Noether's bound for exterior algebras

1 Introduction

Let K be a field of characteristic 0, let G be a finite group of order d and let V be a K-vector space on which G acts. In previous talks, we have seen a proof of the following theorem.

Theorem 1 (Noether's degree bound)**.** *In the situation above, the action of G of $K[V] = \mathrm{Sym}(V^*)$ has $K[V]^G$ generated in degree $\leq d$, where d is the order of G.*

In Daniel's talk, we have even seen that this statement can be generalized a bit, it holds under the milder assumption that $d < \mathrm{char}\, K$.

Today, we want to investigate further generalizations of this result. For one, we want to explain an approach by Derksen [Der99] that yields Noether's degree bound under the even milder assumption that $d \in K^\times$. Afterwards, we explain how this approach makes it possible to proof an analogue of Noether's degree bound for the exterior algebra.

Let us write $K\langle x_1, \dots, x_n\rangle$ for the (non-commutative) free algebra of words in the symbols $x_1, \dots, x_n$. We say that a graded K-algebra R is generated in degree d iff it is generated by $R_{\leq d} = R_1 \oplus \cdots \oplus R_d$ as K-algebra. That is, the aim of the talk is to show the following

Theorem 2 (Noether's degree bound for the exterior algebra)**.** *Let G be a finite group of order d acting on a finite vector space V. This induces an action of G on the exterior algebra $R = \bigwedge V$ of V. Its subalgebra of G-invariant elements R^G is generated in degree at most d.*

1.1 Outline of the talk

As explained above, we want to explain the proof of theorem 2, which has recently been given in a paper by Francesca Gandini [Gan21]. It makes use of an approach to the classical problem by Derksen and levereges structural similarities between the algebras $\bigwedge V$ and $K[V] = \mathrm{Sym}(V^*)$ to transfer the proof to the exterior algebra. In the first half of the talk we will discuss the approach of Derksen, in the second we will investigate how Gandini manages to transfer this approach to the new setting.

2 Derksen's proof for Noether's degree bound

The Idea behind Gandini's proof of theorem 2 is to imitate a proof of Noether's degree bound for algebras of the form $K[V]$. Again, V is a representation over K of some finite group G whose order is invertible in K.

The approach described here was discovered by Harm Derksen in [Der99], where he proofs propositions 7 and 8, and states 9 as a conjecture. The main realization is that certain subspace arrangements carry all the information about the Hilbert ideal associated to a group action, and that understanding the Hilbert ideal suffices to proof Noether's bound. We will review these notions and go on to explain the approach.

Definition 3 (Subspace arrangement)**.** Let W be a finite dimensional K-vector space. A *subspace arrangement* is a finite set of linear subspaces of W, denoted by $\mathcal{A} = \{W_1, \dots, W_t\}$.

Definition 4 (Vanishing ideal)**.** If $S \subset V$ is any subset of a vector space, we define the associated *vanishing ideal* $I(S) = \{f \in K[V] \mid \forall s \in S : f(s) = 0\}$. Let W and $\mathcal{A}$ be as above. In this case, the vanishing ideal $I(\mathcal{A})$ is defined as $I(W_1 \cup \cdots \cup W_t)$, i.e.,

$$I(\mathcal{A}) = \{f \in K[V] \mid \forall x \in \mathcal{A} : f(x) = 0\} = \bigcap_{i=1}^{t} I(W_i).$$

Definition 5 (Subspace Arrangement associated to a group action)**.** Let G be a finite group acting on a finite dimensional K vector space K, and denote this action by π. We define the subspace arrangement $\mathcal{A}_G$ associated to this action via

$$\mathcal{A}_G := \bigcup_{g \in G} \{(v, \pi(g)v) \mid v \in V\} \subseteq V \oplus V.$$

We denote the vanishing Ideal associated to $\mathcal{A}_G$ by $I(\mathcal{A}_G) \subset K[V \oplus V]$. The key observation behind Derksen's proof is that $I(\mathcal{A}_G)$ is related to the Hilbert ideal $J_G \subset K[V]$, which is essentially the ideal in $K[V]$ "generated by $K[V]^G$".

Definition 6 (Hilbert Ideal)**.** The Hilbert ideal is defined as the ideal of $K[V]$ generated by the G-invariants of positive degree, i.e., the ideal $J = (K[V]^G_+)K[V] \subset K[V]$.

