Explicit Aspects of Non-Abelian Lubin-Tate Theory

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June 27, 2024

Contents

1 Introduction

1.1 Notation

We denote the category of sets with (Set) and the category of (unital, commututative) rings with (Ring). If A is a ring, we write (A-Alg) for the category of A-algebras, and (A-Mod) for the category of A-modules.

If $f(T) = c_1T + c_2T^2 + \cdots \in A[\![T]\!]$, we write $f^k(T)$ for the k-fold self composite of f, that is

$$f^k(T) = \underbrace{f(f(\cdots(f}_{k-\text{fold}}(T))\cdots).$$

In order to not confuse this with taking multiplicative powers, we write

$$f(T)^k = \underbrace{f(T)f(T)\cdots f(T)}_{k-\text{fold}}.$$

1.2 Acknowledgements

2 Formal Modules

This section will serve as an introduction to formal groups and formal modules. Formal groups (or rather, formal group laws) were first introduced by Salomon Bochner in 1946 as a natural means of studying Lie Groups over fields of characteristic 0, cf. [Boc46]. The study of formal groups later became interesting for its own right, with pioneering works of Lazard [Laz55]. blabla

2.1 Basic Notions

As promised in the introduction, we begin by defining formal group laws. For now, let A be any ring.

Definition 2.1.1 (Formal Group Laws of arbitrary dimension)**.** A (commutative) formal group law of dimension n over R is a tuple of power series $F = (F_1, \dots, F_n)$ with

$$F_i(X_1, \dots, X_n, Y_1, \dots, Y_n) \in R[\![X_1, \dots, X_n, Y_1, \dots, Y_n]\!], \quad 1 \leq i \leq n$$

such that $F_i(\mathbf{X}, \mathbf{Y}) \equiv X_i + Y_i$ modulo degree ≥ 2 and the following equalities are satisfied:

1. $F(F(\mathbf{X}, \mathbf{Y}), \mathbf{Z}) = F(\mathbf{X}, F(\mathbf{Y}, \mathbf{Z}))$.
2. $F(\mathbf{X}, \mathbf{0}) = \mathbf{X}$.
3. $F(\mathbf{X}, \mathbf{Y}) = F(\mathbf{Y}, \mathbf{X})$.

Here, and in the sequel, we abbreviate $\mathbf{X} = (X_1, \dots, X_n)$, et cetera. Given a formal group F of dimension n and a formal group law G of dimension m, a morphism $F \to G$ is a m-tuple $f = (f_1, \dots, f_m)$ of power series $f_i \in R[\![X_1, \dots, X_n]\!]$ such that $f(0) = 0$ and

$$G(f(\mathbf{X}), f(\mathbf{Y})) = f(F(\mathbf{X}, \mathbf{Y})).$$

For any n-dimensional formal module F, the identity is given by the morphism id_F with components $\mathrm{id}_{F,i}(\mathbf{X}) = X_i$. Composition of morphisms is given by composition of tuples of power-series. This yields the category of formal modules of arbitrary dimension over R, which we denote by $(\mathrm{FGL}^{\mathrm{arb}}/R)$. We will mostly be concerned with the full subcategory of one-dimensional formal groups, which we denote by (FGL/R).

Lemma 2.1.2. *1. The set* $\mathrm{Hom}_{(\mathrm{FGL}^{\mathrm{arb}}/R)}(F, G)$ *is an abelian group with addition* $f + g = G(f, g)$*. In particular,* $(\mathrm{FGL}^{\mathrm{arb}}/R)$ *is pre-additive (cf. [Stacks, Tag 00ZY]).*

2. Furthermore, $(\mathrm{FGL}^{\mathrm{arb}}/R)$ *admits finite products. Thereby it is an additive category (cf. [Stacks, Tag 0104]). The unique final and initial object of* $(\mathrm{FGL}^{\mathrm{arb}}/R)$ *is the unique* 0*-dimensional formal* A*-module law.*

3. In particular $\mathrm{End}_{(\mathrm{FGL}^{\mathrm{arb}}/R)}(F)$ *is a (possibly non-commutative) ring.*

Example. Let us introduce the following two formal group laws.

- *The additive formal group law.* Write $\widehat{\mathbb{G}}_a$ for the formal group law with addition given by $\widehat{\mathbb{G}}_a(X,Y) = X + Y$.
- We write $\widehat{\mathbb{G}}$ for the formal group law associated with the with $\widehat{\mathbb{G}}(X,Y) = X+Y+XY$. Note that $\widehat{\mathbb{G}}(X,Y) = (X+1)(Y+1) - 1$

Next up is the definition of formal A-module laws. Naively, we would like to define formal A-module laws as formal group laws F with A-module structure, i.e. a morphism of rings $[\cdot]_F : A \to \mathrm{End}_{(\mathrm{FGL}^{\mathrm{arb}}/R)}(F)$. But there is a subtlety, which becomes evident after defining the Lie-algebra of a formal group law.

Definition 2.1.3 (Lie-algebra of formal group law)**.** Let Lie : $(\mathrm{FGL}^{\mathrm{arb}}/R) \to (\mathrm{Ab})$ be the functor taking an n-dimensional formal group law F to the R-module

$$\mathrm{Lie}(F) = \mathrm{Hom}_{(R\text{-Mod})}\left(\frac{(X_1,\ldots,X_n)}{(X_1,\ldots,X_n)^2}, R\right)$$

Given an m-dimensional group law G and a morphism $f : F \to G$, $\mathrm{Lie}(f)$ is the induced morphism

$$\mathrm{Lie}(F) \to \mathrm{Lie}(G), \quad \psi \mapsto \left(S_j \mapsto \psi(\overline{f_j})\right) \in \mathrm{Hom}_{(R\text{-Mod})}\left(\frac{(X_1,\ldots,X_n)}{(X_1,\ldots,X_n)^2}, R\right),$$

where $\overline{f_j}$ is the reduction of f_j mod $(\mathbf{X})^2$.

We have a canonical basis on both sides, and writing $\mathrm{Lie}(F) = R^n$, $\mathrm{Lie}(G) \cong R^m$, the induced map $\mathrm{Lie}(f) : R^n \to R^m$ is given by multiplication with the matrix

$$\left(\frac{\partial f_i}{\partial X_j}(0)\right)_{\substack{1\le i\le m\\ 1\le j\le n}}.$$

Given a one-dimensional group law $F \in (\mathrm{FGL}/R)$, the condition that $F(X,Y) \equiv X + Y$ modulo degree ≥ 2 enforces that the induced map $\mathrm{End}(F) \xrightarrow{\mathrm{Lie}} \mathrm{End}(R)$ is a morphism of rings. If we are given $[-]_F : A \to \mathrm{End}_{(\mathrm{FGL}/R)}(F)$, this A-module structure on F yields an A-module structure on R, given by the composition

$$A \xrightarrow{[\cdot]_F} \mathrm{End}(F) \xrightarrow{\mathrm{Lie}} \mathrm{End}(R), \quad a \mapsto \mathrm{Lie}([a]_F)$$

This is a morphism of rings, and we obtain an A-algebra structure on R. This motivates the following definition.

Definition 2.1.4 (Formal A-Module Law of arbitrary dimension)**.** Let R be an A-algebra with structure morphism $j : A \to R$. A formal A-module law over R of dimension n is given by the data of a formal n-dimensional group law F over R and a morphism of rings

$$A \to \mathrm{End}_{(\mathrm{FGL}^{\mathrm{arb}}/R)}(F), \quad a \mapsto ([a]_{F,i})_{1\le i\le n} \in (R[\![X_1,\ldots,X_n]\!])^n$$

such that $[a]_{F,i}(\mathbf{X}) \equiv j(a)X_i$ modulo terms of degree ≥ 2. Morphisms between formal A-modules of arbitrary dimension are morphisms of formal groups respecting the A-module structure. The resulting category is denoted $(A\text{-FML}^{\text{arb}}/R)$. As before, the full subcategory of one-dimensional formal A modules over R is denoted $(A\text{-FML}/R)$.

Note that $(\text{FGL}/R) \cong (\mathbb{Z}\text{-FML}/R)$. Slightly abusing notation, we usually do not explicitely mention the A-structure when referring to formal module laws, simply writing $F \in (A\text{-FML}/R)$, for example.

The following lemma explains the functoriality of the assignment $R \mapsto (A\text{-FML}^{\text{arb}}/R)$.

Lemma 2.1.5. *The assignment $R \mapsto (A\text{-FML}^{\text{arb}}/R)$ is functorial in the following sense. If $i : R \to R'$ is a morphism of A-algebras, we obtain a functor*

$$(A\text{-FML}^{\text{arb}}/R) \to (A\text{-FML}^{\text{arb}}/R'), \quad F \mapsto F \otimes_R R',$$

where $F \otimes_R R'$ is the formal A-module law obtained by applying i to the coefficients of the formal power series representing the A-module structure of F.

Note that every n-dimensional formal module law $F \in (A\text{-FML}^{\text{arb}}/R)$ yields a functor

$$(R\text{-Alg}) \to (A\text{-Mod}), \quad S \mapsto \text{Nil}(S)^n, \tag{2.1}$$

where $\text{Nil}(S)^n$, the set of n-tuples of nilpotent elements of S, is equipped with addition and scalars given by

$$s_1 + s_2 = F(s_1, s_2) \in \text{Nil}(S)^n, \quad as = [a]_F(s) \in \text{Nil}(S)^n.$$

This construction yields a functor

$$(A\text{-FML}/R) \to \text{Fun}((R\text{-Alg}), (A\text{-Mod})), \tag{2.2}$$

where Fun denotes the functor category.

Passing from discrete R-algebras to admissible R-algebras (cf. Definition A.0.1), this construction extends naturally to a functor

$$(A\text{-FML}/R) \to \text{Fun}((R\text{-Adm}), (A\text{-Mod})), \quad F \mapsto \text{Spf}\, R[\![\mathbf{T}]\!],$$

where we equip $\text{Spf}\, R[\![\mathbf{T}]\!]$ with the structure of an A-module object using the endomorphisms coming from F. Following this line of thought leads naturally to the definition of formal modules.

Definition 2.1.6 (Formal Groups and Formal Modules.)**.** Given an A-scheme X, we define the category $(A\text{-FM}^{\text{arb}}/X)$ as follows. Objects are A-module objects $\mathcal{F}$ in the category of formal schemes over X, having the property that there is a cover of X by Zariski-open affine subsets $U_i = \text{Spec}(R_i)$ such that $\mathcal{F} \times_X U_i$ is isomorphic to $\text{Spf}\, R_i[\![X_1, \ldots, X_n]\!]$ and the induced A-module structure on $\text{Spf}\, R_i[\![X_1, \ldots, X_n]\!]$ yields a formal A-module law on R_i. Given $\mathcal{F}, \mathcal{G} \in (A\text{-FML}^{\text{arb}}/X)$, a morphism $\phi : \mathcal{F} \to \mathcal{G}$ is the same as a morphism of A-module objects in the category of formal schemes over X. Again, we denote the full subcategory of one-dimensional formal A-modules over X by $(A\text{-FM}/X)$.

Remark. Formal schemes (over a base an A-scheme X, say) locally isomorphic to $\operatorname{Spf} \mathcal{O}_X(U)[\![\mathbf{T}]\!]$ are sometimes called Formal Lie Varieties . Equivalently to the definition above, we could reference
have defined formal A-modules as A-module objects in the category of Formal Lie Varieties, such that the A-module structure on the tangent space at the identity agrees with the usual one.

Definition 2.1.7 (Coordinate)**.** Let $\mathcal{F}$ be a formal A-module over X. The choice of a cover $\sqcup_{i\in I} \operatorname{Spec}(R_i) \to X$ together with isomorphisms $\mathcal{F} \times_X \operatorname{Spec}(R_i) \cong \operatorname{Spf}(R_i[\![\mathbf{T}]\!])$ will be referred to as a coordinate of $\mathcal{F}$.

Of course there is a functor

$$\mathrm{FG} : (A\text{-FML}^{\mathrm{arb}}/R) \to (A\text{-FM}^{\mathrm{arb}}/R),$$

essentially forgetting the choice of module law. The observation of Lemma 2.1.5 translates to formal modules, a morphism $j : R \to R'$ yields a functor

$$(A\text{-FM}/R) \to (A\text{-FM}/R'), \quad \mathcal{F} \mapsto \mathcal{F} \otimes_R R'.$$

Definition 2.1.8 (Lie functor)**.** The functor Lie descents to a functor

$$\mathrm{Lie} : (A\text{-FM}^{\mathrm{arb}}/X) \to (\mathcal{O}_X\text{-QCoh}),$$

given by locally describing a formal A-module $\mathcal{F}$ via formal group laws and gluing the local data. Alternatively, it arises from sending sending a formal A-module $\mathcal{F}$ to $(\mathcal{I}/\mathcal{I}^2)^\vee$, where $\mathcal{I}$ is the ideal associated to the closed immersion $[0]_\mathcal{F} : X \to \mathcal{F}$.

Lemma 2.1.9. *A map $\phi : \mathcal{F} \to \mathcal{G}$ of formal A-modules (of arbitrary dimension) over X is an isomorphism if and only if the induced morphism of Lie-algebras $\mathrm{Lie}(\phi) : \mathrm{Lie}(\mathcal{F}) \to \mathrm{Lie}(\mathcal{G})$ is an isomorphism.*

Proof. This is easily verified in the one-dimensional situation after choosing coordinates. The general case adds no complication. □

Example. The additive group law $\widehat{\mathbb{G}}_a$ extends to a formal A-module over an affine base $\operatorname{Spec} R$ by setting

$$[a]_{\widehat{\mathbb{G}}_a}(T) = aT$$

for $a \in A$. More generally, we obtain a formal A-module over an arbitrary base scheme X over A.

Over $\mathbb{Z}_p$, the formal group $\widehat{\mathbb{G}}$ extends to a formal $\mathbb{Z}_p$-module as follows. As a functor, $\widehat{\mathbb{G}}$ is isomorphic to the assignment

$$(\mathbb{Z}_p\text{-Adm}) \to (\mathrm{Ab}), \quad S \mapsto 1 + S^{\circ\circ} \subset S^\times.$$

Here, we equipped $\mathbb{Z}_p$ with the p-adic topology. The subgroup $1 + S^{\infty}$ naturally carries the structure of a $\mathbb{Z}_p$-module. Indeed, for $k \in \mathbb{N}$, we have

$$(1+s)^{p^k} = 1 + p^k s + \binom{p^k}{2} s^2 + \cdots + s^{p^k},$$

and given $s \in S^{\infty}$, this is of the form $1 + o(1)$ as k gets large. In particular, if $x = a_0 + a_1 p + a_2 p^2 + \cdots \in \mathbb{Z}_p$, expressions of the form

$$(1+s)^x = \prod_{i=1}^{\infty} (1+s)^{a_k p^k}$$

make sense by Lemma A.0.3. This gives $\widehat{\mathbb{G}}_{m,\mathbb{Z}_p}$ the structure of a formal $\mathbb{Z}_p$-module. In the upcoming subsection, we discuss how this is the simplest example of a whole family of formal modules constructed by Lubin and Tate. In section 3 we explain applications of these formal modules to local class field theory.

Definition 2.1.10 (Formal Module associated to R-module)**.** Suppose that M is a finite projective R-module. Then we write $\widehat{\mathbb{G}}_a \otimes M$ for the additive formal A-module associated to M over R. As a formal scheme, this formal module is given by

$$\widehat{\mathbb{G}}_a \otimes M \cong \operatorname{Spf} R[\![M^{\vee}]\!],$$

where $R[\![M^{\vee}]\!]$ denotes the completion of $\operatorname{Sym}_R(M^{\vee})$ with respect to the ideal generated by $M^{\vee}$. The (formal) A-module structure is the canonical additive one. Note that $\operatorname{Lie}(\widehat{\mathbb{G}}_a \otimes M) = M$ by design. More generally, if X is a quasi-compact and quasi-separated A-scheme and $\mathcal{M}$ is a finite locally free quasi-coherent $\mathcal{O}_X$-module, this construction yields a formal A-module $\widehat{\mathbb{G}}_a \otimes \mathcal{M}$ over X.

Remark. If $R \to R'$ is a morphism of rings that turns R' into a (say) finite free R-algebra, the definition above overloads the expression $\widehat{\mathbb{G}}_a \otimes_R R'$. In order to disambiguate, we usually denote the additive formal A-module over R' by $\widehat{\mathbb{G}}_{a,R'}$.

2.2 Lubin–Tate Formal Module Laws

We sketch the construction of a family of formal modules introduced by Lubin and Tate in [LT65].

Let A be a complete discrete valuation ring with finite residue field k, set $q = \#k$ and let $\varpi \in A$ be a choice of a uniformizer. Write $\mathcal{F}_{\varpi,h}$ for the following set of power series

$$\mathcal{F}_{\varpi} := \{ f \in \mathcal{O}_K[\![T]\!] \mid f \equiv \varpi T \pmod{T^2} \text{ and } f \equiv T^{q^n} \pmod{\varpi} \}.$$

The construction of Lubin–Tate formal module laws rests on the following lemma, which is Lemma 1 in [LT65].

Lemma 2.2.1. *Let $f(T)$ and $g(T)$ be elements of $\mathcal{F}_{\varpi,h}$ and let $L(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_i X_i$ be a linear form with coefficients in A. Then there exists a unique series $F(X_1, \ldots, X_n)$ with* *coefficients in A such that*

$$F(X_1, \dots, X_n) \equiv L(X_1, \dots, X_n) \pmod{T^2},$$
$$\textit{and}$$
$$f(F(X_1, \dots, X_n)) = F(g(X_1), \dots, g(X_n)).$$

As a direct consequence, we obtain the following useful result.

Lemma 2.2.2. *Let $f \in \mathcal{F}_{\varpi,h}$. Then there is a unique formal A-module law F_f over A with $[\varpi]_F(T) = f(T)$.*

Proof. In the above Lemma, set $L(X,Y) = X + Y$ and $g = f$ to uniquely determine the power series F_f. The same Lemma yields unique power series $[a]_{F_f}(T) \in R[\![T]\!]$ by setting $L(T) = aT$, $g = f$. It is routine to check that $(F_f, ([a]_f)_{a\in A})$ is a formal A-module law, cf. [LT65]. □

Definition 2.2.3 (Lubin–Tate Module Law)**.** We refer to module laws arising by the construction above as Lubin–Tate module laws.

Furthermore, attached to each $a \in \mathcal{O}_K$ and $f, g \in \mathcal{F}_{\varpi,h}$, we find unique $[a]_{f,g}(T) \in \mathcal{O}_K[\![T]\!]$ satisfying

$$[a]_{f,g}(T) \equiv aT \pmod{(T)^2} \quad \text{and} \quad f([a]_{f,g}(T)) = [a]_{f,g}(g(T)). \tag{2.3}$$

We now have the following theorem.

Theorem 2.2.4 (Lubin–Tate Formal $\mathcal{O}_K$-Module Laws)**.** *Let K be a local field with ring of integers $\mathcal{O}_K$. For any choice of uniformizer $\varpi \in \mathcal{O}_K$ and any $f \in \mathcal{F}_{\varpi,h}$, the family of power series $(F_f, ([a]_{f,f})_{a\in\mathcal{O}_K})$ gives rise to a formal $\mathcal{O}_K$-module law over $\mathcal{O}_K$. For $f, g \in \mathcal{F}_{\varpi,h}$, the formal $\mathcal{O}_K$-module laws F_f and F_g are canonically isomorphic, via the morphism induced by $[1]_{f,g} \in \mathcal{O}_K[\![T]\!]$.*

Proof. See Theorem 1 of [LT65] and the subsequent discussion. □

In particular, up to canonical isomorphism, there is only one Lubin–Tate formal $\mathcal{O}_K$-module law over $\mathcal{O}_K$ attached to the choice of the uniformizer $\varpi \in \mathcal{O}_K$.

Example. If $K = \mathbb{Q}_p$, this reconstructs the multiplicative formal $\mathbb{Z}_p$ module $\widehat{\mathbb{G}}$ constructed above. Indeed, we have

$$\mathcal{F}_p = \{f \in \mathbb{Z}_p[\![T]\!] \mid f(T) \equiv T^p \bmod p \text{ and } f(T) \equiv pT \bmod (T)^2\},$$

implying that $f(T) = (1+T)^p - 1$ lies in $\mathcal{F}_p$. One quickly checks that

$$F_f(X,Y) = (1+X)(1+Y) - 1 = X + Y + XY \in \mathbb{Z}_p[\![X,Y]\!]$$

is the addition law associated to f, and that for $a \in \mathbb{Z}_p$, the power series

$$[a]_{f,f} = (1+T)^a - 1 \in \mathbb{Z}_p[\![T]\!]$$

satisfies the condition of (2.3).

2.3 Logarithms

Again, A is a complete discrete valuation ring with uniformizing parameter ϖ and finite residue field $k = A/\varpi A$. We write q for the cardinality of k and K for the field of fractions of A. Let R be a local, admissible A-algebra with structure map $i : A \to R$.

We review results from section 2 and 3 of [GH94]. Suppose that $F = (F_1, \dots, F_n)$ is an n-dimensional formal A-module law over a R. We abbreviate $\mathbf{X} = (X_1, \dots, X_n)$, $\mathbf{Y} = (Y_1, \dots, Y_n)$, etc.

Definition 2.3.1 (Invariant Differentials)**.** The module $\omega(F)$ of invariant differentials is the submodule of the R-module of differentials

$$\Omega_{R[\![T_1,\dots,T_n]\!]/R} \cong \bigoplus_{i=1}^{n} R[\![T_1, \dots, T_n]\!] \, \mathrm{d}T_i,$$

consisting of those $\omega \in \omega(F)$ satisfying

$$\omega(F(\mathbf{X}, \mathbf{Y})) = \omega(\mathbf{X}) + \omega(\mathbf{Y}) \quad \text{and} \quad \omega([a]_F(\mathbf{X})) = a\omega(\mathbf{X}). \tag{2.4}$$

for all $a \in A$.

It is possible to explicitly calculate a basis for the R-module $\omega(F)$, which we now explain. Let

$$A(\mathbf{X}, \mathbf{Y}) \in \mathrm{Mat}_{n\times n}(R[\![\mathbf{X}, \mathbf{Y}]\!])$$

denote the matrix $((\partial/\partial X_j)F_i(\mathbf{X}, \mathbf{Y}))_{i,j}$, the derivative of $F(\mathbf{X}, \mathbf{Y})$ with respect to $\mathbf{X}$. Set $B(\mathbf{Y}) = A(0, \mathbf{Y})$. Then B is a unit in $\mathrm{Mat}_{n\times n} R[\![\mathbf{Y}]\!]$; and we write $(C_{ij}(\mathbf{Y}))$ for the components of $B(\mathbf{Y})^{-1}$. We now construct

$$\omega_i := \sum_{j=1}^{n} C_{ij}(\mathbf{X}) \, \mathrm{d}X_j \in \Omega_{R[\![\mathbf{X}]\!]/R}$$

for $1 \le i \le n$. By definition we have

$$C_{ij}(0) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

Checking that ω_i is an invariant differential is a matter of applying the chain rule.

Proposition 2.3.2. *The R-module $\omega(F)$ is free of rank n generated by invariant differentials $\omega_1, \omega_2, \dots, \omega_n$.*

Proof. This is [Hon70, Proposition 1.1]. □

Example. The invariant differentials for $\widehat{\mathbb{G}}_a$ are spanned by the form $\mathrm{d}X$. The invariant differentials for $\widehat{\mathbb{G}}$ are spanned by the form $\omega_1(X) = \frac{1}{1+X}\,\mathrm{d}X$.

By the Proposition above and Equation (2.5), we may define a pairing

$$\omega(F) \times \mathrm{Lie}(F) \to R, \quad \langle X_i, \omega_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

This pairing is independent of the parametrization of F. In particular, it descents to a pairing defined for formal modules $\mathcal{F} \in (A\text{-FM}^{\mathrm{arb}}/R)$, and we have a natural isomorphism $\omega(\mathcal{F}) \cong \mathrm{Hom}_R(R, \mathrm{Lie}(\mathcal{F}))$.
Let $\widehat{\mathbb{G}}_a$ be the additive formal A-module over R. There is a map

$$\mathrm{d}_F : \mathrm{Hom}_{(A\text{-FML}/R)}(F, \widehat{\mathbb{G}}_{a,R}) \to \omega(F), \quad f \mapsto \mathrm{d}f(\mathbf{X}) \tag{2.6}$$

which is a map of R-modules if we equip the left hand side with the R-module structure coming from the natural action of $R \subset \mathrm{End}(\widehat{\mathbb{G}}_a)$.

