Solutions to Sheet 9.

Problem 1&2

Want to estimate

$$S_2(x) \coloneqq \sum_{n\leq x, \Omega(n)=2} 1.$$

Write it as

$$S_2(x) = \sum_{p\leq\sqrt{x}} \sum_{p\leq q\leq x/p} 1 = \sum_{p\leq\sqrt{x}} (\pi(x/p) - \pi(p)) + O(\sqrt{x}).$$

Use PNT, get

$$S_2(x) = \sum_{p\leq\sqrt{x}} \int_p^{x/p} \frac{\mathrm{d}t}{\log t} + O(x\mathrm{e}^{-c\sqrt{\log x}}).$$

for a constant $c > 0$ (not the same as in the PNT). Concept-wise we are done here, as all is left to do is to do partial summation with $g(t) = \mathrm{Li}(x/t) - \mathrm{Li}(t)$ as smooth weight, and a_n the indicator function on primes. Estimating the rest is a bit tedious, but straight-forward:

We have $g(\sqrt{x}) = 0$ and $-g'(t) = \frac{1}{\log t} + \frac{x}{t^2\log(x/t)}$. We obtain

$$S_2(x) = \sum_{p\leq\sqrt{x}} g(p) = \int_2^{\sqrt{x}} \frac{\pi(t)}{\log t} + \frac{\pi(t)x}{t^2\log(x/t)}\,\mathrm{d}t.$$

The integral over $\pi(t)/\log t$ can be dealt with quite quickly. We have $\pi(t) \ll \frac{t}{\log t}$, hence

$$\int_2^{\sqrt{x}} \frac{\pi(t)}{\log t}\,\mathrm{d}t \ll \int_2^{\sqrt{x}} \frac{t}{(\log t)^2}\,\mathrm{d}t \ll \left(\int_2^{x^{1/4}} + \int_{x^{1/4}}^{\sqrt{x}}\right) \frac{t}{(\log t)^2}\,\mathrm{d}t \ll x^{1/2} + \frac{x}{(\log x)^2}.$$

We are left with

$$S_2(x) = \int_2^{\sqrt{x}} \frac{\pi(t)x\,\mathrm{d}t}{t^2\log(x/t)} + O\left(\frac{x}{(\log x)^2}\right) = \int_2^{\sqrt{x}} \frac{x((\log t)^{-1} + O((\log t)^{-2}))}{t\log(x/t)}\,\mathrm{d}t + O\left(\frac{x}{(\log x)^2}\right),$$

where we applied the PNT again, this time with error term $O(x/(\log x)^2)$. The integral over the O-term is also easily handled. We have $\log(x/t) \gg \log x$, and hence find that the contribution is bounded by

$$\frac{x}{\log x}\int_2^{\sqrt{x}} \frac{1}{t(\log t)^2}\,\mathrm{d}t \ll \frac{x}{\log x}.$$

We are left with

$$S_2(x) = x\int_2^{\sqrt{x}} \frac{1}{t(\log t)(\log\frac{x}{t})}\,\mathrm{d}t + O(\frac{x}{\log x}).$$

We can use the geometric series to show that

$$\frac{1}{\log\frac{x}{t}} = \frac{1}{\log x(1-\frac{\log x}{\log t})} = \frac{1}{\log x}\left(1 + O(\tfrac{\log t}{\log x})\right) = \frac{1}{\log x} + O\left(\frac{\log t}{(\log x)^2}\right).$$

Hence we obtain

$$S_2(x) = \frac{x}{\log x}\int_2^{\sqrt{x}} \frac{1}{t\log t}\,\mathrm{d}t + O\left(\frac{x}{(\log x)^2}\int_2^{\sqrt{x}}\frac{1}{t}\right) + O(\tfrac{x}{\log x}) = \frac{x}{\log x}\int_2^{\sqrt{x}} \frac{1}{t\log t}\,\mathrm{d}t + O\left(\frac{x}{\log x}\right).$$

This integral is exactly given by

$$\int_2^{\sqrt{x}} \frac{1}{t\log t}\,\mathrm{d}t = \log\log\sqrt{x} - \log\log 2,$$

which leaves us with

$$S_2(x) = \frac{x\log\log x}{\log x} + O\left(\frac{x}{\log x}\right),$$

as desired.

