TODO

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1 Local Class Field Theory following Lubin-Tate

This section will serve as an introduction to formal groups and formal modules. Formal groups (or rather, formal group laws) were first introduced by Salomon Bochner in 1946 as a natural means of studying Lie Groups over fields of characteristic 0, cf. [1]. The study of formal groups later became interesting for its own right, with pioneering works of Lazard [2]

1.1 Formal Modules

As promised in the introduction, we begin by defining formal group laws.

Definition 1.1.1 (Formal Group Law)**.** Let R be a ring. A (commututative, one-dimensional) formal group law over R is a power series $F(X,Y) \in R[\![X,Y]\!]$ such that $F(X,Y) \equiv X+Y$ modulo terms of degree 2 and the following properties are satisfied:

- $F(F(X,Y),Z) = F(X,F(Y,Z))$,
- $F(X,Y) = F(Y,X)$,
- $F(X,0) = X$.

Given two formal group laws $F, G \in R[\![X,Y]\!]$, a morphism $f : F \to G$ is a power series $f \in R[\![T]\!]$ such that $f(F(X,Y)) = G(f(X), f(Y))$. Such a series is an isomorphism if there is an inverse, that is, a power series $g \in R[\![T]\!]$ with $(f \circ g)(T) = T$. This yields the category of formal group laws over R, which we notate by (FGL/R).

The following statements about morphisms of formal group laws are useful and easily verified.

Lemma 1.1.2. *Let R be a ring and let $F, G \in R[\![X,Y]\!]$ be two formal group laws over R.*

1. *Given two morphisms $f, g : F \to G$, the power series $G(f(T), g(T)) \in R[\![T]\!]$ is a morphism of formal group laws $F \to G$. In particular, $\mathrm{Hom}_{(\mathrm{FGL}/R)}(F,G)$ is an abelian group for any two formal group laws F, G.*
2. *The abelian group $\mathrm{End}_{(\mathrm{FGL}/R)}(F)$ has a natural ring structure with multiplication given by concatenation.*

3. *A morphism $f = c_1T + c_2T^2 + \cdots \in R[\![T]\!]$ between F and G is an isomorphism if and only if $c_1 \in R^\times$.*

Example. Let us introduce the following two formal group laws.

- *The additive formal group law.* Write $\mathbb{G}_a$ for the formal group law with addition given by $\mathbb{G}_a(X, Y) = X + Y$.
- We write $\mathbb{G}_m$ for the formal group law associated with the with $\mathbb{G}_m(X, Y) = X + Y + XY$.

Next up is the definition of formal A-module laws. Naively, we'd like to say that an A-module law is the same as that of a formal group law F plus A-module structure, i.e. a morphism of rings $[\cdot]_F : A \to \mathrm{End}_{(\mathrm{FGL}/R)}(F)$. But there is a subtlety going on here: Let

$$\mathrm{Lie} : (\mathrm{FGL}/R) \to (\mathrm{Ab})$$

be the (constant) functor that sends $F \in (\mathrm{FGL}/R)$ to $(R, +)$, and morphisms $f : G \to H$ given by a formal power series $f = c_1T + c_2T^2 + \cdots \in R[\![T]\!]$ to the endomorphism of R given by multiplication with c_1. The condition that $F(X, Y) \equiv X + Y$ modulo degree 2 enforces that the induced map $\mathrm{End}(F) \to \mathrm{End}(R)$ is a morphism of rings. Now, the A-module structure on F yields an A-module structure on R, given by the concatenation

$$A \xrightarrow{[\cdot]_F} \mathrm{End}(F) \xrightarrow{\mathrm{Lie}} \mathrm{End}(R), \quad a \mapsto \mathrm{Lie}([a]_F)$$

This is a morphism of rings, and we obtain that R is an A-algebra. We'd like the A-algebra structure on R to be uniform. This motivates the following definition.

Definition 1.1.3 (Formal A-module law)**.** Let A be a ring and R be an A-algebra with structure morphism $p : A \to R$. A (one-dimensional) A-module law over an R is a pair $(F, ([a]_F)_{a \in A})$, where $F \in R[\![X, Y]\!]$ is a formal group law and $[a]_F = p(a)X + c_2X^2 + \cdots \in R[\![X]\!]$ yield endomorphisms $F \to F$ such that the induced map

$$A \to \mathrm{End}(F), \quad a \mapsto [a]_F$$

is a morphism of rings.

