Noether's bound for exterior algebras

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1 Introduction

Let K be a field of characteristic 0, let G be a finite group of order d and let V be a K-vector space on which G acts. In previous talks, we have seen a proof of the following theorem.

Theorem 1 (Noether's degree bound)**.** *In the situation above, the action of G of $K[V] = \operatorname{Sym}(V^*)$ has $K[V]^G$ generated in degree $\leq d$, where d is the order of G.*

In Daniel's talk, we have even seen that this statement can be generalized a bit, it holds under the milder assumption that $d < \operatorname{char} K$.

Today, we want to investigate further generalizations of this result. For one, we want to explain an approach by Derksen [Der99] that yields Noether's degree bound under the even milder assumption that $d \in K^\times$. Afterwards, we explain how this approach makes it possible to proof an analogue of Noether's degree bound for the exterior algebra.

Perhaps we should recall the definition of the exterior algebra.

Definition 2 (tensor algebra, symmetric algebra, exterior algebra)**.** If V is a finite dimensional vector space, we define the tensor algebra $\mathcal{T}(V)$ as

$$\mathcal{T}(V) = K \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) + \cdots = \bigoplus_{n \in \mathbb{N}} V^{\otimes n}.$$

The tensor algebra carries the structure of a non-commutative K-algebra with multiplication given by taking tensors.

The *symmetric algebra* $\operatorname{Sym}(V)$ is defined as the universal commutative algebra under $\mathcal{T}(V)$, which is simply the quotient

$$\operatorname{Sym}(V) = \mathcal{T}(V)/(v \otimes w - w \otimes v \mid v, w \in V).$$

Note that $K[V] = \operatorname{Sym}(V^*)$.

The *exterior algebra* $\bigwedge(V)$ is the universal anti-commutative algebra under $\mathcal{T}(V)$, which is

$$\bigwedge(V) = \mathcal{T}(V)/(v \otimes v \mid v \in V).$$

If the characteristic of K is not equal to two, we have

$$\bigwedge(V) \cong \mathcal{T}(V)/(v \otimes w + w \otimes v \mid v, w \in V),$$

which is convenient for calculations.

Note that both $\mathrm{Sym}(V)$ and $\bigwedge(V)$ have a natural grading inherited from $\mathcal{T}(V)$. We say that a graded K-algebra R is generated in degree d iff it is generated by $R_{\leq d} = R_1 \oplus \cdots \oplus R_d$ as (non-commutative) K-algebra. That is, the aim of the talk is to show the following

Theorem 3 (Noether's degree bound for the exterior algebra)**.** *Let G be a finite group of order d acting on a finite vector space V. This induces an action of G on the exterior algebra $R = \bigwedge(V^*)$ of V. Its subalgebra of G-invariant elements R^G is generated in degree at most d.*

1.1 A few examples

If $\dim(V) = n$, there is a (non-canonical) isomorphism

$$\bigwedge(V) \cong \frac{K\langle x_1, \ldots, x_n\rangle}{(x_i x_j + x_j x_i \mid 1 \leq i, j \leq n)}.$$

Here, $K\langle x_1, \ldots, x_n\rangle$ denotes the (non-commutative) algebra generated by the words in the symbols $x_1, \ldots, x_n$. For an integer $k \geq 0$, we write $\bigwedge^k(V)$ for the degree-k-piece of $\bigwedge(V)$. Note that $\bigwedge^k = 0$ if $k > \dim(V)$, as elements of the form $v_1 \otimes v_2 \otimes \cdots \otimes v_r$ vanish if the v_i are not pairwise linearly independent.

Example 1. Theorem 3 is trivial if the order of G is greater or equal to the dimension of V. Indeed, in this case the pieces of $\bigwedge(V)^G \subset \bigwedge(V)$ of degree $> \#G$ vanish, hence everything is generated in degree $\leq \#G$.

Example 2. Consider the group $G = \mathbb{Z}/4\mathbb{Z}$ and let G act on $V = K^4$, with basis $(e_1, \ldots, e_4)$ via shifting these basis vectors. Let $H = \langle 2\rangle \subset G$. We want to compute the invariants of $\bigwedge(V)$ under H. One quickly finds:

- The degree 1 piece of $\bigwedge(V)^H$ is (K-linearly) generated by $\langle x_1 + x_3, x_2 + x_4\rangle$.
- The degree 2 piece of $\bigwedge(V)^H$ is generated by $\langle x_1x_2 + x_3x_4, x_1x_4 + x_3x_2\rangle$.