Proposition 2.3.3. *1. If R is a flat A-algebra, the map d_F is injective.*

2. If R is a K-algebra, the map d_F is an isomorphism.

Proof. This is [GH94, Proposition 3.2]. □

Suppose now that $F \in (A\text{-FML}^{\mathrm{arb}}/R)$ is a formal module law of dimension n over a flat A-algebra R. Let $\omega_1, \ldots, \omega_n$ be the distinguished basis for $\omega(F)$ constructed above. By the previous proposition, there are unique power series $f_i(\mathbf{X}) \in (R \otimes_A K)[\![\mathbf{X}]\!]$ furnishing homomorphisms $F \otimes (R \otimes_A K) \to \widehat{\mathbb{G}}_{a, R\otimes_A K}$ of formal A-module laws and satisfying

$$\mathrm{d}_F f_i(\mathbf{X}) = \omega_i(\mathbf{X}) \in \omega(F).$$

Definition 2.3.4 (Logarithm and Exponential)**.** The induced morphism of formal group laws

$$f = (f_1, \ldots, f_n) : F \otimes (R \otimes_A K) \to \widehat{\mathbb{G}}_a^n \otimes_R (R \otimes K)$$

is called the logarithm attached to F, we write $\log_F(\mathbf{X}) \in ((R \otimes_A K)[\![\mathbf{X}]\!])^n$ for the corresponding collection of power series. The inverse of $\log_F(\mathbf{X})$ is called the exponential attached to F, denoted $\exp_F(\mathbf{X})$. We have $\mathrm{Lie}(\log_F) = \mathrm{Lie}(\exp_F) = \mathrm{id}$, so $\log_F$ and $\exp_F$ are isomorphisms.

Lemma 2.3.5. *Let F and G be formal A-module laws over R, with $\dim F = n$ and $\dim G = m$. Let $\phi : F \to G$ be a morphism. Then the diagram*

$$\begin{tikzcd} F \otimes (R \otimes_A K) \arrow[r, "\log_F"] \arrow[d, "\phi"'] & \widehat{\mathbb{G}}_a \otimes (\mathrm{Lie}(F) \otimes_A K) = \widehat{\mathbb{G}}^n_{a, R\otimes_A K} \arrow[d, "\mathrm{Lie}(\phi)"] \\ G \otimes (R \otimes_A K) \arrow[r, "\log_G"] & \widehat{\mathbb{G}}_a \otimes (\mathrm{Lie}(G) \otimes_A K) = \widehat{\mathbb{G}}^m_{a, R\otimes_A K} \end{tikzcd}$$

commutes. In particular, attached to any $\mathcal{F} \in (A\text{-FM}^{\mathrm{arb}}/R)$ comes a natural morphism

$$\log_{\mathcal{F}} : \mathcal{F} \otimes (R \otimes_A K) \to \widehat{\mathbb{G}}_{a, R\otimes_A K} \otimes (\mathrm{Lie}(\mathcal{F}) \otimes_R (R \otimes_A K)).$$

Proof. The square commutes because $\mathrm{Hom}(\widehat{\mathbb{G}}^n_{a,R\otimes_A K}, \widehat{\mathbb{G}}^m_{a,R\otimes_A K}) = \mathrm{Hom}_{R\otimes_A K}((R\otimes_A K)^n, (R\otimes_A K)^m)$ and $\mathrm{Lie}(\log_G \circ\phi \circ \exp_H) = \mathrm{Lie}(\phi)$. □

Lemma 2.3.6. *Let K be a local field with integers $\mathcal{O}_K$ and a choice of uniformizer $\varpi \in \mathcal{O}_K$, and let F be a Lubin-Tate $\mathcal{O}_K$-module law corresponding to some $f \in \mathcal{F}_\varpi$, cf. Theorem 2.2.4. Let S be an admissible $\mathcal{O}_K$-algebra, and let $s \in S^{\infty}$ be an element such that the series $\log_{\mathcal{F}}(s)$ converges. Then we have $\log_F(s) = 0$ if and only if $[\varpi]^r_F(s) = 0$ for some $r \in \mathbb{N}$.*

Proof. Up to canonical isomorphism, F is a $\mathcal{O}_K$-module law with $[\varpi]_F(T) = \varpi T + T^q$. Now one may check that

$$\log_F(T) = \lim_{r\to\infty} \frac{[\varpi]^r_F(T)}{\varpi^r} = \prod_{i=1}^{\infty} \frac{[\varpi]^i_F(T)}{\varpi[\varpi]^{i-1}_F(T)},$$

where convergence is to be taken coefficient-wise. After plugging in $s \in S^{\infty}$, we see that the product vanishes if and only if $[\varpi]^r_F(s) = 0$ for some $r \in \mathbb{N}$. □

2.4 Formal DVR-Modules over Fields of Finite Characteristic

As above, let A be a discrete valuation ring with uniformizer ϖ and finite residue field k; write q for the cardinality of k. Let K denote the field of fractions of A, and let R be a A-algebra.

We have seen in the previous section that if R is a field extension of K, then any morphism of formal group laws $f : F \to G$ over R is either 0 or an isomorphism, which makes the theory of formal A-modules over R rather bland. If R is an extension of k, the situation becomes more interesting, mainly because there are the non-trivial (relative) Frobenius-endomorphisms $\mathrm{Frob} : F \to F$, sending T to T^q.

We introduce the concept of height, which is an attempt to quantify the disorder introduced by the Frobenius endomorphisms. This also leads to interesting invariants of formal A-modules over local A-algebras R.

Definition 2.4.1 (Height of morphisms of group laws)**.** Assume that R is a field extension of k and $f : F \to G$ is a morphism of formal groups laws over R, given by a formal series $f(T) \in R[\![T]\!]$. If $f = 0$, we say that f has infinite height. If $f \neq 0$, the height of f is defined as the largest integer h such that $f = g(T^{q^h})$ for some power series $g(T) = c_1T + c_2T^2 + \cdots \in R[\![T]\!]$ with $c_1 \neq 0$.

One readily checks that if $f : \mathcal{F} \to \mathcal{G}$ is a morphism of formal groups over a field extension R of k, the height of f does not depend on the choices of group laws on $\mathcal{F}$ and $\mathcal{G}$. This allows us to define the height function attached to f.

Definition 2.4.2 (Height function)**.** Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of formal groups over a scheme X. For a scheme-theoretic point $x \in |X|$, let f_x denote the base-change of f to the residue field of x. The height function attached to f is the upper-semicontinuous function

$$\mathrm{ht}(f) : |X| \to \mathbb{Z}_{\geq 0} \cup \{\infty\}, \quad x \mapsto \mathrm{ht}(f_x). \tag{2.7}$$

It is not hard to see that the height function is additive, that is, we have

$$\mathrm{ht}(f \circ g) = \mathrm{ht}(f) + \mathrm{ht}(g).$$

Definition 2.4.3 (Isogeny)**.** A morphism $f : \mathcal{F} \to \mathcal{G}$ of formal groups over a field k is called an isogeny if $\mathrm{Ker}(f)$ is a represented by a finite free k-scheme. More generally, a morphism of formal A-modules over a base scheme X is an isogeny if and only if $\mathrm{Ker}(f)$ is finite and locally free over X.

Isogenies can be described using the height function.

Lemma 2.4.4. *A morphism $f : \mathcal{F} \to \mathcal{G}$ is a isogeny if and only if the height function* $\mathrm{ht}(f)$ *is locally constant with values in $\mathbb{Z}_{\geq 0}$.*

Definition 2.4.5 (ϖ-divisible A-module)**.** We say that a formal A-module H over X is ϖ-divisible if $[\varpi]_H$ is an isogeny. If X is connected, the height of H is the (constant) height of the endomorphism $[\varpi]_H : H \to H$.

We close this subsection with a discussion about the structure of formal $\mathcal{O}_K$-modules over separably closed field extensions k' of k.

Proposition 2.4.6. *Over k', any two formal $\mathcal{O}_K$-module laws of the same height are isomorphic.*

Proof. [Dri74, Proposition 1.7]. □

In particular, any formal $\mathcal{O}_K$-module of height h is isomorphic to the formal $\mathcal{O}_K$-module F_{norm} with $[\varpi]_{F_{\mathrm{norm}}}(T) = T^{q^h}$. We call this the normalized formal $\mathcal{O}_K$-module.

Proposition 2.4.7. *Suppose that $F \in (\mathcal{O}_K\text{-FML}/k')$. Then $\mathrm{End}_{(A\text{-FM}/k')}(F)$ is isomorphic to the maximal order of the central division algebra D over K of rank h^2 and invariant $\frac{1}{h}$.*

Proof. Also [Dri74, Proposition 1.7]. □

Lemma 2.4.8. *Let $f : F \to G$ be an isogeny of ϖ-divisible formal $\mathcal{O}_K$-module laws over k'. Then there is an integer $n \geq 0$ and an isogeny $g : G \to F$ with*

$$f \circ g = [\varpi^n]_{\mathcal{G}} \quad \textit{and} \quad g \circ f = [\varpi^n]_{\mathcal{F}}.$$

Proof. As the height is additive, we necessarily have $\mathrm{ht}(F) = \mathrm{ht}(G)$, thus by Lemma 2.4.6, we may assume that F and G are given by the normalized formal $\mathcal{O}_K$-module F_{norm}. Write $f(T) = g(T^{q^n})$ for some power series $h(T) = c_1T + c_2T^2 + \dots$, where $c_1 \neq 0$ is a unit in k', and let $g(T) = h^{-1}(T)$ be the formal inverse of h. Now g is a morphism of formal $\mathcal{O}_K$-module laws satisfying $f \circ g(T) = g \circ f(T) = T^{q^n}$. The claim follows. □

2.5 Exact Categories, Extensions of Formal Modules

In this section, we equip the category $(A\text{-FM}^{\mathrm{arb}}/X)$, where A is any ring and X is a quasi-compact and quasi-separated A-scheme, with a notion of short exact sequences. We show that this gives $(A\text{-FM}^{\mathrm{arb}}/X)$ the structure of an exact category in the sense of Quillen [Kel90, Appendix A]. We introduce functors

$$\begin{aligned}\mathrm{Ext}(-,-) &: (A\text{-FM}^{\mathrm{arb}}/X)^{\mathrm{op}} \times (A\text{-FM}^{\mathrm{arb}}/X) \to (\mathrm{Set}) \\ \mathrm{RigExt}(-,-) &: (A\text{-FM}^{\mathrm{arb}}/X)^{\mathrm{op}} \times (A\text{-FM}^{\mathrm{arb}}/X) \to (\mathrm{Set}),\end{aligned}$$

which send a pair $(\mathcal{F}, \mathcal{F}')$ to the set of equivalence classes of extensions (respectively rigidified extensions) of $\mathcal{F}$ by $\mathcal{F}'$. These functors will play a major role in the upcoming discussion.

2.5.1 The Category of Formal Modules is Exact

Before turning our attention to formal modules, we introduce the notion of exact categories, following [Kel90, Appendix A].

Definition 2.5.1 (Exact Category)**.** Let $\mathcal{A}$ be an additive category, and let $\mathcal{E}$ be a class whose members are exact triples of objects connected by arrows

$$X \xrightarrow{i} Y \xrightarrow{d} Z,$$

where i is a kernel of d and d is a cokernel of i. We call a morphism $i : X \to Y$ an inflation if it appears as first component of some $(i, d) \in E$, second components of such pairs are called deflations. We say that the pair $(\mathcal{A}, \mathcal{E})$ is an exact category if $\mathcal{E}$ is closed under isomorphisms and satisfies the following properties.

1. The identity $\mathrm{id}_0 : 0 \to 0$ is a deflation.

2. The composition of two deflations is a deflation.

3. For each $f \in \mathrm{Hom}_{\mathcal{A}}(Z', Z)$, there is a cartesian square

$$\begin{tikzcd} Y' \arrow[r, "d'"] \arrow[d, "f'"'] & Z' \arrow[d, "f"] \\ Y \arrow[r, "d"'] & Z \end{tikzcd}$$

   such that d' is a deflation.

3^{op}. For each $f \in \mathrm{Hom}_{\mathcal{A}}(X, X')$, there is a co-cartesian square

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, "f'"] \\ X' \arrow[r, "i'"'] & Y' \end{tikzcd}$$

   such that i' is an inflation.

As above, suppose that A is any ring and X is a quasi-compact and quasi-separated A-scheme. Let $\mathcal{F}$, $\mathcal{E}$ and $\mathcal{F}'$ be formal A-modules over X. As a primer, we note that the category $(A\text{-FM}^{\mathrm{arb}}/X)$ is additive (essentially by Lemma 2.1.2).

Definition 2.5.2 (Short Exact Sequence)**.** A pair of composable morphisms $\mathcal{F}' \to \mathcal{E} \to \mathcal{F}$ in $(A\text{-FM}^{\text{arb}}/X)$ is called a short exact sequence if the induced sequence

$$0 \to \operatorname{Lie}(\mathcal{F}') \to \operatorname{Lie}(\mathcal{E}) \to \operatorname{Lie}(\mathcal{F}) \to 0$$

is a short exact sequence of $\mathcal{O}_X$-modules. In this case, we write

$$0 \to \mathcal{F}' \to \mathcal{E} \to \mathcal{F}' \to 0.$$

A pair of composable morphisms $F' \to E \to F$ in $(A\text{-FML}^{\text{arb}}/R)$ is called an exact sequence if it is exact after passing to the respective formal modules.

Lemma 2.5.3. *Let R be an A-algebra and let $F, F' \in (A\text{-FML}^{\text{arb}}/R)$ be formal A-module laws of dimensions m and n respectively. Write $\mathcal{F}', \mathcal{F} \in (A\text{-FM}^{\text{arb}}/R)$ for the associated formal A-modules, and suppose that they fit into a exact sequence*

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$$0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{F} \to 0.$$

Write $\mathbf{X}$ for the variables of F' and $\mathbf{Z}$ for those of F. Then there exists a (non-canonical) coordinate on $\mathcal{E}$ giving rise to a formal A-module law E in the variables $(\mathbf{X}, \mathbf{Z})$ such that the induced morphisms of formal module laws are of the form $\alpha(\mathbf{X}) = (\mathbf{X}, 0)$, $\beta(\mathbf{X}, \mathbf{Z}) = \mathbf{Z}$. Furthermore, the formal A-module law E is of the form

$$\begin{gathered} E((\mathbf{X}_1, \mathbf{Z}_1), (\mathbf{X}_2, \mathbf{Z}_2)) = (F'(\mathbf{X}_1, \mathbf{X}_2) +_{F'} \Delta(\mathbf{Z}_1, \mathbf{Z}_2), F(\mathbf{Z}_1, \mathbf{Z}_2)) \\ \textit{and} \\ [a]_E(\mathbf{X}, \mathbf{Z}) = ([a]_{F'}(\mathbf{X}) +_{F'} \delta_a(\mathbf{Z}), [a]_F(\mathbf{Z})). \end{gathered} \tag{2.8}$$

for some m-tuple of power series $\Delta \in (R[\![\mathbf{Z}_1, \mathbf{Z}_2]\!])^m$, $\delta_a \in (R[\![\mathbf{Z}]\!])^m$.

Proof. The construction of E is sketched in [GH94, Proposition 6.5]. We know that $\mathcal{E} \cong \operatorname{Spf} R[\![M]\!]$ for some free R-module M of rank $m+n$. As we have a short exact sequence on Lie-algebras, we may apply the formal implicit function theorem to obtain a section reference/ $\sigma : \mathcal{F} \to \mathcal{E}$ of $\beta : \mathcal{E} \to \mathcal{F}$. The datum of the morphisms α and σ is equivalent to morphisms

$$\alpha^\flat : R[\![M]\!] \to R[\![\mathbf{X}]\!] \quad \text{and} \quad \sigma^\flat : R[\![M]\!] \to R[\![\mathbf{Z}]\!]$$

on affines. Taking their sum, we obtain a morphism $R[\![M]\!] \to R[\![\mathbf{X}, \mathbf{T}]\!]$. On Lie-algebras, this morphism recovers the isomorphism $\operatorname{Lie}(\mathcal{E}) \cong \operatorname{Lie}(\mathcal{F}') \oplus \operatorname{Lie}(\mathcal{F})$ induced by $\operatorname{Lie}(\sigma)$. In particular, $\sigma^\flat + \alpha^\flat$ is an isomorphism in degree 1, hence an isomorphism. This yields the desired coordinate $\mathcal{E} \cong \operatorname{Spf} R[\![\mathbf{X}, \mathbf{Z}]\!]$. The fact about the structure of the formal A-module law E follows quickly from the fact that α and β are morphisms of formal A-module laws. $\square$

Let's turn our attention to the power series $(\Delta, (\delta_a)_{a \in A})$ appearing in the above Lemma. They satisfy certain conditions.

Definition 2.5.4 (Symmetric 2-cocycles)**.** Let $\operatorname{SymCoc}^2(F, F')$ be the set of collections of power series $(\Delta, (\delta_a)_{a \in A})$ satisfying the following properties

- $\Delta(\mathbf{Z}_1, \mathbf{Z}_2) = \Delta(\mathbf{Z}_2, \mathbf{Z}_1)$
- $\Delta(\mathbf{Z}_2, \mathbf{Z}_3) +_{F'} \Delta(\mathbf{Z}_1, F(\mathbf{Z}_2, \mathbf{Z}_3)) = \Delta(F(\mathbf{Z}_1, \mathbf{Z}_2), \mathbf{Z}_3) +_{F'} \Delta(\mathbf{Z}_1, \mathbf{Z}_2)$
- $\delta_a(\mathbf{Z}_1) +_{F'} \delta_a(\mathbf{Z}_2) +_{F'} \Delta([a]_F(\mathbf{Z}_1), [a]_F(\mathbf{Z}_2)) = [a]_{F'}\Delta(\mathbf{Z}_1, \mathbf{Z}_2) +_{F'} \delta_a(F(\mathbf{Z}_1, \mathbf{Z}_2))$
- $\delta_a(\mathbf{Z}_1) +_{F'} \delta_b(\mathbf{Z}_1) +_{F'} \Delta([a]_F(\mathbf{Z}_1), [b]_F(\mathbf{Z}_1)) = \delta_{a+b}(\mathbf{Z}_1)$
- $[a]_{F'}\delta_b(\mathbf{Z}_1) +_{F'} \delta_a([b]_F(\mathbf{Z}_1)) = \delta_{ab}(\mathbf{Z}_1)$.

These objects are called symmetric 2-cocycles. The set $\mathrm{SymCoc}^2(F, F')$ is naturally a left-$\mathrm{End}(F')$-module.

Proposition 2.5.5. *There is a bijection*

$$\mathrm{SymCoc}^2(F, F') \xrightarrow{\sim} \left\{ \begin{array}{c} A\text{-module laws } E \text{ on } R[\![\mathbf{X}, \mathbf{Z}]\!] \text{ fitting into an exact sequence} \\ 0 \to F' \xrightarrow{\alpha} E \xrightarrow{\beta} F \to 0 \\ \text{where } \alpha(\mathbf{X}) = (\mathbf{X}, 0) \text{ and } \beta(\mathbf{X}, \mathbf{Z}) = \mathbf{Z}. \end{array} \right\}$$

The map sends a pair $\{\Delta, (\delta_a)_a\}$ to the A-module law with structure defined following (2.8).

Proof. This is only a matter of calculation, cf. [GH94, Section 6]. □

Lemma 2.5.6. *If $\mathcal{F}'$, $\mathcal{E}$ and $\mathcal{F}$ are formal A-modules over a quasi-compact and quasi-separated A-scheme X, and α and β are morphisms such that $0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{F} \to 0$ is a short exact sequence of formal A-modules, α is a kernel of β and β is a cokernel of α.*

Proof. Let $\psi : \mathcal{G} \to \mathcal{E}$ be a morphism of formal A-modules such that the composition $\mathcal{G} \xrightarrow{\psi} \mathcal{E} \xrightarrow{\beta} \mathcal{F}$ is trivial. We have to show that there is a unique morphism $\overline{\psi} : \mathcal{G} \to \mathcal{F}'$ making the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r, "\alpha"] & \mathcal{E} \arrow[r, "\beta"] & \mathcal{F} \arrow[r] & 0 \\
 & & \mathcal{G} \arrow[ul, dashed, "\exists!\overline{\psi}"] \arrow[u, "\psi"'] \arrow[ur, "0"'] & &
\end{tikzcd}$$

As $\overline{\psi}$ is unique, we may work locally X and assume that $X = \operatorname{Spec} R$ is affine and $\mathcal{F}'$, $\mathcal{F}$ and $\mathcal{G}$ all come from formal A-module laws. We may now assume that the short exact sequence is in the form of Lemma 2.5.3. Write E, F, F', G for the formal A-module laws corresponding to $\mathcal{E}$, $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{G}$. Write $\mathbf{Y}$ for the variables of G. Now, as $\beta \circ \psi = 0$, the induced morphism of formal A-module laws $\psi : G \to E$ is of the form $\psi(\mathbf{Y}) = (\psi_1(\mathbf{Y}), 0)$, and we find that $\psi_1(\mathbf{Y}) \in (R[\![\mathbf{Y}]\!])^m$ yields a morphism of formal A-modules $G \to F'$. It is clearly unique. Similar ideas show that β is a cokernel of α. □

Lemma 2.5.7. *The composition of two deflations of formal A-modules is a deflation.*

Proof. [Proof is simple application of Lemma 2.5.3 but no time to write down] □

Lemma 2.5.8. *Let* $0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{F} \to 0$ *be a short exact sequence in* $(A\text{-FML}^{\text{arb}}/X)$. *If* $f \in \mathrm{Hom}_{(A\text{-FM}^{\text{arb}}/X)}\, \mathcal{G} \to \mathcal{F}$ *is a morphism of formal A-modules, then there is a formal A-module* $f^*\mathcal{E}$ *and a deflation* $f^*\mathcal{E} \to \mathcal{G}$ *fitting into a diagram with short exact sequences as rows*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}' \arrow[r, "\alpha'"] \arrow[d, equal] & f^*\mathcal{E} \arrow[r, "\beta'"] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & \mathcal{F}' \arrow[r, "\alpha"] & \mathcal{E} \arrow[r, "\beta"] & \mathcal{F} \arrow[r] & 0 \end{tikzcd}$$

The square on the right is cartesian.

Proof. Assume first that $X = \operatorname{Spec} R$ is affine and that $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{G}$ come from formal A-module laws over R. Then we assume to be in the situation of Lemma 2.5.3, with $\mathcal{E}$ coming from a formal A-module law E. Using the induced morphism $f : G \to F$ of formal A-module laws, define the A-module law law f^*E via

$$f^*E((\mathbf{X}_1, \mathbf{Y}_1), (\mathbf{X}_2, \mathbf{Y}_2)) = (F'(\mathbf{X}_1, \mathbf{X}_2) +_{F'} \Delta(f(\mathbf{Y}_1), f(\mathbf{Y}_2)), G(\mathbf{Y}_1, \mathbf{Y}_2))$$
and
$$[a]_{f^*E}(\mathbf{X}, \mathbf{Y}) = ([a]_{F'}(\mathbf{X}) +_{F'} \delta_a(f(\mathbf{Y})), [a]_F(\mathbf{Y})).$$

Here, Δ and δ_a are the power series coming from E (cf. Lemma 2.5.3). Now the top-row is exact with $\alpha'(\mathbf{X}) = (\mathbf{X}, 0)$ and $\beta'(\mathbf{X}, \mathbf{Y}) = \mathbf{Y}$. The morphism of A-module laws $f^*E \to E$ is given by $(\mathbf{X}, \mathbf{Y}) \mapsto (\mathbf{X}, f(\mathbf{Y}))$. One readily checks that

$$\begin{tikzcd} f^*E \arrow[r, "\beta'"] \arrow[d] & G \arrow[d, "f"] \\ E \arrow[r, "\beta"] & F \end{tikzcd}$$

is cartesian in the category of formal A-module laws over R. As the data of $\mathcal{E}$ glue, the power series defining f^*E glue to give a formal A-module $f^*\mathcal{E}$, satisfying all of the desired properties. □

The dual statement is also true.

