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This section will serve as an introduction to formal groups and formal modules. Formal groups (or rather, formal group laws) were first introduced by SALOMON BOCHNER in 1946 as a natural means of studying Lie Groups over fields of characteristic 0, cf. **Bochner1946FGrps**. The study of formal groups later became interesting for its own right, with pioneering works of Lazard **Lazard1955FGrps**. blabla

2.1 Basic Notions

As promised in the introduction, we begin by defining formal group laws.

Definition 2.1.1 (Formal Group Law)**.** Let R be a ring. A (commututative, one-dimensional) formal group law over R is a power series $F(X,Y) \in R[\![X,Y]\!]$ such that $F(X,Y) \equiv X+Y$ modulo terms of degree 2 and the following properties are satisfied:

- $F(F(X,Y),Z) = F(X,F(Y,Z))$,
- $F(X,Y) = F(Y,X)$,
- $F(X,0) = X$.

Given two formal group laws $F,G \in R[\![X,Y]\!]$, a morphism $f : F \to G$ is a power series $f \in R[\![T]\!]$ such that $f(0) = 0$ and $f(F(X,Y)) = G(f(X),f(Y))$. Such a series is an isomorphism if there is an inverse, that is, a power series $g \in R[\![T]\!]$ with $(f \circ g)(T) = T$. This yields the category of formal group laws over R, which we notate by (FGL/R).

The following statements about morphisms of formal group laws are useful and easily verified.

Lemma 2.1.2. *Let R be a ring and let $F,G \in R[\![X,Y]\!]$ be two formal group laws over R.*

1. *Given two morphisms $f,g : F \to G$, the power series $G(f(T),g(T)) \in R[\![T]\!]$ is a morphism of formal group laws $F \to G$. In particular, $\mathrm{Hom}_{(\mathrm{FGL}/R)}(F,G)$ is an abelian group for any two formal group laws F,G.*

2. *The abelian group* $\mathrm{End}_{(\mathrm{FGL}/R)}(F)$ *has a natural ring structure with multiplication given by concatenation.*

3. *A morphism* $f = c_1T + c_2T^2 + \cdots \in R[\![T]\!]$ *between* F *and* G *is an isomorphism if and only if* $c_1 \in R^\times$.

Example. Let us introduce the following two formal group laws.

- *The additive formal group law.* Write $\widehat{\mathbb{G}}_a$ for the formal group law with addition given by $\widehat{\mathbb{G}}_a(X,Y) = X + Y$.
- We write $\widehat{\mathbb{G}}_m$ for the formal group law associated with the with $\widehat{\mathbb{G}}_m(X,Y) = X + Y + XY$. Note that $\widehat{\mathbb{G}}_m(X,Y) = (X+1)(Y+1) - 1$

Next up is the definition of formal A-module laws. Naively, we'd like to say that an A-module law is the same as that of a formal group law F plus A-module structure, i.e. a morphism of rings $[\cdot]_F : A \to \mathrm{End}_{(\mathrm{FGL}/R)}(F)$. But there is a subtlety here: Let

$$\mathrm{Lie} : (\mathrm{FGL}/R) \to (\mathrm{Ab})$$

be the (constant) functor that sends $F \in (\mathrm{FGL}/R)$ to $(R,+)$, and morphisms $f : G \to H$ given by a formal power series $f = c_1T + c_2T^2 + \cdots \in R[\![T]\!]$ to the endomorphism of R given by multiplication with c_1. The condition that $F(X,Y) \equiv X + Y$ modulo degree 2 enforces that the induced map $\mathrm{End}(F) \to \mathrm{End}(R)$ is a morphism of rings. Now, the A-module structure on F yields an A-module structure on R, given by the concatenation

$$A \xrightarrow{[\cdot]_F} \mathrm{End}(F) \xrightarrow{\mathrm{Lie}} \mathrm{End}(R), \quad a \mapsto \mathrm{Lie}([a]_F)$$

This is a morphism of rings, and we obtain an A-algebra structure on R. We'd like the A-algebra structure on R to be uniform. This motivates the following definition.

Definition 2.1.3 (Formal A-module law)**.** Let A be a ring and R be an A-algebra with structure morphism $p : A \to R$. A (one-dimensional) A-module law over an R is a pair $(F, ([a]_F)_{a\in A})$, where $F \in R[\![X,Y]\!]$ is a formal group law and $[a]_F = p(a)X + c_2X^2 + \cdots \in R[\![X]\!]$ yield endomorphisms $F \to F$ such that the induced map

$$A \to \mathrm{End}(F), \quad a \mapsto [a]_F$$

is a morphism of rings.

Similarly to above, we obtain a category of formal A-module laws over R, which we denote by $(A\text{-FML}/R)$. Note that $(\mathrm{FGL}/R) \cong (\mathbb{Z}\text{-FML}/R)$. Slightly abusing notation, we usually do not explicitely mention the A-structure when referring to formal module laws, simply writing $F \in (A\text{-FML}/R)$, for example.
The following lemma explains a the functoriality of the assignment $R \mapsto (A\text{-FML}/R)$.

Lemma 2.1.4. *The assignment* $R \mapsto (A\text{-FML}/R)$ *is functorial in the following sense. If* $p : R \to R'$ *is a morphism of* A*-algebras, we obtain a functor*

$$(A\text{-FML}/R) \to (A\text{-FML}/R'), \quad F \mapsto p_*F,$$

where p_*F is the formal A-module law obtained by applying p to the coefficients of the formal power series representing addition and scalar multiplication of F. We sometimes write (with abuse of notation) $p_*F = F \otimes_R R'$.

Note that every formal module law $F \in (A\text{-FML}/R)$ yields a functor

$$(R\text{-Alg}) \to (A\text{-Mod}), \quad S \mapsto \mathrm{Nil}(S), \tag{2.1}$$

where $\mathrm{Nil}(S)$, the set of nilpotent elements of S, is equipped with addition and scalars given by

$$s_1 + s_2 = F(s_1, s_2) \in \mathrm{Nil}(S), \quad as = [a]_F(s) \in \mathrm{Nil}(S).$$

This construction yields a functor (with slight abuse of notation)

$$(A\text{-FML}/R) \to \mathrm{Fun}((R\text{-Alg}), (A\text{-Mod})), \tag{2.2}$$

where Fun denotes the functor category.