Now Noether's degree bound steps in and states that these elements generate $\bigwedge(V)^H$. But note that in this situation, it is feasible to verify this by hand. Only the degree 3 and degree 4 pieces are left, and these can be explicitely calculated as well. For example, the only (up to factor) H-invariant piece of degree 4 is given by $x_1x_2x_3x_4$. One quickly verifies that

$$x_1x_2x_3x_4 = \frac{1}{2}(x_1x_2 + x_3x_4)^2,$$

so this lies in some subring generated in degree 2. Interestingly enough, we used that $\mathrm{char}\, K \nmid \#H = 2$.

Example 3. As it turns out, Noether's bound is false for general graded non-commutative algebras with degree-preserving action of a finite group G. For example, we can inspect $R = K\langle x, y\rangle/(xy + yx)$. Let $G = \mathbb{Z}/2\mathbb{Z}$ and consider the action of G on R that swaps x and y. Now let us take a look at the invariants.

- The degree 1 piece of R^G is given by $\{ax + ay | a \in K\}$.
- The degree 2 piece of R^G is given by $\{ax^2 + ay^2 | a \in K\}$.

Note that $ax^2 + ay^2 = (ax + ay)(x + y)$, so that the degree-1 and degree-2 pieces are generated by $(x + y)$. If Noether's bound were to hold in this setting, we'd be done at this point. We'd have $R^G = K\langle x + y\rangle$. But there is the degree-3 element $x^3 + y^3 \in R^G$, and this is not divisible by $x + y$. Indeed, if $(x + y)f = x^3 + y^3$, we'd find that $f \in R^G_2$, and necessarily $f = x^2 + y^2$. But $(x + y)(x^2 + y^2) = x^3 + xy^2 + yx^2 + y^3 \neq x^3 + y^3$.

1.2 Outline of the Talk

As explained above, we want to explain the proof of Theorem 3, which has recently been given in a paper by Francesca Gandini [Gan21]. It makes use of an approach to the classical problem by Derksen and levereges structural similarities between the algebras $\bigwedge(V^*)$ and $K[V] = \mathrm{Sym}(V^*)$ to transfer the proof to the exterior algebra. In the first half of the talk we will discuss the approach of Derksen, in the second we will investigate how Gandini manages to transfer this approach to the new setting.

Throughout the talk, G is a finite group of order d, K is our base field and V is a finite-dimensional representation of G over K. The assumptions on the characteristic of K will differ from case to case; either we assume characteristic 0 or we assume that $\operatorname{char} K \nmid d$.

2 Derksen's Proof for Noether's Degree Bound

In this chapter, we assume that the characteristic of K does not divide d. Again, G is a finite group acting on a finite dimensinoal K-vector space V. The aim of this chapter is the following

Theorem 4 (Noether's degree bound, more general). *Under these assumptions, the invariants $K[V]^G$ are generated in degree $\leq d$.*

The approach described here was discovered by Harm Derksen in [Der99], where he proves Propositions 9 and 10, and states 11 as a conjecture, this conjecture later was proved in [DS02]. The main realization is that certain subspace arrangements carry all the information about the Hilbert ideal associated to a group action, and that understanding the Hilbert ideal suffices to prove Noether's bound. We will review these notions and go on to explain the approach.

Definition 5 (Subspace arrangement). Let W be a finite dimensional K-vector space. A *subspace arrangement* is a finite set of linear subspaces of W, denoted by $\mathcal{A} = \{W_1, \dots, W_t\}$.

Definition 6 (Vanishing ideal). If $S \subset V$ is any subset of a vector space, we define the associated *vanishing ideal* $I(S) = \{f \in K[V] \mid \forall s \in S : f(s) = 0\}$. Let W and $\mathcal{A}$ be as above. In this case, the vanishing ideal $I(\mathcal{A})$ is defined as $I(W_1 \cup \dots \cup W_t)$, i.e.,

$$I(\mathcal{A}) = \{f \in K[V] \mid \forall i \forall x \in W_i : f(x) = 0\} = \bigcap_{i=1}^{t} I(W_i).$$

This notion is useful to us because we can assign a subset arrangement to the action of G on V.

