# Noether's bound for exterior algebras

Max von Consbruch

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## 1 Introduction

Let $K$ be a field of characteristic 0, let $G$ be a finite group of order $d$ and let $V$ be a $K$-vector space on which $G$ acts. In previous talks, we have seen a proof of the following theorem.

**Theorem 1** (Noether's degree bound)**.** *In the situation above, the action of $G$ of $K[V] = \mathrm{Sym}(V^*)$ has $K[V]^G$ generated in degree $\leq d$, where $d$ is the order of $G$.*

In Daniel's talk, we have even seen that this statement can be generalized a bit, it holds under the milder assumption that $d < \operatorname{char} K$.

Today, we want to investigate further generalizations of this result. For one, we want to explain an approach by Derksen [Der99] that yields Noether's degree bound under the even milder assumption that $d \in K^\times$. Afterwards, we explain how this approach makes it possible to proof an analogue of Noether's degree bound for the exterior algebra.

Perhaps we should recall the definition of the exterior algebra.

**Definition 2** (tensor algebra, symmetric algebra, exterior algebra)**.** If $V$ is a finite dimensional vector space, we define the tensor algebra $\mathcal{T}(V)$ as

$$\mathcal{T}(V) = K \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) + \cdots = \bigoplus_{n \in \mathbb{N}} V^{\otimes n}.$$

The tensor algebra carries the structure of a non-commutative $K$-algebra with multiplication given by taking tensors.

The *symmetric algebra* $\mathrm{Sym}(V)$ is defined as the universal commutative algebra under $\mathcal{T}(V)$, which is simply the quotient

$$\mathrm{Sym}(V) = \mathcal{T}(V)/(v \otimes w - w \otimes v \mid v, w \in V).$$

Note that $K[V] = \mathrm{Sym}(V^*)$.

The *exterior algebra* $\bigwedge(V)$ is the universal anti-commutative algebra under $\mathcal{T}(V)$, which is

$$\bigwedge(V) = \mathcal{T}(V)/(v \otimes w + w \otimes v \mid v, w \in V).$$

Note that both $\mathrm{Sym}(V)$ and $\bigwedge(V)$ have a natural grading inherited from $\mathcal{T}(V)$. We say that a graded $K$-algebra $R$ is generated in degree $d$ iff it is generated by $R_{\leq d} = R_1 \oplus \cdots \oplus R_d$ as (non-commutative) $K$-algebra. That is, the aim of the talk is to show the following

**Theorem 3** (Noether's degree bound for the exterior algebra)**.** *Let $G$ be a finite group of order $d$ acting on a finite vector space $V$. This induces an action of $G$ on the exterior algebra $R = \bigwedge V$ of $V$. Its subalgebra of $G$-invariant elements $R^G$ is generated in degree at most $d$.*

## 1.1 A few examples

If $V \cong K^n$, there is a (non-canonical) isomorphism

$$\bigwedge(V) \cong \frac{K\langle x_1, \ldots, x_n \rangle}{(x_i x_j + x_j x_i \mid 1 \leq i, j \leq n)}.$$

Here, $K\langle x_1, \ldots, x_n \rangle$ denotes the (non-commutative) algebra generated by the words in the symbols $x_1, \ldots, x_n$. For an integer $k \geq 0$, we write $\bigwedge^k(V)$ for the degree-$k$-piece of $\bigwedge(V)$. Note that $\bigwedge^k = 0$ if $k > \dim(V)$.

**Example 1.** Theorem 3 is trivial if the order of $G$ is greater or equal to the dimension of $V$. Indeed, in this case the pieces of $\bigwedge(V)^G \subset \bigwedge(V)$ of degree $> \#G$ vanish, hence everything is generated in degree $\leq \#G$.

