Solution to Sheet 5.

Problem 1

Problem 2

We have to show the bound

$$\int_{(-A+\frac{1}{2})} \Gamma(s)x^s \,\mathrm{d}s \ll \frac{x^{-A+1/2}}{(A-1)!}.$$

Note that the integral exists by the rapid decay of Γ along vertical lines. However, we cannot apply Stirling's formula to bound the integral directly as Stirling a priori only gives uniform bounds in regions of the form $|\arg(s)-\pi| \geq \delta > 0$. We can however apply stirlings formula if we apply the recurrence $s\Gamma(s) = \Gamma(s+1)$ repeatedly:

$$\begin{aligned}\int_{(-A+1/2)} \Gamma(s)x^s \,\mathrm{d}s &\ll \int_{(-A+1/2)} |\Gamma(s)x^s| \,\mathrm{d}s \ll x^{-A+1/2} \int_{(1/2)} |\Gamma(s-A)| \,\mathrm{d}s \\ &= x^{-A+1/2} \int_{(1/2)} \left| \frac{\Gamma(s)}{(s-A+1)\cdots(s-1)} \right| \mathrm{d}s \leq \frac{x^{-A+1/2}}{(A-1)!} \int_{(1/2)} |\Gamma(s)| \,\mathrm{d}s.\end{aligned}$$

Notes. Once we know this inequality, we actually can do better: Remember that Γ has poles at the negative integers, the residue at $-n$ is given by $\frac{(-1)^n}{n!}$. Hence for (large) $T > 0$, we have that

$$\int_{1/2-A-\mathrm{i}T}^{1/2-A+\mathrm{i}T} \Gamma(s)x^s \,\mathrm{d}s = 2\pi\mathrm{i}\frac{(-x)^{-A}}{A!} + \int_{1/2-A-\mathrm{i}T}^{-1/2-A+\mathrm{i}T} \Gamma(s)x^s \,\mathrm{d}s + O\left(\int_{1/2-A-\mathrm{i}T}^{-1/2-A-\mathrm{i}T} \Gamma(s)x^s \,\mathrm{d}s\right).$$

By the rapid decay of Γ, the horizontal integral vanishes as $T \to \infty$, and we can bound the vertical integral using what we showed before, applied to $A+1$. This yields

$$\int_{(-A+1/2)} \Gamma(s)x^s \,\mathrm{d}s = 2\pi\mathrm{i}\frac{(-x)^{-A}}{A!} + O\left(\frac{x^{-A-1/2}}{A!}\right).$$

In fact, as for every $x > 0$ the fraction $x^A/A!$ tends to zero as $A \to \infty$, we may repeat this as often as we want, obtaining

$$\frac{1}{2\pi\mathrm{i}} \int_{(-A+1/2)} \Gamma(s)x^s \,\mathrm{d}s = \sum_{k=A}^{\infty} \frac{(-x)^{-k}}{k!} = \mathrm{e}^{-\frac{1}{x}} - \sum_{k=0}^{A-1} \frac{(-x)^{-k}}{k!}.$$

The equation for $A = 0$ is nothing new! As $\Gamma(s)$ is holomorphic for $\Re s > 0$ we already know that

$$\mathrm{e}^{-1/x} = \frac{1}{2\pi\mathrm{i}} \int_{(1/2)} \mathcal{M}(\mathrm{e}^{1/x})(s)\frac{x^s}{s} \,\mathrm{d}s = \frac{1}{2\pi\mathrm{i}} \int_{(1/2)} \frac{\Gamma(s+1)}{s} x^s \,\mathrm{d}s.$$

Problem 3