Definition 7 (Subspace Arrangement associated to a group action). Let G be a finite group acting on a finite dimensional K-vector space K, and denote this action by π. We define the subspace arrangement $\mathcal{A}_G$ associated to this action via

$$\mathcal{A}_G := \bigcup_{g \in G} \{(v, \pi(g)v) \mid v \in V\} \subseteq V \oplus V.$$

We denote the vanishing Ideal associated to $\mathcal{A}_G$ by $I(\mathcal{A}_G) \subset K[V \oplus V]$. The key observation behind Derksen's proof is that $I(\mathcal{A}_G)$ is related to the Hilbert ideal $J_G \subset K[V]$, which is essentially the ideal in $K[V]$ "generated by $K[V]^G$".

Definition 8 (Hilbert Ideal)**.** The Hilbert ideal is defined as the ideal of $K[V]$ generated by the G-invariants of positive degree. That is, it is the ideal $J = (K[V]^G_+)K[V] \subset K[V]$.

Let $\mathcal{R} : K[V] \to K[V]^G$ denote the *Reynolds operator*, which is the K-vector space homomorphism given by $f \mapsto \frac{1}{\#G}\sum_{g\in G} g.f$. Note that this operator only exists if char $K \nmid \#G$.

Derksen's approach consists of three key observations, which we'll state now. The first observation explains why the Hilbert ideal is interesting for us.

Proposition 9. *Suppose the Hilbert ideal is generated by elements $h_1, \dots, h_r \in J$ (note that these are not necessarily G-invariant). We can assume that all those functions are homogenous. Now the subring of invariants is generated by $\mathcal{R}(h_1), \dots, \mathcal{R}(h_r)$ over K. In formulas:*

$$K[V]^G = K[\mathcal{R}(h_1), \dots, \mathcal{R}(h_r)].$$

In particular, as the Reynolds operator preserves the degree, this shows that it suffices for the proof to show that the Hilbert ideal is generated in degree $\leq \#G$.

The second observation explains why the vanishing ideal of the subspace arrangement $\mathcal{A}_G$ is interesting for us.

Proposition 10. *As $K[V \oplus V]$ is graded and noetherian, we can assume that $I(\mathcal{A}_G) \subset K[V \oplus V]$ is generated by homogenous elements*

$$f_1(\mathbf{x}, \mathbf{y}), \dots, f_r(\mathbf{x}, \mathbf{y}) \in K[V \oplus V].$$

Given such a tuple of elements, the Hilbert ideal is generated by the elements

$$f_1(\mathbf{x}, 0), \dots, f_r(\mathbf{x}, 0) \in K[V].^1$$

This statement can equivalently be stated as

$$(I(\mathcal{A}_G) + (\mathbf{y})) \cap K[\mathbf{x}] = J.$$

Hence, in order to finish the proof of Theorem 4 it suffices to show that $I(\mathcal{A}_G)$ is generated in degree $\leq d$. Note that $I(\mathcal{A}_G)$ is the intersection of d linear ideals in a ring isomorphic to $K[x_1, \dots, x_n]$, so this at least seems plausible: In the "extreme" case where $\bigcap_{g\in G} I(V_g) = \prod_{g\in G} I(V_g)$, which for example is the case if the subspaces V_g have pairwise trivial intersection,[2] this statement is clear: The product is generated by d-fold products of linear polynomials. In general, questions like this are hard to answer. But in this situation, Derksen and Sidman were able to provide an answer in [DS02].

Theorem 11 (Subspace arrangement theorem for the symmetric algebra)**.** *Let $\mathcal{A} = \{W_1, \dots, W_t\}$ be a subspace arrangement. Then the ideal $I(\mathcal{A})$ is generated in degree $\leq t$.*

Technically, they show the stronger statement that $I(\mathcal{A})$ is t-regular, but we will not explain this notion here. See chapter 20.5 of [Eis13] for an introduction.

[1] The notation $f(\mathbf{x}, \mathbf{y})$ makes it seem like $\mathbf{x}$ and $\mathbf{y}$ are coordinates. This really only makes sense once we chose a basis for V, which is something one could wish to avoid. To remedy this, we could take "abstract" elements $f_i \in K[V \oplus V]$, and write $\pi(f)$ instead of $f(\mathbf{x}, 0)$, where $\pi : K[V \oplus V] \to K[V]$ is the K-algebra obtained from the first factor projection $V \oplus V \to V$ by functoriality.

[2] This applies the neat criterion described in the mathoverflow answer [Dao].

2.1 Proof of Proposition 9

We will prove this using a few lemmas. Recall that the Hilbert ideal was denoted by J.

