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1 Introduction

1.1 Notation

We denote the category of sets with (Set) and the category of (unital, commututative) rings with (Ring). If A is a ring, we write (A-Alg) for the category of A-algebras, and (A-Mod) for the category of A-modules.
If $f(T) = c_1T + c_2T^2 + \cdots \in A[\![T]\!]$, we write $f^k(T)$ for the k-fold self composite of f, that is

$$f^k(T) = \underbrace{f(f(\cdots(f}_{k-\text{fold}}(T))\cdots).$$

In order to not confuse this with taking multiplicative powers, we write

$$f(T)^k = \underbrace{f(T)f(T)\cdots f(T)}_{k-\text{fold}}.$$

1.2 Acknowledgements

2 Formal Modules

This section will serve as an introduction to formal groups and formal modules. Formal groups (or rather, formal group laws) were first introduced by Salomon Bochner in 1946 as a natural means of studying Lie Groups over fields of characteristic 0, cf. [Boc46]. The study of formal groups later became interesting for its own right, with pioneering works of Lazard [Laz55]. blabla

2.1 BasicNotions.tex

As promised in the introduction, we begin by defining formal group laws. For now, let A be any ring.

Definition 2.1.1 (Formal Group Laws of arbitrary dimension)**.** A (commutative, one-dimensional) formal group law of dimension n over R is a power series $F(X,Y) \in R[\![X,Y]\!]$ such that $F(X,Y) \equiv X+Y$ modulo degree ≥ 2 and the following equalities are satisfied:

1. $F(F(X,Y),Z) = F(X,F(Y,Z))$.
2. $F(X,0) = X$.
3. $F(X,Y) = F(Y,X)$.

Given two formal group laws F and G a morphism $F \to G$ is a power series $f \in R[\![T]\!]$ such that $f(0) = 0$ and

$$G(f(X), f(Y)) = f(F(X,Y)).$$

For any formal module F, the identity is given by the morphism $\mathrm{id}_F(X) = X$. Composition of morphisms is given by composition of power-series. This yields the category of formal modules of dimension 1 over R, which we denote by (FGL/R).

We encounter higher-dimensional group laws later in this section, but they only play an auxilliary role, see Definition 2.5.1.
The following statements about morphisms of formal group laws are useful and easily verified.

Lemma 2.1.2. *Let R be a ring and let $F, G \in R[\![X,Y]\!]$ be two formal (one-dimensional) group laws over R.*

1. *The set $\mathrm{Hom}_{(\mathrm{FGL}/R)}(F,G)$ is an abelian group with addition $f+g = G(f,g)$, and $\mathrm{End}_{(\mathrm{FGL}/R)}(F)$ has a natural ring structure with multiplication given by composition.*
2. *A morphism $f = c_1T + c_2T^2 + \cdots \in R[\![T]\!]$ between F and G is an isomorphism if and only if $c_1 \in R^\times$.*

Example. Let us introduce the following two formal group laws.

- *The additive formal group law.* Write $\widehat{\mathbb{G}}_a$ for the formal group law with addition given by $\widehat{\mathbb{G}}_a(X,Y) = X+Y$.
- We write $\widehat{\mathbb{G}}_m$ for the formal group law associated with the with $\widehat{\mathbb{G}}_m(X,Y) = X+Y+XY$. Note that $\widehat{\mathbb{G}}_m(X,Y) = (X+1)(Y+1)-1$

Next up is the definition of formal A-module laws. Naively, we would like to define formal A-module laws as formal group laws F with A-module structure, i.e. a morphism of rings $[\cdot]_F : A \to \mathrm{End}_{(\mathrm{FGL}/R)}(F)$. But there is a subtlety: let

$$\mathrm{Lie} : (\mathrm{FGL}/R) \to (\mathrm{Ab})$$

be the (constant) functor that sends $F \in (\mathrm{FGL}/R)$ to $(R, +)$, and morphisms $f : G \to H$ given by a formal power series $f = c_1 T + c_2 T^2 + \cdots \in R[\![T]\!]$ to the endomorphism of R given by multiplication with c_1. The condition that $F(X,Y) \equiv X + Y$ modulo degree 2 enforces that the induced map $\mathrm{End}(F) \to \mathrm{End}(R)$ is a morphism of rings. Now, the A-module structure on F yields an A-module structure on R, given by the composition

$$A \xrightarrow{[\cdot]_F} \mathrm{End}(F) \xrightarrow{\mathrm{Lie}} \mathrm{End}(R), \quad a \mapsto \mathrm{Lie}([a]_F)$$

This is a morphism of rings, and we obtain an A-algebra structure on R. This motivates the following definition.

Definition 2.1.3 (Formal A-module law)**.** Let A be a ring and R be an A-algebra with structure morphism $i : A \to R$. A (one-dimensional) A-module law over R is a pair $(F, ([a]_F)_{a \in A})$, where $F \in R[\![X,Y]\!]$ is a formal group law and $[a]_F = i(a)X + c_2 X^2 + \cdots \in R[\![X]\!]$ yield endomorphisms $F \to F$ such that the induced map

$$A \to \mathrm{End}(F), \quad a \mapsto [a]_F$$

is a morphism of rings. A morphism $f : F \to G$ between formal A-module laws F and G is a morphism $f(T) \in \mathrm{Hom}_{(\mathrm{FGL}/A)R}(F, G)$ satisfying the additional constraint $f([a]_F(T)) = [a]_G(f(T))$ for all $a \in A$.

Similarly to above, we obtain a category of formal A-module laws over R, which we denote by $(A\text{-FML}/R)$. Note that $(\mathrm{FGL}/R) \cong (\mathbb{Z}\text{-FML}/R)$. Slightly abusing notation, we usually do not explicitely mention the A-structure when referring to formal module laws, simply writing $F \in (A\text{-FML}/R)$, for example.
The following lemma explains a the functoriality of the assignment $R \mapsto (A\text{-FML}/R)$.

Lemma 2.1.4. *The assignment $R \mapsto (A\text{-FML}/R)$ is functorial in the following sense. If $i : R \to R'$ is a morphism of A-algebras, we obtain a functor*

$$(A\text{-FML}/R) \to (A\text{-FML}/R'), \quad F \mapsto F \otimes_R R',$$

where $F \otimes_R R'$ is the formal A-module law obtained by applying i to the coefficients of the formal power series representing the A-module structure of F.

Note that every formal module law $F \in (A\text{-FML}/R)$ yields a functor

$$(R\text{-Alg}) \to (A\text{-Mod}), \quad S \mapsto \mathrm{Nil}(S), \tag{2.1}$$

where $\mathrm{Nil}(S)$, the set of nilpotent elements of S, is equipped with addition and scalars given by

$$s_1 + s_2 = F(s_1, s_2) \in \mathrm{Nil}(S), \quad as = [a]_F(s) \in \mathrm{Nil}(S).$$

This construction yields a functor (with slight abuse of notation)

$$(A\text{-FML}/R) \to \mathrm{Fun}((R\text{-Alg}), (A\text{-Mod})), \tag{2.2}$$

where Fun denotes the functor category.

Passing from discrete R-algebras to admissible R-algebras (cf. Definition A.0.1), this construction extends naturally to a functor

$$(A\text{-FML}/R) \to \mathrm{Fun}((R\text{-Adm}), (A\text{-Mod})), \quad F \mapsto \operatorname{Spf} R[\![T]\!],$$

where we equip $\operatorname{Spf} R[\![T]\!]$ with the structure of an A-module object using the endomorphisms coming from F. Following this line of thought leads naturally to the definition of formal modules.

Definition 2.1.5 (Formal Group and Formal Module.)**.** Let X be an A-scheme. A formal A-module $\mathcal{F}$ over X is an A-module object in (FSch/X), the category of formal schemes over X, satisfying the following condition. That there is a Zariski-covering $(\operatorname{Spec}(R_i))_{i\in I}$ of X with $\mathcal{F} \times_X U_i \cong \operatorname{Spf}(R_i[\![T]\!])$ for every $i \in I$ the induced A-module structure on $\operatorname{Spf}(R_i[\![T]\!])$ comes from a formal A-module law F_i over R_i.

Remark. Formal schemes (over a base an A-scheme X, say) locally isomorphic to $\operatorname{Spf} \mathcal{O}_X(U)[\![T]\!]$ are sometimes called (one-dimensional) Formal Lie Varieties . Equivalently to the definition above, we could have defined formal A-modules as A-module objects (of relative dimension one over X) in the category of Formal Lie Varieties, such that the A-module structure on the tangent space at the identity agrees with the usual one.

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Definition 2.1.6 (Coordinate)**.** Let $\mathcal{F}$ be a formal A-module over X. The choice of a cover $\sqcup_{i\in I} \operatorname{Spec}(R_i) \to X$ together with isomorphisms $\mathcal{F} \times_X \operatorname{Spec}(R_i) \cong \operatorname{Spf}(R_i[\![T]\!])$ will be referred to as a coordinate of $\mathcal{F}$.

Of course there is a functor

$$(A\text{-FML}/R) \to (A\text{-FM}/R),$$

essentially forgetting the choice of module law. The observation of Lemma 2.1.4 translates to formal modules, a morphism $p : R \to R'$ yields a functor

$$p_* : (A\text{-FM}/R) \to (A\text{-FM}/R'), \quad \mathcal{F} \mapsto \mathcal{F} \otimes_R R'.$$

Example. The additive group law $\widehat{\mathbb{G}}_a$ extends to a formal A-module over an affine base $\operatorname{Spec} R$ by setting

$$[a]_{\widehat{\mathbb{G}}_a}(T) = aT$$

for $a \in A$. More generally, we obtain a formal A-module over an arbitrary base scheme X over A.

Over $\mathbb{Z}_p$, the formal group $\widehat{\mathbb{G}}_m$ extends to a formal $\mathbb{Z}_p$-moduel as follows. As a functor, $\widehat{\mathbb{G}}_m$ is isomorphic to the assignment

$$(\mathbb{Z}_p\text{-Adm}) \to (\mathrm{Ab}), \quad S \mapsto 1 + S^{\infty} \subset S^{\times}.$$

Here, we equipped $\mathbb{Z}_p$ with the p-adic topology. The subgroup $1 + S^{\infty}$ naturally carries the structure of a $\mathbb{Z}_p$-module. Indeed, for $k \in \mathbb{N}$, we have

$$(1+s)^{p^k} = 1 + p^k s + \binom{p^k}{2} s^2 + \cdots + s^{p^k},$$

and given $s \in S^{\infty}$, this is of the form $1 + o(1)$ as k gets large. In particular, if $x = a_0 + a_1 p + a_2 p^2 + \cdots \in \mathbb{Z}_p$, expressions of the form

$$(1+s)^x = \prod_{i=1}^{\infty} (1+s)^{a_k p^k}$$

make sense by lemma A.0.3. This gives $\widehat{\mathbb{G}}_{m,\mathbb{Z}_p}$ the structure of a formal $\mathbb{Z}_p$-module. In the upcoming subsection, we discuss how this is the simplest example of a whole family of formal modules constructed by Lubin and Tate. In section 3 we explain applications of these formal modules to local class field theory.

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2.2 Lubin–Tate Formal Module Laws

Suppose that K is a local field with ring of integers $\mathcal{O}_K$, with uniformizer π and residue field $\mathbb{F}_q$.
Let H_0 be the formal $\mathcal{O}_K$-module law defined over $\mathbb{F}_q$ by setting

$$H_0(X,Y) = X + Y, \quad [\pi]_{H_0}(X) = X^q, \quad [u]_{H_0} = \overline{u}X.$$

Here, u runs over the units of $\mathcal{O}_K$ and $\overline{u} \in \mathbb{F}_q$ is such that $u \equiv \overline{u} \bmod \pi$. This uniquely determines $[a]_{H_0}$ for $a \in \mathcal{O}_K$ as a may be written as $a = \pi^{\nu} u$ for a unit u and $[\pi]_{H_0}$ and $[u]_{H_0}$ commute.
Lubin–Tate formal module laws are $\mathcal{O}_K$-module laws H over $\mathcal{O}_K$ such that $H \otimes_{\mathcal{O}_K} \mathbb{F}_q = H_0$.
The construction of the Lubin–Tate formal module laws rests on the following lemma, which is Lemma 1 in [LT65].

Lemma 2.2.1. *Let $f(T)$ and $g(T)$ be elements of $\mathcal{F}_\pi$ and let $L(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_i X_i$ be a linear form with coefficients in $\mathcal{O}_K$. Then there exists a unique series $F(X_1, \ldots, X_n)$ with coefficients in $\mathcal{O}_K$ such that*

$$F(X_1, \ldots, X_n) \equiv L(X_1, \ldots, X_n) \pmod{T^2},$$
$$\text{and}$$
$$f(F(X_1, \ldots, X_n)) = F(g(X_1), \ldots, g(X_n)).$$

Restate in more general terms, for arbitrary powers of q, and perhaps for general discrete valuation rings with finite residue field.

Although stated only for the rings of integers of a local field, the proof only uses that $\mathcal{O}_K$ is complete with respect to the π-adic topology, π is not a zero divisor and the map $x \mapsto x^q$ restricts to the identity mod π. In particular, the statement remains true if working over the integers of the completion of a maximal unramified extension $\breve{K}$ of K.

