Exercise 1

1. We may choose $a_n = (2n+2)! + 2$. Note that now $2 \mid (2n+2)! + 2$, $3 \mid (2n+2)! + 3$, etc.

2. We already know that $\pi(x) \leq M\frac{x}{\log(x)}$ for some $M > 0$ and $x > 2$. We solve the exercise by assuming that for all $c > 0$ there are only finitely many $n \in \mathbb{N}$ such that the interval $[n, n + c\log(n)]$ does not contain a prime, which ultimately will result in a contradiction to the statement above.

Let us make a choice for c and count the number of primes in $[x, 2x]$, for some large number x. We trivially obtain

$$\pi(2x) - \pi(x) \leq M\frac{2x}{\log(2x)}.$$

By our assumption, if x is large enough, there is no $n \in \mathbb{N} \cap [x, 2x]$ such the interval $[n, n+c\log(n)]$ does not contain a prime. Let us define numbers a_k such that $a_0 = [x]+1$, $a_{k+1} = a_k + c\log(a_k)$. Further, let $N \in \mathbb{N}$ be defined via $a_{N-1} \leq 2x < a_N$. As every interval $[a_k, a_{k+1}]$ contains a prime, this yields the estimate $N \leq \pi(2x) - \pi(x)$. Also, for $k < N$ we have $a_{k+1} - a_k \leq c\log(2x)$. This yields the estimate

$$\frac{x}{c\log(2x)} \leq N \leq \pi(2x) - \pi(x) \leq 2M\frac{x}{\log(2x)},$$

which is a contradiction once we choose $c < \frac{1}{2M}$.

Notes after correcting.

- Part 1 was relatively easy.
- Main reason for point-loss: Messy write-ups
- Common mistake: Whenever we have inequalities $a \leq b$ and $c \leq d$, we cannot deduce $a - c \leq b - d$. For that reason, we cannot effectively bound $\pi(x+h) - \pi(x)$ for small values of h by only knowing an upper bound for π.
- $f(x) = O(g(x))$ does not imply that $\frac{f(x)}{g(x)}$ approaches some value $C \in \mathbb{R}$ as $x \to \infty$. Rather, it implies that the absolute value of this fraction is bounded.

Exercise 2

1. Via $\alpha \star \alpha = 1$, we obtain $\alpha(1) = \pm 1$. Having defined $\alpha(n)$ for values $n \leq N$, $\alpha(N)$ is uniquely determined by the equation

$$1 = \sum_{d|N} \alpha(d)\alpha(N/d) = 2\alpha(N) + \sum_{d|N, d\neq 1, N} \alpha(d)\alpha(N/d).$$

Any choice of $\alpha(1)$ thereby extends to an arithmetic function with $\alpha \star \alpha = 1$, and α cannot be multiplicative if $\alpha(1) \neq 1$.

2. We set $\alpha(1) = 1$ define $\alpha(p^n)$ via the taylor series expansion of $(1-x)^{\frac{-1}{2}}$:

$$\sum_{n\in\mathbb{N}} \alpha(p^n)x^n = (1-x)^{\frac{-1}{2}}$$

(Note that $(1-x)^{\frac{-1}{2}}$ is holomorphic in some neighbourhood around 0) and extend α to a multiplicative function via $\alpha(n) = \prod_p \alpha(p^{v_p(n)})$. By the formula for multiplying taylor series, we find

$$\sum_{n\in\mathbb{N}} x^n = \frac{1}{1-x} = \left(\frac{1}{1-x}\right)^{2\frac{1}{2}} = \sum_{k\in\mathbb{N}} x^k \sum_{0\le l\le k} \alpha(p^l)\alpha(p^{k-l}).$$

After equating coefficients, this gives

$$\sum_{0\le l\le k} \alpha(p^l)\alpha(p^{k-1}) = 1,$$

i.e. $\alpha \star \alpha = 1$. (Note that α and 1 are multiplicative, so it suffices to check the equality on prime-powers). Basic analysis also reveals that α is now given by $\alpha(p^n) = \frac{(2n)!}{4^n(n!)^2}$, as demanded by the exercise.