Lemma 12. *Suppose that J is generated by elements $f_1, \dots, f_r \in K[V]$. Then the elements $\mathcal{R}(f_1), \dots, \mathcal{R}(f_r)$ generate J.*

Proof. Recall that J was the ideal in $K[V]$ generated by $K[V]^G_+$. Denote

$$H := (\mathcal{R}(f_1), \dots, \mathcal{R}(f_r)).$$

We have $\mathcal{R}(f_i) \in K[V]^G \subset J$ and in particular $H \subset J$. We want to show the reverse inclusion. As a first step, we show that for any $a \in K[V]$ and any $g \in G$, it holds that $g(a) - a \in K[V]^+$. Indeed, suppose $g(a) - a = x \in K \setminus \{0\}$. We find $g(a) = a + x$, hence $a = g^d(a) = a + dx \neq a$. Here we used that $d \in K^\times$. This small result implies that G acts trivially on the quotient $J/K[V]_+J$, and in particular, the residue class of f_i is the same as that of $\mathcal{R}(f_i)$. Hence, $H + K[V]_+J = J$. But now we compute

$$K[V]_+(J/H) = (K[V]_+J + H)/H = J/H,$$

readily implying $J = H$. □

Lemma 13. *Let R be a graded K-algebra and suppose that R_+ is generated as ideal by homogenous elements $f_1, \dots, f_r \in R_+$. Then $R = K[f_1, \dots, f_r]$.*

Proof. It is clear that $K[f_1, \dots, f_r] \subset R$, we show the reverse inclusion. Let $g \in R_+$ be an arbitrary homogenous element. We want to show $g \in K[f_1, \dots, f_r]$, via induction on the degree of g. The base case $\deg g = 0$ is clear. Let's assume $\deg g > 0$. By assumption, we can write $g = \sum_{i=1}^r a_i f_i$. Now (after reduction) all non-trivial summands satisfy $\deg(a_i) + \deg(f_i) = \deg g$. As $\deg(f_i) > 0$, we find $\deg(a_i) < \deg(g)$, so $a_i \in K[f_1, \dots, f_r]$ for all i. This implies $g \in K[f_1, \dots, f_r]$, as required. □

Now we are ready to prove

Lemma 14. *Assume that the Hilbert ideal J is generated by homogenous and G-invariant elements $f_1, \dots, f_r \in K[V]^G$. Then $K[V]^G = K[f_1, \dots, f_r]$.*

Proof. One inclusion is clear. Let $g \in K[V]^G$, we show that $g \in K[f_1, \dots, f_r]$. It suffices to show that $g = \sum f_i a_i$ with $a_i \in K[V]^G$ by the lemma above (applied with $R = K[V]^G$). As $K[V]^G \subset J$, we have a representation $g = \sum f_i a_i$ with $a_i \in K[V]$. Now we apply $\mathcal{R}$, obtaining $g = \mathcal{R}(g) = \sum f_i \mathcal{R}(a_i)$. This is the expression we need. □

Now Proposition 9 follows quickly. Applying Lemma 12, we replace the generators $f_1, \dots, f_r$ of J with their respective images under the Reynolds operator $\mathcal{R}(f_1), \dots, \mathcal{R}(f_r)$. Subsequently, employing Lemma 14 completes the argument.

2.2 Proof of Proposition 10

To ease notation, we'll write $K[V \oplus V]$ as $K[\mathbf{x}, \mathbf{y}]$. Here G acts trivially on the $\mathbf{x}$-part and as usual on the $\mathbf{y}$-part. Let $H = (f_1(\mathbf{x}, 0), \dots, f_r(\mathbf{x}, 0))$. We have to show that $H = J$, and we show both inclusions separately. The easier one is $J \subset H$. As J is generated by homogeneous G-invariant objects, it suffices to show that any such $g \in K[V]^G$ lies in H.

As g is G-invariant, we verify $g(\mathbf{x}) - g(\mathbf{y}) \in I(\mathcal{A}_G)$ (indeed, $g(\mathbf{x}) - g(\sigma(\mathbf{x})) = 0$ for all $\sigma \in G$). In particular, $g(\mathbf{x}) - g(\mathbf{y}) = \sum_{i=1}^r a_i(\mathbf{x},\mathbf{y}) f_i(\mathbf{x},\mathbf{y})$, and the claim follows, as

$$g(\mathbf{x}) = g(\mathbf{x}) - g(0) = \sum_{i=1}^{r} a_i(\mathbf{x}, 0) f_i(\mathbf{x}, 0) \in H.$$

Now for the reverse inclusion. The Reynolds operator for the action of G on $K[\mathbf{x},\mathbf{y}]$ is a morphism of $K[\mathbf{x}]$-modules

$$K[\mathbf{x},\mathbf{y}] \to K[\mathbf{x},\mathbf{y}]^G \cong K[\mathbf{x}] \otimes_K K[\mathbf{y}]^G.$$