Write $\mathcal{F}_\pi$ for the set of power series that may arise as $[\pi]_H$, that is,

$$\mathcal{F}_\pi := \{f \in \mathcal{O}_K[\![T]\!] \mid f \equiv \pi T \pmod{T^2} \text{ and } f \equiv T^q \pmod{\pi}\}.$$

Using Lemma 2.2.1, we can construct formal $\mathcal{O}_K$-modules over $\mathcal{O}_K$ as follows. Attached to $f \in \mathcal{F}_\pi$, we find a unique power series $F_f(X,Y) \in \mathcal{O}_K[\![X,Y]\!]$ satisfying

$$F_f(X,Y) \equiv X + Y \pmod{(X,Y)^2} \quad \text{and} \quad F_f(f(X), f(Y)) = f(F_f(X,Y)). \tag{2.3}$$

Furthermore, attached to each $a \in \mathcal{O}_K$ and $f, g \in \mathcal{F}_\pi$, we find unique $[a]_{f,g}(T) \in \mathcal{O}_K[\![T]\!]$ satisfying

$$[a]_{f,g}(T) \equiv aT \pmod{(T)^2} \quad \text{and} \quad f([a]_{f,g}(T)) = [a]_{f,g}(g(T)). \tag{2.4}$$

We now have

Theorem 2.2.2 (Lubin–Tate Formal $\mathcal{O}_K$-Module Laws). *For $f \in \mathcal{F}_\pi$, the family of power series $(F_f, ([a]_{f,f})_{a \in \mathcal{O}_K})$ gives rise to a formal $\mathcal{O}_K$-module law over $\mathcal{O}_K$. For $f, g \in \mathcal{F}_\pi$, the formal $\mathcal{O}_K$-module laws F_f and F_g are isomorphic, via the morphism induced by $[1]_{f,g} \in \mathcal{O}_K[\![T]\!]$.*

Proof. See Theorem 1 of [LT65] and the succeeding discussion. □

In particular, up to isomorphism, the construction of a Lubin–Tate formal $\mathcal{O}_K$-module law only depends on the choice of the uniformizer $\pi \in \mathcal{O}_K$, not on the choice of $f \in \mathcal{F}_\pi$.

Example. If $K = \mathbb{Q}_p$, this reconstructs the multiplicative formal $\mathbb{Z}_p$ module $\widehat{\mathbb{G}}_m$ constructed above. Indeed, we have

$$\mathcal{F}_p = \{f \in \mathbb{Z}_p[\![T]\!] \mid f(T) \equiv T^p \bmod p \text{ and } f(T) \equiv pT \bmod (T)^2\},$$

implying that $f(T) = (1+T)^p - 1$ lies in $\mathcal{F}_p$. One quickly checks that

$$F_f(X,Y) = (1+X)(1+Y) - 1 = X + Y + XY \in \mathbb{Z}_p[\![X,Y]\!]$$

satisfies the conditions of (2.3), and that for $a \in \mathbb{Z}_p$, the power series

$$[a]_{f,f} = (1+T)^a - 1 \in \mathbb{Z}_p[\![T]\!]$$

satisfies the condition of (2.4).

2.3 Logarithms

[Again, A is a complete discrete valuation ring with uniformizing parameter π and finite residue field $k = A/\pi A$. We write q for the cardinality of k and K for the residue field of A. Let R be a (commutative) A-algebra with structure map $i : A \to R$.]

We review results from section 2 and 3 of [HG94]. Suppose that $\mathbf{F} = (F_1, \ldots, F_n)$ is an n-dimensional formal A-module law over an A-algebra R. Write $\mathbf{X} = (X_1, \ldots, X_n)$, $\mathbf{Y} = (Y_1, \ldots, Y_n)$, etc.

Definition 2.3.1 (Invariant Differentials for module laws.)**.** The module $\omega(\mathbf{F})$ of invariant differentials is the submodule of the R-module of differentials

$$\Omega_{R[\![T_1,\dots,T_n]\!]/R} \cong \bigoplus_{i=1}^{n} R[\![T_1,\dots,T_n]\!]\,\mathrm{d}T_i,$$

cut out by the condition that any $\omega \in \omega(\mathbf{F})$ satisfies

$$\omega(\mathbf{F}(\mathbf{X},\mathbf{Y})) = \omega(\mathbf{X}) + \omega(\mathbf{Y}) \quad \text{and} \quad \omega([a]_{\mathbf{F}}(\mathbf{X})) = a\omega(\mathbf{X}). \tag{2.5}$$

for all $a \in A$.

It is possible to explicitly calculate a basis for the R-module $\omega(\mathbf{F})$, which we now explain. Let

$$A(\mathbf{X},\mathbf{Y}) \in \mathrm{Mat}_{n\times n}(R[\![\mathbf{X},\mathbf{Y}]\!])$$

denote the matrix $((\partial/\partial X_j)F_i(\mathbf{X},\mathbf{Y}))_{i,j}$, the derivative of $\mathbf{F}(\mathbf{X},\mathbf{Y})$ with respect to $\mathbf{X}$. Write $B(\mathbf{Y}) = A(0,\mathbf{Y})$. Then B is a unit in $\mathrm{Mat}_{n\times n} R[\![\mathbf{Y}]\!]$; and we write $(C_{ij}(\mathbf{Y}))$ for the components of $B(\mathbf{Y})^{-1}$. We now construct

$$\omega_i := \sum_{j=1}^{n} C_{ij}(\mathbf{X})\,\mathrm{d}X_j \in \Omega_{R[\![\mathbf{X}]\!]/R}$$

for $1 \le i \le n$. By definition we have

$$C_{ij}(0) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Checking that ω_i is an invariant differential is a matter of applying the chain rule, and we have

Proposition 2.3.2. *The R-module $\omega(\mathbf{F})$ is free of rank n generated by invariant differentials $\omega_1, \omega_2, \dots, \omega_n$.*

Proof. This is [HON70, Proposition 1.1]. □

Example. The invariant differentials for $\widehat{\mathbb{G}}_a$ are spanned by the form $\mathrm{d}X$. The invariant differentials for $\widehat{\mathbb{G}}_m$ are spanned by the form $\omega_1(X) = \frac{1}{1+X}\,\mathrm{d}X$.

By the Proposition above and Equation (2.6), we may define a pairing

$$\omega(\mathbf{F}) \times \mathrm{Lie}(\mathbf{F}) \to R, \quad \langle X_i, \omega_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

This pairing is independent of the parametrization of $\mathbf{F}$. In particular, it descents to a pairing defined for formal modules $\mathcal{F} \in (A\text{-FM}^{\mathrm{arb}}/R)$, and we have a natural isomorphism $\omega(\mathcal{F}) \cong \mathrm{Hom}_R(R, \mathrm{Lie}(\mathcal{F}))$.

Let $\widehat{\mathbb{G}}_a$ be the additive formal A-module over R. There is a map

$$\mathrm{d}_{\mathbf{F}} : \mathrm{Hom}_{(A\text{-FML}/R)}(\mathbf{F}, \widehat{\mathbb{G}}_{a,R}) \to \omega(\mathbf{F}), \quad f \mapsto \mathrm{d}f(\mathbf{X}) \tag{2.7}$$

which is a map of R-modules if we equip the left hand side with the R-module structure coming from the natural action of $R \subset \mathrm{End}(\widehat{\mathbb{G}}_a)$.

Proposition 2.3.3. *1. If R is a flat A-algebra, the map d_F is injective.*

2. If R is a K-algebra, the map d_F is an isomorphism.

Proof. [HG94, Proposition 3.2] [Everything is easy if K has characteristic 0, as we can integrate the differential forms. The proof in positive characteristic is a bit tricky; First it is shown that there is an isomorphism of formal goups $F \cong \widehat{\mathbb{G}}_a$, which is immediate. Then that there is a unique homomorphism $f : \widehat{\mathbb{G}}_a \to \widehat{\mathbb{G}}_a$ that maps to ω_F and behaves well with respect to the A-module structure on F.] PROOF □

Suppose now that $F \in (A\text{-FML}^{\mathrm{arb}}/R)$ is a formal module law of dimension n, and let $\omega_1, \dots, \omega_n$ be the distinguished basis for $\omega(F)$ constructed above. By the previous proposition, there are unique power series $f_i(\mathbf{X}) \in (R \otimes_A K)[\![\mathbf{X}]\!]$ furnishing homomorphisms $F \otimes (R \otimes_A K) \to \widehat{\mathbb{G}}_a$ of formal A-module laws and satisfying

$$\mathrm{d}_F f_i(\mathbf{X}) = \omega_i(\mathbf{X}) \in \omega(F).$$

The induced morphism

$$f = (f_1, \dots, f_n) : F \otimes (R \otimes_A K) \to \widehat{\mathbb{G}}_a$$

satsfies $\mathrm{Lie}(f) = \mathrm{id} \in \mathrm{End}(R^n) = \mathrm{End}(\mathrm{Lie}(F))$. Hence f is an isomorphism. We call f the logarithm attached to F.

Definition 2.3.4 (Logarithm and Exponential)**.** The isomorphism of formal group laws $f : F \otimes (R \otimes_A K) \to \widehat{\mathbb{G}}_a$ is called the logarithm attached to F, we write $\log_F(\mathbf{X}) \in ((R \otimes_A K)[\![\mathbf{X}]\!])^n$ for the corresponding collection of power series. The inverse of $\log_F(\mathbf{X})$ is called the exponential attached to F, denoted $\exp_F(\mathbf{X})$. We have $\mathrm{Lie}(\log_F) = \mathrm{Lie}(\exp_F) = \mathrm{id}$.

Lemma 2.3.5. *Let F and G be formal A-modules of over R, with $\dim F = n$ and $\dim G = m$. Let $\phi : F \to G$ be a morphism. Then the diagram*

$$\begin{tikzcd} F \otimes (R \otimes_A K) \arrow[r, "\log_F"] \arrow[d, "\phi"'] & \widehat{\mathbb{G}}_a \otimes \mathrm{Lie}(F) = \widehat{\mathbb{G}}_a^n \arrow[d, "\mathrm{Lie}(\phi)"] \\ G \otimes (R \otimes_A K) \arrow[r, "\log_G"] & \widehat{\mathbb{G}}_a \otimes \mathrm{Lie}(G) = \widehat{\mathbb{G}}_a^m \end{tikzcd}$$

commutes. In particular, attached to any $\mathcal{F} \in (A\text{-FM}^{\mathrm{arb}}/R)$ comes a natural morphism

$$\log_{\mathcal{F}} : \mathcal{F} \otimes (R \otimes_A K) \to \widehat{\mathbb{G}}_a \otimes \mathrm{Lie}(\mathcal{F}).$$

Proof. The square commutes because $\mathrm{Hom}(\widehat{\mathbb{G}}_a^n, \widehat{\mathbb{G}}_a^m) \cong \mathrm{Hom}_{R \otimes_A K}((R \otimes_A K^n), (R \otimes_A K)^m)$ and $\mathrm{Lie}(\log_G \circ \phi \circ \exp_H) = \mathrm{Lie}(\phi)$. □

2.4 Formal DVR-Modules over Fields

As above, let A be a discrete valuation ring with uniformizer π and finite residue field k; write q for the cardinality of k. Let K denote the field of fractions of A.

We introduce the concept of height, which is an integer attached to morphisms of formal group laws over fields. The height of a formal A-module $\mathcal{F}$ over R will be defined as the height of it's endomorphism $[\pi]_{\mathcal{F}}$.

We have seen in the previous section that if R is a field extension of K, then any morphism of formal group laws $f : F \to G$ over R is either 0, in which case we say it has height ∞, or an isomorphism, in which case we say it has height 0. The height becomes interesting in positive characteristic.

We define the height over field extensions of the residue field.

Definition 2.4.1 (Height of morphisms of group laws)**.** Assume that R is a field extension of k and $f : F \to G$ is a morphism of formal groups laws over R, given by a formal series $f(T) \in R[\![T]\!]$. If $f = 0$, we say that f has infinite height. If $f \neq 0$, the height of f is defined as the largest integer h such that $f = g(T^{q^h})$ for some power series $g(T) = c_1T + c_2T^2 + \cdots \in R[\![T]\!]$ with $c_1 \neq 0$.

One readily checks that if $f : \mathcal{F} \to \mathcal{G}$ is a morphism of formal groups over a field extension R of k, the height of f does not depend on the choices of group laws on $\mathcal{F}$ and $\mathcal{G}$. This allows us to define the height function attached to f.

Definition 2.4.2 (Height function)**.** Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of formal groups over a scheme X. For a (scheme-theoretic) point $x \in |X|$, let f_x denote the base-change of f to the residue field of x. The height function attached to f is the upper-semicontinuous function

$$\mathrm{ht}(f) : |X| \to \mathbb{Z}_{\geq 0} \cup \{\infty\}, \quad x \mapsto \mathrm{ht}(f_x). \tag{2.8}$$

It is not hard to see that the height function is additive, that is, we have

$$\mathrm{ht}(f \circ g) = \mathrm{ht}(f) + \mathrm{ht}(g).$$

Definition 2.4.3 (Isogeny)**.** A morphism $f : \mathcal{F} \to \mathcal{G}$ of formal groups over a field k is called an isogeny if $\mathrm{Ker}(f)$ is a represented by a finite free k-scheme. More generally, a morphism of formal A-modules over a base scheme X is an isogeny if and only if $\mathrm{Ker}(f)$ is finite and locally free over X.

Isogenies can be described using the height function.

Lemma 2.4.4. *A morphism $f : \mathcal{F} \to \mathcal{G}$ is a isogeny if and only if the height function* $\mathrm{ht}(f)$ *is locally constant with values in $\mathbb{Z}_{\geq 0}$.*

Definition 2.4.5 (π-divisible A-module)**.** We say that a formal A-module H over X is π-divisible if $[\pi]_H$ is an isogeny. If X is connected, the height of H is the (constant) height of the endomorphism $[\pi]_H : H \to H$.

We close this subsection with a discussion about the structure of formal $\mathcal{O}_K$-modules over separably closed field extensions k' of k.

Lemma 2.4.6. *Over k', any two formal $\mathcal{O}_K$-module laws of the same height are isomorphic.*

Proof. [Dri74, Proposition 1.7]. □

In particular, any formal $\mathcal{O}_K$-module of height h is isomorphic to the formal $\mathcal{O}_K$-module F_{norm} with $[\pi]_{F_{\mathrm{norm}}}(T) = T^{q^h}$. We call this the normalized formal $\mathcal{O}_K$-module.