Lemma 2.5.9. *Let* $0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{F} \to 0$ *be as above, and let* $g \in \mathrm{Hom}_{(A\text{-FM}^{\text{arb}}/X)}(\mathcal{F}', \mathcal{G}')$ *be a morphism of formal A modules. There is a formal A-module* $g_*\mathcal{E}$ *over* X *and an inflation* $\alpha' : \mathcal{G}' \to g_*\mathcal{E}$ *fitting into a diagram with short exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}' \arrow[r, "\alpha"] \arrow[d, "g"'] & \mathcal{E} \arrow[r, "\beta"] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{G}' \arrow[r, "\alpha'"] & g_*\mathcal{E} \arrow[r, "\beta'"] & \mathcal{F} \arrow[r] & 0 \end{tikzcd}$$

Proof. We proceed as in the proof of the previous lemma and assume that $X = \operatorname{Spec} R$ and that $\mathcal{F}'$, $\mathcal{F}$ and $\mathcal{G}$ come from formal A-module laws over R. Now E is a formal A-module law over R of the form described in Lemma 2.5.3, and using the power series Δ and δ_a we define g_*E via

$$g_*E((\mathbf{Y}_1,\mathbf{Z}_1),(\mathbf{Y}_2,\mathbf{Z}_2)) = (G'(\mathbf{Y}_1,\mathbf{Y}_2) +_{G'} g(\Delta(\mathbf{Z}_1,\mathbf{Z}_2)), F(\mathbf{Z}_1,\mathbf{Z}_2))$$
and
$$[a]_{g_*E}(\mathbf{X},\mathbf{Y}) = ([a]_{G'}(\mathbf{X}) +_{G'} g(\delta_a(\mathbf{Z})), [a]_F(\mathbf{Z})).$$

The morphism $E \to g_*E$ is given by $(\mathbf{X},\mathbf{Z}) \mapsto (g(\mathbf{X}),\mathbf{Z})$. These data glue and give rise to a formal A-module $g_*\mathcal{E}$ over X satisfying the desired properties. $\square$

As a consequence of the previous lemmas, we obtain

Proposition 2.5.10. *Let S be a quasi-compact and quasi-separated S-scheme. Then the category $(A\text{-FML}^{\mathrm{arb}}/S)$, equipped with the notion of exact sequences from Definition 2.5.2, is an exact category.*

The following calculation is convenient.

Lemma 2.5.11. *We have natural isomorpisms*

$$\mathrm{Lie}(f^*\mathcal{E}) \cong \mathrm{Lie}(\mathcal{E}) \times_{\mathrm{Lie}(\mathcal{F})} \mathrm{Lie}(\mathcal{G}) \quad \textit{and} \quad \mathrm{Lie}(g_*\mathcal{E}) \cong \mathrm{Lie}(\mathcal{G}') \sqcup_{\mathrm{Lie}(\mathcal{F}')} \mathrm{Lie}(\mathcal{E}).$$

Proof. This is true locally, and the local descriptions descent to X. $\square$

2.5.2 Extensions and Rigidified Extensions

We now introduce the functors Ext and RigExt. Let $\mathcal{F}$ and $\mathcal{F}'$ be formal A-modules over an A-scheme S.

Definition 2.5.12 (Extension)**.** An extension of $\mathcal{F}$ by $\mathcal{F}'$ is a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{E} \to \mathcal{F} \to 0.$$

We say that this extension is equivalent to another extension

$$0 \to \mathcal{F}' \to \mathcal{E}' \to \mathcal{F} \to 0$$

if and only if there is an isomorphism $\mathcal{E} \to \mathcal{E}'$ making the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{F} \arrow[r] & 0 \end{tikzcd}$$

commute. We denote the set of equivalence classes of extensions of $\mathcal{F}$ by $\mathcal{F}'$ as $\mathrm{Ext}(\mathcal{F},\mathcal{F}')$.

Proposition 2.5.10 turns $\mathrm{Ext}(-,-)$ into a functor. In particular, $\mathrm{Ext}(\mathcal{F},\mathcal{F}')$ carries the structure of a left-$\mathrm{End}(\mathcal{F}')$-module, with zero-object given by the canonical extension $\mathcal{F}\oplus\mathcal{F}'$.

Definition 2.5.13 (Rigifidied Extension)**.** A rigidified extension of $\mathcal{F}$ by $\mathcal{F}'$ is a pair consisting of an extension
$$0 \to \mathcal{F}' \to \mathcal{E} \to \mathcal{F} \to 0$$
and a splitting s of the short exact sequence
$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Lie}(\mathcal{F}') \arrow[r] & \mathrm{Lie}(\mathcal{E}) \arrow[r] & \mathrm{Lie}(\mathcal{F}) \arrow[r] \arrow[l, bend left, "s"] & 0. \end{tikzcd}$$

We say that two rigidified extensions (E, s), (E', s') are isomorphic if there is an isomorphism $i : E \to E'$ of extensions such that $s' = \mathrm{Lie}(i) \circ s$. We denote the set of isomorphism classes of rigidified extensions by $\mathrm{RigExt}(\mathcal{F}, \mathcal{F}')$.

Lemma 2.5.14. *The assignment $(\mathcal{F}, \mathcal{F}') \mapsto \mathrm{RigExt}(\mathcal{F}, \mathcal{F}')$ extends to a functor in both entries (contravariant in the first, covariant in the second).*

Proof. Given a morphism $f : \mathcal{G} \to \mathcal{F}$, the induced morphism $\mathrm{RigExt}(\mathcal{F}, \mathcal{F}') \to \mathrm{RigExt}(\mathcal{G}, \mathcal{F}')$ is given by sending the pair $(\mathcal{E}, s)$ to the pair $(f^*\mathcal{E}, s')$, where
$$s' : \mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(f^*\mathcal{E}) \cong \mathrm{Lie}(\mathcal{E}) \times_{\mathrm{Lie}(\mathcal{F})} \mathrm{Lie}(\mathcal{G}), \quad x \mapsto ((s \circ \mathrm{Lie}(f))(x), x).$$

Here we used the description of $\mathrm{Lie}(f^*\mathcal{E})$ from Lemma 2.5.11. Similarly, given a morphism $g : \mathcal{F}' \to \mathcal{G}'$, the induced morphism $\mathrm{RigExt}(\mathcal{F}, \mathcal{F}') \to \mathrm{RigExt}(\mathcal{F}, \mathcal{G}')$ sends $(\mathcal{E}, s)$ to $(g_*\mathcal{E}, \mathrm{Lie}(g') \circ s)$, where $g' : \mathcal{E} \to g_*\mathcal{E}$ is the canonical morphism. $\square$

In particular, $\mathrm{RigExt}(-, \mathcal{F}')$ carries the structure of an $\mathrm{End}(\mathcal{F}')$-module, the zero-object is given by the equivalence class of the pair $(\mathcal{F}' \oplus \mathcal{F}, s_{\mathrm{triv}})$, where $s_{\mathrm{triv}} : \mathrm{Lie}(\mathcal{F}) \to \mathrm{Lie}(\mathcal{F}') \oplus \mathrm{Lie}(\mathcal{F})$ is the canonical inclusion.
Of course there is a natural transformation $\mathrm{RigExt}(-,-) \to \mathrm{Ext}(-,-)$, forgetting the splitting. It appears as the right-most term of an interesting exact sequence.

Proposition 2.5.15. *There is an exact sequence of Abelian groups, functorial in $\mathcal{F}$ and $\mathcal{F}'$*
$$\mathrm{Hom}_{(A\text{-}\mathrm{FM}^{\mathrm{arb}}/S)}(\mathcal{F}, \mathcal{F}') \xrightarrow{\mathrm{Lie}} \mathrm{Hom}_{(\mathcal{O}_S\text{-}\mathrm{QCoh})}(\mathrm{Lie}(\mathcal{F}), \mathrm{Lie}(\mathcal{F}')) \to \mathrm{RigExt}(\mathcal{F}, \mathcal{F}') \to \mathrm{Ext}(\mathcal{F}, \mathcal{F}').$$

Proof. The kernel of $\mathrm{RigExt}(\mathcal{F}, \mathcal{F}') \to \mathrm{Ext}(\mathcal{F}, \mathcal{F}')$ is given (up to equivalence) by pairs of the form $(\mathcal{F}' \oplus \mathcal{F}, s)$, where s is a morphism of quasi-coherent $\mathcal{O}_S$-modules such that
$$\mathrm{Lie}(\mathcal{F}) \xrightarrow{s} \mathrm{Lie}(\mathcal{F}') \oplus \mathrm{Lie}(\mathcal{F}) \to \mathrm{Lie}(\mathcal{F})$$
is the identity. It is clear that these morphisms s correspond to morphisms $\mathrm{Lie}(\mathcal{F}) \to \mathrm{Lie}(\mathcal{F}')$. The kernel of $\mathrm{Hom}_{(\mathcal{O}_S\text{-}\mathrm{QCoh})}(\mathrm{Lie}(\mathcal{F}), \mathrm{Lie}(\mathcal{F}')) \to \mathrm{RigExt}(\mathcal{F}, \mathcal{F}')$ is spanned by those pairs $(\mathcal{E}, s)$ that are in the same class as $(\mathcal{F}' \oplus \mathcal{F}, s_{\mathrm{triv}})$. Any such $\mathcal{E}$ fits into a diagram
$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d, equal] & \mathcal{F}' \oplus \mathcal{F} \arrow[r] \arrow[d, "\psi"] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{F}' \arrow[r, "\alpha"] & \mathcal{E} \arrow[r, "\beta"] & \mathcal{F} \arrow[r] & 0. \end{tikzcd}$$

Working locally, we assume that $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{F}'$ come from formal module laws E, F and F'. Now ψ is necessarily of the form $\psi(\mathbf{X}, \mathbf{Z}) = (\mathbf{X} +_{F'} g(\mathbf{Z}), \mathbf{Z})$. Hence, the power series g furnishes a morphism of formal module laws $F \to F'$. This construction descents to a morphism of formal A-modules $\mathcal{F} \to \mathcal{F}'$, and we have

$$s(x) = \mathrm{Lie}(\psi) \circ s_{\mathrm{triv}}(x) = \mathrm{Lie}(\alpha) \circ \mathrm{Lie}(g)(x) + x \in \mathrm{Lie}(\mathcal{E}).$$

This explains exactness on the left. $\square$

2.5.3 Explicit Dieudonné Theory

Let $\mathcal{F}$ and $\mathcal{F}'$ be formal A-modules of dimension m and n respectively, over an affine base $\mathrm{Spec}\, R$, coming from formal module laws F and F'. We give an explicit description of $\mathrm{Ext}(\mathcal{F}, \mathcal{F}')$ in terms of terms of the Symmetric 2-Cocycles associated with F and F' (cf. Definition 2.5.4). We also give a related explicit description of $\mathrm{RigExt}(F, \widehat{\mathbb{G}}_a)$ in terms of Quasi-Logarithms, cf. Definition 2.5.18.
Write $\mathbf{X}$ for the variables of F' and $\mathbf{Z}$ for the variables of F.

Definition 2.5.16 (Symmetric 1-Cochain)**.** A symmetric 1-cochain associated to (F, F') is a n-tuple of power series $g = (g_1, \dots, g_m)$, such that $g_i(\mathbf{Z}) \in R[\![\mathbf{Z}]\!]$ has no constant term for all i. We write δg for the coboundary of g, that is, the pair $(\Delta g, (\delta_a g)_{a \in A})$, where

$$\Delta g = g(\mathbf{Z}_1) -_{F'} g(F(\mathbf{Z}_1, \mathbf{Z}_2)) +_{F'} g(\mathbf{Z}_2) \in (R[\![\mathbf{Z}_1, \mathbf{Z}_2]\!])^m$$

and

$$\delta_a g = [a]_{F'} g(\mathbf{Z}) -_{F'} g([a]_F(\mathbf{Z})) \in (R[\![\mathbf{Z}]\!])^m.$$

One readily checks that $\delta g \in \mathrm{SymCoc}^2(F, F')$.

Proposition 2.5.17. *Given two extensions $\mathcal{E}, \mathcal{E}' \in \mathrm{Ext}(\mathcal{F}, \mathcal{F}')$, write E, E' for the respective formal A-module laws coming from Lemma* 2.5.3*, and write Δ_E and $\Delta_{E'}$ for the associated symmetric* 2*-cocycles (cf. Proposition 2.5.5). There is a bijection*

$$\{g \in (R[\![\mathbf{Z}]\!])^m \mid g(0) = 0 \text{ and } \delta g = \Delta_{E'} - \Delta_E\} \xrightarrow{\sim} \{\text{Isomorphisms of extensions } E \to E'\}.$$

Explicitly, this bijection is given by sending g to the morphism $i_g \in \mathrm{Hom}_{(A\text{-}\mathrm{FML}^{\mathrm{arb}}/R)}(E, E')$, where $i_g(\mathbf{X}, \mathbf{Z}) = (\mathbf{X} +_{F'} g(\mathbf{Z}), \mathbf{Z})$. In particular, there is a bijection

$$\mathrm{Ext}(\mathcal{F}, \mathcal{F}') \cong \frac{\mathrm{SymCoc}^2(F, F')}{\{\delta g \mid g \in (R[\![\mathbf{Z}]\!])^m \text{ with } g(0) = 0\}}.$$

This bijection is an isomorphism of $\mathrm{End}(\mathcal{F}')$*-modules.*

For now, this finishes the study of $\mathrm{Ext}(\mathcal{F}, \mathcal{F}')$.
Assume now that $\mathcal{F}' = \widehat{\mathbb{G}}_a$, and that $\mathcal{F}$ comes from a one-dimensional formal A-module $F \in (A\text{-FML}/R)$. For the remainder of this subsection, we will be concerned with the R-module $\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a)$. The notion of Quasi-Logarithms will play a major role.

Definition 2.5.18 (Quasi-Logarithms)**.** A power series $g(T) \in (R \otimes_A K)[\![T]\!]$ is called a Quasi-Logarithm for F, if $g(0) = 0$ and $g'(T)$, as well as all of the power series appearing in δg (with $F' = \widehat{\mathbb{G}}_a$, cf. Definition 2.5.16) have coefficients in R. We define the R-module

$$\mathrm{QLog}(F) = \frac{\{g(T) \in (R \otimes_A K)[\![T]\!] \mid g \text{ is a quasi-logarithm for } F\}}{\{g(T) \in R[\![T]\!] \mid g(0) = 0\}}$$

Let $(\mathcal{E}, s) \in \mathrm{RigExt}(F, \widehat{\mathbb{G}}_a)$ be a rigidified extension. The splitting s yields an isomorphism $\omega(\mathcal{E}) \cong \omega(\widehat{\mathbb{G}}_a) \oplus \omega(\mathcal{F})$ on duals, giving an invariant differential $\omega_{\mathcal{E}} \in \omega(\mathcal{E})$ pulling back to $\mathrm{d}X$ on $\widehat{\mathbb{G}}_a$. Conversely, any such invariant differential $\omega_{\mathcal{E}}$ yields a splitting, so the choice of s is equivalent to the choice of ω_E, and we will henceforth write $(\mathcal{E}, \omega_{\mathcal{E}}) \in \mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a)$.

Theorem 2.5.19 (Classification of Rigidified Extensions in terms of Quasi-Logarithms)**.** *There is a bijection*

$$\{\textit{Quasi-logarithms for } F\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{Pairs } (E, \omega_E), \textit{ where } E \textit{ is an } A\textit{-module law} \\ \textit{fitting into an exact sequence} \\ 0 \to \widehat{\mathbb{G}}_a \xrightarrow{\alpha} E \xrightarrow{\beta} F \to 0 \\ \textit{with } \alpha(X) = (X, 0) \textit{ and } \beta(X, T) = T \textit{ and } \omega_E \\ \textit{is an invariant differential on } E \textit{ with } \alpha^* \omega_E = \mathrm{d}X. \end{array} \right\} \tag{2.9}$$

The map sends any quasi-logarithm $g(T) \in (R \otimes_A K)[\![T]\!]$ *to the pair* $(E_{\delta g}, \mathrm{d}(X + g(T)) \in \mathrm{RigExt}(F, \widehat{\mathbb{G}}_a)$. *Here* $E_{\delta g} \in \mathrm{Ext}(F, \widehat{\mathbb{G}}_a)$ *is the extension corresponding to* $\delta g \in \mathrm{SymCoc}^2(F, \widehat{\mathbb{G}}_a)$. *Furthermore, given two rigidified extensions* $(E, \omega_E), (D, \omega_D)$ *with associated quasi-logarithms* $g(T)$ *and* $h(T)$, *there is a (unique) isomorphism* $(E, \omega_E) \to (D\omega_D)$ *if and only if* $h(T) - g(T) =: f(T)$ *has coefficients in* $R[\![T]\!]$. *In this case, the isomorphism* $i_f(X, T) \in \mathrm{Hom}_{(A\text{-FML}^{\mathrm{arb}}/R)}(E, D)$ *is given by* $i_f(X, T) = (X + f(T), T)$. *In particular, there is a canonical bijection*

$$\mathrm{QLog}(F) \xrightarrow{\sim} \mathrm{RigExt}(F, \widehat{\mathbb{G}}_a).$$

This bijection is an isomorphism of R-modules.

Proof. We construct an inverse of the map in (2.9). Let (E, ω_E) be an element of the set on the right and let $(\Delta, (\delta_a)_{a \in A}) \in \mathrm{SymCoc}^2(F, \widehat{\mathbb{G}}_a)$ be the symmetric 2-cochain corresponding to E. Following Proposition 2.3.3, the datum of $\omega_E \in \omega(E)$ is equivalent to a morphism

$$f_E \in \mathrm{Hom}_{(A\text{-FML}/R \otimes K)}(E \otimes_R (R \otimes_A K), \widehat{\mathbb{G}}_a) \quad \text{satisfying} \quad f_E(X, T) = X + g(T)$$

for some $g(T) \in (R \otimes_A K)[\![T]\!]$. The fact that f_E is a homomorphism implies that

$$\begin{aligned} X_1 + X_2 + \Delta(T_1, T_2) + g(F(T_1, T_2)) = f_E(E((X_1, T_1), (X_2, T_2))) = \\ = f_E(X_1, T_1) + f_E(X_2, T_2)) = X_1 + g(T_1) + X_2 + g(T_2), \end{aligned}$$

thereby $\Delta g = \Delta(T_1, T_2) \in R[\![T_1, T_2]\!]$. Similarly, we find $\delta_a g = \delta_a \in R[\![T]\!]$. Hence, $g(T)$ is a quasi-logarithm with $\delta g = (\Delta, (\delta_a)_a)$. This construction yields the desired inverse. The remaining statements are verified directly, also cf. [GH94, Section 8]. $\square$

Now, let A be a complete, discrete valuation ring with uniformizing parameter ϖ and finite residue field k.

Proposition 2.5.20. *If $\mathcal{F}$ comes from a one-dimensional formal A-module law over a flat, local A-algebra R and $\mathcal{F}' = \widehat{\mathbb{G}}_a$, the short exact sequence of Proposition 2.5.15 fits into a commutative diagram with exact rows and vertical maps (canonical) isomorphisms*

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{F}, \widehat{\mathbb{G}}_a) \arrow[r, hook, "\mathrm{d}_F"] \arrow[d, equal] & \omega(\mathcal{F}) \arrow[r] \arrow[d] & \mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a) \arrow[r, two heads] \arrow[d] & \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a) \arrow[d] \\
\left\{ \begin{matrix} f \in TR[\![T]\!] : \\ \delta f = 0 \end{matrix} \right\} \arrow[r, hook] & \left\{ \begin{matrix} f \in (R \otimes_A K)[\![T]\!] : \\ \delta f = 0,\ f(0) = 0 \\ \text{and } f'(T) \in R[\![T]\!] \end{matrix} \right\} \arrow[r] & \mathrm{QLog}(F) \arrow[r, two heads, "\delta"] & \frac{\mathrm{SymCoc}^2(F, \widehat{\mathbb{G}}_a)}{\{\delta g | g \in TR[\![T]\!]\}}
\end{tikzcd}$$

Proof. Injectivity of d_F is provided by Proposition 2.3.3, and related to the original exact sequence as $\mathrm{Hom}_R(\mathrm{Lie}(\mathcal{F}), \mathrm{Lie}(\widehat{\mathbb{G}}_a)) = \omega(\mathcal{F})$. Surjectivity of $\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a) \to \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a)$ comes from the fact that $\mathrm{Lie}(\mathcal{F})$ is projective. The first vertical map is an equality, cf. Definitions 2.5.16 and 2.1.4. The vertical arrow describing $\omega(F)$ is obtained by identifying the preimage of $\omega(F) \subseteq \omega(F \otimes_R (R \otimes_A K))$ under the isomorphism

$$\{f \in T(R \otimes_A K)[\![T]\!] \mid \delta f = 0\} = \mathrm{Hom}_{(A\text{-FML}/R \otimes_A K)}(F \otimes (R \otimes_A K), \widehat{\mathbb{G}}_a) \xrightarrow{\mathrm{d}_F} \omega(F \otimes_R (R \otimes_A K)).$$

All squares commute by construction. □

We admit the following facts from Section 9 of [GH94].

Proposition 2.5.21. *Let F be a formal A-module law of height h over a local, adic A-algebra R. Write $\mathcal{F}$ for the formal A-module coming from F. Then $\mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a)$ is a free R-module of rank $n - 1$, $\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a)$ is a free R-module of rank n.*

Proof. This is Proposition 9.8 in [GH94]. The authors make use of a description of $\mathrm{Ext}(F, \widehat{\mathbb{G}}_a)$ in terms of deformation theory and combine it with a convenient normal form of formal A-modules, so called A-typical modules (we touch upon the theory in section 2.6), to construct an explicit basis for the corresponding modules. □

As a corollary, the authors obtain

Lemma 2.5.22. *If $R \to R'$ is a homomorphism of local A-algebras, the induced maps of free R'-modules*

$$\mathrm{Ext}_R(\mathcal{F}, \widehat{\mathbb{G}}_a) \otimes_R R' \to \mathrm{Ext}_{R'}(\mathcal{F}, \widehat{\mathbb{G}}_a)$$
$$\mathrm{RigExt}_R(\mathcal{F}, \widehat{\mathbb{G}}_a) \otimes_R R' \to \mathrm{RigExt}_{R'}(\mathcal{F}, \widehat{\mathbb{G}}_a)$$

are isomorphisms.

Proof. [GH94, Corollary 9.13]. □

Definition 2.5.23 (The Dieudonné module of a formal A-module)**.** Given $\mathcal{F} \in (A\text{-FM}/R)$, we define

$$\mathrm{D}(\mathcal{F}) \coloneqq \mathrm{Hom}_R(\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a), R).$$

We call $\mathrm{D}(\mathcal{F})$ the (covariant) Dieudonné-module of $\mathcal{F}$.

Proposition 2.5.24 (Crystalline Nature of $\mathrm{D}(-)$)**.** *The assignment $\mathcal{F} \mapsto \mathrm{D}(\mathcal{F})$ yields a functor*

$$(A\text{-FM}/R) \to (R\text{-Mod}).$$

Given two formal A-modules $\mathcal{F}, \mathcal{G} \in (A\text{-FM}/R)$ and two morphisms ϕ, ψ from $\mathcal{F}$ to $\mathcal{G}$ such that the induced morphisms of their reductions to R/I agree, the induced morphisms $\mathrm{D}(\mathcal{F}) \to \mathrm{D}(\mathcal{G})$ agree.

Proof. □ !!!

2.6 Hazewinkel's Functional Equation Lemma and the Standard Formal Module Law

If, A is an integral domain and R is a flat A-module, the structure of a formal A-module F over R is uniquely determined by its logarithm $\log_H \in R \otimes_A K[\![T]\!]$. Indeed, we find

$$F(X, Y) = \exp_H(X + Y), \quad [a]_F(X) = \exp_H(aX).$$

It is therefore natural to wonder about conditions on power series $f \in (R \otimes_A K)[\![T]\!]$ ensuring that f is the logarithm of some formal group law. Hazewinkel found such a condition in his functional equation lemma.