Passing from discrete R-algebras to admissible R-algebras, this construction extends naturally to a functor

$$\mathrm{Spf}^F : (A\text{-FML}/R) \to \mathrm{Fun}((R\text{-Adm}), (A\text{-Mod})), \quad F \mapsto \mathrm{Spf}\, R[\![T]\!],$$

where we equip $\mathrm{Spf}\, R[\![T]\!]$ with the structure of an A-module object using the endomorphisms coming from F. Following this line of thought leads naturally to the definition of formal modules.

Definition 2.1.5 (Formal Group and Formal Module.)**.** Let X be an A-scheme, and let let $\mathcal{F}$ be an A-module object in (FSch/X), the category of formal schemes over X. Suppose that there is a Zariski-covering $(\mathrm{Spec}(R_i))_{i\in I}$ of X with $\mathcal{F} \times_X U_i \cong \mathrm{Spf}(R_i[\![T]\!])$. If for every $i \in I$ the induced A-module structure on $\mathrm{Spf}(R_i[\![T]\!])$ comes from a formal A-module law F_i over R_i, we say that $\mathcal{F}$ is a formal A-module.

Definition 2.1.6 (Coordinate)**.** Let $\mathcal{F}$ be a formal A-module over X. The choice of a cover $\sqcup_{i\in I} \mathrm{Spec}(R_i) \to X$ together with isomorphisms $\mathcal{F} \times_X \mathrm{Spec}(R_i) \cong \mathrm{Spf}(R_i[\![T]\!])$ will be referred to as a coordinate of $\mathcal{F}$.

Of course there is a functor

$$(A\text{-FML}/R) \to (A\text{-FM}/R),$$

essentially forgetting the choice of module law. The observation of Lemma 2.1.4 translates to formal modules, a morphism $p : R \to R'$ yields a functor

$$p_* : (A\text{-FM}/R) \to (A\text{-FM}/R'), \quad \mathcal{F} \mapsto \mathcal{F} \otimes_R R'.$$

Example. The additive group law $\widehat{\mathbb{G}}_a$ extends to a formal A-module over an affine base $\mathrm{Spec}\, R$ by setting

$$[a]_{\widehat{\mathbb{G}}_a}(T) = aT$$

for $a \in A$. More generally, we obtain a formal A-module over an arbitrary base scheme.

The formal group associated to $\widehat{\mathbb{G}}_m$ over $\mathbb{Z}_p$ is isomorphic to the functor

$$(\mathbb{Z}_p\text{-Adm}) \to (\mathrm{Ab}), \quad S \mapsto 1 + S^{\infty} \subset S^{\times}.$$

Here, we equipped $\mathbb{Z}_p$ with the p-adic topology. The subgroup $1 + S^{\infty}$ naturally carries the structure of a $\mathbb{Z}_p$-module. Indeed, for $k \in \mathbb{N}$, we have

$$(1+s)^{p^k} = 1 + p^k s + \binom{p^k}{2} s^2 + \cdots + s^{p^k},$$

and given $s \in S^{\infty}$, this is of the form $1 + o(1)$ as k gets large. In particular, if $x = a_0 + a_1 p + a_2 p^2 + \cdots \in \mathbb{Z}_p$, expressions of the form

$$(1+s)^x = \prod_{i=1}^{\infty} (1+s)^{a_k p^k}$$

make sense by lemma A.0.3. This gives $\widehat{\mathbb{G}}_{m,\mathbb{Z}_p}$ the structure of a formal $\mathbb{Z}_p$-module. In the upcoming subsection, we discuss how this is the simplest example of a whole family of formal modules constructed by Lubin and Tate. In section 3 we explain applications of these formal modules to local class field theory.

2.2 Lubin-Tate Formal Module Laws

Suppose that K is a local field with ring of integers $\mathcal{O}_K$, with uniformizer π and residue field $\mathbb{F}_q$.

Let H_0 be the formal $\mathcal{O}_K$-module law defined over $\mathbb{F}_q$ by setting

$$H_0(X,Y) = X + Y, \quad [\pi]_{H_0}(X) = X^q, \quad [u]_{H_0} = \overline{u}X.$$

Here, u runs over the units of $\mathcal{O}_K$ and $\overline{u} \in \mathbb{F}_q$ is such that $u \equiv \overline{u} \bmod \pi$. This uniquely determines $[a]_{H_0}$ for $a \in \mathcal{O}_K$ as a may be written as $a = \pi^{\nu} u$ for a unit u and $[\pi]_{H_0}$ and $[u]_{H_0}$ commute.

Lubin-Tate formal module laws are $\mathcal{O}_K$-module laws H over $\mathcal{O}_K$ such that $H \otimes_{\mathcal{O}_K} \mathbb{F}_q = H_0$.

The construction of the Lubin-Tate formal module laws rests on the following lemma, which is Lemma 1 in **LubinTateFormalMult**.

Lemma 2.2.1. *Let $f(T)$ and $g(T)$ be elements of $\mathcal{F}_\pi$ and let $L(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_i X_i$ be a linear form with coefficients in $\mathcal{O}_K$. Then there exists a unique series $F(X_1, \ldots, X_n)$ with coefficients in $\mathcal{O}_K$ such that*

$$F(X_1, \ldots, X_n) \equiv L(X_1, \ldots, X_n) \pmod{T^2},$$

and

$$f(F(X_1, \ldots, X_n)) = F(g(X_1), \ldots, g(X_n)).$$

Restate in more general terms, for arbitrary powers of q, and perhaps for general discrete valuation rings with

Although stated only for the rings of integers of a local field, the proof only uses that $\mathcal{O}_K$ is complete with respect to the π-adic topology, π is not a zero divisor and the map $x \mapsto x^q$

restricts to the identity mod π. In particular, the statement remains true if working over the integers of the completion of a maximal unramified extension $\breve{K}$ of K.
Write $\mathcal{F}_\pi$ for the set of power series that may arise as $[\pi]_H$, that is,

$$\mathcal{F}_\pi := \{f \in \mathcal{O}_K[\![T]\!] \mid f \equiv \pi T \pmod{T^2} \text{ and } f \equiv T^q \pmod{\pi}\}.$$

