Solutions to Sheet 8.

Problem 1

a-4p) I already gave a solution to this on sheet 5.

Problem 2

Problem 3

Problem 4

First, an aside on the weird-looking error term $\psi(x) - x \ll x\mathrm{e}^{-c\sqrt{\log x}}$. On the one side it is better than every error term of the form $x/(\log x)^A$ (for $A \in \mathbb{R}_{>0}$ large), on the other side it is worse than every error term of the form $x^{1-\delta}$ would be (for $\delta \in \mathbb{R}_{>0}$ small).

Our version of the prime number theorem reads

$$\psi(x) = \sum_{p^n \le x} \log p = x + O(x\mathrm{e}^{-c\sqrt{\log x}})$$

for some constant $c > 0$. We deduce a formula for π in two steps. First we show that $\psi(x)$ does not differ too much from the weighted prime-counting function

$$\psi_0(x) := \sum_{p \le x} \log p.$$

Then we use ψ_0 for partial summation, utilizing that

$$\pi(x) = \sum_{p \le x} \frac{\log p}{\log p} = \frac{\psi_0(x)}{\log x} + \int_2^x \frac{\psi_0(t)}{t(\log t)^2}\,\mathrm{d}t. \tag{1}$$

Evaluating this should be possible using the approximation for $\psi_0(x)$.

Let's carry this through, beginning with the estimate for $|\psi(x) - \psi_0(x)|$. We find

$$\psi(x) - \psi_0(x) = \sum_{p^k \le x,\ k \ge 2} \log p \le \left(\sum_{p \le \sqrt{x}} + \sum_{p \le x^{1/3}} + \cdots + \right) \log x$$

Note that there are at most $\log_2 x$ summation signs which don't run over an empty set, and every index set contains (trivially) less than $\sqrt{x}$ primes. We obtain

$$\psi(x) - \psi_0(x) \le (\log_2 x)\sqrt{x}(\log x) \ll x^{1/2+\varepsilon}.$$

Now ψ_0 satisfies the same approximation as ψ, as

$$\psi_0(x) = \psi(x) + O(x^{1/2+\varepsilon}) = x + O(x\mathrm{e}^{-c\sqrt{\log x}}).$$

Inserting this in (1) yields

$$\pi(x) = \frac{x}{\log x} + \int_2^x \frac{1}{(\log t)^2}\,\mathrm{d}t + O(x\mathrm{e}^{-c\sqrt{\log x}}),$$

where we used that $\int_2^x \frac{1}{t(\log t)^2}\,\mathrm{d}t \ll 1$. As

$$\int_2^x \frac{1}{(\log t)^2}\,\mathrm{d}t = \left[\mathrm{Li}(t) - \frac{t}{\log t}\right]_2^x = \mathrm{Li}(x) - \frac{x}{\log x} + O(1),$$

the claim follows.

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