**Example 2.** Consider the group $G = \mathbb{Z}/4\mathbb{Z}$ and let $G$ act on $V = K^4$, with basis $(e_1, \ldots, e_4)$ via shifting these basis vectors. Let $H = \langle 2 \rangle \subset G$. We want to compute the invariants of $\bigwedge(V)$ under $H$. One quickly finds:

- The degree 1 pieces are ($K$-linearly) generated by $\langle x_1 + x_3, x_2 + x_4 \rangle$.
- The degree 2 pieces are ($K$-linearly) generated by $\langle x_1x_2 + x_3x_4, x_1x_4 + x_3x_2 \rangle$.

Now Noether's degree bound steps in and states that these elements generate $\bigwedge(V)$! But note that in this situation, it is feasible to verify this by hand. Only the degree 3 and degree 4 pieces are left, and these can be explicitely calculated as well. For example, the only (up to factor) $H$-invariant piece of degree 4 is given by $x_1x_2x_3x_4$. One quickly verifies that

$$x_1x_2x_3x_4 = \frac{1}{2}(x_1x_2 + x_3x_4)^2,$$

so this lies in some subring generated in degree 2! Interestingly enough, we used that $\operatorname{char} K \nmid \#H = 2$.

**Example 3.** As it turns out, Noether's bound is false for general graded non-commutative algebras with degree-preserving action of a finite group $G$. Now take the algebra $R = K\langle x, y \rangle/(xy + yx)$. Let $G = \mathbb{Z}/2\mathbb{Z}$ and consider the action of $G$ on $R$ that swaps $x$ and $y$. Now let's take a look at the invariants.

- The degree 1 piece of $R^G$ is given by $\{ax + ay | a \in K\}$.
- The degree 2 piece of $R^G$ is given by $\{ax^2 + ay^2 | a \in K\}$.

Note that $ax^2 + ay^2 = (ax + ay)(x + y)$, so that the degree-1 and degree-2 pieces are generated by $(x + y)$. If Noether's bound were to hold in this setting, we'd be done at this point. We'd have $R^G = K\langle x + y \rangle$. But there is the degree-3 element $x^3 + y^3 \in R^G$, and this is not divisible by $x + y$! Indeed, if $(x + y)f = x^3 + y^3$, we'd find that $f \in R_2^G$, and necessarily $f = x^2 + y^2$. But $(x + y)(x^2 + y^2) = x^3 + xy^2 + yx^2 + y^3 \neq x^3 + y^3$.

## 1.2 Outline of the talk

As explained above, we want to explain the proof of theorem 3, which has recently been given in a paper by Francesca Gandini [Gan21]. It makes use of an approach to the classical problem by Derksen and levereges structural similarities between the algebras $\bigwedge(V^*)$ and $K[V] = \operatorname{Sym}(V^*)$ to transfer the proof to the exterior algebra. In the first half of the talk we will discuss the approach of Derksen, in the second we will investigate how Gandini manages to transfer this approach to the new setting.

## 2 Derksen's proof for Noether's degree bound

The Idea behind Gandini's proof of theorem 3 is to imitate a proof of Noether's degree bound for algebras of the form $K[V]$. Again, $V$ is a representation over $K$ of some finite group $G$ whose order is invertible in $K$.

The approach described here was discovered by Harm Derksen in [Der99], where he proofs propositions 8 and 9, and states 10 as a conjecture. The main realization is that certain subspace arrangements carry all the information about the Hilbert ideal associated to a group action, and that understanding the Hilbert ideal suffices to proof Noether's bound. We will review these notions and go on to explain the approach.

**Definition 4** (Subspace arrangement)**.** Let $W$ be a finite dimensional $K$-vector space. A *subspace arrangement* is a finite set of linear subspaces of $W$, denoted by $\mathcal{A} = \{W_1, \ldots, W_t\}$.