Using Lemma 2.2.1, we can construct formal $\mathcal{O}_K$-modules over $\mathcal{O}_K$ as follows. Attached to $f \in \mathcal{F}_\pi$, we find a unique power series $F_f(X,Y) \in \mathcal{O}_K[\![X,Y]\!]$ satisfying

$$F_f(X,Y) \equiv X+Y \pmod{(X,Y)^2} \quad \text{and} \quad F_f(f(X),f(Y)) = f(F_f(X,Y)). \tag{2.3}$$

Furthermore, attached to each $a \in \mathcal{O}_K$ and $f,g \in \mathcal{F}_\pi$, we find unique $[a]_{f,g}(T) \in \mathcal{O}_K[\![T]\!]$ satisfying

$$[a]_{f,g}(T) \equiv aT \pmod{(T)^2} \quad \text{and} \quad f([a]_{f,g}(T)) = [a]_{f,g}(g(T)). \tag{2.4}$$

We now have

Theorem 2.2.2 (Lubin-Tate Formal $\mathcal{O}_K$-Module Laws)**.** *For $f \in \mathcal{F}_\pi$, the family of power series $(F_f, ([a]_{f,f})_{a\in\mathcal{O}_K})$ gives rise to a formal $\mathcal{O}_K$-module law over $\mathcal{O}_K$. For $f,g \in \mathcal{F}_\pi$, the formal $\mathcal{O}_K$-module laws F_f and F_g are isomorphic, via the morphism induced by $[1]_{f,g} \in \mathcal{O}_K[\![T]\!]$.*

Proof. See Theorem 1 of **LubinTateFormalMult** and the succeeding discussion. □

In particular, up to isomorphism, the construction of a Lubin-Tate formal $\mathcal{O}_K$-module law only depends on the choice of the uniformizer $\pi \in \mathcal{O}_K$, not on the choice of $f \in \mathcal{F}_\pi$.

Example. If $K = \mathbb{Q}_p$, this reconstructs the multiplicative formal $\mathbb{Z}_p$ module $\widehat{\mathbb{G}}_m$ constructed above. Indeed, we have

$$\mathcal{F}_p = \{f \in \mathbb{Z}_p[\![T]\!] \mid f(T) \equiv T^p \bmod p \text{ and } f(T) \equiv pT \bmod (T)^2\},$$

implying that $f(T) = (1+T)^p - 1$ lies in $\mathcal{F}_p$. One quickly checks that

$$F_f(X,Y) = (1+X)(1+Y) - 1 = X+Y+XY \in \mathbb{Z}_p[\![X,Y]\!]$$

satisfies the conditions of (2.3), and that for $a \in \mathbb{Z}_p$, the power series

$$[a]_{f,f} = (1+T)^a - 1 \in \mathbb{Z}_p[\![T]\!]$$

satisfies the condition of (2.4).

2.3 Logarithms

We return to the more general framework where A is an integral domain with field of fractions K. We review results from section 2 and 3 of **hopkins1994equivariant**. Suppose that F is a formal A-module law over an A-algebra R.

Definition 2.3.1 (Invariant Differentials for module laws.). The module $\omega(F)$ of invariant differentials is the submodule of the module of differentials

$$\Omega_{R[\![T]\!]/R} \cong R[\![T]\!]\,\mathrm{d}T,$$

cut out by the condition that all $\omega \in \omega(F)$ satisfy

$$\omega(F(X,Y)) = \omega(X) + \omega(Y) \quad \text{and} \quad \omega([a]_F(X)) = a\omega(X). \tag{2.5}$$

for all $a \in A$.

Given a formal group law F, it is possible to explcitely calculate the R-module $\omega(F)$, which we now explain. Let $f(X,Y)$ denote $(\partial_x F)(X,Y)$, the derivative of $F(X,Y)$ with respect to X. Denote $g(Y) = f(0,Y)$. Then g is a unit in $R[\![Y]\!]$; and we construct $\omega_F(X) := \frac{1}{g(X)}\,\mathrm{d}X$. Checking that ω_F is indeed invariant is a matter of applying the chain rule.
All other invariant differentials are scalar multiples of ω_F.

Proposition 2.3.2. *1. The R-module $\omega(R)$ is free of rank 1 generated by ω_F*

2. There is a non-degenerate pairing $\omega(F) \times \mathrm{Lie}(F) \to R$.

Proof. Part one is **hopkins1994equivariant**, Proposition 2.2. □

Example. The invariant differentials for $\widehat{\mathbb{G}}_m$ are spanned by the form $\omega_1(X) = \frac{1}{1+X}\,\mathrm{d}X$.

The conditions imposed on invariant differentials remind of those imposed on morphisms of A-module laws $F \to \widehat{\mathbb{G}}_a$. And indeed, there is a map

$$\mathrm{d}_F : \mathrm{Hom}_{(A\text{-FML}/R)}(F, \widehat{\mathbb{G}}_{a,R}) \to \omega(F), \quad f \mapsto \mathrm{d}f(X) \tag{2.6}$$

One may check that $\mathrm{End}(\widehat{\mathbb{G}}_{a,R}) \supseteq R$, turning d in a map of R-modules.

Proposition 2.3.3. *1. If R is a flat A-algebra, the map d_F is injective.*

2. If R is a K-algebra, the map d_F is an isomorphism.

Proof. **hopkins1994equivariant**, Chapter 3 [Everything is easy if K has characteristic 0, as we can integrate the differential forms. The proof in positive characteristic is a bit tricky; First it is shown that there is an isomorphism of formal goups $F \cong \widehat{\mathbb{G}}_a$, which is immediate. Then that there is a unique homomorphism $f : \widehat{\mathbb{G}}_a \to \widehat{\mathbb{G}}_a$ that maps to ω_F and behaves well with respect to the A-module structure on F.] PROOF □

In particular, if R is a K-algebra, the invariant differential $\omega_F(X)$ constructed above comes from a homomorphism $f(X) = X + c_2X^2 + \dots$, which is an isomorphism by lemma 2.1.2. This allows us to define the logarithm attached to F.

