

todo

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April 5, 2024

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1 Introduction

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This section will serve as an introduction to formal groups and formal modules. Formal groups (or rather, formal group laws) were first introduced by SALOMON BOCHNER in 1946 as a natural means of studying Lie Groups over fields of characteristic 0, cf. [1]. The study of formal groups later became interesting for its own right, with pioneering works of Lazard [2].

2.1 Formal Modules

As promised in the introduction, we begin by defining formal group laws.

Definition 2.1.1 (Formal Group Law). Let R be a ring. A (commututative, one-dimensional) formal group law over R is a power series $F(X, Y) \in R[[X, Y]]$ such that $F(X, Y) \equiv X + Y$ modulo terms of degree 2 and the following properties are satisfied:

- $F(F(X, Y), Z) = F(X, F(Y, Z))$,
- $F(X, Y) = F(Y, X)$,
- $F(X, 0) = X$.

Given two formal group laws $F, G \in R[[X, Y]]$, a morphism $f : F \rightarrow G$ is a power series $f \in R[[T]]$ such that $f(0) = 0$ and $f(F(X, Y)) = G(f(X), f(Y))$. Such a series is an isomorphism if there is an inverse, that is, a power series $g \in R[[T]]$ with $(f \circ g)(T) = T$. This yields the category of formal group laws over R , which we notate by (FGL/R) .

The following statements about morphisms of formal group laws are useful and easily verified.

Lemma 2.1.2. *Let R be a ring and let $F, G \in R[[X, Y]]$ be two formal group laws over R .*

1. Given two morphisms $f, g : F \rightarrow G$, the power series $G(f(T), g(T)) \in R[[T]]$ is a morphism of formal group laws $F \rightarrow G$. In particular, $\text{Hom}_{(\text{FGL}/R)}(F, G)$ is an abelian group for any two formal group laws F, G .
2. The abelian group $\text{End}_{(\text{FGL}/R)}(F)$ has a natural ring structure with multiplication given by concatenation.
3. A morphism $f = c_1T + c_2T^2 + \cdots \in R[[T]]$ between F and G is an isomorphism if and only if $c_1 \in R^\times$.

Example. Let us introduce the following two formal group laws.

- *The additive formal group law.* Write \mathbb{G}_a for the formal group law with addition given by $\mathbb{G}_a(X, Y) = X + Y$.
- We write \mathbb{G}_m for the formal group law associated with the with $\mathbb{G}_m(X, Y) = X + Y + XY$.

Next up is the definition of formal A -module laws. Naively, we'd like to say that an A -module law is the same as that of a formal group law F plus A -module structure, i.e. a morphism of rings $[\cdot]_F : A \rightarrow \text{End}_{(\text{FGL}/R)}(F)$. But there is a subtlety going on here: Let

$$\text{Lie} : (\text{FGL}/R) \rightarrow (\text{Ab})$$

be the (constant) functor that sends $F \in (\text{FGL}/R)$ to $(R, +)$, and morphisms $f : G \rightarrow H$ given by a formal power series $f = c_1T + c_2T^2 + \cdots \in R[[T]]$ to the endomorphism of R given by multiplication with c_1 . The condition that $F(X, Y) \equiv X + Y$ modulo degree 2 enforces that the induced map $\text{End}(F) \rightarrow \text{End}(R)$ is a morphism of rings. Now, the A -module structure on F yields an A -module structure on R , given by the concatenation

$$A \xrightarrow{[\cdot]_F} \text{End}(F) \xrightarrow{\text{Lie}} \text{End}(R), \quad a \mapsto \text{Lie}([a]_F)$$

This is a morphism of rings, and we obtain an A -algebra structure on R . We'd like the A -algebra structure on R to be uniform. This motivates the following definition.

Definition 2.1.3 (Formal A -module law). Let A be a ring and R be an A -algebra with structure morphism $p : A \rightarrow R$. A (one-dimensional) A -module law over an R is a pair $(F, ([a]_F)_{a \in A})$, where $F \in R[[X, Y]]$ is a formal group law and $[a]_F = p(a)X + c_2X^2 + \cdots \in R[[X]]$ yield endomorphisms $F \rightarrow F$ such that the induced map

$$A \rightarrow \text{End}(F), \quad a \mapsto [a]_F$$

is a morphism of rings.

Similarly to above, we obtain a category of formal A -module laws over R , which we denote by $(A\text{-FML}/R)$. Note that $(\text{FGL}/R) \cong (\mathbb{Z}\text{-FML}/R)$. Slightly abusing notation, we usually do not explicitly mention the A -structure when referring to formal module laws, simply writing $F \in (A\text{-FML}/R)$, for example.

