Solution to Sheet 3.

Problem 3

As the hint commands, we apply partial summation on $\tau(\chi)$, obtaining

$$|\tau(\chi)| = \sum_{h=1}^{q} \chi(h)e(h/q) = e(q/q)\sum_{h=1}^{q}\chi(h) - \frac{2\pi \mathrm{i}}{q}\int_1^q e(t/q)\sum_{h\le t}\chi(h)\,\mathrm{d}t.$$

As $\chi \neq \chi_0$, the sum $\sum_{h=1}^{q}\chi(h)$ vanishes. We also know by theorem (1.23) that $|\tau(\chi)| = \sqrt{q}$. Let M deonte the supremum of the absolute values of $\sum_{h\le x}\chi(h)$ for varying x. Then we obtain

$$\frac{q^{3/2}}{2\pi} = \left|\int_1^q e(t/q)\sum_{h\le t}\chi(h)\,\mathrm{d}t\right| \le \int_1^q \left|\sum_{h\le t}\chi(h)\right|\,\mathrm{d}t \le (q-1)M,$$

which is even a tad stronger than what we had to show.

Notes after correcting.

Problem 4

Let's just plug in the definition and look at what we have here.

$$\tau(\chi_1\chi_2) = \sum_{h\ (q)} \chi_1(h)\chi_2(h)e(h/q),$$

where $q = q_1q_2$. By the chinese remainder theorem, taking residues mod q gives a bijection

$$\{h_1q_2 + h_2q_1 \mid 1 \le h_i \le q_i\} \to \mathbb{Z}/q\mathbb{Z}.$$

Thus we may rewrite the sum above as

$$\tau(\chi_1\chi_2) = \sum_{1\le h_1\le q_1}\sum_{1\le h_2\le q_2}\chi_1(h_1q_2+h_2q_1)\chi_2(h_1q_2+h_2q_1)e(\tfrac{h_1q_2+h_2q_1}{q}),$$

and the claim follows after a few manipulations:

$$\begin{aligned}
&\sum_{1\le h_1\le q_1}\sum_{1\le h_2\le q_2}\chi_1(h_1q_2+h_2q_1)\chi_2(h_1q_2+h_2q_1)e(\tfrac{h_1q_2+h_2q_1}{q})\\
&\qquad= \sum_{1\le h_1\le q_1}\sum_{1\le h_2\le q_2}\chi_1(h_1q_2)\chi_2(h_2q_1)e(\tfrac{h_1q_2}{q})e(\tfrac{h_2q_1}{q})\\
&= \left(\chi_1(q_2)\sum_{1\le h_1\le q_1}\chi_1(q_2)e(\tfrac{h_1}{q_1})\right)\left(\chi_2(q_1)\sum_{1\le h_2\le q_2}\chi_2(q_1)e(\tfrac{h_2}{q_2})\right) = \chi_1(q_2)\tau(\chi_1)\chi_2(q_1)\tau(\chi_2).
\end{aligned}$$