Definition 2.3.4 (Logarithm and Exponential). If R is a flat A-algebra, there is a unique power series

$$\log_F(X) = X + c_2X^2 + \dots \in (R \otimes_A K)[\![X]\!]$$

inducing an isomorphism $F \otimes (R \otimes K) \to \widehat{\mathbb{G}}_{a,R\otimes K}$. This power series is called the logarithm attached to F. The inverse of $\log_H$ is called the exponential of F and will be denoted by $\exp_H$.

2.4 Deformations of Formal Modules and the Standard Formal Module

If, A is an integral domain and R is a flat A-module, the structure of a formal A-module F over R is uniquely determined by its logarithm $\log_H \in R \otimes_A K[\![T]\!]$. Indeed, we find

$$F(X,Y) = \exp_H(X+Y), \quad [a]_F(X) = \exp_H(aX).$$

It is therefore natural to wonder about conditions on power series $f \in (R \otimes_A K)[\![T]\!]$ ensuring that f is the logarithm of some formal group law. Hazewinkel found such a condition in his functional equation lemma.

Proposition 2.4.1 (Hazewinkel's Functional Equation Lemma)**.** *Let p be a prime and $q = p^e$. Given an inclusion of rings $B \subseteq L$, an ideal $\mathfrak{a} \subseteq B$ containing p, an endomorphism of rings $\sigma : L \to L$ and elements $s_1, s_2, \dots \in L$ subject to the conditions that*

$$\sigma(b) \equiv b^q \pmod{\mathfrak{a}} \textit{ for all } b \in B \quad \textit{and} \quad \sigma^r(s_i)\mathfrak{a} \subset B \textit{ for all } r, s \geq 1.$$

Suppose now that $f \in L[\![T]\!]$ has $f'(0) \in L^\times$ and satisfies the functional equation condition

$$f(X) - \sum_{i=1}^{\infty} s_i(\sigma_*^i f)(X^{q^i}) \in B[\![X]\!].$$

Then we have

$$F(X,Y) = f^{-1}(f(X) + f(Y)) \in B[\![X,Y]\!],$$

where f^{-1} is the inverse power series as in Lemma 2.1.2. Also, if $g(Z) \in L[\![Z]\!]$ is another power series satisfying the same condition

$$g(Z) - \sum_{i=1}^{\infty} s_i(\sigma_*^i f)(Z^{q^i}) \in B[\![Z]\!],$$

then $f^{-1}(g(Z)) \in B[\![Z]\!]$. Furthermore, if $\alpha(T) \in B[\![T]\!]$ and $\beta(T) \in B[\![T]\!]$, then

$$\alpha(T) \equiv \beta(T) \pmod{\mathfrak{a}^r} \iff f(\alpha(T)) \equiv f(\beta(T)) \pmod{\mathfrak{a}^r} \tag{2.7}$$

Proof. A more general statement can be found in **hazewinkel1979funceqexp**, Section 2. Proofs can be found in **hazewinkel1978formal**, Sections 2 and 10. □

Note that by construction, $F(X,Y)$ as defined above yields a (commutative) formal group law over B. Let B^σ denote the subring of elements in B fixed by σ. Then the second part of the Functional Equation Lemma implies that we even obtain formal B^σ-modules with $[b]_F(T) = f^{-1}(bf(T))$, as $bf(T)$ satisfies the same functional equation if $b \in B^\sigma$.

We use the Functional Equation Lemma to construct Lubin-Tate Formal Group Laws. Hence we now enter the situation where K is a local field with ring of integers $\mathcal{O}_K$ and uniformizer π. Look at the formal power series

$$f(T) = \sum_{i=1}^{\infty} \frac{T^{q^{in}}}{\pi^i} \in K[\![T]\!].$$

It satisfies the functional equation

$$f(T) = T + \frac{1}{\pi} f(T^{q^n}),$$

which is a functional equation of the form above, with $B = \mathcal{O}_K$, $L = K$, $s_1 = \pi^{-1}$, $s_2 = s_3 = \dots = 0$, $\sigma = \mathrm{id}_L$. Hence f arises as the logarithm of a formal $\mathcal{O}_K$-module law H over $\mathcal{O}_K$. The fact that $f^{-1}(X) = X - \frac{1}{\pi} X^{q^n} + \dots$ reveals $[\pi]_H(T) \equiv \pi T \bmod (T^2)$. Additionally, note that

$$f([\pi]_H(T)) = \pi f(T) = \pi T + f(T^{q^n}) \equiv f(T^{q^n}) \pmod{\pi}.$$

Hence, the equivalence in (2.7) implies that $[\pi]_H(T) \equiv T^{q^n} \bmod \pi$. So H is a Lubin-Tate formal $\mathcal{O}_K$-module law of height n, we'll call it the standard Lubin-Tate formal module law of height n.

2.5 Formal DVR-Modules over Fields

As above, let A be a discrete valuation ring with uniformizer π and finite residue field k; write q for the cardinality of k. Let K denote the field of fractions of A.

We introduce the concept of height, which is an integer attached to morphisms of formal group laws over fields. The height of a formal A-module $\mathcal{F}$ over R will be defined as the height of it's endomorphism $[\pi]_{\mathcal{F}}$.

We have seen in the previous section that if R is a field extension of K, then any morphism of formal group laws $f : F \to G$ over R is either 0, in which case we say it has height ∞, or an isomorphism, in which case we say it has height 0. The height becomes interesting in positive characteristic.

We define the height over field extensions of the residue field.

Definition 2.5.1 (Height of morphisms of group laws)**.** Assume that R is a field extension of k and $f : F \to G$ is a morphism of formal groups laws over R, given by a formal series $f(T) \in R[\![T]\!]$. If $f = 0$, we say that f has infinite height. If $f \neq 0$, the height of f is defined as the largest integer h such that $f = g(T^{q^h})$ for some power series $g(T) = c_1 T + c_2 T^2 + \dots \in R[\![T]\!]$ with $c_1 \neq 0$.

One readily checks that if $f : \mathcal{F} \to \mathcal{G}$ is a morphism of formal groups over a field extension R of k, the height of f does not depend on the choices of group laws on $\mathcal{F}$ and $\mathcal{G}$. This allows us to define the height function attached to f.

