Explicit Aspects of Non-Abelian Lubin-Tate Theory

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Introduction

When JONATHAN LUBIN and JOHN T. TATE published their paper *Formal Complex Multiplication in Local Fields* in 1965 (cf. [LT65]), they achieved one of the last major breakthroughs in Class Field Theory. Perhaps inpsired by the theory of complex multiplication of elliptic curves, they show that formal group laws, which can be seen as local analogues of algebraic groups, can be used to construct abelian extensions of arbitrary non-Archimedian local fields E. This approach, commonly known as Lubin–Tate theory, provides a strikingly simple solution to a complicated problem: that of finding generators for the maximal abelian extension of E and describing how the Galois group acts on them.

Two years later, in 1967, ROBERT LANGLANDS wrote his famous letter to ANDRÉ WEIL, introducing the web of conjectures that is now known as the Langlands Program. The "local part" of these conjectures predicts, in vague terms, a correspondence between smooth representations of $\mathrm{GL}_n(E)$ and representations of the absolute Galois group of E, thus conjecturing an analogue of local class field theory for non-Abelian extensions. Inspired by results of PIERRE DELIGNE on the cohomology of certain modular curves with bad reduction at p (where p denotes the residue characteristic of E), HENRI CARAYOL predicted in his 1990 paper *Non-Abelian Lubin–Tate Theory* that this (at the time conjectural) correspondence appears, along with the Jacquet–Langlands Correspondence, in the ℓ-adic cohomology of the Lubin–Tate tower (cf. [Car90]), which is a certain projective system of rigid analytic spaces related to deformation spaces of formal $\mathcal{O}_E$-modules. The cohomology group in question, denoted H_{LT}, carries the structure of a representation of $\mathrm{GL}_n(E) \times D^\times \times \mathrm{W}_E$, where D is a division algebra over E with invariant $1/n$ and W_E is the Weil group attached to E. Carayol's conjecture predicts that for supercuspidal representations π of $\mathrm{GL}_n(E)$, the π-isotypic component of H_{LT} decomposes (up to duals and twists) as

$$H_{\mathrm{LT},\pi} = \pi \boxtimes \mathrm{JL}(\pi) \boxtimes \mathrm{rec}_E(\pi),$$

where $\mathrm{JL}(\pi)$ and $\mathrm{rec}_E(\pi)$ denote the representations of $D^\times$ and W_E corresponding to π under the Jacquet–Langlands and local Langlands correspondences. The study of the representation H_{LT} is colloquially regarded to as Non-Abelian Lubin–Tate Theory.

Roughly a decade later, this approach was used by MICHAEL HARRIS and RICHARD TAYLOR to define the local Langlands correspondence for GL_n over p-adic fields, verifying (a part of) Carayol's conjectures along the way (cf. [HT01]). Around the same time, PIERRE BOYER proved Carayol's conjectures for local function fields (cf. [Boy99]).

Although these results are milestone achievements, they leave a few questions unanswered, two of which we aim to address in the present text. First, the results of Harris–Taylor offer no insight into the question of providing an explicit description of the correspondence. Secondly, and this point is of rather conceptual nature, their proof makes essential use of global arguments, ultimately requiring to realize E as the completion of some global field at one of its places.

In order to have any hope of attacking the latter issue, it appears essential to gain a deeper understanding of the geometry of the Lubin–Tate tower. A step in this direction was taken by TERUYOSHI YOSHIDA in his 2004 paper *On non-abelian Lubin–Tate theory via vanishing*

cycles (cf. [Yos10]). Yoshida uses a series of blow-ups to construct a semi-stable model of the generic fiber of the first layer of the Lubin–Tate space, whose reduction contains a Deligne–Lusztig variety for $\mathrm{GL}_n(\mathbb{F}_q)$ (where $\mathbb{F}_q$ is the residue field of E) as an affine open subscheme. This allowed him to prove, using geometric methods, that a part of non-Abelian Lubin–Tate theory boils down to Deligne–Lusztig theory. Building on this work, JARED WEINSTEIN initiated a program to obtain similar results in 2010.

Around the same time, PETER SCHOLZE developed the theory of perfectoid spaces, providing a framework that allows one to analyze the geometry of the projective limit of the spaces in the Lubin–Tate tower. Although no longer a locally finite-type analytic space, the resulting Lubin–Tate perfectoid space M_∞ turns out to be easier to work with than the individual layers of the Lubin–Tate tower. For example, it admits a coordinate that is not available at finite level. Recent results along the lines of Weinstein's program (cf., for example, [Wei16], [BW16], [IT20] and [Tok20]) make use of this development to construct certain formal models of rational subsets of the Lubin–Tate perfectoid space M_∞ and observe parts of the local Langlands correspondence inside the ℓ-adic cohomology of their reduction. This approach relies on previously known explicit descriptions of the local Langlands correspondence.

Work of YOICHI MIEDA (cf. [Mie16]) reverses this approach. Given a formal model $\mathcal{X}$ of a rational subset $U \subset M_\infty$, his results show that parts of the cohomology of the special fiber of $\mathcal{X}$ appear inside H_{LT}. Using the results of non-Abelian Lubin–Tate theory, this allows, together with the zoo of formal models constructed in the works referenced above, for an explicit description of certain parts of the local Langlands correspondence.

The aim of this thesis is to provide a (mostly) self-contained proof of the following results, which appear in [Mie16, Section 5].

Theorem (Main Results of this Thesis)**.**

1. *There exists an affinoid $U \subset M_\infty$ inside the Lubin–Tate perfectoid space that admits a formal model $\mathcal{X}$ whose reduction is isomorphic to the perfection of a Deligne–Lusztig variety for $\mathrm{GL}_n(\mathbb{F}_q)$. Furthermore, this isomorphism is compatible for certain group actions on both sides.*

2. *For certain (so-called depth zero supercuspidal) representations π of $\mathrm{GL}_n(E)$, one can calculate $\mathrm{rec}_E(\pi)$ and $\mathrm{JL}(\pi)$ explicitly, using only the geometry of the Lubin–Tate tower and the fact that these representations appear inside its ℓ-adic cohomology.*

The first part is an infinite-level analogue of the construction by Yoshida. With this result at hand, the second part is an application of Mieda's ideas. For precise statements, see Proposition 7.1.5, Proposition 7.2.2 and Theorem 7.3.3 below.

The structure of the thesis is as follows.

Section 1 reviews the theory of formal $\mathcal{O}_E$-modules, which will play a significant role in the analysis of the Lubin–Tate tower. It encompasses all the results that we need in later parts of the text.

In **Sections 2 and 3**, we review classical Lubin–Tate theory as in [LT65], and the theorems of non-Abelian Lubin–Tate theory due to Boyer and Harris–Taylor. In particular, we introduce the Lubin–Tate tower $\{\mathcal{M}_m\}_{m\in\mathbb{N}}$ and describe the relevant group actions.

Section 4 introduces the Lubin–Tate space at infinite level. We follow the ideas of Weinstein to define a certain coordinate on $\mathcal{M}_\infty = \lim_{m \in \mathbb{N}} \mathcal{M}_m$. The main result of this section is a description of the group actions on $\mathcal{M}_\infty$ in terms of this coordinate.

Section 5 reviews the technical results of Mieda's paper [Mie16]. We do not go into details, but hope to provide some insight into his methods.

Section 6 deals with Deligne–Lusztig theory for $\mathrm{GL}_n(\mathbb{F}_q)$. We develop the theory exclusively for GL_n, describing in detail the geometry of the moduli of flags and the related spaces that occur as special cases of the varieties considered by PIERRE DELIGNE and GEORGE LUSZTIG in [DL76]. The representation-theoretic results of their paper are stated as facts and applied to our situation.

Finally, in **Section 7**, we collect the results of the previous sections to prove the results announced above.

Notation and Conventions

Fields. Throughout the text, the letter E usually denotes a non-Archimedian local field, for which we make a choice of uniformizer $\varpi \in E$. We then denote by $\mathbb{F}_q$ the residue field of E, whose characteristic we denote by $p \in \mathbb{N}$, and we further fix an algebraic closure $\overline{E}$ of E. The ϖ-adic absolute value $|-|_\varpi$ on E (normalized with $|\varpi|_\varpi = 1/q$) extends uniquely to $\overline{E}$, and we denote the completion of $\overline{E}$ with respect to $|-|_\varpi$ by $\mathbb{C}_\varpi$. If $E \subseteq E' \subseteq \mathbb{C}_\varpi$ is any field extension, we denote by $\mathcal{O}_{E'} = \{x \in E' : |x|_\varpi \leq 1\}$ its ring of integers and by $\mathfrak{m}_{E'} = \{x \in E' : |x|_\varpi < 1\}$ its maximal ideal. Inside $\mathbb{C}_\varpi$, we have for $m \in \mathbb{N}$ the degree m unramified extension E_m of E with residue field $\mathbb{F}_{q^m}$, and the maximal unramified extension E^{nr} of E, whose residue field we denote by $\overline{\mathbb{F}}_q$ and whose closure inside $\mathbb{C}_\varpi$ we denote by $\breve{E}$. We have the maximal abelian extension E^{ab} of E with closure $\widehat{E}^{\mathrm{ab}} \subset \mathbb{C}_\varpi$. In Section 2 we introduce the field extensions $E_{\varpi,m}$ and E_m^{LT} of E.

Categories. We denote the category of sets by (Set) and the category of (unital, commutative) rings by (Ring). If A is a ring, we write (A-Alg) for the category of A-algebras and (A-Mod) for the category of A-modules. In Definition A.1 we introduce the categories (A-Adm) and (A-Adic) of admissible and adic A-algebras, and in Section 1.1 we introduce for A-algebras R the categories (A-FML/R) of formal A-modules over R (with various alterations). We follow this style of notation for almost all categories that occur, and their meaning should be clear from the context, with one exception: inspired by [Dri74], we define $\mathcal{C}$ as the category of local, Noetherian, complete $\mathcal{O}_{\breve{E}}$-algebras with residue field $\overline{\mathbb{F}}_q$, and continuous homomorphisms.

Cohomology. Throughout the text, we also fix a prime number $\ell \neq p$. Assume that X is either

- a scheme; separated and of finite type over $\mathrm{Spec}(\overline{\mathbb{F}}_q)$, or

- an analytic adic space; locally of finite type, separated and taut over $\mathrm{Spa}(\mathbb{C}_\varpi, \mathcal{O}_{\mathbb{C}_\varpi})$.

Then we denote the étale site of X (cf. [Fu11] for schemes and [Hub96] for analytic adic spaces) by $X_{\text{ét}}$. For an ℓ-torsion ring $\Lambda = \mathbb{Z}/\ell^m\mathbb{Z}$ (for some integer $m > 0$) we denote by $\underline{\Lambda}_X$ the associated constant sheaf on $X_{\text{ét}}$. This gives rise to the Grothendieck Abelian

category $\mathrm{Sh}(X_{\text{ét}}, \underline{\Lambda}_X)$ of $\underline{\Lambda}_X$-module objects in $\mathrm{Sh}(X_{\text{ét}})$, whose derived category we denote by $D(X_{\text{ét}}, \underline{\Lambda}_X)$. Denoting the structure morphism of X by f, we write $R\Gamma = Rf_*$ and $R\Gamma_{\text{c}} = Rf_!$ (interpreted as functors $D^+(X_{\text{ét}}, \underline{\Lambda}_X) \to D^+(\Lambda\text{-Mod})$). For any $\mathcal{F} \in D^+(X_{\text{ét}}, \underline{\Lambda}_X)$, we denote by $R^i\Gamma\mathcal{F} = H^i(X, \mathcal{F}) \in (\Lambda\text{-Mod})$ (resp. $R^i\Gamma_{\text{c}}\mathcal{F} = H^i_{\text{c}}(X, \mathcal{F}) \in (\Lambda\text{-Mod})$) its i-th cohomology group (resp, its i-th cohomology with compact support). We also need the ℓ-adic cohomology groups $H^i(X, \overline{\mathbb{Q}}_\ell)$ and $H^i_{\text{c}}(X, \overline{\mathbb{Q}}_\ell)$. The classical ℓ-adic formalism (cf. [Fu11], [Jan88] for schemes and [Hub98] for analytic adic spaces) defines these groups via a limiting procedure in the étale topos. Recent developments by Scholze and Bhatt–Scholze (cf. [Sch22] and [BS15]) allow a definition as honest sheaf cohomology groups over the pro-étale site. We will usually fix an isomorphism $\overline{\mathbb{Q}}_\ell \to \mathbb{C}$. We occasionally need half Tate twists, whose definition requires the choice of a square root of p. We simply pick the one corresponding to $\sqrt{p} > 0$ under the identification above.

Formal Composition. For a ring A and a formal power series $f(T) = c_1T + c_2T^2 + \cdots \in A[[T]]$, we write $f^k(T)$ for the k-fold self composite of f, that is

$$f^k(T) = \underbrace{f(f(\cdots(f(T))\cdots))}_{k-\text{fold}} \quad \text{not to be confused with} \quad f(T)^k = \underbrace{f(T)f(T)\cdots f(T)}_{k-\text{fold}}.$$

We call $f^k(T)$ the k-fold self-composite of f. If $f^{-1}(T) \in A[[T]]$ is a power series such that $f(f^{-1}(T)) = f^{-1}(f(T)) = T$, we call $f^{-1}(T)$ a formal inverse to f.

Other.

- If R is a local ring, we write $\mathfrak{m}_R$ for its maximal ideal and $k_R = R/\mathfrak{m}_R$ for its residue field.

- If S is a topological ring, we write S° for the set of power-bounded elements and $S^{\circ\circ}$ for the set of topologically nilpotent elements.

- If $\mathcal{F}$ is a set (or a presheaf), and $\sigma \colon \mathcal{F} \to \mathcal{F}$ is an automorphism, we denote by $\mathcal{F}^\sigma$ the set (or presheaf) of σ-invariants.

- Starting in Section 2, we fix a positive integer $n \in \mathbb{N}$, along with a division algebra D over E of invariant $1/n$.

Acknowledgements

1 Formal Modules

This section will serve as an introduction to formal groups and formal modules. Formal groups (or, rather, formal group laws) were first introduced by SALOMON BOCHNER in 1946 as a natural means of studying Lie Groups, cf. [Boc46]. The study of formal groups later became interesting in its own right, with the pioneering work of MICHEL LAZARD. One of Lazard's most notable achievements is the observation that the universal (one-dimensional, commutative) formal group law lives over a ring with miraculously simple structure, cf. [Laz55]. Lazard's work inspired Lubin to study formal groups, which ultimately led him, in joint work with Tate, to develop Lubin–Tate theory[1] We are mostly interested in formal $\mathcal{O}_E$-modules, which are (roughly) formal groups whose endomorphism rings contain $\mathcal{O}_E$.

1.1 Fundamentals

For now, let R be any ring, and let n and m be arbitrary integers.

Definition 1.1.1 (Formal Group Laws of arbitrary dimension). A (commutative) *formal group law* of dimension n over R is a tuple of power series $F = (F_1, \ldots, F_n)$ with

$$F_i(X_1, \ldots, X_n, Y_1, \ldots, Y_n) \in R[\![X_1, \ldots, X_n, Y_1, \ldots, Y_n]\!], \quad 1 \leq i \leq n$$

such that the following identities hold.

1. $F(F(\boldsymbol{X}, \boldsymbol{Y}), \boldsymbol{Z}) = F(\boldsymbol{X}, F(\boldsymbol{Y}, \boldsymbol{Z}))$

2. $F(\boldsymbol{X}, \boldsymbol{0}) = \boldsymbol{X}$

3. $F(\boldsymbol{X}, \boldsymbol{Y}) = F(\boldsymbol{Y}, \boldsymbol{X})$

Here, and in the sequel, we abbreviate $\boldsymbol{X} = (X_1, \ldots, X_n)$, etc. From points two and three, it follows that $F_i(\boldsymbol{X}, \boldsymbol{Y}) \equiv X_i + Y_i$ modulo terms of degree ≥ 2. Given a formal group law F of dimension n and a formal group law G of dimension m, a morphism $F \to G$ is an m-tuple $f = (f_1, \ldots, f_m)$ of power series $f_i \in R[\![X_1, \ldots, X_n]\!]$ such that $f(0) = 0$ and

$$f(F(\boldsymbol{X}, \boldsymbol{Y})) = G(f(\boldsymbol{X}), f(\boldsymbol{Y})).$$

Composition of morphisms is given by composition of tuples of power series. Consequently, the identity id_F is the tuple of power series with components $\mathrm{id}_{F,i}(\boldsymbol{X}) = X_i$. This yields the category of formal group laws of arbitrary dimension over R, which we denote by $(\mathrm{FGL}^{\mathrm{arb}}/R)$. We will mostly be concerned with the full subcategory of one-dimensional formal groups, which we denote by (FGL/R).

Lemma 1.1.2. *Attached to any $F \in (\mathrm{FGL}^{\mathrm{arb}}/R)$ of dimension n, there exists a unique power series*

$$[-1]_F(X_1, \ldots, X_n) \in R[\![X_1, \ldots, X_n]\!]$$

satisfying $F(\boldsymbol{X}, [-1]_F(\boldsymbol{X})) = 0$. This is automatically an endomorphism of F.

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[1]Lubin wrote about his experience of the development of the theory in an online forum, cf. [Lub15].

Proof. This statement is an application of the formal implicit function theorem, cf. [Haz78, Theorem A.4.7]. □

Example. Let us introduce the following two formal group laws.

- *The additive formal group law.* Over any ring R, we write $\widehat{\mathbb{G}}_a$ for the formal group law with addition given by $\widehat{\mathbb{G}}_a(X, Y) = X + Y$.

- *The multiplicative formal group law.* We write $\widehat{\mathbb{G}}_m$ for the formal group law associated with the with $\widehat{\mathbb{G}}_m(X, Y) = X + Y + XY$. Note that $\widehat{\mathbb{G}}_m(X, Y) = (X + 1)(Y + 1) - 1$.

Formal group laws may be thought of as infinitesimal Lie groups; thus,it should be of no surprise that to any $F \in (\text{FGL}^{\text{arb}}/R)$, we may associate a Lie algebra.

Definition 1.1.3 (Lie-algebra of formal group law). Let $\text{Lie} \colon (\text{FGL}^{\text{arb}}/R) \to (\text{Ab})$ be the functor taking an n-dimensional formal group law F to the R-module

$$\text{Lie}(F) = \text{Hom}_{(R\text{-Mod})}\left(\frac{(X_1, \ldots, X_n)}{(X_1, \ldots, X_n)^2}, R \right)$$

Given an m-dimensional group law G and a morphism $f \colon F \to G$, $\text{Lie}(f)$ is the induced morphism

$$\text{Lie}(F) \to \text{Lie}(G), \quad \psi \mapsto \left(S_j \mapsto \psi(\overline{f_j}) \right) \in \text{Hom}_{(R\text{-Mod})}\left(\frac{(X_1, \ldots, X_n)}{(X_1, \ldots, X_n)^2}, R \right),$$

where $\overline{f_j}$ is the reduction of f_j mod $(\boldsymbol{X})^2$.

We have a canonical basis on both sides, and writing $\text{Lie}(F) = R^n$, $\text{Lie}(G) \cong R^m$, the induced map $\text{Lie}(f) \colon R^n \to R^m$ is given by multiplication with the matrix

$$\left(\frac{\partial f_i}{\partial X_j}(0) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}.$$

Now let A be any other ring (in all our applications, A be the ring of integers of a local field E). Our next aim is to introduce formal A-module laws. Naively, we would like to define these objects as formal group laws F with A-module structure, i.e. a morphism of rings $[\cdot]_F \colon A \to \text{End}_{(\text{FGL}^{\text{arb}}/R)}(F)$. However, there is a subtlety: given a one-dimensional group law $F \in (\text{FGL}/R)$, the fact that $F(X, Y) \equiv X + Y$ modulo degree ≥ 2 enforces that the induced map $\text{End}(F) \xrightarrow{\text{Lie}} \text{End}(R)$ is a morphism of rings. If we are given $[-]_F \colon A \to \text{End}_{(\text{FGL}/R)}(F)$, this A-module structure on F yields an A-module structure on R, given by the composition

$$A \xrightarrow{[\cdot]_F} \text{End}(F) \xrightarrow{\text{Lie}} \text{End}(R), \quad a \mapsto \text{Lie}([a]_F).$$

This is a morphism of rings, and we obtain an A-algebra structure on R. This motivates the following definition.

Definition 1.1.4 (Formal A-Module Law of arbitrary dimension). Let R be an A-algebra with structure morphism $j\colon A \to R$. A *formal A-module law* over R of dimension n is given by the data of a formal n-dimensional group law F over R and a morphism of rings

$$A \to \mathrm{End}_{(\mathrm{FGL}^{\mathrm{arb}}/R)}(F), \quad a \mapsto ([a]_{F,i})_{1 \leq i \leq n} \in (R[\![X_1, \ldots, X_n]\!])^n$$

such that $[a]_{F,i}(\boldsymbol{X}) \equiv j(a)X_i$ modulo terms of degree ≥ 2. Morphisms between formal A-modules of arbitrary dimension are morphisms of formal groups respecting the A-module structure. The resulting category is denoted $(A\text{-}\mathrm{FML}^{\mathrm{arb}}/R)$. The full subcategory of one-dimensional formal A-module laws over R is denoted by $(A\text{-}\mathrm{FML}/R)$.

Note that $(\mathrm{FGL}/R) \cong (\mathbb{Z}\text{-}\mathrm{FML}/R)$. At the slight cost of precision, we usually do not explicitly mention the A-structure when referring to formal module laws, simply writing $F \in (A\text{-}\mathrm{FML}/R)$ for example. The following facts are readily verified.

Lemma 1.1.5. *Let F and G be formal A-module laws of arbitrary dimension.*

1. *The set $\mathrm{Hom}_{(A\text{-}\mathrm{FML}^{\mathrm{arb}}/R)}(F, G)$ is an A-module with A-module structure given by $f + ag = G(f, [a]_G(g))$ for $a \in A$. Furthermore, the composition of morphisms is bilinear; hence $(A\text{-}\mathrm{FML}^{\mathrm{arb}}/R)$ is an A-linear category (cf. [Stacks, Tag 09MI]).*

2. *Furthermore, $(A\text{-}\mathrm{FML}^{\mathrm{arb}}/R)$ admits finite products. Thereby it is an additive category (cf. [Stacks, Tag 0104]). The unique final and initial object of $(\mathrm{FGL}^{\mathrm{arb}}/R)$ is the unique 0-dimensional formal A-module law.*

3. *In particular, $\mathrm{End}_{(A\text{-}\mathrm{FML}^{\mathrm{arb}}/R)}(F)$ is a (possibly non-commutative) A-algebra.*

Given an additional A-algebra R', a morphsim $i\colon R \to R'$ yields an A-linear functor

$$(A\text{-}\mathrm{FML}^{\mathrm{arb}}/R) \to (A\text{-}\mathrm{FML}^{\mathrm{arb}}/R'), \quad F \mapsto F \otimes_R R',$$

where $F \otimes_R R'$ denotes the formal A-module law obtained by applying i to the coefficients of the formal power series representing the A-module structure of F. This turns the assignment $R \mapsto (A\text{-}\mathrm{FML}^{\mathrm{arb}}/R)$ into a functor $(A\text{-}\mathrm{Alg}) \to (\mathrm{Set})$. The corresponding presheaf on (AffSch/A) is separated for the Zariski topology, but it fails to be a sheaf in general: the Lie algebra of any formal group law over R has to be free, but this condition cannot be ensured Zariski-locally on $\mathrm{Spec}(R)$. It may therefore be advisable to consider the sheafification $R \mapsto (A\text{-}\mathrm{FML}^{\mathrm{arb}}/R)^+$ instead, but in our applications R will almost always be a local A-algebra, so considerations of this type will not play an important role.

Every n-dimensional formal module law $F \in (A\text{-}\mathrm{FML}^{\mathrm{arb}}/R)$ yields a functor

$$(R\text{-}\mathrm{Alg}) \to (A\text{-}\mathrm{Mod}), \quad S \mapsto \mathrm{Nil}(S)^n, \tag{1.1}$$

where $\mathrm{Nil}(S)^n$, the set of n-tuples of nilpotent elements of S, is equipped with addition and scalars given by

$$s_1 + s_2 = F(s_1, s_2) \in \mathrm{Nil}(S)^n, \quad as = [a]_F(s) \in \mathrm{Nil}(S)^n.$$

This construction yields a functor

$$(A\text{-FML}/R) \to \mathrm{Fun}((R\text{-Alg}), (A\text{-Mod})), \tag{1.2}$$

where Fun denotes the functor category.

Passing from discrete R-algebras to admissible R-algebras (cf. Definition A.1), this construction extends naturally to a functor

$$\mathrm{FG}\colon (A\text{-FML}/R) \to \mathrm{Fun}((R\text{-Adm}), (A\text{-Mod})), \quad F \mapsto \mathrm{Spf}(R[\![\boldsymbol{T}]\!]),$$

where we equip $\mathrm{Spf}(R)[\![\boldsymbol{T}]\!]$ with the structure of an A-module object using the endomorphisms coming from F.

Following this line of thought naturally leads to the definition of formal modules.

Definition 1.1.6 (Formal Groups and Formal Modules.)**.** A functor $\mathcal{F}\colon (R\text{-Adm}) \to (A\text{-Mod})$ is called a *formal A-module* if it is isomorphic to a functor of the form $\mathrm{FG}(F)$ for any formal A-module law F over R. Given two formal A-modules $\mathcal{F}$ and $\mathcal{G}$ over R, a morphism $\mathcal{F} \to \mathcal{G}$ is a natural transformation that respects the A-module structure on components.

The functor

$$\mathrm{FG}\colon (A\text{-FML}^{\mathrm{arb}}/R) \to (A\text{-FM}^{\mathrm{arb}}/R),$$

essentially forgets the choice of module law. Given a formal A-module $\mathcal{F}$ over R, any isomorphism $\mathcal{F} \cong \mathrm{FG}(F)$, with F a formal A-module law, will be referred to as a Coordinate for $\mathcal{F}$.

Just as in the case of formal module laws, a morphism $R \to R'$ yields a functor

$$(A\text{-FM}/R) \to (A\text{-FM}/R'), \quad \mathcal{F} \mapsto \mathcal{F} \otimes_R R'.$$

Again, the reader shall be warned that our definition of formal A-modules is not well behaved if one chooses to work over non-local A-algebras R. Here, one should rather pass to the Zariski-stackification of the prestack $\mathrm{Spec}(R') \mapsto (A\text{-FM}/R')$ described above.

The Lie-functor extends to formal A-modules, for any $\mathcal{F} \in (A\text{-FM}^{\mathrm{arb}}/R)$, the R-module $\mathrm{Lie}(\mathcal{F})$ is free of rank n.

Lemma 1.1.7. *A map $\phi\colon \mathcal{F} \to \mathcal{G}$ of formal A-modules (of arbitrary dimension) over R is an isomorphism if and only if the induced morphism of Lie algebras $\mathrm{Lie}(\phi)\colon \mathrm{Lie}(\mathcal{F}) \to \mathrm{Lie}(\mathcal{G})$ is an isomorphism.*

Proof. Choosing coordinates, we may assume that $\mathcal{F}$ and $\mathcal{G}$ come from formal A-module laws F and G. Now, the claim follows from the formal implicit function theorem. $\qquad\square$

Example. Consider the formal group law $\widehat{\mathbb{G}}_m$ over $\mathbb{Z}_p$. It admits the structure of a $\mathbb{Z}_p$-module law as follows. As a functor, $\widehat{\mathbb{G}}_m$ is isomorphic to the assignment

$$(\mathbb{Z}_p\text{-Adm}) \to (\mathrm{Ab}), \quad S \mapsto 1 + S^{\mathrm{oo}} \subset S^{\times}.$$

The subgroup $1 + S^{\infty}$ naturally carries the structure of a $\mathbb{Z}_p$-module. Indeed, for $k \in \mathbb{N}$, we have

$$(1+s)^{p^k} = 1 + p^k s + \binom{p^k}{2} s^2 + \cdots + s^{p^k},$$

and given $s \in S^{\infty}$, this is of the form $1 + o(1)$ as k increases. In particular, if $a = a_0 + a_1 p + a_2 p^2 + \cdots \in \mathbb{Z}_p$, expressions of the form

$$(1+s)^a = \prod_{i=1}^{\infty}(1+s)^{a_k p^k}$$

make sense by Lemma A.3. Using the evident coordinate, we may equip $\widehat{\mathbb{G}}_{m,\mathbb{Z}_p}$ the structure of a formal $\mathbb{Z}_p$-module law, with $[a]_{\widehat{\mathbb{G}}_m}(T) = (1+T)^a - 1$ for $a \in \mathbb{Z}_p$. Lubin and Tate found analogues of this construction for the rings of integers over arbitrary non-Archimedian local fields E. We recall their construction in Section 1.4. In Section 2 we explain the applications of these formal modules to local class field theory.

Definition 1.1.8 (Formal Module associated to R-module). Suppose that M is a finite free R-module. Then we write $\widehat{\mathbb{G}}_a \otimes M$ for the additive formal A-module associated to M over R. As a formal scheme, this formal module is given by

$$\widehat{\mathbb{G}}_a \otimes M \cong \mathrm{Spf}(R[\![M^{\vee}]\!]),$$

where $R[\![M^{\vee}]\!]$ denotes the completion of $\mathrm{Sym}_R(M^{\vee})$ with respect to the ideal generated by $M^{\vee}$. The (formal) A-module structure is the canonical additive one. Note that $\mathrm{Lie}(\widehat{\mathbb{G}}_a \otimes M) = M$ by design.

Remark. If $R \to R'$ is a ring morphism that turns R' into a finite free R-algebra, the above definition overloads the expression $\widehat{\mathbb{G}}_a \otimes_R R'$. In order to disambiguate, we usually denote the additive formal A-module over R' by $\widehat{\mathbb{G}}_{a,R'}$.

1.2 Invariant Differentials and Logarithms

We now let $A = \mathcal{O}_E$. Let R be a local $\mathcal{O}_E$-algebra with structure map $i \colon \mathcal{O}_E \to R$.

We review the results of Sections 2 and 3 of [GH94]. Suppose that $F = (F_1, \ldots, F_n)$ is an n-dimensional formal $\mathcal{O}_E$-module law over R. We abbreviate $\boldsymbol{X} = (X_1, \ldots, X_n)$, $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$, etc.

Definition 1.2.1 (Invariant Differentials). The module $\omega(F)$ of *invariant differentials* is the submodule of the R-module of differentials

$$\Omega_{R[\![T_1,\ldots,T_n]\!]/R} \cong \bigoplus_{i=1}^{n} R[\![T_1, \ldots, T_n]\!] \, \mathrm{d}T_i,$$

consisting of those $\omega \in \omega(F)$ satisfying

$$\omega(F(\boldsymbol{X}, \boldsymbol{Y})) = \omega(\boldsymbol{X}) + \omega(\boldsymbol{Y}) \quad \text{and} \quad \omega([a]_F(\boldsymbol{X})) = a\omega(\boldsymbol{X}). \tag{1.3}$$

for all $a \in \mathcal{O}_E$.

This construction is functorial in F. Any morphism $f\colon F \to G$ of formal $\mathcal{O}_E$-module laws yields a morphism

$$f^*\colon \omega(G) \to \omega(F), \qquad \sum_{1 \leq i \leq \dim(G)} g_i(\boldsymbol{T})\,\mathrm{d}T_i \mapsto \sum_{1 \leq i \leq \dim(F)} g_i(f(\boldsymbol{T}))\,\mathrm{d}f_i(\boldsymbol{T}).$$

It is possible to explicitly calculate a basis for the R-module $\omega(F)$. Let

$$A(\boldsymbol{X}, \boldsymbol{Y}) \in \mathrm{Mat}_{n \times n}(R[\![\boldsymbol{X}, \boldsymbol{Y}]\!])$$

denote the matrix $((\partial/\partial X_j)F_i(\boldsymbol{X}, \boldsymbol{Y}))_{i,j}$, the derivative of $F(\boldsymbol{X}, \boldsymbol{Y})$ with respect to $\boldsymbol{X}$. Set $B(\boldsymbol{Y}) = A(0, \boldsymbol{Y})$. Then B is a unit in $\mathrm{Mat}_{n \times n}(R[\![\boldsymbol{Y}]\!])$; and we write $(C_{ij}(\boldsymbol{Y}))$ for the components of $B(\boldsymbol{Y})^{-1}$. We now construct

$$\omega_i := \sum_{j=1}^{n} C_{ij}(\boldsymbol{X})\,\mathrm{d}X_j \in \Omega_{R[\![\boldsymbol{X}]\!]/R}$$

for $1 \leq i \leq n$. By definition we have

$$C_{ij}(0) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

Checking that ω_i is an invariant differential is a matter of applying the chain rule.

Proposition 1.2.2. *The R-module $\omega(F)$ is free of rank n, generated by the invariant differentials $\omega_1, \omega_2, \ldots, \omega_n$.*

Proof. This is [Hon70, Proposition 1.1]. $\qquad\square$

Example. The invariant differentials for $\widehat{\mathbb{G}}_a$ are spanned by the form $\mathrm{d}X$. The invariant differentials for $\widehat{\mathbb{G}}_{m,\mathbb{Z}_p}$ are spanned by the form $\omega_1(X) = \frac{1}{1+X}\,\mathrm{d}X$.

By the proposition above and (1.4), we maydefine a pairing

$$\omega(F) \times \mathrm{Lie}(F) \to R, \qquad \langle X_i, \omega_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

This pairing is independent of the parameterization of F. In particular, it descends to a pairing defined for formal modules $\mathcal{F} \in (A\text{-FM}^{\mathrm{arb}}/R)$, and we have a natural isomorphism $\omega(\mathcal{F}) \cong \mathrm{Hom}_R(R, \mathrm{Lie}(\mathcal{F}))$.

Recall that $\widehat{\mathbb{G}}_{a,R}$ is the additive formal $\mathcal{O}_E$-module over R. There is a map

$$\mathrm{d}_F\colon \mathrm{Hom}_{(\mathcal{O}_E\text{-FML}/R)}(F, \widehat{\mathbb{G}}_{a,R}) \to \omega(F), \qquad f \mapsto \mathrm{d}f(\boldsymbol{X}) \tag{1.5}$$

which is a map of R-modules if we equip the left hand side with the R-module structure coming from the natural action of $R \subset \mathrm{End}(\widehat{\mathbb{G}}_a)$.

Proposition 1.2.3. *1. If R is a flat $\mathcal{O}_E$-algebra, the map d_F is injective.*

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2. If R is an E-algebra, the map d_F is an isomorphism.

Proof. This is [GH94, Proposition 3.2]. $\qquad\square$

Suppose now that $F \in (\mathcal{O}_E\text{-FML}^{\mathrm{arb}}/R)$ is a formal module law of dimension n over a flat $\mathcal{O}_E$-algebra R. Let $\omega_1, \ldots, \omega_n$ be the distinguished basis for $\omega(F)$ constructed above. By the previous proposition, there are unique power series $f_i(\boldsymbol{X}) \in (R \otimes_{\mathcal{O}_E} E)[\![\boldsymbol{X}]\!]$ that furnish homomorphisms $F \otimes (R \otimes_{\mathcal{O}_E} E) \to \widehat{\mathbb{G}}_{a, R\widehat{\otimes}_{\mathcal{O}_E} E}$ of formal $\mathcal{O}_E$-module laws and satisfying

$$\mathrm{d}_F f_i(\boldsymbol{X}) = \omega_i(\boldsymbol{X}) \in \omega(F).$$

Definition 1.2.4 (Logarithm and Exponential)**.** The induced morphism of formal group laws

$$f = (f_1, \ldots, f_n) \colon F \otimes (R \otimes_{\mathcal{O}_E} E) \to \widehat{\mathbb{G}}_a^n \otimes_R (R\widehat{\otimes}E)$$

is called the *logarithm* attached to F, we write $\log_F(\boldsymbol{X}) \in ((R \otimes_{\mathcal{O}_E} E)[\![\boldsymbol{X}]\!])^n$ for the corresponding collection of power series. The inverse of $\log_F(\boldsymbol{X})$ is called the *exponential* attached to F, denoted $\exp_F(\boldsymbol{X})$. We have $\mathrm{Lie}(\log_F) = \mathrm{Lie}(\exp_F) = \mathrm{id}$, so $\log_F$ and $\exp_F$ are isomorphisms.

Example.

- The logarithm for the formal $\mathbb{Z}_p$-module law $\widehat{\mathbb{G}}_m$ over $\mathbb{Z}_p$ is given by the integral of $\frac{1}{1+T}\,\mathrm{d}T$, which is simply the usual logarithm

$$\log_{\widehat{\mathbb{G}}_m}(T) = -\sum_{i=1}^{\infty} \frac{(-T)^i}{i}.$$

- If F is the formal $\mathbb{F}_q[\![\varpi]\!]$-module law with $F(X, Y) = X + Y$, $[\zeta]_F(T) = \zeta T$ for $\zeta \in \mathbb{F}_q$ and $[\varpi]_F(T) = T^q$, the logarithm for F needs to satisfy $\log_F(\zeta T) = \zeta \log_F(T)$. Furthermore, it satisfies the functional equation

$$\log_F(T^q) = \log_F([\varpi]_F(T)) = \varpi \log_F(T).$$

Hence, it is necessarily of the form

$$\log_F(T) = \sum_{i=0}^{\infty} a_i T^{q^i}$$

with $a_0 = 1$ and the remaining coefficients uniquely determined by the equation above. This example foreshadows some of the results in Section 1.5.

Lemma 1.2.5. *Let F and G be formal $\mathcal{O}_E$-module laws over R, with $\dim F = n$ and $\dim G =$*

m. Let $\phi \colon F \to G$ be a morphism. Then the diagram

$$
\begin{array}{ccc}
F \otimes (R \otimes_{\mathcal{O}_E} E) & \xrightarrow{\ \log_F\ } & \widehat{\mathbb{G}}_{a,R\otimes_{\mathcal{O}_E}E} \otimes (\mathrm{Lie}(F) \otimes_{\mathcal{O}_E} E) = \widehat{\mathbb{G}}^n_{a,R\widehat{\otimes}_{\mathcal{O}_E}E} \\
\downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\mathrm{Lie}(\phi)} \\
G \otimes (R \otimes_{\mathcal{O}_E} E) & \xrightarrow{\ \log_G\ } & \widehat{\mathbb{G}}_{a,R\otimes_{\mathcal{O}_E}E} \otimes (\mathrm{Lie}(G) \otimes_{\mathcal{O}_E} E) = \widehat{\mathbb{G}}^m_{a,R\widehat{\otimes}_{\mathcal{O}_E}E}
\end{array}
$$

commutes. In particular, attached to any $\mathcal{F} \in (\mathcal{O}_E\text{-}\mathrm{FM}^{\mathrm{arb}}/R)$ comes a natural morphism

$$
\log_{\mathcal{F}} \colon \mathcal{F} \otimes (R \otimes_{\mathcal{O}_E} E) \to \widehat{\mathbb{G}}_{a,R\otimes_{\mathcal{O}_E}E} \otimes (\mathrm{Lie}(\mathcal{F}) \otimes_R (R \otimes_{\mathcal{O}_E} E)) \,.
$$

Proof. The square commutes because of the equality

$$
\mathrm{Hom}(\widehat{\mathbb{G}}^n_{a,R\widehat{\otimes}_{\mathcal{O}_E}E}, \widehat{\mathbb{G}}^m_{a,R\widehat{\otimes}_{\mathcal{O}_E}E}) = \mathrm{Hom}_{R\otimes_{\mathcal{O}_E}E}((R \otimes_{\mathcal{O}_E} E)^n, (R \otimes_{\mathcal{O}_E} E)^m),
$$

and the fact that $\mathrm{Lie}(\log_G \circ \phi \circ \exp_F) = \mathrm{Lie}(\phi)$. $\qquad\square$

1.3 Formal Modules over Extensions of the Residue Field

Let R be an $\mathcal{O}_E$-algebra, and let $\mathbb{F}_q$ be the residue field of $\mathcal{O}_E$. We have seen in the previous section that if R is a field extension of E, then any morphism of formal group laws $f \colon F \to G$ over R is either trivial or an isomorphism, which makes the theory of formal $\mathcal{O}_E$-modules over R rather simple. This situation changes if R is a field extension of $\mathbb{F}_q$: there are homomorphisms of formal group laws $f \colon F \to G$ corresponding to power series $f(T) \in R[\![T]\!]$ with vanishing differential. The prototype of such homomorphisms is the relative Frobenius homomorphism $\Pi \colon F \to F^{(q)}$ corresponding to the monomial $\Pi(T) = T^q$. Here, $F^{(q)}$ is the formal $\mathcal{O}_E$-module law obtained by raising each coefficient of the power series belonging to F to the q-th power.