Proposition 2.6.1 (Hazewinkel's Functional Equation Lemma)**.** *Let p be a prime and $q = p^e$. Given an inclusion of rings $B \subseteq L$, an ideal $\mathfrak{a} \subseteq B$ containing p, an endomorphism of rings $\sigma : L \to L$ and elements $s_1, s_2, \dots \in L$ subject to the conditions that*

$$\sigma(b) \equiv b^q \pmod{\mathfrak{a}} \text{ for all } b \in B \quad \text{and} \quad \sigma^r(s_i)\mathfrak{a} \subset B \text{ for all } r, s \geq 1.$$

Suppose now that $f \in L[\![T]\!]$ has $f'(0) \in L^\times$ and satisfies the functional equation condition

$$f(X) - \sum_{i=1}^{\infty} s_i(\sigma^i_* f)(X^{q^i}) \in B[\![X]\!].$$

Then we have

$$F(X, Y) = f^{-1}(f(X) + f(Y)) \in B[\![X, Y]\!],$$

where f^{-1} is the inverse power series as in Lemma 2.1.9. Also, if $g(Z) \in L[\![Z]\!]$ is another power series satisfying the same condition

$$g(Z) - \sum_{i=1}^{\infty} s_i(\sigma^i_* f)(Z^{q^i}) \in B[\![Z]\!],$$

then $f^{-1}(g(Z)) \in B[\![Z]\!]$. Furthermore, if $\alpha(T) \in B[\![T]\!]$ and $\beta(T) \in B[\![T]\!]$, then

$$\alpha(T) \equiv \beta(T) \pmod{\mathfrak{a}^r} \iff f(\alpha(T)) \equiv f(\beta(T)) \pmod{\mathfrak{a}^r} \tag{2.10}$$

Proof. A more general statement can be found in [Haz79, Section 2]. Proofs can be found in [Haz78, Sections 2 and 10]. □

Note that by construction, $F(X,Y)$ as defined above yields a (commutative) formal group law over B. Let B^σ denote the subring of elements in B fixed by σ. Then the second part of the Functional Equation Lemma implies that we even obtain formal B^σ-modules with $[b]_F(T) = f^{-1}(bf(T))$, as $bf(T)$ satisfies the same functional equation if $b \in B^\sigma$.
We now enter the situation where K is a local field with ring of integers $\mathcal{O}_K$ and uniformizer ϖ and use the Functional Equation Lemma to construct Lubin–Tate Formal Group Laws. A special role will play the power series

$$f(T) = \sum_{i=1}^{\infty} \frac{T^{q^{in}}}{\varpi^i} \in K[\![T]\!].$$

It satisfies the functional equation

$$f(T) = T + \frac{1}{\varpi} f(T^{q^n}),$$

which is a functional equation of the form above, with $B = \mathcal{O}_K$, $\mathfrak{a} = (\varpi)$, $L = K$, $s_1 = \varpi^{-1}$, $s_2 = s_3 = \cdots = 0$, $\sigma = \mathrm{id}_L$. Hence f arises as the logarithm of a formal $\mathcal{O}_K$-module law H over $\mathcal{O}_K$. The fact that $f^{-1}(X) = X - \frac{1}{\varpi}X^{q^n} + \dots$ reveals $[\varpi]_H(T) \equiv \varpi T \bmod (T^2)$. Additionally, note that

$$f([\varpi]_H(T)) = \varpi f(T) = \varpi T + f(T^{q^n}) \equiv f(T^{q^n}) \pmod{\varpi}.$$

Hence, the equivalence in (2.10) implies that $[\varpi]_H(T) \equiv T^{q^n} \bmod \varpi$. So H is a Lubin–Tate formal $\mathcal{O}_K$-module law of height n, we call it the standard Lubin–Tate formal module law of height n.

Remark. The formal $\mathcal{O}_K$-module H is a member of the class of so called A-typical formal modules - formal A-modules F with logarithm of the form

$$\log_F(T) = \sum_{i=0}^{\infty} b_i X^{q^i}$$

for elements $b_0, b_1, \dots \in R \otimes_A K$ (cf. [Haz78, Definition 21.5.5 and Criterion 21.5.9]). If R is flat over A, every formal A-module over R is isomorphic to an A-typical one (cf. [Haz78, p. 21.5.6]). Most results about the standard $\mathcal{O}_K$-module H remain true for general $\mathcal{O}_K$-typical formal module laws.

It will be convenient to make the terms in the exact sequence of Proposition 2.5.20 explicit for $\mathcal{F} = \mathrm{FG}(H)$. As H is of height $n > 0$, there is no non-trivial map $H \to \widehat{\mathbb{G}}_a$ and the sequence becomes

$$\begin{tikzcd} 0 \arrow[r] & \omega(H) \arrow[r] \arrow[d] & \mathrm{RigExt}(H, \widehat{\mathbb{G}}_a) \arrow[r] \arrow[d] & \mathrm{Ext}(H, \widehat{\mathbb{G}}_a) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \left\{ \begin{matrix} g \in TK[\![T]\!] : \delta g = 0 \\ \text{and } g'(T) \in \mathcal{O}_K[\![T]\!] \end{matrix} \right\} \arrow[r] & \mathrm{QLog}(H) \arrow[r, "\delta"] & \frac{\mathrm{SymCoc}^2(H, \widehat{\mathbb{G}}_a)}{\{\delta g | g \in T\mathcal{O}_K[\![T]\!]\}} \arrow[r] & 0. \end{tikzcd}$$

We now have

Proposition 2.6.2. *The R-module $\omega(H)$ is free of rank 1, generated by $f(T) = \log_H(T)$. $\mathrm{QLog}(H)$ is free of rank n, generated by the classes of $(f(T), \frac{1}{\varpi} f(T^q), \ldots, \frac{1}{\varpi} f(T^{q^{n-1}}))$. Consequently, the short exact sequence above is given by*

$$0 \to \langle f(T) \rangle \to \left\langle f(T), \frac{1}{\varpi} f(T^q), \ldots, \frac{1}{\varpi} f(T^{q^{n-1}}) \right\rangle \xrightarrow{\delta} \left\langle \delta\left(\frac{1}{\varpi} f(T^q)\right), \ldots, \delta\left(\frac{1}{\varpi} f(T^{q^{n-1}})\right) \right\rangle \to 0.$$

Proof. A simple calculation shows that $\frac{1}{\varpi} f(T^{q^k})$ is a quasi-logarithm for $1 \leq k \leq n-1$. As $\delta f = 0$, we have $f(T) \in \mathrm{QLog}(F)$ as well. The claim is [GH94, Proposition 13.8] which is a special case of [ibid., Proposition 9.8]. □

2.7 The Universal Additive Extension

We follow [GH94, Section 11], and specialize to the situation where A is a complete discrete valuation ring with uniformizer ϖ and finite residue field of characteristic p and R is a local admissible A-algebra with residue field $\overline{\mathbb{F}}_q$.

Lemma 2.7.1. *Let M be a finite free module over R. Then there is a natural bijection, functorial in M and $\mathcal{F}$*

$$\mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a \otimes M) \cong \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a) \otimes_R M.$$

Proof. After choosing coordinates on $\mathcal{F}$, this follows directly from the description of Ext in terms of symmetric 2-cocycles, cf. Propositions 2.5.5 and 2.5.17. □

Let $\mathcal{F}$ be a one-dimensional formal A-module over R. We put $\mathrm{M}(\mathcal{F}) := \mathrm{Hom}_R(\mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a), R)$, which is free of rank $n-1$, and write $\mathcal{V} = \widehat{\mathbb{G}}_a \otimes \mathrm{M}(\mathcal{F})$. Now, by the previous lemma,

$$\mathrm{Ext}(\mathcal{F}, \mathcal{V}) = \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a \otimes \mathrm{M}(\mathcal{F})) = \mathrm{End}_R(\mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a)).$$

Let $0 \to \mathcal{V} \to \mathcal{E} \to \mathcal{F} \to 0$ be the extension corresponding to the identity on the right. This class is unique up to unique isomorphism. Indeed, as R is a local ring we may choose formal module laws F and V giving rise to $\mathcal{F}$ and $\mathcal{V}$, and let E be the module law obtained from Lemma 2.5.3. If $0 \to V \to E' \to F \to 0$ is another extension in this class, we have by construction a commutative square

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle i} & & \| & & \\ 0 & \longrightarrow & V & \longrightarrow & E' & \longrightarrow & F & \longrightarrow & 0, \end{array}$$

and by Proposition 2.5.17 we see that any other isomorphism i' making the diagram above commute differs from i by an element in $\mathrm{Hom}(F, V) = 0$.

Definition 2.7.2 (Universal Additive Extension)**.** The extension

$$0 \to \mathcal{V} \to \mathcal{E} \to \mathcal{F} \to 0$$

constructed above is called the universal additive extension of $\mathcal{F}$.

Proposition 2.7.3. *If N is a finite, free R-module, $\mathcal{G}' = \widehat{\mathbb{G}}_a \otimes N$ and*

$$0 \to \mathcal{G}' \to \mathcal{E}' \to F \to 0$$

is an extension of $\mathcal{F}$ by $\mathcal{G}'$, there are unique homomorphisms $i : \mathcal{E} \to \mathcal{E}'$ and $g' : \mathcal{V} \to \mathcal{G}'$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle i} & & \| & & \\ 0 & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

commute. In particular, we have $\mathcal{E}' = g'_* \mathcal{E}$.

Proof. As $\mathcal{V}$ and $\mathcal{G}'$ are additive, we have

$$\mathrm{Hom}(\mathcal{V}, \mathcal{G}') = \mathrm{Hom}_R(\mathrm{M}(\mathcal{F}), N) = \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a) \otimes N = \mathrm{Ext}(\mathcal{F}, \mathcal{G}').$$

This yields g'. Again, i is unique as by observations similar to Proposition 2.5.17, the difference of two morphisms $i, i' : \mathcal{E} \to \mathcal{E}'$ is given a morphism $\mathcal{F} \to \mathcal{G}'$, which has to be trivial. □

Lemma 2.7.4. *There is a natural isomorphism* $\mathrm{Lie}(\mathcal{E}) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a), R) = \mathrm{D}(\mathcal{F})$.

Proof. We show the equivalent statement $\omega(\mathcal{E}) = \mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a)$. Let $(\mathcal{E}', \omega_{\mathcal{E}'}) \in \mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a)$. Then by universality of $\mathcal{E}$, we obtain a unique homomorphism $i : \mathcal{E} \to \mathcal{E}'$. This yields a homomorphism of R-modules $\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a) \to \omega(\mathcal{E})$, sending a pair $(\mathcal{E}', \omega_{\mathcal{E}'})$ to $i^* \omega_{\mathcal{E}'}$. This morphism fits into the following commutative diagram, where the top row is the short exact sequence from Proposition 2.5.20 and the bottom row is the dual short exact sequence of $0 \to \mathrm{Lie}(\mathcal{V}) \to \mathrm{Lie}(\mathcal{E}) \to \mathrm{Lie}(\mathcal{F}) \to 0$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega(\mathcal{F}) & \longrightarrow & \mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a) & \longrightarrow & \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & \omega(\mathcal{F}) & \longrightarrow & \omega(\mathcal{E}) & \longrightarrow & \omega(\mathcal{V}) & \longrightarrow & 0 \end{array}$$

Thereby, $\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a) \to \omega(\mathcal{E})$ is a natural isomorphism. $\square$

2.8 Tate Modules and the Universal Cover

2.8.1 Useful Calculations

Let p be a prime. Let R be a Noetherian local ring with maximal ideal I such that $p \in I$, R is complete with respect to the I-adic topology and $k_R := R/I$ is an algebraically closed field (necessarily of characteristic p). If q is a power of p, we write $\mathcal{F}_{R,q}$ for the set of power series $f \in R[\![T]\!]$ satisfying

$$f(T) \equiv g(T^q) \pmod{I} \tag{2.11}$$

for some power series $g(T) = c_1T + c_2T^2 + \cdots \in R[\![T]\!]$ with $c_1 \in R^\times$. If $q' > q$ is another power of p, we have injections $\mathcal{F}_{R,q} \hookrightarrow \mathcal{F}_{R,q'}$ given by sending $f(T)$ to its (q'/q)-fold self-composite $f^{q'/q}(T)$. Making use of these transition maps, we define

$$\mathcal{F}_R := \operatorname*{colim}_{n \in \mathbb{N}} \mathcal{F}_{R,p^n},$$

identifying any power series $f \in \mathcal{F}_{R,q}$ with its image in $\mathcal{F}_{R,q'}$ for higher p-powers q'. For any $f \in \mathcal{F}_{R,q}$, we define the functor

$$U_f : (R\text{-Adm}) \to (\mathrm{Set}), \quad S \mapsto \left\{ (x_0, x_1, \dots) \in \prod_{\mathbb{N}} S^{\circ\circ} \mid f(x_{i+1}) = x_i \right\}.$$

This functor does, up to canonical isomorphism, only depend on the equivalence class of f in $\mathcal{F}_R$. We write $U_{0,f}$ for the base change of U_f to k_R, that is

$$U_{0,f} : (k_R\text{-Adm}) \to (\mathrm{Set}), \quad S \mapsto \left\{ (x_0, x_1, \dots) \in \prod_{\mathbb{N}} S^{\circ\circ} \mid \overline{f}(x_{i+1}) = x_i \right\}.$$

Here, $\overline{f}$ is the image of f under the reduction map $R[\![T]\!] \to k_R[\![T]\!]$.

In the sequel, we denote R-algebras by S and write J for an ideal of definition containing the image of I (provided, for example, by A.0.2). Given an element $f \in \mathcal{F}_R$, we do not distinguish between f and a choice of a representative $\tilde{f} \in \mathcal{F}_{R,q}$ for some sufficiently large p-power.

The following observation lays the groundwork for many of the upcoming results.

Lemma 2.8.1. *Let f be any power series in $\mathcal{F}_R$. For any two elements $s_1, s_2 \in S$ with $s_1 \equiv s_2$ mod J such that $f(s_1)$ and $f(s_2)$ exist (for example if f is a polynomial or $s_1, s_2 \in S^{\circ\circ}$), we have*

$$f^k(s_1) \equiv f^k(s_2) \pmod{J^{k+1}}.$$

Here, f^k denotes k-fold composition of f.

Proof. We will show that if $s_1 \equiv s_2$ mod J^k, then $f(s_1) \equiv f(s_2)$ mod J^{k+1}, which suffices to prove the claim. We may write $s_2 = s_1 + r$ for some $r \in J^k$. By the assumptions on f there exist power series $g, h \in R[\![T]\!]$ such that h only has coefficients in I and $f(T) = g(T^q) + h(T)$.

As I is finitely generated, say by elements $(r_1, \dots, r_l)$, we obtain a representation

$$f(s_1) - f(s_2) \in g(s_1^q) - g(s_2^q) + \sum_{i=1}^{l} r_i \left(h_i(s_1) - h_i(s_2)\right).$$

As r divides $(h_i(s_1) - h_i(s_2))$, we find $r_i(h_i(s_1) - h_i(s_2)) \in (r_i r) \subseteq J^{k+1}$. Also note that for any $s \in S$ and $n \in \mathbb{N}$,

$$(s + r)^{nq} = s^{nq} + nqrs^{nq-1}r + \cdots + r^{nq},$$

so after cancellation, all monomials of $g(s_1^q) - g(s_2^q)$ lie in (qr) or (r^2). This implies

$$g(s_1^q) - g\left((s_1 + r)^q\right) \in (qr) + (r^2) \subseteq J^{k+1},$$

and we are done. □

Lemma 2.8.2. *The natural reduction map*

$$U_f(S) \to U_f(S/J) = U_{0,f}(S/J)$$

is bijective.

Proof. We first show surjectivity. Given a sequence $(x_0, x_1, \dots) \in U_f(S/J)$, we can choose a sequence of arbitrary lifts $(y_0, y_1, \dots) \in \prod_{\mathbb{N}} S^{\infty}$ and set

$$z_i = \lim_{r \to \infty} f^r(y_{i+r}).$$

The limit exists, because if $s \geq r$ are two non-negative integers, we calculate

$$f^{s-r}(y_{i+s}) \equiv \overline{f}^{s-r}(x_{i+s}) = x_{i+r} \equiv y_{i+r} \pmod{J},$$

implying by Lemma 2.8.1 that

$$f^s(y_{i+s}) \equiv f^r(y_{i+r}) \pmod{J^r}.$$

This shows that $(f^r(y_{i+r}))_{r \in \mathbb{N}}$ is a Cauchy-sequence for the J-adic topology on S, thereby convergent (cf. Lemma A.0.4). The sequence $(z_0, z_1, \dots)$ now lies in $U_f(S)$ and lifts $(x_0, x_1, \dots)$. It remains to show that the lift is unique. Suppose that $(z_0', z_1', \dots)$ is another lift. Then, for any $i, k \in \mathbb{N}$ we have $z_{i+k} \equiv z_{i+k}' \mod J$, and another application of Lemma 2.8.1 shows that

$$z_i = f^k(z_{i+k}) \equiv f^k(z_{i+k}') = z_i' \pmod{J^k}.$$

Thereby $(z_i - z_i') \in \bigcap_{k \in \mathbb{N}} J^k = \{0\}$. Hence, the lift is unique. □

We write $\mathrm{Nilp}^{\flat}$ for the functor U_{T^q}. That is, $\mathrm{Nilp}^{\flat}(S) = \lim_{x \mapsto x^q} S^{\infty}$ is the set of q-power compatible sequences with values in S^{∞}.

Lemma 2.8.3. *For any $f \in \mathcal{F}_R$, there is a canonical bijection $U_{0,f}(S/J) \to \mathrm{Nilp}^{\flat}(S/J)$. This bijection is functorial in S.* Use different S

Proof. By assumption on f we have $f(T) = g(T^q) \in k_R[\![T]\!]$ for some $g(T) = c_1T + c_2T^2 + \dots$ with $c_1 \neq 0$. For each coefficient c_i, let $d_i \in k_R$ be the unique element such that $d_i^q = c_i$. Let $h(T) \in k_R[\![T]\!]$ be the power series given by $d_1T + d_2T^2 + \dots$. Now $(h(T))^q = f(T)$, and we find that

$$U_f(S/J) \to \mathrm{Nilp}^\flat(S/J): \quad (x_1, x_2, x_3, \dots) \mapsto (x_1, h(x_2), h(h(x_3)), \dots)$$

is a well-defined function, and (trivially) functorial in S. For the inverse, let $h^{-1}(T) \in k_R[\![T]\!]$ be the unique power series with $h^{-1}(h(T)) = h(h^{-1}(T)) = T$, see Lemma 2.1.9. The map

$$\mathrm{Nilp}^\flat(S/J) \to U_f(S/J), \quad (x_1, x_2, \dots) \mapsto (x_1, h^{-1}(x_2), h^{-1}(h^{-1}(x_3)), \dots)$$

is well-defined as

$$f(h^{-1}(T)) = g((h^{-1}(T))^q) = (h(h^{-1}(T)))^q = T^q,$$

and it is readily seen to be inverse to the map constructed above. □

We collect results.

Proposition 2.8.4. *Given $f, g \in \mathcal{F}_R$, we have bijections, functorial in S,*

$$U_f(S) \to U_f(S/J) \to \mathrm{Nilp}^\flat(S/J) \to U_g(S/J) \to U_g(S). \tag{2.12}$$

Explicitly, the bijection $U_f(S) \to U_g(S)$ can be described as follows. Suppose that $f, g \in \mathcal{F}_{R,q}$ for some sufficiently large q. Let $h_f(T)$ and $h_g(T)$ be power series with coefficients in A such that

$$h_f(T)^q \equiv f(T) \pmod I \quad \textit{and} \quad h_g(T)^q \equiv g(T) \pmod I.$$

Write $h_g^{-1}(T)$ for the (formal) inverse power series of h_g. Now the isomorphism is given by the mapping

$$(x_0, x_1, \dots) \mapsto (y_0, y_1, \dots), \quad \textit{where} \quad y_i = \lim_{r\to\infty} g^r(h_g^{-(r+i)}(h_f^{r+i}(x_{i+r}))).$$

Here, the exponents are to be interpreted as iterated composition.

Proof. The first part follows directly from repeated application of the previous two Lemmas. The second part follows by tracing through the previous lemmas. □

2.8.2 The Universal Cover

Let A be an integral domain and R be an A-algebra. Given $H \in (A\text{-FM}/R)$ and $a \in A$, we define the functor

$$\widetilde{H}_a : (R\text{-Adm}) \to (A\text{-Mod}), \quad S \mapsto \left\{ (x_1, x_2, \dots) \in \prod_{\mathbb{N}} H(S) \mid [a]_H(x_{i+1}) = x_i \right\}.$$

Here, the A-module structure is given by $b.(x_1, x_2, \dots) = ([b]_H(x_1), [b]_(x_2), \dots)$. Note that multiplication by a on $\tilde{H}_a(S)$ is an automorphism (it sends $(x_1, x_2, \dots)$ to $([a]_H x_1, x_1, x_2, \dots)$, which has inverse given by shifting to the left) so that $\tilde{H}_a(S)$ is naturally an $A[\frac{1}{a}]$-module.

From now on assume that A is a discrete valuation ring with uniformizer ϖ, finite residue field k and field of fractions K. Write $q = \#k$. Let R be a local A-algebra with maximal ideal I and algebraically closed residue field $k_R = R/I$. Let H be a formal ϖ-divisible A-module over R of height n.

Definition 2.8.5 (The Universal Cover and Tate Module)**.** We write $\tilde{H} = \tilde{H}_\varpi$. This functor takes values in the category of K-vector spaces. Up to natural isomorphism, $\tilde{H}$ does not depend on the choice of ϖ. We call this functor the universal cover of H.
The Tate-Module $T_\varpi H$ is the subfunctor of $\widetilde{H}$ cut out out by the condition that $[\varpi]_H(x_1) = 0$. Note that $T_\varpi H$ does no longer carry the structure of a K-vector space, it is an A-module. The Rational Tate Module $V_\varpi H$ is the subfunctor of $\widetilde{H}$ cut out by the condition that x_1 has $[\varpi]_H$-torsion. Equivalently, we have

$$V_\varpi H(S) = T_\varpi H(S) \otimes_A K.$$

Lemma 2.8.6. *Let H be a ϖ-divisible formal A-module over R and write $H_0 = H \otimes_R k_R$. Now the choice of a coordinate on H_0 gives rise to an isomorphism*

$$\tilde{H}_0 \cong \mathrm{Nilp}^\flat_{k_R}$$

of functors $(k_R\text{-Adm}) \to (\mathrm{Set})$

Proof. Note that given any coordinate on H, we have $[\varpi]_H(T) \in \mathcal{F}_R$. Hence, the statement is an application of Lemma 2.8.3. □

Lemma 2.8.7. *Suppose that S is an admissible R-algebra admitting an ideal of definition J such that $\varpi \in J$. Then the natural reduction map*

$$\tilde{H}(S) \to \tilde{H}(S/J)$$

is an isomorphism.

Proof. After choosing a coordinate on H, we have $[\varpi]_H \in \mathcal{F}_R$ and $\tilde{H}(S) \cong U_{[\varpi]_H}$, and the statement is given by Lemma 2.8.2. □

The following is analogous to Proposition 2.8.4.

Proposition 2.8.8. *Let S be an admissible R-algebra with ideal of definition J such that $\phi(I) \subseteq J$. Then there are canonical isomorphisms (of sets)*

$$\tilde{H}(S) \cong \tilde{H}(S/J) = \tilde{H}_0(S/J) \cong \mathrm{Nilp}^\flat(S/J) \cong \mathrm{Nilp}^\flat(S).$$

In particular, $\tilde{H}(S)$ is, as a functor to (Set)*, representable by* $\mathrm{Spf}(R[\![T^{q^{-\infty}}]\!])$.

We write λ for the isomorphism $\widetilde{H} \to \mathrm{Nilp}^\flat$, and $\lambda_i : \widetilde{H} \to (-)^\infty$ for projection on the i-th component. Similarly, we write $\mu : \mathrm{Nilp}^\flat \to \widetilde{H}$ for the inverse of λ and μ_i for the i-th component of μ.
By Proposition 2.8.8, we obtain an action of $\mathrm{End}(H \otimes_R k_R)$ on $\widetilde{H}$.

Definition 2.8.9 (Frobenius on $\widetilde{H}$)**.** Write $\Pi : \widetilde{H} \to \widetilde{H}$ for the automorphism of $\widetilde{H}$ corresponding to the Frobenius automorphism (which sends T to T^q) of H_0.

Note that $\lambda_i(\Pi x) = \lambda_i(x)^q$ for $x \in \widetilde{H}(S)$ and $i = 0, 1, \dots$.

Remark. In case where $\mathcal{F}$ comes from a $\mathcal{O}_K$-module law F over $\mathcal{O}_K$ with $[\varpi]_F(T) \equiv T^{q^n}$ mod (ϖ), the bijections

$$\tilde{H}(S) \rightleftarrows \mathrm{Nilp}^\flat(S), \quad (x_0, x_1, \dots) \leftrightarrows (y, y^{q^{-n}}, y^{q^{-2n}}, \dots)$$

are, in either direction, given by the equations

$$y^{1/q^{ni}} = \lim_{r\to\infty} x_{r+i}^{q^{nr}} \quad \text{and} \quad x_i = \lim_{s\to\infty} [\varpi^s]_H(y^{q^{-n(i+s)}}).$$

This follows directly from the explicit description of the isomorphism in Proposition 2.8.4, as we may choose $h_{[\varpi]_H}(T) = h_{T^{q^n}}(T) = T$.

We add calculations regarding the interplay of λ and $\log_H$ which will prove useful later.