Let $\mathcal{R} : K[V] \to K[V]^G$ denote the *Reynolds operator*, which is the K-vector space homomorphism given by $f \mapsto \frac{1}{\#G} \sum_{g \in G} g.f$.
Derksen's approach consists of three key observations, which we'll state now. The first observation explains why the Hilbert ideal is interesting for us.

Proposition 7. *Suppose the Hilbert ideal is generated by elements $h_1, \dots, h_r \in J$ (note that these are not necessarily G-invariant). We can assume that all those functions are homogenous. Now the subring of invariants is generated by $\mathcal{R}(h_1), \dots, \mathcal{R}(h_r)$ over K. In formulas:*

$$K[V]^G = K[\mathcal{R}(h_1), \dots, \mathcal{R}(h_r)].$$

In particular, if we can show that the Hilbert ideal is generated in degree $\leq \#G$, we are done.
The second observation explains why the vanishing ideal of the subspace arrangement $\mathcal{A}_G$ is interesting for us.

Proposition 8. *As $K[V \oplus V]$ is graded and noetherian, we can assume that $I(\mathcal{A}_G) \subset K[V \oplus V]$ is generated by homogenous elements*

$$f_1(\mathbf{x}, \mathbf{y}), \dots, f_r(\mathbf{x}, \mathbf{y}) \in K[V \oplus V].$$

Given such a tuple of elements, the Hilbert ideal is generated by the elements

$$f_1(\mathbf{x}, 0), \dots, f_r(\mathbf{x}, 0) \in K[V].^1$$

This statement can equivalently be stated as

$$(I(\mathcal{A}_G) + (\mathbf{y})) \cap K[\mathbf{x}] = J.$$

[1] The notation $f(\mathbf{x}, \mathbf{y})$ makes it seem like $\mathbf{x}$ and $\mathbf{y}$ are coordinates. This really only makes sense once we chose a basis for V, which is something one could wish to avoid. To remedy this, we could take "abstract" elements $f_i \in K[V \oplus V]$, and write $\pi(f)$ instead of $f(\mathbf{x}, 0)$, where $\pi : K[V \oplus V] \to K[V]$ is the K-algebra obtained from the first factor projection $V \oplus V \to V$ by functoriality.

Hence, in order to finish the proof of theorem 1 it suffices to show that $I(\mathcal{A}_G)$ is generated in degree $\leq d$. Note that $I(\mathcal{A}_G)$ is the intersection of d linear ideals in a ring isomorphic to $K[x_1, \ldots, x_n]$, so this at least seems plausible: In the "extreme" case where $\bigcap_{g \in G} I(V_g) = \prod_{g \in G} I(V_g)$, which for example is the case if the have pairwise trivial intersection,[2] this statement is clear: The product is generated by d-fold products of linear polynomials. In general, questions like this are hard to answer. But in this situation, Derksen and Sidman were able to provide an answer in [DS02].

Theorem 9. *Let $\mathcal{A} = \{W_1, \ldots, W_t\}$ be a subspace arrangement. Then $I(\mathcal{A})$ is generated in degree $\leq t$.*

Technically, they show the stronger statement that $I(\mathcal{A})$ is t-regular, but we will not explain this notion here. See chapter 20.5 of [Eis13] for an introduction.

2.1 Proof of proposition 8

We will prove this is using a few lemmas. Remember that the Hilbert ideal was denoted by J.

Lemma 10. *Suppose that J is generated by elements $f_1, \ldots, f_r \in K[V]$. Then the elements $\mathcal{R}(f_1), \ldots, \mathcal{R}(f_r)$ generate J.*

Proof. Remember that J was the ideal in $K[V]$ generated by $K[V]^G_+$. Denote

$$J' := (\mathcal{R}(f_1), \ldots, \mathcal{R}(f_r)).$$

We have $\mathcal{R}(f_i) \in K[V]^G \subset J$ and in particular $J' \subset J$. We want to show the reverse inclusion. As a first step, we show that for any $a \in K[V]$ and any $g \in G$, it holds that $g(a) - a \in K[V]^+$. Indeed, suppose $g(a) - a = x \in K \setminus \{0\}$. We find $g(a) = a + x$, hence $a = g^d(a) = a + dx \neq a$. Here we used that $d \in K^\times$! This small result implies that G acts trivially on the quotient $J/K[V]_+J$, and in particular, the residue class of f_i is the same as that of $\mathcal{R}(f_i)$. Hence, $J' + K[V]_+J = J$. But now we compute

$$K[V]_+(J/J') = (K[V]_+J + J')/J' = J/J',$$

readily implying $J = J'$. □

Lemma 11. *Let R be a graded K-algebra and suppose that R_+ is generated by as ideal by homogenous elements $f_1, \ldots, f_r \in R_+$. Then $R = K[f_1, \ldots, f_r]$.*