Lemma 2.4.7. *Suppose that $F \in (\mathcal{O}_K\text{-FML}/k')$. Then $\mathrm{End}_{(A\text{-FM}/k')}(F)$ is isomorphic to the maximal order of the central division algebra D over K of rank h^2 and invariant $\frac{1}{h}$.*

Proof. Also [Dri74, Proposition 1.7]. □

Lemma 2.4.8. *Let $f : F \to G$ be an isogeny of π-divisible formal $\mathcal{O}_K$-module laws over k'. Then there is an integer $n \geq 0$ and an isogeny $g : G \to F$ with*

$$f \circ g = [\pi^n]_{\mathcal{G}} \quad \textit{and} \quad g \circ f = [\pi^n]_{\mathcal{F}}.$$

Proof. As the height is additive, we necessarily have $\mathrm{ht}(F) = \mathrm{ht}(G)$, thus by Lemma 2.4.6, we may assume that F and G are given by the normalized formal $\mathcal{O}_K$-module F_{norm}. Write $f(T) = g(T^{q^n})$ for some power series $h(T) = c_1T + c_2T^2 + \dots$, where $c_1 \neq 0$ is a unit in k', and let $g(T) = h^{-1}(T)$ be the formal inverse of h. Now g is a morphism of formal $\mathcal{O}_K$-module laws satisfying $f \circ g(T) = g \circ f(T) = T^{q^n}$. The claim follows. □

2.5 Formal Modules of Arbitrary Dimension

We introduce the category $(A\text{-FML}^{\mathrm{arb}}/S)$ of formal A-modules of arbitrary dimension over a base scheme S.

Definition 2.5.1 (Formal Group Laws of arbitrary dimension)**.** A (commutative) formal group law of dimension n over R is a tuple of power series $\mathbf{F} = (F_1, \dots, F_n)$ with

$$F_i(X_1, \dots, X_n, Y_1, \dots, Y_n) \in R[\![X_1, \dots, X_n, Y_1, \dots, Y_n]\!], \quad 1 \leq i \leq n$$

such that $F_i(\mathbf{X}, \mathbf{Y}) \equiv X_i + Y_i$ modulo degree ≥ 2 and the following equalities are satisfied:

1. $\mathbf{F}(\mathbf{F}(\mathbf{X}, \mathbf{Y}), \mathbf{Z}) = \mathbf{F}(\mathbf{X}, \mathbf{F}(\mathbf{Y}, \mathbf{Z}))$.
2. $\mathbf{F}(\mathbf{X}, \mathbf{0}) = \mathbf{X}$.
3. $\mathbf{F}(\mathbf{X}, \mathbf{Y}) = \mathbf{F}(\mathbf{Y}, \mathbf{X})$.

Here, and in the sequel, we abbreviate $\mathbf{F} = (F_1, \dots, F_n)$, $\mathbf{X} = (X_1, \dots, X_n)$, et cetera. Given a formal group $\mathbf{F}$ of dimension n and a formal group law $\mathbf{G}$ of dimension m, a morphism $\mathbf{F} \to \mathbf{G}$ is a m-tuple $\mathbf{f} = (f_1, \dots, f_m)$ of power series $f_i \in R[\![X_1, \dots, X_n]\!]$ such that $\mathbf{f}(0) = 0$ and

$$\mathbf{G}(\mathbf{f}(\mathbf{X}), \mathbf{f}(\mathbf{Y})) = \mathbf{f}(\mathbf{F}(\mathbf{X}, \mathbf{Y})).$$

For any n-dimensional formal module $\mathbf{F}$, the identity is given by the morphism $\mathrm{id}_{\mathbf{F}}$ with components $\mathrm{id}_{\mathbf{F},i}(\mathbf{X}) = X_i$. Composition of morphisms is given by composition of tuples of power-series. This yields the category of formal modules of arbitrary dimension over R, which we denote by $(\mathrm{FGL}^{\mathrm{arb}}/R)$.

One readily checks that $\mathrm{End}_{(\mathrm{FGL}^{\mathrm{arb}}/R)}(\mathbf{F})$ is a (possibly non-commutative) algebra. Note that there is a functor $\mathrm{Lie} : (\mathrm{FGL}^{\mathrm{arb}}/R) \to (\mathrm{Ab})$, taking an n-dimensional formal group law $\mathbf{F}$ to the R-module

$$\mathrm{Lie}(\mathbf{F}) = \mathrm{Hom}_{(R\text{-Mod})}\left(\frac{(X_1,\dots,X_n)}{(X_1,\dots,X_n)^2}, R\right)$$

Given an m-dimensional group law $\mathbf{G}$ and a morphism $\mathbf{f} : \mathbf{F} \to \mathbf{G}$, $\mathrm{Lie}(\mathbf{f})$ is the induced morphism

$$\mathrm{Lie}(\mathbf{F}) \to \mathrm{Lie}(\mathbf{G}), \quad \psi \mapsto \left(S_j \mapsto \psi(\overline{f_j})\right) \in \mathrm{Hom}_{(R\text{-Mod})}\left(\frac{(X_1,\dots,X_n)}{(X_1,\dots,X_n)^2}, R\right),$$

where $\overline{f_j}$ is the reduction of f_j mod $(\mathbf{X})^2$. We have a canonical basis on both sides, and writing $\mathrm{Lie}(\mathbf{F}) \cong R^n$, $\mathrm{Lie}(\mathbf{G}) \cong R^m$, the induced map $\mathrm{Lie}(\mathbf{f}) : R^n \to R^m$ is given by multiplication with the matrix

$$\left(\frac{\partial f_j}{\partial X_i}(0)\right)_{i,j}.$$

Given a ring A and a morphism of rings $A \to \mathrm{End}_{(\mathrm{FGL}^{\mathrm{arb}}/R)}(\mathbf{F})$, we obtain an A-module structure on $\mathrm{Lie}(\mathbf{F}) \cong R^n$. by composing with $\mathrm{Lie}(-) : \mathrm{End}_{(A\text{-}\mathrm{FML}^{\mathrm{arb}}/R)}(\mathbf{F}) \to \mathrm{End}_{(R\text{-Mod})}(\mathrm{Lie}(\mathbf{F}))$. If R is an A-algebra, we want the A-module structures on $\mathrm{Lie}(\mathbf{F})$ to be the same. This motivates the following definition.

Definition 2.5.2 (Formal A-Modules of arbitrary dimension)**.** Let R be an A-algebra with structure morphism $i : A \to R$. A formal A-module over R of dimension n is given by the data of a formal n-dimensional group law $\mathbf{F}$ over R and a morphism of rings

$$A \to \mathrm{End}_{(\mathrm{FGL}^{\mathrm{arb}}/R)}(\mathbf{F}), \quad a \mapsto ([a]_{\mathbf{F},i})_{1\le i\le n} \in (R[\![X_1,\dots,X_n]\!])^n$$

such that $[a]_{\mathbf{F},i}(\mathbf{X}) \equiv i(a)X_i$ modulo terms of degree ≥ 2. Morphisms between formal A-modules of arbitrary dimension are morphisms of formal groups respecting the A-module structure. The resulting category is denoted $(A\text{-}\mathrm{FML}^{\mathrm{arb}}/R)$. As before, the subcategory of one-dimensional formal A modules over R is denoted $(A\text{-}\mathrm{FML}/R)$.

Note that an n-dimensional formal module law $(\mathbf{F}, [-]_{\mathbf{F}})$ equips the formal scheme $\mathrm{FG}(\mathcal{F}) := \mathrm{Spf}(R[\![T_1,\dots,T_n]\!])$ with the structure of an A-module object in the category (FSch/R) of formal schemes over $\mathrm{Spec}\, R$. This leads naturally to the more geometric notion of formal A-modules.

Definition 2.5.3 (Formal A-modules)**.** Given an A-scheme X, we define the category $(A\text{-}\mathrm{FM}^{\mathrm{arb}}/X)$ as follows. Objects are group objects $\mathcal{F}$ in the category of formal schemes over X, such that there is a cover of X by Zariski-open affine subsets $U_i = \mathrm{Spec}(R_i)$ such that $\mathcal{F} \times_X U_i$ is isomorphic to $\mathrm{Spf}\, R_i[\![X_1,\dots,X_n]\!]$ and the induced A-module structure on $R_i[\![X_1,\dots,X_n]\!]$ yields a formal A-module law on R_i.

Definition 2.5.4 (Lie functor)**.** The functor Lie descents to a functor

$$\mathrm{Lie} : (A\text{-}\mathrm{FM}^{\mathrm{arb}}/X) \to (\mathcal{O}_X\text{-QCoh}),$$

given by locally describing a formal A-module $\mathcal{F}$ via formal group laws and glueing the local data. Alternatively, it arises from sending sending a formal A-module $\mathcal{F}$ to $(\mathcal{I}/\mathcal{I}^2)^\vee$, where $\mathcal{I}$ is the ideal associated to the closed immersion $[0]_\mathcal{F} : X \to \mathcal{F}$.

Definition 2.5.5 (Formal Module associated to R-module)**.** Suppose that M is a projective R-module. Then we write $\widehat{\mathbb{G}}_a \otimes M$ for the additive formal A-module associated to M over R. As a formal scheme, this formal module is given by

$$\widehat{\mathbb{G}}_a \otimes_A M \cong \operatorname{Spf} R[\![M^\vee]\!],$$

where $R[\![M^\vee]\!]$ denotes the completion of $\operatorname{Sym}_R(M^\vee)$ with respect to the ideal generated by $M^\vee$. The (formal) A-module structure is the canonical additive one. Note that $\operatorname{Lie}(\widehat{\mathbb{G}}_a \otimes M) = M$ by design. More generally, if X is a quasi-compact and quasi-separated A-scheme and $\mathcal{M}$ is a finite locally free quasi-coherent $\mathcal{O}_X$-module, this construction yields a formal A-module $\widehat{\mathbb{G}}_a \otimes \mathcal{M}$ over X.

2.6 Exact Categories, Extensions of Formal Modules

In this section, we equip the category $(A\text{-FM}^{\text{arb}}/X)$, where A is any ring and X is a quasi-compact and quasi-separated A-scheme, with a notion of short exact sequences. We show that this gives $(A\text{-FM}^{\text{arb}}/X)$ the structure of an exact category in the sense of Quillen [Kel90, Appendix A]. We introduce functors

$$\begin{aligned} \operatorname{Ext}(-,-) &: (A\text{-FM}^{\text{arb}}/X)^{\text{op}} \times (A\text{-FM}^{\text{arb}}/X) \to (\text{Set}) \\ \operatorname{RigExt}(-,-) &: (A\text{-FM}^{\text{arb}}/X)^{\text{op}} \times (A\text{-FM}^{\text{arb}}/X) \to (\text{Set}), \end{aligned}$$

which send a pair $(\mathcal{F}, \mathcal{F}')$ to the set of equialence classses of extensions (resp. rigidified extensions) of $\mathcal{F}$ by $\mathcal{F}'$. These functors will play a major role in the upcoming discussion.

2.6.1 The Category of Formal Modules is Exact

Before turning our attention to formal modules, we introduce the notion of exact categories, following [Kel90, Appendix A].

Definition 2.6.1 (Exact Category)**.** Let $\mathcal{A}$ be an additive category, and let $\mathcal{E}$ be a class whose members are exact triples of objects connected by arrows

$$X \xrightarrow{i} Y \xrightarrow{d} Z,$$

where i is a kernel of d and d is a cokernel of i. We call a morphism $i : X \to Y$ an inflation if it apears as first component of some $(i, d) \in E$, second components are called deflations. We say that the pair $(\mathcal{A}, \mathcal{E})$ is an exact category if $\mathcal{E}$ is closed under isomorphisms and satisfies the following properties.

1. The identity $\mathrm{id}_0 : 0 \to 0$ is a deflation.

2. The composition of two deflations is a deflation.

3. For each $f \in \mathrm{Hom}_{\mathcal{A}}(Z', Z)$, there is a cartesian square

$$\begin{tikzcd} Y' \arrow[r, "d'"] \arrow[d, "f'"'] & Z' \arrow[d, "f"] \\ Y \arrow[r, "d"'] & Z \end{tikzcd}$$

such that d' is a deflation.

3^{op}. For each $f \in \mathrm{Hom}_{\mathcal{A}}(X, X')$, there is a co-cartesian square

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, "f'"] \\ X' \arrow[r, "i'"'] & Y' \end{tikzcd}$$

such that i' is an inflation.

As above, suppose that A is any ring and X is a quasi-compact and quasi-separated A-scheme. Let $\mathcal{F}$, $\mathcal{E}$ and $\mathcal{F}'$ be formal A-modules over X.

Definition 2.6.2 (Short Exact Sequence)**.** A pair of composable morphisms $\mathcal{F}' \to \mathcal{E} \to \mathcal{F}$ in $(A\text{-FM}^{\mathrm{arb}}/X)$ is called a short exact sequence if the induced sequence

$$0 \to \mathrm{Lie}(\mathcal{F}') \to \mathrm{Lie}(\mathcal{E}) \to \mathrm{Lie}(\mathcal{F}) \to 0$$

is a short exact sequence of $\mathcal{O}_X$-modules. In this case, we write

$$0 \to \mathcal{F}' \to \mathcal{E} \to \mathcal{F}' \to 0.$$

A pair of composable morphisms $F' \to E \to F$ in $(A\text{-FML}^{\mathrm{arb}}/R)$ is called an exact sequence if it is exact after passing to the respective formal modules.