Lemma 2.8.10. *Let H be the standard formal $\mathcal{O}_K$-module of height n over $R = \mathcal{O}_{\breve{K}}$. We have a commutative diagram (cf. [BW11, Lemma 2.6.1])*

$$\begin{tikzcd}
(x_0, x_1, \dots) \arrow[d, mapsto] & \in & \widetilde{H}(S) \arrow[rr, "\lambda"] \arrow[dr, "\log_H"'] & & \mathrm{Nil}^\flat(S) \arrow[dl] & \ni & (y, y^{1/q}, \dots) \arrow[d, mapsto] \\
\sum_{i=0}^\infty \frac{x_0^{q^{ni}}}{\varpi^i} & & & S \otimes_{\mathcal{O}_K} K & & & \sum_{i=-\infty}^\infty \frac{y^{q^{ni}}}{\varpi^i}
\end{tikzcd}$$

With this terminology, we have $\log_H((\Pi^j x)_0) = \sum_{i=-\infty}^\infty \frac{y^{ni+j}}{\varpi^i}$.

Proof. This follows directly from the remark above. Let $x \in \widetilde{H}(S)$ and write $(y, y^{1/q}, \dots)$ for $\lambda(x)$. We have $x_0 = \lim_{s\to\infty}[\varpi^s]_H(y^{-ns})$, hence

$$\log_H(x_0) = \lim_{s\to\infty} \left(\varpi^s \log_H(y^{1/q^{ns}})\right) = \lim_{s\to\infty} \left(\sum_{i=0}^\infty \frac{y^{q^{n(i-s)}}}{\varpi^{i-s}}\right) = \sum_{i=-\infty}^\infty \frac{y^{q^{ni}}}{\varpi^i}.$$

The second part is an immediate consequence. $\square$

2.9 The Quasilogarithm Map

We keep the assumptions on A, R and S from the previous subsection. That is, A is a local ring with finite residue field and uniformizer ϖ, R is a local A-algebra with maximal ideal I complete with respect to the I-adic topology and algebraically closed residue field k_R, and S denotes an admissible R-algebra (where $R \to S$ is continuous with the I-adic topology on R) with ideal of definition $J \subseteq S$ containing the image of I.

The aim of this subsection is to define, attached to any ϖ-divisible formal A-module $\mathcal{F}$ over R, a map

$$\mathrm{qlog}_{\mathcal{F}} : \tilde{\mathcal{F}}(S) \to \mathrm{D}(\mathcal{F}) \otimes_R (S \otimes_A K),$$

called the quasi-logarithm map. We give an explicit description of this map if $\mathcal{F} = H$ is the standard $\mathcal{O}_K$-module over $\mathcal{O}_{\breve{K}}$.

We begin with a sequence $(x_1, x_2, \dots) \in \tilde{\mathcal{F}}(S)$. Let $0 \to \mathcal{V} \xrightarrow{\psi} \mathcal{E} \xrightarrow{\phi} \mathcal{F} \to 0$ be the universal additive extension of $\mathcal{F}$, and choose an arbitrary sequence $(y_1, y_2, \dots) \in \tilde{\mathcal{E}}(S)$ such that y_i is a lift of x_i under the map $\mathcal{E}(S) \to \mathcal{F}(S)$. Let y be the limit $y = \lim_{i\to\infty}[\varpi]^i_{\mathcal{E}}(y_i)$ and put

$$\mathrm{qlog}_{\mathcal{F}}((x_1, x_2, \dots)) = \log_{\mathcal{E}}(y) \in \mathrm{D}(\mathcal{F}) \otimes_R (S \otimes_A K).$$

Proposition 2.9.1. *This construction yields a well-defined map.*

Proof. We may assume that $\mathcal{F}$ and $\mathcal{V}$ come from a formal module laws F and V, which canonically leads to a module E for $\mathcal{E}$ by Lemma 2.5.3. Now $(x_1, x_2, \dots)$ is a sequence in S^{∞} and $(y_1, y_2, \dots)$ is a sequence of elements in $(S^{\infty})^n$.

It suffices to show that $y = \lim_{i\to\infty}[\varpi]^i_E(y_i)$ exists and that it is independent of the choice of lifts $(y_1, y_2, \dots)$. Both claims follow from additivity of $\mathcal{V}$, implying that $[\varpi]_V(T) = \varpi T$. The sequence $([\varpi^i]_E(y_i))$ converges, as for positive integers $i \le j$, we have

$$[\varpi^i](y_i) - [\varpi^j](y_j) = [\varpi^i]([\varpi^{i-j}]y_j - y_i) \in \psi(\varpi^i(S^{\infty})^{n-1}) \subseteq J^i(S^{\infty})^n.$$

If $(y_1', y_2', \dots)$ is another sequence of lifts, put $y' = \lim_{i\to\infty}[\varpi^i]_E(y_i') \in S^{\infty}$. Now there exists some $z \in \mathcal{V}(S)$ such that $y - y' = \psi(z)$. But by construction $z \in \bigcap_{i\in\mathbb{N}} \varpi^i(S^{\infty})^{n-1} = 0$. $\square$

Let us now consider the case where $\mathcal{F} = \mathrm{FG}(H)$ comes from the standard formal $\mathcal{O}_K$-module of height n over $\mathcal{O}_{\breve{K}}$. Then from Proposition 2.5.21 we have the distinguished basis elements of $\mathrm{Ext}(H, \widehat{\mathbb{G}}_a)$ corresponding to the symmetric 2-cocycles δf_i, $1 \le i \le n-1$ where $f_i(T) = \frac{1}{\varpi}\log_H(T^{q^i})$. Also recall that, setting $f_0(T) = \log_H(T)$, the elements $(f_0, f_1, \dots, f_{n-1})$ freely generate $\mathrm{QLog}(H)$. The universal additive extension now corresponds to the symmetric 2-cocycle $(\delta f_1, \dots, \delta f_{n-1}) \in \mathrm{SymCoc}^2(H, V)$. We can make the quasi-logarithm map explicit.

Proposition 2.9.2. *Let* $x = (x_0, x_1, \dots) \in \widetilde{H}(S)$. *With respect to the basis* $(\log_H(T), \log_H(T^q), \dots, \log_H(T^{q^{n-1}}))$ *of* $\mathrm{QLog}(H) \otimes_{\mathcal{O}_K} K$*, the quasi-logarithm map is given by*

$$\mathrm{qlog}_H(x) = \Big(\log_H(x_0), \log_H((\Pi x)_0), \dots, \log_H((\Pi^{n-1}x)_0)\Big) \in (S \otimes K)^n.$$

Here, $\Pi x = ((\Pi x)_0, (\Pi x)_1, \dots)$ *is the image of* x *under* Π*, the endomorphism of* $\widetilde{H}(S)$ *induced by the Frobenius automorphism on* H_0*, cf. Definition 2.8.9.*

We postpone the proof to state the following auxiliary result.

Lemma 2.9.3. *Let* $x = (x_0, x_1, \dots) \in \widetilde{H}(S)$. *For positive integers* i *and* j *we have*

$$\log_H((\Pi^j x)_i) = \lim_{r\to\infty} \varpi^r \log_H(x_{r+i}^{q^j}).$$

Proof. Tracing through the commutative square (with λ and μ the isomorphisms from the previous subsection)

$$\begin{tikzcd}
\widetilde{H}(S) \arrow[r, "\lambda"] \arrow[d, "\Pi"] & \mathrm{Nilp}^\flat(S) \arrow[d, "(y_i)_i \mapsto (y_i^q)_i"] \\
\widetilde{H}(S) & \mathrm{Nilp}^\flat(S), \arrow[l, "\mu"']
\end{tikzcd}$$

we find

$$(\Pi^j x)_i = \lim_{s\to\infty} \lim_{r\to\infty} \left([\varpi]_H^s (x_{r+s+i}^{q^{nr+j}}) \right). \tag{2.13}$$

The claim follows after applying $\log_H$ and making repeated use of the functional equation $\log_H(T^{q^n}) = \varpi \log_H(T) + \varpi T$. $\square$

Proof of Proposition 2.9.2. Using the coordinates provided by $(\delta f_1, \dots, \delta f_{n-1})$, the universal additive extension of H is isomorphic to

$$0 \to \widehat{\mathbb{G}}_a^{n-1} \to E \to H \to 0,$$

where E is a module law with

$$[\varpi]_E(\mathbf{X}, T) = \Big(\varpi X_1 + (\delta_\varpi f_1)(T), \dots, \varpi X_{n-1} + (\delta_\varpi f_{n-1})(T), [\varpi]_H(T) \Big).$$

Beginning with $x = (x_0, x_1, \dots) \in \widetilde{H}(S)$, lifting to $(y_0, y_1, \dots) \in E(S)^{\mathbb{N}}$ and writing $y = \lim_{i\to\infty} [\varpi]_E^i(y_i)$, we find

$$y = \Big(\lim_{r\to\infty} (\delta_{\varpi^r} f_1)(x_r), \dots, \lim_{r\to\infty} (\delta_{\varpi^r} f_{n-1})(x_r), x_0 \Big) \in E(S).$$

Now, Lemma 2.9.3 provides the equality

$$\lim_{r\to\infty} \delta_{\varpi^r} f_i(x_r) = \frac{1}{\varpi} \lim_{r\to\infty} \varpi^r \log_H(x_r^{q^{nr+i}}) - \frac{1}{\varpi} \log_H \left(x_0^{q^i} \right) = \frac{1}{\varpi} \left(\log_H((\Pi^i x)_0) - \log_H(x_0^{q^i}) \right).$$

We need to calculate $\log_E(y)$, which calls for an explicit description of $\log_E : E \otimes (R \otimes_A K) \to (\widehat{\mathbb{G}}_a \otimes (R \otimes_A K))^n$. Tracing through the procedure provided in Subsection 2.3, we find

$$\log_E(\mathbf{X}, T) = \left(X_1 + \tfrac{1}{\varpi} \log_H(T^q), \dots, X_{n-1} + \tfrac{1}{\varpi} \log_H(T^{q^{n-1}}), \log_H(T) \right).$$

This representation is with respect to the basis $(f_1, \dots, f_{n-1}, f_0)$. The claim follows. $\square$

2.10 Determinants

Let H be the standard formal $\mathcal{O}_K$-module over $\mathcal{O}_{\breve{K}}$ of height n. Write $\wedge H$ for the formal $\mathcal{O}_K$-module over $\mathcal{O}_{\breve{K}}$ with logarithm

$$\log_{\wedge H}(T) = \sum_{i=0}^{\infty} (-1)^{(n-1)i} \frac{T^{qi}}{\varpi^i}.$$

We have $\mathrm{D}(\wedge H) = \wedge^n \mathrm{D}(H)$. We follow [BW11, Theorem 2.10.3] to describe a map $\delta : \widetilde{H}^n \to \widetilde{\wedge H}$ making the square

$$\begin{tikzcd} \widetilde{H}^n(S) \arrow[r, "\delta"] \arrow[d, "\mathrm{qlog}_H \times \cdots \times \mathrm{qlog}_H"'] & \widetilde{\wedge H}(S) \arrow[d, "\mathrm{qlog}_{\wedge H}"] \\ \mathrm{D}(H)^n \otimes (S \otimes_{\mathcal{O}_K} K) \arrow[r, "\det"] & \mathrm{D}(\wedge H) \otimes (S \otimes_{\mathcal{O}_K} K) \end{tikzcd} \tag{2.14}$$

commute.

Let $(s_1, \dots, s_n) \in \widetilde{H}(S)^n$, and write $x_i = \lambda(s_i) \in \mathrm{Nilp}^\flat(S)$, which are elements in S^∞ with distinguished q-power roots. Here $\lambda : \widetilde{H} \to \mathrm{Nilp}^\flat$ is the isomorphism from Section 2.8 with inverse $\mu = (\mu_0, \mu_1, \dots)$. We set

$$\delta_0(s_1, \dots, s_n) = \sum_{(a_1,\dots,a_n)} \varepsilon(a_1, \dots a_n)\mu_0(x_1^{q^{a_1}} \cdots x_n^{q^{a_n}}) \in \wedge H(S),$$

where

- The sum takes place in $\wedge H(S)$.
- The sum ranges over n-tuples $(a_1, \dots, a_n)$ of (possibly negative) integers satisfying $a_1 + \cdots + a_n = n(n-1)/2$, subject to the condition that each a_i occupies a distinct residue class modulo n.
- The expression $\varepsilon(a_1, \dots, a_n)$ denotes the sign of the permutation $i \mapsto a_{i+1} \pmod{n}$ of $(0, \dots, n-1)$.

Proposition 2.10.1. *The map δ_0 makes the diagram*

$$\begin{tikzcd} \widetilde{H}^n(S) \arrow[r, "\delta_0"] \arrow[d, "\mathrm{qlog}_H^n"'] & \wedge H(S) \arrow[d, "\log_{\wedge H}"] \\ D(H)^n \otimes (S \otimes K) \arrow[r, "\det"] & D(\wedge H) \otimes (S \otimes K) \end{tikzcd}$$

commute. It is $\mathcal{O}_K$-multilinear and alternating.

Proof. This is part of the proof of [BW11, Theorem 2.10.3]. Commutativity follows from

$$\begin{aligned} \log_{\wedge H}(\delta_0(s_1, \dots, s_n)) &= \sum_{(a_1,\dots,a_n)} \varepsilon(\mathbf{a}) \log_{\wedge H} \mu_0(x_1^{q^{a_1}} \cdots x_n^{q^{a_n}}) \\ &= \sum_{(a_1,\dots,a_n)} \varepsilon(\mathbf{a}) \sum_{m \in \mathbb{Z}} (-1)^{(n-1)m} \frac{x_1^{q^{a_1+m}} \cdots x_n^{q^{a_n+m}}}{\varpi^m} = \det\left(\sum_{m \in \mathbb{Z}} \frac{x_i^{q^{mn+j-1}}}{\varpi^m} \right)_{\substack{1 \le i \le n, \\ 1 \le j \le n}}, \end{aligned}$$

which is equal to $\det(\mathrm{qlog}_H^n(s_1, \dots, s_n))$ by Proposition 2.9.2 and Lemma 2.8.10. The fact that δ_0 is multilinear and alternating ultimately follows from the corresponding properties of det, the fact that $\mathrm{Ker}(\log_H) = \wedge H[\varpi^\infty]$ (cf. Lemma 2.3.6) and topological considerations in the induced diagram in the category of adic spaces over $(\breve{K}, \mathcal{O}_{\breve{K}})$. $\square$

This allows us to define the sought for morphism of functors $\delta : \widetilde{H}^n \to \widetilde{\wedge H}$.

Definition 2.10.2. Put $\delta_i(s_1, \ldots, s_n) = \delta_0(\varpi^{-i}s_1, \ldots, s_n)$. Then $\delta = (\delta_0, \delta_1, \ldots)$ yields a map $\widetilde{H}^n \to \widetilde{\wedge H}$. It is K-multilinear and alternating.

Using the canonical identifications $\widetilde{H}^n \cong (\mathrm{Nilp}^\flat)^n$ and $\widetilde{\wedge H} \cong \mathrm{Nilp}^\flat$, the morphism δ yields a map $(\mathrm{Nilp}^\flat)^n \to \mathrm{Nilp}^\flat$, which in turn is the same as a power series

$$\Delta(X_1, \ldots, X_n) \in \mathcal{O}_{\breve{K}}[\![X_1^{q^{-\infty}}, \ldots, X_n^{q^{-\infty}}]\!]$$

together with distinguished q-th power roots. We have the following approximation of Δ.

Lemma 2.10.3. *We have* $\Delta(X_1, \ldots, X_n) \equiv \det(X_i^{q^{j-1}})_{\substack{1\le i\le n,\\ 1\le j\le n}}$ *modulo* $(X_1, \ldots, X_n)^{q^n}$.

Proof. [TODO. This is [BW11, Lemma 2.10.4], but they don't explain the proof.] □

3 Local Class Field Theory following Lubin–Tate

Let K be a local field with residue field k, put $q = \#k$, and denote by $\nu_K : K \to \mathbb{Z} \cup \{\infty\}$ the valuation of K, normalized such that $\nu_K(\varpi) = 1$ for a uniformizer ϖ of K. The aim of this subsection is to describe the maximal abelian extension of a local field K.
The Local Kronecker-Weber theorem gives an explicit description of the abelianization of the absolute Galois group of K only in terms of K:

Theorem 3.0.1 (Local Kronecker-Weber). *There is an isomorphism (canonical up to choice of a uniformizer $\varpi \in K$)*

$$\mathrm{Gal}(\overline{K}/K)^{\mathrm{ab}} \cong \mathrm{Gal}(K^{\mathrm{ab}}/K) \cong \mathcal{O}_K^\times \times \widehat{\mathbb{Z}}.$$

Here, K^{ab} denote the maximal abelian extension of K, which can (after choosing an algebraic closure of K) be described as $\overline{K}^{[G_K,G_K]}$.
The extension K^{ab} consists of two parts, we have $K^{\mathrm{ab}} = K^{\mathrm{rm}} \cdot K^{\mathrm{nr}}$. The field K^{nr}, the maximal unramified extension of K, has relatively simple structure. Describing the field K^{rm} (or rather, it's completion) is the hard part and it is here where we apply the theory of formal modules.
The valuation ν_K extends uniquely to $\overline{K}$, yielding a ϖ-adic norm on $\overline{K}$. Let C denote the completion with respect to this norm. An application of Krasner's Lemma implies that $\mathrm{Gal}(C/K) \cong \mathrm{Gal}(\overline{K}/K) =: G_K$. One readily checks that any $\sigma \in G_K$ yields a continuous automorphism $\mathcal{O}_C \to \mathcal{O}_C$, and we obtain a short exact sequence Ref

$$0 \to I_K \to G_K \to \mathrm{Gal}(\overline{k}/k) \to 0.$$

The subgroup $I_K \subset G_K$ is called the inertia subgroup of K, and we write $\breve{K}$ for the subfield of C fixed by I_K. In particular we have $\mathrm{Gal}(\breve{K}/K) \cong \mathrm{Gal}(\overline{k}/k)$. One readily confirms that $\breve{K}$ is complete with respect to the norm induced by K.

As the Galois group of any finite extension of k is cyclic, we find that $\mathrm{Gal}(\breve{K}/K)$ is abelian. In fact, it is isomorphic to $\widehat{\mathbb{Z}} = \lim_n(\mathbb{Z}/n\mathbb{Z})$. Hence K_∞ decomposes as $\breve{K} \cdot K_\varpi$ for some abelian, complete extension K_ϖ/K such that $K_\varpi \cap \breve{K} = K$. Now K_ϖ is the completion of K^{rm}. Observe that

$$\mathrm{Gal}(K_\infty/K) \cong \mathrm{Gal}(K_\varpi/K) \times \mathrm{Gal}(\breve{K}/K) \cong \mathrm{Gal}(K_\varpi/K) \times \widehat{\mathbb{Z}},$$

so Theorem 3.0.1, the local Kronecker-Weber Theorem, is equivalent to showing that the Galois group of K_ϖ over K is isomorphic to $\mathcal{O}_K^\times$.

4 Non-Abelian Lubin-Tate Theory: An Overview

In the preceeding chapter we used formal $\mathcal{O}_F$-modules to understand the maximial abelian extension of a local field F. The hope of non-Abelian Lubin-Tate theory is to gain insight about the Abelian extensions of F by considering certain moduli spaces of formal $\mathcal{O}_F$-modules. More precisely, attached to a formal $\mathcal{O}_F$-module H_0 over $\overline{\mathbb{F}}_q$ (determined up to isomorphism by its height n), we attach a system of rigid spaces $\{M_K\}_{K \subset \mathrm{GL}_n(\mathcal{O}_F)}$, the so called Lubin-Tate Tower. For $l \neq p$, the system of l-adic compactly supported cohomology groups $\{H^i_c(M_K, \overline{\mathbb{Q}}_l)\}_K$ admits commuting actions by $\mathrm{GL}_n(F)$, W_F and $D^\times$, where the latter denotes the units of the central divison algebra $D = \mathrm{End}_{(\mathcal{O}_K\text{-FM}/\overline{\mathbb{F}}_q)}(H_0) \otimes \mathbb{Q}$. This yields a correspondence of representations of the respective groups, and Harris and Taylor showed in [HT01] that the cohomology of middle degree induces (a version of) the Local Langlands Correspondence for GL_n. Our goal is an explicit description of (a part of) this correspondence, and we obtain such a description by understanding (a part of) the Lubin-Tate tower explicitely.

4.1 The Lubin-Tate Tower

4.1.1 Deformations of Formal Modules

We mostly follow [Str08, Chapter 2] for notation. Let $\mathcal{C}$ denote the category of local, Noetherian $\mathcal{O}_{\breve{K}}$-modules with residue field $\overline{\mathbb{F}}_q$. Let H_0 be a formal $\mathcal{O}_K$-module over $\overline{\mathbb{F}}_q$.

Definition 4.1.1 (Deformation)**.** Let $R \in \mathcal{C}$. A deformation of H_0 to R is a pair (H, ι) where H is a formal $\mathcal{O}_K$-module over R and ι is a quasi-isogeny

$$\iota : H_0 \dashrightarrow H \otimes_R \overline{\mathbb{F}}_q.$$

Two deformations (H, ι) and (H', ι') are isomorphic if there is an isomorphism $\tau : H \to H'$ with $\iota' \circ \tau = \iota$.

The Lubin-Tate space without level structure is the moduli space of such deformations. More precisely, we define it as the functor

$$\mathcal{M}_0 : \mathcal{C} \to (\mathrm{Set}), \quad R \mapsto \{\text{deformations } (H, \iota) \text{ of } H_0\}/\mathrm{iso}.$$

Theorem 4.1.2 (Representability of $\mathcal{M}_0$)**.** *The functor $\mathcal{M}_0$ is (non-canonically) representable, by the noetherian local ring*

$$A_0 \cong \mathcal{O}_{\breve{K}}[\![u_1, \ldots, u_{n-1}]\!].$$

In particular, there is a universal deformation $(F^{\mathrm{univ}}, \iota^{\mathrm{univ}})$, with $F^{\mathrm{univ}} \in (\mathcal{O}_{\breve{K}}\text{-FM}/A_0)$.

4.1.2 Deformations of Formal Modules with Drinfeld Level Structure

Definition 4.1.3 (Drinfeld level $\mathfrak{p}^m$-structure)**.** Let $R \in \mathcal{C}$ and $H \in (\mathcal{O}_K\text{-FM}/R)$. A Drinfeld level $\mathfrak{p}^m$-structure on H is a morphism of R-group schemes

$$(\mathfrak{p}^{-m}/\mathcal{O}_K)^{\oplus n} \to H(R)[\varpi^m]$$

such that after choosing a coordinate $H \cong \operatorname{Spf} R[\![T]\!]$, the power series $[\varpi]_H(T) \in R[\![T]\!]$ satisfies the divisibility constraint

$$\prod_{x \in (\mathfrak{p}^{-1}/\mathcal{O}_K)} (T - \phi(x)) \,\Bigg|\, [\varpi]_H(T).$$

The following examples might shed some light on this definition.

Example.

- $\widehat{\mathbb{G}}$
- Things over $\mathbb{F}_q$.
- Drinfeld Level
- Moduli Problem + Representability
- The Lubin-Tate Tower

4.1.3 The Group actions on the Tower and its Cohomology

- Action By $D^\times$ and GL_n
- Action by W_K via Weil descent Datum.

Definition 4.1.4 (Lubin-Tate rigid space with K-level structure)**.**

For $K \subset \mathrm{GL}_n(\mathcal{O}_F)$, write $M_{K,\varpi^{\mathbb{Z}}}$ for the quotient of M_K (cf. Definition 4.1.4) by the action of the subgroup $\varpi^{\mathbb{Z}} \subset D^\times$. Writing $\mathcal{M}_m = \coprod_{\delta \in \mathbb{Z}} \mathcal{M}_m^{(\delta)}$ induces $M_K = \coprod_{\delta \in \mathbb{Z}} M_K^{(\delta)}$, and the action of ϖ induces for any $\delta \in \mathbb{Z}$ an isomorphism $M_K^{(\delta)} \cong M_K^{(\delta+n)}$. Hence, $M_{K,\varpi^{\mathbb{Z}}}$ is isomorphic to $\coprod_{0 \le \delta \le n-1} M_K^{(\delta)}$.
Let $l \neq p$ be a prime number and fix an isomorphism $\overline{\mathbb{Q}}_l \cong \mathbb{C}$.

Definition 4.1.5 (Cohomology of the Lubin–Tate tower)**.** We write $H_{\mathrm{LT}} = \lim_K H_c^{n-1}(M_{K,\varpi^{\mathbb{Z}}} \otimes_{\breve{F}} C, \overline{\mathbb{Q}}_l)$.