**Definition 5** (Vanishing ideal)**.** If $S \subset V$ is any subset of a vector space, we define the associated *vanishing ideal* $I(S) = \{f \in K[V] \mid \forall s \in S : f(s) = 0\}$. Let $W$ and $\mathcal{A}$ be as above. In this case, the vanishing ideal $I(\mathcal{A})$ is defined as $I(W_1 \cup \cdots \cup W_t)$, i.e.,

$$I(\mathcal{A}) = \{f \in K[V] \mid \forall x \in \mathcal{A} : f(x) = 0\} = \bigcap_{i=1}^{t} I(W_i).$$

**Definition 6** (Subspace Arrangement associated to a group action)**.** Let $G$ be a finite group acting on a finite dimensional $K$ vector space $K$, and denote this action by $\pi$. We define the subspace arrangement $\mathcal{A}_G$ associated to this action via

$$\mathcal{A}_G := \bigcup_{g \in G} \{(v, \pi(g)v) \mid v \in V\} \subseteq V \oplus V.$$

We denote the vanishing Ideal associated to $\mathcal{A}_G$ by $I(\mathcal{A}_G) \subset K[V \oplus V]$. The key observation behind Derksen's proof is that $I(\mathcal{A}_G)$ is related to the Hilbert ideal $J_G \subset K[V]$, which is essentially the ideal in $K[V]$ "generated by $K[V]^G$".

**Definition 7** (Hilbert Ideal)**.** The Hilbert ideal is defined as the ideal of $K[V]$ generated by the $G$-invariants of positive degree, i.e., the ideal $J = (K[V]^G_+)K[V] \subset K[V]$.

Let $\mathcal{R} : K[V] \to K[V]^G$ denote the *Reynolds operator*, which is the $K$-vector space homomorphism given by $f \mapsto \frac{1}{\#G} \sum_{g \in G} g.f$.

Derksen's approach consists of three key observations, which we'll state now. The first observation explains why the Hilbert ideal is interesting for us.

**Proposition 8.** *Suppose the Hilbert ideal is generated by elements $h_1, \ldots, h_r \in J$ (note that these are not necessarily $G$-invariant). We can assume that all those functions are homogenous. Now the subring of invariants is generated by $\mathcal{R}(h_1), \ldots, \mathcal{R}(h_r)$ over $K$. In formulas:*

$$K[V]^G = K[\mathcal{R}(h_1), \ldots, \mathcal{R}(h_r)].$$

In particular, if we can show that the Hilbert ideal is generated in degree $\leq \#G$, we are done.

The second observation explains why the vanishing ideal of the subspace arrangement $\mathcal{A}_G$ is interesting for us.

**Proposition 9.** *As $K[V \oplus V]$ is graded and noetherian, we can assume that $I(\mathcal{A}_G) \subset K[V \oplus V]$ is generated by homogenous elements*

$$f_1(\mathbf{x}, \mathbf{y}), \dots, f_r(\mathbf{x}, \mathbf{y}) \in K[V \oplus V].$$

*Given such a tuple of elements, the Hilbert ideal is generated by the elements*

$$f_1(\mathbf{x}, 0), \dots, f_r(\mathbf{x}, 0) \in K[V].^1$$

*This statement can equivalently be stated as*

$$(I(\mathcal{A}_G) + (\mathbf{y})) \cap K[\mathbf{x}] = J.$$

Hence, in order to finish the proof of theorem 1 it suffices to show that $I(\mathcal{A}_G)$ is generated in degree $\leq d$. Note that $I(\mathcal{A}_G)$ is the intersection of $d$ linear ideals in a ring isomorphic to $K[x_1, \dots, x_n]$, so this at least seems plausible: In the "extreme" case where $\bigcap_{g \in G} I(V_g) = \prod_{g \in G} I(V_g)$, which for example is the case if the have pairwise trivial intersection,[2] this statement is clear: The product is generated by $d$-fold products of linear polynomials. In general, questions like this are hard to answer. But in this situation, Derksen and Sidman were able to provide an answer in [DS02].

**Theorem 10.** *Let $\mathcal{A} = \{W_1, \dots, W_t\}$ be a subspace arrangement. Then $I(\mathcal{A})$ is generated in degree $\leq t$.*

Technically, they show the stronger statement that $I(\mathcal{A})$ is $t$-regular, but we will not explain this notion here. See chapter 20.5 of [Eis13] for an introduction.

## 2.1 Proof of proposition 8

We will prove this is using a few lemmas. Remember that the Hilbert ideal was denoted by $J$.

