

# Solution to Sheet 4.

## Problem 1

a) Let  $g(x) = f(qx + a)$ , so that

$$\sum_{n \equiv a \pmod{q}} f(n) = \sum_{m \in \mathbb{Z}} g(m).$$

We want to apply Poisson summation to  $g$ . The results of lemma (2.3) directly give that

$$\hat{g}(y) = \frac{1}{q} e\left(\frac{ya}{q}\right) \hat{f}\left(\frac{y}{q}\right).$$

The claim follows, as

$$\sum_{m \in \mathbb{Z}} g(m) = \sum_{m \in \mathbb{Z}} \hat{g}(m) = \frac{1}{q} \sum_{m \in \mathbb{Z}} e\left(\frac{ma}{q}\right) \hat{f}\left(\frac{m}{q}\right).$$

b) We would like to apply Poisson summation again, however we cannot calculate the "Fourier transform" of  $f\chi$ , as,  $\chi$  is only defined on integers. We can abuse that  $\chi$  is periodic though, rewriting

$$\sum_{m \in \mathbb{Z}} f(m)\chi(m) = \sum_{a \pmod{q}} \chi(a) \sum_{m \equiv a \pmod{q}} f(m).$$

Applying Poisson summation to the inner sum (we already did this in part a)) gives

$$\sum_{m \in \mathbb{Z}} f(m)\chi(m) = \frac{1}{q} \sum_{a \pmod{q}} \chi(a) \sum_{m \in \mathbb{Z}} e\left(\frac{ma}{q}\right) \hat{f}\left(\frac{m}{q}\right).$$

Reordering sums, we obtain

$$\begin{aligned} \frac{1}{q} \sum_{a \pmod{q}} \chi(a) \sum_{m \in \mathbb{Z}} e\left(\frac{ma}{q}\right) \hat{f}\left(\frac{m}{q}\right) &= \frac{1}{q} \sum_{m \in \mathbb{Z}} \hat{f}\left(\frac{m}{q}\right) \left( \sum_{a \pmod{q}} \chi(a) e\left(\frac{ma}{q}\right) \right) \\ &= \frac{1}{q} \sum_{m \in \mathbb{Z}} \hat{f}\left(\frac{m}{q}\right) \tau(\chi) \bar{\chi}(m) = \frac{\tau(\chi)}{q} \sum_{m \in \mathbb{Z}} \hat{f}\left(\frac{m}{q}\right) \bar{\chi}(m). \end{aligned}$$

### Notes after correcting.

- In part a), instead of using the results from the lecture, we can also obtain the formula for the fourier transform directly. Setting  $g(x) = f(qx + a)$  and substituting  $u = qx + a$ , we obtain

$$\hat{g}(y) = \int_{\mathbb{R}} f(qx + a) e(-xy) dx = \frac{1}{q} \int_{\mathbb{R}} f(u) e(-u\frac{y}{q} + \frac{ay}{q}) du = \frac{1}{q} e(\frac{ay}{q}) \hat{f}\left(\frac{y}{q}\right).$$

## Problem 2

We do as the hint commands. Let

$$f(t) = \begin{cases} e^{-1/t^2} & t > 0 \\ 0 & \text{else.} \end{cases}$$

Then one easily checks that  $f$  is smooth and non-negative. Now we put  $g(t) = \frac{f(t)}{f(t)+f(1-t)}$ , which is still smooth and non-negative. We clearly have  $g(t) = 0$  if  $t < 0$ ,  $g(t) \in [0, 1]$  for  $t \in [0, 1]$  and  $g(t) = 1$  for  $t > 1$ . Finally, define

$$h(t) = g\left(\frac{t - X + Z}{Z}\right) - g\left(\frac{t - X - Y}{Z}\right).$$