Proof. It is clear that $K[f_1, \ldots, f_r] \subset R$, we show the reverse inclusion. Let $g \in R_+$ be an arbitrary homogenous element. We want to show $g \in K[f_1, \ldots, f_r]$, via induction on the degree of g. The base case $\deg g = 0$ is clear. Let's assume $\deg g > 0$. By assumption, we can write $g = \sum_{i=1}^r a_i f_i$. Now (after reduction) all non-trivial summands satisfy $\deg(a_i) + \deg(f_i) = \deg g$. As $\deg(f_i) > 0$, we find $\deg(a_i) < \deg(g)$, so $a_i \in K[f_1, \ldots, f_r]$ for all i. This implies $g \in K[f_1, \ldots, f_r]$, as required. □

As a corollary, we obtain

Lemma 12. *Assume that the Hilbert ideal J is generated by homogenous and G-invariant elements $f_1, \ldots, f_r \in K[V]^G$. Then $K[V]^G = K[f_1, \ldots, f_r]$.*

[2] This applies the neat criterion described in [Dao]

Proof. One inclusion is clear. Let $g \in K[V]^G$, we show that $g \in K[f_1, \ldots, f_r]$. It suffices to show that $g = \sum f_i a_i$ with $a_i \in K[V]^G$ by the lemma above (applied with $R = K[V]^G$). As $K[V]^G \subset J$, we have a representation $g = \sum f_i a_i$ with $a_i \in K[V]$. Now apply $\mathcal{R}$, obtaining $g = \mathcal{R}(g) = \sum f_i \mathcal{R}(a_i)$. This is the expression we need. □

Now proposition 7 follows quickly. Applying Lemma 10, we replace the generators $f_1, \ldots, f_r$ of J with their respective images under the Reynolds operator $\mathcal{R}(f_1), \ldots, \mathcal{R}(f_r)$. Subsequently, employing Lemma 12 completes the argument.

2.2 Proof of proposition 9

To ease notation, we'll write $K[V \oplus V]$ as $K[\mathbf{x}, \mathbf{y}]$. Here G acts trivially on the $\mathbf{x}$-part and as usual on the $\mathbf{y}$-part. Let $J' = (f_1(\mathbf{x}, 0), \ldots, f_r(\mathbf{x}, 0))$. We have to show that $J' = J$, and we show both inclusions separately. The easier one is $J \subset J'$. As J is generated by homogeneous G-invariant objects, it suffices to show that any such $g \in K[V]^G$ lies in J'. As g is G-invariant, we verify $g(\mathbf{x}) - g(\mathbf{y}) \in I(\mathcal{A}_G)$ (indeed, $g(\mathbf{x}) - \gamma(\sigma(\mathbf{x})) = 0$ for all $\sigma \in G$). In particular, $g(\mathbf{x}) - g(\mathbf{y}) = \sum_{i=1}^r a_i(\mathbf{x}, \mathbf{y}) f_i(\mathbf{x}, \mathbf{y})$, and the claim follows, as

$$g(\mathbf{x}) = g(\mathbf{x}) - g(0) = \sum_{i=1}^{r} a_i(\mathbf{x}, 0) f_i(\mathbf{x}, 0) \in J'.$$

Now for the reverse inclusion. The Reynolds operator for the action of G on $K[\mathbf{x}, \mathbf{y}]$ is a morphism of $K[\mathbf{x}]$-modules

$$K[\mathbf{x}, \mathbf{y}] \to K[\mathbf{x}, \mathbf{y}]^G \cong K[\mathbf{x}] \otimes_K K[\mathbf{y}]^G.$$

We also define morphism of algebras

$$\delta : K[\mathbf{x}, \mathbf{y}] \to K[\mathbf{x}], \mathbf{y} \mapsto \mathbf{x}.$$