Lemma 2.6.3. *Let R be an A-algebra and let $F, F' \in (A\text{-FML}^{\mathrm{arb}}/R)$ be formal A-module laws of dimensions m and n respectively. Write $\mathcal{F}', \mathcal{F} \in (A\text{-FM}^{\mathrm{arb}}/R)$ for the associated formal A-modules, and suppose that they fit into a exact sequence*

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$$0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{F} \to 0.$$

Write $\mathbf{X}$ for the variables of F' and $\mathbf{Z}$ for those of F. Then there exists a (non-canonical) coordinate on $\mathcal{E}$ giving rise to a formal A-module law E in the variables $(\mathbf{X}, \mathbf{Z})$ such that the induced morphisms of formal module laws are of the form $\alpha(\mathbf{X}) = (\mathbf{X}, 0)$, $\beta(\mathbf{X}, \mathbf{Z}) = \mathbf{Z}$. Furthermore, the formal A-module law E is of the form

$$\begin{gathered} E((\mathbf{X}_1, \mathbf{Z}_1), (\mathbf{X}_2, \mathbf{Z}_2)) = (F'(\mathbf{X}_1, \mathbf{X}_2) +_{F'} \Delta(\mathbf{Z}_1, \mathbf{Z}_2), F(\mathbf{Z}_1, \mathbf{Z}_2)) \\ \textit{and} \\ [a]_E(\mathbf{X}, \mathbf{Z}) = ([a]_{F'}(\mathbf{X}) +_{F'} \delta_a(\mathbf{Z}), [a]_F(\mathbf{Z})). \end{gathered} \tag{2.9}$$

for some m-tuple of power series $\Delta \in (R[\![\mathbf{Z}_1, \mathbf{Z}_2]\!])^m$, $\delta_a \in (R[\![\mathbf{Z}]\!])^m$.

Proof. The construction of E is sketched in [HG94, Proposition 6.5]. We know that $\mathcal{E} \cong \operatorname{Spf} R[\![M]\!]$ for some free R module M of rank $m+n$. As we have a short exact sequence on Lie-algebras, we may apply the formal implicit function theorem to obtain a section reference/ $\sigma : \mathcal{F} \to \mathcal{E}$ of $\beta : \mathcal{E} \to \mathcal{F}$. The datum of the morphisms α and σ is equivalent to morphisms

$$\alpha^\flat : R[\![M]\!] \to R[\![\mathbf{X}]\!] \quad \text{and} \quad \sigma^\flat : R[\![M]\!] \to R[\![\mathbf{Z}]\!]$$

on affines. Taking their sum, we obtain a morphism $R[\![M]\!] \to R[\![\mathbf{X}, \mathbf{T}]\!]$. On Lie-algebras, this morphism recovers the isomorphism $\operatorname{Lie}(\mathcal{E}) \cong \operatorname{Lie}(\mathcal{F}') \oplus \operatorname{Lie}(\mathcal{F})$ induced by $\operatorname{Lie}(\sigma)$. In particular, $\sigma^\flat + \alpha^\flat$ is an isomorphism in degree 1, hence an isomorphism. This yields the desired coordinate $\mathcal{E} \cong \operatorname{Spf} R[\![\mathbf{X}, \mathbf{Z}]\!]$. The fact about the structure of the formal A-module law E follows quickly from the fact that α and β are morphisms of formal A-module laws. $\square$

Let's turn our attention to the power series $(\Delta, (\delta_a)_{a \in A})$ appearing in the above Lemma. They satisfy certain conditions.

Definition 2.6.4 (Symmetric 2-cocycles)**.** Let $\operatorname{SymCoc}^2(F, F')$ be the set of collections of power series $(\Delta, (\delta_a)_{a \in A})$ satisfying the following properties

- $\Delta(\mathbf{Z}_1, \mathbf{Z}_2) = \Delta(\mathbf{Z}_2, \mathbf{Z}_1)$
- $\Delta(\mathbf{Z}_2, \mathbf{Z}_3) +_{F'} \Delta(\mathbf{Z}_1, F(\mathbf{Z}_2, \mathbf{Z}_3)) = \Delta(F(\mathbf{Z}_1, \mathbf{Z}_2), \mathbf{Z}_3) +_{F'} \Delta(\mathbf{Z}_1, \mathbf{Z}_2)$
- $\delta_a(\mathbf{Z}_1) +_{F'} \delta_a(\mathbf{Z}_2) +_{F'} \Delta([a]_F(\mathbf{Z}_1), [a]_F(\mathbf{Z}_2)) = [a]_{F'}\Delta(\mathbf{Z}_1, \mathbf{Z}_2) +_{F'} \delta_a(F(\mathbf{Z}_1, \mathbf{Z}_2))$
- $\delta_a(\mathbf{Z}_1) +_{F'} \delta_b(\mathbf{Z}_1) +_{F'} \Delta([a]_F(\mathbf{Z}_1), [b]_F(\mathbf{Z}_1)) = \delta_{a+b}(\mathbf{Z}_1)$
- $[a]_{F'}\delta_b(\mathbf{Z}_1) +_{F'} \delta_a([b]_F(\mathbf{Z}_1)) = \delta_{ab}(\mathbf{Z}_1)$.

These objects are called symmetric 2-cocycles. The set $\operatorname{SymCoc}^2(F, F')$ is naturally a left-$\operatorname{End}(F')$-module.

Proposition 2.6.5. *There is a bijection*

$$\operatorname{SymCoc}^2(F, F') \xrightarrow{\sim} \left\{ \begin{array}{c} A\text{-module laws } E \text{ on } R[\![\mathbf{X}, \mathbf{Z}]\!] \text{ fitting into an exact sequence} \\ 0 \to F' \xrightarrow{\alpha} E \xrightarrow{\beta} F \to 0 \\ \text{where } \alpha(\mathbf{X}) = (\mathbf{X}, 0) \text{ and } \beta(\mathbf{X}, \mathbf{Z}) = \mathbf{Z}. \end{array} \right\}$$

The map sends a pair $\{\Delta, (\delta_a)_a\}$ to the A-module law with structure defined following (2.9).

Proof. This is only a matter of calculation, cf. [HG94, Section 6]. $\square$

Lemma 2.6.6. *If $\mathcal{F}'$, $\mathcal{E}$ and $\mathcal{F}$ are formal A-modules over a qcqs A-scheme X, and α and β are morphisms such that $0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{F} \to 0$ is a short exact sequence of formal A-modules, α is a kernel of β and β is a cokernel of α.*

Proof. Let $\psi : \mathcal{G} \to \mathcal{E}$ be a morphism of formal A-modules such that the composition $\mathcal{G} \xrightarrow{\psi} \mathcal{E} \xrightarrow{\beta} \mathcal{F}$ is trivial. We have to show that there is a unique morphism $\overline{\psi} : \mathcal{G} \to \mathcal{F}'$ making the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r, "\alpha"] & \mathcal{E} \arrow[r, "\beta"] & \mathcal{F} \arrow[r] & 0 \\
& & \mathcal{G} \arrow[ul, dashed, "\exists!\overline{\psi}"] \arrow[u, "\psi"'] \arrow[ur, "0"'] & &
\end{tikzcd}$$

As $\overline{\psi}$ is unique, we may work locally X and assume that $X = \operatorname{Spec} R$ is affine and $\mathcal{F}'$, $\mathcal{F}$ and $\mathcal{G}$ all come from formal A-module laws. We may now assume that the short exact sequence is in the form of Lemma 2.6.3. Write E, F, F', G for the formal A-module laws corresponding to $\mathcal{E}$, $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{G}$. Write $\mathbf{Y}$ for the variables of G. Now, as $\beta \circ \psi = 0$, the induced morphism of formal A-module laws $\psi : G \to E$ is of the form $\psi(\mathbf{Y}) = (\psi_1(\mathbf{Y}), 0)$, and we find that $\psi_1(\mathbf{Y}) \in (R[\![\mathbf{Y}]\!])^m$ yields a morphism of formal A-modules $G \to F'$. It is clearly unique.
Similar ideas show that β is a cokernel of α. □

Lemma 2.6.7. *The composition of two deflations of formal A-modules is a deflation.*

Proof. [Proof is simple application of Lemma 2.6.3 but no time to write down] □

Lemma 2.6.8. *Let $0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{F} \to 0$ be a short exact sequence in $(A\text{-FML}^{\mathrm{arb}}/X)$. If $f \in \operatorname{Hom}_{(A\text{-FM}^{\mathrm{arb}}/X)} \mathcal{G} \to \mathcal{F}$ is a morphism of formal A-modules, then there is a formal A-module $f^*\mathcal{E}$ and a deflation $f^*\mathcal{E} \to \mathcal{G}$ fitting into a diagram with short exact sequences as rows*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r, "\alpha'"] \arrow[d, equal] & f^*\mathcal{E} \arrow[r, "\beta'"] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \mathcal{F}' \arrow[r, "\alpha"] & \mathcal{E} \arrow[r, "\beta"] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

The square on the right is cartesian.

Proof. Assume first that $X = \operatorname{Spec} R$ is affine and that $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{G}$ come from formal A-module laws over R. Then we assume to be in the situation of Lemma 2.6.3, with $\mathcal{E}$ coming from a formal A-module law E. Using the induced morphism $f : G \to F$ of formal A-module laws, define the A-module law law f^*E via

$$f^*E((\mathbf{X}_1, \mathbf{Y}_1), (\mathbf{X}_2, \mathbf{Y}_2)) = (F'(\mathbf{X}_1, \mathbf{X}_2) +_{F'} \Delta(f(\mathbf{Y}_1), f(\mathbf{Y}_2)), G(\mathbf{Y}_1, \mathbf{Y}_2))$$
and
$$[a]_{f^*E}(\mathbf{X}, \mathbf{Y}) = ([a]_{F'}(\mathbf{X}) +_{F'} \delta_a(f(\mathbf{Y})), [a]_F(\mathbf{Y})).$$

Here, Δ and δ_a are the power series coming from E (cf. Lemma 2.6.3). Now the top-row is exact with $\alpha'(\mathbf{X}) = (\mathbf{X}, 0)$ and $\beta'(\mathbf{X}, \mathbf{Y}) = \mathbf{Y}$. The morphism of A-module laws $f^*E \to E$ is given by $(\mathbf{X},\mathbf{Y}) \mapsto (\mathbf{X}, f(\mathbf{Y}))$. One readily checks that

$$\begin{array}{ccc} f^*E & \xrightarrow{\beta'} & G \\ \downarrow & & \downarrow f \\ E & \xrightarrow{\beta} & F \end{array}$$

is cartesian in the category of formal A-module laws over R. As the data of $\mathcal{E}$ glue, the power series defining f^*E glue to give a formal A-module $f^*\mathcal{E}$, satisfying all of the desired properties. $\square$

The dual statement is also true.

Lemma 2.6.9. *Let $0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{F} \to 0$ be as above, and let $g \in \mathrm{Hom}_{(A\text{-FM}^{\mathrm{arb}}/X)}(\mathcal{F}', \mathcal{G}')$ be a morphism of formal A modules. There is a formal A-module $g_*\mathcal{E}$ over X and an inflation $\alpha' : \mathcal{G}' \to g_*\mathcal{E}$ fitting into a diagram with short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}' & \xrightarrow{\alpha} & \mathcal{E} & \xrightarrow{\beta} & \mathcal{F} & \longrightarrow & 0 \\ & & g\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{G}' & \xrightarrow{\alpha'} & g_*\mathcal{E} & \xrightarrow{\beta'} & \mathcal{F} & \longrightarrow & 0 \end{array}$$

Proof. We proceed as in the proof of the previous lemma and assume that $X = \operatorname{Spec} R$ and that $\mathcal{F}'$, $\mathcal{F}$ and $\mathcal{G}$ come from formal A-module laws over R. Now E is a formal A-module law over R of the form described in Lemma 2.6.3, and using the power series Δ and δ_a we define g_*E via

$$g_*E((\mathbf{Y}_1,\mathbf{Z}_1),(\mathbf{Y}_2,\mathbf{Z}_2)) = (G'(\mathbf{Y}_1,\mathbf{Y}_2) +_{G'} g(\Delta(\mathbf{Z}_1,\mathbf{Z}_2)), F(\mathbf{Z}_1,\mathbf{Z}_2))$$

and

$$[a]_{g_*E}(\mathbf{X},\mathbf{Y}) = ([a]_{G'}(\mathbf{X}) +_{G'} g(\delta_a(\mathbf{Z})), [a]_F(\mathbf{Z})).$$

The morphism $E \to g_*E$ is given by $(\mathbf{X},\mathbf{Z}) \mapsto (g(\mathbf{X}),\mathbf{Z})$. These data glue and give rise to a formal A-module $g_*\mathcal{E}$ over X satisfying the desired properties. $\square$

As a consequence of the previous lemmas, we obtain

Proposition 2.6.10. *Let S be a quasi-compact and quasi-separated S-scheme. Then the category $(A\text{-FML}^{\mathrm{arb}}/S)$, equipped with the notion of exact sequences from Definition 2.6.2, is an exact category.*

The following calculation is convenient.

Lemma 2.6.11. *We have natural isomorpisms*

$$\mathrm{Lie}(f^*\mathcal{E}) \cong \mathrm{Lie}(\mathcal{E}) \times_{\mathrm{Lie}(\mathcal{F})} \mathrm{Lie}(\mathcal{G}) \quad \textit{and} \quad \mathrm{Lie}(g_*\mathcal{E}) \cong \mathrm{Lie}(\mathcal{G}') \sqcup_{\mathrm{Lie}(\mathcal{F}')} \mathrm{Lie}(\mathcal{E}).$$

Proof. This is true locally, and the local descriptions descent to X. $\square$

2.6.2 Extensions and Rigidified Extensions

We now introduce the functors Ext and RigExt. Let $\mathcal{F}$ and $\mathcal{F}'$ be formal A-modules over an A-scheme S.

Definition 2.6.12 (Extension)**.** An extension of $\mathcal{F}$ by $\mathcal{F}'$ is a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{E} \to \mathcal{F} \to 0.$$

We say that this extension is equivalent to another extension

$$0 \to \mathcal{F}' \to \mathcal{E}' \to \mathcal{F} \to 0$$

if and only if there is an isomorphism $\mathcal{E} \to \mathcal{E}'$ making the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{F} \arrow[r] & 0 \end{tikzcd}$$

commute. We denote the set of equivalence classes of extensions of $\mathcal{F}$ by $\mathcal{F}'$ as $\mathrm{Ext}(\mathcal{F}, \mathcal{F}')$.