We introduce the concept of height, which is in a sense an attempt to quantify the disorder introduced by the Frobenius homomorphisms. This leads to interesting invariants of formal $\mathcal{O}_E$-modules over local $\mathcal{O}_E$-algebras R.

Lemma 1.3.1. *Assume that k is a field extension of $\mathbb{F}_q$ and let $f \colon F \to G$ be a morphism of formal $\mathcal{O}_E$-module laws over k, given by a formal series $f(T) \in k[\![T]\!]$. Then either $f'(0) \neq 0$, or $f(T) = g(T^q)$ for some power series $g(T) \in k[\![T]\!]$.*

Proof. Upon differentiating the equation $f(F(X,Y)) = G(f(X), f(Y))$ with respect to Y, we find

$$
f'(X) \left(\frac{\partial}{\partial Y} F \right)(X, 0) = \left(\frac{\partial}{\partial Y} G(f(X), 0) \right) f'(0).
$$

We find that $f'(0) = 0$ implies $f'(X) = 0$. Now the argument is the same as in [GH94, Lemma 4.1]. $\qquad\square$

This allows us to define the height of $f \colon F \to G$, denoted $\mathrm{ht}(f)$, as follows.

Definition 1.3.2 (Height of morphisms of group laws). If $f \neq 0$, the *height* of f is defined as the unique integer h such that $f = g(T^{q^h})$ for some power series $g(T) = c_1 T + c_2 T^2 + \cdots \in R[\![T]\!]$ with $c_1 \neq 0$. If $f = 0$, we say that f has infinite height.

One readily checks that if $f \colon \mathcal{F} \to \mathcal{G}$ is a morphism of formal $\mathcal{O}_E$-module laws over a field extension k of $\mathbb{F}_q$, the height of f does not depend on the choices of coordinates on $\mathcal{F}$ and $\mathcal{G}$. The following result is easy to verify.

Lemma 1.3.3. *The height function is additive. That is, if $f \colon \mathcal{F} \to \mathcal{G}$ and $g \colon \mathcal{G} \to \mathcal{H}$ are two morphisms of formal $\mathcal{O}_E$-modules, we have*

$$\mathrm{ht}(f \circ g) = \mathrm{ht}(f) + \mathrm{ht}(g).$$

Definition 1.3.4 (Isogeny). A morphism $f \colon \mathcal{F} \to \mathcal{G}$ of formal groups over a field k is called an *isogeny* if $\mathrm{Ker}(f)$ is a representable by a finite free k-scheme.

By Proposition 1.11.5, the set $\mathrm{Hom}_{(\mathcal{O}_E\text{-FML}/k')}(\mathcal{F}, \mathcal{G})$ is an $\mathcal{O}_E$-module.

Definition 1.3.5 (Quasi-Isogeny). A *quasi-isogeny* is an element $\iota \in \mathrm{Hom}_{(\mathcal{O}_E\text{-FML}/k')}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{O}_E} E$ such that $\varpi^n \cdot \iota$ is an isogeny for some $n \geq 1$.

It is not difficult to verify the following result.

Lemma 1.3.6. *A morphism $f \colon \mathcal{F} \to \mathcal{G}$ of formal groups over k is an isogeny if and only if the height of f is finite.*

Definition 1.3.7 (ϖ-divisible A-module). Let R be a local $\mathcal{O}_E$-algebra with maximal ideal containing the image of ϖ. Then the residue field of R is an extension of $\mathbb{F}_q$, and we say that a formal $\mathcal{O}_E$-module $\mathcal{F}$ over R is ϖ-*divisible* if $[\varpi]_{\mathcal{F} \otimes R/\mathfrak{m}_R}$ is an isogeny.

If $\mathcal{F}$ is a ϖ-divisible formal $\mathcal{O}_E$-module over R, we define the *height of $\mathcal{F}$* as the height of the isogeny $[\varpi]_{\mathcal{F} \otimes R/\mathfrak{m}_R}$. The additivity of height quickly implies the following.

Lemma 1.3.8. *There are no non-zero homomorphisms between formal $\mathcal{O}_E$-modules over R of different height.*

Proposition 1.3.9. *Let F and G be one-dimensional formal $\mathcal{O}_E$-module laws over $R \in \mathcal{C}$, and suppose that F is ϖ-divisible. The $\mathcal{O}_E$-module homomorphism*

$$\mathrm{Hom}_{(\mathcal{O}_E\text{-FML}/R)}(F, G) \to \mathrm{Hom}_{(\mathcal{O}_E\text{-FML}/\overline{\mathbb{F}}_q)}(F \otimes \overline{\mathbb{F}}_q, G \otimes \overline{\mathbb{F}}_q),$$

given by sending a polynomial $f \in R[\![T]\!]$ to its reduction modulo $\mathfrak{m}_R$, is injective.

Proof. This is [GH94, Proposition 4.2]. $\qquad\square$

We close this subsection with a discussion about the structure of formal $\mathcal{O}_E$-modules over an algebraic closure $\mathbb{F}_q \hookrightarrow \overline{\mathbb{F}}_q$.

Proposition 1.3.10. *Over $\overline{\mathbb{F}}_q$, any two formal $\mathcal{O}_E$-module laws of the same height are isomorphic.*

Proof. This is part of [Dri74, Proposition 1.7]. $\qquad\square$

We call a formal $\mathcal{O}_E$-module law $\mathbb{X}$ of height n over $\overline{\mathbb{F}}_q$ *normalized* if it is defined over $\mathbb{F}_q$ and furthermore satisfies $[\varpi]_{\mathbb{X}}(T) = T^{q^n}$ and $[\zeta]_{\mathbb{X}}(T) = \overline{\zeta}T$ for every $(q-1)$-th root of unity in $\mathcal{O}_E$. Normalized formal modules of arbitrary height exist, for example as the reduction of the standard formal $\mathcal{O}_E$-module H over $\mathcal{O}_{\breve{E}}$ (cf. Section 1.5). In particular, any formal $\mathcal{O}_E$-module law of height n is isomorphic to a normalized formal $\mathcal{O}_E$-module law $\mathbb{X}$.

We give a more explicit description of $\mathcal{O}_D$ in the case that $F = \mathbb{X}$ is normalized. Then $\mathcal{O}_D = \operatorname{End}(\mathbb{X})$ admits the following description. Let E_n be the degree n unramified extension of $\mathcal{O}_E$, and identify the residue field of $\mathcal{O}_{E_n}$ with $\mathbb{F}_{q^n} \subset \overline{\mathbb{F}}_q$. Let $\Phi\colon \mathcal{O}_{E_n} \to \mathcal{O}_{E_n}$ denote the lift of the q-th power frobenius on $\mathbb{F}_{q^n}$.

Proposition 1.3.11.   *1. The endomorphism ring of a normalized $\mathcal{O}_E$-formal module law of height n is given by*

$$\mathcal{O}_D = \frac{\mathcal{O}_{E_n}\{\Pi\}}{(\Pi^n = \varpi, \Pi a = \Phi(a)\Pi)}. \tag{1.6}$$

2. Suppose that $F \in (\mathcal{O}_E\text{-FML}/\overline{\mathbb{F}}_q)$ is ϖ-divisible of height n. Then $\operatorname{End}_{(\mathcal{O}_E\text{-FML}/\overline{\mathbb{F}}_q)}(F)$ is isomorphic to the maximal order $\mathcal{O}_D$ of the central division algebra D over E of rank n^2 and invariant $\frac{1}{n}$.

Proof. There is an embedding of $\mathcal{O}_{E_n}$ into $\operatorname{End}(\mathbb{X})$, uniquely determined by $[\zeta]_{\mathbb{X}}(T) = \overline{\zeta}T$ for any $(q^n - 1)$-th root of unity $\zeta \in \mathcal{O}_{E_n}$. Furthermore, as $\mathbb{X}$ is defined over $\mathbb{F}_q$, the monomial $\Pi(T) = T^q$ is an endomorphism of $\mathbb{X}$, and one readily checks that $\Pi a = \Phi(a)\Pi$ for $a \in \mathcal{O}_{E_n}$. As any power series $f(T) \in \overline{\mathbb{F}}_q[\![T]\!]$ that furnishes an endomorphisms of $\mathbb{X}$ has to satisfy $f(T^{q^n}) = f(T)^{q^n}$, we find that $f(T)$ is defined over $\mathbb{F}_{q^n}$. But now $g(T) = f(T) -_{\mathbb{X}} [\zeta](T)$ has trivial derivative for some suitable $\zeta \in \mathcal{O}_{E_n}$, so $g(T) = f_1(T^q) = f_1 \circ \Pi(T)$ by Lemma 1.3.1. Iterating this process, we find that $\operatorname{End}(\mathbb{X})$ is indeed of the form given above. The second part follows immediately by Proposition 1.3.10. Also cf. [Dri74, Proposition 1.7]. $\qquad\square$

In particular, reduction modulo Π yields a map $\mathcal{O}_D \to \mathbb{F}_{q^h}$, as well as a map of multiplicative monoids

$$\operatorname{Nrd}\colon D \to E, \quad \Pi \mapsto \varpi, \quad a \mapsto \operatorname{N}_{E_n/E}(a) \text{ for } a \in E_n.$$

Both these maps are invariant under conjugation, and thus extend to maps from $\operatorname{End}(\operatorname{FG}(\mathbb{X}))$. The height of any quasi-isogeny $\tau\colon \operatorname{FG}(\mathbb{X}) \dashrightarrow \operatorname{FG}(\mathbb{X})$ is equal to $\operatorname{val}_{\varpi}(\operatorname{Nrd}(\tau))$.

Lemma 1.3.12. *Let $f\colon F \to G$ be an isogeny of ϖ-divisible formal $\mathcal{O}_E$-module laws over $\overline{\mathbb{F}}_q$. Then there is an integer $c \geq 0$ and an isogeny $g\colon G \to F$ with*

$$f \circ g = [\varpi^c]_G \quad and \quad g \circ f = [\varpi^c]_F.$$

In particular, any quasi-isogeny has an inverse.

Proof. As the height is additive, we necessarily have $\operatorname{ht}(F) = \operatorname{ht}(G) =: h$, thus by Lemma 1.3.10, we may assume that F and G are given by the normalized formal $\mathcal{O}_E$-module $\mathbb{X}$ of height h. Now the claim follows from the description of $\operatorname{End}(\mathbb{X})$ given above. $\qquad\square$

1.4   Lubin–Tate Formal Module Laws

In this section we explain the construction of Lubin–Tate module laws. These $\mathcal{O}_E$-module laws were introduced by Lubin and Tate in [LT65] in order to construct the totally ramified Abelian extensions of E. We come back to this in Section 2.

We fix a positive integer h and write $\mathcal{F}_{\varpi,h}$ for the set of power series

$$\mathcal{F}_{\varpi,h} := \{f \in \mathcal{O}_E[\![T]\!] \mid f \equiv \varpi T \pmod{T^2} \ \text{ and } \ f \equiv T^{q^h} \pmod{\varpi}\}.$$

The construction of the Lubin–Tate formal module laws is based on the following lemma, which is Lemma 1 in [LT65].

Lemma 1.4.1. *Let $f(T)$ and $g(T)$ be elements of $\mathcal{F}_{\varpi,h}$ and let $L(X_1,\ldots,X_n) = \sum_{i=1}^n a_i X_i$ be a linear form with coefficients in $\mathcal{O}_E$. Then there exists a unique series $F(X_1,\ldots,X_n)$ with coefficients in $\mathcal{O}_E$ such that*

$$F(X_1,\ldots,X_n) \equiv L(X_1,\ldots,X_n) \pmod{T^2},$$
$$and$$
$$f(F(X_1,\ldots,X_n)) = F(g(X_1),\ldots,g(X_n)).$$

As a direct consequence, we obtain the following useful result.

Definition and Lemma 1.4.2 (Lubin–Tate Module Laws)**.** Let $f \in \mathcal{F}_{\varpi,h}$. Then there is a unique formal $\mathcal{O}_E$-module law F_f over $\mathcal{O}_E$ with $[\varpi]_F(T) = f(T)$. If $f \in \mathcal{F}_{f,h}$, we refer to F_f as a *Lubin–Tate module law* of height h.

Proof. In the above lemma, set $L(X,Y) = X + Y$ and $g = f$ to uniquely determine the power series F_f. The same Lemma yields unique power series $[a]_{F_f}(T) \in R[\![T]\!]$ by setting $L(T) = aT$, $g = f$. It is routine to check that $(F_f, ([a]_f)_{a \in \mathcal{O}_E})$ is a formal $\mathcal{O}_E$-module law, cf. [LT65]. $\qquad\square$

Another application of Lemma 1.4.1 shows that attached to each $a \in \mathcal{O}_E$ and $f, g \in \mathcal{F}_{\varpi,h}$, there are unique $[a]_{f,g}(T) \in \mathcal{O}_E[\![T]\!]$ satisfying

$$[a]_{f,g}(T) \equiv aT \pmod{T^2} \quad \text{and} \quad f([a]_{f,g}(T)) = [a]_{f,g}(g(T)). \tag{1.7}$$

We now have the following theorem.

Theorem 1.4.3 (Lubin–Tate Formal $\mathcal{O}_E$-Module Laws)**.** *For $f, g \in \mathcal{F}_{\varpi,h}$, the formal $\mathcal{O}_E$-module laws F_f and F_g are canonically isomorphic, via the morphism induced by $[1]_{f,g} \in \mathcal{O}_E[\![T]\!]$.*

Proof. This is Theorem 1 of [LT65] and the subsequent discussion. $\qquad\square$

In particular, up to canonical isomorphism, there is only one Lubin–Tate formal $\mathcal{O}_E$-module law over $\mathcal{O}_E$ attached to the choice of the uniformizer $\varpi \in \mathcal{O}_E$.

Example. If $E = \mathbb{Q}_p$, this reconstructs the multiplicative formal $\mathbb{Z}_p$-module $\widehat{\mathbb{G}}_m$ constructed above. Indeed, we have

$$\mathcal{F}_{p,1} = \{f \in \mathbb{Z}_p[\![T]\!] \mid f(T) \equiv T^p \bmod p \text{ and } f(T) \equiv pT \bmod (T)^2\},$$

so that $f(T) = (1+T)^p - 1$ lies in $\mathcal{F}_{p,1}$. One quickly checks that

$$F_f(X,Y) = (1+X)(1+Y) - 1 = X + Y + XY \in \mathbb{Z}_p[\![X,Y]\!]$$

is the addition law associated to f, and that for $a \in \mathbb{Z}_p$, the power series

$$[a]_{f,f} = (1+T)^a - 1 \in \mathbb{Z}_p[\![T]\!]$$

satisfies the condition of (1.7).

We will need the following result.

Lemma 1.4.4. *Let E be a local field with integers $\mathcal{O}_E$ and a choice of uniformizer $\varpi \in \mathcal{O}_E$, and let F be a Lubin-Tate $\mathcal{O}_E$-module law corresponding to some $f \in \mathcal{F}_\varpi$, cf. Theorem 1.4.3. Let S be an admissible $\mathcal{O}_E$-algebra and let $s \in S^\infty$ be an element such that the series $\log_{\mathcal{F}}(s)$ converges. Then we have $\log_F(s) = 0$ if and only if $[\varpi]_F^r(s) = 0$ for some $r \in \mathbb{N}$.*

Proof. Up to canonical isomorphism, F is an $\mathcal{O}_E$-module law with $[\varpi]_F(T) = \varpi T + T^q$. Now one may check that

$$\log_F(T) = \lim_{r \to \infty} \frac{[\varpi]_F^r(T)}{\varpi^r} = \prod_{i=1}^\infty \frac{[\varpi]_F^i(T)}{\varpi[\varpi]_F^{i-1}(T)}.$$

After inserting $s \in S^\infty$, we see that the product vanishes if and only if $[\varpi]_F^r(s) = 0$ for some $r \in \mathbb{N}$. $\qquad\square$

1.5 Hazewinkel's Functional Equation Lemma and the Standard Formal Module Law

Let R be a flat $\mathcal{O}_E$-algebra. By the results of Section 1.2, the structure of a formal $\mathcal{O}_E$-module law F over R is uniquely determined by its logarithm $\log_F(T) \in R[\frac{1}{\varpi}][\![T]\!]$. Indeed, we find

$$F(X,Y) = \exp_F(\log_F(X) + \log_F(Y)), \quad [a]_F(X) = \exp_F(a \log_F(X)).$$

It is therefore natural to wonder if it is possible to reverse this process, that is, if it is possible to classify the power series arising as logarithms of formal $\mathcal{O}_E$-module laws. Hazewinkel's functional equation lemma is a result in this direction. It provides a sufficient condition on power series $f \in R[\frac{1}{\varpi}][\![T]\!]$ that ensures that f arises as the logarithm of some formal $\mathcal{O}_E$-module law.

Theorem 1.5.1 (Hazewinkel's Functional Equation Lemma)**.** *Let $f \in R[\frac{1}{\varpi}][\![T]\!]$ be a power series with $f'(0) \in R[\frac{1}{\varpi}]^\times$. Let $\sigma \colon R[\frac{1}{\varpi}] \to R[\frac{1}{\varpi}]$ be an endomorphism of rings that restricts*

to an endomorphism of R and suppose that there are elements $s_1, s_2, \cdots \in R[\frac{1}{\varpi}]$ such that

$$f(X) - \sum_{i=1}^{\infty} s_i(\sigma_*^i f)(X^{q^i}) \in R[\![X]\!].$$

Let $\mathfrak{a} \subset R$ be an ideal and suppose that that the conditions

$$\sigma(b) \equiv b^q \pmod{\mathfrak{a}} \text{ for all } b \in R \quad \text{and} \quad \sigma^r(s_i)\mathfrak{a} \subset R \text{ for all } r, i \geq 1$$

are satsified. Now the following statements hold.

1. *We have*
   $$F(X, Y) = f^{-1}(f(X) + f(Y)) \in R[\![X, Y]\!],$$
   where f^{-1} is the inverse power series as in Lemma 1.1.7.

2. *If $g(Z) \in R[\frac{1}{\varpi}][\![Z]\!]$ is another power series satisfying the same condition*
   $$g(Z) - \sum_{i=1}^{\infty} s_i(\sigma_*^i f)(Z^{q^i}) \in R[\![Z]\!],$$
   then $f^{-1}(g(Z)) \in R[\![Z]\!]$.

3. *If $\alpha(T) \in R[\![T]\!]$ and $\beta(T) \in R[\![T]\!]$, then*
   $$\alpha(T) \equiv \beta(T) \pmod{\mathfrak{a}^r} \iff f(\alpha(T)) \equiv f(\beta(T)) \pmod{\mathfrak{a}^r}. \tag{1.8}$$

Proof. A more general statement can be found in [Haz78, Sections 2 and 10]. $\qquad\square$

Note that by construction, $F(X, Y)$ as defined above yields a (commutative) formal group law over R. If σ restricts to the identity on $\mathcal{O}_E \subseteq R$, then the second part of the Functional Equation Lemma implies that we even obtain a formal $\mathcal{O}_E$-module law with $[b]_F(T) = f^{-1}(bf(T))$. Indeed, $bf(T)$ satisfies the same functional equation if $b \in \mathcal{O}_E$.

We now consider a special family of formal $\mathcal{O}_E$-module laws, so called $\mathcal{O}_E$-typical formal module laws.

Definition 1.5.2 ($\mathcal{O}_E$-typical Formal Module Law). We say that a formal module law $F \in (\mathcal{O}_E\text{-FML}/R)$ is $\mathcal{O}_E$-*typical*, if its logarithm is of the form

$$\log_F(T) = \sum_{i=0}^{\infty} b_i X^{q^i}$$

for elements $b_0, b_1, \cdots \in R \otimes_{\mathcal{O}_E} E$.

The family of $\mathcal{O}_E$-typical formal module laws is quite exhaustive. The following result is [Haz78, §21.5.6].

Lemma 1.5.3. *Any formal $\mathcal{O}_E$-module law over $R \in \mathcal{C}$ is isomorphic to an $\mathcal{O}_E$-typical one.*

The Functional Equation Lemma allows us to classify $\mathcal{O}_E$-typical module laws. Let us write $\mathcal{O}_E[\underline{v}] = \mathcal{O}_E[v_1, v_2, \dots]$ and let $f_v(T) \in \mathcal{O}_E[\underline{v}][\frac{1}{\varpi}][\![T]\!]$ be the unique power series satisfying the functional equation

$$f_v(T) = T + \sum_{i \geq 1} \frac{v_i}{\varpi} \sigma^i(f_v)(X^{q^i}),$$

where σ is the endomorphism of $\mathcal{O}_E[\underline{v}]$ obtained by sending v_j to v_j^q for all j. By the Functional Equation Lemma this gives rise to an $\mathcal{O}_E$-module law F_v over $\mathcal{O}_E[\underline{v}]$, which is clearly $\mathcal{O}_E$-typical. By [Haz78, Definition 21.5.5 and Criterion 21.5.9], we have the following.

Lemma 1.5.4. *A formal $\mathcal{O}_E$-module law F over R is $\mathcal{O}_E$-typical if and only if there is a homomorphism $\mathcal{O}_E[\underline{v}] \to R$ such that $F = F_v \otimes R$.*

A special role will play the $\mathcal{O}_E$-typical formal module law H with logarithm given by the power series

$$f(T) = \sum_{i=1}^{\infty} \frac{T^{q^{in}}}{\varpi^i} \in E[\![T]\!].$$

It satisfies the functional equation

$$f(T) = T + \frac{1}{\varpi} f(T^{q^n})$$

and arises from F_v via the homomorphism $\mathcal{O}_E[\underline{v}] \to \mathcal{O}_{\breve{E}}$ sending v_n to 1 and v_j to 0 for $j \neq n$. The fact that $f^{-1}(X) = X - \frac{1}{\varpi} X^{q^n} + \dots$ reveals $[\varpi]_H(T) \equiv \varpi T \mod (T^2)$. Additionally, note that

$$f([\varpi]_H(T)) = \varpi f(T) = \varpi T + f(T^{q^n}) \equiv f(T^{q^n}) \pmod{\varpi}.$$

Hence, the equivalence in (1.8) implies that $[\varpi]_H(T) \equiv T^{q^n} \mod \varpi$. So H is a Lubin–Tate formal $\mathcal{O}_E$-module law of height n. We call it the *standard formal $\mathcal{O}_E$-module law* of height n. Although the coefficients of H lie inside $\mathcal{O}_E$, we usually consider it as a formal $\mathcal{O}_E$-module over $\mathcal{O}_{\breve{E}}$.

Lemma 1.5.5. *Let $\zeta \in \breve{E}$ be a $(q^n - 1)$-th root of unity. Then $[\zeta]_H(T) = \zeta T$ is an automorphism of H. In particular, $\mathrm{End}(H)$ naturally carries the structure of an $\mathcal{O}_{E_n}$-algebra, where E_n is the unramified extension of E with residue field $\mathbb{F}_{q^n}$.*

Proof. This is an immediate consequence of the equality $\zeta \log_H(T) = \log_H(\zeta T)$. We have

$$\zeta T = \exp_H(\log_H(\zeta T)) = \exp_H(\zeta \log_H(T)) = \exp_H([\zeta]_{\widehat{\mathbb{G}}_{a,\breve{E}}}(\log_H(T))).$$

As $\exp_H \colon \widehat{\mathbb{G}}_{a,\breve{E}} \to H \otimes_{\mathcal{O}_{\breve{E}}} \breve{E}$ is an isomorphism of formal module laws, the claim follows. $\square$

1.6 Deformations of Formal Modules

We shortly discuss deformations of formal $\mathcal{O}_E$-modules. Let $\mathcal{C}$ denote the category of complete, local, Noetherian $\mathcal{O}_{\breve{F}}$-algebras with residue field $\overline{\mathbb{F}}_q$. Morphisms in $\mathcal{C}$ are continuous homomorphisms, a homomorphism of rings $\phi : R \to S$ lies in $\mathrm{Hom}_{\mathcal{C}}(R, S)$ if and only if $\phi(\mathfrak{m}_R) \subset \mathfrak{m}_S$.

Let $\mathbb{X} = H \otimes \overline{\mathbb{F}}_q$ be the formal $\mathcal{O}_E$-module law of height n over $\overline{\mathbb{F}}_q$ obtained by reduction of the standard formal $\mathcal{O}_E$-module H and let $R \in \mathcal{C}$ be an $\mathcal{O}_{\breve{E}}$-algebra with maximal ideal $\mathfrak{m}_R$. A *deformation* of $\mathbb{X}$ to R is a pair $(\mathcal{F}, \iota)$ consisting of a formal $\mathcal{O}_{\breve{E}}$-module $\mathcal{F}$ over R and an isomorphism $\iota\colon \mathrm{FG}(\mathbb{X}) \xrightarrow{\sim} \mathcal{F} \otimes \overline{\mathbb{F}}_q$. We say that two deformations $(\mathcal{F}, \iota)$ and $(\mathcal{F}', \iota')$ are isomorphic if there is an isomorphism $\mathcal{F} \to \mathcal{F}'$ whose reduction to $\overline{\mathbb{F}}_q$ translates ι into ι'. We define the deformation functor

$$\mathcal{M}_0^{(0)}\colon \mathcal{C} \to (\mathrm{Set}), \quad R \mapsto \{\text{Deformations } (\mathcal{F}, \iota) \text{ of } \mathbb{X} \text{ to } R\}/\sim \,.$$

We remark that, as all formal module laws of height n over $\overline{\mathbb{F}}_q$ are isomorphic, this functor does not depend on the choice of $\mathbb{X}$.

In the literature one encounters a variant of this functor, defined on the level of formal module laws rather than formal modules. Let F and G be formal $\mathcal{O}_E$-modules over $R \in \mathcal{C}$. We say that F and G are $\star$-isomorphic if there exists a $\star$-isomorphism $F \xrightarrow{\sim} G$, that is, an isomorphism given by a power series $f \in R[\![T]\!]$ satisfying $f(T) \equiv T$ modulo $\mathfrak{m}_R$ (by Proposition 1.3.9, there is at most one such f). The functor in question is now defined as

$$\mathcal{M}_{0,\mathrm{FML}}^{(0)}\colon \mathcal{C} \to (\mathrm{Set}), \quad R \mapsto \{F \in (\mathcal{O}_E\text{-FML}/\overline{\mathbb{F}}_q) \mid F \otimes \overline{\mathbb{F}}_q = \mathbb{X}\}/\star\text{-isom.}$$

Lemma 1.6.1. *The functors $\mathcal{M}_0^{(0)}$ and $\mathcal{M}_{0,\mathrm{FML}}^{(0)}$ are naturally isomorphic.*

Proof. There is a natural map $\mathcal{M}_{0,\mathrm{FML}}^{(0)} \to \mathcal{M}_0^{(0)}$, given on any component $R \in \mathcal{C}$ by sending a formal $\mathcal{O}_E$-module law $F \in \mathcal{M}_{0,\mathrm{FML}}^{(0)}(R)$ to its associated formal module $(\mathrm{FG}(F), \mathrm{FG}(\mathrm{id}))$. We show that this map is bijective.

Surjectivity. Let $(\mathcal{F}, \iota) \in \mathcal{M}_0^{(0)}(R)$, and choose any coordinate on $\mathcal{F}$ so that $\mathcal{F} = \mathrm{FG}(F)$ for some $\mathcal{O}_E$-module law F. Then ι gives rise to an isomorphism of formal laws $\mathbb{X} \xrightarrow{\sim} F \otimes \overline{\mathbb{F}}_q$ which we also denote by ι. Let $\tilde{\iota} \in R[\![T]\!]$ be an arbitrary power series reducing to ι modulo $\mathfrak{m}_R$. Now the constant coefficient of $\tilde{\iota}$ is invertible, hence $\tilde{\iota}$ has a formal inverse power series $\tilde{\iota}^{-1} \in R[\![T]\!]$. Let F' be the formal module law with addition given by $\tilde{\iota}^{-1}F(\tilde{\iota}(S), \tilde{\iota}(T)) \in R[\![S, T]\!]$ and similarly defined $\mathcal{O}_E$-module structure. Now by construction, $\tilde{\iota}^{-1}$ is an isomorphism of module laws $F \to F'$, and we find that F' satisfies $F' \otimes \overline{\mathbb{F}}_q = \mathbb{X}$. Hence, F' maps to the isomorphism class of $(\mathcal{F}, \iota)$. This shows surjectivity.

Injectivity. Let F and G be formal $\mathcal{O}_E$-module laws inside $\mathcal{M}_{0,\mathrm{FML}}^{(0)}(R)$ with isomorphic associated deformations $(\mathcal{F}, \iota_{\mathcal{F}}) \sim (\mathcal{G}, \iota_{\mathcal{G}}) \in \mathcal{M}_0^{(0)}(R)$. This isomorphism yields an isomorphism of formal $\mathcal{O}_E$-module laws $\alpha\colon F \to G$ whose reduction modulo $\mathfrak{m}_R$ fits into the following commutative triangle.

$$F \otimes \overline{\mathbb{F}}_q \xrightarrow{\ \alpha_0\ } G \otimes \overline{\mathbb{F}}_q$$
$$\mathbb{X}$$

Hence, α is a $\star$-isomorphism, implying that F and G lie in the same equivalence class. This proves injectivity. $\square$

By Proposition 1.3.9, every automorphism of objects in $\mathcal{M}_0^{(0)}(R)$ is trivial. This allows for representability of $\mathcal{M}_0^{(0)}$.

Theorem 1.6.2 (Representability of $\mathcal{M}_0^{(0)}$). *The functor $\mathcal{M}_0^{(0)}$ is representable by a ring $A_0 \in \mathcal{C}$, non-canonically isomorphic to*

$$\mathcal{O}_{\breve{E}}[\![u_1, \ldots, u_{n-1}]\!] \in \mathcal{C}.$$

Proof. This statement is due to Lubin–Tate in the case $\mathcal{O}_E = \mathbb{Z}_p$, cf. [LT66]. For the general case, cf. [Dri74, Proposition 4.2]. $\qquad\square$

We may use the universal $\mathcal{O}_E$-typical formal module law F_v over $\mathcal{O}_E[v_1, v_2, \ldots]$ to construct an isomorphism $\mathrm{Hom}_{\mathcal{C}}(\mathcal{O}_{\breve{E}}[\![u_1, \ldots, u_{n-1}]\!], R) \cong \mathcal{M}_0^{(0)}(R)$. Any homomorphism $\alpha \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{O}_{\breve{E}}[\![u_1, \ldots, u_{n-1}]\!], R)$ gives rise to elements $\alpha(u_1), \ldots, \alpha(u_{n-1}) \in \mathfrak{m}_R$. Consider the homorphism of $\mathcal{O}_E$-algebras

$$\tilde{\alpha} \colon \mathcal{O}[v_1, v_2, \ldots] \to R, \quad v_i \mapsto \begin{cases} \alpha(u_i), & \text{if } 1 \le i \le n-1 \\ 1, & \text{if } i = n \\ 0, & \text{if } i \ge n+1. \end{cases}$$

and let $F_\alpha = F_v \otimes R$ be the formal $\mathcal{O}_E$-module law over R obtained by applying $\tilde{\alpha}$ to the coefficients of F_v. By the Functional Equation Lemma (Theorem 1.5.1), we find that $F_\alpha \otimes \overline{\mathbb{F}}_q$ is equal to $\mathbb{X}$. Hence $F_\alpha \in \mathcal{M}_{0,\mathrm{FML}}^{(0)}(R)$, and the mapping $\alpha \mapsto F_\alpha$ can be shown to be an isomorphism, cf. [GH94, Section 12].

1.7 Quotients of Formal Modules by Finite Submodules

Let R be a Noetherian, regular, local $\mathcal{O}_{\breve{E}}$-algebra that is complete with respect to its maximal ideal and has residue field $\overline{\mathbb{F}}_q$. Let $F \in (\mathcal{O}_E\text{-FML}/R)$ be a formal $\mathcal{O}_E$-module law over R. Then F equips $\mathfrak{m}_R$ with the structure of an $\mathcal{O}_E$-module. Let $Q \subset \mathfrak{m}_R$ be a finite subset that is a submodule with respect to this $\mathcal{O}_E$-module structure. Note that the cardinality of Q is a power of q; we write $q^c = \#Q$. We define

$$f_Q(T) \coloneqq \prod_{a \in Q} (T -_F a),$$

where the subscript F denotes subtraction with respect to F.

We now formulate the main result of this section. Let $\mathrm{Frac}(R)$ denote the field of fractions of R, which contains R as a subring as R is an integral domain. Note that $f_Q'(0) = \pm u(0) \prod_{a \in Q} a \ne 0$, so there exists a formal inverse power series $f_Q^{-1}(T) \in \mathrm{Frac}(R)[\![T]\!]$. Denote by (F/Q) the formal $\mathcal{O}_E$-module law over $\mathrm{Frac}(R)$ with

$$(F/Q)(X, Y) = f_Q(F(f_Q^{-1}(X), f_Q^{-1}(Y))) \quad \text{and} \quad [a]_{(F/Q)}(T) = f_Q([a]_F(f_Q^{-1}(T))) \text{ for } a \in \mathcal{O}_E.$$

The power series f_Q yields a morphism $F \to F/Q$ of formal module laws over $\mathrm{Frac}(R)$.

Theorem 1.7.1 (Quotients of Formal Module Laws).

   1. *The power series constituting the formal $\mathcal{O}_E$-module law (F/Q) all have coefficients in R. That is, (F/Q) is a formal module law over R, and $f_Q \colon F \to (F/Q)$ is a morphism*

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of formal module laws over R.

2. *We have $f_Q(T) \equiv T^{q^c}$ modulo $\mathfrak{m}_R$. In particular, $(F/Q) \otimes \overline{\mathbb{F}}_q = (F \otimes \overline{\mathbb{F}}_q)^{(\#Q)}$.*

3. *Denote by $\mathcal{F}$ the formal module corresponding to F. The finite group Q acts on the formal scheme $\mathcal{F} = \mathrm{Spf}(R[\![T]\!])$, and the map $\mathcal{F} \xrightarrow{f_Q} (\mathcal{F}/Q)$, where $(\mathcal{F}/Q)$ denotes the formal module corresponding to (F/Q), is a categorical quotient in the category of formal Lie-varieties of relative dimension 1 over R (i.e., formal schemes isomorphic to $\mathrm{Spf}(R[\![T]\!])$) for this action.*

4. *Furthermore, if $\mathcal{H}$ is another formal $\mathcal{O}_E$-module law over R and $g\colon \mathcal{F} \to \mathcal{H}$ is Q-invariant, the induced morphism $\tilde{g}\colon (\mathcal{F}/Q) \to \mathcal{H}$ is a morphism of formal $\mathcal{O}_E$-modules.*

The proof is rather elementary and only uses commutative ring theory of $R[\![T]\!]$ and $R[\![X, Y]\!]$. These rings are Noetherian regular local rings by [Mat89, Proposition 19.5], implying that they are unique factorization domains by the theorem of Auslander–Buchsbaum, cf. [Stacks, Tag 0AG0]. Furthermore, they are complete with respect to their maximal ideal.

The first important observation is the following lemma. For the notion of Weierstraß degree and the Weierstraß Preparation Theorem, we refer to Appendix A.3 of [Haz78].

Lemma 1.7.2.

1. *The power series f_Q has Weierstraß degree q^c. That is, it is of the form*

$$f_Q(T) = \sum_{i=0}^{\infty} a_i T^i$$

with $a_i \in \mathfrak{m}_R$ for $i < q^c$ and $a_{q^c} \in R^\times$. Furthermore, $f_Q(T) \equiv T^{q^c} \mod \mathfrak{m}_R$.

2. *There exists a unit $u(T) \in R[\![T]\!]^\times$ such that f_Q admits the factorization*

$$f_Q(T) = u(T)g_Q(T), \quad \text{where} \quad g_Q(T) := \prod_{a \in Q} (T - a).$$

Proof. The first statement follows directly from the fact that for $a \in Q$, we have

$$F(T, a) \in T + a + T\mathfrak{m}_R.$$

The second is an application of the Weierstraß Preparation Theorem. By part 1, this theorem provides a decomposition

$$f_Q(T) = u(T)g(T)$$

with $g(T) = a_0 + a_1 T + \cdots + a_{q^c-1}T^{q^c-1} + T^{q^c} \in R[T]$ and $a_i \in \mathfrak{m}_R$. One quickly checks that for $a \in \mathfrak{m}_R$, the elements $(T - a) \in R[\![T]\!]$ are irreducible. For $a \in Q$, we find $f_Q(a) = 0$, implying that each factor $(T - a)$ divides $f_Q(T)$. As $R[\![T]\!]$ is a unique factorization domain, each factor $(T - a)$ divides $g(T)$. For degree-reasons we obtain $g(T) = \prod_{a \in Q} (T - a)$, as desired. $\qquad \square$

We define an action of Q on $R[\![T]\!]$ as follows. For $a \in Q$ and $h(T) \in R[\![T]\!]$, we put

$$a.h(T) = h(T +_F a) \in R[\![T]\!].$$

As $T +_F a$ lies inside the maximal ideal of $R[\![T]\!]$, this definition makes sense. The power series $f_Q(T)$ is Q-invariant, as

$$f_Q(T +_F a) = \prod_{b \in Q} (T +_F a -_F b) = \prod_{b \in Q} (T -_F b) = f_Q(T).$$

Furthermore, for any $w(T) \in R[\![T]\!]$, the power series $w(f_Q(T)) \in R[\![T]\!]$ is Q-invariant. The main observation in the proof of Theorem 1.7.1 is that every Q-invariant power series arises this way.

Proposition 1.7.3. *Let $R[\![T]\!]^Q$ denote the subring of invariants under the action of Q on $R[\![T]\!]$. Any element in $R[\![T]\!]^Q$ arises as a (unique) power series in $f_Q(T)$, i.e.,*

$$R[\![T]\!]^Q = R[\![f_Q(T)]\!] \subseteq R[\![T]\!].$$

Proof. By the short calculation above, only one inclusion remains. Let $h(T) =: h_0(T) \in R[\![T]\!]^Q$ be an arbitrary Q-invariant power series. We want to show that $h(T) = w(f_Q(T))$ for some $w \in R[\![T]\!]$. Put $c_0 = h_0(0)$ and write $v_0(T) = h_0(T) - c_0$. Then v_0 is Q-invariant, hence we find $v_0(a) = v_0(a +_F 0) = 0$ for $a \in Q$. Ranging over $a \in Q$, this implies $g_Q(T) \mid v_0(T)$, where $g_Q(T)$ is the polynomial from Lemma 1.7.2. As g_Q differs from f_Q by a unit, we obtain $v_0(T) = f_Q(T)h_1(T)$ for some power series $h_1(T) \in R[\![T]\!]$. As R is a domain, $h_1(T)$ is again Q-invariant.

We can now iterate this process, writing $h_1(T) = c_1 + v_1(T)$, $v_1(T) = f_Q(T)h_2(T)$, $h_2(T) = c_2 + v_2(T)$, et cetera. This results for any positive integer m in a continued product

$$
\begin{aligned}
h(T) &= c_0 + f_Q(T)(c_1 + f_Q(T)(c_2 + \cdots + f_Q(T)(c_m + v_m(T))\ldots)) \\
&= c_0 + c_1 f_Q(T) + c_2 f_Q(T)^2 + \cdots + f_Q(T)^m v_m(T) \\
&\equiv c_0 + c_1 f_Q(T) + c_2 f_Q(T)^2 + \cdots + c_{m-1} f_Q(T)^{m-1} \pmod{T^m}.
\end{aligned}
$$

In the limit $m \to \infty$, we obtain the desired representation of $h(T)$. $\qquad\square$

As $R[\![f_Q(T)]\!]$ is again a Noetherian, regular, local $\mathcal{O}_{\breve{E}}$-algebra, we also obtain the equality

$$R[\![X, Y]\!]^{Q \times Q} = R[\![f_Q(X), f_Q(Y)]\!],$$

where $Q \times Q$ acts on $R[\![X, Y]\!]$ separately on each variable.

Proof of Theorem 1.7.1. One readily checks that the power series $f_Q(F(X, Y)) \in R[\![X, Y]\!]$ is $Q \times Q$-invariant, and that $f_Q([a]_F(T)) \in R[\![T]\!]$ is Q-invariant. Hence, by Proposition 1.7.3, there exist unique power series $F'(X, Y) \in R[\![X, Y]\!]$ and $[a]_{F'}(T) \in R[\![T]\!]$ such that

$$f_Q(F(X, Y)) = F'(f_Q(X), f_Q(Y)) \quad \text{and} \quad f_Q([a]_F(T)) = [a]_{F'}(f_Q(T)) \text{ for } a \in \mathcal{O}_E.$$

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As $f'_Q(0) \neq 0$, a family of power series with coefficients in $\mathrm{Frac}(R)$ is uniquely determined by these equations. Namely, these power series have to be equal to those defining the formal $\mathcal{O}_E$-module law (F/Q). As R injects into $\mathrm{Frac}(R)$, we find $F' = (F/Q)$, and the first claim follows. The second claim is Lemma 1.7.2. The third claim is a reformulation of Proposition 1.7.3. Indeed, any morphism of formal Lie varieties $\mathcal{F} \to \mathcal{V}$ corresponds, after choosing coordinates, to a power series $g(T) \in R[\![T]\!]$. Invariance under Q translates to $g(T) \in R[\![T]\!]^Q$, we obtain $g(T) = \tilde{g}(f_Q(T))$ for some $\tilde{g}(T) \in R[\![T]\!]$, and $\tilde{g}(T)$ furnishes the morphsim $(\mathcal{F}/Q) \to \mathcal{V}$. The fourth claim follows formally from the description the formal $\mathcal{O}_E$-module law (F/Q) in terms of F and f_Q. $\qquad\square$

1.8 Extensions of Formal Modules

In this section, we equip the category $(\mathcal{O}_E\text{-FM}^{\mathrm{arb}}/R)$, where R is a local $\mathcal{O}_E$-algebra, with a notion of short exact sequences. Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{F}'$ be formal $\mathcal{O}_E$-modules over R.