Let $f(\mathbf{x}, \mathbf{y}) \in I(\mathcal{A}_G)$ be an arbitrary element. We want to show that $f(\mathbf{x}, 0)$ lies in the Hilbert ideal, i.e., that there is a linear combination $f(\mathbf{x}, 0) = \sum c_i(\mathbf{x}) h_i(\mathbf{x})$ with G-invariant elements $h_i \in K[V]^G$. Note that $f(\mathbf{x}, 0) = f(\mathbf{x}, \mathbf{y}) - r(\mathbf{x}, \mathbf{y})$, where $r(\mathbf{x}, \mathbf{y})$ is an element in the ideal $(\mathbf{y}) \subset K[\mathbf{x}, \mathbf{y}]$ and can thereby be written as $\sum c_i(\mathbf{x}) p_i(\mathbf{y})$ with $p_i(0) = 0$. The next step is tricky: Applying the Reynolds operator, we find

$$f(\mathbf{x}, 0) = \mathcal{R}(f(\mathbf{x}, 0)) = \mathcal{R}(f(x, y)) - \sum_i c_i(\mathbf{x}) \mathcal{R}(p_i(\mathbf{y})).$$

Now applying δ gives

$$f(\mathbf{x}, 0) = \delta\mathcal{R}(f(\mathbf{x}, 0)) = \sum_i c_i(\mathbf{x}) \delta\mathcal{R}(p_i(\mathbf{y})).$$

Here we are done, as $\delta\mathcal{R}(p_i(\mathbf{y}))$ is simply $\mathcal{R}(p_i(\mathbf{x}))$ with the usual Reynolds operator on $K[V]$, and in particular G-invariant.

This finishes the proof of proposition 8, and with that of the classical degree bound, theorem 1.

3 Comparing Sym and $\wedge$

We want to imitate the proof given above to the case of the exterior algebra. That is, we want to find analogues of the three propositions above and proof them. Hence we will also need analogues of the Hilbert ideal, and the ideals $I(\mathcal{A})$. The (new) Hilbert ideal J' is simply the left ideal generated by the invariants of positive degree. The new form of proposition 7 is the following

Proposition 13. *Suppose the Hilbert ideal is generated by elements $h_1, \dots, h_r \in J'$ (note that these are not necessarily G-invariant). Now the subring of invariants is generated over K by words in $\mathcal{R}(h_1), \dots, \mathcal{R}(h_r)$ over K. In formulas:*

$$\bigwedge(V^*)^G = K\langle \mathcal{R}(h_1), \dots, \mathcal{R}(h_r)\rangle.$$

This is (a special case of) theorem 18 in [Gan21], and the proof is almost the same as that given above.

In order to generalize proposition 8, we have to transfer the object $I(\mathcal{A})$ for subspace arrangements $\mathcal{A} = \{W_1, \dots, W_t\}$. We'll do so by writing $I'(W_i)$ for the left ideal generated by the linear (degree-1)-functions in $I(W_i)$, and write $I'(\mathcal{A}) = \bigcap_{i=1}^t I'(W_i)$. Now the proof of proposition 8 also readily generalizes, and we obtain theorem 20 of [Gan21], which reads

Proposition 14. *Let J' be the Hilbert ideal for the action of G on $E = \bigwedge(V^*)$. Similarly to above, let G act on $\bigwedge(V^* \oplus V^*) = \bigwedge(\mathbf{x}, \mathbf{y})$, trivially on $\mathbf{x}$ and as usual on $\mathbf{y}$. We now have*

$$(I'(\mathcal{A}_G) + (\mathbf{y})) \cap \bigwedge(\mathbf{x}) = J'.$$

As above, the crux is to show that $I'(\mathcal{A}_G)$ has generators of degree $\leq d$. The fact that this was hard in the commutative case might make it seem as if this is impossible in our (non-commutative) situation. To establish a proof, Gandini used that $\bigwedge(V^*)$ and $K[V] = \mathrm{Sym}(V^*)$ have very similar structure.

3.1 Similarities between the symmetric and the exterior algebra

From now on, V is a finite-dimensional K-vector space on which G acts, and K is a field of characteristic 0. In most introductory courses to linear algebra, $\mathrm{Sym}(V)$ and $\bigwedge V$ are introduced alongside each other, and this is no coincidence, as they look very similar:

- Both Sym and $\bigwedge$ are functors on $\mathsf{FinVec} \to \mathsf{Set}$
- They have decomposition into "homogeneous" degree-parts

$$\mathrm{Sym}(V) = K + \mathrm{Sym}^1(V) + \mathrm{Sym}^2(V) + \dots, \quad \bigwedge(V) = K + \bigwedge{}^1 V + \bigwedge{}^2 V + \dots.$$

- They carry the structure of a K-agebra.