Proposition 2.6.10 turns $\mathrm{Ext}(-,-)$ into a functor. In particular, $\mathrm{Ext}(\mathcal{F}, \mathcal{F}')$ carries the structure of a left-$\mathrm{End}(\mathcal{F}')$-module, with zero-object given by the canonical extension $\mathcal{F} \oplus \mathcal{F}'$.

Definition 2.6.13 (Rigifidied Extension)**.** A rigidified extension of $\mathcal{F}$ by $\mathcal{F}'$ is a pair consisting of an extension

$$0 \to \mathcal{F}' \to \mathcal{E} \to \mathcal{F} \to 0$$

and a splitting s of the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Lie}(\mathcal{F}') \arrow[r] & \mathrm{Lie}(\mathcal{E}) \arrow[r] & \mathrm{Lie}(\mathcal{F}) \arrow[r] \arrow[l, bend left, "s"] & 0. \end{tikzcd}$$

We say that two rigidified extensions (E, s), (E', s') are isomorphic if there is an isomorphism $i : E \to E'$ of extensions such that $s' = \mathrm{Lie}(i) \circ s$. We denote the set of isomorphism classes of rigidified extensions by $\mathrm{RigExt}(\mathcal{F}, \mathcal{F}')$.

Lemma 2.6.14. *The assignment* $(\mathcal{F}, \mathcal{F}') \mapsto \mathrm{RigExt}(\mathcal{F}, \mathcal{F}')$ *extends to a functor in both entries (contravariant in the first, covariant in the second).*

Proof. Given a morphism $f : \mathcal{G} \to \mathcal{F}$, the induced morphism $\mathrm{RigExt}(\mathcal{F}, \mathcal{F}') \to \mathrm{RigExt}(\mathcal{G}, \mathcal{F}')$ is given by sending the pair $(\mathcal{E}, s)$ to the pair $(f^*\mathcal{E}, s')$, where

$$s' : \mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(f^*\mathcal{E}) \cong \mathrm{Lie}(\mathcal{E}) \times_{\mathrm{Lie}(\mathcal{F})} \mathrm{Lie}(\mathcal{G}), \quad x \mapsto ((s \circ \mathrm{Lie}(f))(x), x).$$

Here we used the description of $\mathrm{Lie}(f^*\mathcal{E})$ from Lemma 2.6.11. Similarly, given a morphism $g : \mathcal{F}' \to \mathcal{G}'$, the induced morphism $\mathrm{RigExt}(\mathcal{F}, \mathcal{F}') \to \mathrm{RigExt}(\mathcal{F}, \mathcal{G}')$ sends $(\mathcal{E}, s)$ to $(g_*\mathcal{E}, \mathrm{Lie}(g') \circ s)$, where $g' : \mathcal{E} \to g_*\mathcal{E}$ is the canonical morphism. $\square$

In particular, $\mathrm{RigExt}(-, \mathcal{F}')$ carries the structure of an $\mathrm{End}(\mathcal{F}')$-module, the zero-object is given by the equivalence class of the pair $(\mathcal{F}' \oplus \mathcal{F}, s_{\mathrm{triv}})$, where $s_{\mathrm{triv}} : \mathrm{Lie}(\mathcal{F}) \to \mathrm{Lie}(\mathcal{F}') \oplus \mathrm{Lie}(\mathcal{F})$ is the canonical inclusion.

Of course there is a natural transformation $\mathrm{RigExt}(-,-) \to \mathrm{Ext}(-,-)$, forgetting the splitting. It appears as the right-most term of an interesting exact sequence.

Proposition 2.6.15. *There is an exact sequence of Abelian groups, functorial in $\mathcal{F}$ and $\mathcal{F}'$*

$$\mathrm{Hom}_{(A\text{-}\mathrm{FM}^{\mathrm{arb}}/S)}(\mathcal{F}, \mathcal{F}') \xrightarrow{\mathrm{Lie}} \mathrm{Hom}_{(\mathcal{O}_S\text{-}\mathrm{QCoh})}(\mathrm{Lie}(\mathcal{F}), \mathrm{Lie}(\mathcal{F}')) \to \mathrm{RigExt}(\mathcal{F}, \mathcal{F}') \to \mathrm{Ext}(\mathcal{F}, \mathcal{F}').$$

Proof. The kernel of $\mathrm{RigExt}(\mathcal{F}, \mathcal{F}') \to \mathrm{Ext}(\mathcal{F}, \mathcal{F}')$ is given (up to equivalence) by pairs of the form $(\mathcal{F}' \oplus \mathcal{F}, s)$, where s is a morphism of quasi-coherent $\mathcal{O}_S$-modules such that

$$\mathrm{Lie}(\mathcal{F}) \xrightarrow{s} \mathrm{Lie}(\mathcal{F}') \oplus \mathrm{Lie}(\mathcal{F}) \to \mathrm{Lie}(\mathcal{F})$$

is the identity. It is clear that these morphisms s correspond to morphisms $\mathrm{Lie}(\mathcal{F}) \to \mathrm{Lie}(\mathcal{F}')$. The kernel of $\mathrm{Hom}_{(\mathcal{O}_S\text{-}\mathrm{QCoh})}(\mathrm{Lie}(\mathcal{F}), \mathrm{Lie}(\mathcal{F}')) \to \mathrm{RigExt}(\mathcal{F}, \mathcal{F}')$ is spanned by those pairs $(\mathcal{E}, s)$ that are in the same class as $(\mathcal{F}' \oplus \mathcal{F}, s_{\mathrm{triv}})$. Any such $\mathcal{E}$ fits into a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d, equal] & \mathcal{F}' \oplus \mathcal{F} \arrow[r] \arrow[d, "\psi"] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{F}' \arrow[r, "\alpha"] & \mathcal{E} \arrow[r, "\beta"] & \mathcal{F} \arrow[r] & 0.
\end{tikzcd}$$

Working locally, we assume that $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{F}'$ come from formal module laws E, F and F'. Now ψ is necessarily of the form $\psi(\mathbf{X}, \mathbf{Z}) = (\mathbf{X} +_{F'} g(\mathbf{Z}), \mathbf{Z})$. Hence, the power series g furnishes a morphism of formal module laws $F \to F'$. This construction descents to a morphism of formal A-modules $\mathcal{F} \to \mathcal{F}'$, and we have

$$s(x) = \mathrm{Lie}(\psi) \circ s_{\mathrm{triv}}(x) = \mathrm{Lie}(\alpha) \circ \mathrm{Lie}(g)(x) + x \in \mathrm{Lie}(\mathcal{E}).$$

This explains exactness on the left. $\square$

2.6.3 Explicit Dieudonne Theory

Let $\mathcal{F}$ and $\mathcal{F}'$ be formal A-modules of dimension m and n respectively, over an affine base $\mathrm{Spec}\, R$, coming from formal module laws F and F'. We give an explicit description of $\mathrm{Ext}(\mathcal{F}, \mathcal{F}')$ in terms of terms of the Symmetric 2-Cocycles associated with F and F' (cf. Definition 2.6.4). We also give a related explicit description of $\mathrm{RigExt}(F, \widehat{\mathbb{G}}_a)$ in terms of Quasi-Logarithms, cf. Definition 2.6.18.

Write $\mathbf{X}$ for the variables of F' and $\mathbf{Z}$ for the variables of F.

Definition 2.6.16 (Symmetric 1-Cochain)**.** A symmetric 1-cochain associated to (F, F') is a n-tuple of power series $\mathbf{g} = (g_1, \ldots, g_m)$, such that $g_i(\mathbf{Z}) \in R[\![\mathbf{Z}]\!]$ has no constant term for all i. We write $\delta\mathbf{g}$ for the coboundary of $\mathbf{g}$, that is, the pair $(\Delta\mathbf{g}, (\delta_a\mathbf{g})_{a \in A})$, where

$$\Delta\mathbf{g} = \mathbf{g}(\mathbf{Z}_1) -_{F'} \mathbf{g}(F(\mathbf{Z}_1, \mathbf{Z}_2)) +_{F'} \mathbf{g}(\mathbf{Z}_2) \in (R[\![\mathbf{Z}_1, \mathbf{Z}_2]\!])^m$$

and

$$\delta_a \mathbf{g} = [a]_{F'}\mathbf{g}(\mathbf{Z}) -_{F'} \mathbf{g}([a]_F(\mathbf{Z})) \in (R[\![\mathbf{Z}]\!])^m.$$

One readily checks that $\delta \mathbf{g} \in \mathrm{SymCoc}^2(F, F')$.

Proposition 2.6.17. *Given two extensions $\mathcal{E}, \mathcal{E}' \in \mathrm{Ext}(\mathcal{F}, \mathcal{F}')$, write E, E' for the respective formal A-module laws coming from Lemma 2.6.3, and write Δ_E and $\Delta_{E'}$ for the associated symmetric 2-cocycles (cf. Proposition 2.6.5). There is a bijection*

$$\{\mathbf{g} \in (R[\![\mathbf{Z}]\!])^m \mid \mathbf{g}(0) = 0 \textit{ and } \delta\mathbf{g} = \Delta_{E'} - \Delta_E\} \xrightarrow{\sim} \{\textit{Isomorphisms of extensions } E \to E'\}.$$

Explicitly, this bijection is given by sending $\mathbf{g}$ to the morphism $i_{\mathbf{g}} \in \mathrm{Hom}_{(A\text{-}\mathrm{FML}^{\mathrm{arb}}/R)}(E, E')$, where $i_{\mathbf{g}}(\mathbf{X}, \mathbf{Z}) = (\mathbf{X} +_{F'} \mathbf{g}(\mathbf{Z}), \mathbf{Z})$. In particular, there is a bijection

$$\mathrm{Ext}(\mathcal{F}, \mathcal{F}') \cong \frac{\mathrm{SymCoc}^2(F, F')}{\{\delta\mathbf{g} \mid \mathbf{g} \in (R[\![\mathbf{Z}]\!])^m \textit{ with } \mathbf{g}(0) = 0\}}.$$

This bijection is an isomorphism of $\mathrm{End}(\mathcal{F}')$-modules.

For now, this finishes the study of $\mathrm{Ext}(\mathcal{F}, \mathcal{F}')$.
Assume now that $\mathcal{F}' = \widehat{\mathbb{G}}_a$, and that $\mathcal{F}$ comes from a one-dimensional formal A-module $F \in (A\text{-FML}/R)$. For the remainder of this subsection, we will be concerned with the R-module $\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a)$. The notion of Quasi-Logarithms will play a major role.

Definition 2.6.18 (Quasi-Logarithms)**.** A power series $g(T) \in (R \otimes_A K)[\![T]\!]$ is called a Quasi-Logarithm for F, if $g(0) = 0$ and $g'(T)$, as well as all of the power series appearing in δg (with $F' = \widehat{\mathbb{G}}_a$, cf. Definition 2.6.16) have coefficients in R. We define the R-module

$$\mathrm{QLog}(F) = \frac{\{g(T) \in (R \otimes_A K)[\![T]\!] \mid g \text{ is a quasi-logarithm for } F\}}{\{g(T) \in R[\![T]\!] \mid g(0) = 0\}}$$

Let $(\mathcal{E}, s) \in \mathrm{RigExt}(F, \widehat{\mathbb{G}}_a)$ be a rigidified extension. The splitting s yields an isomorphism $\omega(\mathcal{E}) \cong \omega(\widehat{\mathbb{G}}_a) \oplus \omega(\mathcal{F})$ on duals, giving an invariant differential $\omega_{\mathcal{E}} \in \omega(\mathcal{E})$ pulling back to $\mathrm{d}X$ on $\widehat{\mathbb{G}}_a$. Conversely, any such invariant differential $\omega_{\mathcal{E}}$ yields a splitting, so the choice of s is equivalent to the choice of ω_E, and we will henceforth write $(\mathcal{E}, \omega_{\mathcal{E}}) \in \mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a)$.

Theorem 2.6.19 (Classification of Rigidified Extensions in terms of Quasi-Logarithms)**.** *There is a bijection*

$$\{\textit{Quasi-logarithms for } F\} \xrightarrow{\sim} \left\{\begin{array}{c} \textit{Pairs } (E, \omega_E), \textit{ where } E \textit{ is an } A\textit{-module law} \\ \textit{fitting into an exact sequence} \\ 0 \to \widehat{\mathbb{G}}_a \xrightarrow{\alpha} E \xrightarrow{\beta} F \to 0 \\ \textit{with } \alpha(X) = (X, 0) \textit{ and } \beta(X, T) = T \textit{ and } \omega_E \\ \textit{is an invariant differential on } E \textit{ with } \alpha^*\omega_E = \mathrm{d}X. \end{array}\right\} \quad (2.10)$$

The map sends any quasi-logarithm $g(T) \in (R \otimes_A K)[\![T]\!]$ to the pair $(E_{\delta g}, \mathrm{d}(X + g(T))) \in \mathrm{RigExt}(F, \widehat{\mathbb{G}}_a)$. Here $E_{\delta g} \in \mathrm{Ext}(F, \widehat{\mathbb{G}}_a)$ is the extension corresponding to $\delta g \in \mathrm{SymCoc}^2(F, \widehat{\mathbb{G}}_a)$.

Furthermore, given two rigidified extensions $(E, \omega_E), (D, \omega_D)$ with associated quasi-logarithms $g(T)$ and $h(T)$, there is a (uniqe) isomorphism $(E, \omega_E) \to (D\omega_D)$ if and only if $h(T) - g(T) =: f(T)$ has coefficients in $R[\![T]\!]$. In this case, the isomorphism $i_f(X,T) \in \mathrm{Hom}_{(A\text{-FML}^{\mathrm{arb}}/R)}(E, D)$ is given by $i_f(X,T) = (X + f(T), T)$. In particular, there is a canonical bijection

$$\mathrm{QLog}(F) \xrightarrow{\sim} \mathrm{RigExt}(F, \widehat{\mathbb{G}}_a).$$

This bijection is an isomorphism of R-modules.