The following lemma explains the functoriality of the assignment $R \mapsto (A\text{-FML}/R)$.

Lemma 2.1.4. *The assignment $R \mapsto (A\text{-FML}/R)$ is functorial in the following sense. If $p : R \rightarrow R'$ is a morphism of A -algebras, we obtain a functor*

$$(A\text{-FML}/R) \rightarrow (A\text{-FML}/R'), \quad F \mapsto p_*F,$$

where p_*F is the formal A -module law obtained by applying p to the coefficients of the formal power series representing addition and scalar multiplication of F . We sometimes write (with abuse of notation) $p_*F = F \otimes_R R'$.

Note that every formal module law $F \in (A\text{-FML}/R)$ yields a functor

$$(R\text{-Alg}) \rightarrow (A\text{-Mod}), \quad S \mapsto \text{Nil}(S), \quad (2.1)$$

where $\text{Nil}(S)$, the set of nilpotent elements of S , is equipped with addition and scalars given by

$$s_1 + s_2 = F(s_1, s_2) \in \text{Nil}(S), \quad as = [a]_F(s) \in \text{Nil}(S).$$

This construction yields a functor (with slight abuse of notation)

$$(A\text{-FML}/R) \rightarrow \text{Fun}((R\text{-Alg}), (A\text{-Mod})), \quad (2.2)$$

where Fun denotes the functor category.

Passing from discrete R -algebras to admissible R -algebras, this construction extends naturally to a functor

$$\text{Spf}^F : (A\text{-FML}/R) \rightarrow \text{Fun}((R\text{-Adm}), (A\text{-Mod})), \quad F \mapsto \text{Spf } R[[T]],$$

where we equip $\text{Spf } R[[T]]$ with the structure of an A -module object using the endomorphisms coming from F . Following this line of thought leads naturally to the definition of formal modules.

Definition 2.1.5 (Formal Group and Formal Module.). Let X be an A -scheme, and let \mathcal{F} be an A -module object in (FSch/X) , the category of formal schemes over X . Suppose that there is a Zariski-covering $(\text{Spec}(R_i))_{i \in I}$ of X with $\mathcal{F} \times_X U_i \cong \text{Spf}(R_i[[T]])$. If for every $i \in I$ the induced A -module structure on $\text{Spf}(R_i[[T]])$ comes from a formal A -module law F_i over R_i , we say that \mathcal{F} is a formal A -module.

Definition 2.1.6 (Coordinate). Let \mathcal{F} be a formal A -module over X . The choice of a cover $\sqcup_{i \in I} \text{Spec}(R_i) \rightarrow X$ together with isomorphisms $\mathcal{F} \times_X \text{Spec}(R_i) \cong \text{Spf}(R_i[[T]])$ will be referred to as a coordinate of \mathcal{F} .

Of course there is a functor

$$(A\text{-FML}/R) \rightarrow (A\text{-FM}/R),$$

essentially forgetting the choice of module law. The observation of Lemma ?? translates to formal modules, a morphism $p : R \rightarrow R'$ yields a functor

$$p_* : (A\text{-FM}/R) \rightarrow (A\text{-FM}/R'), \quad \mathcal{F} \mapsto \mathcal{F} \otimes_R R'.$$

Example. The additive group law \mathbb{G}_a extends to a formal A -module over an affine base $\text{Spec } R$ by setting

$$[a]_{\mathbb{G}_a}(T) = aT$$

for $a \in A$. More generally, we obtain a formal A -module over an arbitrary base scheme.

The multiplicative formal group \mathbb{G}_m does not have an obvious generalization to a formal A -module law for general A . In the case where A is the ring of integers of a local field, Lubin and Tate [3] construct such generalizations. This construction, and the application to local class field theory, will be discussed in section ??.

2.1.1 Formal DVR-Modules over Fields of Characteristic 0

As above, let A be a discrete valuation ring with uniformizer π and finite residue field k . Let K denote the field of fractions of A .

2.1.2 Formal DVR-Modules over Residue Fields

Let \mathbb{F}_q denote the finite field with $q = p^n$ elements.

Definition 2.1.7 (Frobenius). Given a formal group F over \mathbb{F}_q , let ϕ denote the Frobenius endomorphism. This is the endomorphism given by $f(T) = T^q$ after choosing a coordinate on F .

Let F and G be formal group laws over \mathbb{F}_q , and let $f : F \rightarrow G$ be a non-zero morphism between F and G given by a formal power series $f(T) = c_1T + c_2T^2 + \dots$.