We are going to see that both Sym and $\bigwedge$ fit into the category GPoly of *graded polynomial functors*, which will be the object of study for the remainder of the talk.

3.2 (graded) polynomial functors

In this subsection we define the category GPoly, the main references are Appendix I.A of Macdonald's book [Mac98] and the expository article [SS12]. Let V be a finite dimeinsional vector space. A *polynomial on* V is an element of $\mathrm{Sym}(V^*)$. After choosing a basis $(x_1, \ldots, x_n)$ of V^*, a polynomial on V is simply a polynomial in the variables x_i. In particular, every polynomial on V yields a map $V \to K$.

Definition 15 (Polynomial maps)**.** Let V and W be a finite dimensional vector spaces. A *polynomial map* $f : V \to W$ is a mapping such that f can be written as

$$f(v) = \sum_{i=1}^{m} \lambda_i(v) w_i,$$

where λ_i are polynomials on V and $w_i \in W$ are points. Equivalently, a polynomial map $V \to W$ is a morphism of K-algebras in the reverse direction, i.e., an element of $\mathrm{Hom}_{K\text{-Alg}}(\mathrm{Sym}(W^*), \mathrm{Sym}(V^*))$.

Remark. The second definition is very terse and does not make it evident how to recreate the "map" $V \to W$. This can be done via

$$V \cong \mathrm{maxSpec}(\mathrm{Sym}(V^*)) \to \mathrm{maxSpec}(\mathrm{Sym}(W^*)) \cong W.$$

We can now define the category of Polynomial functors.

Definition 16 (Category of Polynomial functors)**.** A *polynomial functor* is a functor $\mathsf{FinVec} \to \mathsf{FinVec}$, such that for all finite vector spaces V and W, the induced map

$$\mathrm{Hom}(V, W) \to \mathrm{Hom}(F(V), F(W))$$

is a polynomial map. That is, a polynomial functor F assigns to any finite dimensional vector space V a new finite dimensional vector space $F(V)$.

Examples.

1. For any $k \geq 0$, Sym^k is an element of Poly. Indeed, assume for simplicity that $V = K^n$ (with basis (x_i)), and $W = K^m$ (with basis (y_j)). Let $\phi : V \to W$ be any linear map and let a_{ij} be the entries of the corresponding matrix, i.e., $\phi(x_i) = \sum_i a_{ij} y_j$. The induced map $\mathrm{Sym}^k(\phi) : \mathrm{Sym}^k(V) \to \mathrm{Sym}^k(W)$ is the map defined by

$$\mathrm{Sym}^k(\phi)(x_{i_1} \cdots x_{i_k}) = \phi(x_{i_1}) \cdots \phi(x_{i_k}) = \left(\sum a_{i_1 j} y_j\right) \cdots \left(\sum a_{i_k j} y_j\right).$$

Note that the space $\mathrm{Hom}(V, W)^*$ is generated by the functions $\phi \mapsto a_{ij}$. The expression above is a polynomial in a_{ij} for each coefficient of the monomials $y_{j_1} \cdots y_{j_k}$. Hence Sym^k is a polynomial functor.

2. The same is true for $\bigwedge^k$.

Even more is true, we have $\mathrm{Sym}^k(\lambda\phi) = \lambda^k \, \mathrm{Sym}^k(\phi)$. Mappings like this are called *homogenous*.

Definition 17 (Homogenous polynomial map)**.** We say that a polynomial map $f : V \to W$ is homogenous of degree t if $f(\lambda x) = \lambda^t f(x)$ for all $x \in V$.

We are ready to define the category of *graded polynomial functors*

Definition 18 (Graded polynomial functors)**.** A graded polynomial functor is a functor $F : \mathsf{FinVec} \to \mathsf{Vec}$ such that F has a decomposition $F = \bigoplus_{d \in \mathbb{N}_0} F_d$, where each summand F_d is a homogenous polynomial functor of degree d.