Proof. We construct an inverse of the map in (2.10). Let (E, ω_E) be an element of the set on the right and let $(\Delta, (\delta_a)_{a \in A}) \in \mathrm{SymCoc}^2(F, \widehat{\mathbb{G}}_a)$ be the symmetric 2-cochain corresponding to E. Following Proposition 2.3.3, the datum of $\omega_E \in \omega(E)$ is equivalent to a morphism

$$f_E \in \mathrm{Hom}_{(A\text{-FML}/R \otimes K)}(E \otimes_R (R \otimes_A K), \widehat{\mathbb{G}}_a) \quad \text{satisfying} \quad f_E(X,T) = X + g(T)$$

for some $g(T) \in (R \otimes_A K)[\![T]\!]$. The fact that f_E is a homomorphism implies that

$$\begin{aligned} X_1 + X_2 + \Delta(T_1, T_2) + g(F(T_1, T_2)) &= f_E(E((X_1, T_1), (X_2, T_2))) = \\ &= f_E(X_1, T_1) + f_E(X_2, T_2)) = X_1 + g(T_1) + X_2 + g(T_2), \end{aligned}$$

thereby $\Delta g = \Delta(T_1, T_2) \in R[\![T_1, T_2]\!]$. Similarly, we find $\delta_a g = \delta_a \in R[\![T]\!]$. Hence, $g(T)$ is a quasi-logarithm with $\delta g = (\Delta, (\delta_a)_a)$. This construction yields the desired inverse. The remaining statements are verified directly, also cf. [HG94, Section 8]. □

Now, let A be a complete, discrete valuation ring with uniformizing parameter π and finite residue field k.

Proposition 2.6.20. *If $\mathcal{F}$ comes from a one-dimensional formal A-module law over a flat A-algebra R and $\mathcal{F}' = \widehat{\mathbb{G}}_a$, the short exact sequence of Proposition 2.6.15 fits into a commutative diagram with exact rows and vertical maps (canonical) isomorphisms*

$$\begin{tikzcd} \mathrm{Hom}(\mathcal{F}, \widehat{\mathbb{G}}_a) \arrow[r, hook, "\mathrm{d}_F"] \arrow[d, equal] & \omega(\mathcal{F}) \arrow[r] \arrow[d] & \mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a) \arrow[r, two heads] \arrow[d] & \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a) \arrow[d] \\ \left\{ \begin{matrix} f \in TR[\![T]\!] : \\ \delta f = 0 \end{matrix} \right\} \arrow[r, hook] & \left\{ \begin{matrix} f \in (R \otimes_A K)[\![T]\!] : \\ \delta f = 0,\ f(0) = 0 \\ \text{and } f'(T) \in R[\![T]\!] \end{matrix} \right\} \arrow[r] & \mathrm{QLog}(F) \arrow[r, two heads, "\delta"] & \frac{\mathrm{SymCoc}^2(F, \widehat{\mathbb{G}}_a)}{\{\delta g \mid g \in TR[\![T]\!]\}} \end{tikzcd}$$

Proof. Injectivity of d_F is provided by Proposition 2.3.3, and related to the original exact sequence as $\mathrm{Hom}_R(\mathrm{Lie}(\mathcal{F}), \mathrm{Lie}(\widehat{\mathbb{G}}_a)) = \omega(\mathcal{F})$. Surjectivity of $\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a) \to \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a)$ comes from the fact that $\mathrm{Lie}(\mathcal{F})$ is projective. The first vertical map is an equality, cf. Definitions 2.6.16 and 2.1.3. The vertical arrow describing $\omega(F)$ is obtained by identifying the preimage of $\omega(F) \subseteq \omega(F \otimes_R (R \otimes_A K))$ under the isomorphism

$$\{f \in T(R \otimes_A K)[\![T]\!] \mid \delta f = 0\} = \mathrm{Hom}_{(A\text{-FML}/R \otimes_A K)}(F \otimes (R \otimes_A K), \widehat{\mathbb{G}}_a) \xrightarrow{\mathrm{d}_F} \omega(F \otimes_R (R \otimes_A K)).$$

All squares commute by construction. □

We admit the following facts from Section 9 of [HG94].

Proposition 2.6.21. *Let F be a formal A-module law of height h over a local, adic A-algebra R. Write $\mathcal{F}$ for the formal A-module coming from F. Then $\mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a)$ is a free R-module of rank $n-1$, $\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a)$ is a free R-module of rank n.*

Proof. This is Proposition 9.8 in the beforementioned source. The authors make use of a description of $\mathrm{Ext}(F, \widehat{\mathbb{G}}_a)$ in terms of deformation theory and combine it with a convenient normal form of formal A-modules, so called A-typical modules (we touch upon the theory in section 2.7), to construct an explicit basis for the corresponding modules. □

As a corollary, the authors obtain

Lemma 2.6.22. *If $R \to R'$ is a homomorphism of local A-algebras, the induced maps of free R'-modules*

$$\mathrm{Ext}_R(\mathcal{F}, \widehat{\mathbb{G}}_a) \otimes_R R' \to \mathrm{Ext}_{R'}(F, \widehat{\mathbb{G}}_a)$$
$$\mathrm{RigExt}_R(\mathcal{F}, \widehat{\mathbb{G}}_a) \otimes_R R' \to \mathrm{RigExt}_{R'}(F, \widehat{\mathbb{G}}_a)$$

are isomorphisms.

Proof. [HG94, Corollary 9.13]. □

2.7 Hazewinkel's Functional Equation Lemma and the Standard Formal Module Law

If, A is an integral domain and R is a flat A-module, the structure of a formal A-module F over R is uniquely determined by its logarithm $\log_H \in R \otimes_A K[\![T]\!]$. Indeed, we find

$$F(X, Y) = \exp_H(X + Y), \quad [a]_F(X) = \exp_H(aX).$$

It is therefore natural to wonder about conditions on power series $f \in (R \otimes_A K)[\![T]\!]$ ensuring that f is the logarithm of some formal group law. Hazewinkel found such a condition in his functional equation lemma.

Proposition 2.7.1 (Hazewinkel's Functional Equation Lemma)**.** *Let p be a prime and $q = p^e$. Given an inclusion of rings $B \subseteq L$, an ideal $\mathfrak{a} \subseteq B$ containing p, an endomorphism of rings $\sigma : L \to L$ and elements $s_1, s_2, \dots \in L$ subject to the conditions that*

$$\sigma(b) \equiv b^q \pmod{\mathfrak{a}} \text{ for all } b \in B \quad \text{and} \quad \sigma^r(s_i)\mathfrak{a} \subset B \text{ for all } r, s \geq 1.$$

Suppose now that $f \in L[\![T]\!]$ has $f'(0) \in L^\times$ and satisfies the functional equation condition

$$f(X) - \sum_{i=1}^{\infty} s_i (\sigma^i_* f)(X^{q^i}) \in B[\![X]\!].$$

Then we have

$$F(X, Y) = f^{-1}(f(X) + f(Y)) \in B[\![X, Y]\!],$$

where f^{-1} is the inverse power series as in Lemma 2.1.2. Also, if $g(Z) \in L[\![Z]\!]$ is another power series satisfying the same condition

$$g(Z) - \sum_{i=1}^{\infty} s_i(\sigma_*^i f)(Z^{q^i}) \in B[\![Z]\!],$$

then $f^{-1}(g(Z)) \in B[\![Z]\!]$. Furthermore, if $\alpha(T) \in B[\![T]\!]$ and $\beta(T) \in B[\![T]\!]$, then

$$\alpha(T) \equiv \beta(T) \pmod{\mathfrak{a}^r} \iff f(\alpha(T)) \equiv f(\beta(T)) \pmod{\mathfrak{a}^r} \tag{2.11}$$

Proof. A more general statement can be found in [Haz79, Section 2]. Proofs can be found in [Haz78, Sections 2 and 10]. □

Note that by construction, $F(X,Y)$ as defined above yields a (commutative) formal group law over B. Let B^σ denote the subring of elements in B fixed by σ. Then the second part of the Functional Equation Lemma implies that we even obtain formal B^σ-modules with $[b]_F(T) = f^{-1}(bf(T))$, as $bf(T)$ satisfies the same functional equation if $b \in B^\sigma$.

We use the Functional Equation Lemma to construct Lubin–Tate Formal Group Laws. Hence we now enter the situation where K is a local field with ring of integers $\mathcal{O}_K$ and uniformizer π. A special role will play the power series

$$f(T) = \sum_{i=1}^{\infty} \frac{T^{q^{in}}}{\pi^i} \in K[\![T]\!].$$

It satisfies the functional equation

$$f(T) = T + \frac{1}{\pi} f(T^{q^n}),$$

which is a functional equation of the form above, with $B = \mathcal{O}_K$, $\mathfrak{a} = (\pi)$, $L = K$, $s_1 = \pi^{-1}$, $s_2 = s_3 = \cdots = 0$, $\sigma = \mathrm{id}_L$. Hence f arises as the logarithm of a formal $\mathcal{O}_K$-module law H over $\mathcal{O}_K$. The fact that $f^{-1}(X) = X - \frac{1}{\pi}X^{q^n} + \ldots$ reveals $[\pi]_H(T) \equiv \pi T \bmod (T^2)$. Additionally, note that

$$f([\pi]_H(T)) = \pi f(T) = \pi T + f(T^{q^n}) \equiv f(T^{q^n}) \pmod{\pi}.$$

Hence, the equivalence in (2.11) implies that $[\pi]_H(T) \equiv T^{q^n} \bmod \pi$. So H is a Lubin–Tate formal $\mathcal{O}_K$-module law of height n, we call it the standard Lubin–Tate formal module law of height n.

Remark. The formal $\mathcal{O}_K$-module H is a member of the set of so called A-typical formal modules - formal A-modules F with logarithm of the form

$$\log_F(T) = \sum_{i=0}^{\infty} b_i X^{q^i}$$

for elements $b_0, b_1, \dots \in R \otimes_A K$. If R is flat over A, every formal A-module over R is isomorphic to an A-typical one (cf. [Haz78, p. 21.5.6]). The following discussion remains valid for $\mathcal{O}_K$-typical formal modules.

It will be convenient to make the terms in the exact sequence of Proposition 2.6.20 explicit for $F = H$. As F is of height $n > 0$, there is no non-trivial map $F \to \widehat{\mathbb{G}}_a$ and the sequence becomes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega(H) & \longrightarrow & \mathrm{RigExt}(H, \widehat{\mathbb{G}}_a) & \longrightarrow & \mathrm{Ext}(H, \widehat{\mathbb{G}}_a) & \xrightarrow{0} & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \left\{ \begin{matrix} g \in TK[\![T]\!] : \delta g = 0 \\ \text{and } g'(T) \in \mathcal{O}_K[\![T]\!] \end{matrix} \right\} & \longrightarrow & \mathrm{QLog}(H) & \xrightarrow{\delta} & \frac{\mathrm{SymCoc}^2(H, \widehat{\mathbb{G}}_a^n)}{\{\delta g \mid g \in T\mathcal{O}_K[\![T]\!]\}} & \xrightarrow{0} & 0. \end{array}$$

We now have

Proposition 2.7.2. *The R-module $\omega(H)$ is free of rank 1, generated by $f(T) = \log_H(T)$. $\mathrm{QLog}(H)$ is free of rank n, generated by the classes of $(f(T), \frac{1}{\pi}f(T^q), \ldots, \frac{1}{\pi}f(T^{q^{n-1}}))$. Consequently, the short exact sequence above is given by*

$$0 \to \langle f(T) \rangle \to \left\langle f(T), \frac{1}{\pi}f(T^q), \ldots, \frac{1}{\pi}f(T^{q^{n-1}}) \right\rangle \xrightarrow{\delta} \left\langle \delta\left(\frac{1}{\pi}f(T^q)\right), \ldots, \delta\left(\frac{1}{\pi}f(T^{q^{n-1}})\right) \right\rangle \to 0.$$

Proof. It is easily checked that $\frac{1}{\pi}f(T^{q^k})$ is a quasi-logarithm for $1 \le k \le n-1$. As $\delta f = 0$, we have $f(T) \in \mathrm{QLog}(F)$ as well. The claim is [HG94, Proposition 13.8] which is a special case of [ibid., Proposition 9.8]. □

2.8 The Universal Additive Extension

We follow [HG94, Section 11], and specialize to the situation where A is a complete discrete valuation ring with uniformizer π and finite residue field of characteristic p and R is a local admissible A-algebra with residue field $\overline{\mathbb{F}}_q$.

Lemma 2.8.1. *Let M be a finite free module over R. Then there is a natural bijection, functorial in M and $\mathcal{F}$*

$$\mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a \otimes M) \cong \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a) \otimes_R M.$$

Proof. After choosing coordinates on $\mathcal{F}$, this follows directly from the description of Ext in terms of symmetric 2-cocycles, cf. Propositions 2.6.5 and 2.6.17. □

Let $\mathcal{F}$ be a one-dimensional formal A-module over R of height n, coming from a formal A-module law F over R. We put $M := \mathrm{Hom}_R(\mathrm{Ext}(F, \widehat{\mathbb{G}}_a), R)$, which is free of rank $n-1$, and write $\mathcal{F}' = \widehat{\mathbb{G}}_a \otimes M$. Now, by the previous lemma,

$$\mathrm{Ext}(\mathcal{F}, \mathcal{F}') = \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a \otimes M) = \mathrm{End}_R(\mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a)).$$

Let $0 \to \mathcal{F}' \to \mathcal{E} \to \mathcal{F} \to 0$ be the extension corresponding to the identity on the right. This class is unique up to unique isomrphism. Indeed, as R is a local ring we may choose formal module laws F and F' giving rise to $\mathcal{F}$ and $\mathcal{F}'$, and let E be the module law obtained from Lemma 2.6.3. If $0 \to F' \to E' \to F \to 0$ is another extension in this class, we have by construction a commutative square

$$\begin{tikzcd}
0 \arrow[r] & F' \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "i"] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F' \arrow[r] & E' \arrow[r] & F \arrow[r] & 0,
\end{tikzcd}$$

and by Proposition 2.6.17 we see that any other isomorphisms i' making the diagram above diagram commutes differes from i by an element in $\mathrm{Hom}(F, F') = 0$.

