Solution to Sheet 3.

Problem 1

a) Let $g(x) = f(qx + a)$, so that

$$\sum_{n \equiv a \pmod q} f(n) = \sum_{m \in \mathbb{Z}} g(m).$$

We want to apply Poisson summation to g. The results of lemma (2.3) directly give that

$$\hat{g}(y) = \frac{1}{q} e\left(\frac{ya}{q}\right) \hat{f}\left(\frac{y}{q}\right).$$

The claim follows, as

$$\sum_{m \in \mathbb{Z}} g(m) = \sum_{m \in \mathbb{Z}} \hat{g}(m) = \frac{1}{q} \sum_{m \in \mathbb{Z}} e\left(\frac{ma}{q}\right) \hat{f}\left(\frac{m}{q}\right).$$

b) We would like to apply Poisson summation again, however we cannot calculate the "Fourier transform" of $f\chi$, as, χ is only defined on integers. We can abuse that χ is periodic though, rewriting

$$\sum_{m \in \mathbb{Z}} f(m)\chi(m) = \sum_{a \pmod q} \chi(a) \sum_{m \equiv a \pmod q} f(m).$$

Applying Poisson summation to the inner sum (we already did this in part a)) gives

$$\sum_{m \in \mathbb{Z}} f(m)\chi(m) = \frac{1}{q} \sum_{a \pmod q} \chi(a) \sum_{m \in \mathbb{Z}} e\left(\frac{ma}{q}\right) \hat{f}\left(\frac{m}{q}\right).$$

Reordering sums, we obtain

$$\begin{aligned}
\frac{1}{q} \sum_{a \pmod q} \chi(a) \sum_{m \in \mathbb{Z}} e\left(\frac{ma}{q}\right) \hat{f}\left(\frac{m}{q}\right) &= \frac{1}{q} \sum_{m \in \mathbb{Z}} \hat{f}\left(\frac{m}{q}\right) \left(\sum_{a \pmod q} \chi(a) e\left(\frac{ma}{q}\right)\right) \\
&= \frac{1}{q} \sum_{m \in \mathbb{Z}} \hat{f}\left(\frac{m}{q}\right) \tau(\chi)\overline{\chi}(m) = \frac{\tau(\chi)}{q} \sum_{m \in \mathbb{Z}} \hat{f}\left(\frac{m}{q}\right) \overline{\chi}(m).
\end{aligned}$$

Problem 2

I really don't like this problem as it has not much to do with number theory. I might add a solution later, but I am sure one can find this in most books on real analysis.

Problem 3

As the hint commands, we apply partial summation to the definition of $\tau(\chi)$, obtaining

$$|\tau(\chi)| = \sum_{h=1}^{q} \chi(h) e(h/q) = e(q/q) \sum_{h=1}^{q} \chi(h) - \frac{2\pi \mathrm{i}}{q} \int_1^q e(t/q) \sum_{h \le t} \chi(h)\,\mathrm{d}t.$$

As $\chi \neq \chi_0$, the sum $\sum_{h=1}^{q} \chi(h)$ vanishes. We also know by theorem (1.23) that $|\tau(\chi)| = \sqrt{q}$. Let M deonte the supremum of the absolute values of $\sum_{h \leq x} \chi(h)$ for varying x (By Polya-Vinogradov, $M < \infty$). Then we obtain

$$\frac{q^{3/2}}{2\pi} = \left| \int_1^q e(t/q) \sum_{h \leq t} \chi(h) \,\mathrm{d}t \right| \leq \int_1^q \left| \sum_{h \leq t} \chi(h) \right| \,\mathrm{d}t \leq (q-1)M,$$

which is even a tad stronger than what we had to show.

Notes after correcting.

Problem 4

Let's just plug in the definition and look at what we have here.

$$\tau(\chi_1\chi_2) = \sum_{h\ (q)} \chi_1(h)\chi_2(h)e(h/q),$$

where $q = q_1 q_2$. By the chinese remainder theorem, taking residues mod q gives a bijection

$$\{h_1 q_2 + h_2 q_1 \mid 1 \leq h_i \leq q_i\} \to \mathbb{Z}/q\mathbb{Z}.$$

Thus we may rewrite the sum above as

$$\tau(\chi_1\chi_2) = \sum_{1 \leq h_1 \leq q_1} \sum_{1 \leq h_2 \leq q_2} \chi_1(h_1 q_2 + h_2 q_1)\chi_2(h_1 q_2 + h_2 q_1) e(\tfrac{h_1 q_2 + h_2 q_1}{q}),$$

and the claim follows after a few manipulations:

$$\begin{aligned}
&\sum_{1 \leq h_1 \leq q_1} \sum_{1 \leq h_2 \leq q_2} \chi_1(h_1 q_2 + h_2 q_1)\chi_2(h_1 q_2 + h_2 q_1) e(\tfrac{h_1 q_2 + h_2 q_1}{q}) \\
&\qquad = \sum_{1 \leq h_1 \leq q_1} \sum_{1 \leq h_2 \leq q_2} \chi_1(h_1 q_2)\chi_2(h_2 q_1) e(\tfrac{h_1 q_2}{q}) e(\tfrac{h_2 q_1}{q}) \\
&= \left(\chi_1(q_2) \sum_{1 \leq h_1 \leq q_1} \chi_1(q_2) e(\tfrac{h_1}{q_1}) \right) \left(\chi_2(q_1) \sum_{1 \leq h_2 \leq q_2} \chi_2(q_1) e(\tfrac{h_2}{q_2}) \right) = \chi_1(q_2)\tau(\chi_1)\chi_2(q_1)\tau(\chi_2).
\end{aligned}$$