Our calculation above shows that Sym and $\bigwedge$ are objects in GPoly. They even are K-*algebra objects* in this category, which is to say, they carry a *natural* K-algebra structure evaluated at any $V \in \mathsf{FinVec}$. There is one more important example we'll need: Let W be a finite dimensional vector space and let $\mathcal{A} = \{W_1, \ldots, W_t\}$ be a subspace arrangement on W. Let $\mathcal{A} \otimes V$ be the subspace arrangement on $W \otimes V$ given by $\{W_1 \otimes V, \ldots, W_t \otimes V\}$. Then the functors $\mathcal{I}$ and $\mathcal{I}'$, given by

$$\mathcal{I}_{\mathcal{A}}(V) = I(\mathcal{A} \otimes V) \subset \mathrm{Sym}(V^*) \quad \text{ and } \quad \mathcal{I}'_{\mathcal{A}}(V) = I'(\mathcal{A} \otimes V) \subset \bigwedge(V^*)$$

are graded polynomial functors (this is proposition 7.2 in [Gan21]).

3.3 The structure of GPoly

In this last part of the talk, we state some results, and do not give any proofs.
We first talk a bit about partitions.

Definition 19 (Partition)**.** Let k be an integer. A partition of k is a multiset of integers $\lambda = \{l_1, \ldots, l_r\}$ with $\sum_{i=1}^r l_i = k$. In this situation we write $\lambda \vdash k$.

Definition 20 (Young Diagrams)**.** Let $l_1, \ldots, l_r$ be a decending sequence of (positive) natural numbers, which is the same as a partition of $k = \sum l_i$. The associated Young diagram is the following alignment of boxes. [IMAGE]Note that any Young diagram can be transposed, which yields a new partition $\lambda^\dagger \vdash k$.

Theorem 21 (Structure of GPoly, [Mac98])**.** *The category* GPoly *is abelian and semisimple. The simple objects are indexed by partitions: For each integer k and each partition λ of k, there is a* Schur-functor S_λ, *which is a homogenous polynomial functor of degree k.*

In practice, this means that any graded polynomial functor F can be written as

$$F = \bigoplus_{k \in \mathbb{N}} \bigoplus_{\lambda \vdash k} S_\lambda^{\oplus a_\lambda}.$$

Remark. Categories with this structure seem to appear a lot. For example, the category of S_*-representations (with objects $(V_n)_{n \in \mathbb{N}}$, where V_n is a representation of S_n) is equivalent to GPoly. For more on this, see [SS12].
Example. The functors Sym^k and $\bigwedge^k$ are Schur-functors! The functor Sym^k belongs to the partition (k), the functor $\bigwedge^k$ belongs to $(1, \ldots, 1)$.
One might hope that transposition of partitions somehow descents to a transposition operation on GPoly. This hope is fulfilled!

Theorem 22. *There is an additive endofunctor* $\Omega : \mathsf{GPoly} \to \mathsf{GPoly}$ *such that* $\Omega(S_\lambda) = S_{\lambda^\dagger}$.

One construction of this functor can be found in Gandini's paper, [Gan22], but there are many interesting remarks in [SS12]. In this last paper, the authors show that Ω is even an equivalence of symmetric monoidal categories. In particular, it is exact and preserves module- and algebra objects. Remember that we want to proof that the ideal $I'(\mathcal{A}_G) = \mathcal{I}'_{\mathcal{A}_G}(K)$ is generated in degree $\leq \#G$. We know the corresponding statement for $I(\mathcal{A}_G)$, and Gandini shows that we only have to apply Ω in order to obtain the statement for $I'(\mathcal{A}_G)$:

Proposition 23. *([Gan22, Proposition 6.1]) If $\mathcal{R} \in \mathsf{GPoly}$ is an algebra object and $\mathcal{M}$ is a $\mathcal{R}$-module object generated of regularity $\leq t$, then $\Omega(\mathcal{M})$ is generated in degree $\leq t$.*

Here again we couldn't get around *regularity.* We need a final lemma to conclude.

Lemma 24. *We have $\Omega(\mathcal{I}_{\mathcal{A}}) \cong \mathcal{I}'_{\mathcal{A}}$.*

Proof. [I didn't find a proof of this, Gandini simply uses this statement in [Gan21]. Perhaps we can write $\mathcal{I}_{\mathcal{A}}$ and $\mathcal{I}'_{\mathcal{A}}$ as some Kernels that get mapped into one another.] $\square$

This concludes the proof of Noether's degree bound for the exterior algebra.

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