Solution to Sheet 5.

Problem 1

Problem 2

We have to show the bound

$$\int_{(-A+\frac{1}{2})} \Gamma(s)x^s \,\mathrm{d}s \ll \frac{x^{-A+1/2}}{(A-1)!}.$$

Note that the integral exists by the rapid decay of Γ along vertical lines. However, we cannot apply Stirling's formula to bound the integral directly as Stirling a priori only gives uniform bounds in regions of the form $|\arg(s) - \pi| \geq \delta > 0$. We can however apply stirlings formula if we apply the recurrence $s\Gamma(s) = \Gamma(s+1)$ repeatedly:

$$\begin{aligned}\int_{(-A+1/2)} |\Gamma(s)x^s| \,\mathrm{d}s &\ll x^{-A+1/2} \int_{(1/2)} |\Gamma(s-A)| \,\mathrm{d}s \\ &= x^{-A+1/2} \int_{(1/2)} \left| \frac{\Gamma(s)}{(s-A+1)\cdots(s-1)} \right| \mathrm{d}s \leq \frac{x^{-A+1/2}}{(A-1)!} \int_{(1/2)} |\Gamma(s)| \,\mathrm{d}s.\end{aligned}$$

Problem 3