Similarly to above, we obtain a category of formal A-module laws over R, which we denote by $(A\text{-FML}/R)$. Note that $(\mathrm{FGL}/R) \cong (\mathbb{Z}\text{-FML}/R)$. Slightly abusing notation, we usually do not explicitely mention the A-structure when referring to formal module laws, simply writing $F \in (A\text{-FML}/R)$, for example.
The following lemma explains a the functoriality of the assignment $R \mapsto (A\text{-FML}/R)$.

Lemma 1.1.4. *The assignment $R \mapsto (A\text{-FML}/R)$ is functorial in the following sense. If $p : R \to R'$ is a morphism of A-algebras, we obtain a functor*

$$(A\text{-FML}/R) \to (A\text{-FML}/R'), \quad F \mapsto p_*F,$$

where p_*F is the formal A-module law obtained by applying p to the coefficients of the formal power series representing addition and scalar multiplication of F. We sometimes write (with abuse of notation) $p_*F = F \otimes_R R'$.

Note that every formal module law $F \in (A\text{-FML}/R)$ yields a functor

$$(R\text{-Alg}) \to (A\text{-Mod}), \quad S \mapsto \mathrm{Nil}(S),$$

where $\mathrm{Nil}(S)$, the set of nilpotent elements of S, is equipped with addition and scalars given by

$$s_1 + s_2 = F(s_1, s_2) \in \mathrm{Nil}(S), \quad as = [a]_F(s) \in \mathrm{Nil}(S).$$

This construction yields a functor (with slight abuse of notation)

$$\mathrm{Spf} : (A\text{-FML}/R) \to \mathrm{Fun}((R\text{-Alg}), (A\text{-Mod})),$$

where Fun denotes the functor category. We will mostly think of formal module laws in terms of this functor. Following this line of thought leads naturally to the definition of formal modules.

Definition 1.1.5 (Formal Group and Formal Module.)**.** A formal A-module $\mathcal{F}$ over an A-scheme X is given by an equivalence class of systems $(U_i, F_i)_{i\in I}$, where $(U_i)_{i\in I}$ is an affine Zariski cover of X with $U_i = \mathrm{Spec}(R_i)$ and $F_i \in (A\text{-FML}/R_i)$ are compatible A-module laws. Here, compatible means that there are isomorphisms

$$F_i \otimes_{R_i} (R_i \otimes_{\mathcal{O}_X(U_i \cup U_j)} R_j) \cong F_j \otimes_{R_j} (R_i \otimes_{\mathcal{O}_X(U_i \cup U_j)} R_j).$$

Two systems (U_i, F_i) and (V_j, G_j) are isomorphic if the corresponding group laws F_i and G_j are isomorphic after refining the respective covers. We denote the resulting categories by $(R\text{-FM}/X)$ and (FG/X). A representing sysstem $(U_i, F_i)_{i\in I}$ will be referred to as a coordinate of $\mathcal{F}$.

In short, a formal A-module over an A-scheme X is an object in the stackification of the prestack $R \mapsto (A\text{-FML}/R)$ on the Zariski-site of schemes over $\mathrm{Spec}\, A$.
Of course there is a functor

$$(A\text{-FML}/R) \to (A\text{-FM}/R),$$

essentially forgetting the choice of module law. The observation of Lemma 1.1.4 translates to formal modules, a morphism $p : R \to R'$ yields a functor

$$p_* : (A\text{-FM}/R) \to (A\text{-FM}/R'), \quad \mathcal{F} \mapsto \mathcal{F} \otimes_R R'.$$

A priori, a formal A-module $\mathcal{F}$ over R does not give rise to a functor constantly valued in $\mathrm{Nil}(R)$. However, it does so locally, and it turns out that there exists a line bundle $\mathcal{L}/\mathcal{O}_X$ (dependent on $\mathcal{F}$) such that $\mathcal{F}$ is equivalent to a functor

$$\mathcal{F} : (\mathcal{O}_X\text{-Alg}) \to (A\text{-Mod}), \quad \mathcal{A} \mapsto \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathrm{Nil}(\mathcal{A})).$$

Here ($\mathcal{O}_X$-Alg) denotes the category of quasi-coherent $\mathcal{O}_X$-algebras, and $\mathrm{Nil}(\mathcal{O}_X)$ is the quasi-coherent $\mathcal{O}_X$-module given on opens subsets by $U \mapsto \mathrm{Nil}(\mathcal{O}_X(U))$. The A-module structure of the right-hand side can be described locally on X after choosing a system of coordinates.

