Solutions to Sheet 8.

Problem 1

a-4p) Look at sheet 5.

b-6p) The approximation in (4.9) reads

$$\sum_{n\leq x} d_k(n) = xP_k(\log x) + O(x^{1-\delta}).$$

Reading the proof reveals that the main term is given by the residue

$$R := \mathrm{Res}_{s=1}\frac{\zeta^k(s)x^s}{s} = \mathrm{Res}_{s=1}F_k(s),$$

where for convenience $F_k(s) := \frac{\zeta^k(s)x^s}{s}$. Of course $\zeta^k(s)$ has a singularity of degree k at 1, and we only need to calculate the (-1)st term of the Laurent expansion of F at 1. We have the Taylor expansions

$$\frac{1}{s} = \sum_{n=0}^{\infty}(-1)^n(s-1)^n = 1-(s-1)+(s-1)^2+O((s-1)^3)$$

and

$$x^s = \sum_{n=0}^{\infty}\frac{x(\log x)^n}{n!}(s-1)^n = x + x(\log x)(s-1) + \frac{1}{2}x(\log x)^2(s-1)^2 + O((s-1)^3).$$

Let a_n denote the coefficients of the Laurent series of ζ at 1, i.e.

$$\zeta(s) = \sum_{n=-1}^{\infty} a_n(s-1)^n.$$

Calculating P_2 and P_3 now is pure calculation.

Calculating P_2. We find

$$\zeta^2(s) = \left(\sum_{n=-1}^{\infty} a_n s^n\right)^2 = a_{-1}^2(s-1)^{-2} + 2a_{-1}a_0(s-1)^{-1} + O(1).$$

We multiply this with the Taylor series above and find that the coefficient of $(s-1)^{-1}$ is given by

$$2a_{-1}a_0x + a_{-1}^2(x\log x - x) = x(a_{-1}^2\log x + 2a_{-1}a_0 - a_{-1}^2).$$

Remark. It is possible to show by elementary means that

$$\sum_{n\leq x}^{\infty} d_2(n) = x\log x + (2\gamma-1)x + O(x^{1/2}),$$

which shows that $a_0 = \gamma$. (I just realized that $a_{-1} = 1$ is already known haha, but we could probably also derive this with a similar approach and $k = 1$). This shows

$$P_2(X) = X + 2a_0 - 1.$$

Calculating P_3. We find similarly to above

$$\zeta^3(s) = (s-1)^{-3} + 3a_0(s-1)^{-2} + 3(a_0^2 + a_1)(s-1)^{-1} + O(1).$$

and again use this to figure out the coeffitient of $(s-1)^{-1}$ in the Laurent expansion of F_3 around $s = 1$. We find that this coefficient is given by

$$\begin{aligned}
&3(a_0^2 + a_1)x + 3a_0 x(\log x - 1) + x((\log x)^2 - \log x + 1) \\
&\qquad\qquad\qquad = x(\frac{1}{2}(\log x)^2 + (\log x)(3\gamma - 1) + 3(\gamma^2 + a_1 - \gamma) + 1),
\end{aligned}$$

i.e.

$$P_3(X) = \frac{1}{2}x^2 + (3\gamma - 1)x + 3(a_1 + a_0^2 - a_0) + 1.$$

Problem 2

a) To calculate $\zeta'(0)$, we make use of the functional equation, in the form

$$\zeta(1-s) = \zeta(s) \cdot \frac{2\Gamma(s)}{(2\pi)^s} \sin((\pi(1-s)/2).$$

(This can be derived from the usual functional equation using the reflection formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$. For s close to 1 we find

$$(s-1)\zeta(s)\Gamma(s) = 1 + O((s-1)^2),$$

and

$$\sin(\pi(1-s)/2) = -\tfrac{\pi}{2}(s-1) + O((s-1)^3).$$

Hence

$$\zeta(1-s) = -\frac{\pi}{(2\pi)^s} + O((s-1)^2) = -\frac{1}{2} - \frac{1}{2}\log(2\pi)(s-1) + O((s-1)^2).$$

This gives $\zeta'(0) = -\frac{\log 2\pi}{2}$. We now insert this into (5.2). For $L(s) = \zeta(s)$, this reads as

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{\log \pi}{2} + \frac{1}{2}\frac{\Gamma'(s/2)}{\Gamma(s/2)} - b + \frac{1}{s} + \frac{1}{s-1} - \sum_{\rho \neq 0,1} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

