Solution to Sheet 5.

Problem 1

We have basically solved this already on sheet 3. Note that as $d_1 \mid q_1$ and $d_2 \mid q_2$, we have $(d_1, d_2) = 1$, so (by sheet 3) there are primitive characters ψ_i mod d_i with $\chi_i = \psi_i \chi_{0,q_i}$ (here again χ_{0,q_i} is the principal character mod q_i) whose product $\psi = \psi_1\psi_2$ is a primitive character mod $d_1 d_2$. Modulo q, this reveals

$$\chi_1\chi_2 = (\chi_{0,q_1}\psi_1)(\chi_{0,q_2}\psi_2) = \chi_{0,q_1q_2}\psi,$$

hence ψ is induced by a primitive character mod d_1d_2.

It is easily seen that the coprimality condition is necessary. Take any real character χ mod q for example, then $\chi^2 = 1$ and has conducter $1 \neq q$.

Problem 2

We have to show the bound

$$\int_{(-A+\frac{1}{2})} \Gamma(s)x^s \,\mathrm{d}s \ll \frac{x^{-A+1/2}}{(A-1)!}.$$

Note that the integral exists by the rapid decay of Γ along vertical lines. However, we cannot apply Stirling's formula to bound the integral directly as Stirling a priori only gives uniform bounds in regions of the form $|\arg(s) - \pi| \geq \delta > 0$. We can however apply stirlings formula if we apply the recurrence $s\Gamma(s) = \Gamma(s+1)$ repeatedly:

$$\begin{aligned}\int_{(-A+1/2)} \Gamma(s)x^s \,\mathrm{d}s &\ll \int_{(-A+1/2)} |\Gamma(s)x^s| \,\mathrm{d}s \ll x^{-A+1/2}\int_{(1/2)} |\Gamma(s-A)| \,\mathrm{d}s \\ &= x^{-A+1/2}\int_{(1/2)} \left|\frac{\Gamma(s)}{(s-A+1)\cdots(s-1)}\right| \mathrm{d}s \leq \frac{x^{-A+1/2}}{(A-1)!}\int_{(1/2)} |\Gamma(s)| \,\mathrm{d}s.\end{aligned}$$

Notes. Once we know this inequality, we actually can do better: Remember that Γ has poles at the negative integers, the residue at $-n$ is given by $\frac{(-1)^n}{n!}$. Hence for (large) $T > 0$, we have that

$$\int_{1/2-A-\mathrm{i}T}^{1/2-A+\mathrm{i}T} \Gamma(s)x^s \,\mathrm{d}s = 2\pi\mathrm{i}\frac{(-x)^{-A}}{A!} + \int_{1/2-A-\mathrm{i}T}^{-1/2-A+\mathrm{i}T} \Gamma(s)x^s \,\mathrm{d}s + O\left(\int_{1/2-A-\mathrm{i}T}^{-1/2-A-\mathrm{i}T} \Gamma(s)x^s \,\mathrm{d}s\right).$$

By the rapid decay of Γ, the horizontal integral vanishes as $T \to \infty$, and we can bound the vertical integral using what we showed before, applied to $A+1$. This yields

$$\int_{(-A+1/2)} \Gamma(s)x^s \,\mathrm{d}s = 2\pi\mathrm{i}\frac{(-x)^{-A}}{A!} + O\left(\frac{x^{-A-1/2}}{A!}\right).$$

In fact, as for every $x > 0$ the fraction $x^A/A!$ tends to zero as $A \to \infty$, we may repeat this as often as we want, obtaining

$$\frac{1}{2\pi\mathrm{i}}\int_{(-A+1/2)} \Gamma(s)x^s \,\mathrm{d}s = \sum_{k=A}^{\infty} \frac{(-x)^{-k}}{k!} = \mathrm{e}^{-\frac{1}{x}} - \sum_{k=0}^{A-1} \frac{(-x)^{-k}}{k!}.$$

The equation for $A = 0$ is nothing new! As $\Gamma(s)$ is holomorphic for $\Re s > 0$ we already know that

$$\mathrm{e}^{-1/x} = \frac{1}{2\pi\mathrm{i}} \int_{(1/2)} \mathcal{M}(\mathrm{e}^{1/x})(s) \frac{x^s}{s} \,\mathrm{d}s = \frac{1}{2\pi\mathrm{i}} \int_{(1/2)} \frac{\Gamma(s+1)}{s} x^s \,\mathrm{d}s.$$

Problem 3

We substitute $p^{-s} = x$ to find the equivalent

$$\sum_{k=0}^{\infty} \beta(k) x^k = \frac{1}{P(x)}.$$

Where $P(x) = \prod_{j=1}^{d}(1 - \alpha_j x) = \sum_{i=0}^{d} a_i x^i$ (in particular, $a_0 = 1$). Multiply both sides with P, revealing

$$\sum_{d=0}^{\infty} x^d \sum_{k=0}^{d} \beta(d-k) a_k = 1.$$

Equating coeffiecients gives that for $k > 0$,

$$\sum_{k=0}^{d} a_k \beta(d-k) = 0,$$

which, after subtracting $\beta(d)$ on both sides and setting $c_i = -a_{i+1}$, gives the desired.