Definition 1.8.1 (Short Exact Sequence). A pair of composable morphisms $\mathcal{F}' \to \mathcal{G} \to \mathcal{F}$ in $(\mathcal{O}_E\text{-FM}^{\mathrm{arb}}/R)$ is called a *short exact sequence* if the induced sequence

$$0 \to \mathrm{Lie}(\mathcal{F}') \to \mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(\mathcal{F}) \to 0$$

is a short exact sequence of R-modules. In this case, we write

$$0 \to \mathcal{F}' \to \mathcal{G} \to \mathcal{F}' \to 0.$$

A pair of composable morphisms $F' \to G \to F$ in $(\mathcal{O}_E\text{-FML}^{\mathrm{arb}}/R)$ is called an exact sequence if it is exact after passing to the respective formal modules.

Definition 1.8.2 (Extension). An extension of $\mathcal{F}$ by $\mathcal{F}'$ is a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{G} \to \mathcal{F} \to 0.$$

We say that this extension is equivalent to another extension

$$0 \to \mathcal{F}' \to \mathcal{G}' \to \mathcal{F} \to 0$$

if and only if there is an isomorphism $\mathcal{G} \to \mathcal{G}'$ making the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

commute. We denote the set of equivalence classes of extensions of $\mathcal{F}$ by $\mathcal{F}'$ as $\mathrm{Ext}(\mathcal{F}, \mathcal{F}')$.

Definition 1.8.3 (Rigidfidied Extension). A *rigidified extension* of $\mathcal{F}$ by $\mathcal{F}'$ is a pair consisting of an extension

$$0 \to \mathcal{F}' \to \mathcal{G} \to \mathcal{F} \to 0$$

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and a splitting s of the short exact sequence

$$0 \longrightarrow \mathrm{Lie}(\mathcal{F}') \longrightarrow \mathrm{Lie}(\mathcal{G}) \xrightarrow{} \mathrm{Lie}(\mathcal{F}) \longrightarrow 0.$$

$$\underset{s}{\longleftarrow}$$

We say that two rigidified extensions $(\mathcal{G}, s)$ and $(\mathcal{G}', s')$ are isomorphic if there is an isomorphism $i\colon \mathcal{G} \to \mathcal{G}'$ of extensions such that $s' = \mathrm{Lie}(i) \circ s$. We denote the set of isomorphism classes of rigidified extensions by $\mathrm{RigExt}(\mathcal{F}, \mathcal{F}')$.

Note that by forgetting the splitting, there is a natural map $\mathrm{RigExt}(\mathcal{F}, \mathcal{F}') \to \mathrm{Ext}(\mathcal{F}, \mathcal{F}')$. We summarize the main results of this section.

Theorem 1.8.4 (Extensions of Formal Modules)**.**

1. *The assignments $(\mathcal{F}, \mathcal{F}') \mapsto \mathrm{Ext}(\mathcal{F}, \mathcal{F}')$ and $(\mathcal{F}, \mathcal{F}') \mapsto \mathrm{RigExt}(\mathcal{F}, \mathcal{F}')$ are contravariantly functorial in $\mathcal{F}$ and covariantly functorial in $\mathcal{F}'$. In particular, both $\mathrm{Ext}(\mathcal{F}, \mathcal{F}')$ and $\mathrm{RigExt}(\mathcal{F}, \mathcal{F}')$ become $(\mathrm{End}(\mathcal{F}'), \mathrm{End}(\mathcal{F}))$-bimodules; the two induced $\mathcal{O}_E$-module structures agree.*

2. *The natural map $\mathrm{RigExt}(\mathcal{F}, \mathcal{F}') \to \mathrm{Ext}(\mathcal{F}, \mathcal{F}')$ fits into an exact sequence of Abelian groups, functorial in $\mathcal{F}$ and $\mathcal{F}'$*

$$\mathrm{Hom}(\mathcal{F}, \mathcal{F}') \xrightarrow{\mathrm{Lie}} \mathrm{Hom}_{(R\text{-}\mathrm{Mod})}(\mathrm{Lie}(\mathcal{F}), \mathrm{Lie}(\mathcal{F}')) \to \mathrm{RigExt}(\mathcal{F}, \mathcal{F}') \twoheadrightarrow \mathrm{Ext}(\mathcal{F}, \mathcal{F}').$$

We remark that the methods developed in this section can also be used to prove that the category $(\mathcal{O}_E\text{-}\mathrm{FM}^{\mathrm{arb}}/R)$, together with the definition of exact sequences, is an exact category (in the sense of Quillen, cf. [Kel90]).

The main tool for studying extensions is the following Lemma.

Lemma 1.8.5. *Let $F, F' \in (A\text{-}\mathrm{FML}^{\mathrm{arb}}/R)$ be formal A-module laws of dimensions m and n, respectively. Write $\mathcal{F}'$ and $\mathcal{F}$ for their associated formal $\mathcal{O}_E$-modules, and suppose that they fit into a exact sequence*

$$0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{F} \to 0.$$

Write $\boldsymbol{X}$ for the variables of F' and $\boldsymbol{Z}$ for those of F. Then there exists a (non-canonical) coordinate on $\mathcal{G}$ giving rise to a formal $\mathcal{O}_E$-module law G in the variables $(\boldsymbol{X}, \boldsymbol{Z})$ such that the induced morphisms of formal module laws are of the form $\alpha(\boldsymbol{X}) = (\boldsymbol{X}, 0)$, $\beta(\boldsymbol{X}, \boldsymbol{Z}) = \boldsymbol{Z}$. Furthermore, the formal $\mathcal{O}_E$-module law G is of the form

$$G((\boldsymbol{X}_1, \boldsymbol{Z}_1), (\boldsymbol{X}_2, \boldsymbol{Z}_2)) = (F'(\boldsymbol{X}_1, \boldsymbol{X}_2) +_{F'} \Delta(\boldsymbol{Z}_1, \boldsymbol{Z}_2), F(\boldsymbol{Z}_1, \boldsymbol{Z}_2))$$

$$and \tag{1.9}$$

$$[a]_G(\boldsymbol{X}, \boldsymbol{Z}) = ([a]_{F'}(\boldsymbol{X}) +_{F'} \delta_a(\boldsymbol{Z}), [a]_F(\boldsymbol{Z})).$$

for some m-tuple of power series $\Delta \in (R[\![\boldsymbol{Z}_1, \boldsymbol{Z}_2]\!])^m$, $\delta_a \in (R[\![\boldsymbol{Z}]\!])^m$.

Proof. The construction of G is sketched in [GH94, Proposition 6.5]. We know that $\mathcal{G} \cong \mathrm{Spf}(R[\![M]\!])$ for some free R-module M of rank $m + n$. As we have a short exact sequence on

Lie-algebras, we may apply the formal implicit function theorem to obtain a section $\sigma\colon \mathcal{F}\to\mathcal{G}$ of $\beta\colon \mathcal{G}\to\mathcal{F}$. The datum of the morphisms α and σ is equivalent to morphisms

$$\alpha^\flat\colon R[\![M]\!]\to R[\![\boldsymbol{X}]\!] \quad\text{and}\quad \sigma^\flat\colon R[\![M]\!]\to R[\![\boldsymbol{Z}]\!]$$

on affines. Taking their sum, we obtain a morphism $R[\![M]\!]\to R[\![\boldsymbol{X},\boldsymbol{T}]\!]$. On Lie-algebras, this morphism recovers the isomorphism $\mathrm{Lie}(\mathcal{G})\cong\mathrm{Lie}(\mathcal{F}')\oplus\mathrm{Lie}(\mathcal{F})$ induced by $\mathrm{Lie}(\sigma)$. In particular, $\sigma^\flat+\alpha^\flat$ is an isomorphism in degree 1, hence an isomorphism. This yields the desired coordinate $\mathcal{G}\cong\mathrm{Spf}(R[\![\boldsymbol{X},\boldsymbol{Z}]\!])$, which gives rise to the formal $\mathcal{O}_E$-module law G. The fact about the structure of G follows quickly from the fact that α and β are morphisms of formal $\mathcal{O}_E$-module laws. $\qquad\square$

We further investigate the power series $(\Delta,(\delta_a)_{a\in\mathcal{O}_E})$ appearing in the Lemma 1.8.5. They satisfy certain conditions.

Definition 1.8.6 (Symmetric 2-cocycles)**.** Let $\mathrm{SymCoc}^2(F,F')$ be the set of collections of power series $(\Delta,(\delta_a)_{a\in\mathcal{O}_E})$ satisfying the following properties

- $\Delta(\boldsymbol{Z}_1,\boldsymbol{Z}_2)=\Delta(\boldsymbol{Z}_2,\boldsymbol{Z}_1)$

- $\Delta(\boldsymbol{Z}_2,\boldsymbol{Z}_3)+_{F'}\Delta(\boldsymbol{Z}_1,F(\boldsymbol{Z}_2,\boldsymbol{Z}_3))=\Delta(F(\boldsymbol{Z}_1,\boldsymbol{Z}_2),\boldsymbol{Z}_3)+_{F'}\Delta(\boldsymbol{Z}_1,\boldsymbol{Z}_2)$

- $\delta_a(\boldsymbol{Z}_1)+_{F'}\delta_a(\boldsymbol{Z}_2)+_{F'}\Delta([a]_F(\boldsymbol{Z}_1),[a]_F(\boldsymbol{Z}_2))=[a]_{F'}\Delta(\boldsymbol{Z}_1,\boldsymbol{Z}_2)+_{F'}\delta_a(F(\boldsymbol{Z}_1,\boldsymbol{Z}_2))$

- $\delta_a(\boldsymbol{Z}_1)+_{F'}\delta_b(\boldsymbol{Z}_1)+_{F'}\Delta([a]_F(\boldsymbol{Z}_1),[b]_F(\boldsymbol{Z}_1))=\delta_{a+b}(\boldsymbol{Z}_1)$

- $[a]_{F'}\delta_b(\boldsymbol{Z}_1)+_{F'}\delta_a([b]_F(\boldsymbol{Z}_1))=\delta_{ab}(\boldsymbol{Z}_1)$.

These objects are called *symmetric 2-cocycles*.

Proposition 1.8.7. *There is a bijection*

$$\mathrm{SymCoc}^2(F,F')\xrightarrow{\sim}\left\{\begin{array}{c}\mathcal{O}_E\text{-module laws } G \text{ in variables } (\boldsymbol{X},\boldsymbol{Z}) \text{ fitting into an exact sequence}\\[4pt] 0\to F'\xrightarrow{\alpha}G\xrightarrow{\beta}F\to 0\\[4pt] \text{where }\alpha(\boldsymbol{X})=(\boldsymbol{X},0)\text{ and }\beta(\boldsymbol{X},\boldsymbol{Z})=\boldsymbol{Z}.\end{array}\right\}$$

The map sends a pair $\{\Delta,(\delta_a)_a\}$ to the $\mathcal{O}_E$-module law with structure defined following (1.9).

Proof. This is only a matter of calculation, cf. [GH94, Section 6]. $\qquad\square$

For the moment, let us fix a short exact sequence

$$0\to F'\xrightarrow{\alpha}G\xrightarrow{\beta}F\to 0,$$

corresponding to a symmetric 2-cocycle $(\Delta,(\delta_a)_{a\in\mathcal{O}_E})$. Given a morphism $f\colon H\to F$ from a formal module law $H\in(\mathcal{O}_E\text{-FML}^{\mathrm{arb}}/R)$, we define the formal $\mathcal{O}_E$-module law f^*G via

$$f^*G((\boldsymbol{X}_1,\boldsymbol{Y}_1),(\boldsymbol{X}_2,\boldsymbol{Y}_2))=(F'(\boldsymbol{X}_1,\boldsymbol{X}_2)+_{F'}\Delta(f(\boldsymbol{Y}_1),f(\boldsymbol{Y}_2)),G(\boldsymbol{Y}_1,\boldsymbol{Y}_2))$$

$$\text{and}$$

$$[a]_{f^*G}(\boldsymbol{X},\boldsymbol{Y})=([a]_{F'}(\boldsymbol{X})+_{F'}\delta_a(f(\boldsymbol{Y})),[a]_F(\boldsymbol{Y})).$$

Now, setting $\alpha'(\boldsymbol{X}) = (\boldsymbol{X}, 0)$ and $\beta'(\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{Y}$, the formal $\mathcal{O}_E$-module law f^*G fits into a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F' & \xrightarrow{\alpha'} & f^*G & \xrightarrow{\beta'} & H & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & F' & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & F & \longrightarrow & 0.
\end{array}
$$

The morphism of A-module laws $f^*G \to G$ is given by $(\boldsymbol{X}, \boldsymbol{Y}) \mapsto (\boldsymbol{X}, f(\boldsymbol{Y}))$. Furthermore, it is routine to check that the diagram

$$
\begin{array}{ccc}
f^*G & \xrightarrow{\beta'} & H \\
\downarrow & & \downarrow{\scriptstyle f} \\
G & \xrightarrow{\beta} & F
\end{array}
$$

is cartesian in the category of formal $\mathcal{O}_E$-module laws over R. Hence, this construction is independent from the choice of coordinate an descents to the category of formal $\mathcal{O}_E$-modules. Dually, given a morphism $g \colon G \to H'$, we may define g_*G via

$$
g_*G((\boldsymbol{Y}_1, \boldsymbol{Z}_1), (\boldsymbol{Y}_2, \boldsymbol{Z}_2)) = (H'(\boldsymbol{Y}_1, \boldsymbol{Y}_2) +_{H'} g(\Delta(\boldsymbol{Z}_1, \boldsymbol{Z}_2)), F(\boldsymbol{Z}_1, \boldsymbol{Z}_2))
$$

$$
\text{and}
$$

$$
[a]_{g_*G}(\boldsymbol{X}, \boldsymbol{Y}) = ([a]_{H'}(\boldsymbol{X}) +_{H'} g(\delta_a(\boldsymbol{Z})), [a]_F(\boldsymbol{Z})).
$$

There is a morphism $G \to g_*G$ is given by $(\boldsymbol{X}, \boldsymbol{Z}) \mapsto (g(\boldsymbol{X}), \boldsymbol{Z})$, that makes the evident diagram commute. Furthermore, the resulting square is co-cartesian, which allows for a functorial description of g_*G. This shows that the assignment $(F, F') \to \mathrm{SymCoc}(F, F')$ is functorial. Another consequence is the following lemma.

Lemma 1.8.8. *Let* $0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{F} \to 0$ *be a short exact sequence in* $(\mathcal{O}_E\text{-}\mathrm{FML}^{\mathrm{arb}}/R)$. *Given morphisms* $f \in \mathrm{Hom}_{(\mathcal{O}_E\text{-}\mathrm{FM}^{\mathrm{arb}}/R)}(\mathcal{H}, \mathcal{F})$ *and* $g \in \mathrm{Hom}_{(\mathcal{O}_E\text{-}\mathrm{FM}^{\mathrm{arb}}/R)}(\mathcal{F}', \mathcal{H}')$ *there is a formal* $\mathcal{O}_E$-*module* $f^*\mathcal{G}$ *and a formal* $\mathcal{O}_E$-*module* $g_*\mathcal{G}$ *fitting into a commutative diagram with short exact sequences as rows*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}' & \longrightarrow & f^*\mathcal{G} & \longrightarrow & \mathcal{H} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & \mathcal{F}' & \xrightarrow{\alpha} & \mathcal{G} & \xrightarrow{\beta} & \mathcal{F} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{H}' & \longrightarrow & g_*\mathcal{G} & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

The formal $\mathcal{O}_E$-*modules* $f^*\mathcal{G}$ *and* $g_*\mathcal{G}$ *are uniquely determined by the property that the upper right square is cartesian, and the lower left square is co-cartesian (in the category of formal* $\mathcal{O}_E$-*modules).*

In particular, this shows the functoriality result about $\mathrm{Ext}(-, -)$. Functoriality of RigExt follows quickly from the following result.

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Lemma 1.8.9. *We have natural isomorpisms*

$$\mathrm{Lie}(f^*\mathcal{G}) \cong \mathrm{Lie}(\mathcal{G}) \times_{\mathrm{Lie}(\mathcal{F})} \mathrm{Lie}(\mathcal{H}) \quad and \quad \mathrm{Lie}(g_*\mathcal{G}) \cong \mathrm{Lie}(\mathcal{H}') \sqcup_{\mathrm{Lie}(\mathcal{F}')} \mathrm{Lie}(\mathcal{G}).$$

Proof. This is true independently of the choice of coordinates. □

Lemma 1.8.10. *The assignment $(\mathcal{F}, \mathcal{F}') \mapsto \mathrm{RigExt}(\mathcal{F}, \mathcal{F}')$ is functorial in both entries (contravariant in the first, covariant in the second).*

Proof. Given a morphism $f \colon \mathcal{G} \to \mathcal{F}$, the induced morphism $\mathrm{RigExt}(\mathcal{F}, \mathcal{F}') \to \mathrm{RigExt}(\mathcal{G}, \mathcal{F}')$ is given by sending the pair $(\mathcal{E}, s)$ to the pair $(f^*\mathcal{E}, s')$, where

$$s' \colon \mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(f^*\mathcal{E}) \cong \mathrm{Lie}(\mathcal{E}) \times_{\mathrm{Lie}(\mathcal{F})} \mathrm{Lie}(\mathcal{G}), \quad x \mapsto ((s \circ \mathrm{Lie}(f))(x), x).$$

Here we used the description of $\mathrm{Lie}(f^*\mathcal{E})$ from Lemma 1.8.9. Similarly, given a morphism $g \colon \mathcal{F}' \to \mathcal{G}'$, the induced morphism $\mathrm{RigExt}(\mathcal{F}, \mathcal{F}') \to \mathrm{RigExt}(\mathcal{F}, \mathcal{G}')$ sends $(\mathcal{E}, s)$ to $(g_*\mathcal{E}, \mathrm{Lie}(g') \circ s)$, where $g' \colon \mathcal{E} \to g_*\mathcal{E}$ is the canonical morphism. □

Thus, we have shown the first part of Theorem 1.8.4. The second part is the following.

Proposition 1.8.11. *There is an exact sequence of Abelian groups, functorial in $\mathcal{F}$ and $\mathcal{F}'$*

$$\mathrm{Hom}(\mathcal{F}, \mathcal{F}') \xrightarrow{\mathrm{Lie}} \mathrm{Hom}_{(R\text{-Mod})}(\mathrm{Lie}(\mathcal{F}), \mathrm{Lie}(\mathcal{F}')) \to \mathrm{RigExt}(\mathcal{F}, \mathcal{F}') \twoheadrightarrow \mathrm{Ext}(\mathcal{F}, \mathcal{F}').$$

Proof. The kernel of $\mathrm{RigExt}(\mathcal{F}, \mathcal{F}') \to \mathrm{Ext}(\mathcal{F}, \mathcal{F}')$ is given (up to equivalence) by pairs of the form $(\mathcal{F}' \oplus \mathcal{F}, s)$, where s is a morphism of R-modules such that

$$\mathrm{Lie}(\mathcal{F}) \xrightarrow{s} \mathrm{Lie}(\mathcal{F}') \oplus \mathrm{Lie}(\mathcal{F}) \to \mathrm{Lie}(\mathcal{F})$$

is the identity. It is clear that these morphisms s correspond to morphisms $\mathrm{Lie}(\mathcal{F}) \to \mathrm{Lie}(\mathcal{F}')$. For any $\lambda \colon \mathrm{Lie}(\mathcal{F}) \to \mathrm{Lie}(\mathcal{F}')$, denote by $s_\lambda \colon \mathrm{Lie}(\mathcal{F}) \to \mathrm{Lie}(\mathcal{F}') \oplus \mathrm{Lie}(\mathcal{F})$ the splitting given by $y \mapsto (\lambda(y), y)$. The kernel of $\mathrm{Hom}_{(R\text{-Mod})}(\mathrm{Lie}(\mathcal{F}), \mathrm{Lie}(\mathcal{F}')) \to \mathrm{RigExt}(\mathcal{F}, \mathcal{F}')$ is spanned by those $\lambda \colon \mathrm{Lie}(\mathcal{F}) \to \mathrm{Lie}(\mathcal{F}')$ such that $(\mathcal{F}' \oplus \mathcal{F}, s_\lambda)$ is in the same equivalence class as $(\mathcal{F}' \oplus \mathcal{F}, s_{\mathrm{triv}})$. Any such pair fits into a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F}' \oplus \mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \psi} & & \| & & \\
0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F}' \oplus \mathcal{F} & \xrightarrow{\beta} & \mathcal{F} & \longrightarrow & 0,
\end{array}
$$

where ψ is an isomorphism satisfying $\mathrm{Lie}(\psi)(x, y) = (x + \lambda(y), y)$. This shows that the composition

$$\mathrm{Hom}_{(\mathcal{O}_E\text{-FM}^{\mathrm{arb}}/R)}(\mathcal{F}, \mathcal{F}') \xrightarrow{\mathrm{Lie}} \mathrm{Hom}_{(R\text{-Mod})}(\mathrm{Lie}(\mathcal{F}), \mathrm{Lie}(\mathcal{F}')) \to \mathrm{RigExt}(\mathcal{F}, \mathcal{F}')$$

is the zero-morphism. Indeed, given $g \in \mathrm{Hom}_{(\mathcal{O}_E\text{-FM}^{\mathrm{arb}}/R)}(\mathcal{F}, \mathcal{F}')$, we may choose ψ as the map $(x, y) \mapsto (x +_{\mathcal{F}'} g(y), y)$. Conversely, any morphism ψ fitting into the diagram above is necessarily of this form. This proves exactness on the left. □

1.9 Explicit Dieudonné Theory

Let $\mathcal{F}$ and $\mathcal{F}'$ be formal $\mathcal{O}_E$-modules of dimension m and n respectively, over a local $\mathcal{O}_E$-algebra $R \in \mathcal{C}$, with coordinates $\mathcal{F} \cong \mathrm{FG}(F)$ and $\mathcal{F}' \cong \mathrm{FG}(F')$ for formal $\mathcal{O}_E$-module laws $F, F' \in (\mathcal{O}_E\text{-FML}^{\mathrm{arb}}/R)$. We give an explicit description of $\mathrm{Ext}(\mathcal{F}, \mathcal{F}')$ in terms of the Symmetric 2-Cocycles associated with F and F' (cf. Definition 1.8.6). We also give a related explicit description of $\mathrm{RigExt}(F, \widehat{\mathbb{G}}_a)$ in terms of Quasi-Logarithms, cf. Definition 1.9.3. Write $\boldsymbol{X}$ for the variables of F' and $\boldsymbol{Z}$ for the variables of F.

Definition 1.9.1 (Symmetric 1-Cochain). A *symmetric 1-cochain* for the pair (F, F') is an n-tuple of power series $g = (g_1, \ldots, g_m)$, such that $g_i(\boldsymbol{Z}) \in R[\![\boldsymbol{Z}]\!]$ satisfying $g_i(0) = 0$ for all i. We write δg for the coboundary of g, that is, the pair $(\Delta g, (\delta_a g)_{a \in A})$, where

$$\Delta g = g(\boldsymbol{Z}_1) -_{F'} g(F(\boldsymbol{Z}_1, \boldsymbol{Z}_2)) +_{F'} g(\boldsymbol{Z}_2) \in (R[\![\boldsymbol{Z}_1, \boldsymbol{Z}_2]\!])^m$$

and

$$\delta_a g = [a]_{F'} g(\boldsymbol{Z}) -_{F'} g([a]_F(\boldsymbol{Z})) \in (R[\![\boldsymbol{Z}]\!])^m.$$

One readily checks that $\delta g \in \mathrm{SymCoc}^2(F, F')$.

Proposition 1.9.2. *Given two extensions $\mathcal{G}, \mathcal{G}' \in \mathrm{Ext}(\mathcal{F}, \mathcal{F}')$, write G and G' for two choices of respective formal $\mathcal{O}_E$-module laws coming from Lemma 1.8.5, and write Δ_G and $\Delta_{G'}$ for the associated symmetric 2-cocycles (cf. Proposition 1.8.7). There is a bijection*

$$\{g \in (R[\![\boldsymbol{Z}]\!])^m \mid g(0) = 0 \text{ and } \delta g = \Delta_{G'} - \Delta_G\} \xrightarrow{\sim} \{\text{Isomorphisms of extensions } G \to G'\}.$$

Explicitly, this bijection is given by sending g to the morphism $i_g \in \mathrm{Hom}_{(\mathcal{O}_E\text{-FML}^{\mathrm{arb}}/R)}(G, G')$, given by $i_g(\boldsymbol{X}, \boldsymbol{Z}) = (\boldsymbol{X} +_{F'} g(\boldsymbol{Z}), \boldsymbol{Z})$. In particular, there is a bijection

$$\mathrm{Ext}(\mathcal{F}, \mathcal{F}') \cong \frac{\mathrm{SymCoc}^2(F, F')}{\{\delta g \mid g \in (R[\![\boldsymbol{Z}]\!])^m \text{ with } g(0) = 0\}}.$$

This bijection is an isomorphism of left $\mathrm{End}(F')$-modules.

For now, this finishes the study of $\mathrm{Ext}(\mathcal{F}, \mathcal{F}')$.

Assume now that $\mathcal{F}' = \widehat{\mathbb{G}}_a$, and that $\mathcal{F}$ comes from a one-dimensional formal $\mathcal{O}_E$-module $F \in (\mathcal{O}_E\text{-FML}/R)$. For the remainder of this subsection, we will be concerned with the R-module $\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a)$. The notion of Quasi-Logarithms will play a major role.

Definition 1.9.3 (Quasi-Logarithms). A power series $g(T) \in (R \otimes_{\mathcal{O}_E} E)[\![T]\!]$ is called a Quasi-Logarithm for F, if $g(0) = 0$ and $g'(T)$, as well as all of the power series appearing in δg (with $F' = \widehat{\mathbb{G}}_a$, cf. Definition 1.9.1) have coefficients in R. We define the R-module

$$\mathrm{QLog}(F) = \frac{\{g(T) \in (R \otimes_{\mathcal{O}_E} E)[\![T]\!] \mid g \text{ is a quasi-logarithm for } F\}}{\{g(T) \in R[\![T]\!] \mid g(0) = 0\}}$$

Let $(\mathcal{G}, s) \in \mathrm{RigExt}(F, \widehat{\mathbb{G}}_a)$ be a rigidified extension. The splitting s yields an isomorphism $\omega(\mathcal{G}) \cong \omega(\widehat{\mathbb{G}}_a) \oplus \omega(\mathcal{F})$ on duals, giving an invariant differential $\omega_{\mathcal{G}} \in \omega(\mathcal{G})$ pulling back to

dX on $\widehat{\mathbb{G}}_a$. Conversely, any such invariant differential $\omega_{\mathcal{G}}$ yields a splitting, so the choice of s is equivalent to the choice of $\omega_{\mathcal{G}}$, and we will henceforth write $(\mathcal{G}, \omega_{\mathcal{G}}) \in \mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a)$.

Theorem 1.9.4 (Classification of Rigidified Extensions in terms of Quasi-Logarithms). *There is a bijection*

$$\{Quasi\text{-}logarithms \text{ for } F\} \xrightarrow{\sim} \left\{ \begin{array}{c} Pairs \ (G, \omega_G), \ where \ G \ is \ an \ A\text{-}module \ law \\ fitting \ into \ an \ exact \ sequence \\ 0 \to \widehat{\mathbb{G}}_a \xrightarrow{\alpha} G \xrightarrow{\beta} F \to 0 \\ with \ \alpha(X) = (X, 0) \ and \ \beta(X, T) = T \ and \ \omega_G \\ is \ an \ invariant \ differential \ on \ G \ with \ \alpha^* \omega_G = \mathrm{d}X. \end{array} \right\} \quad (1.10)$$

The map sends any quasi-logarithm $g(T) \in (R \otimes_{\mathcal{O}_E} E)[\![T]\!]$ to the pair $(G_{\delta g}, \mathrm{d}(X + g(T)) \in \mathrm{RigExt}(F, \widehat{\mathbb{G}}_a)$. Here $G_{\delta g} \in \mathrm{Ext}(F, \widehat{\mathbb{G}}_a)$ is the extension corresponding to $\delta g \in \mathrm{SymCoc}^2(F, \widehat{\mathbb{G}}_a)$. Furthermore, given two rigidified extensions (G, ω_G) and (D, ω_D) with associated quasi-logarithms $g(T)$ and $h(T)$, there is a (unique) isomorphism $(G, \omega_G) \to (D, \omega_D)$ if and only if $h(T) - g(T) =: f(T)$ has coefficients in $R[\![T]\!]$. In this case, the isomorphism $i_f(X, T) \in \mathrm{Hom}(G, D)$ is given by $i_f(X, T) = (X + f(T), T)$. In particular, there is a canonical bijection

$$\mathrm{QLog}(F) \xrightarrow{\sim} \mathrm{RigExt}(F, \widehat{\mathbb{G}}_a).$$

This bijection is an isomorphism of R-modules.

Proof. We construct an inverse of the map in (1.10). Let (G, ω_G) be an element of the set on the right and let $(\Delta, (\delta_a)_{a \in A}) \in \mathrm{SymCoc}^2(F, \widehat{\mathbb{G}}_a)$ be the symmetric 2-cochain corresponding to G. Following Proposition 1.2.3, the datum of $\omega_G \in \omega(G)$ is equivalent to a morphism

$$f_G \in \mathrm{Hom}_{(A\text{-}\mathrm{FML}/R \otimes E)}(G \otimes_R (R \otimes_{\mathcal{O}_E} E), \widehat{\mathbb{G}}_a) \quad \text{satisfying} \quad f_G(X, T) = X + g(T)$$

for some $g(T) \in (R \otimes_{\mathcal{O}_E} E)[\![T]\!]$. The fact that f_G is a homomorphism implies that

$$X_1 + X_2 + \Delta(T_1, T_2) + g(F(T_1, T_2)) = f_G(G((X_1, T_1), (X_2, T_2))) =$$
$$= f_G(X_1, T_1) + f_G(X_2, T_2)) = X_1 + g(T_1) + X_2 + g(T_2),$$

thereby $\Delta g = \Delta(T_1, T_2) \in R[\![T_1, T_2]\!]$. Similarly, we find $\delta_a g = \delta_a \in R[\![T]\!]$. Hence, $g(T)$ is a quasi-logarithm with $\delta g = (\Delta, (\delta_a)_a)$. This construction yields the desired inverse. The remaining statements are verified directly, also cf. [GH94, Section 8]. $\qquad\square$

Proposition 1.9.5. *Let $\mathcal{F}$ be a one-dimensional formal $\mathcal{O}_E$-module law over a flat, local $\mathcal{O}_E$-algebra R, and suppose that $\mathcal{F}' = \widehat{\mathbb{G}}_a$. The short exact sequence of Proposition 1.8.11 fits into a commutative diagram with exact rows and vertical maps isomorphisms induced by any*

choice of coordinate $\mathcal{F} = \mathrm{FG}(F)$.

$$
\begin{array}{ccccccc}
\mathrm{Hom}(\mathcal{F}, \widehat{\mathbb{G}}_a) & \xhookleftarrow{\quad \mathrm{d}_F \quad} & \omega(\mathcal{F}) & \longrightarrow & \mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a) & \twoheadrightarrow & \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\left\{ \begin{array}{c} f \in TR[\![T]\!]: \\ \delta f = 0 \end{array} \right\} & \hookrightarrow & \left\{ \begin{array}{c} f \in (R \otimes_{\mathcal{O}_E} E)[\![T]\!]: \\ \delta f = 0,\ f(0) = 0 \\ \text{and } f'(T) \in R[\![T]\!] \end{array} \right\} & \longrightarrow & \mathrm{QLog}(F) & \xrightarrow{\ \delta\ } & \frac{\mathrm{SymCoc}^2(F, \widehat{\mathbb{G}}_a)}{\{\delta g \mid g \in TR[\![T]\!]\}}
\end{array}
$$

Proof. Injectivity of d_F is provided by Proposition 1.2.3, and related to the original exact sequence as $\mathrm{Hom}_R(\mathrm{Lie}(\mathcal{F}), \mathrm{Lie}(\widehat{\mathbb{G}}_a)) = \omega(\mathcal{F})$. Surjectivity of $\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a) \to \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a)$ comes from the fact that $\mathrm{Lie}(\mathcal{F})$ is projective. The first vertical map is an isomorphism by definition. The vertical arrow describing $\omega(F)$ is obtained by identifying the preimage of $\omega(F) \subseteq \omega(F \otimes_R (R \otimes_{\mathcal{O}_E} E))$ under the isomorphism

$$
\{ f \in T(R \otimes_{\mathcal{O}_E} E)[\![T]\!] \mid \delta f = 0 \} = \mathrm{Hom}_{(\mathcal{O}_E\text{-FML}/R[\frac{1}{\varpi}])}(F \otimes (R \otimes_{\mathcal{O}_E} E), \widehat{\mathbb{G}}_a) \xrightarrow{\mathrm{d}_F} \omega(F \otimes_R (R \otimes_{\mathcal{O}_E} E)).
$$

All squares commute by construction. $\qquad\square$

We admit the following facts from Section 9 of [GH94].

Proposition 1.9.6. *Let $\mathcal{F}$ be a formal $\mathcal{O}_E$-module of height h over $R \in \mathcal{C}$. Then $\mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a)$ is a free R-module of rank $n - 1$, and $\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a)$ is a free R-module of rank n.*

Proof. This is Proposition 9.8 in [GH94]. $\qquad\square$

In the case that R is flat over $\mathcal{O}_E$, and $\mathcal{F}$ comes from a $\mathcal{O}_E$-typical formal module, [GH94, Section 9] also gives an explicit description of a basis for the modules in question. If $\mathcal{F}$ is isomorphic to the standard formal $\mathcal{O}_E$-module H of height n, this allows an explicit description of the exact sequence in Proposition 1.8.11. There is no non-trivial map $H \to \widehat{\mathbb{G}}_a$, so the sequence becomes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \omega(H) & \longrightarrow & \mathrm{RigExt}(H, \widehat{\mathbb{G}}_a) & \longrightarrow & \mathrm{Ext}(H, \widehat{\mathbb{G}}_a) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \left\{ \begin{array}{c} g \in TE[\![T]\!]: \delta g = 0 \\ \text{and } g'(T) \in \mathcal{O}_E[\![T]\!] \end{array} \right\} & \longrightarrow & \mathrm{QLog}(H) & \xrightarrow{\ \delta\ } & \frac{\mathrm{SymCoc}^2(H, \widehat{\mathbb{G}}_a)}{\{\delta g \mid g \in T\mathcal{O}_E[\![T]\!]\}} & \longrightarrow & 0.
\end{array}
$$

Now [GH94, Proposition 9.8] implies the following result.

Proposition 1.9.7. *The $\mathcal{O}_{\breve{E}}$-module $\omega(H)$ is free of rank 1, generated by $f(T) = \log_H(T)$. $\mathrm{QLog}(H)$ is free of rank n, generated by the classes of $(f(T), \frac{1}{\varpi}f(T^q), \ldots, \frac{1}{\varpi}f(T^{q^{n-1}}))$. Consequently, the short exact sequence above is given by*

$$
0 \to \langle f(T) \rangle \to \left\langle f(T), \frac{1}{\varpi}f(T^q), \ldots, \frac{1}{\varpi}f(T^{q^{n-1}}) \right\rangle \xrightarrow{\ \delta\ } \left\langle \delta\left(\frac{1}{\varpi}f(T^q)\right), \ldots, \delta\left(\frac{1}{\varpi}f(T^{q^{n-1}})\right) \right\rangle \to 0.
$$

We remark that the functors $\mathrm{Ext}_R(-, \widehat{\mathbb{G}}_a)$ and $\mathrm{RigExt}_R(-, \widehat{\mathbb{G}}_a)$ are also functorial in R.

Lemma 1.9.8. *If $R \to R'$ is a homomorphism of local $\mathcal{O}_E$-algebras, the induced maps of free R'-modules*

$$\mathrm{Ext}_R(\mathcal{F}, \widehat{\mathbb{G}}_a) \otimes_R R' \to \mathrm{Ext}_{R'}(\mathcal{F} \otimes R', \widehat{\mathbb{G}}_a)$$
$$\mathrm{RigExt}_R(\mathcal{F}, \widehat{\mathbb{G}}_a) \otimes_R R' \to \mathrm{RigExt}_{R'}(\mathcal{F} \otimes R', \widehat{\mathbb{G}}_a)$$

are isomorphisms.

Proof. [GH94, Corollary 9.13]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

By the results of [De 95], the assignment $\mathcal{F} \mapsto \mathrm{D}(\mathcal{F})$ is crystalline, that is, it only depends on the reduction of $\mathcal{F}$ modulo ϖ.

1.10 The Universal Additive Extension

We follow [GH94, Section 11]. Let $R \in \mathcal{C}$ be a local $\mathcal{O}_E$-algebra.

Lemma 1.10.1. *Let M be a finite free module over R. Then there is a natural bijection, functorial in M and $\mathcal{F}$*

$$\mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a \otimes M) \cong \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a) \otimes_R M.$$

Proof. After choosing coordinates on $\mathcal{F}$, this follows directly from the description of Ext in terms of symmetric 2-cocycles, cf. Propositions 1.8.7 and 1.9.2. $\qquad\qquad\qquad$ $\square$

Let $\mathcal{F}$ be a one-dimensional formal $\mathcal{O}_E$-module over R. We put $\mathrm{M}(\mathcal{F}) := \mathrm{Hom}_R(\mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a), R)$, which is free of rank $n-1$, and write $\mathcal{V} = \widehat{\mathbb{G}}_a \otimes \mathrm{M}(\mathcal{F})$. Now, by the previous lemma,

$$\mathrm{Ext}(\mathcal{F}, \mathcal{V}) = \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a \otimes \mathrm{M}(\mathcal{F})) = \mathrm{End}_R(\mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a)).$$

Let $0 \to \mathcal{V} \to \mathcal{U} \to \mathcal{F} \to 0$ be the extension corresponding to the identity on the right. This class is unique up to unique isomorphism. Indeed, as R is a local ring we may choose formal module laws F and V giving rise to $\mathcal{F}$ and $\mathcal{V}$, and let E be the module law obtained from Lemma 1.8.5. If $0 \to V \to E' \to F \to 0$ is another extension in this class, we have by construction a commutative square

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V & \longrightarrow & U & \longrightarrow & F & \longrightarrow & 0 \\
& & \| & & \downarrow{\scriptstyle i} & & \| & & \\
0 & \longrightarrow & V & \longrightarrow & U' & \longrightarrow & F & \longrightarrow & 0,
\end{array}
$$

and by Proposition 1.9.2 we see that any other isomorphism i' making the diagram above commute differs from i by an element in $\mathrm{Hom}(F, V) = 0$.

Definition 1.10.2 (Universal Additive Extension)**.** The extension

$$0 \to \mathcal{V} \to \mathcal{U} \to \mathcal{F} \to 0$$

constructed above is called the *universal additive extension* of $\mathcal{F}$.

Proposition 1.10.3. *If N is a finite, free R-module, $\mathcal{G}' = \widehat{\mathbb{G}}_a \otimes N$ and*

$$0 \to \mathcal{G}' \to \mathcal{U}' \to F \to 0$$

is an extension of $\mathcal{F}$ by $\mathcal{G}'$, there are unique homomorphisms $i \colon \mathcal{U} \to \mathcal{U}'$ and $g' \colon \mathcal{V} \to \mathcal{G}'$ making the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{U} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & {\scriptstyle g'}\downarrow & & {\scriptstyle i}\downarrow & & \parallel & & \\
0 & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{U}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

commute. In particular, we have $\mathcal{U}' = g'_* \mathcal{U}$.

