^{3s}} = \zeta(2s) \sum_b \mu^2(b) b^{-3s}.$$

We can extend the Dirichlet series of μ^2 into an Euler product, obtaining

$$\sum_{n \in \mathbb{N}} \mu^2(n) n^{-s} = \prod_p (1 + p^{-s}) = \prod_p \frac{(1 - p^{-s})^{-1}}{(1 - p^{-2s})^{-1}} = \frac{\zeta(s)}{\zeta(2s)}.$$

(In the second-to-last equality we used $(1 + x)(1 - x) = 1 - x^2$.) We find

$$\sum_b \mu^2(b) b^{-3s} = \frac{\zeta(3s)}{\zeta(6s)},$$

done.

Exercise 2

This is just a messy calculation. We somehow want to get of the (a, b)-symbol in the sum. We do so by using that given $a, b \in \mathbb{N}$, we find unique coprime numbers k, l with $a = k(a, b)$ and $b = k(a, b)$. Now summing over all possible gcds d yields

$$\sum_{a,b} \frac{(a,b)}{a^s b^t} = \sum_d \frac{d}{d^{s+t}} \sum_{k,l \text{ coprime}} k^{-s} l^{-t} = \zeta(s + t - 1) \sum_{k,l} k^{-s} l^{-t} \sum_{e|(k,l)} \mu(e)$$

where we rephrased the coprimality condition on k and l using the trick from the last sheet. Now we rewrite

$$\sum_{k,l} k^{-s} l^{-t} \sum_{e|(k,l)} \mu(e) = \sum_e \mu(e) \sum_{k,l} (ke)^{-s} (le)^{-t} = \frac{\zeta(s)\zeta(t)}{\zeta(s+t)},$$

obtaining

$$\sum_{a,b} \frac{(a,b)}{a^s b^t} = \frac{\zeta(s + t - 1)\zeta(s)\zeta(t)}{\zeta(s + t)}.$$

Tracing through this calculation, we find that it is sufficient for absolute convergence to have $\Re(s) > 1$ and $\Re(t) > 1$. These conditions are easily seen to be necessary too (the sub-sums with $a = 1$ or $b = 1$ diverge otherwise).

Notes after correcting.

- Even though it is easily seen that the double sum cannot converge absolutely whenever (say) $\Re(s) \le 1$, this does immediately follow from the divergence of the series in the ζ-representation! The reason is that it is that we split the series in the first equality. It is possible to split a convergent series into divergent ones, as for example

$$\sum_{n\in\mathbb{N}} 0 = \sum_{n\in\mathbb{N}} (1-1) \neq \sum_n 1 - \sum_n 1.$$

Exercise 3

1. We have

$$\psi(s) = \sum_n n^{-s} - 2\sum_n (2n)^{-s}$$

and

$$\tilde{\psi}(s) = \sum_n n^{-s} - 3\sum_n (3n)^{-s}.$$

2. Using the Leibniz criterion, we see that the series converge conditionally on the positive real line, and thereby for $\Re s > 0$ by theorem (1.10). Alternatively, one can use (1.11) to see that the abscissa of convergence is given by

$$\sigma_0 = \limsup_{N\to\infty} \frac{\log|\sum_{n\le N}(-1)^n|}{\log N} = 0.$$

3. As both ψ and $\tilde{\psi}$ are holomorphic in $\Re s > 0$, ζ can only have a pole whenever $(1-2^{1-s})$ and $(1-3^{1-s})$ vanish. But this is the case whenever

$$1 = 2^{1-s} = \mathrm{e}^{(\log 2)(1-s)} \quad \Leftrightarrow \quad (\log 2)(1-s) \in 2\pi\mathrm{i}\mathbb{Z}$$

and

$$1 = 3^{1-s} = \mathrm{e}^{(\log 3)(1-s)} \quad \Leftrightarrow \quad (\log 3)(1-s) \in 2\pi\mathrm{i}\mathbb{Z}.$$

4. If $\log 2/\log 3 = p/q$ was rational, we'd find that $2^q = 3^p$, contradiction. Hence the two sets $(\log 2)^{-1}(2\pi i\mathbb{Z})$ and $(\log 3)^{-1}(2\pi i\mathbb{Z})$ have intersection the set $\{0\}$. Thereby, ζ cannot have a pole away from $s=1$. There it has a pole from a theorem in the lecture, and it is a simple pole as $(2^{1-s}-1)$ has a simple zero at $s=1$.

Exercise 4

We know that the d-th cyclotomic polynomial $\Phi_d(x)$ has degree $\varphi(d)$, and that $\prod_{d|n}\Phi_d(x) = x^n - 1$. Hence

$$\sum_{d|n}\varphi(d) = \sum_{d|n}\deg\Phi_d = \deg\left(\prod_{d|n}\Phi_d\right) = \deg(x^n-1) = n,$$

hence (by Möbius-inversion)

$$\varphi(n) = (\mu \star \mathrm{id})(n) = \sum_{d|n}\frac{n}{d}\mu(d).$$

Now we find

$$\sum_{n\le x}\varphi(n)/n=\sum_{n\le x}\frac{1}{n}\sum_{d|n}\frac{n}{d}\mu(d)=\sum_{d\le x}\frac{\mu(d)}{d}\sum_{k:kd\le x}1=\sum_{d\le x}\frac{\mu(d)}{d}\left[\frac{x}{d}\right].$$

We write $[x/d]=x/d+O(1)$ and use that $\mu(d)\in\{-1,0,1\}$. This gives

$$\sum_{d\le x}\frac{\mu(d)}{d}\left[\frac{x}{d}\right]=\sum_{d\le x}\frac{\mu(d)}{d}\frac{x}{d}+O\left(\sum_{d\le x}\frac{1}{d}\right)=\sum_{d\le x}\frac{\mu(d)}{d}\frac{x}{d}+O(\log x)$$

(by approximating the n-th harmonic number with the logarithm) and we have

$$\sum_{d\le x}\frac{\mu(d)}{d}\frac{x}{d}=x\sum_{d=1}^{\infty}\mu(d)d^{-2}+O\left(x\sum_{x<d<\infty}d^{-2}\right)=x\zeta(2)^{-1}+O(1).$$

One can show the estimate $\sum_{x<d<\infty}d^{-2}\ll x^{-1}$ using the inequality

$$\sum_{x<d<\infty}d^{-2}\le\int_{x-1}^{\infty}t^{-2}\,\mathrm{d}t=O((x-1)^{-1})=O(x^{-1}).$$

Done.

Notes after correcting.

- The convolution formula can also be obtained formally by writing

  $$\varphi(n)=\sum_{k\le n\text{ and }(k,n)=1}1=\sum_{k\le n}\sum_{d|(k,n)}\mu(d)$$

  and reordering sums.