Definition 2.5.2 (Height function)**.** Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of formal groups over a scheme X. For a (scheme-theoretic) point $x \in |X|$, let f_x denote the base-change of f to the residue field of x. The height function attached to f is the upper-semicontinuous function

$$\mathrm{ht}(f) : |X| \to \mathbb{Z}_{\geq 0} \cup \{\infty\}, \quad x \mapsto \mathrm{ht}(f_x). \tag{2.8}$$

It is not hard to see that the height function is additive, that is, we have

$$\mathrm{ht}(f \circ g) = \mathrm{ht}(f) + \mathrm{ht}(g).$$

Let R be a local A-algebra and let k' be a separable closure of $R/\mathfrak{m}_R$. Let F be a formal A-module law over R of height h. Then we have the following result on the endomorphisms of $F \otimes_R k'$.

Lemma 2.5.3. *The A-module $\mathrm{End}_{(A\text{-FM}/k')}(F \otimes_R k')$ is isomorphic to the ring of integers of a central division algebra D over K of invariant $\frac{1}{h}$.*

Proof. **drinfel1974elliptic**, Proposition 1.7 [The proof uses techniques from deformations of formal modules. Hence perhaps it would make sense to have this in the next chapter.] □

Definition 2.5.4 (Isogeny)**.** A morphism $f : \mathcal{F} \to \mathcal{G}$ of formal groups over a field k is called an isogeny if $\mathrm{Ker}(f)$ is a represented by a finite free k-scheme. More generally, a morphism of formal A-modules over a base scheme X is an isogeny if and only if $\mathrm{Ker}(f)$ is finite and locally free over X.

Isogenies can be described using the height function.

Lemma 2.5.5. *A morphism $f : \mathcal{F} \to \mathcal{G}$ is a isogeny if and only if the height function $\mathrm{ht}(f)$ is locally constant with values in $\mathbb{Z}_{\geq 0}$.*

Definition 2.5.6 (π-divisible A-module)**.** We say that a formal A-module H over X is π-divisible if $[\pi]_H$ is an isogeny. If X is connected, the height of H is the (constant) height of the endomorphism $[\pi]_H : A \to A$.

Lemma 2.5.7. *Over sepearbly closed fields, the formal group laws are classified by their heights.*

Proof. **hazewinkel1978formal**, Theorem 19.4.1, this is originally due to Drinfeld. Note that this is only interesting in positive characteristic. □

The following lemma allows us to invert quasi-isogenies.

Lemma 2.5.8. *Let $f : \mathcal{F} \to \mathcal{G}$ be an isogeny of π-divisible formal A-modules over a quasi-compact* [quasi-separated?]*A-scheme X. Then there is an integer $n \geq 0$ and an isogeny $g : \mathcal{G} \to \mathcal{F}$ with*

$$f \circ g = [\pi^n]_{\mathcal{G}} \quad \text{and} \quad g \circ f = [\pi^n]_{\mathcal{F}}.$$

2.6 The Dieudonné functor

We explain construct a functor $\mathrm{M}^\vee : (A\text{-FM}/R) \to (R\text{-Mod})$, satisfying the following condition.

1. bla

Definition 2.6.1 (Quasi-Logarithm)**.** Let F be a formal A-module law over R. Given a power series $f \in (R \otimes_A K)[\![T]\!]$ with $f(0) = 0$, we write

$$\Delta_F(f) = f(F(X,Y)) - f(X) - f(Y) \quad \text{and} \quad \delta_a(f) = f([a]_F(T)) - af(T)$$

for $a \in A$. A Quasi-Logarithm for F is a power series $f \in (R \otimes_A K)[\![T]\!]$ with $f(0) = 0$ such that $\Delta_F(f)$ and $\delta_a(f)$ have coefficients in A. We define the R-module

$$\mathrm{QLog}(F) = \frac{\{\text{Quasi-Logarithms for } F\}}{\{f \in A[\![T]\!] \mid f(0) = 0\}}.$$

Proposition 2.6.2. *The R-module* $\mathrm{QLog}(F)$ *satisfies the properties listed above.*

Proof. □

- Definition using quasi-logarithms
- Definition with rigidified extensions as in **hopkins1994equivariant** (?)

2.7 Tate Modules and the Universal Cover

Let A be an integral domain and R be an A-algebra. Given $H \in (A\text{-FM}/R)$ and $a \in A$, we define the functor

$$\tilde{H}_a : (R\text{-Adm}) \to (A\text{-Mod}), \quad S \mapsto \left\{ (x_1, x_2, \dots) \in \prod_{\mathbb{N}} H(S) \mid [a]_H(x_{i+1}) = x_i \right\}.$$

Here, the A-module structure is given by $b.(x_1, x_2, \dots) = ([b]_H(x_1), [b]_(x_2), \dots)$. Note that if $a \in A$ is not a zero-divisor, multiplication by a on $\tilde{H}_a(S)$ is an automorphism, so that $\tilde{H}_a(S)$ is naturally an $A[\frac{1}{a}]$-module.
From now on assume that A is a discrete valuation ring with uniformizer π, finite residue field k and field of fractions K.

Definition 2.7.1 (The Universal Cover and Tate Module)**.** We omit π from notation and write $\tilde{H} = \tilde{H}_\pi$. This functor takes values in the category of K-vector spaces. Up to isomorphism, $\tilde{H}$ is uniquely determined. We call this functor the universal cover of H.
The Tate-Module $T_\pi H$ is the subfunctor of $\tilde{H}$ cut out out by the condition that $[\pi]_H(x_1) = 0$. Note that $T_\pi H$ does no longer carry the structure of a K-vector space, it is an A-module. The Rational Tate Module $V_\pi H$ is the subfunctor of $\tilde{H}$ cut out by the condition that x_1 has $[\pi]_H$-torsion. Equivalently, we have

$$V_\pi H(S) = T_\pi H(S) \otimes_A K.$$

Let R be an adic A-algebra with finitely generated ideal of definition I such $k_R = R/I$ is a separably closed field of characteristic p (in particular, $p \in I$). We write $\mathrm{Nilp}^\flat_R$ for the functor sending on admissible R-algebras sending S to the set $\lim_{s \mapsto s^q} S^{\infty}$. This functor is representable by $\mathrm{Spf}\, R[\![T^{\frac{1}{q^\infty}}]\!]$ (with the T-adic topology).