Proof. As $\mathcal{V}$ and $\mathcal{G}'$ are additive, we have

$$\operatorname{Hom}(\mathcal{V}, \mathcal{G}') = \operatorname{Hom}_R(\mathrm{M}(\mathcal{F}), N) = \operatorname{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a) \otimes N = \operatorname{Ext}(\mathcal{F}, \mathcal{G}').$$

This yields g'. Again, i is unique as by observations similar to Proposition 1.9.2, the difference of two morphisms $i, i' \colon \mathcal{U} \to \mathcal{U}'$ is given a morphism $\mathcal{F} \to \mathcal{G}'$, which has to be trivial. $\qquad\square$

Lemma 1.10.4. *There is a natural isomorphism $\operatorname{Lie}(\mathcal{U}) \xrightarrow{\sim} \operatorname{Hom}(\operatorname{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a), R) = \mathrm{D}(\mathcal{F})$.*

Proof. We show the equivalent statement $\omega(\mathcal{U}) = \operatorname{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a)$. Let $(\mathcal{U}', \omega_{\mathcal{U}'}) \in \operatorname{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a)$. Then by universality of $\mathcal{U}$, we obtain a unique homomorphism $i \colon \mathcal{U} \to \mathcal{U}'$. This yields a homomorphism of R-modules $\operatorname{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a) \to \omega(\mathcal{U})$, sending a pair $(\mathcal{U}', \omega_{\mathcal{U}'})$ to $i^* \omega_{\mathcal{U}'}$. This morphism fits into the following commutative diagram, where the top row is the short exact sequence from Proposition 1.9.5 and the bottom row is the dual short exact sequence of $0 \to \operatorname{Lie}(\mathcal{V}) \to \operatorname{Lie}(\mathcal{U}) \to \operatorname{Lie}(\mathcal{F}) \to 0$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \omega(\mathcal{F}) & \longrightarrow & \operatorname{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a) & \longrightarrow & \operatorname{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a) & \longrightarrow & 0 \\
& & \parallel & & \downarrow & & \parallel & & \\
0 & \longrightarrow & \omega(\mathcal{F}) & \longrightarrow & \omega(\mathcal{U}) & \longrightarrow & \omega(\mathcal{V}) & \longrightarrow & 0
\end{array}
$$

Thereby, $\operatorname{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a) \to \omega(\mathcal{U})$ is a natural isomorphism. $\qquad\square$

1.11 The Universal Cover

Let $R \in \mathcal{C}$ and let $\mathcal{F}$ be a formal ϖ-divisible $\mathcal{O}_E$-module over R of height n.

Definition 1.11.1 (The Universal Cover). We denote by $\tilde{\mathcal{F}}$ the *universal cover* of $\mathcal{F}$, which we define as the functor

$$\tilde{\mathcal{F}} \colon (R\text{-Adm}) \to (E\text{-Vec}), \quad S \mapsto \left\{ (x_1, x_2, \dots) \in \prod_{\mathbb{N}} \mathcal{F}(S) \mid \varpi(x_{i+1}) = x_i \right\}.$$

Note that for $S \in (R\text{-Adm})$, the set $\tilde{\mathcal{F}}(S)$ has a natural $\mathcal{O}_E$-module structure with multiplication by ϖ an isomorphism, making it a E-vector space.

We remark that the Tate module

$$T_\varpi \mathcal{F} \colon (R\text{-Adm}) \to (\mathcal{O}_E\text{-Mod}), \quad S \mapsto \{(x_1, x_2, \dots) \in \tilde{\mathcal{F}} \mid \varpi x_1 = 0\} \qquad (1.11)$$

as well as the rational Tate module

$$V_\varpi \mathcal{F} \colon (R\text{-Adm}) \to (E\text{-Vec}), \quad S \mapsto \{(x_1, x_2, \dots) \in \tilde{\mathcal{F}} \mid \exists n \in \mathbb{N} : \varpi^n x_1 = 0\} \qquad (1.12)$$

arise as subfunctors of $\tilde{\mathcal{F}}$.

1.11.1   Useful Calculations

Let $R \in \mathcal{C}$, and denote by $I \subset R$ its maximal ideal. Recall that $R/I = \overline{\mathbb{F}}_q$. If q is a power of p, we write $\mathcal{P}_{R,q}$ for the set of power series $f \in R[\![T]\!]$ satisfying

$$f(T) \equiv g(T^q) \pmod{I} \qquad (1.13)$$

for some power series $g(T) = c_1 T + c_2 T^2 + \cdots \in R[\![T]\!]$ with $c_1 \in R^\times$. If $q' > q$ is another power of p, we have injections $\mathcal{P}_{R,q} \hookrightarrow \mathcal{P}_{R,q'}$ given by sending $f(T)$ to its (q'/q)-fold self-composite $f^{q'/q}(T)$. Making use of these transition maps, we define

$$\mathcal{P}_R := \operatorname*{colim}_{n \in \mathbb{N}} \mathcal{P}_{R,p^n},$$

identifying any power series $f \in \mathcal{P}_{R,q}$ with its image in $\mathcal{P}_{R,q'}$ for higher p-powers q'. For any $f \in \mathcal{P}_{R,q}$, we define the functor

$$U_f \colon (R\text{-Adm}) \to (\text{Set}), \quad S \mapsto \left\{ (x_0, x_1, \dots) \in \prod_{\mathbb{N}} S^{\infty} \mid f(x_{i+1}) = x_i \right\}.$$

This functor does, up to canonical isomorphism, only depend on the equivalence class of f in $\mathcal{P}_R$. We write $U_{0,f}$ for the base change of U_f to k_R, that is

$$U_{0,f} \colon (k_R\text{-Adm}) \to (\text{Set}), \quad S \mapsto \left\{ (x_0, x_1, \dots) \in \prod_{\mathbb{N}} S^{\infty} \mid \overline{f}(x_{i+1}) = x_i \right\}.$$

Here, $\overline{f}$ is the image of f under the reduction map $R[\![T]\!] \to k_R[\![T]\!]$.

In the sequel, we denote R-algebras by S and write J for an ideal of definition containing the image of I (provided, for example, by A.2). Given an element $f \in \mathcal{P}_R$, we do not distinguish between f and a choice of a representative $\tilde{f} \in \mathcal{P}_{R,q}$ for some sufficiently large p-power. The following observation lays the groundwork for many of the upcoming results.

Lemma 1.11.2. *Let f be any power series in $\mathcal{P}_R$. For any two elements $s_1, s_2 \in S$ with $s_1 \equiv s_2 \mod J$ such that $f(s_1)$ and $f(s_2)$ exist (for example if f is a polynomial or $s_1, s_2 \in S^{\infty}$),*

we have

$$f^k(s_1) \equiv f^k(s_2) \pmod{J^{k+1}}.$$

Here, f^k denotes k-fold composition of f.

Proof. We will show that if $s_1 \equiv s_2 \bmod J^k$, then $f(s_1) \equiv f(s_2) \bmod J^{k+1}$, which suffices to prove the claim. We may write $s_2 = s_1 + r$ for some $r \in J^k$. By the assumptions on f there exist power series $g, h \in R[\![T]\!]$ such that h only has coefficients in I and $f(T) = g(T^q) + h(T)$. As I is finitely generated, say by elements $(r_1, \ldots, r_l)$, we obtain a representation

$$f(s_1) - f(s_2) = g(s_1^q) - g(s_2^q) + \sum_{i=1}^{l} r_i \left(h_i(s_1) - h_i(s_2) \right).$$

As r divides $(h_i(s_1) - h_i(s_2))$, we find $r_i(h_i(s_1) - h_i(s_2)) \in (r_i r) \subseteq J^{k+1}$. Also note that for any $s \in S$ and $n \in \mathbb{N}$,

$$(s + r)^{nq} = s^{nq} + nqrs^{nq-1}r + \cdots + r^{nq},$$

so after cancellation, all monomials of $g(s_1^q) - g(s_2^q)$ lie in (qr) or (r^2). This implies

$$g(s_1^q) - g\left((s_1 + r)^q \right) \in (qr) + (r^2) \subseteq J^{k+1},$$

and we are done. $\qquad\square$

Lemma 1.11.3. *The natural reduction map*

$$U_f(S) \to U_f(S/J) = U_{0,f}(S/J)$$

is bijective.

Proof. We first show surjectivity. Given a sequence $(x_0, x_1, \ldots) \in U_f(S/J)$, we can choose a sequence of arbitrary lifts $(y_0, y_1, \ldots) \in \prod_{\mathbb{N}} S^\infty$ and set

$$z_i = \lim_{r \to \infty} f^r(y_{i+r}).$$

The limit exists, because if $s \geq r$ are two non-negative integers, we calculate

$$f^{s-r}(y_{i+s}) \equiv \overline{f}^{s-r}(x_{i+s}) = x_{i+r} \equiv y_{i+r} \pmod{J},$$

implying by Lemma 1.11.2 that

$$f^s(y_{i+s}) \equiv f^r(y_{i+r}) \pmod{J^r}.$$

This shows that $(f^r(y_{i+r}))_{r \in \mathbb{N}}$ is a Cauchy-sequence for the J-adic topology on S, thereby convergent (cf. Lemma A.4). The sequence $(z_0, z_1, \ldots)$ now lies in $U_f(S)$ and lifts $(x_0, x_1, \ldots)$. It remains to show that the lift is unique. Suppose that $(z_0', z_1', \ldots)$ is another lift. Then, for any $i, k \in \mathbb{N}$ we have $z_{i+k} \equiv z_{i+k}' \bmod J$, and another application of Lemma 1.11.2 shows that

$$z_i = f^k(z_{i+k}) \equiv f^k(z_{i+k}') = z_i' \pmod{J^k}.$$

Thereby $(z_i - z_i') \in \bigcap_{k \in \mathbb{N}} J^k = \{0\}$. Hence, the lift is unique. $\qquad \square$

We write $\mathrm{Nilp}^\flat$ for the functor U_{T^q}. That is, $\mathrm{Nilp}^\flat(S) = \lim_{x \mapsto x^q} S^{\infty\infty}$ is the set of q-power compatible sequences with values in $S^{\infty\infty}$.

Lemma 1.11.4. *For any $f \in \mathcal{P}_R$, there is a canonical bijection $U_{0,f}(S/J) \to \mathrm{Nilp}^\flat(S/J)$.*

Proof. By assumption on f we have $f(T) = g(T^q) \in k_R[\![T]\!]$ for some $g(T) = c_1 T + c_2 T^2 + \dots$ with $c_1 \neq 0$. For each coefficient c_i, let $d_i \in k_R$ be the unique element such that $d_i^q = c_i$. Let $h(T) \in k_R[\![T]\!]$ be the power series given by $d_1 T + d_2 T^2 + \dots$. Now $h(T)^q = f(T)$, and we find that

$$U_f(S/J) \to \mathrm{Nilp}^\flat(S/J) \quad (x_1, x_2, x_3, \dots) \mapsto (x_1, h(x_2), h(h(x_3)), \dots)$$

is a well-defined function, and functorial in S. For the inverse, let $h^{-1}(T) \in k_R[\![T]\!]$ be the unique power series with $h^{-1}(h(T)) = h(h^{-1}(T)) = T$, see Lemma 1.1.7. The map

$$\mathrm{Nilp}^\flat(S/J) \to U_f(S/J), \quad (x_1, x_2, \dots) \mapsto (x_1, h^{-1}(x_2), h^{-1}(h^{-1}(x_3)), \dots)$$

is well-defined as

$$f(h^{-1}(T)) = g(h^{-1}(T)^q) = h(h^{-1}(T))^q = T^q,$$

and it is readily seen to be inverse to the map constructed above. $\qquad \square$

We collect results.

Proposition 1.11.5. *Given $f, g \in \mathcal{P}_R$, we have bijections, functorial in S,*

$$U_f(S) \to U_f(S/J) \to \mathrm{Nilp}^\flat(S/J) \to U_g(S/J) \to U_g(S). \tag{1.14}$$

Explicitly, the bijection $U_f(S) \to U_g(S)$ can be described as follows. Suppose that $f, g \in \mathcal{P}_{R,q}$ for some sufficiently large q. Let $h_f(T)$ and $h_g(T)$ be power series with coefficients in $\mathcal{O}_E$ such that

$$h_f(T)^q \equiv f(T) \pmod{I} \quad and \quad h_g(T)^q \equiv g(T) \pmod{I}.$$

Write $h_g^{-1}(T)$ for the (formal) inverse power series of h_g. Now the isomorphism is given by the mapping

$$(x_0, x_1, \dots) \mapsto (y_0, y_1, \dots), \quad where \quad y_i = \lim_{r \to \infty} g^r(h_g^{-(r+i)}(h_f^{r+i}(x_{i+r}))).$$

Here, the exponents are to be interpreted as iterated composition.

Proof. The first part follows directly from repeated application of the previous two Lemmas. The second part follows by tracing through the previous lemmas. $\qquad \square$

1.11.2 Applications to the Universal Cover

Fix a coordinate $\mathcal{F} \cong \mathrm{Spf}(R[\![T]\!])$ so that $\mathcal{F} = \mathrm{FG}(F)$ for some $\mathcal{O}_E$-module law $F \in (\mathcal{O}_E\text{-}\mathrm{FML}/R)$. Then $[\varpi]_F(T) \in \mathcal{P}_R$, and we obtain an isomorphism $\tilde{\mathcal{F}} \cong U_{[\varpi]_F} =: \tilde{F}$. Write $F_0 = F \otimes k_R$, $\mathcal{F}_0 = \mathcal{F} \otimes k_R$ and $\tilde{F}_0 = U_{0,[\varpi]_F}$.

Lemma 1.11.6. *We have an isomorphism*

$$\tilde{\mathcal{F}}_0 \cong \mathrm{Nilp}^{\flat}_{k_R}$$

of functors $(k_R\text{-Adm}) \to (\mathrm{Set})$

Proof. This statement is an application of Lemma 1.11.4. □

Lemma 1.11.7. *Suppose that S is an admissible R-algebra with an ideal of definition J such that $\varpi \in J$. Then the natural reduction map*

$$\tilde{\mathcal{F}}(S) \to \tilde{\mathcal{F}}(S/J) = \tilde{\mathcal{F}}_0(S/J)$$

is an isomorphism.

Proof. After choosing a coordinate $\mathcal{F} = \mathrm{FG}(F)$, we have $[\varpi]_F \in \mathcal{P}_R$ and hence $\tilde{\mathcal{F}}(S) \cong U_{[\varpi]_F}$. Thereby the statement is given by Lemma 1.11.3. □

The following is analogous to Proposition 1.11.5.

Proposition 1.11.8. *Let S be an admissible R-algebra with ideal of definition J such that $\phi(I) \subseteq J$. Then there are canonical isomorphisms (of sets)*

$$\tilde{\mathcal{F}}(S) \cong \tilde{\mathcal{F}}(S/J) = \tilde{\mathcal{F}}_0(S/J) \cong \mathrm{Nilp}^{\flat}(S/J) \cong \mathrm{Nilp}^{\flat}(S).$$

In particular, $\tilde{\mathcal{F}}(S)$ is, as a functor to (Set), *representable by* $\mathrm{Spf}(R[\![T^{q^{-\infty}}]\!])$.

We write λ for the isomorphism $\tilde{\mathcal{F}} \to \mathrm{Nilp}^{\flat}$, and $\lambda_i \colon \tilde{\mathcal{F}} \to (-)^{\infty}$ for projection on the i-th component. Similarly, we write $\mu \colon \mathrm{Nilp}^{\flat} \to \tilde{\mathcal{F}}$ for the inverse of λ and μ_i for the i-th component of μ.

By the proposition above, quasi-isogenies on $\mathcal{F}_0$ induce isomorphisms on $\tilde{\mathcal{F}}$. This will be used to construct an action of $D^{\times}$ (the algebra of quasi-isongenies of $\mathcal{F}_0$, cf. Section 1.3) on $\tilde{\mathcal{F}}$ below.

If $R = \mathcal{O}_{E'} \subseteq \mathcal{O}_{\breve{E}}$ is the ring of integers of a unramified extension of E, the relative Frobenius morphism lifts as well.

Definition 1.11.9 (Relative Frobenius on $\tilde{\mathcal{F}}$). We define $\Phi^{-1,*}\mathcal{F} := \mathcal{F} \otimes_{\mathcal{O}_{E'}, \Phi} \mathcal{O}_{E'}$ and define $\Pi \colon \tilde{\mathcal{F}} \to \Phi^{-1,*}\tilde{\mathcal{F}}$ as the isomorphism coming from the Frobenius quasi-isogeny

$$\mathrm{Frob}_q \colon \mathcal{F}_0 \to \mathcal{F}_0^{(q)} = \Phi^{-1,*}\mathcal{F}_0.$$

We finally note the following auxiliary result.

Lemma 1.11.10. *Let S be a discrete admissible R-algebra. Then*

$$V_{\varpi}\mathcal{F}(S) = \tilde{\mathcal{F}}(S).$$

Proof. As S is discrete, each $s \in S^{\infty}$ is nilpotent. From here it is easy to see that there is some $m \in \mathbb{N}$ such that $[\varpi^m]_F(s) = 0$. The desired statement follows directly. □

Finally, we remark how to express the logarithm map in terms of the isomorphism λ.

Lemma 1.11.11. *Let H be the standard formal $\mathcal{O}_E$-module of height n over $\mathcal{O}_{\breve{E}}$. We have a commutative diagram (cf. [BW16, Lemma 2.6.1])*

$$
\begin{array}{ccccc}
(x_0, x_1, \dots) & \in & \widetilde{H}(S) \xrightarrow{\quad\lambda\quad} \mathrm{Nil}^{\flat}(S) & \ni & (y, y^{1/q}, \dots) \\
\downarrow & & \searrow^{\log_H} \quad \swarrow & & \downarrow \\
\sum_{i=0}^{\infty} \frac{x_0^{q^{ni}}}{\varpi^i} & & S \otimes_{\mathcal{O}_E} E & & \sum_{i=-\infty}^{\infty} \frac{y^{q^{ni}}}{\varpi^i}
\end{array}
$$

With this terminology, we have $\log_H((\Pi^j x)_0) = \sum_{i=-\infty}^{\infty} \frac{y^{ni+j}}{\varpi^i}$.

Proof. This follows directly from the remark above. Let $x \in \widetilde{H}(S)$ and write $\lambda(x) = (y, y^{1/q}, \dots)$. We have $x_0 = \lim_{s\to\infty} [\varpi^s]_H(y^{-ns})$, hence

$$
\log_H(x_0) = \lim_{s\to\infty} \left(\varpi^s \log_H(y^{1/q^{ns}}) \right) = \lim_{s\to\infty} \left(\sum_{i=0}^{\infty} \frac{y^{q^{n(i-s)}}}{\varpi^{i-s}} \right) = \sum_{i=-\infty}^{\infty} \frac{y^{q^{ni}}}{\varpi^i}.
$$

The second part is an immediate consequence. $\qquad\square$

1.12 Determinants of Formal modules

Determinants of formal $\mathcal{O}_E$-modules will play a central role in the analysis of the Lubin–Tate tower. Let $R \in \mathcal{C}$ and let $\mathcal{F}$ be a formal $\mathcal{O}_E$-module over R. Write $\mathcal{F}_0 = \mathcal{F} \otimes_R \overline{\mathbb{F}}_q$ and let $\wedge^n \mathcal{F}_0$ be the formal $\mathcal{O}_E$-module with Diodonné module isomorphic to $\wedge^n \mathrm{D}(\mathcal{F}_0)$; it is the unique formal $\mathcal{O}_E$-module of height 1 over $\overline{\mathbb{F}}_q$ and admits, by 1.6, a unique deformation $\wedge\mathcal{F}$ to $R \in \mathcal{C}$.

Theorem 1.12.1 (Determinants of Formal Modules). *For every $m \geq 1$, there exists an alternating and multilinear map of*

$$
\delta_m \colon \mathcal{F}[\varpi^m]^n \to \wedge^n \mathcal{F}[\varpi^m] \tag{1.15}
$$

of formal schemes over R with $\mathcal{O}_E$-module structure. This map is universal in the sense that any alternating multilinear map from $\mathcal{F}[\varpi^m]^n$ into another $\mathcal{O}_E$-module factors through δ_m.

Proof. In the case of equal characteristic, this statement is due to Weinstein (cf. [Wei10]), whose construction uses an explicit description of the universal deformation only available in characteristic p. The characteristic 0 case is much more complicated; it was carried out in the PhD thesis of Hedayatzadeh. His construction makes use of the classification of p-divisible groups in terms of Zink's category of displays (cf. [Hed15, Theorem 4.34]). $\qquad\square$

We will need the following result.

Lemma 1.12.2. *Let L/E be a separable extension of degree n and suppose that there is an action $\mathcal{O}_L \hookrightarrow \mathrm{End}(\mathcal{F})$ turning $\mathcal{F}$ into a formal $\mathcal{O}_L$-module of height 1. Then, for all $m \geq 1$,*

the identity

$$\delta_m(\alpha x_1, \ldots, \alpha x_n) = \mathrm{N}_{L/E}(\alpha)\delta_m(x_1, \ldots, x_n)$$

holds.

Proof. This is [Wei16, Lemma 2.12]. $\qquad\square$

We remark that the Lemma above in particular applies to the standard formal $\mathcal{O}_E$-module H over $\mathcal{O}_{\breve{E}}$. We turn our attention to the determinant of the standard formal $\mathcal{O}_E$-module in the following example.

Example 1.12.3. The determinant $\wedge^n H$ is canonically isomorphic to the formal $\mathcal{O}_E$-module law over $\mathcal{O}_{\breve{E}}$ with logarithm given by the power series

$$f_\wedge(T) = \sum_{i=0}^{\infty}(-1)^{(n-1)i}\frac{T^{q^i}}{\varpi^i}.$$

We do not prove this, but we note that it can be witnessed on the corresponding Dieuodonné-modules: $\mathrm{D}(\wedge^n H)$ and $\wedge^n \mathrm{D}(H)$ are naturally isomorphic. The module $\mathrm{QLog}(\wedge^n H)$ is generated by $f_\wedge(T)$, and $\wedge^n \mathrm{QLog}(H)$ is generated by the element

$$w(T) = \log_H(T) \wedge \frac{1}{\varpi}\log_H(T^q) \wedge \cdots \wedge \frac{1}{\varpi}\log_H(T^{q^{n-1}}) \in \wedge^n \mathrm{D}(H).$$

We have $f_\wedge(T^q) = (-1)^{n-1}\varpi f(T) - \varpi T$, which equals $(-1)^{n-1}\varpi f_\wedge(T)$ in $\mathrm{QLog}(\wedge^n H) \cong \mathrm{D}(\wedge^n H)^\vee$. This readily implies

$$\phi(w) = w(T^q) = (-1)^{n-1}\varpi w(T) \quad \text{and} \quad \phi(f_\wedge(T)) = f_\wedge(T^q) = (-1)^{n-1}\varpi f_\wedge(T).$$

Thereby, $\mathrm{D}(\wedge^n H)$ and $\wedge^n \mathrm{D}(H)$ are free of rank one with isomorphic Frobenius structure, hence isomorphic as Dieudonné-modules. We also note that by Theorem 1.5.1, one finds

$$[\varpi]_{(\wedge^n H)_0}(T) = (-1)^{n-1}T^q. \tag{1.16}$$

We also need a version of the determinant mapping for the universal cover. Taking the inverse limit with respect to m, the morphsims δ_m from (1.15) yield, for any admissible R-algebra S, a determinant map on Tate modules

$$\delta\colon T_\varpi\mathcal{F}(S)^n \to T_\varpi\wedge\mathcal{F}(S) \quad \text{and} \quad \delta\colon V_\varpi\mathcal{F}(S)^n \to V_\varpi\wedge\mathcal{F}(S),$$

as $V_\varpi\mathcal{F}(S) = (T_\varpi\mathcal{F}(S)) \otimes_{\mathcal{O}_E} E$. This allows us to define a determinant map on the universal cover as follows. Let $J \subset S$ be an ideal of definition containing the image of I. Then S/J is discrete, and by Lemma 1.11.10, we may define $\delta\colon \tilde{\mathcal{F}}^n(S) \to \widetilde{\wedge\mathcal{F}}(S)$ as the composite

$$\tilde{\mathcal{F}}^n(S) \xrightarrow{\sim} \tilde{\mathcal{F}}(S/J)^n = V_\varpi\mathcal{F}^n(S/J) \xrightarrow{\delta} V_\varpi\wedge\mathcal{F}(S/J) = \widetilde{\wedge\mathcal{F}}(S/J) \cong \widetilde{\wedge\mathcal{F}}(S).$$

1.13 The Quasilogarithm Map

We keep the assumption on $R \in \mathcal{C}$, it is a local $\mathcal{O}_E$-algebra with maximal ideal I such that $R/I = \overline{\mathbb{F}}_q$, and we let S denote an admissible R-algebra (where $R \to S$ is continuous with the I-adic topology on R) with ideal of definition $J \subseteq S$ containing the image of I.

The aim of this subsection is to define, attached to any ϖ-divisible formal $\mathcal{O}_E$-module $\mathcal{F}$ over R, a map

$$\mathrm{qlog}_{\mathcal{F}} \colon \tilde{\mathcal{F}}(S) \to \mathrm{D}(\mathcal{F}) \otimes_R (S \otimes_{\mathcal{O}_E} E),$$

which we call the *quasi-logarithm map*. We give an explicit description of this map if $\mathcal{F} = \mathrm{FG}(H)$ is the standard $\mathcal{O}_E$-module over $\mathcal{O}_{\breve{E}}$.

The construction of $\mathrm{qlog}_{\mathcal{F}}$ is as follows. Let $0 \to \mathcal{V} \xrightarrow{\psi} \mathcal{U} \xrightarrow{\phi} \mathcal{F} \to 0$ be the universal additive extension of $\mathcal{F}$. For any sequence $(x_1, x_2, \dots) \in \tilde{\mathcal{F}}(S)$, choose an arbitrary sequence $(y_1, y_2, \dots) \in \tilde{\mathcal{U}}(S)$ such that y_i is a lift of x_i under the map $\mathcal{U}(S) \to \mathcal{F}(S)$. Let y be the limit $y = \lim_{i \to \infty} [\varpi]_{\mathcal{U}}^i (y_i)$ and put

$$\mathrm{qlog}_{\mathcal{F}}((x_1, x_2, \dots)) = \log_{\mathcal{U}}(y) \in \mathrm{D}(\mathcal{F}) \otimes_R (S \otimes_{\mathcal{O}_E} E).$$

Proposition 1.13.1. *This construction yields a well-defined map.*

Proof. We may assume that $\mathcal{F}$ and $\mathcal{V}$ come from formal module laws F and V, and we may furthermore assume that $\mathcal{U} = \mathrm{FG}(U)$ for an $\mathcal{O}_E$-module law U obtained by Lemma 1.8.5. Now $(x_1, x_2, \dots)$ is a sequence in S^{∞} and $(y_1, y_2, \dots)$ is a sequence of elements in $(S^{\infty})^n$. It suffices to show that $y = \lim_{i \to \infty} [\varpi]_U^i (y_i)$ exists and that it is independent of the choice of lifts $(y_1, y_2, \dots)$. Both claims follow from the additivity of $\mathcal{V}$, implying that $[\varpi]_V(T) = \varpi T$. The sequence $([\varpi^i]_U(y_i))$ converges, as for positive integers $i \le j$, we have

$$[\varpi^i](y_i) - [\varpi^j](y_j) = [\varpi^i]([\varpi^{i-j}]y_j - y_i) \in \psi(\varpi^i (S^{\infty})^{n-1}) \subseteq J^i(S^{\infty})^n.$$

If $(y_1', y_2', \dots)$ is another sequence of lifts, put $y' = \lim_{i \to \infty} [\varpi^i]_U(y_i') \in S^{\infty}$. Now there exists some $z \in \mathcal{V}(S)$ such that $y - y' = \psi(z)$. But by construction $z \in \bigcap_{i \in \mathbb{N}} \varpi^i (S^{\infty})^{n-1} = 0$. $\qquad\square$

Let us now consider the case where $\mathcal{F} = \mathrm{FG}(H)$ comes from the standard formal $\mathcal{O}_E$-module of height n over $\mathcal{O}_{\breve{E}}$. Then from Proposition 1.9.6 we have the distinguished basis elements of $\mathrm{Ext}(H, \widehat{\mathbb{G}}_a)$ corresponding to the symmetric 2-cocycles δf_i, $1 \le i \le n-1$ where $f_i(T) = \frac{1}{\varpi} \log_H(T^{q^i})$. Also recall that, setting $f_0(T) = \log_H(T)$, the elements $(f_0, f_1, \dots, f_{n-1})$ freely generate $\mathrm{QLog}(H)$. The universal additive extension now corresponds to the symmetric 2-cocycle $(\delta f_1, \dots, \delta f_{n-1}) \in \mathrm{SymCoc}^2(H, V)$. We can make the quasi-logarithm map explicit.

Proposition 1.13.2. *Let* $x = (x_0, x_1, \dots) \in \widetilde{H}(S)$. *With respect to the basis* $(\log_H(T),$ $\log_H(T^q), \dots, \log_H(T^{q^{n-1}}))$ *of* $\mathrm{QLog}(H) \otimes_{\mathcal{O}_E} E$, *the quasi-logarithm map is given by*

$$\mathrm{qlog}_H(x) = \left(\log_H(x_0), \log_H((\Pi x)_0), \dots, \log_H((\Pi^{n-1} x)_0) \right) \in (S \otimes E)^n.$$

Here, $\Pi x = ((\Pi x)_0, (\Pi x)_1, \dots)$ *is the image of* x *under* Π, *the automorphism of* $\widetilde{H}(S)$ *induced by the (relative) Frobenius quasi-isogeny on* H_0, *cf. Definition 1.11.9.*

We postpone the proof to state the following auxiliary result.

Lemma 1.13.3. *Let $x = (x_0, x_1, \dots) \in \widetilde{H}(S)$. For positive integers i and j we have*

$$\log_H((\Pi^j x)_i) = \lim_{r \to \infty} \varpi^r \log_H(x_{r+i}^{q^j}).$$

Proof. Tracing through the commutative square (with λ and μ the isomorphisms from the previous subsection)

$$
\begin{array}{ccc}
\widetilde{H}(S) & \xrightarrow{\ \lambda\ } & \mathrm{Nilp}^\flat(S) \\
{\scriptstyle \Pi}\downarrow & & \downarrow {\scriptstyle (y_i)_i \mapsto (y_i^q)_i} \\
\widetilde{H}(S) & \xleftarrow{\ \mu\ } & \mathrm{Nilp}^\flat(S),
\end{array}
$$

we find

$$(\Pi^j x)_i = \lim_{s \to \infty} \lim_{r \to \infty} \left([\varpi]_H^s (x_{r+s+i}^{q^{nr+j}}) \right). \tag{1.17}$$

The claim follows after applying $\log_H$ and making repeated use of the functional equation $\log_H(T^{q^n}) = \varpi \log_H(T) + \varpi T$. $\qquad \square$

Proof of Proposition 1.13.2. Using the coordinates provided by $(\delta f_1, \dots, \delta f_{n-1})$, the universal additive extension of H is isomorphic to

$$0 \to \widehat{\mathbb{G}}_a^{n-1} \to U \to H \to 0,$$

where U is a module law with

$$[\varpi]_U(\boldsymbol{X}, T) = \Big(\varpi X_1 + (\delta_\varpi f_1)(T), \dots, \varpi X_{n-1} + (\delta_\varpi f_{n-1})(T), [\varpi]_H(T) \Big).$$

Beginning with $x = (x_0, x_1, \dots) \in \widetilde{H}(S)$, lifting to $(y_0, y_1, \dots) \in U(S)^{\mathbb{N}}$ and writing $y = \lim_{i \to \infty} [\varpi]_U^i(y_i)$, we find

$$y = \Big(\lim_{r \to \infty} (\delta_{\varpi^r} f_1)(x_r), \dots, \lim_{r \to \infty} (\delta_{\varpi^r} f_{n-1})(x_r), x_0 \Big) \in U(S).$$

Now, Lemma 1.13.3 provides the equality

$$\lim_{r \to \infty} \delta_{\varpi^r} f_i(x_r) = \frac{1}{\varpi} \lim_{r \to \infty} \varpi^r \log_H(x_r^{q^{nr+i}}) - \frac{1}{\varpi} \log_H\left(x_0^{q^i}\right) = \frac{1}{\varpi} \Big(\log_H((\Pi^i x)_0) - \log_H(x_0^{q^i}) \Big).$$

We need to calculate $\log_U(y)$, which calls for an explicit description of

$$\log_U : U \otimes (R \otimes_{\mathcal{O}_E} E) \to (\widehat{\mathbb{G}}_{a \otimes (R \otimes_{\mathcal{O}_E} E)})^n.$$

Tracing through the procedure provided in Subsection 1.2, we find

$$\log_U(\boldsymbol{X}, T) = \Big(X_1 + \tfrac{1}{\varpi} \log_H(T^q), \dots, X_{n-1} + \tfrac{1}{\varpi} \log_H(T^{q^{n-1}}), \log_H(T) \Big).$$

This representation is with respect to the basis $(f_1, \dots, f_{n-1}, f_0)$. The claim follows. $\qquad \square$

1.14 An Approximation of the Determinant Morphism

Let H be the standard formal $\mathcal{O}_E$-module over $\mathcal{O}_{\breve{E}}$ of height n, and let $S \in (\mathcal{O}_{\breve{E}}\text{-Adm})$ such that S admits an ideal of definition J containing the image of ϖ. Write $\wedge H$ for the formal $\mathcal{O}_E$-module over $\mathcal{O}_{\breve{E}}$ with logarithm

$$\log_{\wedge H}(T) = \sum_{i=0}^{\infty} (-1)^{(n-1)i} \frac{T^{qi}}{\varpi^i}.$$

By Hazewinkel's integrality Lemma (cf. Theorem 1.5.1), such a module law exists, and we have $D(\wedge H) = \wedge^n D(H)$ (cf. the discussion at the end of Section 1.12). We follow [BW16, Theorem 2.10.3] to describe a map $\delta \colon \widetilde{H}^n \to \widetilde{\wedge H}$ making the square

$$
\begin{array}{ccc}
\widetilde{H}^n(S) & \xrightarrow{\ \ \delta\ \ } & \widetilde{\wedge H}(S) \\
{\scriptstyle \mathrm{qlog}_H \times \cdots \times \mathrm{qlog}_H} \downarrow & & \downarrow {\scriptstyle \mathrm{qlog}_{\wedge H}} \\
D(H)^n \otimes (S \otimes_{\mathcal{O}_E} E) & \xrightarrow{\ \det\ } & D(\wedge H) \otimes (S \otimes_{\mathcal{O}_E} E)
\end{array}
\tag{1.18}
$$

commute.

Let $(s_1, \ldots, s_n) \in \widetilde{H}(S)^n$, and write $x_i = \lambda(s_i) \in \mathrm{Nilp}^{\flat}(S)$, which are elements in $S^{\circ\circ}$ with distinguished q-power roots. Here $\lambda \colon \widetilde{H} \to \mathrm{Nilp}^{\flat}$ is the isomorphism from 1.11 with inverse $\mu = (\mu_0, \mu_1, \ldots)$. We set

$$\delta_0(s_1, \ldots, s_n) = \sum_{(a_1, \ldots, a_n)} [\varepsilon(a_1, \ldots a_n)]_{\wedge H} \mu_0(x_1^{q^{a_1}} \cdots x_n^{q^{a_n}}) \in \wedge H(S),$$

where

- The expression $\varepsilon(a_1, \ldots, a_n)$ denotes the sign of the permutation $i \mapsto a_{i+1} \pmod{n}$ of $(0, \ldots, n-1)$.

- The sum takes place in $\wedge H(S)$.

- The sum ranges over n-tuples $(a_1, \ldots, a_n)$ of (possibly negative) integers satisfying $a_1 + \cdots + a_n = n(n-1)/2$, subject to the condition that each a_i occupies a distinct residue class modulo n.

Proposition 1.14.1. *The map δ_0 makes the diagram*

$$
\begin{array}{ccc}
\widetilde{H}^n(S) & \xrightarrow{\ \ \delta_0\ \ } & \wedge H(S) \\
{\scriptstyle \mathrm{qlog}_H^n} \downarrow & & \downarrow {\scriptstyle \log_{\wedge H}} \\
D(H)^n \otimes (S \otimes E) & \xrightarrow{\ \det\ } & D(\wedge H) \otimes (S \otimes E)
\end{array}
$$

commute. It is $\mathcal{O}_E$-multilinear and alternating.

Sketch of proof. This is part of the proof of [BW16, Theorem 2.10.3]. Commutativity follows

from

$$\log_{\wedge H}(\delta_0(s_1, \ldots, s_n)) = \sum_{(a_1, \ldots, a_n)} \varepsilon(\boldsymbol{a}) \log_{\wedge H} \mu_0(x_1^{q^{a_1}} \cdots x_n^{q^{a_n}})$$

$$= \sum_{(a_1, \ldots, a_n)} \varepsilon(\boldsymbol{a}) \sum_{m \in \mathbb{Z}} (-1)^{(n-1)m} \frac{x_1^{q^{a_1+m}} \cdots x_n^{q^{a_n+m}}}{\varpi^m} = \det \left(\sum_{m \in \mathbb{Z}} \frac{x_i^{q^{mn+j-1}}}{\varpi^m} \right)_{\substack{1 \le i \le n, \\ 1 \le j \le n}},$$

which is equal to $\det(\mathrm{qlog}_H^n(s_1, \ldots, s_n))$ by Proposition 1.13.2 and Lemma 1.11.11. The fact that δ_0 is multilinear and alternating ultimately follows from the corresponding properties of det, the fact that $\mathrm{Ker}(\log_H) = \wedge H[\varpi^\infty]$ (cf. Lemma 1.4.4) and topological considerations in the induced diagram in the category of adic spaces over $\mathrm{Spa}(\check{E}, \mathcal{O}_{\check{E}})$. $\qquad \square$

This allows us to define the sought for morphism of functors $\delta \colon \widetilde{H^n} \to \widetilde{\wedge H}$.

Definition 1.14.2. Put $\delta_i(s_1, \ldots, s_n) = \delta_0(\varpi^{-i} s_1, \ldots, s_n)$. Then $\delta = (\delta_0, \delta_1, \ldots)$ yields a map $\widetilde{H^n} \to \widetilde{\wedge H}$. It is E-multilinear and alternating, hence it agrees, up to a unit, with the determinant morphism constructed in Section 1.12.

Using the canonical identifications $\widetilde{H^n} \cong (\mathrm{Nilp}^\flat)^n$ and $\widetilde{\wedge H} \cong \mathrm{Nilp}^\flat$, the morphism δ yields a map $(\mathrm{Nilp}^\flat)^n \to \mathrm{Nilp}^\flat$, which in turn is the same as a power series

$$\Delta(X_1, \ldots, X_n) \in \mathcal{O}_{\check{E}}[\![X_1^{q^{-\infty}}, \ldots, X_n^{q^{-\infty}}]\!]$$

together with distinguished q-th power roots. We have the following approximation of Δ, cf. [BW16, Lemma 2.10.4].

Lemma 1.14.3. *We have*

$$\Delta(X_1, \ldots, X_n) \equiv \det(X_i^{q^{j-1}})_{\substack{1 \le i \le n, \\ 1 \le j \le n}}$$

modulo terms of degree greater than $1 + q + \cdots + q^{n-1}$.

Proof. By Proposition 1.14.1 and the explicit description of the quasi-logarithm map in Proposition 1.13.2, we have the equality

$$\sum_{k=-\infty}^{\infty} (-1)^{(n-1)k} \frac{\Delta(X_1, \ldots, X_n)^{q^k}}{\varpi^k} = \det \left(\sum_{k=-\infty}^{\infty} \frac{X_i^{q^{nk+j-1}}}{\varpi^k} \right)_{1 \le i,j \le n}$$

of elements inside $\check{E}[\![X_1^{q^{-\infty}}, \ldots, X_n^{q^{-\infty}}]\!]$ (equipped with the topology induced from the $(\varpi, X_1, \ldots, X_n)$-adic topology on $\mathcal{O}_{\check{E}}[\![X_1^{q^{-\infty}}, \ldots, X_n^{q^{-\infty}}]\!]$). The claim follows after comparing coefficients of the respective series. $\qquad \square$

2 Lubin–Tate Theory

We review the approach to local class field theory developed by Lubin and Tate in [LT65].

