Solutions to Sheet 10.

Reminder: $\mathrm{Li}(n) \coloneqq \int_2^n \frac{1}{\log t}\,\mathrm{d}t$.

Problem 1

One might think that this problem is incredibly complicated, but in reality it is terribly simple. Let $P = P(q,a)$ be the smallest prime congruent to a mod q. The idea is now to plug this into to use the that

$$\psi_0(P(q,a)-1,q,a) = \sum_{p \equiv a \pmod q} \log p = 0,$$

Siegel-Walfisz says that $\psi(x,q,a) \approx \psi_0(x,q,a) \approx \frac{x}{\varphi(q)}$ if x is large, so we'd expect to find that $P(q,a)$ cannot be too large if the equality above holds.

Okay. If

Problem 2

Problem 3

Before solving this, we should maybe try to figure out why we would expect this result. Given some number n, we are supposed to evaluate the counting function

$$R(n) = \#\{p \leq n \mid n-p \text{ is square free}\}.$$

Naively, one might be think that

$$R(n) \approx \zeta(2)^{-1}\pi(n) = \prod_p (1-p^{-2})\pi(n),$$

as the propability of a random number to be square-free is (in a suitable sense) given by $\zeta(2)^{-1}$, and we inspect numbers (which seem random) in a set of cardinality $\pi(n)$. This heuristic is not too far off, but it is wrong! The main term of the asymptotic is clearly different.
To see what goes wrong, let q be any prime number. First assume that $q \nmid n$. What is the probability that q^2 divides $n-p$ for some prime $p \neq q$? Neither n nor p are divisible by q, so the residue classes of these numbers mod q^2 are invertible, and there are $\varphi(q^2)$ such residue classes. So the probability is given by $\varphi(q^2)^{-1}$. Now assume $q \mid n$. One quickly checks that q^2 cannot divide $n-p$ (unless $p=q$, but this case does not contribute much). Now we can explain the asymptotic: There are $\approx \mathrm{Li}(n)$ primes $\leq n$, and the probability for $n-p$ not being divisible by some prime q is given by $(1-\varphi(q^2)^{-1})$ if $q \nmid n$ and by 1 if $q \mid n$. As $n-p$ is square-free iff no square of a prime divides it, we should expect

$$R(n) \approx \prod_{q \nmid n} (1-\varphi(q^2)^{-1})\mathrm{Li}(n) = \prod_{q \nmid n} \left(1 - \frac{1}{q(q-1)}\right)^{-1} \mathrm{Li}(n),$$

and this is what we have to prove.

Proof. Clearly, we have $R(n) = \sum_{p \le n} \mu^2(n-p)$. A standard trick to deal with μ^2 is writing it as $\mu(k) = \sum_{d^2 | k} \mu(d)$. Applying this gives

$$R(n) = \sum_{p \le n} \mu^2(n-p) = \sum_{p \le n} \sum_{d^2 | n-p} \mu(d) = \sum_{d \le \sqrt{n}} \mu(d) \sum_{p \le n,\ p \equiv n \bmod d^2} 1.$$

This is now basically an issue of counting primes in an arithmetic progression! Hence it really smells like Siegel-Walfisz, but this is not applicable right away. One issue is that we can only apply Siegel-Walfisz if $(d,n) = 1$. But restricting to those d does not really affect our main term, as whenever $(d,n) > 1$ there is at most one prime number in that arithmetic progression, and the contribution of those is bounded by $\omega(n) \ll n^{\varepsilon}$. Furthermore, and more seriously, Siegel-Walfisz is only applicable if d is small compared to n, more precisely, only if $d < (\log n)^A$. But again, we can elementarily bound the terms with $d > (\log n)^A$. Given some d, the amount of numbers $< n$ congruent to n mod d^2 can be bounded by $\ll \frac{n}{d^2}$. We obtain

$$R(n) = \sum_{d \le (\log n)^A,\ (d,n)=1} \psi(n; n, d^2) + O\left(\sum_{(\log n)^A < d < \sqrt{n}} \frac{n}{d^2} \right) + O(\sqrt{n}),$$

and the O-terms can be bound by $\ll \frac{n}{(\log n)^A}$. Also, we can now apply Siegel-Walfisz! We find

$$R(n) = \sum_{d \le (\log n)^A,\ (d,n)=1} \frac{1}{\varphi(d^2)} \mathrm{Li}(n) + O\left(\frac{n}{(\log n)^A} \right).$$

The sum can be completed, as $\varphi(d^2) \gg \frac{d^2}{\log \log d} \gg d^{2-\varepsilon}$, so that

$$\sum_{d > (\log n)^A} \frac{1}{\varphi(d^2)} \ll \frac{1}{(\log n)^{A(1-\varepsilon)}}.$$

This allows us to conclude (for any A, not the choice we made before)

$$R(n) = \sum_{d \in \mathbb{N},\ (d,n)=1} \frac{1}{\varphi(d^2)} \mathrm{Li}(n) + O_A\left(\frac{n}{(\log n)^A} \right) = \prod_{p \nmid n} \left(1 - \frac{1}{\varphi(p^2)} \right) \mathrm{Li}(n) + O_A\left(\frac{n}{(\log n)^A} \right).$$

Problem 4

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