Solutions to Sheet 5.

Problem 1

Okay, we just go through everyting. For $\zeta(s)$ we have degree $d = 1$, conductor $N = 1$, root number $\eta = 1$, $\kappa_1 = 0$ and hence $L_\infty(s) = \pi^{-s/2}\Gamma(s/2)$. For $L(s, \chi)$ with a primitive Dirichlet character χ mod $q > 1$ we have degree $d = 1$, conductor $N = q$, root number $\eta = \mathrm{i}^{-\kappa}\tau(\chi)q^{-1/2}$, $\kappa_1 = \kappa$ and $L_\infty(s) = \pi^{-s/2}\Gamma(\frac{s+\kappa}{2})$ where $\kappa = 0$ if χ is even and $\kappa = 1$ if χ is odd.

The functional equation now reads as follows.

Theorem 1 (Approximate functional equation for ζ)**.** *Let $G(u)$ be any even function which is holomorphic and bounded in $|\Re(u)| < 4$ and normalized by $G(0) = 1$. Let $X > 0$. Then for $0 < \sigma < 1$ we have*

$$\zeta(s) = \sum_n n^{-s}V_s\left(\frac{n}{X}\right) + \pi^{s-1/2}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\sum_n n^{s-1}V_{1-s}(nX) - R$$

where

$$V_s(y) = \frac{1}{2\pi \mathrm{i}}\int_{(3)} G(u)\frac{\Gamma((s+u)/2)}{\Gamma(s/2)}(y\sqrt{\pi})^{-u}\frac{\mathrm{d}u}{u}$$

and

$$R = \frac{\pi^{s/2}}{\Gamma(s/2)}\frac{G(1-s)}{1-s}X^{1-s} - \frac{\pi^{s/2}}{\Gamma(s/2)}\frac{G(-s)}{-s}X^{-s}.$$

Completed Dirichlet L-functions are entire, so we get rid of R. As above, in the following κ depends on the parity of χ.

Theorem 2 (Approximate functional equation for Dirichlet L-functions)**.** *Let $G(u)$ be any even function which is holomorphic and bounded in $|\Re(u)| < 4$ and normalized by $G(0) = 1$. Let $X > 0$. Then for $0 < \sigma < 1$ we have*

$$\zeta(s) = \sum_n \chi(n)n^{-s}V_s\left(\frac{n}{X\sqrt{q}}\right) + \epsilon(s)\sum_n \overline{\chi(n)}n^{s-1}V_{1-s}\left(\frac{nX}{\sqrt{q}}\right)$$

where

$$V_s(y) = \frac{1}{2\pi \mathrm{i}}\int_{(3)} G(u)\frac{\Gamma((s+u+\kappa)/2)}{\Gamma((s+\kappa)/2)}(y\sqrt{\pi})^{-u}\frac{\mathrm{d}u}{u}$$

and

$$\epsilon(s) = \mathrm{i}^{-\kappa}\tau(\chi)q^{-s}\pi^{s-1/2}\frac{\Gamma((1-s+\kappa)/2)}{\Gamma((s+\kappa)2)}.$$

As for (3.11), we have for ζ that $\mathcal{C}(s) = \mathcal{C}_0(s) = |s+2|$, for Dirichlet L-functions we find $\mathcal{C}_0(s) = |s+\kappa| + 2$ and $\mathcal{C}(s) = q(|s+\kappa|+2)$. As an aside, the 2 here is quite arbitrary and is only there to make sure everything works out when $|s|$ is small. We can plug this into (3.11), finding (with $G(u) = \mathrm{e}^{u^2}$) that for ζ we have that

$$y^a V_s^{(a)}(y) \ll_{a,A} \left(1 + \frac{y}{\sqrt{|s|+2}}\right)^{-A}$$

for $\Re(s) > 0$ whereas for $L(s, \chi)$ we find

$$y^a V_s^{(a)}(y) \ll_{a,A} \left(1 + \frac{y}{\sqrt{|s+\kappa|+2}}\right)^{-A}$$

for $\Re(s) > -\kappa$.

Lastly, the conditions for (3.12) are satisfied for both $\zeta(s)$ and $L(s,\chi)$, we have the convexity bound

$$\zeta(s) \ll_{\varepsilon,\delta} (|s|+2)^{\frac{1-\sigma}{2}+\varepsilon}$$

whenever $|s-1| \geq \delta$ (i.e., away from the pole) and similarly

$$L(s,\chi) \ll_{\varepsilon} (q\,|s+\kappa|+2)^{\frac{1-\sigma}{2}+\varepsilon}.$$

Again, it should be noted that the 2 is added artificially to have small $|s|$ not mess everything up. For large s, these vanish and we obtain (and should really read these as)

$$\zeta(s) \ll |s|^{\frac{1-\sigma}{2}+\varepsilon} \quad \text{and} \quad L(s,\chi) \ll |qs|^{\frac{1-\sigma}{2}+\varepsilon}.$$

Also, if we fix L, we can absorb the factor q into the implicit constant from $\ll$.