We also define morphism of algebras

$$\delta : K[\mathbf{x},\mathbf{y}] \to K[\mathbf{x}], \quad \mathbf{y} \mapsto \mathbf{x}.$$

Let $f(\mathbf{x},\mathbf{y}) \in I(\mathcal{A}_G)$ be an arbitrary element. We want to show that $f(\mathbf{x},0)$ lies in the Hilbert ideal, i.e., that there is a linear combination $f(\mathbf{x},0) = \sum c_i(\mathbf{x}) h_i(\mathbf{x})$ with G-invariant elements $h_i \in K[V]^G$. Note that $f(\mathbf{x},0) = f(\mathbf{x},\mathbf{y}) - r(\mathbf{x},\mathbf{y})$, where $r(\mathbf{x},\mathbf{y})$ is an element in the ideal $(\mathbf{y}) \subset K[\mathbf{x},\mathbf{y}]$ and can thereby be written as $\sum c_i(\mathbf{x}) p_i(\mathbf{y})$ with $p_i(0) = 0$. The next step is tricky: Applying the Reynolds operator, we find

$$f(\mathbf{x},0) = \mathcal{R}(f(\mathbf{x},0)) = \underbrace{\mathcal{R}(f(x,y))}_{=0} - \sum_i c_i(\mathbf{x}) \mathcal{R}(p_i(\mathbf{y})).$$

Now applying δ gives

$$f(\mathbf{x},0) = \delta\mathcal{R}(f(\mathbf{x},0)) = \sum_i c_i(\mathbf{x}) \delta\mathcal{R}(p_i(\mathbf{y})).$$

Here we are done, as $\delta\mathcal{R}(p_i(\mathbf{y}))$ is simply $\mathcal{R}(p_i(\mathbf{x}))$ with the usual Reynolds operator on $K[V]$, and in particular G-invariant.

This finishes the proof of Proposition 10, and with that of the classical degree bound, Theorem 4.

3 Comparing Sym and $\bigwedge$

We want to imitate the proof given above to the case of the exterior algebra. That is, we want to find analogues of the three propositions above and prove them. Hence we will also need analogues of the Hilbert ideal, and the ideals $I(\mathcal{A})$. For now, we keep the assumptions on the characteristic of K. The (new) Hilbert ideal J' is simply the left ideal generated by the invariants of positive degree. Let us write $K\langle x_1, \dots, x_n\rangle$ for the (non-commutative) free algebra of words in the symbols $x_1, \dots, x_n$. The new form of Proposition 9 is the following

Proposition 15. *Suppose the Hilbert ideal is generated by elements $h_1, \dots, h_r \in J'$ (note that these are not necessarily G-invariant). Now the subring of invariants is generated over K by words in $\mathcal{R}(h_1), \dots, \mathcal{R}(h_r)$ over K. In formulas:*

$$\bigwedge(V^*)^G = K\langle \mathcal{R}(h_1), \dots, \mathcal{R}(h_r)\rangle.$$

This is (a special case of) Theorem 18 in [Gan21], and the proof is almost the same as that given above.

In order to generalize Proposition 10, we have to transfer the object $I(\mathcal{A})$ for subspace arrangements $\mathcal{A} = \{W_1, \dots, W_t\}$. We'll do so by writing $I'(W_i)$ for the left ideal generated by the linear (degree-1)-functions in $I(W_i)$, and write $I'(\mathcal{A}) = \bigcap_{i=1}^t I'(W_i)$. Now the proof of Proposition 10 also readily generalizes, and we obtain Theorem 20 of [Gan21], which reads

Proposition 16. *Let J' be the Hilbert ideal for the action of G on $E = \bigwedge(V^*)$. Similarly to above, let G act on $\bigwedge(V^* \oplus V^*) = \bigwedge(\mathbf{x}, \mathbf{y})$, trivially on $\mathbf{x}$ and as usual on $\mathbf{y}$. We now have*

$$(I'(\mathcal{A}_G) + (\mathbf{y})) \cap \bigwedge(\mathbf{x}) = J'.$$

Having these propositions, the crux is again to show that $I'(\mathcal{A}_G)$ has generators of degree $\leq d$. This theorem is due to Gandini [Gan21, Theorem 9].