Definition 2.1.8 (Height). In the above situation, the height of f , denoted $\text{ht}(f)$, is the greatest integer h such that f factors through $\phi^h : F \rightarrow F$. In case that $f = 0$, we write $\text{ht}(f) = \infty$.

2.2 Application: The Local Class Field Theory

Let K be a local field with residue field k , put $q = \#k$, and denote by $\nu_K : K \rightarrow \mathbb{Z} \cup \{\infty\}$ the valuation of K , normalized such that $\nu_K(\pi) = 1$ for a uniformizer π of K . The aim of this subsection is to describe the maximal abelian extension of a local field K .

The Local Kronecker-Weber theorem gives an explicit description of the abelianization of the absolute Galois group of K only in terms of K :

Theorem 2.2.1 (Local Kronecker-Weber). *There is an isomorphism (canonical up to choice of a uniformizer $\pi \in K$)*

$$\text{Gal}(\overline{K}/K)^{\text{ab}} \cong \text{Gal}(K^{\text{ab}}/K) \cong \mathcal{O}_K^\times \times \widehat{\mathbb{Z}}.$$

Here, K^{ab} denote the maximal abelian extension of K , which can (after choosing an algebraic closure of K) be described as $\overline{K}^{[G_K, G_K]}$.

The extension K^{ab} consists of two parts, we have $K^{\text{ab}} = K^{\text{rm}} \cdot K^{\text{nr}}$. The field K^{nr} , the maximal unramified extension of K , has relatively simple structure. Describing the field K^{rm}

(or rather, it's completion) is the hard part and it is here where we apply the theory of formal modules.

The valuation ν_K extends uniquely to \overline{K} , yielding a π -adic norm on \overline{K} . Let C denote the completion with respect to this norm. An application of Krasner's Lemma implies that $\text{Gal}(C/K) \cong \text{Gal}(\overline{K}/K) =: G_K$. One readily checks that any $\sigma \in G_K$ yields a continuous automorphism $\mathcal{O}_C \rightarrow \mathcal{O}_C$, and we obtain a short exact sequence

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$$0 \rightarrow I_K \rightarrow G_K \rightarrow \text{Gal}(\overline{k}/k) \rightarrow 0.$$

The subgroup $I_K \subset G_K$ is called the inertia subgroup of K , and we write \check{K} for the subfield of C fixed by I_K . In particular we have $\text{Gal}(\check{K}/K) \cong \text{Gal}(\overline{k}/k)$. One readily confirms that \check{K} is complete with respect to the norm induced by K .

As the Galois group of any finite extension of k is cyclic, we find that $\text{Gal}(\check{K}/K)$ is abelian. In fact, it is isomorphic to $\widehat{\mathbb{Z}} = \varprojlim_n (\mathbb{Z}/n\mathbb{Z})$. Hence K_∞ decomposes as $\check{K} \cdot K_\pi$ for some abelian, complete extension K_π/K such that $K_\pi \cap \check{K} = K$. Now K_π is the completion of K^{rm} . Observe that

$$\text{Gal}(K_\infty/K) \cong \text{Gal}(K_\pi/K) \times \text{Gal}(\check{K}/K) \cong \text{Gal}(K_\pi/K) \times \widehat{\mathbb{Z}},$$

so Theorem ??, the local Kronecker-Weber Theorem, is equivalent to showing that the Galois group of K_π over K is isomorphic to \mathcal{O}_K^\times .

3 Non-Abelian Lubin-Tate Theory: An Overview

3.1 The Lubin-Tate Tower

3.2 The Local Langlands Correspondence for the General Linear Group

3.3 The Lubin-Tate Perfectoid Space

4 Mieda's Approach to the Explicit Local Langlands Correspondence

5 The Explicit Local Langlands Correspondence for Depth Zero Supercuspidal Representations

5.1 The Special Affinoid

5.2 Deligne-Lusztig Theory for Depth Zero Representations

5.3 Proof

A Topological Rings

To deal with the topological rings showing up, the notion of admissible rings will be convenient (taken from [Stacks, Tag 07E8]).

Definition A.0.1. Let A be a topological ring. We say that A is admissible if

- The element $0 \in A$ has a fundamental system of neighbourhoods consisting of ideals.
- There exists an ideal of definition, that is, an ideal $I \subset A$ such that every open neighbourhood of 0 contains I^n for some n .
- It is complete, that is, the natural map

$$A \rightarrow \varprojlim_{J \subset A \text{ open ideal}} A/J$$

is an isomorphism.

We say that A is adic if it admits an open ideal of definition. Given a topological ring A , we denote the category of admissible and adic A -algebras (algebras S with continuous morphism $A \rightarrow S$) by $(A\text{-Adm})$ and $(A\text{-Adic})$, respectively.