Class field theory is, in general terms, concerned with the study of abelian extensions of local or global fields. This area of number theory has a rich history, first results towards this direction date back as far as 1801, when Gauß proved the famous quadratic reciprocity law: if $\ell \neq p$ are odd prime numbers, we have the equality

$$\left(\frac{p}{\ell}\right) = (-1)^{\frac{p-1}{2}\frac{l-1}{2}}\left(\frac{\ell}{p}\right).$$

This unintuitive statement about the interplay of the multiplicative structures of $\mathbb{F}_p$ and $\mathbb{F}_\ell$ finds a deeper and more conceptual meaning when inspected through the lens of cyclotomic fields. This connection was first made by Gauß himself when he provided a proof of quadratic reciprocity, using so-called Gauß sums. Admitting the theory of cyclotomic fields (as developed in Chapter 1 of [Neu06]), and denoting by $\mathbb{Q}(\zeta_\ell)$ the ℓ-th cyclotomic extension, quadratic reciprocity is essentially equivalent to the following statement.

$$\left(\frac{\ell}{p}\right) = (-1)^{\frac{l-1}{2}\frac{p-1}{2}} \iff \begin{array}{l}\text{The prime } p \in \mathbb{Z} \text{ decomposes into an even}\\ \text{number of prime ideals in } \mathbb{Q}(\zeta_\ell).\end{array}$$

This foreshadows class field theory. Indeed, one major achievement of class field theory is the Artin reciprocity law, which in particular yields a description of the decomposition behavior of unramified primes inside abelian extensions of number fields, cf. [Neu06, Theorem 7.3]. The appearance of cyclotomic fields above is also no coincidence.

Theorem (Kronecker–Weber[2])**.** *Every finite extension of* $\mathbb{Q}$ *embeds into* $\mathbb{Q}(\zeta_m)$ *for some* $m \in \mathbb{Z}$. *That is, after choosing an algebraic closure* $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}$, *the maximal Abelian subextension is given by*

$$\mathbb{Q}^{\mathrm{ab}} = \bigcup_{m \in \mathbb{N}} \mathbb{Q}(\zeta_m) \subset \overline{\mathbb{Q}}.$$

The problem of finding similar descriptions for the maximal Abelian extension of general number fields seems to be very difficult. There is a similar description of Abelian extensions of imaginary quadratic fields (cf. [Sil94, Chapter II, §5]; these extensions arise by adjoining certain values attained by the so-called Weber function at the torsion points of an elliptic curve with complex multiplication by the field in question), but in general this problem seems to be open.

2.1 The Statements of Local Class Field Theory

The situation simplifies a lot for non-Archimedian local fields, and the aim of this section is to discuss the analogues of the Kronecker–Weber theorem and the Artin map for a non-Archimedian local field E. By definition of the *intertia subgroup* I_E (cf. [Ser13, §1]), we have

---

[2]Despite the name, the first complete proof of this theorem is (apparently) due to Hilbert, cf. [Hil96].

the short exact sequence

$$0 \to I_E = \mathrm{Gal}(\overline{E}/E^{\mathrm{nr}}) \to \mathrm{Gal}(\overline{E}/E) \to \mathrm{Gal}(E^{\mathrm{nr}}/E) = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to 0.$$

Recall that $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is topologically generated by the q-th power Frobenius automorphism $\mathrm{Frob}_q \colon x \mapsto x^q$. Let $\Phi \in \mathrm{Gal}(E^{\mathrm{nr}}/E)$ denote the corresponding Frobenius automorphism of E^{nr}. The *Weil group* is defined as

$$\mathrm{W}_E := \{\sigma \in \mathrm{Gal}(\overline{E}/E) : \sigma|_{E^{\mathrm{nr}}} = \Phi^m \text{ for some } m \in \mathbb{Z}\},$$

endowed with the unique topology such that the inertia subgroup $I_E \subset \mathrm{W}_E$ (with its usual topology) is an open subgroup. As above, we have the short exact sequence of topological groups

$$0 \to I_E \to \mathrm{W}_E \to \mathbb{Z} \to 0.$$

The natural map $\mathrm{W}_E^{\mathrm{ab}} \to \mathrm{Gal}(\overline{E}/E)^{\mathrm{ab}} = \mathrm{Gal}(E^{\mathrm{ab}}/E)$ induced by taking (topological) abelianization can be shown to be injective and yields a bijection

$$\mathrm{W}_E^{\mathrm{ab}} \xrightarrow{\sim} \{\sigma \in \mathrm{Gal}(E^{\mathrm{ab}}/E) : \sigma|_{E^{\mathrm{nr}}} \in \Phi^{\mathbb{Z}}\}. \tag{2.1}$$

We use this bijection to identify elements of $\mathrm{W}_E^{\mathrm{ab}}$ with the corresponding field automorphism of E^{ab}. In this section, we work towards the following Theorem.

Theorem 2.1.1 (Local Class Field Theory).

1. The Local Kronecker–Weber Theorem. *For any choice of uniformizer $\varpi \in E^{\times}$, there exists a totally ramified field extensions E_{ϖ} of E such that the maximal abelian subextension $E^{\mathrm{ab}} \subset \overline{E}$ of E decomposes as $E^{\mathrm{ab}} = E_{\varpi} E^{\mathrm{nr}}$.*

2. Local Artin Reciprocity. *There is a unique isomorphism (the* Artin *map) of topological groups*

$$\mathrm{Art}_E \colon E^{\times} \xrightarrow{\sim} \mathrm{W}_E^{\mathrm{ab}}$$

   such that

   (a) *for any choice of uniformizer $\varpi \in E^{\times}$, we have $\mathrm{Art}_E(\varpi)|_{E^{\mathrm{nr}}} = \Phi$.*

   (b) *for any finite Abelian extension E' of E, the map $E^{\times} \xrightarrow{\mathrm{Art}_E} \mathrm{W}_E^{\mathrm{ab}} \xrightarrow{\sigma \mapsto \sigma|_{E'}} \mathrm{Gal}(E'/E)$ is surjective and has kernel given by $\mathrm{N}_{E'/E}(E'^{\times}) \subset E^{\times}$.*

3. Functoriality of the Artin Map. *Let L/E be a finite extension of local fields. Then there is a commutative diagram*

$$\begin{array}{ccc} L^{\times} & \xrightarrow{\mathrm{Art}_L} & \mathrm{W}_L^{\mathrm{ab}} \\ {\scriptstyle \mathrm{N}_{L/E}}\downarrow & & \downarrow \\ E^{\times} & \xrightarrow{\mathrm{Art}_E} & \mathrm{W}_E^{\mathrm{ab}} \end{array}$$

   where $\mathrm{W}_L^{\mathrm{ab}} \to \mathrm{W}_E^{\mathrm{ab}}$ is the morphism induced by the inclusion $\mathrm{W}_L \hookrightarrow \mathrm{W}_E$.

We follow the construction of E_ϖ in [LT65]. The construction resembles the case of imaginary quadratic fields; E_ϖ is obtained from E by adjoining ϖ-power-torsion points of the Lubin–Tate formal module laws constructed in Section 1.4. We sketch a proof of the equality $E^{\mathrm{ab}} = E_\varpi E^{\mathrm{nr}}$. Lubin and Tate showed the latter making use of previously known results in local class field theory. We avoid arguments resting on results of local class field theory by referring to a proof found by Gold[3] (cf. [Gol81]). Similarly to Hilbert's proof of the classical Kronecker–Weber theorem, the proof makes essential use of the theory of higher ramification groups, in particular the theorem of Hasse–Arf. Once all of this is known, the isomorphism Art_E and property (a) arise naturally. We do not prove that this isomorphism satisfies property (b), but we do note briefly how to see that an isomorphism satisfying properties (a) and (b) has to be unique. We also present a proof for the functoriality statement.

2.2 The Lubin–Tate Approach to Local Class Field Theory

We now explain the interplay of Lubin–Tate formal module laws (as discussed in Section 1.4) with local class field theory. The main focus will be on the construction of the totally ramified abelian extension E_ϖ from Theorem 2.1.1. As announced, it arises as an infinite union of finite extensions $E_{\varpi,m}$, which are constructed as follows. Let $f(T) \in \mathcal{O}_E[\![T]\!]$ be a power series in $\mathcal{F}_{\varpi,1}$ (cf. Section 1.4). Now there is a unique Lubin–Tate formal module law F satisfying $[\varpi]_F(T) = f(T)$, and we define

$$\Lambda_{f,m} := \{x \in \mathfrak{m}_{\overline{E}} \mid [\varpi^m]_F(x) = 0\}.$$

Note that $\overline{E}$ is not complete, but any $x \in \mathfrak{m}_{\overline{E}}$ lies inside some finite (hence complete) subextension, so the power series $[\varpi]_F(x)$ converges, and the definition of $\Lambda_{f,m}$ makes sense. With the $\mathcal{O}_E$-module structure from F, the set $\Lambda_{f,m}$ is an $\mathcal{O}_E$-module. The absolute Galois group of E acts $\mathcal{O}_E$-linearly on $\Lambda_{f,m}$, as the power series that make up the $\mathcal{O}_E$-structure all have coefficients in E. This yields a homomorphism

$$\mathrm{Gal}(\overline{E}/E) \to \mathrm{Aut}_{(\mathcal{O}_E\text{-Mod})}(\Lambda_{f,m}). \tag{2.2}$$

We now define

$$E_{\varpi,m} := E(\Lambda_{f,m})$$

and call it the Lubin–Tate extension of degree m. Omitting the choice of f from notation is justified. Indeed, given another choice $f' \in \mathcal{F}_\varpi$, Theorem 1.4.3 implies the existence of a power series $[1]_{f,f'} \in \mathcal{O}_E[\![T]\!]$ inducing an isomorphism $\Lambda_{f,m} \to \Lambda_{f',m}$. As any finite extension of E is complete, we obtain $E(\Lambda_{f,m}) = E(\Lambda_{f',m})$, showing the following.

Lemma 2.2.1. *The Lubin–Tate extension of degree m depends only on the choice of ϖ, not on the choice of power series $f \in \mathcal{F}_\varpi$.*

Hence, we shall henceforth choose $f(T) = \varpi T + T^q$. The fact that the corresponding formal module law is ϖ-divisible quickly implies the following.

---

[3]Appearently, a similar (unpublished) proof was earlier found by Leopoldt.

Lemma 2.2.2. *For any positive integer m, there is an isomorphism of $\mathcal{O}_E$-modules*

$$\Lambda_{f,m} \cong \mathcal{O}_E/\varpi^m\mathcal{O}_E.$$

Proof. It is easily seen that $\Lambda_{f,m}$ is a finite set, hence of the form

$$\Lambda_{f,m} = \prod_{i=1}^{k} (\mathcal{O}_E/\varpi^{m_i}\mathcal{O}_E)^{e_i}$$

by the structure theorem for finitely generated modules over principal ideal domains. For $m = 1$ the claim is now easy to verify, we have $\#\Lambda_{f,1} = q$ and hence $\Lambda_{f,1} \cong \mathcal{O}_E/\varpi\mathcal{O}_E$.

Now let m be arbitrary. For any $\alpha \in \Lambda_{f,m}$, the roots of $f(T) - \alpha$ must have positive ϖ-adic valuation, and it follows that f (i.e. multiplication by ϖ) yields a surjective morphism $\Lambda_{f,m} \to \Lambda_{f,m-1}$. Via induction this implies $\#\Lambda_{f,m} = q^m$. By further inductive reasoning, we may assume that $\Lambda_{f,m-1} \cong \mathcal{O}_E/\varpi^{m-1}\mathcal{O}_E$, implying

$$\Lambda_{f,m} \cong \mathcal{O}_E/\varpi^m\mathcal{O}_E \quad \text{or} \quad \Lambda_{f,m} \cong \mathcal{O}_E/\varpi^{m-1}\mathcal{O}_E \times \mathcal{O}_E/\varpi\mathcal{O}_E.$$

As the multiplication-by-ϖ-map $\Lambda_{f,m} \to \Lambda_{f,m-1}$ is surjective, this enforces $\Lambda_{f,m} \cong \mathcal{O}_E/\varpi^m\mathcal{O}_E$, as desired. $\qquad\square$

Lemma 2.2.3. *For any positive integer m, the extension $E_{\varpi,m}/E$ is totally ramified of degree $[E_{\varpi,m} : E] = (q-1)q^{m-1}$. Furthermore, the morphism (2.2) induces an isomorphism*

$$\mathrm{Gal}(E_{\varpi,m}/E) \cong \mathrm{Aut}_{(\mathcal{O}_E\text{-Mod})}(\Lambda_{f,m}) = \mathcal{O}_E^{\times}/(1 + \varpi^m\mathcal{O}_E).$$

Proof. Given a positive integer $i \in \mathbb{N}$, we write $f^i(T)$ for the i-fold self-composite of f (with $f^0(T) = T$) and define

$$\eta_i(T) := \frac{f^i(T)}{f^{i-1}(T)} = \varpi + f^{i-1}(T)^{q-1}.$$

By induction, this is seen to be an Eisenstein polynomial of degree $(q-1)q^{i-1}$. The roots of $\eta_m(T)$ are exactly the elements in the difference $\Lambda_{f,m} \setminus \Lambda_{f,m-1}$, and any choice of such a root yields an inclusion of fields

$$\frac{E[T]}{(\eta_m(T))} \hookrightarrow E_{\varpi,m}. \tag{2.3}$$

As $\Lambda_{f,m}$ generates $E_{\varpi,m}$, the homomorphism $\mathrm{Gal}(E_{\varpi,m}/E) \to \mathrm{Aut}(\Lambda_{f,m})$ obtained from the homomorphism in (2.2) is injective. By (2.3), the source of this morphism has cardinality $\geq (q-1)q^{m-1}$, while the target is a set with exactly $(q-1)q^{m-1}$ elements. Hence, the homomorphism is bijective. For degree-reasons, it follows that (2.3) is an isomorphism; in particular, $E_{\varpi,m}$ is totally ramified of the specified degree. $\qquad\square$

We define the Lubin–Tate extension of E as the union

$$E_{\varpi} := \bigcup_{m\in\mathbb{N}} E_{\varpi,m} = E(\Lambda_f) \quad \text{with} \quad \Lambda_f := \bigcup_{m\in\mathbb{N}} \Lambda_{f,m}.$$

This is the totally ramified field from Theorem 2.1.1. We could prove at this point that

$E^{\mathrm{ab}} = E^{\mathrm{nr}} E_\varpi$, but we postpone the proof to the end of this section in order to define the Artin map. Lemma 2.2.3 implies the existence of an isomorphism

$$\mathcal{O}_E^\times \cong \varprojlim_{m\in\mathbb{N}} \mathcal{O}_E^\times/(1 + \varpi^m \mathcal{O}_E) \cong \varprojlim_{m\in\mathbb{N}} \mathrm{Gal}(E_{\varpi,m}/E) = \mathrm{Gal}(E_\varpi/E),$$

sending $u \in \mathcal{O}_E^\times$ to the unique E-linear automorphism σ of E_ϖ satisfying $\sigma(x) = [u]_F(x)$ for $x \in \Lambda_f$. As E_ϖ and E^{nr} have trivial intersection, we have

$$\mathrm{Gal}(E^{\mathrm{ab}}/E) = \mathrm{Gal}(E_\varpi E^{\mathrm{nr}}/E) = \mathrm{Gal}(E_\varpi/E) \times \mathrm{Gal}(E^{\mathrm{nr}}/E) \cong \mathcal{O}_E^\times \times \widehat{\mathbb{Z}}.$$

Using (2.1), this isomorphism identifies $\mathrm{W}_E^{\mathrm{ab}}$ with $\mathcal{O}_E^\times \times \mathbb{Z}$, which we may identify with $E^\times$ via $(u, m) \mapsto u\varpi^m$. We obtain the Artin map

$$\mathrm{Art}_E \colon E^\times \cong \mathcal{O}_E^\times \times \mathbb{Z} \to \mathrm{Gal}(E_\varpi/E) \times \Phi^{\mathbb{Z}} \cong \mathrm{W}_E^{\mathrm{ab}}.$$

By definition, this isomorphism satisfies $\mathrm{Art}_E(\varpi)|_{E^{\mathrm{nr}}} = \Phi$. This implies property (a) in Section 2.

Example. Consider the case $E = \mathbb{Q}_p$. As the polynomial $f(T) = (1 + T)^p - 1$ lies in $\mathcal{F}_p$, we find that the extension E_p is generated by the p power torsion points of the multiplicative formal $\mathbb{Z}_p$-module law $\widehat{\mathbb{G}}_m$ over $\mathcal{O}_{\overline{\mathbb{Q}}_p}$. Hence, the maximal abelian extension of $\mathbb{Q}_p$ is given by

$$\mathbb{Q}_p^{\mathrm{ab}} = \bigcup_{m\in\mathbb{N}} \mathbb{Q}_p(\zeta_{p^m}) \times \mathbb{Q}_p^{\mathrm{nr}} = \bigcup_{k\in\mathbb{N}} \mathbb{Q}_p(\zeta_k),$$

with Galois group $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times \times \widehat{\mathbb{Z}}$.

The construction of the extensions $E_{\varpi,m}$ also allows us to show the uniqueness of the Artin map.

Proof of uniqueness of the Artin Map. Let $\alpha \colon E^\times \to \mathrm{W}_E^{\mathrm{ab}}$ be an isomorphism that satisfies properties (a) and (b). Let $\varpi' \in E^\times$ be an arbitrary uniformizer. We show that properties (a) and (b) determine $\alpha(\varpi')$ uniquely. Let $E \subset E_{\varpi',m} \subset E_{\varpi'}$ be the finite subextension defined above. The field $E_{\varpi',m}$ is the splitting field of the polynomial η_m (as defined above, but with ϖ' in place of ϖ), which is irreducible and of even degree, and whose constant term is ϖ'. Hence $\varpi' \in \mathrm{N}_{E_{\varpi',m}/E}(E_{\varpi',m}^\times)$, so $\alpha(\varpi')|_{E_{\varpi',m}}$ is equal to the identity on $E_{\varpi',m}$ by property (b). Varying m, this implies $\alpha(\varpi')|_{E_{\varpi'}} = \mathrm{id}_{E_{\varpi'}}$. Using property (a), we find that $\alpha(\varpi')$ is the unique automorphism of E^{ab} with $\alpha(\varpi')|_{E_{\varpi'}} = \mathrm{id}_{E_{\varpi'}}$ and $\alpha(\varpi')|_{E^{\mathrm{nr}}} = \Phi$. As $E^\times$ is generated by the set of its uniformizers, it follows that α is uniquely determined by the properties (a) and (b). $\qquad\square$

Along the same lines, we can obtain a proof for the functoriality statement in Theorem 2.1.1.

Proof of functoriality of the Artin Map. It suffices to prove the theorem for unramified and totally ramified extensions L/E separately, and then point-wise for each uniformizer $\varpi \in L^\times$. If L/E is unramified, the statement just says that the Frobenius element of L is the $[L : E]$-th power of the Forbenius element of E, which is true. If L/E is totally ramified, we have to

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show that if $\varpi \in L^\times$ is a uniformizer mapping to the uniformizer $\varpi' = \mathrm{N}_{L/E}(\varpi)$ of $E^\times$, then $\mathrm{Art}_E(\varpi') = \mathrm{Art}_L(\varpi)|_{E^{\mathrm{ab}}} = \mathrm{id}$. As L/E is totally ramified, we have $\mathrm{Art}_L(\varpi)|_{E^{\mathrm{nr}}} = \Phi$. Hence, we have $\mathrm{Art}_L(\varpi)|_{E^{\mathrm{ab}}} = \mathrm{Art}_E(\varpi'')$ for some uniformizer $\varpi'' \in E^\times$. By the definition of ϖ', we have $(\varpi')^{\mathbb{Z}} = \mathrm{N}_{L_\varpi/E}(L_\varpi^\times)$. By the definition of ϖ'' we have $E_{\varpi''} \subseteq L_\varpi$, so that $\mathrm{N}_{L_\varpi/E}(L_\varpi^\times) = (\varpi'')^{\mathbb{Z}}$. This implies $\varpi'' = \varpi'$. This is what we wanted to show. $\qquad\square$

The remainder of this section is devoted to the proof of the local Kronecker–Weber Theorem in Theorem 2.1.1. The proof makes heavy use of the upper numbering for the ramification filtration. We don't explain these here; instead we refer to [Ser13, Chapter IV] for a survey of the theory. We do however note that in the case of Lubin–Tate extension, Lemma 2.2.3 allows us to calculate the ramification filtration explicitly (cf. [Gol81, Theorem 2]): we have

$$\mathrm{Gal}(E_{\varpi,m}/E)^v \cong \begin{cases} \mathcal{O}_E^\times/(1+\varpi^m\mathcal{O}_E) & \text{if } -1 < v \le 0 \\ (1+\varpi^i\mathcal{O}_E)/(1+\varpi^m\mathcal{O}_E) & \text{if } 0 \le i < v \le i+1 \le m \quad \text{for } i \in \mathbb{N}_0 \\ \{0\} & \text{if } m < v. \end{cases} \quad (2.4)$$

We need the following variant of Herbrand's theorem (cf. [Ser13, Chapter IV, Proposition 14]).

Theorem 2.2.4 (Herbrand's theorem)**.** *Let L be a finite Galois extension of E and let $H \subseteq \mathrm{Gal}(L/K)$ be a normal subgroup corresponding to the subextension $L^H \subseteq L$. Then, for any real number $v \ge -1$, we have*

$$\mathrm{Gal}(L/E)^v H/H \cong \mathrm{Gal}(L^H/K)^v = (\mathrm{Gal}(L/E)/H)^v.$$

Using Herbrand's Theorem, one quickly finds that the upper numbering filtration of $\mathrm{Gal}(E_\varpi/E)$ has jumps exactly at the non-negative integers, with

$$q - 1 \le [\mathrm{Gal}(E_\varpi/E)^v : \mathrm{Gal}(E_\varpi/E)^{v+1}].$$

By the theorem of Hasse–Arf, the same phenomenon occurs for any abelian extension (cf. [Arf40]).

Theorem 2.2.5 (Hasse–Arf)**.** *Suppose that E'/E is an abelian extension. If $v \in \mathbb{R}$ is a jump in the filtration $\mathrm{Gal}(E'/E)^v$, then v is an integer.*

These results suffice to show the following.

Lemma 2.2.6. *Let E'/E_ϖ be a totally ramified extension with E'/E Abelian. Then $E' = E_\varpi$.*

Sketch of Proof. We write $G = \mathrm{Gal}(E'/E)$ and $H = \mathrm{Gal}(E'/E_\varpi)$. We want to show that H is trivial. This will follow from the following two observations.

1. We have $\cap_{v \ge 0} G^v = \{0\}$. Indeed, for any open subgroup $J \subset G$ we have $\cap_{v \ge 0}(G/J)^v = 0$, and by definition $G^v = \lim_J (G/J)^v$.

2. Also by Herbrand's theorem, we have for integers $v \ge 0$ the equality

$$[G^v : G^{v+1}] = [(G/H)^v : (G/H)^{v+1}][G^v \cap H : G^{v+1} \cap H]. \quad (2.5)$$

The fact that E'/E is totally ramified implies that $\inf_{\varepsilon>0}[G^v : G^{v+\varepsilon}] \le q$ for any $v \ge 0$. By the Theorem of Hasse–Arf, these numbers are nonzero only if v is integral, so the left-hand side of the equation (2.5) is bounded from above by q (cf. [Gol81, Lemma 3]). Meanwhile, by (2.4), the first factor on the right-hand side of (2.5) is bounded from below by $q - 1$. This implies $[G^v \cap H : G^{v+1} \cap H] = 1$ for all $v \ge 0$. As E'/E is totally ramified, we have $H \subseteq G = G^0$, so the first observation allows us to deduce that $H = H \cap \left(\bigcap_{v \ge 0} G^v\right) = \{0\}$, as desired. $\qquad\square$

Proof of the local Kronecker–Weber theorem. Let E'/E be an Abelian extension, identified inside the algebraic closure $\overline{E}$. We show that $E' \subset E_\varpi E^{\mathrm{nr}}$. Consider the short exact sequence of Galois groups

$$\{1\} \to \mathrm{Gal}(E'E_\varpi E^{\mathrm{nr}}/E_\varpi E^{\mathrm{nr}}) \to \mathrm{Gal}(E'E_\varpi E^{\mathrm{nr}}/E_\varpi) \to \mathrm{Gal}(E_\varpi E^{\mathrm{nr}}/E_\varpi) \to \{1\}.$$

As the intersection of E_ϖ and E^{nr} is trivial, the group on the right is isomorphic to $\mathrm{Gal}(E^{\mathrm{nr}}/E)$, which is a free Abelian group. Hence the sequence admits a splitting $\mathrm{Gal}(E_\varpi E^{\mathrm{nr}}/E_\varpi) \to \mathrm{Gal}(E'E_\varpi E^{\mathrm{nr}}/E_\varpi)$. The fixed field of the image of this splitting is an extension L/E_ϖ with $LE^{\mathrm{nr}}E_\varpi = E'E^{\mathrm{nr}}E_\varpi$ and L/E Abelian. By the above lemma this implies $L = E_\varpi$, hence $E' \subset E^{\mathrm{nr}}E_\varpi$. This is what we wanted to show. $\qquad\square$

3 Non-Abelian Lubin-Tate Theory

In the preceeding section, we used formal $\mathcal{O}_E$-modules to understand the maximial abelian extension of a local field E. The hope of non-Abelian Lubin-Tate theory is to gain insight about the structure of non-Abelian extensions of E by studying certain deformation spaces of formal $\mathcal{O}_E$-modules. More precisely, attached to a formal $\mathcal{O}_E$-module $\mathbb{X}$ over $\overline{\mathbb{F}}_q$ (determined up to isomorphism by its height n), we attach a system $\{M_K\}_{K \subset \mathrm{GL}_n(\mathcal{O}_E)}$ of rigid analytic spaces over $\breve{E}$, the so-called Lubin-Tate Tower. Here, each space M_K arises as the generic fiber of a certain formal scheme $\mathcal{M}_K$ that parameterizes deformations of $\mathbb{X}$ with Drinfeld level K structure. Taking geometric fiber, each component of this system admits commuting right actions by the units of a division algebra D of invariant $1/n$ over E, and the Weil group W_E. Furthermore, $\mathrm{GL}_n(E)$ acts on the tower as a pro-object. Hence, For $\ell \neq p$, the colimit of compactly supported ℓ-adic cohomology groups

$$H_{\mathrm{LT}} = \operatorname*{colim}_K H_{\mathrm{c}}^{n-1}(M_K \otimes_{\breve{E}} \mathbb{C}_\varpi, \overline{\mathbb{Q}}_\ell)$$

admits an action by the group

$$G = \mathrm{GL}_n(E) \times \mathrm{W}_E \times D^\times.$$

By the results of non-Abelian Lubin–Tate Theory (cf. [HT01], [Mie10] and [Boy99]), the representation H_{LT} encodes the local Langlands correspondence and the Jacquet–Langlands correspondence for $\mathrm{GL}_n(E)$. The aim of this section is to give a precise statement of these results. Along the way, we introduce the important deformation functors and group actions.

3.1 Lubin–Tate Deformation Spaces

Let $\mathbb{X} \in (\mathcal{O}_E\text{-FML}/\overline{\mathbb{F}}_q)$ be the reduction of the standard formal $\mathcal{O}_E$-module law H of height n (cf. Section 1.5). The aim of this section is to constuct a certain moduli problem $\mathcal{M}_m$, parameterizing deformations of $\mathbb{X}$ with so-called Drinfeld level ϖ^m-structure. By results of VLADIMIR DRINFELD (cf. [Dri74]), these moduli problems turn out to be representable inside the category $\mathcal{C}$.

3.1.1 The Tower of Deformation Spaces

We mostly follow [Str08, Chapter 2] in the following exposition. We introduce a variant of the deformation functor introduced in Section 1.6. Recall the notion of quasi-isogenies from Section 1.3. We define a functor $\mathcal{M}_0 \colon \mathcal{C} \to (\mathrm{Set})$ on components $R \in \mathcal{C}$ via

$$\mathcal{M}_0(R) = \{(\mathcal{F}, \iota) \mid \mathcal{F} \in (\mathcal{O}_E\text{-FM}/R) \text{ and } \iota \colon \mathrm{FG}(\mathbb{X}) \to \mathcal{F} \otimes_R \overline{\mathbb{F}}_q \text{ a quasi-isogeny}\}/\sim,$$

where we say that two pairs $(\mathcal{F}, \iota)$ and $(\mathcal{F}', \iota')$ are isomorphic if and only if there is an isomorphism $\alpha \colon \mathcal{F} \to \mathcal{F}'$, such that $\iota' = \alpha_0 \circ \iota$, where α_0 denotes the reduction of α. Note that, since all formal $\mathcal{O}_E$-module laws over $\overline{\mathbb{F}}_q$ of the same height are isomorphic, this functor only depends on the height n.

We have a stratification
$$\mathcal{M}_0 = \coprod_{j \in \mathbb{Z}} \mathcal{M}_0^{(j)},$$

where $\mathcal{M}_0^{(j)}$ parameterizes deformations $(\mathcal{F}, \iota)$ with $\mathrm{ht}(\iota) = j$. Any quasi-isogeny $\tau \colon \mathbb{X} \dashrightarrow \mathbb{X}$ yields an isomorphism, functorial in R,

$$\mathcal{M}_0^{(j)}(R) \to \mathcal{M}_0^{(j+\mathrm{ht}(\tau))}(R), \quad (\mathcal{F}, \iota) \mapsto (\mathcal{F}, \iota \circ \tau).$$

In particular, for any two integers $j, j' \in \mathbb{Z}$, the two spaces $\mathcal{M}_0^{(j)}$ and $\mathcal{M}_0^{(j')}$ are isomorphic. Hence, by Theorem 1.6.2, $\mathcal{M}_0^{(j)}$ is represented by a local $\mathcal{O}_{\breve{E}}$-algebra $A_0^{(j)} \in \mathcal{C}$, noncanonically isomorphic to $\mathcal{O}_{\breve{E}}[\![u_1, \ldots, u_{n-1}]\!]$.

The functor $\mathcal{M}_0$ can be seen as the '0'-th layer of the Lubin–Tate tower. In order to define the higher layers, we need the notion of Drinfeld level structures.

Definition 3.1.1 (Drinfeld level ϖ^m structure). Let $\mathcal{F} \in (\mathcal{O}_E\text{-FM}/R)$ be a ϖ-divisible formal $\mathcal{O}_E$-module of height $n > 0$ and let m be a nonnegative integer. A *Drinfeld level ϖ^m structure* on $\mathcal{F}$ is a morphism of $\mathcal{O}_E$-modules

$$\phi \colon (\varpi^{-m}\mathcal{O}_E/\mathcal{O}_E) \to \mathcal{F}(R)$$

such that after choosing a coordinate $\mathcal{F} \cong \mathrm{Spf}(R[\![T]\!])$, the power series $[\varpi]_H(T) \in R[\![T]\!]$ satisfies the divisibility constraint

$$\prod_{x \in (\varpi^{-1}\mathcal{O}_E/\mathcal{O}_E)^n} (T - \phi(x)) \; \Big| \; [\varpi]_H(T).$$

Definition 3.1.2 (Lubin–Tate Deformation Space with Level Structure). Let $\mathcal{M}_m \colon \mathcal{C} \to$ (Set) be the functor assigning to $R \in \mathcal{C}$ the set

$$\mathcal{M}_m(R) \coloneqq \{(\mathcal{F}, \iota, \phi) \mid (\mathcal{F}, \iota) \in \mathcal{M}_0(R) \text{ and } \phi \text{ a Drinfeld level } \varpi^m\text{-structure on } \mathcal{F}\}/\simeq.$$

Just as in the case without Drinfeld level, we have a stratification $\mathcal{M}_m = \coprod_{j \in \mathbb{Z}} \mathcal{M}_m^{(j)}$, and any two functors $\mathcal{M}_m^{(j)}$ and $\mathcal{M}_m^{(j')}$ are (non-canonically) isomorphic. Furthermore, for nonnegative integers $m' \le m$, we have natural morphisms $\mathcal{M}_m \to \mathcal{M}_{m'}$ by restricting the level structure. By results of Drinfeld, each functor $\mathcal{M}_m^{(0)}$ is representable.

Theorem 3.1.3 (Representability of the Lubin–Tate Deformation Space with Level Structure). *The functor $\mathcal{M}_m^{(0)}$ is represented by a regular local ring $A_m \in \mathcal{C}$ of dimension $m - 1$. Writing $(\mathcal{F}^{\mathrm{univ}}, \iota^{\mathrm{univ}}, \phi_m^{\mathrm{univ}}) \in \mathcal{M}_m(A_m)$ for a universal triple, a system of local parameters is given by the images of the basis vectors in ϕ_m^{univ}. Furthermore, for any $m' < m$, the morphism $A_{m'} \to A_m$ is finite and flat, and it is étale after inverting ϖ.*

Proof. This is [Dri74, Proposition 4.3], also see [Str08, Proposition 2.2.5]. $\square$

Our main interest will not lie in the individual spaces $\mathcal{M}_m$, but rather in the resulting tower $\{\mathcal{M}_m\}_{m \ge 0}$, which we call the tower of Lubin–Tate deformation spaces.

3.1.2 The Case of Height One

Let $\widehat{\breve{E}}^{\mathrm{ab}}$ denote the closure of the maximal abelian extension of E inside $\mathbb{C}_\varpi$, and let $E_m^{\mathrm{LT}} :=$ $E_{\varpi,m}\breve{E} \subset E^{\mathrm{ab}}$ be the composition of $\breve{E}$ with the totally ramified extension constructed in Section 2. As all Lubin–Tate formal $\mathcal{O}_E$-modules of height one are isomorphic (this follows either by Theorem 1.6.2 or directly by [LT65, Lemma 2]), this extension is independent of ϖ.

Suppose that $\mathbb{X} \in (\mathcal{O}_E\text{-FML}/\overline{\mathbb{F}}_q)$ is the reduction of the standard $\mathcal{O}_E$-module law H over $\overline{\mathbb{F}}_q$ of height one. By Theorem 1.6.2, the corresponding deformation functor $\mathcal{M}_0^{(0)}$ is isomorphic to $\mathcal{O}_{\breve{E}}$, so for any $R \in \mathcal{C}$, all deformations $(\mathcal{F}, \iota)$ of $\mathbb{X}$ to R differ by a (unique) isomorphism. We may therefore choose a coordinate on $\mathcal{F}$ that gives rise to the unique Lubin–Tate module law F with $[\varpi]_F(T) = \varpi T + T^q$. Now, a morphism

$$\phi \colon \varpi^{-m}\mathcal{O}_E/\mathcal{O}_E \to \mathfrak{m}_R = F(R)$$

constitutes a Drinfeld ϖ^m level structure if and only if ϕ maps the generator $e = \varpi^{-m} \in \varpi^{-m}\mathcal{O}_E/\mathcal{O}_E$ to a root of the Eisenstein polyonomial

$$\eta_1([\varpi^{m-1}](T)) = \frac{[\varpi^m](T)}{[\varpi^{m-1}](T)} = \eta_m(T)$$

(cf. Section 2). Hence, we find that

$$\mathcal{M}_m^{(0)} = \frac{\mathcal{O}_{\breve{E}}[T]}{\eta_m(T)} \cong \mathcal{O}_{E_m^{\mathrm{LT}}}, \tag{3.1}$$

where the choice of an isomorphism is equivalent to the choice of a level structure on $F(\mathcal{O}_{E_m^{\mathrm{LT}}})$ by Lemma 2.2.3.

Now, let $n \geq 1$ and let $\mathbb{X}$ be the reduction of the standard formal $\mathcal{O}_E$-module of height n. Let $\wedge\mathbb{X}$ be the determinant of $\mathbb{X}$, that is, the formal $\mathcal{O}_E$-module with $\mathrm{D}(\wedge\mathbb{X}) = \wedge^n\mathrm{D}(\mathbb{X})$ (it arises as the reduction of the formal $\mathcal{O}_E$-module constructed in Example 1.12.3). Write $\mathcal{M}_m^{(0)}$ for the deformation space of $\mathbb{X}$ with Drinfeld ϖ^m level structure and $\mathcal{M}_{m,\wedge}^{(0)}$ for the deformation space of $\wedge\mathbb{X}$.

Let R be an arbitrary object in $\mathcal{C}$ and let $(\mathcal{F}, \iota) \in \mathcal{M}_0^{(0)}(R)$. Recall the construction of determinants of formal $\mathcal{O}_E$-modules from Section 1.12. We have the following result.

Lemma 3.1.4. *We have $(\wedge^n\mathcal{F}, \wedge^n\iota) \in \mathcal{M}_{0,\wedge}^{(0)}(R)$. Moreover, let $\phi \colon \varpi^{-m}\mathcal{O}_E^n/\mathcal{O}_E^n \to \mathcal{F}^n(R)$ be a Drinfeld level ϖ^m structure on $\mathcal{F}$. Then*

$$\wedge^n\phi \colon \varpi^{-m}\mathcal{O}_E/\mathcal{O}_E \to \wedge^n\mathcal{F}(R), \quad \varpi^{-m} \mapsto \delta_m(\varpi^{-m}e_1, \ldots, \varpi^{-m}e_n)$$

is a Drinfeld level ϖ^m structure on $\wedge^n\mathcal{F}$.

Proof. This is [Wei16, Proposition 2.11]. $\square$

In particular, we obtain a natural map

$$\mathcal{M}_m^{(0)}(R) \to \mathcal{M}_{m,\wedge}^{(0)}(R), \quad (\mathcal{F}, \iota, \phi) \mapsto (\wedge^n\mathcal{F}, \wedge^n\iota, \wedge^n\phi).$$

In particular, as $\wedge\mathbb{X}$ is of height 1, we obtain a morphism $\mathcal{M}_m^{(0)} \to \mathcal{O}_{E_m^{LT}}$ upon choosing a Drinfeld ϖ^m level structure on $F(\mathcal{O}_{E_m^{LT}})$.

3.1.3 Group Actions on the Tower of Lubin–Tate Deformation Spaces

We describe actions of $\mathrm{GL}_n(E) \times D^\times$ on the tower $\{\mathcal{M}_m\}_{m\geq 0}$. More precisely, given an element $d \in D^\times$ and an element $g \in \mathrm{GL}_n(E)$, we construct, for sufficiently large $m \geq 0$, morphisms

$$d_m \colon \mathcal{M}_m^{(j)} \to \mathcal{M}_m^{(j')} \quad \text{and} \quad g_{m,m''} \colon \mathcal{M}_m^{(j)} \to \mathcal{M}_{m''}^{(j'')},$$

where $j' = j + \mathrm{val}_\varpi(\mathrm{Nrd}(d))$, $j'' = j - \mathrm{val}_\varpi(\det g)$ and $m'' = m - \delta$ is an integer differeing from m by an integer depending on g.

The action of $D^\times$ is easy to describe. Given $R \in \mathcal{C}$ and $d \in D^\times$, we put

$$(\mathcal{F}, \iota, \phi).d = (\mathcal{F}, \iota \circ d, \phi).$$

The group $\mathrm{GL}_n(E)$ acts in a less simple matter. Similarly to the action of $D^\times$, we would like to define the action as $(\mathcal{F}, \iota, \phi).g = (\mathcal{F}, \iota, \phi \circ g)$, but this only makes sense if $g \in \mathrm{GL}_n(\mathcal{O}_E)$. To extend this action to all of $\mathrm{GL}_n(E)$, we allow ourselves to change the underlying formal group. We need the following constructions.

Definition 3.1.5 (Quotients with respect to Level Structure). Let $R \in \mathcal{C}$ and let $(\mathcal{F}, \iota, \phi) \in \mathcal{M}_m(R)$. Let $P \subseteq (\varpi^{-m}\mathcal{O}_E/\mathcal{O}_E)^n$ be a submodule. We define the quotient $(\mathcal{F}/\phi(P))$ as follows. Let $(\mathcal{F}^{\mathrm{univ}}, \iota^{\mathrm{univ}}, \phi^{\mathrm{univ}}) \in \mathcal{M}_m(A_m)$ be the universal triple, and let $\alpha \colon A_m \to R$ be the morphism giving rise to $(\mathcal{F}, \iota, \phi)$. Then $\phi^{\mathrm{univ}}(P) \subset \mathcal{F}^{\mathrm{univ}}(A_m)$ is a finite subset. As A_m satisfies all the conditions imposed in Section 1.7, we may take the quotient $(\mathcal{F}^{\mathrm{univ}}/\phi^{\mathrm{univ}}(P))$ as in Theorem 1.7.1. We define $(\mathcal{F}/\phi(P))$ as $(\mathcal{F}^{\mathrm{univ}}/\phi^{\mathrm{univ}}(P)) \otimes_{A_m,\alpha} R$.

Lemma 3.1.6. *If $\#P = q^c$, the induced morphism $\mathcal{F} \to (\mathcal{F}/\phi(P))$ of formal module laws over R is of height c.*

Proof. It suffices to check this for the universal triple, where it is part 2 of Theorem 1.7.1. $\square$

Lemma 3.1.7. *As above, let $R \in \mathcal{C}$, let $(\mathcal{F}, \iota, \phi) \in \mathcal{M}_m(R)$ and let $P \subseteq (\varpi^{-m}\mathcal{O}_E/\mathcal{O}_E)^n$ be a submodule.*

1. *There is a unique natural morphism of $\mathcal{O}_E$-modules*

$$\overline{\phi} \colon \frac{(\varpi^{-m}\mathcal{O}_E/\mathcal{O}_E)^n}{P} \to (\mathcal{F}/\phi(P))(R)$$

compatible with the maps $(\varpi^{-m}\mathcal{O}_E/\mathcal{O}_E)^n \to \frac{(\varpi^{-m}\mathcal{O}_E/\mathcal{O}_E)^n}{P}$ and $\mathcal{F} \to (\mathcal{F}/\phi(P))$.