Taking the logarithmic derivative of the recurrece relation $\Gamma(s+1) = s\Gamma(s)$ reveals

$$\frac{\Gamma'(s+1)}{\Gamma(s+1)} = \frac{1}{s} + \frac{\Gamma'(s)}{\Gamma(s)}.$$

We can use this to simplify our equation, leaving us with

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{\log \pi}{2} + \frac{1}{2}\frac{\Gamma'(s/2+1)}{\Gamma(s/2+1)} - b + \frac{1}{s-1} - \sum_{\rho \neq 0,1} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

When inserting $s = 0$, this sum vanishes, and we find

$$-\frac{\zeta'(0)}{\zeta(0)} = -\frac{\log \pi}{2} + \frac{1}{2}\frac{\Gamma'(1)}{\Gamma(1)} - b - 1.$$

This proves the claim, as we know $\zeta(0) = -\frac{1}{2}$, $\Gamma(1) = 1$ and the values for ζ' and Γ' from above.

Remark. The formula $\Gamma'(1) = -\gamma$ can be derived from the Weierstraß product

$$\frac{1}{\Gamma(z)} = z\mathrm{e}^{\gamma z} \prod_{j=1}^{\infty} \left(1 + \frac{z}{j}\right) \mathrm{e}^{-z/j}$$

by taking logarithmic derivatives on both sides and inserting $z = 1$.

b) This statement is false, but of course, we are supposed to show $|\mathrm{Im}(\rho)| \geq 6$ for all the *non-trivial* roots of the zeta function. By (5.3a), we have that

$$-b = -\operatorname{Re} b = \sum_{\rho} \operatorname{Re}(\tfrac{1}{\rho}).$$

The idea is that if $\operatorname{Im} \rho_1$ was small, then this sum would be so large that this equality cannot hold (remember that $b = -0.023$ is quite small). Let $\rho = \sigma + it$ be a root with smallest possible imaginary value. Note that with ρ, we also have roots $1 - \sigma \pm it$ and $\sigma - it$, so we may assume that $\sigma \geq \frac{1}{2}$ and that $t > 0$. As all contributions in the sum of (5.3a) are positive, we find

$$-b \geq \frac{1}{\sigma + \mathrm{i}t} + \frac{1}{\sigma - \mathrm{i}t} = \frac{2\sigma}{\sigma^2 + t^2} \geq \frac{1}{1 + q^2}.$$

This shows $t \geq \sqrt{-b^{-1} - 1} \approx 6.5036$.

Problem 3

a-5p) The "only" thing left to made precise is the contour shift. The main ingrediants are (5.3b) and (5.3c).

Just for convenience, let's summarize the bounds we need for $\frac{\zeta'}{\zeta}$:

$$\frac{\zeta'}{\zeta}(s) = \begin{cases} O(1) & \operatorname{Re} s \geq 2, \\ O(\log|s|) & \operatorname{Re} s \leq -\frac{1}{2},\ |s + 2m| \geq \frac{1}{4} \text{ for all } m \in \mathbb{N}, \\ \sum_{|\rho - s| \leq 1} \frac{1}{s - \rho} + O(1 + \log|s|), & -1 \leq \operatorname{Re} s \leq 3. \end{cases}$$

The first bound follows from $\frac{\zeta'}{\zeta}(s) = \sum_n \Lambda(n) n^{-s}$, the second part follows from the first bound, the functional equation and Stirling's formula. The third part is (5.3c).

By (5.3b), there are approximately $\log T$ roots of the zeta-function with imaginary part close to T (i.e., $|T - \operatorname{Im} \rho| \leq 1$). Hence given some $n \in \mathbb{Z}$, the pigeonhole principle assures that it is possible to find some $T = T_n$ with $|n - T| \leq 1$ and $\min_\rho |T - \operatorname{Im} \rho| \leq \frac{1}{\log|n|}$. Together with (5.3$c$), this gives that $\frac{\zeta'}{\zeta}(\sigma + \mathrm{i}T) \ll (\log|T|)^2$ on that line.