Theorem 4.1.6 (Non-Abelian Lubin–Tate theory)**.** *Let π be an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$ whose central character is trivial on $\varpi^{\mathbb{Z}}$. We write $\mathrm{rec}_F(\pi)$ for the irreducible smooth representation of W_F corresponding to π undet the local Langlands correspondence, and $\mathrm{JL}(\pi)$ for the irreducible smooth representation of $D^\times$ corresponding to π under the local Jacquet-Langlands correspondence. Then we have*

$$H_{\mathrm{LT}}[\pi^\vee] = \pi^\vee \boxtimes \mathrm{JL}(\pi) \boxtimes \mathrm{rec}_F(\pi)(\tfrac{1-n}{2})$$

as representations of $\mathrm{GL}_n(F) \times D^\times \times W_F$.

Proof. □

4.2 The Local Langlands Correspondence for the General Linear Group

We set

$$H_{\mathrm{LT}} := \lim_K H_c^{n-1}(M_{K,\varpi^{\mathbb{Z}}} \otimes_{\breve{F}} C, \overline{Q}_l) \quad \text{and} \quad H'_{\mathrm{LT}} := \lim_K H_c^{n-1}(M_K^{(0)} \otimes_{\breve{F}} C, \overline{Q}_l). \tag{4.1}$$

Also, we set

$$G := \mathrm{GL}_n(F) \times D^\times/\varpi^{\mathbb{Z}} \times W_F \quad \text{and} \quad G^1 := \{(g,d,\sigma) \in \mathrm{GL}_n(F) \times D^\times \times W_F \mid \det(g)^{-1}\,\mathrm{Nrd}(d)\mathrm{Art}_F^{-1}(\sigma) = 1\}. \tag{4.2}$$

Lemma 4.2.1. *The natural map $G^1 \to G$ is injective and realizes G^1 as a co-compact closed normal subgroup of G.*

Proof. The morphism $G^1 \to G$ is clearly injective, Further, the image of the natural homomorphism is isomorphic to the kernel of the map $\nu : G \to F^\times/\varpi^{n\mathbb{Z}}$, given by $\nu(g,\overline{d},\sigma) = \det(g)^{-1}\,\mathrm{Nrd}(d)\mathrm{Art}_F^{-1}(\sigma)$. The claim follows. □

We have actions $G \curvearrowright H_{\mathrm{LT}}$ and $G^1 \curvearrowright H'_{\mathrm{LT}}$. [Again, this uses Weil-Descent Data; make this precise.]

Theorem 4.2.2 (Non-Abelian Lubin-Tate Theory)**.** *Let π be a irreducible supercuspidal representation of GL_n whose cetral character is trivial on $\varpi^{\mathbb{Z}}$. Then, as representations of $\mathrm{GL}_n(F) \times D^\times \times \mathrm{W}_F$, the $\pi^\vee$-supercuspidal part of H_{LT} has the form*

$$H_{\mathrm{LT},\pi^\vee} = \pi^\vee \boxtimes \mathrm{JL}(\pi) \boxtimes \mathrm{rec}_F(\pi)(\tfrac{1-n}{2}), \tag{4.3}$$

$\mathrm{JL}(\pi)$ is a representation of $D^\times$ and $\mathrm{rec}_F(\pi)$ is a representation of W_F. The assignments $\pi \mapsto \mathrm{JL}(\pi)$ and $\pi \mapsto \mathrm{rec}_F(\pi)$ satisfy the conditions imposed on the Jacquet–Langlands and local Langlands correspondeces for GL_n.

Lemma 4.2.3. *These actions are smooth.*

Proof. [TODO] □

Lemma 4.2.4. *The G-representation* $\text{c-Ind}_{G^1}^{G}(H'_{\mathrm{LT}})$ *is isomorphic to* H_{LT}.

Proof. [TODO] □

4.3 The Lubin-Tate Perfectoid Space

5 Mieda's Approach to the Explicit Local Langlands Correspondence

We follow [Mie16].
Still, let F denote a local field with uniformizer ϖ and residue field $\mathbb{F}_q$.

6 Deligne–Lusztig Theory for Depth Zero Representations

The aim of this section is to outline the construction of a correspondence between certain characters of $\mathbb{F}_{q^n}^\times$ (with values in $\mathbb{C}^\times$) and cuspidal representations of $\mathrm{GL}_n(\mathbb{F}_q)$. The correspondence we construct here is an application of a more general theory developed by Deligne–Lusztig. In [DL76], they construct for any connected reductive algebraic group G over $\mathbb{F}_q$ and any Frobenius-stable maximal torus $T \subseteq G$ a correspondence associating to certain characters θ of T^F a virtual representation $R_{T,\theta}$ of G^F. These virtual representations arise from the l-adic cohomology (with $l \neq p$) of a certain variety $\mathrm{DL}_{G,T}$ admitting commuting actions by G^F and T^F. In this section, we give explicit descriptions of the occuring spaces in the situation where $G = \mathrm{GL}_{V_0 \otimes \overline{\mathbb{F}}_q}$ for some n-dimensional $\mathbb{F}_q$-vector space V_0 and $T \subset G$ is a maximal Frobenius-stable torus with $T(\overline{\mathbb{F}}_q) = \mathbb{F}_{q^n}^\times$. The main theorems of the theory are stated as facts, proofs are omitted.

6.1 Deligne–Lusztig Varieties for the General Linear Group

We begin by introducing (full) flags and their classifying objects, flag varieties. Let k be a field and let V be a finite dimensional k-vector space of dimension n. We write $\tilde{V}$ for the corresponding quasi-coherent sheaf on $\operatorname{Spec} k$, and GL_V for the general linear group scheme of $\tilde{V}$.

Definition 6.1.1 (Flag Variety)**.** Let $X : (\mathrm{Sch}/k)^{\mathrm{op}} \to (\mathrm{Set})$ be the functor assigning to each k-scheme $f : S \to \operatorname{Spec} k$ the set

$$X(S) = \left\{ \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_{n-1} \subset f^*\tilde{V} \;\middle|\; \begin{array}{c} \text{For all } i,\ \mathcal{F}_i \text{ is locally a direct summand} \\ \text{of } f^*\tilde{V},\ \text{locally free of rank } i \end{array} \right\}.$$

Recall that a subsheaf $\mathcal{F}_i \subset f^*\tilde{V}$ is locally a direct summand if it is quasi-coherent, and for each $s \in S$ there is some neighbourhood U of s such that $\mathcal{F}_i|_U$ is a direct summand of $f^*\tilde{V}|_U$. The S-valued points of S are called families of flags over S.

Elements of $X(k)$ are called (full) Flags. They are given by an increasing $n-1$-tuple of vector spaces

$$F_\bullet = (F_1 \subsetneq F_2 \subsetneq \dots F_{n-1} \subsetneq V) \in X(k).$$

There are natural morphisms

$$\nu_i : X \to \mathrm{Grass}_{V,i}, \quad (\mathcal{F}_1 \subset \dots \subset \mathcal{F}_{n-1} \subset f^*\tilde{V}) \mapsto (f^*\tilde{V}^\vee \twoheadrightarrow \mathcal{F}_i^\vee),$$

where $\mathrm{Grass}_{V,i}$ denotes the Grassmannian with respect to the finite free $\mathcal{O}_{\mathrm{Spec}\, k}$-module $\tilde{V}^\vee$, as defined in [Ric22].

Proposition 6.1.2. *The induced morphism of functors*

$$X \to \mathrm{Grass}_{V,1} \times_{\mathrm{Spec}\, k} \cdots \times_{\mathrm{Spec}\, k} \mathrm{Grass}_{V,n-1}$$

is representable by a closed embedding. In particular, as $\mathrm{Grass}_{V,d}$ *is representable by a projective scheme for integers* $1 \leq d \leq n-1$*, the functor* X *is representable by a projective scheme.*

Proof. Upon picking a basis of V, the claim can be checked directly on the standard affine cover of the Grassmannians, where the condition that $\mathcal{F}_i$ is contained in $\mathcal{F}_{i+1}$ is a polynomial condition. For representability of $\mathrm{Grass}_{V,d}$, cf. [Ric22, Theorem 5.1.4]. □

There is a natural GL_V-action on X, induced by the natural action of $\mathrm{GL}_V(S)$ on $f^*\tilde{V}$. Given a flag $F_\bullet \in X(k)$, we write $B_{F_\bullet} \subset \mathrm{GL}_V$ for the isotropy subgroup of $F_\bullet$ under this action. In [DL76], the authors work with schemes arising as quotients G/B where B is some Borel subgroup of a connected, reductive algebraic group G. It is an elementary fact that $B_{F_\bullet}$ is a Borel subgroup of GL_V, and the following proposition shows that X is of this form.

Proposition 6.1.3. *The morphism of schemes* $\mu_{F_\bullet} : \mathrm{GL}_V \to X$, $g \mapsto g.F_\bullet$ *yields an isomorphism* $\mathrm{GL}_V / B_{F_\bullet} \to X$.

Proof. We show that Zariski-locally, $\mu_{F_\bullet}$ induces an isomorphism $\mathrm{GL}_V(S)/B_{F_\bullet}(S) \to X(S)$. Let $(v_1, v_2, \dots, v_n)$ be a basis of V such that each F_i is generated by the first i basis vectors. Given any k-scheme S and a family of flags $F'_\bullet \in X(S)$, there is a Zariski-cover $\phi : S' \to S$ (with structure map to k denoted by f') trivializing all of the quotients F'_i/F'_{i-1} for $i = 1, \dots, n$. Hence we may choose generators $w_i \in \Gamma(S', \phi^*(F'_i/F'_{i-1}))$, and lift them to elements $\tilde{w}_i \in \Gamma(S', f'^*\tilde{V})$. The global sections w_i generate $f'^*\tilde{V}$, and the $\mathcal{O}_{S'}$-linear map $f'^*v_i \mapsto w_i$ yields an element in $\mathrm{GL}_V(S')$, unique up to an element in $B_{F_\bullet}(S')$. Thereby $X(S') \cong \mathrm{GL}_V(S')/B_{F_\bullet}(S')$. □

Remark. The proof shows that the quotient GL_V / B is trivialized Zariski-locally (as opposed to fppf-locally).

Corollary 6.1.4. *The scheme* X *is smooth over* k*, of dimension* $\frac{n(n-1)}{2}$.

Proof. This follows as quotients of smooth algebraic groups by algebraic subgroups are smooth (cf. [Mil17, Corollary 5.26]), and the fact that

$$n^2 = \dim \mathrm{GL}_V = \dim B + \dim X = \frac{n(n+1)}{2} + \dim X,$$

cf. [Mil17, Proposition 5.23]. □

Write $\mathbb{T}_X$ for the sheaf assigning to an X-scheme $S \to X$ (corresponding to a family of flags $(\mathcal{F}_i)_i \in X(S)$) the group

$$\mathbb{T}_X(S) = \mathrm{Aut}_{\mathcal{O}_S}(\mathcal{F}_1/\mathcal{F}_0) \times \cdots \times \mathrm{Aut}_{\mathcal{O}_S}(\mathcal{F}_n/\mathcal{F}_{n-1}) = \mathbb{G}^n_{m,X}(S).$$

Definition 6.1.5 (Functor of Marked Flags)**.** Let Y be the functor $(\mathrm{Sch}/X)^{\mathrm{op}} \to (\mathrm{Set})$ given by sending a morphism $S \to X$, corresponding to a family of flags $(\mathcal{F}_i)_i \in X(S)$, to the set

$$Y(S) = \left\{(e_1, \dots, e_n) \mid e_i : \mathcal{O}_S \xrightarrow{\sim} \mathcal{F}_i/\mathcal{F}_{i-1}\right\}.$$

Here, $\mathcal{F}_0$ is the zero-sheaf.

Just like X, the functor Y comes with a natural action by GL_V and the natural morphism

$$Y(S) \to X(S), \quad (\mathcal{F}_i, e_i)_i \mapsto (\mathcal{F}_i)_i \tag{6.1}$$

is equivariant for this action. One readily checks that Y is a sheaf in $(\mathrm{Sch}/X)_{\mathrm{Zar}}$, the big Zariski site of schemes over X. By design, it is a $\mathbb{T}_X$-torsor and thereby admits a Zariski-cover of open subfunctors isomorphic to $\mathbb{T}_X$. Hence it is representable (cf. [GW20, Theorem 8.9]), and the morphism $Y \to X$ is smooth and affine.

Furthermore, the scheme Y is also isomorphic to certain quotients of algebraic groups. Let $F_\bullet \in X(k)$ be a flag and $e_\bullet$ be a marking. Let $U_{F_\bullet, e_\bullet} \subset \mathrm{GL}_V$ be the (unipotent) isotropy subgroup of the corresponding element $(F_\bullet, e_\bullet) \in Y(k)$ under the action of GL_V.

Lemma 6.1.6. *In this situation,* $Y \cong \mathrm{GL}_V / U_{F_\bullet, e_\bullet}$.

Proof. This can be shown using the same arguments as in the proof of Proposition 6.1.3. □

If $(v_1, \dots, v_n)$ is a basis for V, we write $F(v_1, \dots, v_n)$ for the (marked) flag spanned by the vectors $(v_1, \dots, v_n)$. More generally, if S is a k-scheme $(v_1, \dots, v_n)$ is a tuple of elements in $\Gamma(S, f^*\tilde{V})$ such that the induced map $(v_1, \dots, v_n) : \mathcal{O}_S^n \to f^*\tilde{V}$ is an isomorphism (in which case we call $(v_i)_i$ a basis), we write $F(v_1, \dots, v_n)$ for the corresponding family of (marked) flags.

Recall the Bruhat decomposition for GL_V. Fixing a basis $(e_1, \dots, e_n)$ of V, we obtain an injection $\Sigma_n \hookrightarrow \mathrm{GL}_V$ (assigning to each $w \in \Sigma_n$ the corresponding permutation matrix), and a (marked) flag $F_\bullet^{\mathrm{std}} = F(e_1, \dots, e_n) \in X(k)$. For any such choice of a basis, we define O_w as the GL_V-orbit of the pair of flags $(F_\bullet^{\mathrm{std}}, w.F_\bullet^{\mathrm{std}}) \in (X \times X)(k)$. Note that this does not depend on the choice of basis. The Bruhat decomposition states that all GL_V-orbits inside $X \times X$ are of this form.

Proposition 6.1.7 (Bruhat Decomposition). *There is a decomposition of $X \times X$ into GL_V-stable locally closed subschemes*

$$X \times X = \dot{\bigcup_{w \in \Sigma_n}} O_w.$$

Each O_w is smooth of dimension $\dim(X) + l(w)$*, where $l(w)$ denotes the Coxeter-length of w.*

Proof. By construction, O_w is locally closed and smooth (orbits are locally closed by [Mil17, Proposition 1.65 b)]). The remaining claims boil down to classical theory (in particular, the classical Bruhat decomposition), cf. [Mil17, Chapter 21], and elementary considerations about the dimensions of the isotropy subgroups of pairs $(F_\bullet, F'_\bullet) \in X_w(k)$. $\square$

Let $(F_\bullet, F'_\bullet) \in (X \times X)(S)$ be a pair of flags over a k-scheme S. We say that $(F_\bullet, F'_\bullet)$ is in relative position $w \in \Sigma_n$ if it lies inside $O_w(S)$.
Similarly, we can characterize the GL_V-orbits in $Y \times_{\operatorname{Spec} k} Y$. For any choice of elements $w \in \Sigma_n \subset \mathrm{GL}_V(k)$ and $t \in \mathbb{T}_X(k)$, we define $\tilde{O}_{w,t}$ as the GL_V-orbit of the element

$$((F_\bullet^{\mathrm{std}}, e_\bullet^{\mathrm{std}}), (w.F_\bullet^{\mathrm{std}}, w.te_\bullet^{\mathrm{std}})) \in (Y \times Y)(k).$$

A pair of marked flags over a k-scheme S is said to be in relative position $(w,t) \in \Sigma_n \times (k^\times)^n$ if it appears in $\tilde{O}_{w,t}$. The following proposition gives a convenient characterization of relative position.

Lemma 6.1.8. *1. A pair of families of flags $(F_\bullet, F'_\bullet) \in (X \times X)(S)$ is in relative position $w \in \Sigma_n$ if and only if there exists a Zariski-cover $\phi : S' \to S$ (with structure map to k denoted by f') and a basis $(v_1, \dots, v_n) \in \Gamma(S', f'^*\tilde{V})$ such that*

$$\phi^* F_i = \langle v_1, \dots, v_i \rangle \quad \text{and} \quad \phi^* F'_i = \langle v_{w(1)}, \dots, v_{w(i)} \rangle \quad \text{for all } i = 1, \dots, n-1.$$

2. A pair of families of marked flags $((F_\bullet, e_\bullet), (F'_\bullet, e'_\bullet)) \in (Y \times Y)(S)$ is in relative position (w,t) if and only if $(F_\bullet, F'_\bullet)$ is in relative position w and there is a basis as above furthermore satisfying

$$\phi^* e_i \equiv v_i \mod \phi^* F_{i-1} \quad \text{and} \quad \phi^* e'_i \equiv t_{w(i)} v_{w(i)} \mod \phi^* F'_{w(i)-1} \quad \text{for all } i = 1, \dots, n.$$

Here, ϕ^* denotes the natural pullback of sections $\Gamma(S, f^*\tilde{V}) \to \Gamma(S', f'^*\tilde{V})$.

Proof. This is a mere reformulation of what it means to be in the corresponding GL_V-orbits. Given any choice of 'standard' basis $(e_1, \dots, e_n)$ of V and a section $(F_\bullet, F'_\bullet)$ in the orbit of $(F_\bullet^{\mathrm{std}}, w.F_\bullet^{\mathrm{std}})$, we may choose S' such that there exists a $g \in \mathrm{GL}_V(S')$ satisfying $g.(F_\bullet^{\mathrm{std}}|_{S'}, w.F_\bullet^{\mathrm{std}}|_{S'}) = (F_\bullet, F'_\bullet)$. Now it is easily seen that the global sections $v_i = g(e_i) \in \Gamma(S', f^*\tilde{V})$ satisfy the desired conditions. Conversely, any such basis yields an element in $\mathrm{GL}_V(S')$. The same ideas lead to the second statement. Note that here we only need to lift the sections e_i to sections in $\phi^*\tilde{V}$, which is possible once S' is affine. $\square$

We now specialize to the case where $k = \overline{\mathbb{F}}_q$ is an algebraic closure of the finite field with q elements, and $V = V_0 \otimes_{\mathbb{F}_q} k$ for some $\mathbb{F}_q$-vector space V_0. This equips V with a $\mathrm{Gal}(k/\mathbb{F}_q)$-action, and in particular the Frobenius automorphism of k (given on k by $x \mapsto x^q$) yields a k-semilinear automorphism $\mathrm{Frob} : V \to V$. As this automorphism sends subspaces to subspaces, we obtain automorphisms

$$\mathrm{Frob} : X \to X \quad \text{and} \quad \mathrm{Frob} : Y \to Y.$$

Note that X and Y are defined over $\mathbb{F}_q$, and these automorphisms come from the respective (relative) frobenii of X and Y over k. We write γ_{Frob} for the corresponding graph-morphisms $X \to X \times_{\operatorname{Spec} k} X$ and $Y \to Y \times_{\operatorname{Spec} k} Y$.

For $w \in \Sigma_n$ and $t \in \mathbb{T}_X$, we define the spaces

$$X_w := O_w \times_{X \times_{\operatorname{Spec} k} X, \gamma_{\mathrm{Frob}}} X \quad \text{and} \quad Y_{w,t} := \tilde{O}_{w,t} \times_{Y \times_{\operatorname{Spec} k} Y, \gamma_{\mathrm{Frob}}} Y \tag{6.2}$$

As γ_{Frob} admits a section, X_w is naturally a subscheme of X, parametrizing those families of flags that are pointwise in relative position w to their Frobenius twist. Similarly, $Y_{w,t}$ is naturally a subscheme of Y, parametrizing families of marked flags in relative position (w,t) with their Frobenius twist. We have natural maps $Y_{w,t} \to X_w$. As a pair of marked flags $((F_\bullet, e_\bullet), (F'_\bullet, e'_\bullet))$ over S lies in the same GL_V-orbit as $((F_\bullet, t_\bullet e_\bullet), (F'_\bullet, t'_\bullet e'_\bullet))$ for $t_\bullet, t'_\bullet \in \mathbb{G}^n_{m,X}(S)$ if and only if $t_\bullet = t'_\bullet$, we find that $Y_{w,t}$ is a Zariski-torsor over X_w for an affine group scheme $\mathbb{T}_w^{\mathrm{Frob}} \times_k X_w$, where $\mathbb{T}_w^{\mathrm{Frob}}$ is the k-group scheme with S-rational points given by

$$\mathbb{T}_w^{\mathrm{Frob}}(S) = \{(t_1, \dots, t_n) \in \mathbb{G}_m^n(S) \mid t_i^q = t_{w(i)}\}.$$

Furthermore, an element $g \in \mathrm{GL}_V(S)$ stabilizes $Y_{w,t}(S) \subset Y(S)$ if $g \in \mathrm{GL}_V^{\mathrm{Frob}}(S)$, so we obtain a $\mathrm{GL}_V^{\mathrm{Frob}}$ action on $Y_{w,t}$ and X_w. The morphism $Y_{w,t} \to X_w$ is equivariant for the $\mathrm{GL}_V^{\mathrm{Frob}}$-action.

One can show that the scheme X_w is smooth (of pure dimension $l(w)$, as the intersection in (6.2) is transverse, cf. [DL76]), so $Y_{w,t}$ is smooth and affine over X_w. To this end, we have constructed the spaces in the commutative diagram

$$\begin{tikzcd}
Y_{w,t} \arrow[r, hook] \arrow[d, "\mathbb{T}_w^{\mathrm{Frob}}\text{-torsor}"'] & Y \cong \mathrm{GL}_V / U \arrow[d, "\mathbb{T}_X\text{-torsor}"] \\
X_w \arrow[r, hook] & X \cong \mathrm{GL}_V / B.
\end{tikzcd} \tag{6.3}$$

The interesting space is $Y_{w,t}$. It comes with commuting (left-)actions of $\mathrm{GL}_V^{\mathrm{Frob}}(k) = \mathrm{GL}_V(\mathbb{F}_q)$ and $\mathbb{T}_w^{\mathrm{Frob}}(k) = \mathbb{F}_{q^n}^\times$.

6.2 An Explicit Example

We keep the notation from the previous subsection. That is, $k = \overline{\mathbb{F}}_q$, $V = V_0 \otimes_{\mathbb{F}_q} k$, X is the flag variety of V, and Y is the variety of marked flags. In this subsection, we fix $w = (1\ 2\ \dots\ n) \in \Sigma_n$ and $t = (1, \dots, 1) \in \mathbb{G}^n_{X,m}$, and give explicit descriptions of the resulting varieties appearing in the square (6.3). To clearify notation, we write $\mathrm{DL}_V = Y_{w,t}$ in this situation.

First, note that $\mathbb{T}_w^{\mathrm{Frob}}(S) = \mathbb{G}_m(S)^{\mathrm{Frob}^n}$, implying that DL_V admits commuting actions by

$\mathrm{GL}(V_0)$ and $\mathbb{T}_w^{\mathrm{Frob}}(k) = \mathbb{F}_{q^n}^\times$.

Lemma 6.2.1. *A pair of flags $(F_\bullet, F'_\bullet) \in (X \times X)(S)$ is in relative position $(1\ 2\ \dots\ n)$ if and only if for all $i = 1, \dots, n-1$ the condition $F_i + F'_i = F_{i+1}$ is satisfied. Here, the sum denotes the Zariski-sheafification of the corresponding presheaf.*

Proof. It is easily seen that the criterion may be checked Zariski-locally. Here it follows quickly from Lemma 6.1.8 and the Bruhat decomposition, Proposition 6.1.7. $\square$

For a linear form $\mu \in V^\vee$, we write $D^+(\mu)$ for the affine open subscheme of $\mathbb{P}(V)$ parametrizing lines in V that do not lie in the hyperplane defined by the equation $\mu(v) = 0$.

Proposition 6.2.2. *If $w = (1\ 2\ \dots\ n)$, the morphism of functors defined on k-schemes S by*

$$\Phi(S) : X_w(S) \to \mathbb{P}(V)(S) \quad F_\bullet \mapsto F_1$$

yields an isomorphism

$$X_w \cong \bigcap_{\mu \in V_0^\vee} D^+(\mu) \subset \mathbb{P}(V).$$

That is, X_w parametrizes lines in V that do not lie inside any $\mathbb{F}_q$-rational hyperplane. In particular X_w is equal to a finite intersection of affine subschemes, hence an affine scheme itself.