2. *Suppose that there is an injection*

$$\left(\varpi^{-m'}\mathcal{O}_E/\mathcal{O}_E\right)^n \to \frac{(\varpi^{-m}\mathcal{O}_E/\mathcal{O}_E)^n}{P}.$$

Then, the induced morphism

$$\phi' : (\varpi^{-m'} \mathcal{O}_E / \mathcal{O}_E)^n \to (\mathcal{F}/\phi(P))(R)$$

is a Drinfeld ϖ^m level structure.

Proof. This is [Dri74, Proposition 4.4]. Again, it suffices to assume that $R = A_m$ and that $\mathcal{F}$ is the universal formal module with level ϖ^m structure. We may choose a coordinate $\mathcal{F} \cong \mathrm{FG}(F)$. The first statement follows as over A_m, we have $f_{\phi(P)}(\phi(a)) = 0$ for $a \in P$, where $f_{\phi(P)}$ is the power series arising in the construction of $(F/\phi(P))$, cf. Theorem 1.7.1. The second claim follows because the morphism $(\varpi^{-m'} \mathcal{O}_E^n / \mathcal{O}_E^n) \to \mathfrak{m}_{A_m}$, induced by ϕ', is injective. $\qquad\square$

This allows us to construct the $\mathrm{GL}_n(E)$-action. Assume that $g \in \mathrm{GL}_n(E)$ is such that $g^{-1} \in \mathrm{Mat}_{n \times n}(\mathcal{O}_E)$ and $g \in \varpi^{-\delta} \mathrm{Mat}_{n \times n}(\mathcal{O}_E)$ for some nonnegative integer δ. In this case, we construct for all integers $m \geq \delta$ a natural transformation

$$g_{m,m-\delta} : \mathcal{M}_m \to \mathcal{M}_{m-\delta}.$$

Note that $g\mathcal{O}_E^n \subset \varpi^{-\delta}\mathcal{O}_E^n$, and that multiplication with g yields an injection

$$(\varpi^{m-\delta}\mathcal{O}_E/\mathcal{O}_E)^n \xrightarrow{g} (\varpi^{-m}\mathcal{O}_E^n/g\mathcal{O}_E^n) = \frac{(\varpi^{-m}\mathcal{O}_E^n/\mathcal{O}_E^n)}{(g\mathcal{O}_E^n/\mathcal{O}_E^n)}.$$

Now, given a tuple $(\mathcal{F}, \iota, \phi) \in \mathcal{M}_m(R)$, we put

$$(\mathcal{F}, \iota, \phi).g = (\mathcal{F}', \iota', \phi'),$$

where

$$\mathcal{F}' = \mathcal{F}/\phi(g\mathcal{O}_E^n/\mathcal{O}_E^n)$$

is a the quotient of $\mathcal{F}$ as in Definition 3.1.5,

$$\iota' : \mathcal{F}_0 \to \mathcal{F} \otimes \overline{\mathbb{F}}_q \to \mathcal{F}' \otimes \overline{\mathbb{F}}_q$$

is the corresponding quasi-isogeny of height $(\mathrm{ht}(\iota) - \mathrm{val}_\varpi(\det g))$, and

$$\phi' : (\varpi^{m-\delta}\mathcal{O}_E/\mathcal{O}_E)^n \to \mathcal{F}'$$

is the Drinfeld $\varpi^{m-\delta}$ level structure obtained by Lemma 3.1.7. For varying choices of δ, this gives a system of maps compatible with the transition functions $\mathcal{M}_m \to \mathcal{M}_{m'}$. Indeed, given integers $m \geq \delta' \geq \delta$ with δ as above, the triangle

$$
\begin{array}{ccc}
\mathcal{M}_m & \xrightarrow{g_{m,m-\delta}} & \mathcal{M}_{m-\delta} \\
& {\scriptstyle g_{m,m-\delta'}} \searrow & \downarrow \\
& & \mathcal{M}_{m-\delta'}
\end{array}
$$

commutes.

If $g \in \mathrm{GL}_n(E)$ is an arbitrary element, we may choose an integer r such that $(\varpi^{-r}g)^{-1} \in \mathrm{Mat}_{n \times n}(\mathcal{O}_E)$. We now pick $\delta > 0$ such that $\varpi^{-r}g \in \varpi^{-\delta}\mathrm{Mat}_{n \times n}(\mathcal{O}_E)$. Now, for $m \geq \delta$, we obtain natural transformations

$$g_{m,m-\delta} \colon \mathcal{M}_m \to \mathcal{M}_{m-\delta}, \quad (\mathcal{F}, \iota, \phi).g = (\mathcal{F}, \iota \circ \varpi^{-r}, \phi).(\varpi^{-r}g).$$

By the same reason as above, this yields, for varying choices of δ, a compatible system of natural transformations. This construction is independent of the choice of r, as $(\mathcal{F}, \iota, \phi).(\varpi \cdot \mathrm{id}) \simeq (\mathcal{F}, \iota \circ [\varpi^{-1}], \phi)$. Furthermore, given $g, g' \in \mathrm{GL}_n(E)$ with suitable choices of integers δ, δ', we obtain a commutative triangle

$$
\begin{array}{ccc}
\mathcal{M}_m & \xrightarrow{g_{m,m-\delta}} & \mathcal{M}_{m-\delta} \\
& \searrow{\scriptstyle (gg')_{m,m-\delta-\delta'}} & \downarrow{\scriptstyle g'_{m-\delta,m-\delta-\delta'}} \\
& & \mathcal{M}_{m-\delta-\delta'}
\end{array}
$$

This finishes the construction of the $\mathrm{GL}_n(E)$-action.

We will need to consider a refinement of the tower $\{\mathcal{M}_m\}_{m \in \mathbb{N}}$. We define the m-th congruence subgroups

$$K_m = \mathrm{Ker}(\mathrm{GL}_n(\mathcal{O}_E) \to \mathrm{GL}_n(\mathcal{O}_E/(\varpi^m))). \tag{3.2}$$

For any compact open subgroup $K \subset \mathrm{GL}_n(\mathcal{O}_E)$, we may choose integers $m > m'$ such that $K_m \subseteq K \subseteq K_{m'}$. We define

$$A_K^{(j)} := (A_m^{(j)})^K, \quad \mathcal{M}_K^{(j)} := \mathrm{Spf}(A_K^{(j)}), \quad \mathcal{M}_K = \coprod_{j \in \mathbb{Z}} \mathcal{M}_K^{(j)}.$$

By [Str08, Proposition 2.2.5], these constructions are well defined and yield the refined tower of deformation spaces $\{\mathcal{M}_K\}_{K \subset \mathrm{GL}_n(\mathcal{O}_E)}$, where K ranges over the compact open subgroups of $\mathrm{GL}_n(\mathcal{O}_E)$. Any ring $A_K^{(j)}$ is local, noetherian, integrally closed and finite over $A_{m'}^{(j)}$, and complete with respect to the topology induced by its maximal ideal. Furthermore, $A_K^{(j)}[\frac{1}{\varpi}]$ is étale over $A_m^{(j)}[\frac{1}{\varpi}]$, and it is Galois with Galois group K/K_m. Conversely, $A_m^{(j)}[\frac{1}{\varpi}]$ is étale over $A_K^{(j)}$, and if K is normal in $K_{m'}$ then it is also Galois with Galois group $K_{m'}/K$.

3.1.4 The Weil Descent Datum on the Deformation Space

The aim of this section is to define the right action of the Weil group on the formal scheme $\mathcal{M}_m \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathrm{Spf}(\mathcal{O}_{\mathbb{C}_\varpi})$, which will be induced from a certain Weil descent datum on $\mathcal{M}_m$. We recall the notion of Weil descent data and show how they induce an action of the Weil group.

Let $\mathcal{G}$ be a functor $(\mathcal{O}_{\breve{E}}\text{-Adm}) \to (\text{Set})$. As usual, we denote by $\Phi \in \mathrm{Gal}(E^{\mathrm{nr}}/E)$ the automorphism corresponding to the q-th power Frobenius on the residue field. We denote by $\Phi^\flat \colon \mathrm{Spf}(\mathcal{O}_{\breve{E}}) \to \mathrm{Spf}(\mathcal{O}_{\breve{E}})$ the corresponding morphism of formal schemes, and define

$$\Phi^*\mathcal{G} := \mathcal{G} \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}}),\Phi^\flat} \mathrm{Spf}(\mathcal{O}_{\breve{E}}).$$

Note that for any admissible $\mathcal{O}_{\breve{E}}$-algebra $\mathcal{O}_{\breve{E}} \xrightarrow{i} R$, we have $\Phi^*\mathcal{G}(R) = \mathcal{G}(\Phi_*R)$, where Φ_*R is the admissible $\mathcal{O}_{\breve{E}}$-algebra with structure morphism $\mathcal{O}_{\breve{E}} \xrightarrow{i \circ \Phi} R$.

Definition 3.1.8 (Weil Descent Datum). A *Weil descent datum* is an isomorphism of functors

$$\alpha \colon \mathcal{G} \to \Phi^*\mathcal{G}.$$

Now let E'/E^{nr} be a finite Galois field extension and let $w \in \mathrm{Gal}(E'/E)$ be an element with $w|_{\breve{E}} = \Phi^m$ for some $m \in \mathbb{Z}$. Now w yields a morphism

$$\mathcal{G} \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathrm{Spf}(\mathcal{O}_{\widehat{E'}}) \xrightarrow{(1, w^\flat)} \Phi^{-m,*}\mathcal{G} \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathrm{Spf}(\mathcal{O}_{\widehat{E'}}).$$

Write α^m for the morphism $\mathcal{G} \to \Phi^{m,*}\mathcal{G}$ induced by iterating the Weil descent datum. We obtain a commutative diagram

$$
\begin{array}{ccc}
\mathcal{G} \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathrm{Spf}(\mathcal{O}_{\widehat{E'}}) & \xrightarrow{(\mathrm{id}, w^\flat)} & \Phi^{-m,*}\mathcal{G} \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathrm{Spf}(\mathcal{O}_{\widehat{E'}}) \\
{\scriptstyle(\alpha^m, \mathrm{id})} \downarrow & \searrow {\scriptstyle \delta_\alpha(w)} & \downarrow {\scriptstyle(\Phi^{-m,*}\alpha, \mathrm{id})} \\
\Phi^{m,*}\mathcal{G} \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathrm{Spf}(\mathcal{O}_{\widehat{E'}}) & \xrightarrow[(\mathrm{id}, w^\flat)]{} & \mathcal{G} \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathrm{Spf}(\mathcal{O}_{\widehat{E'}}).
\end{array}
$$

With $\delta_\alpha(w)$ defined this way, one quickly verifies the following result.

Lemma 3.1.9. *The assignment $w \mapsto \delta_\alpha(w)$ defines a right action of $\mathrm{Gal}(E'/E)$ on $\mathcal{G} \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathrm{Spf}(\mathcal{O}_{\widehat{E'}})$.*

Through the surjective morphsim $\mathrm{W}_E \to \mathrm{Gal}(E'/E)$, the Weil group acts on $\mathcal{G} \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathrm{Spf}(\mathcal{O}_{\widehat{E'}})$.

Lemma 3.1.10. *The action defined this way is functorial in E'. That is, if $E^{\mathrm{nr}} \subset E' \subset E''$ is a tower of finite extensions inside $\overline{E}$, then the induced morphism of functors $\mathcal{G} \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathrm{Spf}(\mathcal{O}_{\widehat{E''}}) \to \mathcal{G} \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathrm{Spf}(\mathcal{O}_{\widehat{E'}})$ is W_E-equivariant.*

As a direct consequence, we obtain an action of W_E on

$$\varprojlim_{E'/E \text{ finite Galois}} \mathcal{G} \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathrm{Spf}(\mathcal{O}_{\widehat{E'}}) = \mathcal{G} \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathrm{Spf}(\mathcal{O}_{\mathbb{C}_\varpi}).$$

Note that this action only respects the $\mathcal{O}_E$-structure.

Example. Assume that $\mathcal{G} = \mathrm{Spf}(R)$, where R is an $\mathcal{O}_{\breve{E}}$-algebra of the form

$$R = \frac{\mathcal{O}_{\breve{E}}[\![X_1, \ldots, X_k]\!]}{(f_1, \ldots, f_l)}.$$

Then $\Phi^*\mathcal{G} = \mathrm{Spf}(R \widehat{\otimes}_{\mathcal{O}_{\breve{E}}, \Phi} \mathcal{O}_{\breve{E}})$, hence a Weil descent datum on $\mathcal{G}$ is the same as an isomorphism of $\mathcal{O}_{\breve{E}}$-algebras

$$\alpha \colon \frac{\mathcal{O}_{\breve{E}}[\![X_1, \ldots, X_k]\!]}{(f_1^\Phi, \ldots, f_l^\Phi)} \to \frac{\mathcal{O}_{\breve{E}}[\![X_1, \ldots, X_k]\!]}{(f_1, \ldots, f_l)}.$$

The action of W_E on $\mathcal{G} \times_{\mathcal{O}_{\breve{E}}} \mathcal{O}_{\mathbb{C}_\varpi}$ is corresponds to a left action on $R \widehat{\otimes}_{\mathcal{O}_{\breve{E}}} \mathcal{O}_{\mathbb{C}_\varpi}$. Given $w \in W_E$ with $w|_{\breve{E}} = \Phi^m$, the corresponding isomorphism takes the form

$$\delta_\alpha(w)^\sharp \colon \frac{\mathcal{O}_{\mathbb{C}_\varpi}[\![X_1, \ldots, X_k]\!]}{(f_1, \ldots, f_l)} \to \frac{\mathcal{O}_{\mathbb{C}_\varpi}[\![X_1, \ldots, X_k]\!]}{(f_1, \ldots, f_l)}; \quad a \mapsto w(a), \quad X_i \mapsto \alpha^m(X_i).$$

This isomorphism only respects the $\mathcal{O}_E$-structure on both sides.

We now define a Weil descent datum α on the deformation spaces with Drinfeld ϖ^m level structure $\mathcal{M}_m$. As $\mathcal{M}_m$ is representable in $\mathcal{C}$, it suffices to construct the isomorphism $\mathcal{M}_m \to \Phi^* \mathcal{M}_m$ on objects $R \in \mathcal{C}$.

Given $R \in \mathcal{C}$ and $(\mathcal{F}, \iota, \phi) \in \mathcal{M}_m(R)$, we put

$$\alpha(R)\,(\mathcal{F}, \iota, \phi) = (\mathcal{F}', \iota', \phi'),$$

where

- the formal $\mathcal{O}_E$-module $\mathcal{F}'$ is given by $\mathcal{F}' = \mathcal{F} \otimes_{R, \mathrm{id}} \Phi_* R = \Phi^* \mathcal{F}$.

- the quasi-isogeny ι' is defined as

$$\mathbb{X} \overset{\mathrm{Frob}_q^{-1}}{\dashrightarrow} \mathbb{X}^{(-q)} = \mathbb{X} \otimes_{\mathbb{F}_q, \mathrm{Frob}^{-1}} \overline{\mathbb{F}}_q \dashrightarrow \mathcal{F} \otimes_{R, \mathrm{id}} (\Phi_* R / \mathfrak{m}_R) = (\Phi^* \mathcal{F}) \otimes_R \overline{\mathbb{F}}_q.$$

  Here Frob_q^{-1} denotes the inverse quasi-isogeny of the relative Frobenius morphism $\mathrm{Frob}_q \colon \mathbb{X}^{(-q)} \to \mathbb{X}$. In particular, $\mathrm{ht}(\iota') = \mathrm{ht}(\iota) - 1$.

- the level structure ϕ' is

$$\left(\varpi^{-m} \mathcal{O}_E / \mathcal{O}_E\right)^m \overset{\phi}{\to} \mathcal{F}(R) = \Phi^* \mathcal{F}(R).$$

As explained above, this yields a right action of W_E on $\mathcal{M}_{m, \mathcal{O}_{E'}}$. Given $w \in W_E$ such that $w|_{\breve{E}} = \Phi^m$ for $m \in \mathbb{Z}$, we find that δ_w restricts to an isomorphism $\mathcal{M}_{m, \mathcal{O}_{E'}}^{(j)} \to \mathcal{M}_{m, \mathcal{O}_{E'}}^{(j-m)}$.

Example. The action of W_E on $(\mathcal{M}_m \widehat{\otimes}_{\mathcal{O}_{\breve{E}}} \mathcal{O}_{E'})(\mathcal{O}_{E'})$ admits a simple description. Suppose that we are given (an equivalence class of) a triple $(\mathcal{F}, \iota, \phi) \in \mathcal{M}_m(\mathcal{O}_{E'})$ and an element $w \in W_E$ such that $w|_{\breve{E}} = \Phi^m$ for $m \in \mathbb{Z}$. We may choose a coordinate $\mathcal{F} = \mathrm{FG}(F)$, where F is some formal $\mathcal{O}_E$-module law over E'. Now

$$\delta_\alpha(w)(\mathcal{F}, \iota, \phi) = (\mathrm{FG}(F^w), \iota^w, \phi^w),$$

where F^w is the formal group law over E' obtained by applying $w^{-1}|_{E'}$ to F coefficient-wise. The quasi-isogeny ι^w may be described in these coordinates as the composite

$$\mathbb{X} \overset{\mathrm{Frob}_q^{-m}}{\longrightarrow} \mathbb{X}^{(q^{-m})} \overset{\iota^{(q^{-m})}}{\longrightarrow} (F \otimes \overline{\mathbb{F}}_q)^{(q^{-m})} = F^w \otimes \overline{\mathbb{F}}_q,$$

while the level structure ϕ^w is obtained by post-composing ϕ with w^{-1}:

$$\phi^w = w^{-1}|_{\mathfrak{m}_{E'}} \circ \phi \colon (\varpi^{-m} \mathcal{O}_E / \mathcal{O}_E)^n \to \mathfrak{m}_{\mathcal{O}_{E'}}.$$

3.2 The Étale Cohomology of the Lubin–Tate Tower

Let us fix an identification $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. In the preceding paragraphs, we explained that $\{\mathcal{M}_K\}_{K \subset \mathrm{GL}_n(\mathcal{O}_E)}$ is a tower of locally noetherian formal schemes with finite and generically étale transition maps, equipped with an action by $\mathrm{GL}_n(E) \times D^\times$ and a Weil descent datum. Utilizing a classical construction of Raynauld (cf. [Ray74]), we have the rigid generic fiber M_K of $\mathcal{M}_K$, which we may consider as an analytic adic space over $\mathrm{Spa}(\breve{E}, \mathcal{O}_{\breve{E}})$ (cf. [Hub96, Section 1.9]). Ranging over m, this yields a tower of spaces $\{M_K\}_{K \subset \mathrm{GL}_n(\mathcal{O}_E)}$ whose transition maps are finite and étale. Furthermore, each component admits a Weil descent datum and an action by $D^\times$, and $\mathrm{GL}_n(E)$ acts on the tower as a pro-object.

This section is concerned with the representation-theoretic aspects of the compactly supported étale cohomology groups

$$\mathrm{colim}_K H_c^{n-1}\left(M_K \times_{\mathrm{Spa}(\breve{E}, \mathcal{O}_{\breve{E}})} \mathrm{Spa}(\mathbb{C}_\varpi, \mathcal{O}_{\mathbb{C}_\varpi}), \overline{\mathbb{Q}}_\ell\right).$$

The right-action of $G := \mathrm{GL}_n(E) \times D^\times \times W_E$ on the tower $\{\mathcal{M}_{K, \mathcal{O}_{\mathbb{C}_\varpi}}\}_{K \subset \mathrm{GL}_n(\mathcal{O}_E)}$ induces the structure of a G-representation on this vector space. The supercuspidal part of this representation realizes the local Langlands correspondence.

Definition 3.2.1 (Supercuspidal Representation). An admissible (cf. Definition B.2) representation $(\pi, V) \in (\mathrm{GL}_n(E)\text{-Rep})$ is called *supercuspidal* if all of its matrix coefficients are constant modulo the center $E^\times \subseteq \mathrm{GL}_n(E)$.

Here, a matrix coefficient is a function of the form $m(g) = \langle \pi(g)\phi, \phi^\vee \rangle$, where $(\phi, \phi^\vee)$ ranges over the elements of $V \times V^\vee$ (where $\vee$ denotes the contragradient). For details and the important relation to the method of parabolic induction, we refer to [GH23, Section 8].

We briefly recall the local Langlands are Jacquet–Langlands correspondences. The local Langlands correspondence gives a bijection

$$\mathrm{rec}_E \colon \left\{\begin{matrix} \text{supercuspidal representations} \\ \text{of } \mathrm{GL}_n(E) \end{matrix}\right\}\Big/ \cong \xrightarrow{1:1} \left\{\begin{matrix} \text{irreducible representations} \\ \text{of } W_E \end{matrix}\right\}\Big/ \cong .$$

Of course, not any such bijection is interesting; the local Langlands correspondence predicts the existence of a unique bijection rec_E compatible with various structures on both sides. See [GH23, Section 12.4] for a detailed account of the axioms rec_E should satisfy. In particular, case $n = 1$ recovers local class field theory: $\mathrm{rec}_E(\pi) = \pi \circ \mathrm{Art}_E^{-1}$.

The local Jacquet–Langlands correspondence gives a bijection

$$\mathrm{JL} \colon \left\{\begin{matrix} \text{irreducible discrete series} \\ \text{representations of } \mathrm{GL}_n(E) \end{matrix}\right\}\Big/ \cong \xrightarrow{1:1} \left\{\begin{matrix} \text{irreducible representations} \\ \text{of } D^\times. \end{matrix}\right\}\Big/ \cong,$$

characterized by a natural character identity. This result is due to Deligne–Bernstein–Kazhdan–Vigneras cf. [Del+84] and Rogawski [Rog83]. Here, neither the definition of discrete series representations nor the precise statement is of major importance to us; we only need the fact that all supercuspidal representations of $\mathrm{GL}_n(E)$ are discrete series representations.

Let π be a supercuspidal representation of $\mathrm{GL}_n(E)$. By results of Boyer (cf. [Boy99]) in the equal characteristic case and Harris–Taylor (cf. [HT01]) in the mixed characteristic case, both $\mathrm{rec}_F(\pi)$ and $\mathrm{JL}(\pi)$ occur (up to duals and twists) in the π isotypic component of the cohomology of the tower $\{M_K \otimes_{\breve{E}} \mathbb{C}_\varpi\}_K$. Moreover, Boyer proved that the correspondence takes place in the middle degree cohomology if E is a local function field and the same result for mixed characteristic is due to Mieda; cf. [Mie10]. We state a precise version of this result below.

As the Jacquet–Langlands and local Langlands correspondences behave well with respect to character twist, it will be sufficient to consider supercuspidal representations whose central character is trivial on $\varpi^{\mathbb{Z}}$. For these representations, the correspondence already takes place in the cohomology of the quotients $M_{K,\varpi^{\mathbb{Z}}} := M_K/\varpi^{\mathbb{Z}}$ by the action of the subgroup $\varpi^{\mathbb{Z}} \subseteq D^\times$. Recall that the action of ϖ restricts to an isomorphism $\mathcal{M}_K^{(j)} \to \mathcal{M}_K^{(j+n)}$, so we find $M_{K,\varpi^{\mathbb{Z}}} \cong \coprod_{0 \le j < n} \mathcal{M}_K^{(j)}$. We now define

$$H_{\mathrm{LT}} := \operatorname*{colim}_K H_c^{n-1}(M_{K,\varpi^{\mathbb{Z}}} \otimes_{\breve{E}} \mathbb{C}_\varpi, \overline{\mathbb{Q}}_\ell), \tag{3.3}$$

where K ranges over the compact open subgroups of $\mathrm{GL}_n(\mathcal{O}_E)$. For an irreducible representation π of GL_n, we denote by $H_{\mathrm{LT},\pi}$ the π-isotypical component of H_{LT}, which is

$$H_{\mathrm{LT},\pi} = \Big\langle v \in H_{\mathrm{LT}} : \langle (g,1,1).v \mid g \in \mathrm{GL}_n(E) \rangle \cong \pi \Big\rangle \subseteq H_{\mathrm{LT}}.$$

The precise statement of Non-Abelian Lubin–Tate Theory is the following.

Theorem 3.2.2 (Non-Abelian Lubin–Tate Theory). *Let π be an irreducible supercuspidal representation of $\mathrm{GL}_n(E)$ whose central character is trivial on $\varpi^{\mathbb{Z}}$. Then we have*

$$H_{\mathrm{LT},\pi^\vee} = \pi^\vee \boxtimes \mathrm{JL}(\pi) \boxtimes \mathrm{rec}_F(\pi)(\tfrac{1-n}{2})$$

as representations of $\mathrm{GL}_n(E) \times D^\times \times W_F$.

Let $\wedge\mathbb{X}$ denote the determinant of $\mathbb{X}$ as defined in Section 1.12 (it is a formal $\mathcal{O}_E$-module law over $\overline{\mathbb{F}}_q$ of height 1, thus unique up to isomorphism) and denote by $\mathcal{M}_{m,\wedge}^{(0)}$ the corresponding deformation functors of $\wedge\mathbb{X}$. In Section 3.1.2, we constructed a natural determinant map $\mathcal{M}_m^{(0)} \to \mathcal{M}_{m,\wedge}^{(0)}$, and we may choose compatible isomorphisms $\mathcal{M}_{m,\wedge}^{(0)} \cong \mathcal{O}_{E_m^{\mathrm{LT}}}$. We define $\mathcal{M}'_{m,\mathcal{O}_{\mathbb{C}_\varpi}} := \mathcal{M}_m^{(0)} \widehat{\otimes}_{\mathcal{O}_{E_m^{\mathrm{LT}}}} \mathcal{O}_{\mathbb{C}_\varpi}$. As the determinant map is multilinear and alternating, we obtain an action of $\mathrm{SL}_n(\mathcal{O}_E)$ on each space $\mathcal{M}'_{m,\mathcal{O}_{\mathbb{C}_\varpi}}$. This allows us to construct a tower

$$\{\mathcal{M}'_{K,\mathcal{O}_{\mathbb{C}_\varpi}}\}_{K \subset \mathrm{SL}_n(\mathcal{O}_E)},$$

where K ranges over the open subgroups of $\mathrm{SL}_n(\mathcal{O}_E)$, by the same method as sketched at the end of Section 3.1.3. Furthermore, it is a consequence of Lemma 1.12.2 that the group

$$G^1 := \{(g,d,\sigma) \in \mathrm{GL}_n(E) \times D^\times \times W_F \mid \det(g)\,\mathrm{Nrd}(d)^{-1}\,\mathrm{Art}_F^{-1}(\sigma) = 1\} \tag{3.4}$$

acts on this tower.

We define $M'_{K,C}$ as the generic fiber of $\mathcal{M}'_{K,\mathcal{O}_{\mathbb{C}_\varpi}}$, which is an analytic adic space over $\mathrm{Spa}(\mathbb{C}_\varpi,$

$\mathcal{O}_{\mathbb{C}_\varpi}$). We obtain a tower of adic spaces $\{M'_{K,\mathbb{C}_\varpi}\}_{K \subset \mathrm{SL}_n(\mathcal{O}_E)}$ with finite étale transition maps. Much of our study of the Lubin–Tate tower will evolve around this tower. It will therefore be convenient to define

$$H'_{\mathrm{LT}} := \underset{K \subseteq \mathrm{SL}_n(\mathcal{O}_E)}{\mathrm{colim}} \; H^{n-1}_{\mathrm{c}}(M'_{K,\mathbb{C}_\varpi}, \overline{\mathbb{Q}}_\ell).$$

By construction, H'_{LT} is a representation of the group G^1. The vector spaces H_{LT} and H'_{LT} are closely related. As $\varpi^{\mathbb{Z}}$ acts trivially on H_{LT}, it is really a representation of the group

$$\overline{G} := \mathrm{GL}_n(E) \times (D^\times / \varpi^{\mathbb{Z}}) \times \mathrm{W}_E.$$

We note the following relation between G^1 and $\overline{G}$.

Lemma 3.2.3. *The natural map $G^1 \to \overline{G}$ is injective and realizes G^1 as a co-compact closed normal subgroup of $\overline{G}$.*

Proof. The morphism $G^1 \to \overline{G}$ is clearly injective. Further, its image is isomorphic to the kernel of the map

$$\theta \colon \overline{G} \to F^\times / \varpi^{n\mathbb{Z}}, \quad \nu(g, \overline{d}, \sigma) \mapsto \overline{\det(g) \, \mathrm{Nrd}(d)^{-1} \, \mathrm{Art}_F^{-1}(\sigma)}.$$

The claim follows. $\qquad \square$

We have the following crucial result, cf. [Mie16, Section 4].

Proposition 3.2.4. *The $\overline{G}$-representation $\mathrm{Ind}_{G^1}^{\overline{G}}(H'_{\mathrm{LT}})$ is isomorphic to H_{LT}.*

Finally, we need the following result, which is also in Section 4 of [Mie16].

Lemma 3.2.5. *The action of W_E on H_{LT} is smooth.*

4 The Lubin–Tate Space at Infinite Level

In this section, we introduce, attached to a formal $\mathcal{O}_E$-module $\mathbb{X} \in (\mathcal{O}_E\text{-FM}/\overline{\mathbb{F}}_q)$ of height $n \in \mathbb{N}$, the inverse limit

$$\mathcal{M}_\infty^{(0)} = \lim_{m \in \mathbb{N}} \left(\mathcal{M}_m^{(0)} \right).$$

The first question is in what category this objects lives. It is certainly no longer interesting as a covariant functor on $\mathcal{C}$: let $(y_1, y_2, \dots)$ be a system of $[\varpi]_F$-torsion points, for an arbitrary ϖ-divisible formal $\mathcal{O}_E$-module law F over $R \in \mathcal{C}$. As R is Noetherian, the ideal generated by the sequence $(y_1, y_2, \dots)$ is finitely generated, hence it has to be trivial.

However, we can make sense of the limit as a presheaf on the category of formal schemes over $\mathcal{O}_{\breve{E}}$. We have seen in Section 3 that $\mathcal{M}_m^{(0)}$ is representable by the formal spectrum of a local ring A_m which is finite, flat and generically étale over $\mathrm{Spf}(A_0) = \mathrm{Spf}(\mathcal{O}_{\breve{E}}\llbracket u_1, \dots, u_{n-1}\rrbracket)$. In particular, $\mathcal{M}_\infty^{(0)}$ is the formal spectrum of the completed colimit

$$A_\infty = (\operatorname*{colim}_m A_m)_{\mathfrak{m}}^{\wedge}.$$

Here, $\mathfrak{m}$ denotes the ideal generated by the image of the maximal ideal of A_0 (or of any A_m, it doesn't matter). As this ideal is finitely generated, A_∞ is complete with respect to the topology induced by this ideal (in general, this is not the case. See [Stacks, Tag 05JA] for an example). A priori, it is not clear if there is a good moduli description of $\mathcal{M}_\infty^{(0)}$ as a functor $(\mathrm{FSch}/\mathcal{O}_{\breve{E}})^{\mathrm{op}} \to (\mathrm{Set})$.

This section is concerned with the study of the ring A_∞. The main results are summarized as follows. First, we review some of the constructions in Chapter 2 of [Wei16]. As a first step, making use of the determinants of formal $\mathcal{O}_E$-modules constructed in [Hed15], we obtain a natural homomorphism

$$\mathcal{O}_{\widehat{E^{\mathrm{ab}}}} \to A_\infty.$$

We then introduce the notion of universal covers of formal $\mathcal{O}_E$-modules; the passage from H to its universal cover $\tilde{H}$ can be interpreted as a sort of tilting procedure. Still following [Wei16], this notion and its relation with the moduli problem $\mathcal{M}_\infty^{(0)}$ makes it possible to construct an isomorphism

$$A_{\infty, \mathcal{O}_{\mathbb{C}_\varpi}} := A_\infty \widehat{\otimes}_{\mathcal{O}_{\widehat{E^{\mathrm{ab}}}}} \mathcal{O}_{\mathbb{C}_\varpi} \cong \mathcal{O}_{\mathbb{C}_\varpi}\llbracket X_1^{q^{-\infty}}, \dots, X_n^{q^{-\infty}}\rrbracket / (\Delta(X_1^{q^{-\infty}}, \dots, X_n^{q^{-\infty}})^{q^{-m}} - \tau^{q^{-m}} \mid m \in \mathbb{N})^{-},$$
(4.1)

where $\Delta \in \mathcal{O}_{\breve{E}}\llbracket X_1^{q^{-\infty}}, \dots, X_n^{q^{-\infty}}\rrbracket$ denotes the power series constructed in Section 1.14, and a certain element $\tau \in \mathrm{Nilp}^\flat(\mathcal{O}_{\mathbb{C}_\varpi})$ (suspect to some choice). The superscript minus denotes the completion of the ideal.

The main effort is taken in an description of the various group actions on $\mathcal{M}_\infty \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathrm{Spf}(\mathcal{O}_{\mathbb{C}_\varpi})$ in terms of this isomorphism, following [IT20, p. 1.2]. Together with the approximation of the determinant map in Lemma 1.14.3, this gives the necessary information to observe the perfection of a Deligne–Lusztig variety as the special fiber of some affinoid inside the Lubin–Tate perfectoid space, in a certain way compatible with the respective group actions.

4.1 The Case of Height One

Let $\mathbb{X}$ be a formal $\mathcal{O}_E$-module law over $\overline{\mathbb{F}}_q$ of height one, let $E_m^{\mathrm{LT}} = \breve{E}E_{\varpi,m}$ denote the degree m Lubin–Tate extension of E and let $(\mathcal{F}, \iota) \in \mathcal{M}_m^{(0)}(\mathcal{O}_{E_m^{\mathrm{LT}}})$ be the unique deformation of $\mathbb{X}$. We have seen in Section 3.1.2 that any choice of Drinfeld level ϖ^m structure of $\mathcal{F}$ yields an isomorphism $\mathcal{M}_M^{(0)} \cong \mathcal{O}_{E_m^{\mathrm{LT}}}$. By the construction of the maximal Abelian extension of E in Section 2, we have we have $(\mathrm{colim}_m \mathcal{O}_{E_m^{\mathrm{LT}}})_\varpi^\wedge = \mathcal{O}_{\widehat{E}^{\mathrm{ab}}}$. The following is a direct consequence.

Proposition 4.1.1. *Assume that $\mathbb{X}$ is of height one and let $(\mathcal{F}, \iota) \in \mathcal{M}_0(\mathcal{O}_{\breve{E}})$ be a deformation of $\mathbb{X}$. Then any choice of compatible system of level structures on $\mathcal{F}(\mathcal{O}_{\widehat{E}^{\mathrm{ab}}})$ yields an isomorphism*

$$\mathcal{M}_\infty^{(0)} \xrightarrow{\sim} \mathcal{O}_{\widehat{E}^{\mathrm{ab}}},$$

where $\widehat{E}^{\mathrm{ab}}$ is the closure of the maximal Abelian extension of E inside $\mathbb{C}_\varpi$.

4.2 The Deformation Space in terms of the Universal Cover

Let $(e_1, \ldots, e_n)$ denote the standard basis of $\mathcal{O}_E^n$. By Theorem 3.1.3, there is for every positive integer m a universal triple

$$(\mathcal{F}^{\mathrm{univ}}, \iota^{\mathrm{univ}}, \phi_m^{\mathrm{univ}}) \in \mathcal{M}_m^{(0)}(A_m)$$

and the Drinfeld level ϖ^m-structure

$$\phi_m^{\mathrm{univ}} \colon (\varpi^{-m}\mathcal{O}_E/\mathcal{O}_E)^n \to \mathcal{F}^{\mathrm{univ}}(A_m)$$

gives rise to elements $x_i^{(m)} = \phi_m(e_i) \in \mathcal{F}^{\mathrm{univ}}(A_\infty)$ for $i = 1, \ldots, n$. This gives rise to an n-tuple of compatible systems

$$(x_1, \ldots, x_n) \in \tilde{\mathcal{F}}^{\mathrm{univ}}(A_\infty).$$

Now let $\mathcal{F}$ be an arbitrary deformation of $\mathbb{X}$ to $\mathcal{O}_{\breve{E}}$ and let $I \subset A_\infty$ be an ideal of definition containing ϖ. By Proposition 1.11.8, we have isomorphisms

$$\tilde{\mathcal{F}}^{\mathrm{univ},n}(A_\infty) \cong \tilde{\mathcal{F}}_0^{\mathrm{univ},n}(A_\infty/I) \xrightarrow{\iota} \tilde{\mathcal{F}}_0^n(A_\infty/I) \cong \tilde{\mathcal{F}}^n(A_\infty).$$

This constructs a morphism of formal schemes over $\mathrm{Spf}(\mathcal{O}_{\breve{E}})$

$$\mathcal{M}_\infty^{(0)} \to \tilde{\mathcal{F}}^n.$$

Similarly, we obtain natural maps $\mathcal{M}_\infty^{(j)} \to \tilde{\mathcal{F}}^n$, yielding in the colimit over j a morphism

$$\mathcal{M}_\infty \to \tilde{\mathcal{F}}^n. \tag{4.2}$$

Suppose we are given a system $S_0 \to S_1 \to S_2 \to \ldots$ of objects in $\mathcal{C}$, together with a system of compatible triples $(\mathcal{G}, \iota, \phi_m) \in \mathcal{M}_m(S_m)$. Furthermore, assume that there is an $\mathcal{O}_{\breve{E}}$-algebra S_∞ with compatible morphisms $S_m \to S_\infty$. This data determines a morphism $S_\infty \to \mathcal{M}_\infty$, and the morphism above yields an n-tuple $(X_1, \ldots, X_n) \in \tilde{\mathcal{F}}^n(S_\infty)$, which has the following

description. Let $J \subset S_\infty$ be an ideal of definition with $\varpi \in J$. Then X_i is exactly the image of the i-th basis vector in the composition

$$\mathcal{O}_E^n \xrightarrow{(\phi_m)_{m\in\mathbb{N}}} T_\varpi \mathcal{G}(S_\infty) \to \tilde{\mathcal{G}}_0(S_\infty/J) \xrightarrow{\iota^{-1}} \tilde{\mathbb{X}}(S_\infty/J) \xrightarrow{\sim} \tilde{\mathcal{F}}(S_\infty).$$

By naturality of the determinant map, this yields, for $j \in \mathbb{Z}$, a commutative square

$$\begin{array}{ccc}
\mathcal{M}_\infty^{(j)} & \xrightarrow{\det} & \mathcal{M}_{\wedge,\infty}^{(j)} \\
\downarrow & & \downarrow \\
\tilde{\mathcal{F}}^n & \xrightarrow{\delta} & \widetilde{\wedge^n \mathcal{F}}.
\end{array}$$

All of the formal schemes in this square are affine, and with careful analysis of the underlying adic algebras, Weinstein obtained the following key result.

Theorem 4.2.1 (Structure of $\mathcal{M}_\infty$)**.** *The square above is cartesian.*

Proof. This is [Wei16, Theorem 2.17]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

By Proposition 1.11.8, any coordinate $\mathcal{F} \cong \mathrm{FG}(F)$ yields isomorphisms $\tilde{\mathcal{F}} \cong \mathrm{Spf}(\mathcal{O}_{\breve{E}}[\![X^{q^{-\infty}}]\!])$ and $\widetilde{\wedge^n \mathcal{F}} \cong \mathrm{Spf}(\mathcal{O}_{\breve{E}}[\![X^{q^{-\infty}}]\!])$. Let $t = (t_1, t_2, \dots) \in T_\varpi F(\mathcal{O}_{\widehat{\bar{E}^{\mathrm{ab}}}})$ be a primitive element, that is, an element such that $t_1 \neq 0$ and $[\varpi]_F(t_1) = 0$. This is equivalent to a system of compatible level structure $(\phi_m)_{m\in\mathbb{N}}$ with codomain $\wedge^n F(\mathcal{O}_{\widehat{\bar{E}^{\mathrm{ab}}}})$. The element t yields an isomorphism $\mathcal{M}_{\wedge,\infty}^{(j)} \cong \mathcal{O}_{\widehat{\bar{E}^{\mathrm{ab}}}}$. We obtain the following description of $A_{\infty, \mathcal{O}_{\mathbb{C}_\varpi}}$.