Definition 2.8.2 (Universal Additive Extension)**.** The extension

$$0 \to \mathcal{F}' \to \mathcal{E} \to \mathcal{F} \to 0$$

constructed above is called the universal additive extension of $\mathcal{F}$.

Proposition 2.8.3. *If N is a free R-module, $\mathcal{G}' = \widehat{\mathbb{G}}_a \otimes N$ and*

$$0 \to \mathcal{G}' \to \mathcal{E}' \to F \to 0$$

is an extension of $\mathcal{F}$ by $\mathcal{G}'$, there are unique homomorphisms $i : \mathcal{E} \to \mathcal{E}'$ and $g' : \mathcal{F}' \to \mathcal{G}'$ making the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d, "g'"'] & \mathcal{E} \arrow[r] \arrow[d, "i"] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{G}' \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

commute. In particular, we have $\mathcal{E}' = g'_*\mathcal{E}$.

Proof. As $\mathcal{F}'$ and $\mathcal{G}'$ are additive, we have

$$\mathrm{Hom}(\mathcal{F}', \mathcal{G}') = \mathrm{Hom}_R(M, N) = \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a) \otimes N = \mathrm{Ext}(\mathcal{F}, \mathcal{G}').$$

This yields g'. Again, i is unique as by observations similar to Proposition 2.6.17, the difference of two morphisms $i, i' : \mathcal{E} \to \mathcal{E}'$ is given a morphism $\mathcal{F} \to \mathcal{G}'$, which has to be trivial. $\square$

Lemma 2.8.4. *We have* $\mathrm{Lie}(\mathcal{E}) = \mathrm{Hom}(\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a), R) = \mathrm{D}(\mathcal{F})$.

Proof. We show equivalently that $\omega(\mathcal{E}) = \mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a)$. Let $(\mathcal{E}', \omega_{\mathcal{E}'}) \in \mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a)$. Then by universality of $\mathcal{E}$, we obtain a unique homomorphism $i : \mathcal{E} \to \mathcal{E}'$. This yields a homomorphism of R-modules $\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a) \to \omega(\mathcal{E})$, $(\mathcal{E}', \omega_{\mathcal{E}'}) \mapsto i^*\omega_{\mathcal{E}'}$. This morphism fits into the following commutative diagram, where the top row is the short exact sequence from Proposition 2.6.20 and the bottom row is the dual short exact sequence of $0 \to \mathrm{Lie}(\mathcal{F}') \to \mathrm{Lie}(\mathcal{E}) \to \mathrm{Lie}(\mathcal{F}) \to 0$.

$$\begin{tikzcd}
0 \arrow[r] & \omega(\mathcal{F}) \arrow[r] \arrow[d, equal] & \mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a) \arrow[r] \arrow[d] & \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \omega(\mathcal{F}) \arrow[r] & \omega(\mathcal{E}) \arrow[r] & \omega(\mathcal{F}') \arrow[r] & 0
\end{tikzcd}$$

Hence, it is a natural isomorphism. □

2.9 Tate Modules and the Universal Cover

2.9.1 Useful Calculations

Let p be a prime. Let R be a Noetherian local ring with maximal ideal I such that $p \in I$, R is complete with respect to the I-adic topology and $k_R := R/I$ is an algebraically closed field (necessarily of characteristic p). If q is a power of p, we write $\mathcal{F}_{R,q}$ for the set of power series $f \in R[\![T]\!]$ satisfying

$$f(T) \equiv g(T^q) \pmod{I} \tag{2.12}$$

for some power series $g(T) = c_1T + c_2T^2 + \cdots \in R[\![T]\!]$ with $c_1 \in R^\times$. If $q' > q$ is another power of p, we have injections $\mathcal{F}_{R,q} \hookrightarrow \mathcal{F}_{R,q'}$ given by sending $f(T)$ to its (q'/q)-fold self-composite $f^{q'/q}(T)$. Making use of these transition maps, we define

$$\mathcal{F}_R := \operatorname*{colim}_{n\in\mathbb{N}} \mathcal{F}_{R,p^n},$$

identifying any power series $f \in \mathcal{F}_{R,q}$ with its image in $\mathcal{F}_{R,q'}$ for higher p-powers q'. For any $f \in \mathcal{F}_{R,q}$, we define the functor

$$U_f : (R\text{-Adm}) \to (\text{Set}), \quad S \mapsto \left\{ (x_0, x_1, \dots) \in \prod_{\mathbb{N}} S^{\circ\circ} \mid f(x_{i+1}) = x_i \right\}.$$

This functor does, up to canonical isomorphism, only depend on the equivalence class of f in $\mathcal{F}_R$. We write $U_{0,f}$ for the base change of U_f to k_R, that is

$$U_{0,f} : (k_R\text{-Adm}) \to (\text{Set}), \quad S \mapsto \left\{ (x_0, x_1, \dots) \in \prod_{\mathbb{N}} S^{\circ\circ} \mid \overline{f}(x_{i+1}) = x_i \right\}.$$

Here, $\overline{f}$ is the image of f under the reduction map $R[\![T]\!] \to k_R[\![T]\!]$.

In the sequel, we denote R-algebras by S and write J for an ideal of definition containing the image of I (provided, for example, by A.0.2). Given an element $f \in \mathcal{F}_R$, we do not distinguish between f and a choice of a representative $\tilde{f} \in \mathcal{F}_{R,q}$ for some sufficiently large p-power.

The following observation lays the groundwork for many of the upcoming results.

Lemma 2.9.1. *Let f be any power series in $\mathcal{F}_R$. For any two elements $s_1, s_2 \in S$ with $s_1 \equiv s_2$* mod J *such that $f(s_1)$ and $f(s_2)$ exist (for example if f is a polynomial or $s_1, s_2 \in S^{\circ\circ}$), we have*

$$f^k(s_1) \equiv f^k(s_2) \pmod{J^{k+1}}.$$

Here, f^k denotes k-fold composition of f.

Proof. We will show that if $s_1 \equiv s_2 \bmod J^k$, then $f(s_1) \equiv f(s_2) \bmod J^{k+1}$, which suffices to prove the claim. We may write $s_2 = s_1 + r$ for some $r \in J^k$. By the assumptions on f there exist power series $g, h \in R[\![T]\!]$ such that h only has coefficients in I and $f(T) = g(T^q) + h(T)$.

As I is finitely generated, say by elements $(r_1, \dots, r_l)$, we obtain a representation

$$f(s_1) - f(s_2) \in g(s_1^q) - g(s_2^q) + \sum_{i=1}^{l} r_i \left(h_i(s_1) - h_i(s_2)\right).$$

As r divides $(h_i(s_1) - h_i(s_2))$, we find $r_i(h_i(s_1) - h_i(s_2)) \in (r_i r) \subseteq J^{k+1}$. Also note that for any $s \in S$ and $n \in \mathbb{N}$,

$$(s + r)^{nq} = s^{nq} + nqrs^{nq-1}r + \cdots + r^{nq},$$

so after cancellation, all monomials of $g(s_1^q) - g(s_2^q)$ lie in (qr) or (r^2). This implies

$$g(s_1^q) - g\left((s_1 + r)^q\right) \in (qr) + (r^2) \subseteq J^{k+1},$$

and we are done. $\square$

Lemma 2.9.2. *The natural reduction map*

$$U_f(S) \to U_f(S/J) = U_{0,f}(S/J)$$

is bijective.

Proof. We first show surjectivity. Given a sequence $(x_0, x_1, \dots) \in U_f(S/J)$, we can choose a sequence of arbitrary lifts $(y_0, y_1, \dots) \in \prod_{\mathbb{N}} S^{\infty}$ and set

$$z_i = \lim_{r \to \infty} f^r(y_{i+r}).$$

The limit exists, because if $s \geq r$ are two non-negative integers, we calculate

$$f^{s-r}(y_{i+s}) \equiv \overline{f}^{s-r}(x_{i+s}) = x_{i+r} \equiv y_{i+r} \pmod{J},$$

implying by Lemma 2.9.1 that

$$f^s(y_{i+s}) \equiv f^r(y_{i+r}) \pmod{J^r}.$$

This shows that $(f^r(y_{i+r}))_{r \in \mathbb{N}}$ is a Cauchy-sequence for the J-adic topology on S, thereby convergent (cf. Lemma A.0.4). The sequence $(z_0, z_1, \dots)$ now lies in $U_f(S)$ and lifts $(x_0, x_1, \dots)$. It remains to show that the lift is unique. Suppose that $(z'_0, z'_1, \dots)$ is another lift. Then, for any $i, k \in \mathbb{N}$ we have $z_{i+k} \equiv z'_{i+k} \bmod J$, and another application of Lemma 2.9.1 shows that

$$z_i = f^k(z_{i+k}) \equiv f^k(z'_{i+k}) = z'_i \pmod{J^k}.$$

Thereby $(z_i - z'_i) \in \bigcap_{k \in \mathbb{N}} J^k = \{0\}$. Hence, the lift is unique. $\square$

We write $\mathrm{Nilp}^\flat$ for the functor U_{T^q}. That is, $\mathrm{Nilp}^\flat(S) = \lim_{x \mapsto x^q} S^{\infty}$ is the set of q-power compatible sequences with values in S^{∞}.

Lemma 2.9.3. *For any $f \in \mathcal{F}_R$, there is a canonical bijection $U_{0,f}(S/J) \to \mathrm{Nilp}^\flat(S/J)$. This bijection is functorial in S.* Use different S

Proof. By assumption on f we have $f(T) = g(T^q) \in k_R[\![T]\!]$ for some $g(T) = c_1 T + c_2 T^2 + \dots$ with $c_1 \neq 0$. For each coefficient c_i, let $d_i \in k_R$ be the unique element such that $d_i^q = c_i$. Let $h(T) \in k_R[\![T]\!]$ be the power series given by $d_1 T + d_2 T^2 + \dots$. Now $(h(T))^q = f(T)$, and we find that

$$U_f(S/J) \to \mathrm{Nilp}^\flat(S/J): \quad (x_1, x_2, x_3, \dots) \mapsto (x_1, h(x_2), h(h(x_3)), \dots)$$

is a well-defined function, and (trivially) functorial in S. For the inverse, let $h^{-1}(T) \in k_R[\![T]\!]$ be the the unique power series with $h^{-1}(h(T)) = h(h^{-1}(T)) = T$, see Lemma 2.1.2. The map

$$\mathrm{Nilp}^\flat(S/J) \to U_f(S/J), \quad (x_1, x_2, \dots) \mapsto (x_1, h^{-1}(x_2), h^{-1}(h^{-1}(x_3)), \dots)$$

is well-defined as

$$f(h^{-1}(T)) = g((h^{-1}(T))^q) = (h(h^{-1}(T)))^q = T^q$$

and it is readily seen to be inverse to the map constructed above. □

We collect results.

Proposition 2.9.4. *Given $f, g \in \mathcal{F}_R$, we have bijections, functorial in S,*

$$U_f(S) \to U_f(S/J) \to \mathrm{Nilp}^\flat(S/J) \to U_g(S/J) \to U_g(S). \tag{2.13}$$

Explicitely, the bijection $U_f(S) \to U_g(S)$ can be described as follows. Suppose that $f, g \in \mathcal{F}_{R,q}$ for some sufficiently large q. Let $h_f(T)$ and $h_g(T)$ be power series with coefficients in A such that

$$h_f(T)^q \equiv f(T) \pmod{I} \quad \textit{and} \quad h_g(T)^q \equiv g(T) \pmod{I}.$$

Write $h_g^{-1}(T)$ for the (formal) inverse power series of h_g. Now the isomorphism is given by the mapping

$$(x_0, x_1, \dots) \mapsto (y_0, y_1, \dots), \quad \textit{where} \quad y_i = \lim_{r\to\infty} g^r(h_g^{-(r+i)}(h_f^{r+i}(x_{i+r}))).$$

Here, the exponents are to be interpreted as iterated composition.

Proof. The first part follows directly from repeated application of the previous two Lemmas. The second part follows by tracing through the previous lemmas. □

2.9.2 The Universal Cover

Let A be an integral domain and R be an A-algebra. Given $H \in (A\text{-FM}/R)$ and $a \in A$, we define the functor

$$\tilde{H}_a : (R\text{-Adm}) \to (A\text{-Mod}), \quad S \mapsto \left\{ (x_1, x_2, \dots) \in \prod_{\mathbb{N}} H(S) \mid [a]_H(x_{i+1}) = x_i \right\}.$$

Here, the A-module structure is given by $b.(x_1, x_2, \dots) = ([b]_H(x_1), [b]_(x_2), \dots)$. Note that multiplication by a on $\tilde{H}_a(S)$ is an automorphism (it sends $(x_1, x_2, \dots)$ to $([a]_H x_1, x_1, x_2, \dots)$, which has inverse given by shifting to the left) so that $\tilde{H}_a(S)$ is naturally an $A[\frac{1}{a}]$-module.

From now on assume that A is a discrete valuation ring with uniformizer π, finite residue field k and field of fractions K. Let R be a local A-algebra with maximal ideal I and algebraically closed residue field $k_R = R/I$. Let H be a formal A-module over R.

Definition 2.9.5 (The Universal Cover and Tate Module)**.** We write $\tilde{H} = \tilde{H}_\pi$. This functor takes values in the category of K-vector spaces. Up to natural isomorphism, $\tilde{H}$ does not depend on the choice of π. We call this functor the universal cover of H.