The following calculation is helpful.

Lemma 2.7.2. *Let S be a ring, let J be an ideal of S containing p, and let q be a power of p. Then for any two elements $s_1, s_2 \in S$ with $s_1 \equiv s_2 \bmod J$, we have $s_1^{q^k} \equiv s_2^{q^k} \bmod J^{k+1}$.*

Proof. It suffices to check that if $s_1 \equiv s_2 \bmod J^k$, then $s_1^q \equiv s_2^q \bmod J^{k+1}$. Write $s_1 + r = s_2$ for some $r \in J^k$. Then

$$s_2^q = (s_1 + r)^q = s_1^q + \underbrace{q s_1^{q-1} r + \binom{q}{2} s_1^{q-2} r^2 + \cdots + r^q}_{\in J^{k+1}} \in s_1^q + J^{k+1},$$

and done. □

Corollary 2.7.3. *Let S and J be as above and let $f \in S[\![T]\!]$ be a power series such that*

$$f(T) \equiv g(T^q) \pmod J$$

for some $g \in S[\![T]\!]$ with $g(0) = 0$. Then, if $s_1, s_2 \in S$ are elements for which $f(s_1)$ and $f(s_2)$ exist (for example if f is a polynomial or if S is an admissible ring with $s_1, s_2 \in S^{\infty}$) and $s_1 \equiv s_2$ mod I, we have

$$f^k(s_1) \equiv f^k(s_2) \pmod{J^{k+1}}.$$

Here f^k is k-fold concatenation of f.

Lemma 2.7.4 (Crystalline nature of $\mathrm{Nilp}^\flat$.). *Suppose that S is an admissible R-algebra admitting an ideal of definition J containing p. Then there is a bijection*

$$\mathrm{Nilp}^\flat(S) \cong \mathrm{Nilp}^\flat(S/J).$$

This bijection equips S with the structure of an (additive) abelian group.

Proof. Let $(x_1, x_2, \dots)$ be an element of the right hand side. Let $(y_1, y_2, \dots)$ is a sequence of arbitrary lifts. From this choice of lifts we obtain a sequence $(z_1, z_2, \dots) \in \mathrm{Nilp}^\flat(S)$ by setting

$$z_i = \lim_{r \to \infty} y_{i+r}^{q^r}.$$

The limit exists, because if $s \geq r$ are two non-negative integers, we calculate

$$y_{i+s}^{q^{s-r}} \equiv x_{i+s}^{q^{s-r}} = x_{i+r} \equiv y_{i+r} \pmod J,$$

implying by 2.7.2 that

$$y_{i+s}^{q^s} \equiv y_{i+r}^{q^r} \pmod{J^r}.$$

Now the limit exists by completeness of S. The sequence $(z_1, z_2, \dots)$ is readily seen to be a lift of $(x_1, x_2, \dots)$. We show that this lift is unique. Suppose that $(z_1', z_2', \dots) \in \mathrm{Nil}^\flat(S)$ is another lift. Now

$$z_i' \in \bigcap_{r\in\mathbb{N}} (z_{i+r}+J)^{q^r} = \bigcap_{r\in\mathbb{N}} (z_i + q^r z_{i+1} J + \cdots + J^{q^r}) = \{z_i\}.$$

Here we used that for $s \in \mathbb{N}$, $\liminf_{r\to\infty, 1<m<s} \nu_p\left(\binom{q^r}{m}\right) = \infty$. This is what we wanted to show. $\square$

Lemma 2.7.5. *Let H be a π-divisible formal A-module over R and write $H_0 = H \otimes_R k_R$. Now the choice of a coordinate on H_0 gives rise to a an isomorphism*

$$\tilde{H}_0 \cong \mathrm{Nilp}^\flat_{k_R}$$

of functors $(k_R\text{-Adm}) \to (\mathrm{Set})$

Proof. We can write $[\varpi]_{H_0}(X) = g(X^{q^n}) \in k_R[\![X]\!]$ for some $g(X) = c_1X + c_2X^2 + \dots$ with $c_1 \neq 0$. For each coefficient c_i, let $d_i \in k_R$ be the unique element such that $d_i^{q^n} = c_i$. Let $h \in k_R[\![X]\!]$ be the power series given by $d_1X + d_2X^2 + \dots$. Now $(h(X))^{q^n} = [\varpi]_G(X)$, and we find that

$$\tilde{G}(S) \to \lim_{x\mapsto x^{q^n}} S^\infty : \quad (x_1, x_2, x_3, \dots) \mapsto (x_1, h(x_2), h(h(x_3)), \dots)$$

is a well-defined function, and (trivially) functorial in S. For the inverse, let $h^{-1}(X) \in k_R[\![X]\!]$ be the the unique power series with $h^{-1}(h(X)) = h(h^{-1}(X)) = X$, see Lemma 2.1.2. The map

$$\lim_{x\mapsto x^{q^n}} S^\infty \to \tilde{G}(S), \quad (x_1, x_2, \dots) \mapsto (x_1, h^{-1}(x_2), h^{-1}(h^{-1}(x_3)), \dots)$$

is well-defined as

$$[\varpi]_G(h^{-1}(X)) = g((h^{-1}(X))^{q^n}) = (h(h^{-1}(X)))^{q^n} = X^{q^n}$$

and it is readily seen to be inverse to the map constructed above. Note that the functors $\mathrm{Nilp}^\flat$ and $\lim_{x\mapsto x^{q^n}} S^\infty$ are isomorphic. $\square$

We keep the assumptions on H.

Lemma 2.7.6. *Suppose that S is an admissible R-algebra admitting an ideal of definition J such that $\pi \in J$. Then the natural reduction map*

$$\tilde{H}(S) \to \tilde{H}(S/J)$$

is an isomorphism.