Corollary 4.2.2. *Let $(\tau^{1/q^m})_{m\in\mathbb{N}} \in \mathrm{Nilp}^\flat(\mathcal{O}_{\widehat{\bar{E}^{\mathrm{ab}}}})$ be the system of q-th power roots corresponding to t under the inclusion $T_\varpi F \hookrightarrow \tilde{F}$. Then we have*

$$A_\infty \cong \mathcal{O}_{\widehat{\bar{E}^{\mathrm{ab}}}}[\![X_1^{q^{-\infty}}, \dots, X_n^{q^{-\infty}}]\!]/(\Delta^{q^{-m}} - \tau^{q^{-m}} \mid m \in \mathbb{N})^-.$$

We note that any other primitive $t' \in T_\varpi(F)(\mathcal{O}_{\widehat{\bar{E}^{\mathrm{ab}}}})$ satisfies $t' = [\alpha]_F(t)$ for some unique $\alpha \in \mathcal{O}_E^\times$. By definition of the Artin map, the element $\tau' \in \mathrm{Nilp}^\flat(\mathcal{O}_{\widehat{\bar{E}^{\mathrm{ab}}}})$ corresponding to $t' \in \tilde{F}(\mathcal{O}_{\widehat{\bar{E}^{\mathrm{ab}}}})$ is given by $\mathrm{Art}_E(\alpha)(\tau)$.

Let $M'_{\infty, \mathbb{C}_\varpi}$ be the generic fiber of $\mathcal{M}'_{\infty, \mathcal{O}_{\mathbb{C}_\varpi}}$, it is an adic space over $\mathrm{Spa}(\mathbb{C}_\varpi, \mathcal{O}_{\mathbb{C}_\varpi})$. We call $M'_{\infty, \mathbb{C}_\varpi}$ the Lubin–Tate perfectoid space. This name is justified by the following result.

Theorem 4.2.3 (The Lubin–Tate Perfectoid Space)**.** *The adic space $M'_{\infty, \mathbb{C}_\varpi}$ can be covered by perfectoid affinoids, hence it is a perfectoid space.*

Sketch of Proof. We sketch the proof in [Wei16, Lemma 2.32]. Let I denote a finitely generated ideal of definition for $A_{\mathcal{O}_{\mathbb{C}_\varpi}} := A_\infty \widehat{\otimes}_{\mathcal{O}_{\widehat{\bar{E}^{\mathrm{ab}}}}} \mathcal{O}_{\mathbb{C}_\varpi}$ and let $(f_1, \dots, f_k)$ be a tuple of generators for I.

Let $x \in M'_{\infty, \mathbb{C}_\varpi}$ be an arbitrary point. We have $|\varpi(x)| \neq 0$ by definition of the generic fiber. The element f_i is topologically nilpotent in $A_{\infty, \mathcal{O}_{\mathbb{C}_\varpi}}$, and it follows that there is some $r \in \mathbb{N}$ with $|f_i(x)|^r < |\varpi(x)|$ for all $i = 1, \dots, k$. We may now define the subspace $U_r = \mathrm{Spa}(R_r, R_r^+)$, where R_r is defined as

$$R_r := A_{\mathcal{O}_{\mathbb{C}_\varpi}} \left\langle \frac{f_1^r}{\varpi}, \dots, \frac{f_k^r}{\varpi} \right\rangle [\tfrac{1}{\varpi}],$$

and R_r^+ is defined as the integral closure of $A_{\mathcal{O}_{\mathbb{C}_\varpi}} \left\langle \frac{f_1^r}{\varpi}, \ldots, \frac{f_k^r}{\varpi} \right\rangle$ inside R_r. Now $R_r^+ = R_r^\circ$. As the Frobenius map is surjective on $A_{\mathcal{O}_{\mathbb{C}_\varpi}}/\varpi$, it is surjective on R_r^+/ϖ, and we find that R_r is perfectoid. The valuation belonging to x extends uniquely to R_r and satisfies $|R_r^+| \leq 1$, so it is point in $\mathrm{Spa}(R_r, R_r^+)$. The claim follows. $\qquad\square$

4.3   Reviewing the Group Actions

Write

$$G := \mathrm{GL}_n(E) \times D^\times \times \mathrm{W}_E,$$

and define a homomorphism

$$\theta \colon G \to E^\times, \quad (g, d, \sigma) \mapsto \det(g) \, \mathrm{Nrd}(d)^{-1} \, \mathrm{Art}_E^{-1}(\sigma).$$

Let G^{u} be the preimage of $\mathcal{O}_E^\times$ under θ, so that G^{u} acts on $\mathcal{M}_\infty^{(0)} \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathrm{Spf}(\mathcal{O}_{\mathbb{C}_\varpi})$. Given a ϖ-divisible formal $\mathcal{O}_E$-module $\mathcal{F} \in (\mathcal{O}_E\text{-FM}/\mathcal{O}_{\breve{E}})$ of height n, we write $\mathcal{F}_{\mathcal{O}_{\mathbb{C}_\varpi}} = \mathcal{F} \otimes_{\mathcal{O}_{\breve{E}}} \mathcal{O}_{\mathbb{C}_\varpi}$ and describe a natural right action on $\tilde{\mathcal{F}}_{\mathcal{O}_{\mathbb{C}_\varpi}}^n$ by G such that the map $\mathcal{M}_\infty^{(0)} \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathrm{Spf}(\mathcal{O}_{\mathbb{C}_\varpi}) \to \tilde{\mathcal{F}}_{\mathcal{O}_{\mathbb{C}_\varpi}}^n$, induced by the map constructed above is equivariant for the respective G^{u}-actions.

The action of G on $\tilde{\mathcal{F}}^n$ is easy to describe. For the action of $\mathrm{GL}_n(E)$, note that $\tilde{\mathcal{F}}$ carries the structure of a E-vector space object. Hence $\tilde{\mathcal{F}}^n$ obtains a natural right action by $\mathrm{GL}_n(E)$: an object $g \in \mathrm{GL}_n(E)$ with entries $g = (a_{ij})_{i,j}$ acts by matrix multiplication from the right, as in

$$(x_1, \ldots, x_n).g = (y_1, \ldots, y_n), \quad \text{where} \quad y_j = \sum_{i=1}^n a_{ij} x_i. \tag{4.3}$$

For the action of $D^\times$, note that by Proposition 1.11.8, we have a natural left action of $D^\times$ on $\tilde{\mathcal{F}}$. Indeed, given $0 \neq d_0 \in \mathrm{End}_{(A\text{-FM}/\mathbb{F}_q)}(F_0) = \mathcal{O}_D$, an integer $r \in \mathbb{Z}$ and any $S \in (\mathcal{O}_{\breve{E}}\text{-Adm})$ with ideal of definition containing ϖ, we let the element $d = \varpi^r d_0 \in D^\times$ act on $\tilde{\mathcal{F}}$ via the automorphism

$$\tilde{\mathcal{F}}(S) \to \tilde{\mathcal{F}}_0(S/J) \xrightarrow{\varpi^r} \mathcal{F}_0(S/J) \xrightarrow{d_0} \tilde{\mathcal{F}}_0(S/J) \to \tilde{\mathcal{F}}(S), \quad x \mapsto dx.$$

Note that multiplication by ϖ and d_0 commute as ϖ lies in the center of the multiplicative monoid $\mathcal{O}_D$, so this yields a well-defined left action. We define the right action of $D^\times$ on $\tilde{\mathcal{F}}^n$ via

$$(x_1, \ldots, x_n).d = (d^{-1}x_1, \ldots, d^{-1}x_n).$$

The map $\Pi : \tilde{\mathcal{F}} \to \Phi^{-1,*}\tilde{\mathcal{F}}$ from Definition 1.11.9 equips $\mathcal{F}$ with the Weil descent datum

$$(\Phi^*\Pi)^{-1} : \mathcal{F} \to \Phi^*\mathcal{F},$$

and in particular yields an action of W_E on $\tilde{\mathcal{F}}_{\mathcal{O}_{\mathbb{C}_\varpi}}^n$.

It is easy to see that the actions of $\mathrm{GL}_n(E)$ and $D^\times$ commute, and that both these actions commute with the Weil descent datum. Hence we obtain a right action by G on $\tilde{\mathcal{F}}_{\mathcal{O}_{\mathbb{C}_\varpi}}^n$.

The maps constructed in (4.2) induce a map $\mathcal{M}_\infty \to \tilde{\mathcal{F}}^n$.

Lemma 4.3.1. *The map $\mathcal{M}_\infty \to \tilde{\mathcal{F}}^n$ is equivariant for the action of $D^\times$ on both sides. More precisely, given any $d \in D^\times$ and any $j \in \mathbb{Z}$, the diagram*

$$
\begin{array}{ccc}
\mathcal{M}_\infty^{(j)} & \longrightarrow & \tilde{\mathcal{F}}^n \\
\downarrow{\scriptstyle \cdot d} & & \downarrow{\scriptstyle \cdot d} \\
\mathcal{M}_\infty^{(j')} & \longrightarrow & \tilde{\mathcal{F}}^n,
\end{array}
$$

with $j' = j + \mathrm{val}_\varpi(\mathrm{Nrd}\, d)$, commutes.

Proof. This follows directly from the description of the $D^\times$-action on $\mathcal{M}_\infty$ and $\tilde{\mathcal{F}}^n$. $\qquad\square$

The proof of the corresponding statement for the action of $\mathrm{GL}_n(E)$ is more tedious.

Lemma 4.3.2. *The map $\mathcal{M}_\infty \to \tilde{\mathcal{F}}^n$ is equivariant for the action of $\mathrm{GL}_n(E)$ on both sides. More precisely, given any $g \in \mathrm{GL}_n(E)$ and any $j \in \mathbb{Z}$, the diagram*

$$
\begin{array}{ccc}
\mathcal{M}_\infty^{(j)} & \longrightarrow & \tilde{\mathcal{F}}^n \\
\downarrow{\scriptstyle \cdot g} & & \downarrow{\scriptstyle \cdot g} \\
\mathcal{M}_\infty^{(j')} & \longrightarrow & \tilde{\mathcal{F}}^n,
\end{array}
$$

with $j' = j - \mathrm{val}_\varpi(\det g)$, commutes.

Proof. We denote by $(X_1, \ldots, X_n) \in \tilde{\mathcal{F}}^n(A_\infty^{(j)})$ the elements corresponding to the morphism $\mathcal{M}_\infty^{(j)} \to \tilde{\mathcal{F}}^n$ and denote by $(Z_1, \ldots, Z_n)$ the elements corresponding to the composition $\mathcal{M}_\infty^{(j)} \xrightarrow{\cdot g} \mathcal{M}_\infty^{(j')} \to \tilde{\mathcal{F}}^n$. We need to show that $(X_1, \ldots, X_n).g = (Z_1, \ldots, Z_n)$.

This essentially boils down to the description of the group action on $\mathcal{M}_\infty$. As the elements $(\varpi, \varpi, 1)$ act trivially on both sides, we may assume that $g \in \mathrm{GL}_n(E)$ satisfies $g^{-1} \in \mathrm{Mat}_{n \times n}(\mathcal{O}_E)$. Let $k \in \mathbb{N}$ be an integer such that $\varpi^k g \in \mathrm{Mat}_{n \times n}(\mathcal{O}_E)$. Let $m > k$ be another integer and take an arbitrary triple $(\mathcal{G}, \iota, \phi) \in \mathcal{M}_m^{(j)}(R)$ for any $R \in \mathcal{C}$. Let $S \in (R\text{-Adm})$ be an admissible R-algebra and let $J \subset S$ be an ideal of definition with $\varpi \in J$. Write

$$
(\mathcal{G}', \iota', \phi') = (\mathcal{G}, \iota, \phi).g \in \mathcal{M}_m^{(j')}(R),
$$

and let $\mathcal{G}_0$ and $\mathcal{G}_0'$ be the reductions of $\mathcal{G}$ and $\mathcal{G}'$ to formal $\mathcal{O}_E$-modules over $\overline{\mathbb{F}}_q$. By construction of the $\mathrm{GL}_n(E)$-action on $\mathcal{M}_m$, we have a commutative diagram

$$
\begin{array}{ccccccc}
(\varpi^{-m}\mathcal{O}_E^n/\mathcal{O}_E^n) & \xrightarrow{\phi_m} & \mathcal{G}(S) & \longrightarrow & \mathcal{G}_0(S/J) & \dashrightarrow{\iota^{-1}} & \mathbb{X}(S/J) \\
\downarrow & & \downarrow & & \downarrow & & \| \\
(\varpi^{m-k}\mathcal{O}_E^n/\mathcal{O}_E^n) & \xrightarrow{g} & (\varpi^{-m}\mathcal{O}_E^n/g\mathcal{O}_E^n) & \xrightarrow{\bar{\phi}_m} & \mathcal{G}'(S) \longrightarrow \mathcal{G}_0'(S/J) & \dashrightarrow{\iota'^{-1}} & \mathbb{X}(S/J).
\end{array}
$$

$$
\underbrace{\phantom{(\varpi^{m-k}\mathcal{O}_E^n/\mathcal{O}_E^n) \to (\varpi^{-m}\mathcal{O}_E^n/g\mathcal{O}_E^n)}}_{\phi'_{m-k}}
$$

Let us now consider the diagrams arising with $R = A_m^{(j)}$, $S = A_\infty^{(j)}$ with any ideal of definition $I \subset A_\infty^{(j)}$ with $\varpi \in I$ and the universal triple $(\mathcal{F}^{\mathrm{univ}}, \iota^{\mathrm{univ}}, \phi_m^{\mathrm{univ}}) \in \mathcal{M}_m^{(j)}(A_m^{(j)})$. In the limit

over m, these diagrams yield a commutative square of Abelian groups

$$\begin{array}{ccccccc}
\mathcal{O}_E^n & \xrightarrow{(\phi_m)_{m\in\mathbb{N}}} & T_\varpi\mathcal{F}^{\mathrm{univ}}(A_\infty^{(j)}) & \longrightarrow & \tilde{\mathcal{F}}_0^{\mathrm{univ}}(A_\infty^{(j)}/I) & \xrightarrow{\iota^{-1}} & \tilde{\mathbb{X}}(A_\infty^{(j)}/I) \\
{\scriptstyle g^{-1}}\downarrow & & \downarrow & & \downarrow & & \| \\
\mathcal{O}_E^n & \xrightarrow{(\phi_m')_{m\in\mathbb{N}}} & T_\varpi\mathcal{F}'(A_\infty^{(j)}) & \longrightarrow & \tilde{\mathcal{F}}_0'(A_\infty^{(j)}/I) & \xrightarrow{\iota'^{-1}} & \tilde{\mathbb{X}}(A_\infty^{(j)}/I).
\end{array} \tag{4.4}$$

Under the natural bijection $\tilde{\mathbb{X}}(A_\infty^{(j)}/I) \xrightarrow{\sim} \tilde{\mathcal{F}}(A_\infty)$, the images of the basis vectors $(e_1, \ldots, e_n)$ of the top row are mapped to the the elements $(X_1, \ldots, X_n) \in \tilde{\mathcal{F}}^n(A_\infty^{(j)})$, while the basis vectors of the bottom row map to $(Z_1, \ldots, Z_n) \in \tilde{\mathcal{F}}^n(A_\infty^{(j)})$. Commutativity of the diagram now implies

$$(X_1, \ldots, X_n).g = (Z_1, \ldots, Z_n),$$

and this is what we had to show. $\qquad\square$

Lemma 4.3.3. *The map $\mathcal{M}_\infty \to \tilde{\mathcal{F}}^n$ is equivariant for the Weil descent datum on both sides. More precisely, for any $j \in \mathbb{Z}$, the square*

$$\begin{array}{ccc}
\mathcal{M}_\infty^{(j)} & \longrightarrow & \tilde{\mathcal{F}}^n \\
\downarrow & & \downarrow \\
\Phi^*\mathcal{M}_\infty^{(j-1)} & \longrightarrow & \Phi^*\tilde{\mathcal{F}}^n
\end{array}$$

with vertical maps given by the respective Weil descent data, commutes.

Proof. As in the proof of Lemma 4.3.2, we denote by $(X_1, \ldots, X_n) \in \tilde{\mathcal{F}}^n(A_\infty^{(j)})$ the elements corresponding to the morphism $\mathcal{M}_\infty^{(j)} \to \tilde{\mathcal{F}}^n$ and denote by $(Z_1, \ldots, Z_n)$ the elements corresponding to the composition $\mathcal{M}_\infty^{(j)} \to \Phi^*\mathcal{M}_\infty^{(j)} \to \Phi^*\tilde{\mathcal{F}}^n$. We need to show that $(\Phi^*\Pi)^{-1}(X_1, \ldots, X_n) = (Z_1, \ldots, Z_n)$. Let $J \subset A_\infty^{(j)}$ be an ideal of definition with $\varpi \in J$, and denote by $x_i \in \tilde{\mathcal{F}}_0(A_\infty/J) = \mathbb{X}_0(A_\infty/J)$ the reduction of X_i, and similarly denote by $z_i \in \tilde{\mathbb{X}}_0^{(-q)}(A_\infty/J)$ the reduction of Z_i. Now, the statement $Z_i = (\Phi^*\Pi)^{-1}(X_i)$ is equivalent to $\mathrm{Frob}_q(z_i) = x_i$.

As above, let $(\mathcal{F}^{\mathrm{univ}}, \iota, \phi_m) \in \mathcal{M}_m^{(j)}(A_m^{(j)})$ denote a universal triple and write $(\Phi^*\mathcal{F}^{\mathrm{univ}}, \iota', \Phi^*\phi) \in \Phi^*\mathcal{M}_m^{(j-1)}(A_m^{(j)})$ for the triple obtained by the Weil descent datum on $\mathcal{M}_m$. By construction, we have the following commutative diagram.

$$\begin{array}{ccccccc}
\varpi^{-m}\mathcal{O}_E^n/\mathcal{O}_E^n & \xrightarrow{\phi_m} & \mathcal{F}^{\mathrm{univ}}(A_\infty) & \longrightarrow & \mathcal{F}_0^{\mathrm{univ}}(A_\infty/J) & \dashrightarrow{\iota^{-1}} & \mathbb{X}(A_\infty/J) \\
\| & & & & {\scriptstyle\mathrm{Frob}_q}\uparrow & \nearrow & {\scriptstyle\mathrm{Frob}_q}\uparrow \\
\varpi^{-m}\mathcal{O}_E^n/\mathcal{O}_E^n & \xrightarrow{\Phi^*\phi_m} & \Phi^*\mathcal{F}^{\mathrm{univ}}(A_\infty) & \longrightarrow & \mathcal{F}_0^{\mathrm{univ},(-q)}(A_\infty/J) & \xdashrightarrow{(\iota^{(-q)})^{-1}} & \mathbb{X}^{(-q)}(A_\infty/J)
\end{array}$$

In the limit over m, we obtain the diagram

$$
\begin{array}{ccccccc}
\mathcal{O}_E^n & \xrightarrow{(\phi_m)_m} & T_{\varpi}\mathcal{F}^{\mathrm{univ}}(A_\infty) & \longrightarrow & \tilde{\mathcal{F}}_0^{\mathrm{univ}}(A_\infty/J) & \xrightarrow{\iota^{-1}} & \tilde{\mathbb{X}}(A_\infty/J) \\
\| & & & & \big\uparrow \mathrm{Frob}_q & \nearrow \iota'^{-1} & \big\uparrow \mathrm{Frob}_q \\
\mathcal{O}_E^n & \xrightarrow{(\Phi^*\phi_m)_m} & T_{\varpi}[\Phi^*\mathcal{F}^{\mathrm{univ}}](A_\infty) & \longrightarrow & \tilde{\mathcal{F}}_0^{\mathrm{univ},(-q)}(A_\infty/J) & \xrightarrow{(\iota^{(-q)})^{-1}} & \tilde{\mathbb{X}}^{(-q)}(A_\infty/J)
\end{array}
$$

and the claim follows immediately. $\qquad\square$

As a direct corollary from the three previous lemmas, we obtain the following result.

Proposition 4.3.4. *The morphism* $\mathcal{M}_\infty^{(0)} \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathrm{Spf}(\mathcal{O}_{\mathbb{C}_\varpi}) \to \tilde{\mathcal{F}}_{\mathcal{O}_{\mathbb{C}_\varpi}}^n$ *in* (4.2) *is equivariant for the action of* G^{u} *on both sides.*

4.4 Description of the Group Actions in Coordinates

We now choose coordinates for $\mathcal{M}_\infty^{(0)}$ and make the group action of G^{u} explicit in terms of these coordinates. Let H be the standard formal $\mathcal{O}_E$-module over $\breve{E}$ of height n, as defined in Section 1.5. By the monomorphism constructed in (4.2) and Proposition 4.3.4, the action of G^{u} on $\mathcal{M}_\infty^{(0)}$ is determined by the action of G^{u} on $\tilde{H}^n$. As canonically $\tilde{H}^n \cong \mathrm{Nilp}^{\flat,n}$, the right action of G on $\tilde{H}^n$ is equivalent to a left action on the algebra

$$
\Xi_n := \mathcal{O}_{\mathbb{C}_\varpi}[\![X_1^{q^{-\infty}}, \ldots, X_n^{q^{-\infty}}]\!].
$$

Our first aim is to make this action explicit.

We begin with an explicit description of the isomorphisms μ and λ of Proposition 1.11.8.

Lemma 4.4.1. *The bijections*

$$
\lambda \colon \tilde{H}(S) \rightleftarrows \mathrm{Nilp}^{\flat}(S) \colon \mu, \quad (x_0, x_1, \ldots) \leftrightarrows (y, y^{q^{-1}}, y^{q^{-2}}, \ldots)
$$

are, in either direction, given by the equations

$$
y^{1/q^{ni}} = \lim_{r\to\infty} x_{r+i}^{q^{nr}} \quad \text{and} \quad x_i = \lim_{s\to\infty} [\varpi^s]_H(y^{q^{-n(i+s)}}).
$$

Proof. This follows directly from the fact that $[\varpi]_H(T) \equiv T^{q^n}$ modulo ϖ and the explicit description of the isomorphism in Proposition 1.11.5. $\qquad\square$

This lemma allows us to make the $\mathcal{O}_E$-module structure on $\mathrm{Nilp}^{\flat}$ explicit. Let S be an adic $\mathcal{O}_{\breve{E}}$-algebra and J be an ideal of definition that contains ϖ.

Lemma 4.4.2. *The E-vector space structure on* $\mathrm{Nilp}^{\flat}(S)$ *takes on the following form.*

- *Given two q-th power compatible systems $y_1, y_2 \in \mathrm{Nilp}^{\flat}(S)$ corresponding to compatible systems $\mu(y_1) = x_1$, $\mu(y_2) = x_2 \in \tilde{H}(S)$, the sum $x_1 + x_2 \in \tilde{H}(S)$ corresponds to the element $\lambda(x_1 + x_2) = y_1 +_H y_2 \in \mathrm{Nilp}^{\flat}(S)$, where*

$$
(y_1 +_H y_2)^{1/q^j} = \lim_{r\to\infty} H(y_1^{q^{-r}}, y_2^{q^{-r}})^{q^{r-j}}.
$$

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If $G \in (\mathcal{O}_E\text{-FML}/\mathcal{O}_{\breve{E}})$ with $G \otimes \overline{\mathbb{F}}_q = H \otimes \overline{\mathbb{F}}_q$, the systems of q-th power roots $(y_1 +_H y_2)$ and $(y_1 +_G y_2)$ agree.

- *Similarly, given $a \in \mathcal{O}_E$ and $y \in \mathrm{Nilp}^\flat(S)$ with $\mu(y) = x \in \widetilde{H}(S)$, we have*

$$a_H(y)^{1/q^j} = \lambda([a]_H(x)) = \lim_{r \to \infty} [a]_H(y^{q^{-r}})^{q^{r-j}}.$$

For G as above, we have $a_H(x) = a_G(x)$.

Proof. The first statement follows directly from the lemma above, after tracing through the commutative diagram

$$
\begin{array}{ccc}
\widetilde{H}(S)^2 & \xrightarrow{H(-,-)} & \widetilde{H}(S) \\
{\scriptstyle \mu}\nearrow \quad \downarrow & & \downarrow \quad \searrow {\scriptstyle \lambda} \\
\mathrm{Nilp}^\flat(S)^2 \longrightarrow \widetilde{H}_0(S/J)^2 & \xrightarrow{H_0(-,-)} & \widetilde{H}_0(S/J) \longrightarrow \mathrm{Nilp}^\flat(S).
\end{array}
$$

Similarly one proves the third statement. $\qquad\square$

We obtain the following description of the $E^\times \times D^\times$-action.

Corollary 4.4.3. *Let $a \in E$ and $d \in D^\times$ be elements so that*

- *the element a is, for some $l \in \mathbb{Z}$, of the form*

$$a = \sum_{i=l}^{\infty} a_i \varpi^i \quad with \quad a_i \in \mu_{q-1}(E) \cup \{0\}.$$

- *the element d is, for some $l' \in \mathbb{Z}$ and $\vartheta \in \mathrm{End}_{(\mathcal{O}_E\text{-FML}/\overline{\mathbb{F}}_q)}(H_0)$ the endomorphism given by $\vartheta(T) = T^q$ (cf. the description of $D^\times$ at the end of Section 1.3) of the form*

$$d = \sum_{i=l'}^{\infty} d_i \vartheta^i \quad with \quad d_i \in \mu_{q^n-1}(E_n) \cup \{0\}.$$

Here E_n denotes the unramified extension of E with residue field $\mathbb{F}_{q^n}$.

Then, given any $x = (x_1, x_2, \dots) \in \tilde{H}(S)$ with $\lambda(x)^{1/q^i} = y^{1/q^i} \in \mathrm{Nilp}^\flat(S)$, we have the explicit descriptions

$$a_H(y)^{1/q^j} = \sum_{i=l}{}_H a_i y^{q^{ni-j}} \quad and \quad d_H(y)^{1/q^j} := \lambda(dx)^{1/q^j} = \sum_{i=l'}^{\infty}{}_H d_i y^{q^{i-j}}$$

Here the symbol $\sum_H$ denotes addition of q-th power root systems with respect to the addition in H, as defined in the previous lemma.

Proof. This is an immediate corollary of the lemma above, Lemma 1.5.5 and the fact that $[\varpi]_H(T) \equiv T^{q^n} \pmod{\varpi}$. $\qquad\square$

It is now not hard to deduce an explicit description of the $\mathrm{GL}_n(E)$-action on $\tilde{H}^n(S)$, we write it down in terms of Ξ_n below. Finally, we describe the action of W_E on $(\tilde{H} \otimes \mathcal{O}_{\mathbb{C}_\varpi})$ in terms of $\mathrm{Nilp}^\flat \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathrm{Spf}(\mathcal{O}_{\mathbb{C}_\varpi}) =: \mathrm{Nilp}^\flat_{\mathcal{O}_{\mathbb{C}_\varpi}}$.

Lemma 4.4.4. *Let $\sigma \in W_E$ and let $m \in \mathbb{Z}$ be an integer such that $\sigma|_{\breve{E}} = \Phi^m$. Let $(y, y^{1/q}, \dots)$ be an element in $\mathrm{Nilp}^\flat_{\mathcal{O}_{\mathbb{C}_\varpi}}(S)$. Then*

$$\sigma.(y_1, y_2, \dots) = (\sigma(y)^{q^{-m}}, \sigma(y)^{q^{-(m+1)}}, \dots).$$

Proof. This follows from the definition of the Weil descent datum and the fact that the square

$$
\begin{array}{ccc}
\tilde{H}(S) & \xrightarrow{\ \sim\ } & \mathrm{Nilp}^\flat(S) \\
\scriptstyle\Pi \downarrow & & \downarrow \scriptstyle y \mapsto y^q \\
\tilde{H}(S) & \xrightarrow{\ \sim\ } & \mathrm{Nilp}^\flat(S)
\end{array}
$$

is commutative. $\qquad\square$

Remark. Note that as Ξ_n is defined over $\mathcal{O}_E$, it comes with a natural action by W_E. This action differs from the one defined above and is not compatible with the Weil group action on $\mathcal{M}'_{\infty, \mathcal{O}_{\mathbb{C}_\varpi}}$.

We obtain the following description of the left action of G on $\Xi_n = \mathcal{O}_{\mathbb{C}_\varpi}[\![X_1^{q^{-\infty}}, \dots, X_n^{q^{-\infty}}]\!]$.

Lemma 4.4.5. *Let $g = (a_{ij})_{i,j} \in \mathrm{GL}_n(E)$, $d \in D^\times$ and $\sigma \in W_E$. Let $m \in \mathbb{Z}$ be such that $\sigma|_{\breve{E}} = \Phi^m$. Then the morphism $\Xi_n \to \Xi_n$ induced by the element $(g, d, \sigma) \in G$ is given by the composition of the morphsims*

- *$g^* \colon \Xi_n \to \Xi_n$, the morphism of $\mathcal{O}_{\mathbb{C}_\varpi}$ algebras given by $X_i \mapsto \sum_{j=1}^n{}_H\, a_{ji}{}_H(X_j)$. Writing $a_{ij} = \sum_{k=l}^\infty a_{ij}^{(k)} \varpi^k$ for $a_{ij}^{(k)}$ either vanishing or a $(q-1)$th root of unity and a sufficiently chosen integer l, we obtain, by Corollary 4.4.3, the description*

$$X_i \mapsto \gamma_{g,i}(X_1, \dots, X_n) := \sum_{k=l}^\infty{}_H \sum_{j=1}^n{}_H\, a_{ji}^{(k)} X_j^{q^{nk}}.$$

  Note that both the left and the right hand side are systems of q-th power roots.

- *$d^{-1,*} \colon \Xi_n \to \Xi_n$, the isomorphism of $\mathcal{O}_{\mathbb{C}_\varpi}$-algebras given by $X_i \mapsto d_H(X_i)$. Writing $d^{-1} = \sum_{k=l}^\infty d_i \vartheta^k$ for a sufficiently small integer l, and ϑ and d_i as in Corollary 4.4.3, we obtain the description*

$$X_i \mapsto \delta_{d,i}(X_i) := \sum_{k=l}^\infty{}_H\, d_k X_i^{q^k}.$$

  Note that both the left and the right hand side are systems of q-th power roots.

- *$\sigma^* \colon \Xi_n \to \Xi_n$, the isomorphism of $\mathcal{O}_E$-algebras given by $X_i \mapsto X_i^{q^{-m}}$ and $a \mapsto \sigma(a)$ for $a \in \mathcal{O}_{\mathbb{C}_\varpi}$.*

Let us now turn our attention to the space $\mathcal{M}'_{\infty, \mathcal{O}_{\mathbb{C}_\varpi}} = \mathrm{Spf}(A_\infty \widehat{\otimes}_{\mathcal{O}_{\widehat{E}^{\mathrm{ab}}}} \mathcal{O}_{\mathbb{C}_\varpi})$. We choose a primitive element $t = (t_1, t_2, \dots) \in T_\varpi H(\mathcal{O}_{\widehat{E}^{\mathrm{ab}}})$. Denote by τ the corresponding element of $\mathrm{Nilp}^\flat(\mathcal{O}_{\widehat{E}^{\mathrm{ab}}})$, so that

$$\tau^{1/q^m} = \lim_{r \to \infty} (-1)^{m(n-1)} t_{r+m}^{q^m} \in \mathrm{Nilp}^\flat(\mathcal{O}_{\widehat{E}^{\mathrm{ab}}})$$

by Proposition 1.11.8 and the description of the determinant at the end of Section 1.14. By Corollary 4.2.2, this datum yields an isomorphism

$$A_\infty \widehat{\otimes}_{\mathcal{O}_{\widehat{E}^{\mathrm{ab}}}} \mathcal{O}_{\mathbb{C}_\varpi} \cong \frac{\mathcal{O}_{\mathbb{C}_\varpi}[\![X_1^{q^{-\infty}}, \dots, X_n^{q^{-\infty}}]\!]}{(\Delta(X_1, \dots, X_n)^{q^{-m}} - \tau^{q^{-m}} \mid m \in \mathbb{N})^-} =: A^1_{\infty, \mathcal{O}_{\mathbb{C}_\varpi}}.$$

For $\alpha \in \mathcal{O}_E^\times$, we define

$$A^\alpha_{\infty, \mathcal{O}_{\mathbb{C}_\varpi}} := \frac{\mathcal{O}_{\mathbb{C}_\varpi}[\![X_1^{q^{-\infty}}, \dots, X_n^{q^{-\infty}}]\!]}{(\Delta(X_1, \dots, X_n)^{q^{-m}} - \mathrm{Art}_E(\alpha)(\tau)^{q^{-m}} \mid m \in \mathbb{N})^-}.$$

Note that for any admissible $\mathcal{O}_{\mathbb{C}_\varpi}$-algebra R, we have

$$\mathrm{Hom}(A_\infty \widehat{\otimes}_{\mathcal{O}_{\breve{E}}} \mathcal{O}_{\mathbb{C}_\varpi}, R) \cong \coprod_{\alpha \in \mathcal{O}_E^\times} \mathrm{Hom}(A^\alpha_{\infty, \mathcal{O}_{\mathbb{C}_\varpi}}, R)$$

as sets. We obtain the following description of the G^{u} action on $A_\infty \widehat{\otimes}_{\mathcal{O}_{\breve{E}}} \mathcal{O}_{\mathbb{C}_\varpi}$.

Proposition 4.4.6. *The group G^{u} is generated by elements of the form*

- $(a, a, 1) \in G$ *for* $a \in E^\times$.

- $(g, d, 1) \in G$ *such that* $\det(g) \mathrm{Nrd}(d)^{-1} \in \mathcal{O}_E^\times$.

- $(1, \vartheta^{-m}, \sigma)$ *for* $\sigma \in \mathrm{W}_E$ *with* $\sigma|_{\breve{E}} = \Phi^m$.

These elements act on $A_\infty \widehat{\otimes}_{\mathcal{O}_{\breve{E}}} \mathcal{O}_{\mathbb{C}_\varpi}$ as follows.

- $(a, a, 1)$ *acts trivially.*

- $(g, d, 1)$ *acts by the morphism of $\mathcal{O}_{\mathbb{C}_\varpi}$-algebras*

$$A^\alpha_{\infty, \mathcal{O}_{\mathbb{C}_\varpi}} \to A^{\det(g)^{-1} \mathrm{Nrd}(d)\alpha}_{\infty, \mathcal{O}_{\mathbb{C}_\varpi}}, \quad X_i \mapsto (g, d^{-1})^*(X_i) \text{ for } i = 1, \dots, n.$$

- $(1, \vartheta^{-m}, \sigma)$ *acts by the morphism of $\mathcal{O}_E$-algebras $A^\alpha_{\infty, \mathcal{O}_{\mathbb{C}_\varpi}} \to A^{\mathrm{Art}_E^{-1}(\sigma)\alpha}_{\infty, \mathcal{O}_{\mathbb{C}_\varpi}}$ given by $X_i \mapsto X_i$ for $i = 1, \dots, n$ and $a \mapsto \sigma(a)$ for $a \in \mathcal{O}_{\mathbb{C}_\varpi}$.*

Proof. This follows from the description of the action on Ξ_n and the fact that $\delta((g, d)^* (X_1, \dots, X_n)) = \tau$ is equivalent to $\delta(X_1, \dots, X_n) = [\det(g)^{-1} \mathrm{Nrd}(d)]_{\wedge H}(t)$ by the fact that δ is multilinear and alternating, and Lemma 1.12.2. By the construction of the Artin map and the fact that the components of τ lie inside $\widehat{E}^{\mathrm{ab}}$, the corresponding statement on systems of q-th power roots is $\Delta(X_1, \dots, X_n) = \mathrm{Art}_E(\det(g)^{-1} \mathrm{Nrd}(d))(\tau)$. The remaining statements are immediate. $\square$

In particular, this gives an explicit description of the action of the group $G^1 = \{(g, d, \sigma) \in G \mid \theta(g, d, \sigma) = 1\}$ on $A^1_{\infty, \mathcal{O}_{\mathbb{C}_\varpi}}$.

5 Mieda's Approach to the Explicit Local Langlands Correspondence

We give a brief summary of the content of results in [Mie16].

5.1 The Specialization Map

Let $\mathcal{Y}$ be an affine, flat and topologically of finite type formal scheme over $\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_{\varpi}})$. By [Ray74] and [Hub96, §1.9], we have the corresponding rigid generic fiber $d(\mathcal{Y})$, which we may consider as an analytic adic space over $\mathrm{Spa}(\mathbb{C}_{\varpi}, \mathcal{O}_{\mathbb{C}_{\varpi}})$. We denote by $\mathcal{Y}_s = \mathcal{Y} \times_{\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_{\varpi}})} \mathrm{Spec}(\overline{\mathbb{F}}_q)$ the special fiber of $\mathcal{Y}$, which is a finite-type scheme over $\mathrm{Spec}(\overline{\mathbb{F}}_q)$. By the discussion at the beginning of [Hub96, §3.5], we have a morphism of sites

$$\lambda_{\mathcal{Y}} \colon d(\mathcal{Y})_{\text{ét}} \to (\mathcal{Y}_{\text{red}})_{\text{ét}} \cong \mathcal{Y}_{s,\text{ét}}.$$

For a positive integer m, let us fix the torsion ring $\Lambda = \mathbb{Z}/\ell^m\mathbb{Z}$ and write $\underline{\Lambda}_{d(\mathcal{Y})}$ (respectively, $\underline{\Lambda}_{\mathcal{Y}_s}$) for the corresponding constant sheaf on $d(\mathcal{Y})_{\text{ét}}$ (respectively, $\mathcal{Y}_{s,\text{ét}}$). Then push-forward along $\lambda_{\mathcal{Y}}$ induces a left exact functor of Grothendieck Abelian categories

$$\lambda_{\mathcal{Y},*} \colon \mathrm{Sh}(d(\mathcal{Y})_{\text{ét}}, \underline{\Lambda}_{d(\mathcal{Y})}) \to \mathrm{Sh}(\mathcal{Y}_{s,\text{ét}}, \underline{\Lambda}_{\mathcal{Y}_s}),$$

which is right-adjoint to the exact pullback functor $\lambda_{\mathcal{Y}}^*$. This allows for the following definitions.

Definition 5.1.1 (Formal Nearby Cycle Functor and Specialization Map)**.** We denote by

$$R\Psi_{\mathcal{Y}} \colon D^+(d(\mathcal{Y})_{\text{ét}}, \underline{\Lambda}_{d(\mathcal{Y})}) \to D^+(\mathcal{Y}_{s,\text{ét}}, \underline{\Lambda}_{\mathcal{Y}_s})$$

the right derived functor of $\lambda_{\mathcal{Y},*}$, which we call the formal *nearby cycle functor*. The unit of the adjunction $\lambda_{\mathcal{Y}}^* \dashv R\Psi_{\mathcal{Y}}$ induces, evaluated at $\underline{\Lambda}_{\mathcal{Y}_s}$, a morphism

$$\mathrm{sp}^* \colon \underline{\Lambda}_{\mathcal{Y}_s} \to R\Psi_{\mathcal{Y}}(\lambda_{\mathcal{Y}}^* \underline{\Lambda}_{\mathcal{Y}_s}) = R\Psi_{\mathcal{Y}}(\underline{\Lambda}_{d(\mathcal{Y})}).$$

which we shall call *specialization map*.