Proof. We first show that the image of any family of flags $F_\bullet \in X_w(S)$ lies inside $\cap_\mu D^+(\mu)(S)$. As the latter is an open subscheme, it suffices to show that any $s \in |S|$ maps into $\cap_\mu D^+(\mu)$, which is the case if any only if $F_1(s) = F_{1,s} \otimes_{\mathcal{O}_{S,s}} \kappa(s)$ does not lie inside any $\mathbb{F}_q$-rational hyperplane in $\kappa(s) \otimes_k V$. By Lemma 6.2.1, we find

$$F_1(s) \oplus \mathrm{Frob}(F_1(s)) \oplus \cdots \oplus \mathrm{Frob}^{n-1}(F_1(s)) = V \otimes_k \kappa(s),$$

so $F_1(s)$ cannot lie inside any non-trivial Frobenius-stable linear subspace of $V \otimes_k \kappa(s)$. The claim follows.

To see bijectivity of Φ, note that the inverse is, if well-defined, given by

$$\Psi : \bigcap_{\mu \in V_0^\vee} D^+(\mu)(S) \to X_w(S), \quad L \mapsto \left(L \oplus \mathrm{Frob}(L) \oplus \cdots \oplus \mathrm{Frob}^{i-1} L\right)_{i=1,\dots,n}.$$

To see that this is indeed well-defined, choose a basis $(e_1, \dots, e_n)$ of V_0 and take any section $\mathcal{L} \in \cap_{\mu \in V_0^\vee} D^+(\mu)(S)$, interpreted as a locally direct summand of $f^*\tilde{V}$. Hence for any $s \in S$ there is some open affine $\operatorname{Spec} R \subset S$ trivializing $\mathcal{L}$. Write L for the corresponding free rank 1 direct summand of R^n. Then $L = \langle v \rangle$ for some $v = (v_1, \dots, v_n) \in R^n$. Now L constitutes a flag if and only if $(v, \mathrm{Frob}(v), \dots, \mathrm{Frob}^{n-1}(v))$ is a basis for R^n if and only if $\det(\mathrm{Frob}^{j-1}(v_i))_{i,j} \in R^\times$. The last condition is satisfied. Indeed, if not, we may choose a maximal ideal $\mathfrak{m} \in \operatorname{Spec} R$ containing $\det(\mathrm{Frob}^{j-1}(v_i))_{i,j}$. Let $\overline{v}$ denote the residue of v in $(R/\mathfrak{m})^n$. Now, the subspace

$$\langle \overline{v}, \mathrm{Frob}(\overline{v}), \dots, \mathrm{Frob}^{n-1}(\overline{v}) \rangle \subset (R/\mathfrak{m})^n$$

is non-trivial and Frobenius-stable, and in particular contained in some $\mathbb{F}_q$-rational hyperplane. This contradicts $\mathcal{L} \in \bigcap_{\mu \in V_0^\vee} D^+(\mu)(S)$. As both sides are Zariski-sheaves, the morhpism Ψ is well-defined. $\square$

We write $\Delta : \mathrm{Sym}(\bigwedge V^\vee) \to \mathrm{Sym}(V^\vee)$ for the morphism corresponding to the k-linear morphism

$$\bigwedge V^\vee \to \mathrm{Sym}(V^\vee), \quad \mu \mapsto \left[v \mapsto \mu(v \wedge \mathrm{Frob}(v) \wedge \dots \mathrm{Frob}^{n-1}(v))\right].$$

Proposition 6.2.3. *The map of functors $\mathrm{DL}_V \to \operatorname{Spec}\mathrm{Sym}(V^\vee)$ given by $(F_\bullet, e_\bullet) \mapsto e_1$ yields an isomorphism of DL_V and the subfunctor of $\operatorname{Spec}\mathrm{Sym}(V^\vee)$ given on affine schemes by*

$$\operatorname{Spec} R \mapsto \left\{ v \in R \otimes_k V \;\middle|\; \begin{array}{c} (v, \mathrm{Frob}\, v, \dots, \mathrm{Frob}^{n-1} v) \textit{ is a basis and} \\ v \wedge \dots \wedge \mathrm{Frob}^{n-1} v = (-1)^{n-1} \mathrm{Frob}(v \wedge \dots \wedge \mathrm{Frob}^{n-1} v) \end{array} \right\}.$$

Writing S_1 for the degree-1 part of $\mathrm{Sym}(\bigwedge V^\vee)$, this functor is readily seen to representable by the k-scheme

$$\mathrm{DL}_V := \operatorname{Spec}\left(\frac{\mathrm{Sym}(V^\vee)[\Delta(S_1 \setminus \{0\})^{-1}]}{(\mathrm{Frob}(\Delta(\lambda)) - (-1)^{n-1}\Delta(\lambda) \mid \lambda \in S_1)} \right).$$

Upon choosing a basis of $V_0 \cong \mathbb{F}_q^n$, this takes on the form

$$\mathrm{DL}_n := \operatorname{Spec}\left(\frac{k[x_1, \dots, x_n]}{(\det D(\underline{x})^{q-1} - (-1)^{n-1})} \right), \quad \textit{where} \quad D(\underline{x}) = \begin{pmatrix} x_1 & \dots & x_1^{q^{n-1}} \\ \vdots & \ddots & \vdots \\ x_n & \dots & x_n^{q^{n-1}} \end{pmatrix}.$$

Proof. Let $S = \operatorname{Spec} R$ be an affine k-scheme and let $(F_\bullet, e_\bullet) \in \mathrm{DL}_V(S)$. By Lemma 6.1.8, there is a basis $(v_1, \dots, v_n)$ of $R \otimes_k V$ such that

$$\begin{aligned} v_i &\equiv e_i \mod F_{i-1} \quad &\text{and} \quad v_{i+1} &\equiv \mathrm{Frob}(e_i) \mod \mathrm{Frob}(F_{i-1}) \quad \text{for } 1 \le i \le n-1, \\ v_n &\equiv e_n \mod F_{n-1} \quad &\text{and} \quad v_1 &\equiv \mathrm{Frob}(e_n) \mod \mathrm{Frob}(F_{n-1}). \end{aligned} \tag{6.4}$$

From here we quickly find

$$\mathrm{Frob}(v_1 \wedge v_2 \wedge \dots \wedge v_n) = \mathrm{Frob}\, v_1 \wedge \dots \wedge \mathrm{Frob}\, v_n = v_2 \wedge v_3 \wedge \dots \wedge v_n \wedge v_1.$$

The equivalences in (6.4) also imply that for integers $2 \le m \le n$, we have $\mathrm{Frob}^{m-1} v_1 \equiv v_m \mod \mathrm{Frob}(F_{m-2})$. Also, we find $v_1 \equiv \mathrm{Frob}^n v_1 \mod \mathrm{Frob}(F_{n-1})$. Altogether, writing $v = v_1 = e_1$, this yields

$$\mathrm{Frob}(v \wedge \mathrm{Frob}(v) \wedge \dots \wedge \mathrm{Frob}^{n-1} v) = (-1)^{n-1}(v \wedge \dots \wedge \mathrm{Frob}^{n-1} v). \tag{6.5}$$

This shows that the map given in the statement of the proposition is well-defined. To see that it is bijective, note that it has an inverse. Indeed, given any $v \in R \otimes_k V$ such that $(v, \mathrm{Frob}\, v, \dots, \mathrm{Frob}^{n-1} v)$ is a basis and v satisfies the equation (6.5), Gaussian elimination shows that the corresponding marked flag is in relative position $(w, 1)$ to its Frobenius-twist.

If we are given a basis of V_0, we may write $v = (x_1, \dots, x_n)$ and identify $v \wedge \mathrm{Frob}\, v \wedge \dots \mathrm{Frob}^{n-1} v$ with $\det \left[(x_i^{q^{j-1}})_{1 \le i,j \le n} \right]$. Thereby v gives a marked flag in DL_n if and only if

$$\det((x_i^{q^{j-1}})_{i,j})^{q-1} = (-1)^{n-1}.$$

This gives the representability statement of DL_n. $\square$

Note that DL_n has $q-1$ disjoint irreducible components, parametrized by the set of solutions $b \in k$ to the equation $z^{q-1} = (-1)^{n-1}$. For any such $b \in k$ we write (in accordance with notation in [Mie16])

$$Y_b := \operatorname{Spec} \left(\frac{k[x_1, \dots, x_n]}{(\det D(\underline{x}) - b)} \right), \tag{6.6}$$

and obtain $\mathrm{DL}_n = \sqcup_{b^{q-1} = (-1)^{n-1}} Y_b$.
The (left-)action of $\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times$ on DL_n has the following explicit description.

Lemma 6.2.4. *The action of* $\mathrm{GL}_n(\mathbb{F}_q)$ *on* DL_n *is given by*

$$(a_{ij})_{i,j}.(x_1, \dots, x_n) = \left(\sum_{j=1}^n a_{ij} x_j \right)_{i=1,\dots,n}$$

The action of $\mathbb{F}_{q^n}^\times$ *is given by*

$$d.(x_1, \dots, x_n) = (dx_1, \dots, dx_n).$$

Through the relative Frobenius, we may also define an action of $\mathbb{Z}$ on DL_n, sending 1 to the automorphism given by $x_i \mapsto \mathrm{Frob}(x_i) = x_i^q$. By construction, the actions of $\mathrm{GL}_n(\mathbb{F}_q)$, $\mathbb{F}_{q^n}^\times$ and $\mathbb{Z}$ commute.
Note that $g \in \mathrm{GL}_n(\mathbb{F}_q)$ induces a morphism of schemes $Y_b \to Y_{b \det(g)}$. Similarly, $\zeta \in \mathbb{F}_{q^\times}$ restricts to $Y_b \to Y_{b\mathrm{N}(\zeta)}$, where $\mathrm{N} = \mathrm{N}_{\mathbb{F}_{q^n}/\mathbb{F}_q}$ denotes the norm map of the extension $\mathbb{F}_{q^n}/\mathbb{F}_q$. The relative Frobenius morphism $\mathrm{DL}_n \to \mathrm{DL}_n$ stabilizes the components Y_b. In particular, the stabilizer subgroup of $\mathrm{GL}_n \times \mathbb{F}_{q^n}^\times \times \mathbb{Z}$ for each component Y_b is given by

$$(\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times)^1 \times \mathbb{Z} := \{(g, d, n) \in \mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times \times \mathbb{Z} \mid \det(g)\mathrm{N}(d) = 1\}.$$

We write H_{DL} for the $\overline{\mathbb{Q}}_l$-vector space

$$H_{\mathrm{DL}} = H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_l).$$

By the above, this is a representation of $\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times \times \mathbb{Z}$. The following subsection is concerned with the study of this representation.

6.3 An Example of the Deligne–Lusztig Correspondence

In this subsection, we are concerned with the étale cohomology of the variety DL_n introduced in the previous subsection. As before, k denotes an algebraic closure of $\mathbb{F}_q$.

We give a short review of [DL76]. For a connected, reductive algebraic group G defined over $\mathbb{F}_q$ and a maximal Frobenius-stable torus $T \subset G$ contained in a Borel subgroup $B \subset G$, Deligne and Lusztig construct varieties with right G^F-actions $X_{T\subset B}$ and $\tilde{X}_{T\subset B}$, constituting a G^F-equivariant Galois-covering

$$\tilde{X}_{T\subset B} \to X_{T\subset B}$$

with Galois-group T^F. See (6.8) for explicit descriptions of these spaces. The space $\tilde{X}_{T\subset B}$ comes with commuting actions of G^F and T^F, and for characters θ of $T^F(k)$, the main concern of [DL76] is the study of the resulting virtual representations

$$R_T^\theta = \sum_i (-1)^i H_c^i(\tilde{X}_{T\subset B}, \overline{\mathbb{Q}}_l)_\theta \in \mathcal{R}(T^F(k) \times G^F(k)). \tag{6.7}$$

Here, the subscript θ denotes the direct summand of $H_c^i(\tilde{X}_{T\subset B}, \overline{\mathbb{Q}}_l)$ where $T^F(k)$ acts by θ. These representations do only depend on the choice of the torus, not on the choice of the Borel sugroup ([DL76, Corollary 4.3]).

Recall that for a finite group H, the Grothendieck group $\mathcal{R}(H)$ of finite dimensional H-representations in $\overline{\mathbb{Q}}_l$ comes with a natural inner product. Indeed, as $\rho(h)^{\#H} = 1$, any representation takes values in the maximal cyclotomic subfield $\cup_r \mathbb{Q}(\zeta_r) \subset \overline{\mathbb{Q}}_l$, which has the unique "complex conjugation" automorphism given by $\zeta_r \mapsto \zeta_r^{-1}$ for $r \in \mathbb{N}$. The inner product is now defined on representations $\rho, \rho' \in (H\text{-Rep})$ as

$$\langle \rho, \rho' \rangle = \frac{1}{\#H} \sum_{h\in H} \mathrm{Tr}(\rho(h))\overline{\mathrm{Tr}(\rho'(h))} \in \overline{\mathbb{Q}}_l.$$

This definition extends linearly to $\mathcal{R}(H)$. An orthogonal basis for $\mathcal{R}(H) \otimes \mathbb{Q}$ is given by the irreducible finite-dimensional representations of H. If ρ is an irreducible representation of H and R is an element in $\mathcal{R}(H) \otimes \mathbb{Q}$, we say that ρ occurs in R if $\langle \rho, R \rangle \neq 0$.

Definition 6.3.1 (Geometrically Conjugates and Regular Position)**.** Let T be a Frobenius-stable maximal torus of G and let θ be a character of $T^F(k)$.

1. We say that θ is in general position if it is not fixed by any non-trivial element of $\mathrm{W}_G(T)^F$. Here $\mathrm{W}_G(T)$ denotes the Weyl group $(\mathrm{N}_G(T)/T)$.

2. If (T', θ') is another pair consisting of a Frobenius-stable maximal torus and a character of $T^F(k)$, we say that the pairs (T, θ) and (T', θ') are geometrically conjugate if there exists an integer n such that the pairs $(T, \theta \circ N_n)$, $(T', \theta' \circ N_n)$ are conjugate, where N_n is the norm function $T^{\mathrm{Frob}^n}(k) \to T^{\mathrm{Frob}}(k)$.

We summarize a few of the results of the theory.

Theorem 6.3.2 (Some Results of Deligne-Lusztig Theory)**.**

1. [DL76, Corollary 1.22] *Let $Z \subset G$ denote the center of G. Then Z^{Frob} acts on $H_c^{n-1}(\tilde{X}_{T\subset B}, \overline{\mathbb{Q}}_l)_\theta$ through $\theta|_{Z^F}$.*

2. [DL76, Corollary 7.7] *For any irreducible representation ρ of G^F, there exists an F-stable maximal torus T and a character θ of T^F such that $\langle \rho, R_T^\theta \rangle \neq 0$.*

3. [DL76, Corollary 6.3] *If (T,θ) and (T',θ') are not geometrically conjugate, no irreducible representation of $G^F(k)$ occurs in both R_T^θ and $R_{T'}^{\theta'}$.*

4. [DL76, Corollary 7.3] *If (T,θ) is in general position, one of $\pm R_T^\theta$ is an irreducible representation.*

5. [DL76, Corollary 8.3] *If furthermore T is not contained in any proper Frobenius-stable parabolic subgroup, one of $\pm R_T^\theta$ is a cuspidal representation of $G^F(k)$.*

6. [DL76, Corollary 9.9] *If furthermore $\tilde{X}_{T\subset G}$ (or equivalently, $X_{T\subset G}$) is affine, and we denote by $w \in \mathrm{W}_G(T)$ the relative position of B and* $\mathrm{Frob}(B)$ *(cf. the Bruhat decomposition for G), we have*

$$H_c^i(\tilde{X}_{T\subset G},\overline{\mathbb{Q}}_l)_\theta = 0 \quad \textit{if} \quad i \neq l(w).$$

Here, $l(w)$ denotes the Coxeter-length of w.

7. [DL76, Theorem 9.8] *For a character θ in general position, the natural map*

$$H_c^i(Y_{T\subset B},\overline{\mathbb{Q}}_l)_\theta \to H^i(Y_{T\subset B},\overline{\mathbb{Q}}_l)_\theta$$

is an isomorphism.

We next explain how these results apply to the comactly supported étale cohomology of DL_n. Obviously we set $G = \mathrm{GL}_n$, and we fix the standard torus T^{std} (of diagonal matrices), the standard Borel B^{std} (of upper diagonal matrices) and the standard unipotent subgroup U^{std}. Choose a flag $F_\bullet \in X_w(k)$, where w is, as usual, $(1\ 2\ \ldots\ n) \in \Sigma_n \cong \mathrm{W}_G(T^{\mathrm{std}})$ (although the discussion remains valid for any choice of permutations). Let $B \subset \mathrm{GL}_n$ denote the isotropy subgroup of $F_\bullet$, and let $T \subset B$ be a Frobenius-stable torus in B. Write U for the unipotent radical of B. We have the explicit descriptions (cf. [DL76, Definition 1.17])

$$\begin{gathered} X_{T\subset B} = \{g \in \mathrm{GL}_n \mid g^{-1}F(g) \in F(U)\}/(T^F(U\cap F(U))) \\ \text{and} \\ \tilde{X}_{T\subset B} = \{g \in \mathrm{GL}_n \mid g^{-1}F(g) \in F(U)\}(U \cap F(U)). \end{gathered} \tag{6.8}$$

Given any marked flag $(F'_\bullet, e'_\bullet) \in X(S)$, we write $g(F'_\bullet, e'_\bullet) \in \mathrm{GL}_n/U^{\mathrm{std}}(S)$ for the corresponding section under the isomorphism of Proposition 6.1.6 (i.e., $g.(F^{\mathrm{std}}_\bullet, e^{\mathrm{std}}_\bullet) = (F'_\bullet, e'_\bullet)$). Also, we may choose $h \in \mathrm{GL}_n(k)$ such that $h(T^{\mathrm{std}}, B^{\mathrm{std}})h^{-1} = (T,B)$. Then by [DL76, Proposition 1.19], the map

$$\mathrm{DL}_n \to \tilde{X}_{T\subset B}, \quad (F'_\bullet, e'_\bullet) \mapsto g(F'_\bullet, e'_\bullet)h^{-1}$$

gives an isomorphism of $\mathrm{GL}_n^{\mathrm{Frob}}$-equivariant torsors

$$\begin{tikzcd} \mathrm{DL}_n \arrow[d, "\mathbb{T}_w^{\mathrm{Frob}}\text{-torsor}"'] & \tilde{X}_{T\subset B} \arrow[d, "T^{\mathrm{Frob}}\text{-torsor}"] \\ X_w \arrow[r, phantom, ""] & X_{T\subset B}. \end{tikzcd} \qquad \xrightarrow{\sim}$$

Given any representation θ of $\mathbb{T}_w^{\mathrm{Frob}}(k)$, we write

$$R_w^\theta = \sum_i (-1)^i H_c^i(\mathrm{DL}_n, \overline{\mathbb{Q}}_l)_\theta.$$

We may embed the torus $\mathbb{T}_w$, which is the standard torus with rational structure such that Frob acts by $\operatorname{ad} w \circ \mathrm{Frob}$, into GL_n. Now it is conjugated to T^{std}, and this identification yields isomorphisms $R_w^{\theta \circ \operatorname{ad} h} \cong R_{T \subset B}^\theta$, where h is chosen as above. If θ is a character of $\mathbb{T}_w^{\mathrm{Frob}}(k) \cong \mathbb{F}_{q^n}^\times$, the pair $(T, w \circ \operatorname{ad} h)$ is in regular position if and only if θ is regular, in the following sense.

Definition 6.3.3 (Regular Character on $\mathbb{F}_{q^n}^\times$)**.** We say that a character $\theta : \mathbb{F}_{q^n}^\times \to \mathbb{C}^\times$ is regular if it does not factor through the norm morphism

$$\mathrm{N}_{\mathbb{F}_{q^n}/\mathbb{F}_{q^m}} : \mathbb{F}_{q^n}^\times \to \mathbb{F}_{q^m}^\times$$

for any $m \leq n$.

The statements of Theorem 6.3.2 reduce to the following.

Theorem 6.3.4 (Deligne–Lusztig Correspondence)**.**

1. *If θ is a regular character of $\mathbb{F}_{q^n}^\times$, we have*

$$H_{\mathrm{DL},\theta} = H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_l)_\theta = R_\theta \boxtimes \theta$$

   as representations of $\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times$. The representation R_θ is irreducible and cuspidal, and it's central character is given by $\theta|_{\mathbb{F}_q^\times}$ under the identification $\mathbb{F}_q^\times \cong Z_{\mathrm{GL}_n}$.

2. *The natural map*

$$H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_l) \to H^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_l)$$

   is an isomorphism.

This finishes the discussion about the $\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times$-action on H_{DL}. Recall that the Frobenius-automorphism yields an action of $\mathbb{Z}$ on H_{DL}. This action admits the following partial description.

Proposition 6.3.5. *Let θ be a regular character. The subgroup $n\mathbb{Z} \subset \mathbb{Z}$ acts on $H_{\mathrm{DL},\theta}$ through the character $\gamma : n\mathbb{Z} \to \mathbb{Q}^\times$, given by*

$$\gamma(nm) = (-1)^{(n-1)m} q^{m\frac{n(n-1)}{2}}.$$

That is, Frob_q^n *induces multiplication by the scalar $(-1)^{n-1} q^{\frac{n(n-1)}{2}}$ on $H_{\mathrm{DL},\theta}$.*

Proof. The result is due to Digne–Michel, cf. [DM85]. Also see the proof of [Wan14, Théorème 3.1.12]. □

Concludingly, we obtain the following structural result about the $\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times \times \mathbb{Z}$-representation H_{DL}.

Theorem 6.3.6 (Structure of H_{DL}). *For a regular character $\theta : \mathbb{F}_{q^n}^\times \to \overline{\mathbb{Q}}_l^\times$, $H_{\mathrm{DL},\theta}$ is, as a representation of $\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times \times n\mathbb{Z}$, given by*

$$H_{\mathrm{DL},\theta} = R_\theta \boxtimes \theta \boxtimes \gamma.$$

Here, R_θ is a irreducible cuspidal representation, and γ is the character defined in the previous proposition.

7 Explicit Non-Abelian Lubin–Tate Theory for Depth Zero Supercuspidal Representations

7.1 The Special Affinoid

Definition 7.1.1 (The special Affinoid).

7.2 The Explicit Correspondence

Fix, for the remainder of the section, an isomorphism $\overline{\mathbb{Q}}_l \cong \mathbb{C}$ and a regular character $\theta : \mathbb{F}_{q^n}^\times \to \mathbb{C}^\times$. The datum of θ can be used to construct representations of W_F and $D^\times$ and, making use of Deligne–Lusztig theory, a representation of $\mathrm{GL}_n(F)$. We proceed as follows.

- Let $\overline{\tau}_\theta$ be the character of W_{F_n} given by the composition

  $$\mathrm{W}_{F_n} \to \mathrm{W}_{F_n}^{\mathrm{ab}} \xrightarrow{\mathrm{Art}_{F_n}^{-1}} F_n^\times \cong \mathbb{Z} \times \mathcal{O}_{F_n}^\times \twoheadrightarrow \mathbb{F}_{q^n}^\times \xrightarrow{\theta} \mathbb{C}^\times$$

  and put $\tau_\theta = \text{c-Ind}_{\mathrm{W}_{F_n}}^{\mathrm{W}_F}(\overline{\tau}_\theta)$.

- Let $\overline{\rho}_\theta$ be the character on $F^\times \mathcal{O}_D^\times$ given by the composition

  $$F^\times \mathcal{O}_D^\times \cong \varpi^{\mathbb{Z}} \times \mathcal{O}_D^\times \twoheadrightarrow \mathbb{F}_{q^n} \xrightarrow{\theta} \mathbb{C}^\times$$

  and let $\rho_\theta = \text{c-Ind}_{F^\times \mathcal{O}_D^\times}^{D^\times}(\overline{\rho}_\theta)$.

- Let $\overline{\pi}_\theta$ be the representation of $F^\times \mathrm{GL}_n(\mathcal{O}_F)$ arising from post-composing R_θ (cf. Theorem 6.3.4) with the composition

  $$F^\times \mathrm{GL}_n(\mathcal{O}_F) \cong \varpi^{\mathbb{Z}} \times \mathrm{GL}_n(\mathcal{O}_F) \twoheadrightarrow \mathrm{GL}_n(\mathcal{O}_F) \twoheadrightarrow \mathrm{GL}_n(\mathbb{F}_q).$$

  Let $\pi_\theta = \text{c-Ind}_{F^\times \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(F)}(\overline{\pi}_\theta)$.