Proof. Given a coordinate $H \cong \mathrm{Spf}\, R[\![T]\!]$, a sequence $(x_0, x_1, \dots) \in H(S/J)$, and an arbitrary lift to a sequence $(y_0, y_1, \dots) \in \prod_{\mathbb{N}} H(S)$, we may, akin to the calculations in Lemma 2.7.4 and with allusion to Corollary 2.7.3, define a sequence $(z_0, z_1, \dots)$ in $H(S)$ by setting

$$z_i = \lim_{r\to\infty} [\pi^r]_H(y_{i+r}).$$

This shows surjectivity of the reduction map. For injectivity, as $H(S) \to H(S/J)$ is a morphism of groups, it suffices to show that the kernel is trivial. But if $(z_0, z_1, \dots)$ lies in the kernel, every element z_i lies in J and we find that $z_i \in \bigcap_{r\in\mathbb{N}} J^r = \{0\}$. This is what we had to show. $\square$

We collect the results.

Proposition 2.7.7. *Let A be a local ring with finite residue field and uniformizer π, and let R be a noetherian local A-algebra with maximal ideal I containing π and p , such that R is complete with respect to the I-adic topology and $k_R = R/I$ is a separably closed field. Let H be a π-divisible formal A-module over R and fix a coordinate $H_0 = H \otimes_R k_R \cong k_R[\![T]\!]$. Let S be an admissible R-algebra with ideal of definition J such that $\phi(I) \subseteq J$. Then there are canonical isomorphisms (of sets)*

this might be redundant

$$\tilde{H}(S) \cong \tilde{H}(S/J) = \tilde{H}_0(S/J) \cong \mathrm{Nilp}^\flat(S/J) \cong \mathrm{Nilp}^\flat(S).$$

In particular, $\tilde{H}(S)$ is, as a functor to (Set), *representable by* $\mathrm{Spf}(R[\![T^{q^{-\infty}}]\!])$.

2.8 The Quasilogarithm Map

- See **BoyarchenkoWeinstein2011MaxVar**, section 2.6.

2.9 Determinants of Formal Modules

- "Functorial" description of the determinant. Either as in **BoyarchenkoWeinstein2011MaxVar**, or as in **weinstein2016semistable**.
- Construction.
- Approximations.

3 Local Class Field Theory following Lubin-Tate

Let K be a local field with residue field k, put $q = \#k$, and denote by $\nu_K : K \to \mathbb{Z} \cup \{\infty\}$ the valuation of K, normalized such that $\nu_K(\pi) = 1$ for a uniformizer π of K. The aim of this subsection is to describe the maximal abelian extension of a local field K.

The Local Kronecker-Weber theorem gives an explicit description of the abelianization of the absolute Galois group of K only in terms of K:

Theorem 3.0.1 (Local Kronecker-Weber)**.** *There is an isomorphism (canonical up to choice of a uniformizer $\pi \in K$)*

$$\mathrm{Gal}(\overline{K}/K)^{\mathrm{ab}} \cong \mathrm{Gal}(K^{\mathrm{ab}}/K) \cong \mathcal{O}_K^\times \times \widehat{\mathbb{Z}}.$$

Here, K^{ab} denote the maximal abelian extension of K, which can (after choosing an algebraic closure of K) be described as $\overline{K}^{[G_K,G_K]}$.

The extension K^{ab} consists of two parts, we have $K^{\mathrm{ab}} = K^{\mathrm{rm}} \cdot K^{\mathrm{nr}}$. The field K^{nr}, the maximal unramified extension of K, has relatively simple structure. Describing the field K^{rm} (or rather, it's completion) is the hard part and it is here where we apply the theory of formal modules.

The valuation ν_K extends uniquely to $\overline{K}$, yielding a π-adic norm on $\overline{K}$. Let C denote the completion with respect to this norm. An application of Krasner's Lemma implies that $\mathrm{Gal}(C/K) \cong \mathrm{Gal}(\overline{K}/K) =: G_K$. One readily checks that any $\sigma \in G_K$ yields a continuous automorphism $\mathcal{O}_C \to \mathcal{O}_C$, and we obtain a short exact sequence

$$0 \to I_K \to G_K \to \mathrm{Gal}(\overline{k}/k) \to 0.$$

The subgroup $I_K \subset G_K$ is called the inertia subgroup of K, and we write $\breve{K}$ for the subfield of C fixed by I_K. In particular we have $\mathrm{Gal}(\breve{K}/K) \cong \mathrm{Gal}(\overline{k}/k)$. One readily confirms that $\breve{K}$ is complete with respect to the norm induced by K.

As the Galois group of any finite extension of k is cyclic, we find that $\mathrm{Gal}(\breve{K}/K)$ is abelian. In fact, it is isomorphic to $\widehat{\mathbb{Z}} = \lim_n(\mathbb{Z}/n\mathbb{Z})$. Hence K_∞ decomposes as $\breve{K} \cdot K_\pi$ for some abelian, complete extension K_π/K such that $K_\pi \cap \breve{K} = K$. Now K_π is the completion of K^{rm}. Observe that

$$\mathrm{Gal}(K_\infty/K) \cong \mathrm{Gal}(K_\pi/K) \times \mathrm{Gal}(\breve{K}/K) \cong \mathrm{Gal}(K_\pi/K) \times \widehat{\mathbb{Z}},$$

so Theorem 3.0.1, the local Kronecker-Weber Theorem, is equivalent to showing that the Galois group of K_π over K is isomorphic to $\mathcal{O}_K^\times$.

4 Non-Abelian Lubin-Tate Theory: An Overview

In the preceeding chapter we used formal $\mathcal{O}_K$-modules to understand the maximial abelian extension of a local field K. The hope of non-Abelian Lubin-Tate theory is to gain insight about the Abelian extensions of K by considering certain moduli spaces of formal $\mathcal{O}_K$-modules. More precisely, attached to a formal $\mathcal{O}_K$-module H_0 over $\overline{\mathbb{F}}_q$ (determined up to isomorphism by its height n), we attach a system of rigid spaces $\{M_K\}_{K \subset \mathrm{GL}_n(\mathcal{O}_K)}$, the so called Lubin-Tate Tower. For $l \neq p$, the system of l-adic compactly supported cohomology groups $\{H_c^i(M_K, \overline{\mathbb{Q}}_l)\}_K$ admits commuting actions by $\mathrm{GL}_n(K)$, W_K and $D^\times$, where the latter denotes the units of the central divison algebra $D = \mathrm{End}_{(\mathcal{O}_K\text{-FM}/\overline{\mathbb{F}}_q)}(H_0) \otimes \mathbb{Q}$. This yields a correspondence of representations of the respective groups, and Harris and Taylor showed in **HTShimura** that the cohomology of middle degree induces (a version of) the Local Langlands Correspondence for GL_n. Our goal is an explicit description of this correspondence, and we obtain such descriptions by understanding the Lubin-Tate tower explicitely. As it turns out, the the limit $\lim_K M_K$ is representable by a perfectoid space which is is easier to describe than its individual layers.