**Lemma 11.** *Suppose that $J$ is generated by elements $f_1, \dots, f_r \in K[V]$. Then the elements $\mathcal{R}(f_1), \dots, \mathcal{R}(f_r)$ generate $J$.*

*Proof.* Remember that $J$ was the ideal in $K[V]$ generated by $K[V]^G_+$. Denote

$$J' := (\mathcal{R}(f_1), \dots, \mathcal{R}(f_r)).$$

We have $\mathcal{R}(f_i) \in K[V]^G \subset J$ and in particular $J' \subset J$. We want to show the reverse inclusion. As a first step, we show that for any $a \in K[V]$ and any $g \in G$, it holds that $g(a) - a \in K[V]^+$. Indeed, suppose $g(a) - a = x \in K \setminus \{0\}$. We find $g(a) = a + x$, hence $a = g^d(a) = a + dx \neq a$. Here we used that $d \in K^\times$! This small result implies that $G$ acts trivially on the quotient $J/K[V]_+ J$, and in particular, the residue class of $f_i$ is the same as that of $\mathcal{R}(f_i)$. Hence, $J' + K[V]_+ J = J$. But now we compute

$$K[V]_+(J/J') = (K[V]_+ J + J')/J' = J/J',$$

readily implying $J = J'$. $\square$

[1] The notation $f(\mathbf{x}, \mathbf{y})$ makes it seem like $\mathbf{x}$ and $\mathbf{y}$ are coordinates. This really only makes sense once we chose a basis for $V$, which is something one could wish to avoid. To remedy this, we could take "abstract" elements $f_i \in K[V \oplus V]$, and write $\pi(f)$ instead of $f(\mathbf{x}, 0)$, where $\pi : K[V \oplus V] \to K[V]$ is the $K$-algebra obtained from the first factor projection $V \oplus V \to V$ by functoriality.

[2] This applies the neat criterion described in [Dao]

**Lemma 12.** *Let $R$ be a graded $K$-algebra and suppose that $R_+$ is generated by as ideal by homogenous elements $f_1, \dots, f_r \in R_+$. Then $R = K[f_1, \dots, f_r]$.*

*Proof.* It is clear that $K[f_1, \dots, f_r] \subset R$, we show the reverse inclusion. Let $g \in R_+$ be an arbitrary homogenous element. We want to show $g \in K[f_1, \dots, f_r]$, via induction on the degree of $g$. The base case $\deg g = 0$ is clear. Let's assume $\deg g > 0$. By assumption, we can write $g = \sum_{i=1}^r a_i f_i$. Now (after reduction) all non-trivial summands satisfy $\deg(a_i) + \deg(f_i) = \deg g$. As $\deg(f_i) > 0$, we find $\deg(a_i) < \deg(g)$, so $a_i \in K[f_1, \dots, f_r]$ for all $i$. This implies $g \in K[f_1, \dots, f_r]$, as required. □

As a corollary, we obtain

**Lemma 13.** *Assume that the Hilbert ideal $J$ is generated by homogenous and $G$-invariant elements $f_1, \dots, f_r \in K[V]^G$. Then $K[V]^G = K[f_1, \dots, f_r]$.*

*Proof.* One inclusion is clear. Let $g \in K[V]^G$, we show that $g \in K[f_1, \dots, f_r]$. It suffices to show that $g = \sum f_i a_i$ with $a_i \in K[V]^G$ by the lemma above (applied with $R = K[V]^G$). As $K[V]^G \subset J$, we have a representation $g = \sum f_i a_i$ with $a_i \in K[V]$. Now apply $\mathcal{R}$, obtaining $g = \mathcal{R}(g) = \sum f_i \mathcal{R}(a_i)$. This is the expression we need. □

Now proposition 8 follows quickly. Applying Lemma 11, we replace the generators $f_1, \dots, f_r$ of $J$ with their respective images under the Reynolds operator $\mathcal{R}(f_1), \dots, \mathcal{R}(f_r)$. Subsequently, employing Lemma 13 completes the argument.