Lemma 7.2.1. *The representations $\overline{\pi}_\theta$, $\overline{\rho}_\theta$ and $\overline{\tau}_\theta$ are smooth, in particular π_θ, ρ_θ and τ_θ are smooth as well. Additionally, the representations π_θ and ρ_θ are irreducible, and π_θ is supercuspidal.*

Proof. By design, $\overline{\pi}_\theta$ is trivial on the compact open subgroup $1+\varpi\,\mathrm{Mat}_{n\times n}(\mathcal{O}_F)$ of $F^\times \mathrm{GL}_n(\mathcal{O}_F)$. Similar statements hold for $\overline{\rho}_\theta$ and $\overline{\tau}_\theta$. [why is ρ_θ irreducible? Why is π_θ supercuspidal and irreducible?] □

The aim of this section is to prove the following statement.

Theorem 7.2.2 (Explicit Non-Abelian Lubin–Tate Theory for Depth Zero Supercuspidal Representations)**.** *The representation* $\mathrm{JL}(\pi_\theta)$ *of* $D^\times$ *and the representation* $\mathrm{rec}_F(\pi_\theta)$ *of* W_F *take the form*

$$\mathrm{JL}(\pi_\theta) = \rho_\theta \quad \text{and} \quad \mathrm{rec}_F(\pi_\theta) = \mathrm{Ind}_{\mathrm{W}_{F_n}}^{\mathrm{W}_F}(\tau_\theta\, \delta^{n-1}),$$

where $\delta : \mathrm{W}_{F_n} \to \{\pm 1\}$ *is the unramified quadratic character. This is the character corresponding to* $a \mapsto (-1)^{\mathrm{val}_{F_n}(a)}$ *under the isomorphism* $\mathrm{Art}_{F_n} : F_n^\times \to \mathrm{W}_{F_n}^{\mathrm{ab}}$.

We set

$$J := F^\times \mathrm{GL}_n(\mathcal{O}_F) \times \mathcal{O}_D^\times \times \mathrm{W}_{F_n} \text{ and } J^1 = J \cap G^1. \tag{7.1}$$

Also, we define a morphism

$$\Theta : J \to \mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times \times \mathrm{Frob}_q^{n\mathbb{Z}}, \quad (\varpi^m g, d, \sigma) \mapsto (\overline{g}, \overline{d^{-1} u_\sigma^{-1}}, \overline{\sigma}).$$

Recall that H_{DL} denotes the middle l-adic cohomology of DL_n, cf. Section 6.3.

Lemma 7.2.3. *The morphism* Θ *makes* J *act on* $H_{\mathrm{DL},\theta}$*. This representation is of the form*

$$(g, d, \sigma) \mapsto \overline{\pi}_\theta(g) \otimes \overline{\rho}_{\theta^{-1}}(d) \otimes \left(\overline{\tau}_\theta\, \delta^{n-1}\right)^{-1} (\tfrac{1-n}{2})(\sigma).$$

This representation is smooth.

Proof. This is a direct calculation. □

The input we get from Mieda's result is the following.

Proposition 7.2.4. *There is an injective morphism of* J^1*-representations*

$$\mathrm{Res}_{J^1}^{J}(H_{\mathrm{DL},\theta}) \hookrightarrow \mathrm{Res}_{J^1}^{G^1}(H'_{\mathrm{LT}}).$$

Proof. [This is [Mie16, Proposition 5.11].] □

Lemma 7.2.5. *The morphism in Proposition 7.2.4 naturally gives rise to an injective* J*-equivariant morphism*

$$H_{\mathrm{DL},\theta} \hookrightarrow \mathrm{Res}_J^G H_{\mathrm{LT}}.$$

Proof. We construct a sequence of J-equivariant injections

$$H_{\mathrm{DL},\theta} \hookrightarrow \mathrm{Ind}_{J^1}^{J}(\mathrm{Res}_{J^1}^{J} H_{\mathrm{DL},\theta}) \hookrightarrow \mathrm{Ind}_{J^1}^{J}(\mathrm{Res}_{J^1}^{G^1} H'_{\mathrm{LT}}) \xrightarrow{\sim} \mathrm{Res}_J^{G^1 J}(\mathrm{Ind}_{G^1}^{G^1 J} H'_{\mathrm{LT}}) \hookrightarrow \mathrm{Res}_J^G H_{\mathrm{LT}}.$$

The first morphism. This is the unit of the adjunction $\mathrm{Res}_{J^1}^{J} \dashv \mathrm{Ind}_{J^1}^{J}$ applied at $H_{\mathrm{DL},\theta}$, which is injective by Lemma B.0.13.

The second morphism. This is $\mathrm{Ind}_{J^1}^{J}$ applied to the injective morphism in Proposition 7.2.4. The resulting morphism is injective because $\mathrm{Ind}_{J^1}^{J}$ is exact, cf. Proposition B.0.10.

The third morphism. The morphism is given by the inverse of the base-change morphism constructed in Lemma B.0.14, which is applied with $H = G^1$, $N = J$. Note that G^1 is normal in G, so the assumptions of the Lemma are satisfied. As J is open in G, the map is an isomorphism.

The fourth morphism. Since G^1J is open in G, the unit of the adjunction $\text{c-Ind}_{G^1J}^G \dashv \mathrm{Res}_{G^1J}^G$ yields a monomorphism of G^1J-representations

$$\mathrm{Ind}_{G^1}^{G^1J} H'_{\mathrm{LT}} \to \mathrm{Res}_{G^1J}^G(\text{c-Ind}_{G^1J}^G(\mathrm{Ind}_{G^1}^{G^1J} H'_{\mathrm{LT}})). \tag{7.2}$$

As G^1J co-compact in G, we have $\text{c-Ind}_{G^1J}^G = \mathrm{Ind}_{G^1J}^G$, so the right-hand side is isomorphic to $\mathrm{Res}_{G^1J}^G(\mathrm{Ind}_{G^1}^G H'_{\mathrm{LT}}) \cong \mathrm{Res}_{G^1J}^G(H_{\mathrm{LT}})$ by Proposition B.0.12 and Lemma 4.2.4. Hence, applying $\mathrm{Res}_J^{G^1J}$ to the morphism in (7.2) yields the desired map. □

The morphism constructed in Lemma 7.2.5 yields, by Frobenius reciprocity, a non-zero map of G-representations

$$\mathrm{Ind}_J^G(H_{\mathrm{DL},\theta}) \cong \pi_\theta \boxtimes \rho_{\theta^{-1}} \boxtimes (\tau_\theta\, \delta^{n-1})^{-1}(\tfrac{1-n}{2}) \to H_{\mathrm{LT}}. \tag{7.3}$$

As π_θ is supercuspidal and its central character is trivial on $\varpi^{\mathbb{Z}}$, Theorem 4.2.2 yields a non-zero map

$$\rho_{\theta^{-1}} \boxtimes (\tau_\theta\, \delta^{n-1}) \to \mathrm{JL}(\pi_\theta)^\vee \boxtimes \mathrm{rec}_F(\pi_\theta)^\vee.$$

As $\rho_{\theta^{-1}}$ and $\mathrm{JL}(\pi_\theta)^\vee$ are irreducible, this implies $\mathrm{JL}(\pi_\theta) = \rho_{\theta^{-1}}^\vee = \rho_\theta$. As $\mathrm{rec}_F(\pi_\theta)$ is irreducible and $\dim(\tau_\theta) = n = \dim(\mathrm{rec}_F(\pi_\theta))$, this also implies $\tau_\theta\, \delta^{n-1} = \mathrm{rec}_F(\pi_\theta)$. Admitting Proposition 7.2.4, this concludes the proof of Theorem 7.2.2.

7.2.1 Proof of Proposition 7.2.4

Lemma 7.2.6. *The group J^1 is generated by the following elements.*

- $(g,1,1)$ *for* $g \in \mathrm{GL}_n(\mathcal{O}_F)$ *with* $\deg g = 1$,
- $(1,d,1)$ *for* $d \in \mathcal{O}_D^\times$ *with* $\mathrm{Nrd}\, d = 1$,
- $(a,a,1)$ *for* $a \in F_n^\times$,
- $(1, \mathrm{Art}_{F_n}^{-1}(\sigma)^{-1}, \sigma)$ *for* $\sigma \in W_{I_n}$,
- *and* $(1, \varpi^{-1}, \sigma)$ *for* $\sigma \in W_{F_n}$ *with* $\mathrm{Art}_{F_n}^{-1}(\sigma) = \varpi$.

The image of J^1 under the homomorphism Θ lies inside

$$\{(g,d,\sigma) \in \mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times \times \mathrm{Frob}_q^{n\mathbb{Z}} \mid \det(g)\mathrm{N}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(d) = 1\}.$$

Proof. □

Lemma 7.2.7. *Through the homomorphism Θ, J acts on DL_n over Y_b.*

Proof. □

Proposition 7.2.8. *The action of J^1 on $M_{\infty,C}^{(0)}$ stabilizes U and extends to $\mathcal{X}$. The induced action on the special fiber $\mathcal{X}_s$ is compatible with the action of J^1 on Y_b.*

Proof. □

A Topological Rings

To deal with the topological rings showing up, the notion of admissible rings will be convenient (taken from [Stacks, Tag 07E8]).

Definition A.0.1. Let A be a topological ring. We say that A is admissible if

- The element $0 \in A$ has a fundamental system of neighbourhoods consisting of ideals.
- There exists an ideal of definition, that is, an open ideal $I \subset A$ such that every open neighbourhood of 0 contains I^n for some n.
- It is complete, that is, the natural map

$$A \to \lim_{J \subset A \text{ open ideal}} A/J$$

  is an isomorphism.

We say that A is adic if it admits an ideal of definition I such that I^n is open for all n. Given a topological ring A, we denote the category of admissible and adic A-algebras (algebras S with continuous morphism $A \to S$) by (A-Adm) and (A-Adic), respectively.

[The following results might be not interesting enough to make it into the final draft]

Lemma A.0.2. *Let $\phi : R \to S$ be a morphism of admissible rings, and let $I \subset R$ be an admissible ideal. Then the ideal $J = \phi(I) \cdot S$ is an ideal of definition in S.*

Proof. Let U be an open ideal of S. By continuity of ϕ, it's preimage $U' = \phi^{-1}(U)$ is open in R. Hence there is some n with $I^n \subset U'$. But now

$$\phi(I)^n = \phi(I^n) \subseteq \phi(\phi^{-1}(U)) \subseteq U$$

and the claim follows. □

Lemma A.0.3. *Let S be an admissible ring, and let $(s_1, s_2, \dots)$ be a sequence with elements in S. Then $\sum_{i=1}^{\infty} s_i$ converges if and only if $\lim_{i\to\infty} s_i = 0$. In this case, the product $\prod_{i=1}^{\infty}(1+s_i)$ exists in S.*

Proof. If the sum converges, $(s_i)_{i\in\mathbb{N}}$ has to be a null-sequence. The reverse implication and the convergence of the product follows after writing $S \cong \lim_J S/J$ for a system of open ideals $J \subset S$. □

The topology on an admissible ring R with ideal of definition I is coarser than the I-adic topology on R

Lemma A.0.4. *Let R be an admissible ring with ideal of definition I. Let R' be the same ring, but equipped with the I-adic topology. Then the identity map $R' \to R$ is continuous. In particular, if a sequence converges with respect to the I-adic topology, it also converges in R'.*

Proof. It suffices to check that open ideals of R are open in R'. Let $J \subset R$ an open ideal. By assumption, there is some n with $I^n \subset J$. But now, for any $x \in J$, we have $x + I^n \subset J$. Hence, J is open in R'. □

B Smooth Representations of Locally Profinite Groups

We review some aspects of the representation theory (over complex vector spaces) of locally profinite groups. If G is an arbitrary group, we denote the category of complex representations, (that is, morphisms $G \to \mathrm{GL}(V)$, where V is a $\mathbb{C}$-vector space) as (G-Rep). At the slight cost of precision, we also allow ourselves to refer to an element of $\pi : G \to \mathrm{GL}(V) \in$ (G-Rep) by the underlying vector space V, or the pair (π, V).

Definition B.0.1 (Locally Profinite Group)**.** A locally profinite group is a Hausdorff topological group such that there exists a neighbourhood of $1 \in G$ consisting of compact open subgroups.

Throughout this section, if not stated otherwise, G is a locally profinite group and $H \subset G$ is a closed subgroup of G.

Definition B.0.2 (Smooth Representation)**.** A smooth representation of G is a representation $\pi : G \to \mathrm{GL}(V) \in$ (G-Rep), such that for any $v \in V$, the stabilizer G_v of v is an open subgroup of G. We define (G-Rep$^{\mathrm{sm}}$), the category of smooth G-representations, as the full subcategory of (G-Rep) with objects given by smooth G-representations.

The forgetful functor (G-Rep) $\to$ (G-Rep$^{\mathrm{sm}}$) has a left adjoint, given by taking smooth parts.

Definition B.0.3 (Smooth Part of a Representation)**.** Let $(\pi, V) \in$ (G-Rep). We write

$$V^{\mathrm{sm}} = \bigcup_{K \subseteq G} V^K,$$

where K runs over the compact open subgroups of G and $V^K \subseteq V$ denotes the subspace of elements fixed by K. Now V^{sm} is a G-stable subspace of V, and we write $(\pi^{\mathrm{sm}}, V^{\mathrm{sm}})$ for the induced representation $G \to \mathrm{GL}(V^{\mathrm{sm}})$ of π. We call π^{sm} the smooth part of π.

Definition B.0.4 (Algebraic Induction)**.** Let G be any group and let H be a subgroup of G. We define the Algebraic Induction Functor algInd_H^G : (H-Rep) $\to$ (G-Rep) as follows. Given an H-representation $(\pi, V) \in$ (H-Rep), consider the vector space

$$\mathrm{algInd}_H^G(V) = \{\phi : G \to V \mid \phi(hg) = \pi(h)g\}.$$

Now G acts naturally on $\mathrm{algInd}_H^G(V)$ by right-translation (that is, $g.\phi(x) = \phi(xg)$), and we write $(\mathrm{algInd}_H^G(\pi), \mathrm{algInd}_H^G(V))$ for the corresponding representation of G.

Remark. We have $\mathrm{algInd}_H^G(V) = \mathrm{Hom}_{(\mathbb{C}[H]\text{-Mod})}(\mathbb{C}[G], V)$. As $\mathbb{C}[G]$ has a natural $(\mathbb{C}[H], \mathbb{C}[G])$-bimodule structure, we obtain a natural left-G-action on $\mathrm{algInd}_H^G(V)$. This action is precisely the one described above.

Definition B.0.5 (Restriction Functor)**.** Let G be any group and let H be a subgroup of G. If $\pi : G \to \mathrm{GL}(V)$ is a representation of G, we define the restriction of π from G to H as

$$\mathrm{Res}_H^G(\pi) : H \hookrightarrow G \xrightarrow{\pi} \mathrm{GL}(V)$$

and call Res_H^G : (G-Rep) $\to$ (H-Rep) the restriction functor.

Lemma B.0.6. *Let G be any group and let H be any subgroup of G. Then Res_H^G is left-adjoint to algInd_H^G.*

Proof. By the Remark above, this statement readily reduces to the Tensor-Hom-Adjunction. □

Lemma B.0.7. *If G is locally profinite and H is a closed subgroup, for any $(\pi, V) \in (G\text{-Rep})$ we have an H-equivariant split inclusion*

$$\mathrm{Res}_H^G(V^{\mathrm{sm}}) \subseteq \left(\mathrm{Res}_H^G(V)\right)^{\mathrm{sm}},$$

with equality if H is open. In particular, Res_H^G restricts to a functor

$$\mathrm{Res}_H^G : (G\text{-Rep}^{\mathrm{sm}}) \to (H\text{-Rep}^{\mathrm{sm}}).$$

Proof. The first part follows from

$$\mathrm{Res}_H^G(V^{\mathrm{sm}}) = \bigcup_{K \subset G} V^K \subseteq \bigcup_{K \subset G} V^{K \cap H} = \left(\mathrm{Res}_H^G(V)\right)^{\mathrm{sm}},$$

where K runs over the compact open subsets of G. This is an equality if H is open. The canonical splitting (sending everything outside the image to zero) is H-equivariant. □

Definition B.0.8 (Smooth Induction)**.** We define the smooth indction functor $\mathrm{Ind}_H^G : (H\text{-Rep}^{\mathrm{sm}}) \to (G\text{-Rep}^{\mathrm{sm}})$ as the smooth part of the algebraic induction functor. That is, for any smooth representation $\pi : G \to \mathrm{GL}(V)$, we set

$$\mathrm{Ind}_H^G(\pi) := \left(\mathrm{algInd}_H^G(\pi)\right)^{\mathrm{sm}}.$$

Definition B.0.9 (Compact Induction)**.** Let $\pi : H \to \mathrm{GL}(V)$ be a smooth representation of H. Then we define $\text{c-Ind}_H^G(\pi)$, the compactly induced representation of π, as the subrepresentation of $\mathrm{Ind}_H^G(\pi)$ with underlying vector space

$$\{\phi \in \mathrm{Ind}_H^G(\pi) \mid \mathrm{Supp}(\phi) \subseteq G \text{ is compact in } H\backslash G\}.$$

This construction yields a functor $\text{c-Ind}_H^G : (H\text{-Rep}^{\mathrm{sm}}) \to (G\text{-Rep}^{\mathrm{sm}})$.

Note that if H is co-compact in G, we have $\text{c-Ind}_H^G = \mathrm{Ind}_H^G$.

Remark. If H is an open subgroup of G, the quotient $H\backslash G$ is discrete. Now given $(\pi, V) \in (H\text{-Rep}^{\mathrm{sm}})$, an element $\phi \in \mathrm{Ind}_H^G(\pi)$ lies in $\text{c-Ind}_H^G(\pi)$ if and only if the image of $\mathrm{Supp}(\phi)$ is finite in $H\backslash G$. In this case there is an isomorphism

$$\Psi : \text{c-Ind}_H^G(V) \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V, \quad \phi \mapsto \sum_{[g] \in \mathrm{Supp}(\phi)} g^{-1} \otimes \phi(g) \tag{B.1}$$

which does not depend on the choice of representative $g \in [g]$ as $(hg)^{-1} \otimes \phi(hg) = g^{-1} \otimes \phi(g)$. Giving $\mathbb{C}[G]$ the structure of an $(\mathbb{C}[G], \mathbb{C}[H])$-bimodule, the natural left-G-action on $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ is compatible with the one on $\text{c-Ind}_H^G(V)$ under the isomorphism Ψ.

Proposition B.0.10. *For H a closed subgroup of G, the functors* algInd_H^G, Ind_H^G *and* c-Ind_H^G *are exact.*

Proof. The statement for algInd_H^G is implied by the fact that $\mathbb{C}[G]$ is a free (thereby projective) $\mathbb{C}[H]$-module. For the remaining statements, see [BH06, p. 18f]. □

Theorem B.0.11 (Smooth Frobenius Reciprocity)**.** *Let G be a locally profinite group and let $H \subseteq G$ be a closed subgroup. Then there is an adjunction*

$$\mathrm{Res}_H^G \dashv \mathrm{Ind}_H^G.$$

If H is additionally assumed to be open in G, there is an adjunction

$$\text{c-Ind}_H^G \dashv \mathrm{Res}_H^G.$$

In particular, if H is co-compact and open in G, Ind_H^G *is both left- and right-adjoint to* Res_H^G*.*

Proof. Making use of the remarks above, both adjunctions are the Tensor-Hom-Adjunction in disguise. For the adjunction $\text{c-Ind}_H^G \dashv \mathrm{Res}_H^G$, this is immediate. For the second we observe that

$$\mathrm{Hom}_{(H\text{-}\mathrm{Rep}^{\mathrm{sm}})}(\mathrm{Res}_H^G V, W) \cong \mathrm{Hom}_{(G\text{-}\mathrm{Rep})}(V, \mathrm{algInd}_H^G(W)) = \mathrm{Hom}_{(G\text{-}\mathrm{Rep}^{\mathrm{sm}})}(V, \mathrm{Ind}_H^G(W)).$$

Here the first isomorphism is by Tensor-Hom-adjunction, the second equality uses that V is a smooth representation of G. □

Proposition B.0.12. *Let I be a closed subgroup of H. There is a natural isomorphism* $\mathrm{Ind}_H^G \circ \mathrm{Ind}_I^H \xrightarrow{\sim} \mathrm{Ind}_I^G$*. The same statement is true for compact and algebraic induction.*

Proof. Trivially, $\mathrm{Res}_I^G = \mathrm{Res}_I^H \circ \mathrm{Res}_H^G$. The claim follows as the functors in question are adjoints to the left or the right hand side of this equation, thereby isomorphic. □

Lemma B.0.13. *Let H be a closed subgroup of G. The functor* Res_H^G *is faithful. Equivalently, the unit* $\mathrm{id}_{(G\text{-}\mathrm{Rep}^{\mathrm{sm}})} \to \mathrm{Ind}_H^G \circ \mathrm{Res}_H^G$ *of the adjunction* $\mathrm{Res}_H^G \dashv \mathrm{Ind}_H^G$ *is injective on components. If H is additionally assumed to be an open subgroup of G, The functor* c-Ind_H^G *is faithful. Equivalently, the components of the unit* $\mathrm{id}_{(H\text{-}\mathrm{Rep}^{\mathrm{sm}})} \to \mathrm{Res}_H^G \circ \text{c-Ind}_H^G$ *coming from the adjunction* $\text{c-Ind}_H^G \dashv \mathrm{Res}_H^G$ *are injective.*

Proof. Faithfulness of Res_H^G is clear. For faithfulness of c-Ind_H^G, note that the unit of the adjunction $\text{c-Ind}_H^G \dashv \mathrm{Res}_H^G$ is given on components $(\pi, V) \in (H\text{-}\mathrm{Rep}^{\mathrm{sm}})$ by the map $v \mapsto \phi_v$, where $\phi_v \in \text{c-Ind}_H^G(V)$ is defined as

$$\phi_v : G \to V, \quad g \mapsto \begin{cases} \pi(g)v & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

The resulting morphism $V \to \mathrm{Res}_H^G(\text{c-Ind}_H^G(V))$ is injective. Now all claims follow since faithfulness of the left-adjoint is equivalent to the unit being a monomorphism on components, cf. [Rie17, Lemma 4.5.13]. □

Remark. For the sake of completeness, we note that the unit of the adjunction $\mathrm{Res}_H^G \dashv \mathrm{Ind}_H^G$ is given on components $(\pi, V) \in (G\text{-Rep}^{\mathrm{sm}})$ by

$$V \to \mathrm{Ind}_H^G(\mathrm{Res}_H^G(V)), \quad v \mapsto \psi_v; \quad \text{where} \quad \psi_v(g) = \pi(g)v.$$

The following Lemma is an instance of base-change.

Lemma B.0.14. *Let H and N be closed subgroups of G satisfying $NH = HN$. Let (π, V) be a smooth representation of H. Then there is a natural split monomorphism of N-representations*

$$\mathrm{Res}_N^{HN}(\mathrm{Ind}_H^{HN}\pi) \to \mathrm{Ind}_{H\cap N}^N(\mathrm{Res}_{H\cap N}^H\pi).$$

If N is open in HN, this map is an isomorphism.

Proof. One quickly checks that the map

$$\mathrm{Res}_N^{HN}(\mathrm{algInd}_H^{HN}V) \to \mathrm{algInd}_{H\cap N}^N(\mathrm{Res}_{H\cap N}^H V)$$

given by sending $\phi : HN \to V$ to its restriction $\phi|_N$, is an isomorphism. Now the claim follows by taking smooth parts and applying Lemma B.0.7. $\square$

Remark. There are multiple ways to construct the map above. Applying $\mathrm{Ind}_H^{HN}(-)$ to the unit of the adjunction $\mathrm{Res}_{H\cap N}^H \dashv \mathrm{Ind}_{H\cap N}^H$ yields for any $\pi \in (H\text{-Rep}^{\mathrm{sm}})$ a natural morphism

$$\mathrm{Ind}_H^{HN}(\pi) \to \mathrm{Ind}_{H\cap N}^H\mathrm{Res}_{H\cap N}^H(\pi) \cong \mathrm{Ind}_N^{HN}\mathrm{Ind}_{H\cap N}^N\mathrm{Res}_{H\cap N}^H(\pi),$$

which is equivalent to a map

$$\mathrm{Res}_N^{HN}(\mathrm{Ind}_H^{HN}\pi) \to \mathrm{Ind}_{H\cap N}^N(\mathrm{Res}_{H\cap N}^H\pi).$$

This gives the same map as in the proof. The dual construction (starting with the co-unit) also yields the same map.

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