Problem 2

We first calculate $\zeta(0)$. The simple pole of $\zeta(s)$ at $s=1$ has residue 1, so we know that $\lim_{s\to 1}(s-1)\zeta(s) = 1$. Writing the functional equation as $\zeta(s) = \Delta(s)\zeta(1-s)$ gives

$$1 = \lim_{s\to 1}(s-1)\zeta(s) = \lim_{s\to 1}(s-1)\Delta(s)\zeta(0),$$

so we only need to evaluate the remaining term $\lim_{s\to 1}(s-1)\Delta(s)$. We have

$$\Delta(s) = \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\pi^{s-1/2}.$$

It follows that $\zeta(0) = -\frac{1}{2}$ as $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma((1-s)/2)$ has residue -2 at 1 (think about the Laurent expansion at 1 and remember that Γ has residue 1 at 0).

$\zeta(s) < 0$ *for* $s \in (0,1)$. We have that

$$\zeta(s) = \frac{s}{s-1} - s\int_0^\infty \{t\}t^{-s-1}\,\mathrm{d}t.$$

This is negative. Hence ζ is negative in the interval $[0,1)$.

Problem 3

We want to follow the proof from (3.14) as closely as possible. The first difference is that we sum over all characters, not just the primitive ones, but this does not make a difference: If we know that

$$\sum_{\chi (\mathrm{mod}\ q)}^{*} |L(1/2,\chi)|^2 \ll q^{1+\varepsilon}$$

(where the star in the sum means that we sum over primitive characters, this notation is quite common), we can easily deduce

$$\sum_{\chi (\text{mod } q)} |L(1/2,\chi)|^2 = \sum_{d|q} \sideset{}{^*}\sum_{\chi (\text{mod } d)} |L(1/2,\chi)|^2 \leq \tau(q) \max_{d|q} \sideset{}{^*}\sum_{\chi (\text{mod } d)} |L(1/2,\chi)|^2 \ll q^{1+\varepsilon}$$

as $\tau(q)$ the number of divisors of q, satisfies $\tau(q) \ll q^{\varepsilon}$. The L_∞ factor occuring in V_s only depends on the parity of χ, so we further split the sum into odd and even parts. We want to use the approximate functional equation with $X = 1$, $s = 1/2$ and $G(u) = \mathrm{e}^{u^2}$ as in (3.11). Let's check what happens. We find

$$L(1/2,\chi) = \sum_n \frac{\chi(n)}{n^{1/2}} V_{1/2}(n/\sqrt{N}) + \epsilon(1/2) \sum_n \frac{\overline{\chi}(n)}{n^{1/2}} V_{1/2}(n/\sqrt{N}) + R$$

where

- $R = 0$ as the completed L-function $\Lambda(s,\chi)$ is entire.
- The root number $\epsilon(1/2)$ has absolute value 1.
- The terms involving $V = V_{1/2}$ can be bounded by $V(y) \ll_A (1+y)^{-A}$. For all $A > 0$.

Also note that both sums are equal in absolute value. This is not too complicated! We plug it in, using this time that $|a+b|^4 \leq 8(|a|^4 + |b|^4)$ (this can be seen using Hölder's inequality for example), obtaining

$$\sideset{}{^*}\sum_{\chi (\text{mod q}) \text{ even}} |L(1/2,\chi)|^4 \leq 16 \sideset{}{^*}\sum_{\chi(q) \text{ even}} \left| \sum_n \frac{\chi(n)}{n^{1/2}} V(n/\sqrt{q}) \right|^4.$$

Similar to the proof of (3.14), we can complete the sum to go over all characters and open up the sum, obtaining a fourfold sum which we can simplify using orthogonality relations on sums over characters. In short, we get

$$\cdots \leq 16 \sum_{\chi(q)} \left| \sum_n \frac{\chi(n)}{n^{1/2}} V(n/\sqrt{q}) \right|^4 = 16 \sum_{n_1,n_2,m_1,m_2} \frac{V_{n_1} V_{n_2} \overline{V}_{m_1} \overline{V}_{m_2}}{(n_1 n_2 m_1 m_2)^{1/2}} \sum_{\chi(q)} \chi(n_1 + n_2 - m_1 - m_2), \quad (1)$$

where we wrote $V_n = V(n/\sqrt{q})$. The sum over χ does not vanish iff $n_1 n_2 \equiv m_1 m_2 \pmod{q}$, where it equals $\varphi(q)$. As the hint suggests, we glue together n_1 and n_2, m_1 and m_2, which leaves us with the taks of bounding terms of the form

$$(V * V)(n) = \sum_{n_1 n_2 = n} V_{n_1} V_{n_2}.$$

We find for any $A \geq 1$

$$(V * V)(n) \ll \sum_{n_1 n_2 = n} \left(1 + \frac{n_1}{\sqrt{q}}\right)^{-A} \left(1 + \frac{n_2}{\sqrt{q}}\right)^{-A} \leq \sum_{n_1 n_2 = n} \left(1 + \frac{n}{q}\right)^{-A} \ll_\varepsilon n^{\varepsilon} \left(1 + \frac{n}{q}\right)^{-A}.$$

With $A = 1 + \varepsilon$ we calculate

$$(1) \ll \varphi(q) \sum_{n,m} \frac{(V \star V)(n) \overline{(V \star V)(m)}}{(nm)^{1/2}} \ll \varphi(q) \sum_n \sum_{n \equiv m \geq n} (1 + \tfrac{m}{q})^{-1} (1 + \tfrac{n}{q})^{-1} (mn)^{-1/2}, \quad (2)$$

and upon applying the bound $\varphi(q) < q$ this is exactly the sum that arises in the end of the proof of (3.14)! (I might add lines on how to bound this once I have time).

Max von Consbruch, email: s6mavonc@uni-bonn.de. Date: November 30, 2022