## 2.2 Proof of proposition 9

To ease notation, we'll write $K[V \oplus V]$ as $K[\mathbf{x}, \mathbf{y}]$. Here $G$ acts trivially on the $\mathbf{x}$-part and as usual on the $\mathbf{y}$-part. Let $J' = (f_1(\mathbf{x}, 0), \dots, f_r(\mathbf{x}, 0))$. We have to show that $J' = J$, and we show both inclusions separately. The easier one is $J \subset J'$. As $J$ is generated by homogeneous $G$-invariant objects, it suffices to show that any such $g \in K[V]^G$ lies in $J'$. As $g$ is $G$-invariant, we verify $g(\mathbf{x}) - g(\mathbf{y}) \in I(\mathcal{A}_G)$ (indeed, $g(\mathbf{x}) - \gamma(\sigma(\mathbf{x})) = 0$ for all $\sigma \in G$). In particular, $g(\mathbf{x}) - g(\mathbf{y}) = \sum_{i=1}^r a_i(\mathbf{x}, \mathbf{y}) f_i(\mathbf{x}, \mathbf{y})$, and the claim follows, as

$$g(\mathbf{x}) = g(\mathbf{x}) - g(0) = \sum_{i=1}^r a_i(\mathbf{x}, 0) f_i(\mathbf{x}, 0) \in J'.$$

Now for the reverse inclusion. The Reynolds operator for the action of $G$ on $K[\mathbf{x}, \mathbf{y}]$ is a morphism of $K[\mathbf{x}]$-modules

$$K[\mathbf{x}, \mathbf{y}] \to K[\mathbf{x}, \mathbf{y}]^G \cong K[\mathbf{x}] \otimes_K K[\mathbf{y}]^G.$$

We also define morphism of algebras

$$\delta : K[\mathbf{x}, \mathbf{y}] \to K[\mathbf{x}], \mathbf{y} \mapsto \mathbf{x}.$$

Let $f(\mathbf{x}, \mathbf{y}) \in I(\mathcal{A}_G)$ be an arbitrary element. We want to show that $f(\mathbf{x}, 0)$ lies in the Hilbert ideal, i.e., that there is a linear combination $f(\mathbf{x}, 0) = \sum c_i(\mathbf{x}) h_i(\mathbf{x})$ with $G$-invariant elements $h_i \in K[V]^G$. Note that $f(\mathbf{x}, 0) = f(\mathbf{x}, \mathbf{y}) - r(\mathbf{x}, \mathbf{y})$, where $r(\mathbf{x}, \mathbf{y})$ is an element in the ideal $(\mathbf{y}) \subset K[\mathbf{x}, \mathbf{y}]$ and can thereby be written as $\sum c_i(\mathbf{x}) p_i(\mathbf{y})$ with $p_i(0) = 0$. The next step is tricky: Applying the Reynolds operator, we find

$$f(\mathbf{x}, 0) = \mathcal{R}(f(\mathbf{x}, 0)) = \underbrace{\mathcal{R}(f(x, y))}_{=0} - \sum_i c_i(\mathbf{x}) \mathcal{R}(p_i(\mathbf{y})).$$

Now applying $\delta$ gives

$$f(\mathbf{x},0) = \delta\mathcal{R}(f(\mathbf{x},0)) = \sum_i c_i(\mathbf{x})\delta\mathcal{R}(p_i(\mathbf{y})).$$

Here we are done, as $\delta\mathcal{R}(p_i(\mathbf{y}))$ is simply $\mathcal{R}(p_i(\mathbf{x}))$ with the usual Reynolds operator on $K[V]$, and in particular $G$-invariant.
This finishes the proof of proposition 9, and with that of the classical degree bound, theorem 1.