The plan is now to choose some large n, and shifting the truncated integral

$$-\frac{1}{2\pi\mathrm{i}} \int_{2-\mathrm{i}n}^{2+\mathrm{i}n} \frac{\zeta'}{\zeta}(s) \widehat{\omega}(s)\, \mathrm{d}s \tag{1}$$

to $\operatorname{Re} s = -1/2$. This leaves us with the exercise to bound the horizontal integrals along the segments $[2 \pm \mathrm{i}n, -1/2 \pm \mathrm{i}n]$. By the rapid decay of $\widehat{\omega}$ and the bounds for ζ'/ζ, changing the boundaries of the integral in (1) from $[2 - \mathrm{i}n, 2 + \mathrm{i}n]$ to $[2 - \mathrm{i}T_n, 2 + \mathrm{i}T_n]$ comes only with a small cost of $o(1)$. So we may also assume that uniformly $\zeta(\sigma + \mathrm{i}T_n) \ll (\log n)^2$ along the horizontal segments. This justifies the first contour shift, and we obtain

$$\sum_n \Lambda(n)\omega(n) = \frac{1}{2\pi\mathrm{i}} \int_{2-\mathrm{i}n}^{2+\mathrm{i}n} \frac{\zeta'(s)}{\zeta(s)} \widehat{\omega}(s)\,\mathrm{d}s + o(1)$$

$$= \sum_{|\mathrm{Im}\,\rho| \le T_n} \widehat{\omega}(\rho) + \frac{1}{2\pi\mathrm{i}} \int_{-1/2-\mathrm{i}T_n}^{-1/2+\mathrm{i}T_n} \frac{\zeta'(s)}{\zeta(s)} \widehat{\omega}(s)\,\mathrm{d}s + o(1).$$

We may let $n \to \infty$, obtaining

$$\sum_n \Lambda(n)\omega(n) - \sum_\rho \widehat{\omega}(\rho) = \int_{(-1/2)} \frac{\zeta'(s)}{\zeta(s)} \widehat{\omega}(s)\,\mathrm{d}s.$$

In the proof of (4.4c) we can abuse the fact that $\mathrm{Supp}(\omega) \subset [2, \infty)$ to show that $\widehat{\omega}(s) \ll \frac{2^{-s}}{|\mathrm{Im}(s)|^N}$ for all N. This shows the bound

$$\int_{(-A+1/2)} \frac{\zeta'(s)}{\zeta(s)} \widehat{\omega}(s)\,\mathrm{d}s \ll 2^{-A},$$

and the shift is justified.

b-5p) The observation is that this function has large negative peaks at the primes (and smaller ones with sign $(-1)^k$ when $n = p^k$ is a prime power). Although the solution will (implicitely) assume the Riemann conjecture, this illustrates the fact that ζ *knows everything about the primes.*

Okay, let's analyze what's happening here. First, we have

$$\cos(\gamma_j \log x) = \mathrm{Re}(\mathrm{e}^{\mathrm{i}\gamma_j \log x}) = \mathrm{Re}(x^{\mathrm{i}\gamma_j}),$$

so we are plotting the real part of the sum of $x^{\mathrm{i}\gamma}$ over the first few zeroes. If we want to interpret this as a sum $\sum_\rho \widehat{\omega}(\rho)$, we would like to choose ω in way such that $\widehat{\omega}(1/2 + \mathrm{i}\gamma) \approx x^{\mathrm{i}\gamma}$ for the zeroes we want to consider, and $\widehat{\omega}$ decaying rapidly after that range (ignoring the contribution of the trivial zeroes).

Apparently (not clear to me how to come up with this but hey it works) a convenient seems to be

$$\widehat{\omega}(1/2 + \mathrm{i}\gamma) = x^{\mathrm{i}\gamma} \exp(-(\gamma/S)^2) = x^{s-\frac{1}{2}} \exp\left(\left(\frac{s - 1/2}{S}\right)^2\right),$$

as this vanishes quickly once $\gamma > S$. We choose S to be a parameter roughly of the size of the largest zero we want to consider, which in our case is $\gamma_{30} \approx 100$. That's why we choose $S = 100$. We will later show that the weight

$$\omega(y) = \omega_{S,x}(y) = \frac{S}{2(\pi y)^{1/2}} \exp\left(-\left(\frac{S}{2} \log\left(\frac{y}{x}\right)\right)^2\right)$$

is the inverse Mellin-transform of $\widehat{\omega}$. Now this is large if the part in the exponential vanishes, i.e., if $x \approx y$. On the other side, if y is not close to x then $\log(y/x)$ becomes large (say of size $\approx \frac{10}{S}$), then the factor $\exp(-S^2/4 \log(y/x)^2)$ makes ω decay quickly. So at least for x not too large (for large x we need a further distance between y and x to make $\log(y/x)$ become large), ω essentially looks like a peak at $y = x$. We are now ready to explain what's going on. With the explicit formula (which we are technically not even allowed to use as ω is not compactly supported, but whatever), we find

$$\sum_n \Lambda(n)\omega_{S,x}(n) \approx -\sum_{|\operatorname{Im}\rho|\le S} \widehat{\omega}(\rho) \approx \sum_{j=1}^{S} \cos(\gamma_j \log x).$$

If now $x \approx p^k$ is close to a prime power, the LHS is $\approx \Lambda(n)\frac{S}{y^{1/2}}$, large. If not, there is no term on the LHS that contributes much, so we would expect the RHS to be small.