Problem 3

This is a consequence of Merten's theorem, which states that for $x > 1$,

$$\sum_{p\le x}\frac{1}{p} = \log\log x + C + O((\log x)^{-1})$$

for some constant C.

Note that

$$\frac{\varphi(n)}{n} = \prod_{p\mid n}(1-p^{-1}),$$

so we really want to show that the RHS is $\gg (\log\log n)^{-1}$. The product over the prime divisors of n is hard to get a hold on. It would be much easier if we could somehow relate this to products of the form $\prod_{p\le x}(1-p^{-1})$, as these products can be bounded with Merten's formula:

$$\begin{aligned}\prod_{p\le x}\left(1-\frac{1}{p}\right) &= \exp\left(\sum_{p\le x}\log\left(1-\frac{1}{p}\right)\right) = \exp\left(-\sum_{p\le x}\frac{1}{p} - \sum_{p\le x}\sum_{k\ge 2}\frac{1}{kp^k}\right)\\ &= \exp\left(-\log\log x - C + O((\log x)^{-1}) - \sum_{p}\sum_{k\ge 2}\frac{1}{kp^k} + O\left(\sum_{p>x}\sum_{k\ge 2}\frac{1}{p^k}\right)\right)\\ &= \frac{\mathrm{e}^{-C'}}{\log x}\exp\left(O\left(\frac{1}{\log x}\right)\right) = \frac{\mathrm{e}^{-C'}}{\log x}(1+O((\log x)^{-1}) \gg \frac{1}{\log x}.\end{aligned}$$

(This also was on sheet 0). In particular, if we choose $x = \log n$, we obtain

$$\prod_{p\le\log n}\left(1-\frac{1}{p}\right) \gg (\log\log n)^{-1}.$$

This is nice, because the prime divisors $p \mid n$ with $p \ge \log n$ don't contribute anything:

$$\prod_{p\mid n,\ p\le\log n}\left(1-\frac{1}{p}\right) \ge \left(1-\frac{1}{\log n}\right)^{\omega(n)} \ge \left(1-\frac{1}{\log n}\right)^{2\log n} \gg 1.$$

(Here we used $\omega(n) \le \log_2(n) \le 2\log n$ and that one formula for e). Hence we can conclude

$$\frac{\varphi(n)}{n} \ge \left(1-\frac{1}{\log n}\right)^{\omega(n)}\prod_{p\le\log n}\left(1-\frac{1}{p}\right) \gg \frac{1}{\log\log n}.$$

Notes after correcting. I just realized that the long calculation can be replaced by a reference to (5.9). This also makes the reference to Merten's theorem dispensable, but technically uses the (much stronger) prime number theorem.

Problem 4

Okay, let $c > 0$ and let q and q' be two exceptional moduli with zeroes characters χ, χ' and real zeroes β, β' satisfying the condition of the exercise. Let's compare the assumptions with the statement of (5.12).

(A) We have $1 - \frac{c}{\log q} < \beta$, and similar for q'.

(5.12) There is some small $d > 0$ (independent of q and q') such that we have $\min(\beta, \beta') \leq 1 - \frac{d}{\log(qq')}$.

If we assume $q < q'$, we certainly obtain

$$1 - \frac{c}{\log q} < 1 - \frac{d}{\log(qq')}, \quad \text{i.e.} \quad \frac{d}{c} < \frac{\log(qq')}{\log q}, \quad \text{i.e.} \quad q' > q^{d/c-1}.$$

Thus, any $c < d/3$ does the job.

This shows that there are $O(\log\log n)$ exceptional moduli up to n.

Aside: There is nothing special about the 2 in the exponent, if we choose c small enough we can get arbtirarily large exponents. But gives stronger conditions on what it means to be exceptional.

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