4.1 The Lubin-Tate Tower

4.1.1 Deformations of Formal Modules

We mostly follow **Strauch2008DefSp**, Chapter 2 for notation. Let $\mathcal{C}$ denote the category of local, Noetherian $\mathcal{O}_{\breve{K}}$-modules with distinguished isomorphisms $R/\mathfrak{m}_R \to \overline{\mathbb{F}}_q$. Let H_0 be a formal $\mathcal{O}_K$-module over $\overline{\mathbb{F}}_q$.

Definition 4.1.1 (Deformation)**.** Let $R \in \mathcal{C}$. A deformation of H_0 to R is a pair (H, ι) where H is a formal $\mathcal{O}_K$-module over R and ι is a quasi-isogeny

$$\iota : H_0 \dashrightarrow H \otimes_R \overline{\mathbb{F}}_q.$$

Two deformations (H, ι) and (H', ι') are isomorphic if there is an isomorphism $\tau : H \to H'$ with $\iota' \circ \tau = \iota$.

The Lubin-Tate space without level structure is the moduli space of such deformations. More precisely, we define it as the functor

$$\mathcal{M}_0 : \mathcal{C} \to (\mathrm{Set}), \quad R \mapsto \{\text{deformations } (H, \iota) \text{ of } H_0\}/\mathrm{iso}.$$

Theorem 4.1.2 (Representability of $\mathcal{M}_0$)**.** *The functor $\mathcal{M}_0$ is (non-canonically) representable, by the noetherian local ring*

$$A_0 \cong \mathcal{O}_{\breve{K}}[\![u_1, \ldots, u_{n-1}]\!].$$

In particular, there is a universal deformation $(F^{\mathrm{univ}}, \iota^{\mathrm{univ}})$, with $F^{\mathrm{univ}} \in (\mathcal{O}_{\breve{K}}\text{-FM}/A_0)$.

4.1.2 Deformations of Formal Modules with Drinfeld Level Structure

Definition 4.1.3 (Drinfeld level $\mathfrak{p}^m$-structure)**.** Let $R \in \mathcal{C}$ and $H \in (\mathcal{O}_K\text{-FM}/R)$. A Drinfeld level $\mathfrak{p}^m$-structure on H is a morphism of R-group schemes

$$(\mathfrak{p}^{-m}/\mathcal{O}_K)^{\oplus n} \to H(R)[\pi^m]$$

such that after choosing a coordinate $H \cong \operatorname{Spf} R[\![T]\!]$, the power series $[\pi]_H(T) \in R[\![T]\!]$ satisfies the divisibility constraint

$$\prod_{x \in (\mathfrak{p}^{-1}/\mathcal{O}_K)} (T - \phi(x)) \mid [\pi]_H(T).$$

The following examples might shed some light on this definition.

Example.

- $\widehat{\mathbb{G}}_m$
- Things over $\mathbb{F}_q$.
- Drinfeld Level

- Moduli Problem + Representability
- The Lubin-Tate Tower

4.1.3 The Group actions on the Tower and its Cohomology

- Action By $D^\times$ and GL_n
- Action by W_K via Weil descent Datum.

4.2 The Local Langlands Correspondence for the General Linear Group

4.3 The Lubin-Tate Perfectoid Space

5 Mieda's Approach to the Explicit Local Langlands Correspondence

6 The Explicit Local Langlands Correspondence for Depth Zero Supercuspidal Representations

6.1 The Special Affinoid

6.2 Deligne-Lusztig Theory for Depth Zero Representations

6.3 Proof

A Topological Rings

To deal with the topological rings showing up, the notion of admissible rings will be convenient (taken from **stacks-project**, Tag 07E8).

Definition A.0.1. Let A be a topological ring. We say that A is admissible if

- The element $0 \in A$ has a fundamental system of neighbourhoods consisting of ideals.
- There exists an ideal of definition, that is, an ideal $I \subset A$ such that every open neighbourhood of 0 contains I^n for some n.
- It is complete, that is, the natural map

  $$A \to \lim_{J \subset A \text{ open ideal}} A/J$$

  is an isomorphism.

We say that A is adic if it admits an open ideal of definition. Given a topological ring A, we denote the category of admissible and adic A-algebras (algebras S with continuous morphism $A \to S$) by $(A\text{-Adm})$ and $(A\text{-Adic})$, respectively.

[The following results might be not interesting enough to make it into the final draft]

Lemma A.0.2. *Let $\phi : R \to S$ be a morphism of admissible rings, and let $I \subset R$ be an admissible ideal. Then the ideal $J = \phi(I) \cdot S$ is an ideal of definition in S.*

Proof. Let U be an open ideal of S. By continuity of ϕ, it's preimage $U' = \phi^{-1}(U)$ is open in R. Hence there is some n with $I^n \subset U'$. But now

$$\phi(I)^n = \phi(I^n) \subseteq \phi(\phi^{-1}(U)) \subseteq U$$

and the claim follows. □

Lemma A.0.3. *Let S be an admissible ring, and let $(s_1, s_2, \dots)$ be a sequence with elements in S. Then $\sum_{i=1}^{\infty} s_i$ converges if and only if $\prod_{i=1}^{\infty}(1+s_i)$ converges if and only if $\lim_{i\to\infty} s_i = 0$.*

Proof. If sum and product converge, $(s_i)_{i\in\mathbb{N}}$ has to converge to zero. The reverse implication follows after writing $S \cong \lim_J S/J$ for a system of open ideals $J \subset S$. □

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