Theorem 17 (Subspace arrangement theorem for the exterior algebra)**.** *If $\mathcal{A}$ is an arrangement of t subspaces in $V \cong K^n$, then the ideal $I'(\mathcal{A})$ in the exterior algebra $\bigwedge(V^*)$ is t-regular, and in particular, generated in degree $\leq t$.*

The fact that this was hard in the commutative case might make it seem as if this is impossible in our (non-commutative) situation. To establish a proof, Gandini used that $\bigwedge(V^*)$ and $K[V] = \mathrm{Sym}(V^*)$ have very similar structure.

3.1 Similarities between the Symmetric and the Exterior Algebra

From now on, V is a finite-dimensional K-vector space on which G acts, and K is a field of characteristic 0. In most introductory courses to linear algebra, $\mathrm{Sym}(V)$ and $\bigwedge V$ are introduced alongside each other, and this is no coincidence, as they look very similar:

- Both Sym and $\bigwedge$ are functors on $\mathsf{FinVec} \to \mathsf{Set}$
- They have decomposition into "homogeneous" degree-parts

$$\mathrm{Sym}(V) = K + \mathrm{Sym}^1(V) + \mathrm{Sym}^2(V) + \dots, \quad \bigwedge(V) = K + \bigwedge{}^1 V + \bigwedge{}^2 V + \dots.$$

- They carry the structure of a K-agebra.

We are going to see that both Sym and $\bigwedge$ fit into the category GPoly of *graded polynomial functors*, which will be the object of study for the remainder of the talk.

3.2 (Graded) Polynomial Functors

In this subsection we define the category GPoly, the main references are Appendix I.A of Macdonald's book [Mac98] and the expository article [SS12]. Let V be a finite dimeinsional vector space. A *polynomial on V* is an element of $\mathrm{Sym}(V^*)$. After choosing a basis $(x_1, \dots, x_n)$ of V^*, a polynomial on V is simply a polynomial in the variables x_i. In particular, every polynomial on V yields a map $V \to K$.

Definition 18 (Polynomial maps)**.** Let V and W be a finite dimensional vector spaces. A *polynomial map* $f : V \to W$ is a mapping such that f can be written as

$$f(v) = \sum_{i=1}^{m} \lambda_i(v) w_i,$$

where λ_i are polynomials on V and $w_i \in W$ are points. Equivalently, a polynomial map $V \to W$ is a morphism of K-algebras in the reverse direction. That is, an element of $\mathrm{Hom}_{K\text{-Alg}}(\mathrm{Sym}(W^*), \mathrm{Sym}(V^*))$.

Remark. The second definition is very terse and does not make it evident how to recreate the "map" $V \to W$. This can be done via

$$V \cong \mathrm{maxSpec}(\mathrm{Sym}(V^*)) \to \mathrm{maxSpec}(\mathrm{Sym}(W^*)) \cong W.$$

We can now define the category of Polynomial functors.

Definition 19 (Category of Polynomial functors)**.** A *polynomial functor* is a functor $\mathsf{FinVec} \to \mathsf{FinVec}$, such that for all finite vector spaces V and W, the induced map

$$\mathrm{Hom}(V, W) \to \mathrm{Hom}(F(V), F(W))$$

is a polynomial map. That is, a polynomial functor F assigns to any finite dimensional vector space V a new finite dimensional vector space $F(V)$.

Examples.
1. For any $k \geq 0$, Sym^k is an element of Poly. Indeed, assume for simplicity that $V = K^n$ (with basis (x_i)), and $W = K^m$ (with basis (y_j)). Let $\phi : V \to W$ be any linear map and let a_{ij} be the entries of the corresponding matrix, i.e., $\phi(x_i) = \sum_i a_{ij} y_j$. The induced map $\mathrm{Sym}^k(\phi) : \mathrm{Sym}^k(V) \to \mathrm{Sym}^k(W)$ is the map defined by

$$\mathrm{Sym}^k(\phi)(x_{i_1} \cdots x_{i_k}) = \phi(x_{i_1}) \cdots \phi(x_{i_k}) = \left(\sum a_{i_1 j} y_j\right) \cdots \left(\sum a_{i_k j} y_j\right).$$

Note that the space $\mathrm{Hom}(V, W)^*$ is generated by the functions $\phi \mapsto a_{ij}$. The expression above is a polynomial in a_{ij} for each coefficient of the monomials $y_{j_1} \cdots y_{j_k}$. Hence Sym^k is a polynomial functor.
2. The same is true for $\bigwedge^k$.
Even more is true, we have $\mathrm{Sym}^k(\lambda\phi) = \lambda^k \mathrm{Sym}^k(\phi)$. Mappings like this are called *homogenous.*

Definition 20 (Homogenous polynomial map)**.** We say that a polynomial map $f : V \to W$ is homogenous of degree k if $f(\alpha x) = \alpha^k f(x)$ for all $x \in V$, and all scalars $\alpha \in K$.