Notes after correcting.

- Part 1 was relatively easy.
- For part 2, one can also use that $\alpha(p^n) = (-1)^n\binom{-\frac{1}{2}}{n}$ and deduce $\alpha\star\alpha = 1$ using formulas for binomial coefficients. This does not use generating functions, but it is messy.

Exercise 3

1. It is easily seen that both sides are multiplicative, and we may reduce to the case $n = p^k$, p prime. The LHS becomes $1 + ak$, the RHS becomes $1 + ak$ too, and we are done.

2. Again, both sides are multiplicative. (For the RHS, note that the product and the convolution of any two multiplicative functions is multiplicative, and that RHS $= 1 \star (\mu\tau)$.) For $n = 1$, we find LHS = RHS = 1. For prime powers $n = p^k$ with $k \ge 1$, we find

$$\text{LHS} = \mu(p^0)\tau(p^0) + \mu(p^1)\tau(p^1) + \underbrace{\mu(p^2)\tau(p^2) + \cdots + \mu(p^n)\tau(p^n)}_{=0 \text{ as } \mu(p^k)=0 \text{ for } k\ge 2.} = 1 - 2 = -1.$$

As in this case we also have RHS $= -1$, we are done.

3. We write $e(\theta)$ for $\mathrm{e}^{2\pi\mathrm{i}\theta}$. We first get rid of the condition $(m,n) = 1$ via adding the term

$$\eta((m,n)) = (1\star\mu)((m,n))$$

to each summand, obtaining

$$\text{LHS} = \sum_{1\le m\le n \text{ and } (m,n)=1} e(m/n) \sum_{d|(m,n)} \mu(d).$$

We change the order of summation, bringing d to the outer sum, writing $m = dk$ for $d \mid n$. This gives

$$\text{LHS} = \sum_{d|n} \mu(d) \sum_{k\le n/d} e(\tfrac{k}{n/d}).$$

Now the inner sum goes over all n/d-th roots of unity, and thereby equals 0 whenever $n/d > 1$. Hence we find LHS = RHS, as desired.

Notes after correcting.

- Part 2 can be done in multiple ways, one can for example use binomial coefficient stuff to check the identity directly (for general n and not only prime-powers).
- The trick used in part 3 is quite commonly used and should be added to your Analytic number theory toolkit!

Exercise 4

We use summation by parts, setting $a_n = 1$ and $g(x) = \frac{1}{\sqrt{x}}$. We find

$$\sum_{1\le n\le x} \frac{1}{\sqrt{n}} = \frac{[x]}{\sqrt{x}} + \frac{1}{2}\int_1^x \frac{[t]}{t^{\frac{3}{2}}}\,\mathrm{d}t = \sqrt{x} - \frac{\{x\}}{\sqrt{x}} + \frac{1}{2}\int_1^x \frac{1}{\sqrt{t}} - \frac{\{t\}}{t^{3/2}}\,\mathrm{d}t.$$

We have $\{x\}/\sqrt{x} = O(x^{-\frac{1}{2}})$,

$$\frac{1}{2}\int_1^x \frac{1}{\sqrt{t}}\,\mathrm{d}t = [\sqrt{t}]_1^x = \sqrt{x} - 1$$

and

$$\frac{1}{2}\int_1^x \frac{\{t\}}{t^{3/2}}\,\mathrm{d}t = \frac{1}{2}\int_1^\infty \frac{\{t\}}{t^{3/2}}\,\mathrm{d}t - \frac{1}{2}\int_x^\infty \frac{\{t\}}{t^{3/2}}\,\mathrm{d}t.$$

Here the first integral converges, and the second integral lies within $O(x^{-1/2})$. The claim follows, with

$$C = \frac{1}{2}\int_1^\infty \frac{\{t\}}{t^{3/2}}\,\mathrm{d}t - 1.$$

Notes after correcting.

- Common mistake: Errors while calculating the integral (but I am sure this will get better as the course progresses).