# 3 Comparing Sym and $\bigwedge$

We want to imitate the proof given above to the case of the exterior algebra. That is, we want to find analogues of the three propositions above and proof them. Hence we will also need analogues of the Hilbert ideal, and the ideals $I(\mathcal{A})$. The (new) Hilbert ideal $J'$ is simply the left ideal generated by the invariants of positive degree. Let us write $K\langle x_1,\ldots,x_n\rangle$ for the (non-commutative) free algebra of words in the symbols $x_1,\ldots,x_n$. The new form of proposition 8 is the following

**Proposition 14.** *Suppose the Hilbert ideal is generated by elements $h_1,\ldots,h_r \in J'$ (note that these are not necessarily $G$-invariant). Now the subring of invariants is generated over $K$ by words in $\mathcal{R}(h_1),\ldots,\mathcal{R}(h_r)$ over $K$. In formulas:*

$$\bigwedge(V^*)^G = K\langle\mathcal{R}(h_1),\ldots,\mathcal{R}(h_r)\rangle.$$

This is (a special case of) theorem 18 in [Gan21], and the proof is almost the same as that given above.
In order to generalize proposition 9, we have to transfer the object $I(\mathcal{A})$ for subspace arrangements $\mathcal{A} = \{W_1,\ldots,W_t\}$. We'll do so by writing $I'(W_i)$ for the left ideal generated by the linear (degree-1)-functions in $I(W_i)$, and write $I'(\mathcal{A}) = \bigcap_{i=1}^t I'(W_i)$. Now the proof of proposition 9 also readily generalizes, and we obtain theorem 20 of [Gan21], which reads

**Proposition 15.** *Let $J'$ be the Hilbert ideal for the action of $G$ on $E = \bigwedge(V^*)$. Similarly to above, let $G$ act on $\bigwedge(V^* \oplus V^*) = \bigwedge(\mathbf{x},\mathbf{y})$, trivially on $\mathbf{x}$ and as usual on $\mathbf{y}$. We now have*

$$(I'(\mathcal{A}_G) + (\mathbf{y})) \cap \bigwedge(\mathbf{x}) = J'.$$

As above, the crux is to show that $I'(\mathcal{A}_G)$ has generators of degree $\leq d$. The fact that this was hard in the commutative case might make it seem as if this is impossible in our (non-commutative) situation. To establish a proof, Gandini used that $\bigwedge(V^*)$ and $K[V] = \mathrm{Sym}(V^*)$ have very similar structure.

## 3.1 Similarities between the symmetric and the exterior algebra

From now on, $V$ is a finite-dimensional $K$-vector space on which $G$ acts, and $K$ is a field of characteristic 0. In most introductory courses to linear algebra, $\mathrm{Sym}(V)$ and $\bigwedge V$ are introduced alongside each other, and this is no coincidence, as they look very similar:

- Both Sym and $\bigwedge$ are functors on $\mathsf{FinVec} \to \mathsf{Set}$
- They have decomposition into "homogeneous" degree-parts

$$\mathrm{Sym}(V) = K + \mathrm{Sym}^1(V) + \mathrm{Sym}^2(V) + \ldots, \quad \bigwedge(V) = K + \bigwedge{}^1 V + \bigwedge{}^2 V + \ldots.$$

- They carry the structure of a $K$-agebra.

We are going to see that both Sym and $\bigwedge$ fit into the category $\mathsf{GPoly}$ of *graded polynomial functors*, which will be the object of study for the remainder of the talk.

### 3.2 (graded) polynomial functors

In this subsection we define the category $\mathsf{GPoly}$, the main references are Appendix I.A of Macdonald's book [Mac98] and the expository article [SS12]. Let $V$ be a finite dimeinsional vector space. A *polynomial on $V$* is an element of $\mathrm{Sym}(V^*)$. After choosing a basis $(x_1, \ldots, x_n)$ of $V^*$, a polynomial on $V$ is simply a polynomial in the variables $x_i$. In particular, every polynomial on $V$ yields a map $V \to K$.

**Definition 16** (Polynomial maps)**.** Let $V$ and $W$ be a finite dimensional vector spaces. A *polynomial map* $f : V \to W$ is a mapping such that $f$ can be written as

$$f(v) = \sum_{i=1}^{m} \lambda_i(v) w_i,$$

where $\lambda_i$ are polynomials on $V$ and $w_i \in W$ are points. Equivalently, a polynomial map $V \to W$ is a morphism of $K$-algebras in the reverse direction, i.e., an element of $\mathrm{Hom}_{K\text{-Alg}}(\mathrm{Sym}(W^*), \mathrm{Sym}(V^*))$.