The Tate-Module $T_\pi H$ is the subfunctor of $\tilde{H}$ cut out out by the condition that $[\pi]_H(x_1) = 0$. Note that $T_\pi H$ does no longer carry the structure of a K-vector space, it is an A-module. The Rational Tate Module $V_\pi H$ is the subfunctor of $\tilde{H}$ cut out by the condition that x_1 has $[\pi]_H$-torsion. Equivalently, we have

$$V_\pi H(S) = T_\pi H(S) \otimes_A K.$$

Lemma 2.9.6. *Let H be a π-divisible formal A-module over R and write $H_0 = H \otimes_R k_R$. Now the choice of a coordinate on H_0 gives rise to a an isomorphism*

$$\tilde{H}_0 \cong \mathrm{Nilp}^\flat_{k_R}$$

of functors (k_R-Adm) $\to$ (Set)

Proof. Note that given any coordinate on H, we have $[\pi]_H(T) \in \mathcal{F}_R$. Hence, the statement is an application of Lemma 2.9.3. □

Lemma 2.9.7. *Suppose that S is an admissible R-algebra admitting an ideal of definition J such that $\pi \in J$. Then the natural reduction map*

$$\tilde{H}(S) \to \tilde{H}(S/J)$$

is an isomorphism.

Proof. After choosing a coordinate on H, we have $[\pi]_H \in \mathcal{F}_R$ and $\tilde{H}(S) \cong U_{[\pi]_H}$, and the statement is given by Lemma 2.9.2. □

The following is analogous to Proposition 2.9.4.

Proposition 2.9.8. *Let S be an admissible R-algebra with ideal of definition J such that $\phi(I) \subseteq J$. Then there are canonical isomorphisms (of sets)*

$$\tilde{H}(S) \cong \tilde{H}(S/J) = \tilde{H}_0(S/J) \cong \mathrm{Nilp}^\flat(S/J) \cong \mathrm{Nilp}^\flat(S).$$

In particular, $\tilde{H}(S)$ is, as a functor to (Set)*, representable by* $\mathrm{Spf}(R[\![T^{q^{-\infty}}]\!])$.

Remark. In case where H comes from a Lubin–Tate $\mathcal{O}_K$-module law, the bijections

$$\tilde{H}(S) \rightleftarrows \mathrm{Nilp}^\flat(S), \quad (x_0, x_1, \dots) \leftrightarrows (y_0, y_1, \dots)$$

are, in either direction, given by the equations

$$y_i = \lim_{r\to\infty} x_{i+r}^{q^r} \quad \text{and} \quad x_i = \lim_{s\to\infty} [\pi^s]_H(y_{i+s}).$$

This follows directly from the explicit description of the isomorphism in Proposition 2.9.4, as we may choose $h_{[\pi]_H}(T) = h_{T^q}(T) = T$.

2.10 The Quasilogarithm Map

We keep the assumptions on A, R and S from the previous subsection. That is, A is a local ring with finite residue field and uniformizer π, R is a local A-algebra with maximal ideal I complete with respect to the I-adic topology and algebraically closed residue field k_R, and S denotes an admissible R-algebra (where $R \to S$ is continuous with the I-adic topology on R) with ideal of definition $J \subseteq S$ containing the image of I.

The aim of this subsection is to define, attached to any π-divisible formal A-module H over R, a quasi-logarithm map

$$\mathrm{qlog}_H : \tilde{H}(S) \to (M(H_0) \otimes \widehat{\mathbb{G}}_a)(S)$$

and give an explicit description of this map if H is the standard $\mathcal{O}_K$-module over $\mathcal{O}_{\breve{K}}$.

2.11 Determinants of Formal Modules

- "Functorial" description of the determinant. Either as in [BW11], or as in [Wei16].
- Construction.
- Approximations.

3 Local Class Field Theory following Lubin–Tate

Let K be a local field with residue field k, put $q = \#k$, and denote by $\nu_K : K \to \mathbb{Z} \cup \{\infty\}$ the valuation of K, normalized such that $\nu_K(\pi) = 1$ for a uniformizer π of K. The aim of this subsection is to describe the maximal abelian extension of a local field K.

The Local Kronecker-Weber theorem gives an explicit description of the abelianization of the absolute Galois group of K only in terms of K:

Theorem 3.0.1 (Local Kronecker-Weber). *There is an isomorphism (canonical up to choice of a uniformizer $\pi \in K$)*

$$\mathrm{Gal}(\overline{K}/K)^{\mathrm{ab}} \cong \mathrm{Gal}(K^{\mathrm{ab}}/K) \cong \mathcal{O}_K^\times \times \widehat{\mathbb{Z}}.$$

Here, K^{ab} denote the maximal abelian extension of K, which can (after choosing an algebraic closure of K) be described as $\overline{K}^{[G_K,G_K]}$.

The extension K^{ab} consists of two parts, we have $K^{\mathrm{ab}} = K^{\mathrm{rm}} \cdot K^{\mathrm{nr}}$. The field K^{nr}, the maximal unramified extension of K, has relatively simple structure. Describing the field K^{rm} (or rather, it's completion) is the hard part and it is here where we apply the theory of formal modules.

The valuation ν_K extends uniquely to $\overline{K}$, yielding a π-adic norm on $\overline{K}$. Let C denote the completion with respect to this norm. An application of Krasner's Lemma implies that $\mathrm{Gal}(C/K) \cong \mathrm{Gal}(\overline{K}/K) =: G_K$. One readily checks that any $\sigma \in G_K$ yields a continuous automorphism $\mathcal{O}_C \to \mathcal{O}_C$, and we obtain a short exact sequence Ref

$$0 \to I_K \to G_K \to \mathrm{Gal}(\overline{k}/k) \to 0.$$

The subgroup $I_K \subset G_K$ is called the inertia subgroup of K, and we write $\breve{K}$ for the subfield of C fixed by I_K. In particular we have $\mathrm{Gal}(\breve{K}/K) \cong \mathrm{Gal}(\overline{k}/k)$. One readily confirms that $\breve{K}$ is complete with respect to the norm induced by K.

As the Galois group of any finite extension of k is cyclic, we find that $\mathrm{Gal}(\breve{K}/K)$ is abelian. In fact, it is isomorphic to $\widehat{\mathbb{Z}} = \lim_n(\mathbb{Z}/n\mathbb{Z})$. Hence K_∞ decomposes as $\breve{K} \cdot K_\pi$ for some abelian, complete extension K_π/K such that $K_\pi \cap \breve{K} = K$. Now K_π is the completion of K^{rm}. Observe that

$$\mathrm{Gal}(K_\infty/K) \cong \mathrm{Gal}(K_\pi/K) \times \mathrm{Gal}(\breve{K}/K) \cong \mathrm{Gal}(K_\pi/K) \times \widehat{\mathbb{Z}},$$

so Theorem 3.0.1, the local Kronecker-Weber Theorem, is equivalent to showing that the Galois group of K_π over K is isomorphic to $\mathcal{O}_K^\times$.

4 Non-Abelian Lubin-Tate Theory: An Overview

In the preceeding chapter we used formal $\mathcal{O}_K$-modules to understand the maximial abelian extension of a local field K. The hope of non-Abelian Lubin-Tate theory is to gain insight about the Abelian extensions of K by considering certain moduli spaces of formal $\mathcal{O}_K$-modules. More precisely, attached to a formal $\mathcal{O}_K$-module H_0 over $\overline{\mathbb{F}}_q$ (determined up to isomorphism by its height n), we attach a system of rigid spaces $\{M_K\}_{K \subset \mathrm{GL}_n(\mathcal{O}_K)}$, the so called Lubin-Tate Tower. For $l \neq p$, the system of l-adic compactly supported cohomology groups $\{H^i_c(M_K, \overline{\mathbb{Q}}_l)\}_K$ admits commuting actions by $\mathrm{GL}_n(K)$, W_K and $D^\times$, where the latter denotes the units of the central divison algebra $D = \mathrm{End}_{(\mathcal{O}_K\text{-FM}/\overline{\mathbb{F}}_q)}(H_0) \otimes \mathbb{Q}$. This yields a correspondence of representations of the respective groups, and Harris and Taylor showed in [HT01] that the cohomology of middle degree induces (a version of) the Local Langlands Correspondence for GL_n. Our goal is an explicit description of this correspondence, and we obtain such descriptions by understanding the Lubin-Tate tower explicitely. As it turns out, the the limit $\lim_K M_K$ is representable by a perfectoid space which is is easier to describe than its individual layers.

4.1 The Lubin-Tate Tower

4.1.1 Deformations of Formal Modules

We mostly follow [Str08, Chapter 2] for notation. Let $\mathcal{C}$ denote the category of local, Noetherian $\mathcal{O}_{\breve{K}}$-modules with distinguished isomorphisms $R/\mathfrak{m}_R \to \overline{\mathbb{F}}_q$. Let H_0 be a formal $\mathcal{O}_K$-module over $\overline{\mathbb{F}}_q$.

Definition 4.1.1 (Deformation)**.** Let $R \in \mathcal{C}$. A deformation of H_0 to R is a pair (H, ι) where H is a formal $\mathcal{O}_K$-module over R and ι is a quasi-isogeny

$$\iota : H_0 \dashrightarrow H \otimes_R \overline{\mathbb{F}}_q.$$

Two deformations (H, ι) and (H', ι') are isomorphic if there is an isomorphism $\tau : H \to H'$ with $\iota' \circ \tau = \iota$.

The Lubin-Tate space without level structure is the moduli space of such deformations. More precisely, we define it as the functor

$$\mathcal{M}_0 : \mathcal{C} \to (\mathrm{Set}), \quad R \mapsto \{\text{deformations } (H, \iota) \text{ of } H_0\}/\text{iso}.$$

Theorem 4.1.2 (Representability of $\mathcal{M}_0$)**.** *The functor $\mathcal{M}_0$ is (non-canonically) representable, by the noetherian local ring*

$$A_0 \cong \mathcal{O}_{\breve{K}}[\![u_1, \ldots, u_{n-1}]\!].$$

In particular, there is a universal deformation $(F^{\mathrm{univ}}, \iota^{\mathrm{univ}})$, with $F^{\mathrm{univ}} \in (\mathcal{O}_{\breve{K}}\text{-FM}/A_0)$.

4.1.2 Deformations of Formal Modules with Drinfeld Level Structure

Definition 4.1.3 (Drinfeld level $\mathfrak{p}^m$-structure)**.** Let $R \in \mathcal{C}$ and $H \in (\mathcal{O}_K\text{-FM}/R)$. A Drinfeld level $\mathfrak{p}^m$-structure on H is a morphism of R-group schemes

$$(\mathfrak{p}^{-m}/\mathcal{O}_K)^{\oplus n} \to H(R)[\pi^m]$$

such that after choosing a coordinate $H \cong \operatorname{Spf} R[\![T]\!]$, the power series $[\pi]_H(T) \in R[\![T]\!]$ satisfies the divisibility constraint

$$\prod_{x \in (\mathfrak{p}^{-1}/\mathcal{O}_K)} (T - \phi(x)) \mid [\pi]_H(T).$$

The following examples might shed some light on this definition.

Example.

- $\widehat{\mathbb{G}}_m$
- Things over $\mathbb{F}_q$.
- Drinfeld Level
- Moduli Problem + Representability
- The Lubin-Tate Tower

4.1.3 The Group actions on the Tower and its Cohomology

- Action By $D^\times$ and GL_n
- Action by W_K via Weil descent Datum.

4.2 The Local Langlands Correspondence for the General Linear Group

4.3 The Lubin-Tate Perfectoid Space

5 Mieda's Approach to the Explicit Local Langlands Correspondence

6 The Explicit Local Langlands Correspondence for Depth Zero Supercuspidal Representations

6.1 The Special Affinoid

6.2 Deligne-Lusztig Theory for Depth Zero Representations

6.3 Proof

A Topological Rings

To deal with the topological rings showing up, the notion of admissible rings will be convenient (taken from [Stacks, Tag 07E8]).

Definition A.0.1. Let A be a topological ring. We say that A is admissible if

- The element $0 \in A$ has a fundamental system of neighbourhoods consisting of ideals.
- There exists an ideal of definition, that is, an open ideal $I \subset A$ such that every open neighbourhood of 0 contains I^n for some n.
- It is complete, that is, the natural map

  $$A \to \lim_{J \subset A \text{ open ideal}} A/J$$

  is an isomorphism.

We say that A is adic if it admits an ideal of definition I such that I^n is open for all n. Given a topological ring A, we denote the category of admissible and adic A-algebras (algebras S with continuous morphism $A \to S$) by (A-Adm) and (A-Adic), respectively.

[The following results might be not interesting enough to make it into the final draft]

Lemma A.0.2. *Let $\phi : R \to S$ be a morphism of admissible rings, and let $I \subset R$ be an admissible ideal. Then the ideal $J = \phi(I) \cdot S$ is an ideal of definition in S.*

Proof. Let U be an open ideal of S. By continuity of ϕ, it's preimage $U' = \phi^{-1}(U)$ is open in R. Hence there is some n with $I^n \subset U'$. But now

$$\phi(I)^n = \phi(I^n) \subseteq \phi(\phi^{-1}(U)) \subseteq U$$

and the claim follows. □

Lemma A.0.3. *Let S be an admissible ring, and let $(s_1, s_2, \dots)$ be a sequence with elements in S. Then $\sum_{i=1}^{\infty} s_i$ converges if and only if $\lim_{i\to\infty} s_i = 0$. In this case, the product $\prod_{i=1}^{\infty}(1+s_i)$ exists in S.*

Proof. If the sum converges, $(s_i)_{i\in\mathbb{N}}$ has to be a null-sequence. The reverse implication and the convergence of the product follows after writing $S \cong \lim_J S/J$ for a system of open ideals $J \subset S$. □

The topology on an admissible ring R with ideal of definition I is coarser than the I-adic topology on R

Lemma A.0.4. *Let R be an admissible ring with ideal of definition I. Let R' be the same ring, but equipped with the I-adic topology. Then the identity map $R' \to R$ is continuous. In particular, if a sequence converges with respect to the I-adic topology, it also converges in R'.*

Proof. It suffices to check that open ideals of R are open in R'. Let $J \subset R$ an open ideal. By assumption, there is some n with $I^n \subset J$. But now, for any $x \in J$, we have $x + I^n \subset J$. Hence, J is open in R'. □

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