We are ready to define the category of *graded polynomial functors*

Definition 21 (Graded polynomial functors)**.** A graded polynomial functor is a functor $F : \mathsf{FinVec} \to \mathsf{Vec}$ such that F has a decomposition $F = \bigoplus_{d \in \mathbb{N}_0} F_d$, where each summand F_d is a homogenous polynomial functor of degree d.

Our calculation above shows that Sym and $\bigwedge$ are objects in GPoly. They even are K-*algebra objects* in this category, which is to say, they carry a *natural* K-algebra structure evaluated at any $V \in \mathsf{FinVec}$. There is one more important example we'll need: Let W be a finite dimensional vector space and let $\mathcal{A} = \{W_1, \ldots, W_t\}$ be a subspace arrangement on W. Let $\mathcal{A} \otimes V$ be the subspace arrangement on $W \otimes V$ given by $\{W_1 \otimes V, \ldots, W_t \otimes V\}$. Then the functors $\mathcal{I}$ and $\mathcal{I}'$, given by

$$\mathcal{I}_{\mathcal{A}}(V) = I(\mathcal{A} \otimes V) \subset \mathrm{Sym}(V^*) \quad \text{and} \quad \mathcal{I}'_{\mathcal{A}}(V) = I'(\mathcal{A} \otimes V) \subset \bigwedge(V^*)$$

are graded polynomial functors (this is Proposition 7.2 in [Gan22]).

3.3 The Structure of GPoly

In this last part of the talk, we state some results, and do not give any proofs.
We first talk a bit about partitions.

Definition 22 (Partition)**.** Let k be an integer. A partition of k is a multiset of integers $\lambda = \{l_1, \dots, l_r\}$ with $\sum_{i=1}^{r} l_i = k$. In this situation we write $\lambda \vdash k$.

Definition 23 (Young Diagrams)**.** Let $l_1, \dots, l_r$ be a decending sequence of (positive) natural numbers, which is the same as a partition of $k = \sum l_i$. The associated Young diagram is a certain alignment of boxes, with l_1 boxes in the first row, l_2 boxes in the second, and so on. Here are a few examples.

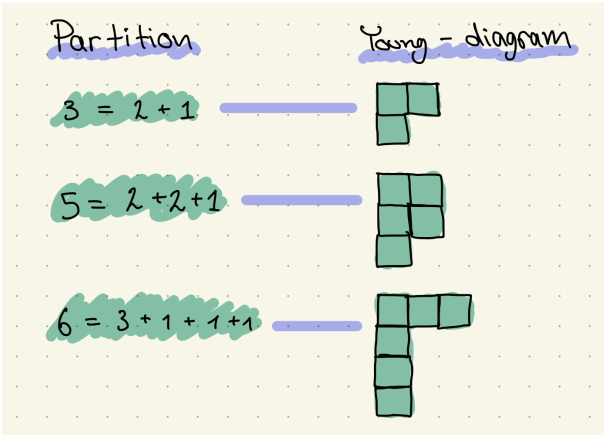

Figure 1: Examples for young diagrams associated to partitions.

One easily checks that partitions of an integer k correspond one-to-one to Young diagrams with k boxes. There is an involution on the set of Young diagrams, given by *transposing* the diagram. This yields an involution on the set of partitions.

Definition 24 (Transpose partition)**.** Given a partition $\lambda = (l_1 \geq l_2 \geq \cdots \geq l_r)$, the *transpose partition* is given by $\lambda^\dagger = (l_1^\dagger \geq l_2^\dagger \geq \cdots \geq l_s^\dagger)$, where

$$l_i^\dagger = \#\{j \mid l_j \geq i\}.$$

In particular, we find $s = l_1$. This definition makes much more sense in terms of Young diagrams.