This satisfies  $\text{supp}(h) \subset [X - Z, X + Y + Z]$  and  $h(t) = 1$  for  $t \in [X, X + Y]$ . We still need to check that  $\|f^{(j)}\|_1 \ll Z^{1-j}$  for all  $j \in \mathbb{N}$ . One could expect this to be really messy as calculating the higher derivatives of  $h$  seems horrible. However, we just need that the  $j$ -th derivative of  $h$  is given by

$$h^{(j)}(t) = Z^{-j} \left( g^{(j)}\left(\frac{t-X+Z}{Z}\right) - g^{(j)}\left(\frac{t-X-Y}{Z}\right) \right).$$

As  $h^{(j)}$  vanishes everywhere except  $[X - Z, X]$  and  $[X + Y, X + Y + Z]$ , we obtain by a linear change of variables

$$\|h^{(j)}\|_1 = \left( \int_{X+Z}^X + \int_{X+Y}^{X+Y+Z} \right) |h^{(j)}(t)| dt = 2Z^{1-j} \int_0^1 |g^{(j)}(t)| dt \ll_j Z^{1-j}.$$

## Problem 3

As the hint commands, we apply partial summation to the definition of  $\tau(\chi)$ , obtaining

$$|\tau(\chi)| = \sum_{h=1}^q \chi(h) e(h/q) = e(q/q) \sum_{h=1}^q \chi(h) - \frac{2\pi i}{q} \int_1^q e(t/q) \sum_{h \leq t} \chi(h) dt.$$

As  $\chi \neq \chi_0$ , the sum  $\sum_{h=1}^q \chi(h)$  vanishes. We also know by theorem (1.23) that  $|\tau(\chi)| = \sqrt{q}$ . Let  $M$  denote the supremum of the absolute values of  $\sum_{h \leq x} \chi(h)$  for varying  $x$  (By Polya-Vinogradov,  $M < \infty$ ). Then we obtain

$$\frac{q^{3/2}}{2\pi} = \left| \int_1^q e(t/q) \sum_{h \leq t} \chi(h) dt \right| \leq \int_1^q \left| \sum_{h \leq t} \chi(h) \right| dt \leq (q-1)M,$$

which is even a tad stronger than what we had to show.

## Problem 4

Let's just plug in the definition and look at what we have here.

$$\tau(\chi_1 \chi_2) = \sum_{h \pmod{q}} \chi_1(h) \chi_2(h) e(h/q),$$

where  $q = q_1 q_2$ . By the chinese remainder theorem, taking residues mod  $q$  gives a bijection

$$\{h_1 q_2 + h_2 q_1 \mid 1 \leq h_i \leq q_i\} \rightarrow \mathbb{Z}/q\mathbb{Z}.$$

Thus we may rewrite the sum above as

$$\tau(\chi_1 \chi_2) = \sum_{1 \leq h_1 \leq q_1} \sum_{1 \leq h_2 \leq q_2} \chi_1(h_1 q_2 + h_2 q_1) \chi_2(h_1 q_2 + h_2 q_1) e\left(\frac{h_1 q_2 + h_2 q_1}{q}\right),$$

and the claim follows after a few manipulations:

$$\begin{aligned} & \sum_{1 \leq h_1 \leq q_1} \sum_{1 \leq h_2 \leq q_2} \chi_1(h_1 q_2 + h_2 q_1) \chi_2(h_1 q_2 + h_2 q_1) e\left(\frac{h_1 q_2 + h_2 q_1}{q}\right) \\ &= \sum_{1 \leq h_1 \leq q_1} \sum_{1 \leq h_2 \leq q_2} \chi_1(h_1 q_2) \chi_2(h_2 q_1) e\left(\frac{h_1 q_2}{q}\right) e\left(\frac{h_2 q_1}{q}\right) \\ &= \left( \chi_1(q_2) \sum_{1 \leq h_1 \leq q_1} \chi_1(q_2) e\left(\frac{h_1}{q_1}\right) \right) \left( \chi_2(q_1) \sum_{1 \leq h_2 \leq q_2} \chi_2(q_1) e\left(\frac{h_2}{q_2}\right) \right) = \chi_1(q_2) \tau(\chi_1) \chi_2(q_1) \tau(\chi_2). \end{aligned}$$