**Remark.** The second definition is very terse and does not make it evident how to recreate the "map" $V \to W$. This can be done via

$$V \cong \mathrm{maxSpec}(\mathrm{Sym}(V^*)) \to \mathrm{maxSpec}(\mathrm{Sym}(W^*)) \cong W.$$

We can now define the category of Polynomial functors.

**Definition 17** (Category of Polynomial functors)**.** A *polynomial functor* is a functor $\mathsf{FinVec} \to \mathsf{FinVec}$, such that for all finite vector spaces $V$ and $W$, the induced map

$$\mathrm{Hom}(V, W) \to \mathrm{Hom}(F(V), F(W))$$

is a polynomial map. That is, a polynomial functor $F$ assigns to any finite dimensional vector space $V$ a new finite dimensional vector space $F(V)$.

**Examples.**

1. For any $k \geq 0$, $\mathrm{Sym}^k$ is an element of $\mathsf{Poly}$. Indeed, assume for simplicity that $V = K^n$ (with basis $(x_i)$), and $W = K^m$ (with basis $(y_j)$). Let $\phi : V \to W$ be any linear map and let $a_{ij}$ be the entries of the corresponding matrix, i.e., $\phi(x_i) = \sum_i a_{ij} y_j$. The induced map $\mathrm{Sym}^k(\phi) : \mathrm{Sym}^k(V) \to \mathrm{Sym}^k(W)$ is the map defined by

$$\mathrm{Sym}^k(\phi)(x_{i_1} \cdots x_{i_k}) = \phi(x_{i_1}) \cdots \phi(x_{i_k}) = \left(\sum a_{i_1 j} y_j\right) \cdots \left(\sum a_{i_k j} y_j\right).$$

Note that the space $\mathrm{Hom}(V, W)^*$ is generated by the functions $\phi \mapsto a_{ij}$. The expression above is a polynomial in $a_{ij}$ for each coefficient of the monomials $y_{j_1} \cdots y_{j_k}$. Hence $\mathrm{Sym}^k$ is a polynomial functor.

2. The same is true for $\bigwedge^k$.

Even more is true, we have $\operatorname{Sym}^k(\lambda\phi) = \lambda^k \operatorname{Sym}^k(\phi)$. Mappings like this are called *homogenous.*

**Definition 18** (Homogenous polynomial map)**.** We say that a polynomial map $f : V \to W$ is homogenous of degree $t$ if $f(\lambda x) = \lambda^t f(x)$ for all $x \in V$.

We are ready to define the category of *graded polynomial functors*

**Definition 19** (Graded polynomial functors)**.** A graded polynomial functor is a functor $F : \mathsf{FinVec} \to \mathsf{Vec}$ such that $F$ has a decomposition $F = \bigoplus_{d\in\mathbb{N}_0} F_d$, where each summand $F_d$ is a homogenous polynomial functor of degree $d$.

Our calculation above shows that Sym and $\bigwedge$ are objects in $\mathsf{GPoly}$. They even are $K$-*algebra objects* in this category, which is to say, they carry a *natural* $K$-algebra structure evaluated at any $V \in \mathsf{FinVec}$. There is one more important example we'll need: Let $W$ be a finite dimensional vector space and let $\mathcal{A} = \{W_1, \dots, W_t\}$ be a subspace arrangement on $W$. Let $\mathcal{A}\otimes V$ be the subspace arrangement on $W\otimes V$ given by $\{W_1\otimes V, \dots, W_t\otimes V\}$. Then the functors $\mathcal{I}$ and $\mathcal{I}'$, given by

$$\mathcal{I}_{\mathcal{A}}(V) = I(\mathcal{A}\otimes V) \subset \operatorname{Sym}(V^*) \quad \text{and} \quad \mathcal{I}'_{\mathcal{A}}(V) = I'(\mathcal{A}\otimes V) \subset \bigwedge(V^*)$$

are graded polynomial functors (this is proposition 7.2 in [Gan22]).