1.1.1 Formal Modules over Finite Fields

1.1.2 Formal DVR-Modules

Let K be a local field with residue field k, put $q = \#k$, and denote by $\nu_K : K \to \mathbb{Z} \cup \{\infty\}$ the valuation of K, normalized such that $\nu_K(\pi) = 1$ for a uniformizer π of K. The aim of this subsection is to describe the maximal abelian extension of a local field K. Local fields naturally carry the topology induced from the π-adic topology on its rings of integers $\mathcal{O}_K$. To deal with the topological rings showing up, the notion of admissible rings will be convenient (taken from [Stacks, Tag 07E8]).

Definition 1.1.6. Let A be a topological ring. We say that A is admissible if

- The element $0 \in A$ has a fundamental system of neighbourhoods consisting of ideals.
- There exists an ideal of definition, that is, an ideal $I \subset A$ such that every open neighbourhood of 0 contains I^n for some n.
- It is complete, that is, the natural map

  $$A \to \lim_{J \subset A \text{ open ideal}} A/J$$

  is an isomorphism.

We say that A is adic if it admits an open ideal of definition. Given a topological ring A, we denote the category of admissible and adic A-algebras (algebras S with continuous morphism $A \to S$) by (A-Adm) and (A-Adic), respectively.

Suppose that R is a local, adic $\mathcal{O}_K$-algebra. Then any $\mathcal{F} \in (\mathcal{O}_K\text{-FM}/R)$ yields a functor

$$(R\text{-Adm}) \to (\mathcal{O}_K\text{-Mod}), \quad S \mapsto S^{\circ\circ} := \{s \in S \mid \lim_{n\to\infty} s^n = 0\},$$

with A-module structure on $S^{\circ\circ}$ given explicitly after choosing a coordinate of $\mathcal{F}$. Indeed, infinite sums over null sequences converge by completeness of admissible algebras.

1.2 The Local Kronecker-Weber Theorem

Let K^{ab} denote the maximal abelian extension of K, which can explicitely be described as $\overline{K}^{[G_K,G_K]}$. Write K_∞ for it's completion, then $K_\infty = C^{[G_K,G_K]}$.
Recall the definition of $\breve{K}$, the completion of the maximal closure of K. First we choose an algebraic closure $\overline{K}$ of K. The valuation ν_K extends uniquely to $\overline{K}$, yielding a π-adic norm on $\overline{K}$. Let C denote the completion with respect to this norm. An application of Krasner's Lemma implies that $\mathrm{Gal}(C/K) \cong \mathrm{Gal}(\overline{K}/K) =: G_K$. One readily checks that any $\sigma \in G_K$ Ref

yields a continuous automorphism $\mathcal{O}_C \to \mathcal{O}_C$, and we obtain a short exact sequence

$$0 \to I_K \to G_K \to \mathrm{Gal}(\overline{k}/k) \to 0.$$

The subgroup $I_K \subset G_K$ is called the inertia subgroup of K, and $\breve{K}$ is the subfield of C fixed by I_K. In particular we have $\mathrm{Gal}(\breve{K}/K) \cong \mathrm{Gal}(\overline{k}/k)$. One readily confirms that $\breve{K}$ is complete with respect to the norm induced by K.

As the Galois group of any finite extension of k is cyclic, we find that $\mathrm{Gal}(\breve{K}/K)$ is abelian. In fact, it is isomorphic to $\widehat{\mathbb{Z}} = \lim_n(\mathbb{Z}/n\mathbb{Z})$. Hence K_∞ decomposes as $\breve{K} \cdot K_\pi$ for some abelian extension K_π/K such that $K_\pi \cap \breve{K} = K$. Now necessarily

$$\mathrm{Gal}(K_\infty/K) \cong \mathrm{Gal}(K_\pi/K) \times \mathrm{Gal}(\breve{K}/K) \cong \mathrm{Gal}(K_\pi/K) \times \widehat{\mathbb{Z}}.$$

We will give an explicit construction of K_π using formal modules.

References

[1] S. Bochner. "Formal Lie Groups". In: *Annals of Mathematics* 47.2 (1946), pp. 192–201. ISSN: 0003486X. URL: `http://www.jstor.org/stable/1969242` (visited on 03/30/2024).

[2] Michel Lazard. "Sur les groupes de Lie formels à un paramètre". fr. In: *Bulletin de la Société Mathématique de France* 83 (1955), pp. 251–274. DOI: `10.24033/bsmf.1462`. URL: `http://www.numdam.org/articles/10.24033/bsmf.1462/`.

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