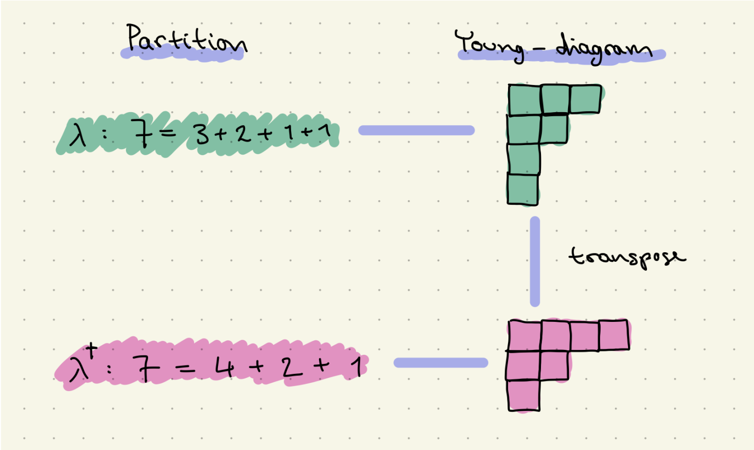

Figure 2: Example of a transpose partition

We care about partitions because they are in natural (whatever this means here) bijection with the "fundamental building blocks" of GPoly.

Theorem 25 (Structure of GPoly, [Mac98], [SS12])**.** *The category* GPoly *is an abelian and semisimple tensor category. The simple objects are indexed by partitions: For each integer k and each partition λ of k, there is a* Schur-functor S_λ*, which is a homogenous polynomial functor of degree k.*

In practice, this means that any graded polynomial functor F can uniquely be written as

$$F = \bigoplus_{k \in \mathbb{N}} \bigoplus_{\lambda \vdash k} S_\lambda^{\oplus a_\lambda}.$$

Remark. Categories with this structure seem to appear a lot. For example, the category of S_*-representations (with objects $(V_n)_{n\in\mathbb{N}}$, where V_n is a representation of S_n) is equivalent to GPoly. For more on this, see [SS12].
The reason this is interesting for us is that the functors Sym^k and $\bigwedge^k$ are Schur-functors! Even better, the functor Sym^k belongs to the partition (k), the functor $\bigwedge^k$ belongs to $(1, \ldots, 1)$. These partitions are mutual transpose of one another, which is a deep connection between Sym and $\bigwedge$, a lot deeper than the list of similarities we found before.
One might hope that transposition of partitions somehow descents to a transposition operation on GPoly. This hope is fulfilled!

Theorem 26. *There is an additive endofunctor $\Omega : \mathsf{GPoly} \to \mathsf{GPoly}$ such that $\Omega(S_\lambda) = S_{\lambda^\dagger}$.*

One construction of this functor can be found in Gandini's paper, [Gan22], but there are many interesting remarks in [SS12]. In this last paper, the authors show that Ω is even an equivalence of symmetric monoidal categories. In particular, it is exact and preserves module- and algebra objects. Remember that we want to prove that the ideal $I'(\mathcal{A}_G) = \mathcal{I}'_{\mathcal{A}_G}(K)$ is generated in degree $\leq \#G$. We know the corresponding statement for $I(\mathcal{A}_G)$, and Gandini shows that we only have to apply Ω in order to obtain the statement for $I'(\mathcal{A}_G)$:

Proposition 27. *([Gan22, Proposition 6.1]) If $\mathcal{R} \in \mathsf{GPoly}$ is an algebra object and $\mathcal{M}$ is a $\mathcal{R}$-module object "of regularity $\leq t$",[3] then $\Omega(\mathcal{M})(V)$ is generated in degree $\leq t$ over $\mathcal{R}(V)$ for every $V \in \mathsf{FinVec}$.*

[3] Again, this needs a suitable adaptation of *regularity* in this setting. We won't go into this here.

Here again we couldn't get around *regularity*. We need a final lemma to conclude.

Lemma 28. *We have* $\Omega(\mathcal{I}_\mathcal{A}) \cong \mathcal{I}'_\mathcal{A}$.

Proof. [I didn't find a proof of this, Gandini simply uses this statement in [Gan21]. Perhaps we can write $\mathcal{I}_\mathcal{A}$ and $\mathcal{I}'_\mathcal{A}$ as some Kernels that get mapped into one another.] $\square$

It can be shows that the functor $\mathcal{I}_\mathcal{A}$ is of regularity $\leq t$ if $\mathcal{A}$ is a subspace arrangement of t subspaces. Using the previous two results, this shows that $I'(\mathcal{A}) = \mathcal{I}'_\mathcal{A}(K)$ is generated over $\bigwedge(V)$ in degree $\leq t$. Applying this result to the subspace arrangement associated with the action of G on V, we obtain Theorem 17. This concludes the proof of the subspace arrangement theorem, which also finishes the proof of Noether's degree bound for the exterior algebra.

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