### 3.3 The structure of $\mathsf{GPoly}$

In this last part of the talk, we state some results, and do not give any proofs.
We first talk a bit about partitions.

**Definition 20** (Partition)**.** Let $k$ be an integer. A partition of $k$ is a multiset of integers $\lambda = \{l_1, \dots, l_r\}$ with $\sum_{i=1}^r l_i = k$. In this situation we write $\lambda \vdash k$.

**Definition 21** (Young Diagrams)**.** Let $l_1, \dots, l_r$ be a decending sequence of (positive) natural numbers, which is the same as a partition of $k = \sum l_i$. The associated Young diagram is the following alignment of boxes. [IMAGE]Note that any Young diagram can be transposed, which yields a new partition $\lambda^\dagger \vdash k$.

**Theorem 22** (Structure of $\mathsf{GPoly}$, [Mac98])**.** *The category* $\mathsf{GPoly}$ *is abelian and semisimple. The simple objects are indexed by partitions: For each integer $k$ and each partition $\lambda$ of $k$, there is a* Schur-functor $S_\lambda$*, which is a homogenous polynomial functor of degree $k$.*

In practice, this means that any graded polynomial functor $F$ can be written as

$$F = \bigoplus_{k\in\mathbb{N}} \bigoplus_{\lambda\vdash k} S_\lambda^{\oplus a_\lambda}.$$

**Remark.** Categories with this structure seem to appear a lot. For example, the category of $S_*$-representations (with objects $(V_n)_{n\in\mathbb{N}}$, where $V_n$ is a representation of $S_n$) is equivalent to $\mathsf{GPoly}$. For more on this, see [SS12].

**Example.** The functors $\operatorname{Sym}^k$ and $\bigwedge^k$ are Schur-functors! The functor $\operatorname{Sym}^k$ belongs to the partition $(k)$, the functor $\bigwedge^k$ belongs to $(1, \dots, 1)$.

One might hope that transposition of partitions somehow descents to a transposition operation on $\mathsf{GPoly}$. This hope is fulfilled!

**Theorem 23.** *There is an additive endofunctor* $\Omega : \mathsf{GPoly} \to \mathsf{GPoly}$ *such that* $\Omega(S_\lambda) = S_{\lambda^\dagger}$.

One construction of this functor can be found in Gandini's paper, [Gan22], but there are many interesting remarks in [SS12]. In this last paper, the authors show that $\Omega$ is even an equivalence of symmetric monoidal categories. In particular, it is exact and preserves module- and algebra objects. Remember that we want to proof that the ideal $I'(\mathcal{A}_G) = \mathcal{I}'_{\mathcal{A}_G}(K)$ is generated in degree $\leq \#G$. We know the corresponding statement for $I(\mathcal{A}_G)$, and Gandini shows that we only have to apply $\Omega$ in order to obtain the statement for $I'(\mathcal{A}_G)$:

**Proposition 24.** *([Gan22, Proposition 6.1]) If* $\mathcal{R} \in \mathsf{GPoly}$ *is an algebra object and* $\mathcal{M}$ *is a* $\mathcal{R}$*-module object generated of regularity* $\leq t$*, then* $\Omega(\mathcal{M})$ *is generated in degree* $\leq t$.

Here again we couldn't get around *regularity.* We need a final lemma to conclude.

**Lemma 25.** *We have* $\Omega(\mathcal{I}_\mathcal{A}) \cong \mathcal{I}'_\mathcal{A}$.

*Proof.* [I didn't find a proof of this, Gandini simply uses this statement in [Gan21]. Perhaps we can write $\mathcal{I}_\mathcal{A}$ and $\mathcal{I}'_\mathcal{A}$ as some Kernels that get mapped into one another.] $\square$

This concludes the proof of Noether's degree bound for the exterior algebra.

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