Solution to Sheet 3.

Problem 1

a) Let $g(x) = f(qx + a)$, so that

$$\sum_{n \equiv a \pmod q} f(n) = \sum_{m \in \mathbb{Z}} g(m).$$

We want to apply Poisson summation to g. The results of lemma (2.3) directly give that

$$\hat{g}(y) = \frac{1}{q} e\left(\frac{ya}{q}\right) \hat{f}\left(\frac{y}{q}\right).$$

The claim follows, as

$$\sum_{m \in \mathbb{Z}} g(m) = \sum_{m \in \mathbb{Z}} \hat{g}(m) = \frac{1}{q} \sum_{m \in \mathbb{Z}} e\left(\frac{ma}{q}\right) \hat{f}\left(\frac{m}{q}\right).$$

b) We would like to apply Poisson summation again, however we cannot calculate the "Fourier transform" of $f\chi$, as, χ is only defined on integers. We can abuse that χ is periodic though, rewriting

$$\sum_{m \in \mathbb{Z}} f(m)\chi(m) = \sum_{a \pmod q} \chi(a) \sum_{m \equiv a \pmod q} f(m).$$

Applying Poisson summation to the inner sum (we already did this in part a)) gives

$$\sum_{m \in \mathbb{Z}} f(m)\chi(m) = \frac{1}{q} \sum_{a \pmod q} \chi(a) \sum_{m \in \mathbb{Z}} e\left(\frac{ma}{q}\right) \hat{f}\left(\frac{m}{q}\right).$$

Reordering sums, we obtain

$$\begin{aligned}\frac{1}{q} \sum_{a \pmod q} \chi(a) \sum_{m \in \mathbb{Z}} e\left(\frac{ma}{q}\right) \hat{f}\left(\frac{m}{q}\right) &= \frac{1}{q} \sum_{m \in \mathbb{Z}} \hat{f}\left(\frac{m}{q}\right) \left(\sum_{a \pmod q} \chi(a) e\left(\frac{ma}{q}\right)\right) \\ &= \frac{1}{q} \sum_{m \in \mathbb{Z}} \hat{f}\left(\frac{m}{q}\right) \tau(\chi)\overline{\chi}(m) = \frac{\tau(\chi)}{q} \sum_{m \in \mathbb{Z}} \hat{f}\left(\frac{m}{q}\right) \overline{\chi}(m).\end{aligned}$$

Notes after correcting.

- In part a), instead of using the results from the lecture, we can also obtain the formula for the fourier transform directly. Setting $g(x) = f(qx + a)$ and substituting $u = qx + a$, we obtain

$$\hat{g}(y) = \int_{\mathbb{R}} f(qx + a) e(-xy) \,\mathrm{d}x = \frac{1}{q} \int_{\mathbb{R}} f(u) e(-u\tfrac{y}{q} + \tfrac{ay}{q}) \,\mathrm{d}u = \frac{1}{q} e(\tfrac{ay}{q}) \hat{f}(\tfrac{y}{q}).$$

Problem 2

I really don't like this problem as it has not much to do with number theory. I might add a solution later, but I am sure one can find this in most books on real analysis.

Problem 3

As the hint commands, we apply partial summation to the definition of $\tau(\chi)$, obtaining

$$|\tau(\chi)| = \sum_{h=1}^{q} \chi(h)e(h/q) = e(q/q)\sum_{h=1}^{q} \chi(h) - \frac{2\pi \mathrm{i}}{q}\int_1^q e(t/q)\sum_{h\le t}\chi(h)\,\mathrm{d}t.$$

As $\chi \neq \chi_0$, the sum $\sum_{h=1}^{q}\chi(h)$ vanishes. We also know by theorem (1.23) that $|\tau(\chi)| = \sqrt{q}$. Let M deonte the supremum of the absolute values of $\sum_{h\le x}\chi(h)$ for varying x (By Polya-Vinogradov, $M < \infty$). Then we obtain

$$\frac{q^{3/2}}{2\pi} = \left|\int_1^q e(t/q)\sum_{h\le t}\chi(h)\,\mathrm{d}t\right| \le \int_1^q \left|\sum_{h\le t}\chi(h)\right|\,\mathrm{d}t \le (q-1)M,$$

which is even a tad stronger than what we had to show.

Notes after correcting.

Problem 4

Let's just plug in the definition and look at what we have here.

$$\tau(\chi_1\chi_2) = \sum_{h\ (q)} \chi_1(h)\chi_2(h)e(h/q),$$

where $q = q_1q_2$. By the chinese remainder theorem, taking residues mod q gives a bijection

$$\{h_1q_2 + h_2q_1 \mid 1 \le h_i \le q_i\} \to \mathbb{Z}/q\mathbb{Z}.$$

Thus we may rewrite the sum above as

$$\tau(\chi_1\chi_2) = \sum_{1\le h_1\le q_1}\sum_{1\le h_2\le q_2} \chi_1(h_1q_2+h_2q_1)\chi_2(h_1q_2+h_2q_1)e(\tfrac{h_1q_2+h_2q_1}{q}),$$

and the claim follows after a few manipulations:

$$\begin{aligned}
&\sum_{1\le h_1\le q_1}\sum_{1\le h_2\le q_2} \chi_1(h_1q_2+h_2q_1)\chi_2(h_1q_2+h_2q_1)e(\tfrac{h_1q_2+h_2q_1}{q}) \\
&\qquad = \sum_{1\le h_1\le q_1}\sum_{1\le h_2\le q_2} \chi_1(h_1q_2)\chi_2(h_2q_1)e(\tfrac{h_1q_2}{q})e(\tfrac{h_2q_1}{q}) \\
&= \left(\chi_1(q_2)\sum_{1\le h_1\le q_1}\chi_1(q_2)e(\tfrac{h_1}{q_1})\right)\left(\chi_2(q_1)\sum_{1\le h_2\le q_2}\chi_2(q_1)e(\tfrac{h_2}{q_2})\right) = \chi_1(q_2)\tau(\chi_1)\chi_2(q_1)\tau(\chi_2).
\end{aligned}$$