Prove that "$\widehat{\omega} = \widehat{\omega}$". We put $\widehat{\omega}$ in the inverse mellin transform to find

$$\omega(y) = \frac{1}{2\pi\mathrm{i}} \int_{(c)} x^{s-1/2} \exp\left(\left(\frac{s-1/2}{S}\right)^2\right) y^{-s}\,\mathrm{d}s$$

for all real numbers c. We substitute $u = \frac{s-1/2}{S}$ and find

$$\omega(y) = \frac{S}{y^{1/2}} \cdot \frac{1}{2\pi\mathrm{i}} \int_{(c)} \exp(u^2)\left(\frac{y}{x}\right)^{-Su} \mathrm{d}u.$$

Abbreviating $v = S\log\frac{y}{x}$ shows further that

$$\frac{S}{y^{1/2}}\frac{1}{2\pi\mathrm{i}} \int_{(c)} \exp(u^2)\exp(-uv)\,\mathrm{d}u = \frac{S\exp(-v^2/4)}{y^{1/2}} \cdot \frac{1}{2\pi\mathrm{i}} \int_{(c)} \exp((u-v/2)^2)\,\mathrm{d}u.$$

This integral does not depend on c, hence we may wlog assume $c = v/2$, which reveals that this integral equals

$$\frac{1}{2\pi\mathrm{i}} \int_{(0)} \exp(u^2)\,\mathrm{d}u = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-t^2)\,\mathrm{d}t = \frac{1}{2\sqrt{\pi}},$$

just what we wanted.

Problem 4

First, an aside on the weird-looking error term $\psi(x) - x \ll x\mathrm{e}^{-c\sqrt{\log x}}$. On the one side it is better than every error term of the form $x/(\log x)^A$ (for $A \in \mathbb{R}_{>0}$ large), on the other side it is worse than every error term of the form $x^{1-\delta}$ would be (for $\delta \in \mathbb{R}_{>0}$ small).

Our version of the prime number theorem reads

$$\psi(x) = \sum_{p^n \le x} \log p = x + O(x\mathrm{e}^{-c\sqrt{\log x}})$$

for some constant $c > 0$. We deduce a formula for π in two steps. First we show that $\psi(x)$ does not differ too much from the weighted prime-counting function

$$\psi_0(x) \coloneqq \sum_{p\le x} \log p.$$

Then we use ψ_0 for partial summation, utilizing that

$$\pi(x) = \sum_{p\le x} \frac{\log p}{\log p} = \frac{\psi_0(x)}{\log x} + \int_2^x \frac{\psi_0(t)}{t(\log t)^2}\,\mathrm{d}t. \tag{2}$$

Evaluating this should be possible using the approximation for $\psi_0(x)$.

Let's carry this through, beginning with the estimate for $|\psi(x) - \psi_0(x)|$. We find

$$\psi(x) - \psi_0(x) = \sum_{p^k \le x,\ k \ge 2} \log p \le \left(\sum_{p \le \sqrt{x}} + \sum_{p \le x^{1/3}} + \cdots + \right) \log x$$

Note that there are at most $\log_2 x$ summation signs which don't run over an empty set, and every index set contains (trivially) less than $\sqrt{x}$ primes. We obtain

$$\psi(x) - \psi_0(x) \le (\log_2 x)\sqrt{x}(\log x) \ll x^{1/2+\varepsilon}.$$

Now ψ_0 satisfies the same approximation as ψ, as

$$\psi_0(x) = \psi(x) + O(x^{1/2+\varepsilon}) = x + O(x\mathrm{e}^{-c\sqrt{\log x}}).$$

Inserting this in (1) yields

$$\pi(x) = \frac{x}{\log x} + \int_2^x \frac{1}{(\log t)^2}\,\mathrm{d}t + O(x\mathrm{e}^{-c\sqrt{\log x}}),$$

where we used that $\int_2^x \frac{1}{t(\log t)^2}\,\mathrm{d}t \ll 1$. As

$$\int_2^x \frac{1}{(\log t)^2}\,\mathrm{d}t = \left[\mathrm{Li}(t) - \frac{t}{\log t}\right]_2^x = \mathrm{Li}(x) - \frac{x}{\log x} + O(1),$$

the claim follows.

Max von Consbruch, email: s6mavonc@uni-bonn.de. Date: December 12, 2022