By [Hub96, Theorem 5.7.6], there is a canonical isomorphism

$$R\Gamma_c(\mathcal{Y}_s, R\Psi_{\mathcal{Y}}\underline{\Lambda}_{d(\mathcal{Y})}) \xrightarrow{\sim} R\Gamma_c(d(\mathcal{Y}), \underline{\Lambda}_{d(\mathcal{Y})}). \tag{5.1}$$

Upon choosing a compactification $j \colon \mathcal{Y} \hookrightarrow \overline{\mathcal{Y}}$, this morphism is essentially constructed by adjunction from the map

$$\lambda_{\overline{\mathcal{Y}}}^* d(j)_! R\Psi_{\mathcal{Y}} \to j_! \lambda_{\mathcal{Y}}^* R\Psi_{\mathcal{Y}} \to j_!.$$

Thus, the formal nearby cycle functor provides sheaves on $\mathcal{Y}_s$ that allow us to calculate the cohomology of torsion sheaves on $d(\mathcal{Y})$. We obtain the morphism (which we also call specialization map)

$$\mathrm{sp}^* \colon R\Gamma_c(\mathcal{Y}_s, \underline{\Lambda}_{\mathcal{Y}_s}) \to R\Gamma_c(d(\mathcal{Y}), \underline{\Lambda}_{d(\mathcal{Y})}).$$

Let us assume that $d(\mathcal{Y})$ is smooth over $\mathrm{Spa}(\mathbb{C}_\varpi, \mathcal{O}_{\mathbb{C}_\varpi})$. Then, by [Hub96, Theorem 7.3.4], we have the trace map

$$\mathrm{Tr}_{d(\mathcal{Y})}\colon R\Gamma_c(d(\mathcal{Y}), \underline{\Lambda}_{d(\mathcal{Y})}) \to \Lambda(-d)[-2d].$$

Let us write $f\colon \mathcal{Y}_s \to \mathrm{Spec}(\overline{\mathbb{F}}_q)$ for the structure morphism. Then by the adjunction $Rf_! \dashv f^!$ and the isomorphism in (5.1), we obtain a map

$$\mathrm{cosp}^*\colon R\Psi_{\mathcal{Y}}\underline{\Lambda}_{d(\mathcal{Y})} \to f^!\underline{\Lambda}_{\mathrm{Spec}(\overline{\mathbb{F}}_q)}(-d)[-2d].$$

Together with the specialization map, we obtain a morphism

$$\underline{\Lambda}_{\mathcal{Y}_s} \xrightarrow{\mathrm{sp}^*} R\Psi_{\mathcal{Y}}\underline{\Lambda}_{d(\mathcal{Y})} \xrightarrow{\mathrm{cosp}^*} f^!\underline{\Lambda}_{\mathrm{Spec}(\overline{\mathbb{F}}_q)}(-d)[-2d].$$

There is another natural map with the same source and target, namely the *Gysin map*, which we define as the map obtained by adjunction from the trace map

$$\mathrm{Tr}_f\colon Rf_!\underline{\Lambda}_{\mathcal{Y}_s} \to \underline{\Lambda}_{\mathrm{Spec}(\overline{\mathbb{F}}_q)}(-d)[-2d].$$

As f is not necessarily smooth, this is not simply the trace map of Poincar'e duality. However, the trace map exists (cf. [SGA4, XVIII, Théorème 2.9]), and we briefly recall how to construct it from the usual trace map. We may assume that $\mathcal{Y}_s$ is reduced. Then, by [Stacks, Tag 056V] there exists a dense open subscheme $j\colon U \hookrightarrow \mathcal{Y}_s$ that is smooth over $\mathrm{Spec}(\overline{\mathbb{F}}_q)$, and we denote by i the closed complement $i\colon Z \hookrightarrow \mathcal{Y}_s$. Any component of Z has dimension $< d$, which implies that $H^{2d}_c(Z, \underline{\Lambda}_U) = 0$ by [Ful1, Theorem 7.4.5]. From the excision exact triangle (cf. [Ful1, Theorem 7.4.4 (iii)]), we obtain a natural isomorphism

$$R^{2d}(fj)_!\underline{\Lambda}_U \xrightarrow{\sim} R^{2d}f_!\underline{\Lambda}_{\mathcal{Y}_s}.$$

As U is smooth over $\mathrm{Spec}(\overline{\mathbb{F}}_q)$, there is the usual trace morphism from Poincaré duality

$$\mathrm{Tr}_{fj}\colon R^{2d}(fj)_!\underline{\Lambda}_U \to \underline{\Lambda}_{\mathrm{Spec}(\overline{\mathbb{F}}_q)}(-d),$$

constructed, for example, in [Ful1, §8.2]. By the isomorphism above, this gives a map $R^{2d}f_!\underline{\Lambda}_{\mathcal{Y}_s} \to \underline{\Lambda}_{\mathrm{Spec}(\overline{\mathbb{F}}_q)}(-d)$, and as $Rf_!\underline{\Lambda}_{\mathcal{Y}_s}$ is (up to quasi-isomorphism) concentrated in degree $\leq 2d$, this suffices to construct the desired map. As the trace map commutes with base change, this construction does not depend on the choice of U.

With the trace map defined, we can formulate Mieda's main result about the specialization map.

Proposition 5.1.2 (Mieda's Result about the Specialization Map)**.** *The composite*

$$\underline{\Lambda}_{\mathcal{Y}_s} \xrightarrow{\mathrm{sp}^*} R\Psi_{\mathcal{Y}}\underline{\Lambda}_{d(\mathcal{Y})} \xrightarrow{\mathrm{cosp}^*} f^!\underline{\Lambda}_{\mathrm{Spec}(\overline{\mathbb{F}}_q)}(-d)[-2d]$$

is equal to the Gysin map with respect to f, that is, the map

$$\mathrm{Gys}_f\colon \underline{\Lambda}_{\mathcal{Y}_s} \to f^!\underline{\Lambda}_{\mathrm{Spec}(\overline{\mathbb{F}}_q)}(-d)[-2d]$$

obtained by the trace map from the adjunction $Rf_! \dashv f^!$.

Proof. This is [Mie16, Theorem 2.1]. □

In practice, $\mathcal{Y}$ will serve as a "finite level version" of certain affinoids constructed inside the Lubin–Tate perfectoid space, which in our application admit surjections to certain smooth varieties over $\overline{\mathbb{F}}_q$. Let X be a purely d-dimensional separated smooth scheme of finite type over $\mathrm{Spec}(\overline{\mathbb{F}}_q)$, with structure morphism denoted by g. Assume that there is a finite surjective morphism

$$\pi \colon \mathcal{Y}_s \to X.$$

As a corollary to the theorem above, Mieda constructs a commutative diagram

$$(5.2)$$

cf. [Mie16, Corollary 2.7]. In this diagram, the maps π_* and $\times \deg \pi$ are given as follows.

- For the map π_*, note that, as the structure map $g \colon X \to \mathrm{Spec}(\overline{\mathbb{F}}_q)$ is smooth, the Gysin map

$$\mathrm{Gys}_g \colon \underline{\Lambda}_X \to g^! \underline{\Lambda}_{\mathrm{Spec}(\overline{\mathbb{F}}_q)}(-d)[-2d]$$

is an isomorphism. Furthermore, as π is finite, we have $R\pi_* = \pi_* = R\pi_!$. This allows us to define π_* as the map induced by the composition

$$\pi_* f^! \underline{\Lambda}_{\mathrm{Spec}(\overline{\mathbb{F}}_q)}(-d)[-2d] \cong R\pi_! \pi^! g^! \underline{\Lambda}_{\mathrm{Spec}(\overline{\mathbb{F}}_q)}(-d)[-2d] \xrightarrow{\sim} R\pi_! \pi^! \underline{\Lambda}_X \to \underline{\Lambda}_X,$$

where the last map is the counit of the adjunction $R\pi_! \dashv \pi^!$ evaluated at $\underline{\Lambda}_X$.

- For the map $\times \deg \pi$, decompose X into connected components $X = \coprod_j X_j$ and write $\delta_j \in \mathbb{Z}$ for the generic degree of π over X_j. Then $\times \deg \pi$ is the map obtained from the morphism of étale sheaves $\underline{\Lambda}_X \to \underline{\Lambda}_X$ admitting the local description

$$\underline{\Lambda}_X(X_j) \xrightarrow{\times \delta_j} \underline{\Lambda}_X(X_j).$$

Commutativity of the square in the upper right hand side is the content of Proposition 5.1.2, and commutativity of the square in the upper left is essentially a consequence of the naturality of the trace map. The lower two squares commute by definition. Furthermore, the square

$$
\begin{array}{ccc}
H^i_{\mathrm{c}}(\mathcal{Y}_s, R\Psi_{\mathcal{Y}}\underline{\Lambda}_{d(\mathcal{Y})}) & \xrightarrow{\ \sim\ } & H^i_{\mathrm{c}}(d(\mathcal{Y}), \underline{\Lambda}_{d(\mathcal{Y})}) \\
\downarrow & & \downarrow \\
H^i(\mathcal{Y}_s, R\Psi_{\mathcal{Y}}\underline{\Lambda}_{d(\mathcal{Y})}) & =\!=\!= & H^i(d(\mathcal{Y}), \underline{\Lambda}_{d(\mathcal{Y})})
\end{array}
$$

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is commutative. The following statement is a direct consequence.

Proposition 5.1.3. *Let X and $\mathcal{Y}$ be as above, that is, let X be a purely d-dimensional separated smooth scheme of finite type over $\overline{\mathbb{F}}_q$, and let $\mathcal{Y}$ be a flat and topologically finite-type formal scheme over $\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_\varpi})$ with generic fiber $d(\mathcal{Y})$ smooth over $\mathrm{Spa}(\mathbb{C}_\varpi, \mathcal{O}_{\mathbb{C}_\varpi})$. Let $\pi\colon \mathcal{Y}_s \to X$ be a finite surjective morphism. Furthermore, assume that there is an open immersion $d(\mathcal{Y}) \hookrightarrow Z$, where Z is an adic space that is locally of finite type, separated, and taut over $\mathrm{Spa}(\mathbb{C}_\varpi, \mathcal{O}_{\mathbb{C}_\varpi})$. Then, the map*

$$H_c^i(X, \underline{\Lambda}_X) \xrightarrow{\times \deg \pi} H_c^i(X, \underline{\Lambda}_X) \to H^i(X, \underline{\Lambda}_X)$$

factors through the composition

$$H_c^i(X, \underline{\Lambda}_X) \xrightarrow{\pi^*} H_c^i(\mathcal{Y}_s, \underline{\Lambda}_{\mathcal{Y}_s}) \xrightarrow{\mathrm{sp}^*} H_c^i(d(\mathcal{Y}), \underline{\Lambda}_{d(\mathcal{Y})}) \to H_c^i(Z, \underline{\Lambda}_Z).$$

Sketch of Proof. By construction of the lower shriek for taut morphisms (cf. [Hub96, §0.4 D)]), formation of the lower shriek functor is covariant with respect open embeddings, and we have the commutative square

$$
\begin{array}{ccc}
H_c^i(d(\mathcal{Y}), \underline{\Lambda}_{d(\mathcal{Y})}) & \longrightarrow & H_c^i(Z, \underline{\Lambda}_Z) \\
\downarrow & & \downarrow \\
H^i(d(\mathcal{Y}), \underline{\Lambda}_{d(\mathcal{Y})}) & \longleftarrow & H^i(Z, \underline{\Lambda}_Z).
\end{array}
$$

The claim now essentially follows from gluing together all the commutative diagrams above. $\square$

Passing from torsion cohomology to ℓ-adic cohomology, we quickly arrive at the following result.

Theorem 5.1.4 (Mieda's Injectivity Result at Finite Level). *Let X and Z be as in Proposition 5.1.3. If $V \subseteq H_c^i(X, \overline{\mathbb{Q}}_\ell)$ is a subspace such that the composition $V \hookrightarrow H_c^i(X, \overline{\mathbb{Q}}_\ell) \to H^i(X, \overline{\mathbb{Q}}_\ell)$ is injective, the map*

$$V \hookrightarrow H_c^i(X, \overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{sp}^* \circ \pi^*} H_c^i(d(\mathcal{Y}), \overline{\mathbb{Q}}_\ell) \to H_c^i(Z, \overline{\mathbb{Q}}_\ell)$$

is also injective.

Sketch of Proof. We have

$$H_c^i(X, \mathbb{Z}_\ell) = \varprojlim_{m \in \mathbb{N}} H_c^i(X, \underline{\mathbb{Z}/\ell^m\mathbb{Z}}) \quad \text{and} \quad H_c^i(d(\mathcal{Y}), \mathbb{Z}_\ell) = \varprojlim_{m \in \mathbb{N}} H_c^i(d(\mathcal{Y}), \underline{\mathbb{Z}/\ell^m\mathbb{Z}})$$

by [Fu11, Corollary 9.3.3 and Proposition 10.1.15] and [Hub98, Theorem 3.1]. The corresponding statement for Z is in general wrong. In the limit over $m \in \mathbb{N}$, Proposition 5.1.3 thus yields that the map

$$H_c^i(X, \overline{\mathbb{Q}}_\ell) \xrightarrow{\times \deg \pi} H_c^i(X, \overline{\mathbb{Q}}_\ell) \to H^i(X, \overline{\mathbb{Q}}_\ell)$$

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factors over the composition

$$H_c^i(X, \overline{\mathbb{Q}}_\ell) \xrightarrow{\pi^*} H_c^i(\mathcal{Y}_s, \overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{sp}^*} H_c^i(d(\mathcal{Y}), \overline{\mathbb{Q}}_\ell) \to \left(\varprojlim_{m \in \mathbb{N}} H_c^i(Z, \underline{\mathbb{Z}/\ell^m \mathbb{Z}}_Z) \right) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell.$$

The right-most map above factors over the natural homomorphism

$$H_c^i(Z, \overline{\mathbb{Q}}_\ell) \to \left(\varprojlim_{m \in \mathbb{N}} H_c^i(Z, \underline{\mathbb{Z}/\ell^m \mathbb{Z}}_Z) \right) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell.$$

The claim follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

5.2 Application to the Lubin–Tate Tower

We now explain how the results of the previous subsection apply to the cohomology of the Lubin–Tate tower. In order to set the stage, recall the definition of the Lubin–Tate perfectoid space $M'_{\infty, \mathbb{C}_\varpi}$ and its right action by the group G^1, as defined in the previous two sections. Assume that we are in the following situation.

Situation 5.2.1. We are given

- a rational subset $U \subset M'_{\infty, \mathbb{C}_\varpi}$ and an open subgroup $K_U \subset \mathrm{SL}_n(\mathcal{O}_E)$ stabilizing U,

- a ϖ-adically complete, flat $\mathcal{O}_{\mathbb{C}_\varpi}$-algebra Q such that $\mathrm{Spf}(Q) =: \mathcal{X}$ is a formal model of U and the action of K_U extends to $\mathcal{X}$ (which is to say, the induced action of K_U on $Q[\frac{1}{\varpi}]$ stabilizes Q),

- an affine scheme $Y = \mathrm{Spec}(R)$, pure-dimensional, smooth and of finite type over $\overline{\mathbb{F}}_q$,

- an isomorphism
$$R^{perf} := \varinjlim_{x \mapsto x^p} R \cong Q \otimes \overline{\mathbb{F}}_q, \qquad\qquad (5.3)$$

- a subgroup J^1 of G^1 whose action on $M'_{\infty, \mathbb{C}_\varpi}$ stabilizes U and extends to $\mathcal{X} := \mathrm{Spf}(Q)$,

- and a left action of J^1 on R such that the isomorphism (5.3) is compatible with respect to the induced J^1-actions on both sides.

We want to use the ideas of Section 5.1 to compare the J^1-representations $H_c^{n-1}(Y, \overline{\mathbb{Q}}_\ell)$ and $\mathrm{colim}_K H_c^{n-1}(M'_{K, \mathbb{C}_\varpi}, \overline{\mathbb{Q}}_\ell) = H'_{\mathrm{LT}}$. We will need to construct finite level versions of the rational subset U. For $m \in \mathbb{N}$, recall the definition of the m-th congruence subgroup K_m from (3.2) and denote $K'_m := K_m \cap \mathrm{SL}_n(\mathcal{O}_E)$. The following is part of [Mie16, Proposition 4.5].

Proposition 5.2.2. *There exists an integer $m \geq 0$ such that $K'_m \subset K_U$ and an open affinoid U_m in $M'_{m, \mathbb{C}_\varpi}$ such that the inverse image of U_m under the projection $M'_{\infty, \mathbb{C}_\varpi} \to M'_{m, \mathbb{C}_\varpi}$ is equal to U.*

With m as in the proposition above, we write $K_0 = K'_m$ and $U_{K_0} = U_m$. For an open normal subgroup $K \subseteq K_0$, we write $U_K = \mathrm{Spa}(B_K, B_K^\circ) \subset M'_{K, \mathbb{C}_\varpi}$ for the preimage of U_{K_0} under $M'_{K, \mathbb{C}_\varpi} \to M'_{K_0, \mathbb{C}_\varpi}$. For any such K, we find that B_K is a complete Huber $\mathbb{C}_\varpi$-algebra, topologically of finite type and endowed with an action of K_0/K. Furthermore, for

$K' \subseteq K$ we have K_0-equivariant transition maps $B_K \to B_{K'}$, and $\{B_K\}_{K \subseteq K_0}$ becomes an inductive system (with K running over open normal subgroups). For each such K, we have K_0-equivariant morphisms of Huber pairs $(B_K, B_K^\circ) \to (Q[\frac{1}{\varpi}], \tilde{Q})$.

It can be shown that the morphism $B_K \to Q[\frac{1}{\varpi}]$ is injective (cf. [Mie16, Lemma 3.4]), and we define $B' = \bigcup_{K \subseteq K_0} B_K \subseteq Q[\frac{1}{\varpi}]$ as the union of the images of $\{B_K\}_K$ inside $Q[\frac{1}{\varpi}]$. Now B' is a dense, K_0-stable subgalgebra of $Q[\frac{1}{\varpi}]$, cf. [Mie16, Proposition 4.5]. We furthermore define $Q' = Q \cap B'$ and $Q_K = (Q')^K \subset Q$ for any open normal subgroup $K \subseteq K_0$.

Lemma 5.2.3. *Inside $Q[\frac{1}{\varpi}]$, the algebra Q_K is a subset of B_K°, and B_K° is integral over Q_K.*

Proof. This follows from Definition 3.5, Lemma 3.7 and Lemma 3.8 of [Mie16]. □

By [Mie16, Lemma 3.16], it is possible to find a normal subgroup $K \subseteq K_0$, an $\mathcal{O}_{\mathbb{C}_\varpi}$-subalgebra Q_0 which is a ring of definition for B_K and topologically finitely generated (that is, admitting a surjection from $\mathcal{O}_{\mathbb{C}_\varpi}\langle T_1, \dots, T_r \rangle$ for some $r \in \mathbb{N}$), and a finite surjective morphism $\pi \colon \mathrm{Spec}(Q_0 \otimes \overline{\mathbb{F}}_q) \to \mathrm{Spec}(R) = Y$ that fits into a commutative square

$$(5.4)$$
$$
\begin{array}{ccc}
\mathrm{Spec}(Q \otimes_{\mathcal{O}_{\mathbb{C}_\varpi}} \overline{\mathbb{F}}_q) & \longrightarrow & \mathrm{Spec}(Q_0 \otimes_{\mathcal{O}_{\mathbb{C}_\varpi}} \overline{\mathbb{F}}_q) \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\pi} \\
Y^{perf} & \longrightarrow & Y.
\end{array}
$$

We set $\mathcal{Y} := \mathrm{Spf}(Q_0)$ and note that $d(\mathcal{Y}) = U_K$. Using the specialization map introduced in Section 5.1, we define the specialization map at infinite level as the composition

$$\mathrm{sp}^* \colon H_c^i(Y, \overline{\mathbb{Q}}_\ell) \xrightarrow{\pi^*} H_c^i(\mathcal{Y}_s, \overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{sp}^*} H_c^i(d(\mathcal{Y}), \overline{\mathbb{Q}}_\ell) = H_c^i(U_K, \overline{\mathbb{Q}}_\ell) \to \operatorname*{colim}_K H_c^i(U_K, \overline{\mathbb{Q}}_\ell). \quad (5.5)$$

A priori, it is not clear that this definition is independent of the choices made above. In order to get a hold of these choices, we introduce the set

$$\mathcal{D} := \left\{ (K, Q_0, \pi) \,\middle|\, \begin{array}{c} K \subseteq K_0 \text{ is a normal open subgroup} \\ Q_0 \subset Q \text{ is a topologically finitely generated ring of definition for } B_K \\ \pi \colon \mathrm{Spec}(Q_0 \otimes_{\mathcal{O}_{\mathbb{C}_\varpi}} \overline{\mathbb{F}}_q) \to Y \text{ is finite surjective and fits into } (5.4) \end{array} \right\}.$$

This set comes equipped with a natural partial order, we say that $(K, Q_0, \pi) \leq (K', Q_0', \pi')$ if $K' \subseteq K$ and $Q_0 \subseteq Q_0'$, and moreover π' arises as the composition of π with the induced morphism $\mathrm{Spec}(Q_0' \otimes_{\mathcal{O}_{\mathbb{C}_\varpi}} \overline{\mathbb{F}}_q) \to \mathrm{Spec}(Q_0 \otimes_{\mathcal{O}_{\mathbb{C}_\varpi}} \overline{\mathbb{F}}_q)$.

Lemma 5.2.4. *This partial order turns $\mathcal{D}$ into a filtered category. Moreover, for any triple $(K, Q_0, \pi) \in \mathcal{D}$ and any open normal subgroup $K' \subset K_0$ contained in K, there is a triple $(K', Q_0', \pi') \in \mathcal{D}$ with $(K, Q_0, \pi) \leq (K', Q_0', \pi')$.*

Proof. This is a combination of Lemma 3.11, Corollaries 3.12 and 3.13 and Lemma 3.14 in [Mie16]. □

Proposition 5.2.5. *The specialization map as defined in (5.5) is independent of the choice of triple $(K, Q_0, \pi) \in \mathcal{D}$.*

Proof. This is Lemma 3.18 in [Mie16]. □

The condition that Q_0 must be topologically finitely generated is imposed because it allows to show that the specialization map is equivariant for the J^1-action on both sides.

Definition 5.2.6 (Morphism of Finite Level). An automorphism $\sigma\colon Q \to Q$ is said to be of finite level if it preserves $Q' = (\bigcup_K B_K) \cap Q$.

Lemma 5.2.7. *Any automorphism of finite level induces a natural automorphism of the vector space* $\operatorname{colim}_K H_c^i(U_K, \overline{\mathbb{Q}}_\ell)$.

Sketch of Proof. This is [Mie16, Lemma 3.20]. The crucial point is that for each open normal subgroup $K \subset K_0$, and any $(K, Q_0, \pi) \in \mathcal{D}$, there is an open subgroup $K' \subset K$ fixing the finitely many topological generators of Q_0. Now, it can be shown that $\sigma(B_K) \subset B_{K'}$, and that the resulting morphism $\sigma\colon B_K \to B_{K'}$ is finite. This induces finite morphisms $U_{K'} \to U_K$, and we obtain the desired automorphism after taking compactly supported cohomology and colimit over K. □

By [Mie16, Proposition 3.21], the specialization map respects automorphisms of finite level.

Proposition 5.2.8. *Let σ be an automorphism of finite level of Q and let $\overline{\sigma}$ be an automorphism of R. Assume that the isomorphism (5.3) is compatible with the automorphisms induced by $\overline{\sigma}$ and σ. Then, the specialization map*

$$\operatorname{sp}^*\colon H_c^i(Y, \overline{\mathbb{Q}}_\ell) \to \operatorname*{colim}_K H_c^i(U_K, \overline{\mathbb{Q}}_\ell)$$

is compatible with the automorphisms induced by $\overline{\sigma}$ and σ.

Proposition 5.2.9. *The group J^1 acts on Q through automorphisms of finite level. By Lemma 5.2.7, this endows $\operatorname{colim}_K H_c^i(U_K, \overline{\mathbb{Q}}_\ell)$ with the structure of a J^1-representation. The natural morphism $\operatorname{colim}_K H_c^i(U_K, \overline{\mathbb{Q}}_\ell) \to \operatorname{colim}_K H_c^i(M'_{K,\mathbb{C}_\varpi}, \overline{\mathbb{Q}}_\ell)$ is equivariant for the J^1-action on both sides.*

Proof. This is [Mie16, Propsition 4.5]. □

We are now in a position to apply Theorem 5.1.4.

Theorem 5.2.10 (Mieda's Injectivity Theorem for the Lubin–Tate Tower). *Assume that we are in Situation 5.2.1. Let $V \subset H_c^i(Y, \overline{\mathbb{Q}}_\ell)$ be a subspace such that the composition $V \hookrightarrow H_c^i(Y, \overline{\mathbb{Q}}_\ell) \to H^i(Y, \overline{\mathbb{Q}}_\ell)$ is injective. Then, the composition*

$$V \hookrightarrow H_c^i(Y, \overline{\mathbb{Q}}_\ell) \xrightarrow{\operatorname{sp}^*} \operatorname*{colim}_K H_c^i(U_K, \overline{\mathbb{Q}}_\ell) \to \operatorname*{colim}_K H_c^i(M'_{K,\mathbb{C}_\varpi}, \overline{\mathbb{Q}}_\ell) = H'_{\mathrm{LT}} \qquad (5.6)$$

is injective as well. If V is J^1-stable, this injection is equivariant for the action of J^1 on both sides.

Proof. Fix a triple $(K, Q_0, \pi) \in \mathcal{D}$, let K' be an open normal subgroup of K_0 contained in K. Then there is a triple $(K, Q_0, \pi) \leq (K', Q'_0, \pi') \in \mathcal{D}$. We write $\mathcal{Y}' = \operatorname{Spf}(Q'_0)$, which is flat

and topologically of finite type over $\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_\varpi})$. Furthermore, it has generic fiber $U_{K'}$, which is smooth over $\mathrm{Spa}(\mathbb{C}_\varpi, \mathcal{O}_{\mathbb{C}_\varpi})$, and $\pi' \colon Y \to \mathcal{Y}_s$ is a finite surjective morphism. Thereby, Theorem 5.1.4 yields an injection

$$V \hookrightarrow H_c^i(Y, \overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{sp}^* \circ \pi'^*} H_c^i(d(\mathcal{Y}'), \overline{\mathbb{Q}}_\ell) = H_c^i(U_{K'}, \overline{\mathbb{Q}}_\ell) \to H_c^i(M'_{K', \mathbb{C}_\varpi}, \overline{\mathbb{Q}}_\ell).$$

Taking colimit over K', we obtain a map

$$V \hookrightarrow H_c^i(Y, \overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{sp}^*} \operatorname*{colim}_K H_c^i(U_K, \overline{\mathbb{Q}}_\ell) \to \operatorname*{colim}_K H_c^i(M'_{K, \mathbb{C}_\varpi}, \overline{\mathbb{Q}}_\ell) = H'_{\mathrm{LT}}$$

which is injective since filtered colimits are exact. Moreover, this map is the same as the one defined in (5.6).

The map $\operatorname{colim}_K H_c^i(U_K, \overline{\mathbb{Q}}_\ell) \to \operatorname{colim}_K H_c^i(M'_{K, \mathbb{C}_\varpi}, \overline{\mathbb{Q}}_\ell)$ is J^1-equivariant by Proposition 5.2.9, and together with Proposition 5.2.8 it follows that (5.6) is J^1-equivariant if V is J^1-stable. This finishes the proof. $\square$

6 Deligne–Lusztig Theory

The aim of this section is to outline the construction of a correspondence between certain characters of $\mathbb{F}_{q^n}^\times$ (with values in $\mathbb{C}^\times$) and cuspidal representations of $\mathrm{GL}_n(\mathbb{F}_q)$. The correspondence we construct here is an instance of a more general theory developed by Deligne–Lusztig. In [DL76], they construct for any connected reductive algebraic group $G = G_0 \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{\mathbb{F}}_q)$ and any Frobenius-stable maximal torus $T \subseteq G$ a correspondence associating to certain characters θ of $T^{\mathrm{Frob}}(\overline{\mathbb{F}}_q)$ a virtual representation $R_{T,\theta}$ of $G^{\mathrm{Frob}}(\overline{\mathbb{F}}_q)$. These virtual representations arise from the compactly supported ℓ-adic cohomology (with $\ell \neq p$) of a certain variety $\mathrm{DL}_{G,T}$ admitting commuting actions by G^{Frob} and T^{Frob}. In this section, we give explicit descriptions of the spaces that occur in the situation where $G = \mathrm{GL}_{V_0 \otimes \overline{\mathbb{F}}_q}$ for some n-dimensional $\mathbb{F}_q$-vector space V_0 and $T \subset G$ is a maximal Frobenius-stable torus with $T^{\mathrm{Frob}}(\overline{\mathbb{F}}_q) = \mathbb{F}_{q^n}^\times$. The representation-theoretic results of the theory are stated as facts, and proofs are omitted.

6.1 Flag Varieties

We begin by introducing (full) flags and their moduli spaces, flag varieties. Let k be a field, and let V be a finite-dimensional k-vector space of dimension n. We write $\tilde{V}$ for the corresponding quasi-coherent sheaf on $\mathrm{Spec}(k)$, and GL_V for the general linear group scheme of $\tilde{V}$.

Definition 6.1.1 (Flag Variety). Let $X \colon (\mathrm{Sch}/k)^{\mathrm{op}} \to (\mathrm{Set})$ be the functor assigning to each k-scheme $f \colon S \to \mathrm{Spec}(k)$ the set

$$X(S) = \left\{ \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_{n-1} \subset f^*\tilde{V} \ \middle| \ \begin{array}{c} \mathcal{F}_i \text{ is, for all } i, \text{ a locally direct summand} \\ \text{of } f^*\tilde{V}, \text{ locally free of rank } i \end{array} \right\}.$$

Recall that a subsheaf $\mathcal{F}_i \subset f^*\tilde{V}$ is locally a direct summand if it is quasicoherent, and for each $s \in S$ there is some neighborhood U of s such that $\mathcal{F}_i|_U$ is a direct summand of $f^*\tilde{V}|_U$. We refer to the S-valued points of X as families of flags over S.

There is a natural GL_V-action on X, induced by the natural action of $\mathrm{GL}_V(S)$ on $f^*\tilde{V}$. Given $(\mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_{n-1}) \in X(S)$ and $g \in \mathrm{GL}_V(S)$, we set

$$g.(\mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_{n-1}) = (g\mathcal{F}_1 \subsetneq g\mathcal{F}_2 \subsetneq \cdots \subsetneq g\mathcal{F}_{n-1}).$$

The k-rational points of X are called *(full) flags*. They are given by an increasing $n-1$-tuple of vector spaces

$$F_\bullet = (F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{n-1} \subsetneq V) \in X(k).$$

There are natural morphisms

$$\nu_i \colon X \to \mathrm{Grass}_{V,i}, \quad (\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{n-1} \subset f^*\tilde{V}) \mapsto (f^*\tilde{V}^\vee \twoheadrightarrow \mathcal{F}_i^\vee),$$

where $\mathrm{Grass}_{V,i}$ denotes the Grassmannian parameterizing surjections of $f^*\tilde{V}^\vee$ to locally free coherent modules of rank i, as defined in [GW20, p. 8].

We have the following classical result about flag varieties.

Proposition 6.1.2. *The induced morphism of functors*

$$X \to \mathrm{Grass}_{V,1} \times_{\mathrm{Spec}(k)} \cdots \times_{\mathrm{Spec}(k)} \mathrm{Grass}_{V,n-1}$$

is representable by a closed embedding. In particular, as for integers $1 \leq d \leq n-1$ the functor $\mathrm{Grass}_{V,d}$ is represented by a projective scheme, X is represented by a projective scheme.

Proof. Upon choosing a basis of V, the claim can be checked directly on the standard affine cover of the Grassmannians, where the condition that $\mathcal{F}_i$ is contained in $\mathcal{F}_{i+1}$ is cut out by a polynomial equation. $\square$

Given a flag $F_\bullet \in X(k)$, we write $B_{F_\bullet} \subset \mathrm{GL}_V$ for the isotropy subgroup of $F_\bullet$ under this action. In [DL76], the authors work with schemes that arise as quotients G/B where B is some Borel subgroup of a connected reductive algebraic group G. The following proposition shows that X is isomorphic to the quotient of $\mathrm{GL}_V / B_{F_\bullet}$.

Proposition 6.1.3. *Zariski-locally, the action of GL_V on X is transitive. In particular, the orbit map $\mu_{F_\bullet} \colon \mathrm{GL}_V \to X$, $g \mapsto g.F_\bullet$ yields an isomorphism of schemes $\mathrm{GL}_V / B_{F_\bullet} \to X$.*

Proof. We show that Zariski-locally, $\mu_{F_\bullet}$ induces an isomorphism $\mathrm{GL}_V(S)/B_{F_\bullet}(S) \to X(S)$. Let $(v_1, v_2, \ldots, v_n)$ be a basis of V such that each F_i is generated by the first i basis vectors. Given any k-scheme S and a family of flags $F_\bullet' \in X(S)$, there is a Zariski cover $\phi \colon S' \to S$ (with structure map to $\mathrm{Spec}(k)$ denoted by f') that trivializes all the quotients F_i'/F_{i-1}' for $i = 1, \ldots, n$. Hence, we may choose generators $w_i \in \Gamma(S', \phi^*(F_i'/F_{i-1}'))$, and lift them to elements $\tilde{w}_i \in \Gamma(S', f'^*\tilde{V})$. The global sections w_i generate $f'^*\tilde{V}$, and the $\mathcal{O}_{S'}$-linear map $f'^*v_i \mapsto w_i$ produces an element in $\mathrm{GL}_V(S')$, unique up to an element in $B_{F_\bullet}(S')$. Hence $X(S') \cong \mathrm{GL}_V(S')/B_{F_\bullet}(S')$.

The quotient sheaf GL_V / B is defined as the fppf sheafification of the quotient presheaf $S \mapsto \mathrm{GL}_V(S)/B(S)$. The argument above shows that the Zariski-sheafification of this presheaf is representable, and in particular, already an fppf-sheaf. $\square$

Corollary 6.1.4. *The scheme X is smooth over k, of dimension $\frac{n(n-1)}{2}$.*

Proof. This follows because quotients of smooth algebraic groups by algebraic subgroups are smooth (cf. [Mil17, Corollary 5.26]), and the fact that

$$n^2 = \dim \mathrm{GL}_V = \dim B + \dim X = \frac{n(n+1)}{2} + \dim X,$$

cf. [Mil17, Proposition 5.23]. $\square$

Write $\mathbb{T}_X$ for the sheaf assigning to an X-scheme $S \to X$ (corresponding to a family of flags $(\mathcal{F}_i)_i \in X(S)$) the group

$$\mathbb{T}_X(S) = \mathrm{Aut}_{\mathcal{O}_S}(\mathcal{F}_1/\mathcal{F}_0) \times \cdots \times \mathrm{Aut}_{\mathcal{O}_S}(\mathcal{F}_n/\mathcal{F}_{n-1}) = \mathbb{G}_{m,X}^n(S).$$

Definition 6.1.5 (Classifying Space of Marked Flags)**.** Let Y be the functor $(\mathrm{Sch}/X)^{\mathrm{op}} \to$ (Set) given by sending a morphism $S \to X$, corresponding to a family of flags $(\mathcal{F}_i)_i \in X(S)$, to the set
$$Y(S) = \left\{ (e_1, \ldots, e_n) \mid e_i \colon \mathcal{O}_S \xrightarrow{\sim} \mathcal{F}_i/\mathcal{F}_{i-1} \text{ for } i = 1, \ldots, n-1 \right\}.$$
Here, $\mathcal{F}_0$ is the zero-sheaf.

Just like X, the functor Y comes with a natural action by GL_V and the natural morphism

$$Y(S) \to X(S), \quad (\mathcal{F}_i, e_i)_i \mapsto (\mathcal{F}_i)_i \tag{6.1}$$

is equivariant for this action. One readily checks that Y is a sheaf on $(\mathrm{Sch}/X)_{\mathrm{Zar}}$, the big Zariski site of schemes over X. By design, it is a $\mathbb{T}_X$-torsor and thereby admits a Zariski cover of open subfunctors. Hence, it is representable (cf. [GW20, Theorem 8.9]), and the morphism $Y \to X$ is smooth and affine.

Similarly to X, the scheme Y is isomorphic to certain quotients of algebraic groups. Let $(F_\bullet, e_\bullet) \in Y(k)$ be a marked flag and write $U_{F_\bullet, e_\bullet} \subset \mathrm{GL}_V$ for the (unipotent) isotropy subgroup of $(F_\bullet, e_\bullet)$ under the action of GL_V.

Lemma 6.1.6. *In this situation, $Y \cong \mathrm{GL}_V / U_{F_\bullet, e_\bullet}$.*

Proof. This can be shown using the same arguments as in the proof of Proposition 6.1.3. $\quad\square$

If $(v_1, \ldots, v_n)$ is a basis for V, we write $F(v_1, \ldots, v_n)$ for the (marked) flag spanned by the vectors $(v_1, \ldots, v_n)$. More generally, if S is a k-scheme and $(v_1, \ldots, v_n)$ is a tuple of elements in $\Gamma(S, f^*\tilde{V})$ such that the induced map $(v_1, \ldots, v_n) \colon \mathcal{O}_S^n \to f^*\tilde{V}$ is an isomorphism (in which case we call $(v_i)_i$ a basis), we write $F(v_1, \ldots, v_n)$ for the corresponding family of (marked) flags.

Recall the Bruhat decomposition for GL_V. Fixing a basis $(e_1, \ldots, e_n)$ of V, we obtain an injection $\Sigma_n \hookrightarrow \mathrm{GL}_V$ (assigning to each $w \in \Sigma_n$ the corresponding permutation matrix), and a (marked) flag $F_\bullet^{\mathrm{std}} = F(e_1, \ldots, e_n) \in X(k)$. We define O_w as the GL_V-orbit of the pair of flags $(F_\bullet^{\mathrm{std}}, w.F_\bullet^{\mathrm{std}}) \in (X \times X)(k)$. Note that this does not depend on the choice of basis. The Bruhat decomposition states that all GL_V-orbits inside $X \times X$ are of this form.

Proposition 6.1.7 (Bruhat Decomposition)**.** *There is a decomposition of $X \times X$ into GL_V-stable locally closed subschemes*

$$X \times X = \dot{\bigcup_{w \in \Sigma_n}} O_w.$$

Each O_w is smooth of dimension $\dim(X) + l(w)$, where $l(w)$ denotes the Coxeter length of w.

Proof. For each w, the scheme O_w is locally closed as orbits are locally closed by [Mil17, Proposition 1.65 b)] and smooth, as it is isomorphic to a quotient of GL_V. The remaining claims boil down to classical theory (in particular, the classical Bruhat decomposition), cf. [Mil17, Chapter 21], and elementary considerations about the dimensions of the isotropy subgroups of pairs $(F_\bullet, F_\bullet') \in X_w(k)$. $\quad\square$

Let $(\mathcal{F}_\bullet, \mathcal{F}'_\bullet) \in (X \times X)(S)$ be a pair of flags over a k-scheme S. We say that $(F_\bullet, F'_\bullet)$ is in relative position $w \in \Sigma_n$ if it lies inside the subset $O_w(S) \subset (X \times X)(S)$.

Similarly, we can characterize the GL_V-orbits in $Y \times_{\mathrm{Spec}(k)} Y$. For any choice of elements $w \in \Sigma_n \subset \mathrm{GL}_V(k)$ and $t \in \mathbb{T}_X(k)$, we define $\tilde{O}_{w,t}$ as the GL_V-orbit of the element

$$((F_\bullet^{\mathrm{std}}, e_\bullet^{\mathrm{std}}), (w.F_\bullet^{\mathrm{std}}, w.te_\bullet^{\mathrm{std}})) \in (Y \times Y)(k).$$

A pair of marked flags over a k-scheme S is said to be in relative position $(w, t) \in \Sigma_n \times (k^\times)^n$ if it lies inside $\tilde{O}_{w,t}$. The following proposition gives a convenient characterization of relative position.

Lemma 6.1.8.   *1. A pair of families of flags $(\mathcal{F}_\bullet, \mathcal{F}'_\bullet) \in (X \times X)(S)$ is in relative position $w \in \Sigma_n$ if and only if there exists an fppf-cover $\phi\colon S' \to S$ (with structure map $f'\colon S' \to \mathrm{Spec}(k)$) and a basis $(v_1, \ldots, v_n) \in \Gamma(S', f'^* \tilde{V})$ such that*

$$\phi^* \mathcal{F}_i = \langle v_1, \ldots, v_i \rangle \quad \text{and} \quad \phi^* \mathcal{F}'_i = \langle v_{w(1)}, \ldots, v_{w(i)} \rangle \quad \text{for all } i = 1, \ldots, n-1.$$

2. A pair of families of marked flags $((\mathcal{F}_\bullet, e_\bullet), (\mathcal{F}'_\bullet, e'_\bullet)) \in (Y \times Y)(S)$ is in relative position (w, t) if and only if $(\mathcal{F}_\bullet, \mathcal{F}'_\bullet)$ is in relative position w and there is a basis as above furthermore satisfying

$$\phi^* e_i \equiv v_i \mod \phi^* \mathcal{F}_{i-1} \quad \text{and} \quad \phi^* e'_i \equiv t_{w(i)} v_{w(i)} \mod \phi^* \mathcal{F}'_{w(i)-1} \quad \text{for all } i = 1, \ldots, n.$$

Here, ϕ^* denotes the natural pullback of sections $\Gamma(S, f^* \tilde{V}) \to \Gamma(S', f'^* \tilde{V})$.

Proof. This is a simple reformulation of what it means to be in the corresponding GL_V-orbits. Given any choice of "standard" basis $(e_1, \ldots, e_n)$ of V and a section $(\mathcal{F}_\bullet, \mathcal{F}'_\bullet)$ in the orbit of $(F_\bullet^{\mathrm{std}}, w.F_\bullet^{\mathrm{std}})$, we may choose an fppf cover $S' \to S$ such that there exists an element $g \in \mathrm{GL}_V(S')$ satisfying $g.(F_\bullet^{\mathrm{std}}|_{S'}, w.F_\bullet^{\mathrm{std}}|_{S'}) = (\mathcal{F}_\bullet, \mathcal{F}'_\bullet)$. Now it is easily seen that the global sections $v_i = g(e_i) \in \Gamma(S', f^* \tilde{V})$ satisfy the desired conditions. Conversely, any such basis yields an element in $\mathrm{GL}_V(S')$. The same ideas lead to the second statement. Note that here we only need to lift the sections e_i to sections in $\phi^* \tilde{V}$, which is possible once S' is affine. $\square$

We now specialize to the case where $k = \overline{\mathbb{F}}_q$ is an algebraic closure of the finite field with q elements, and $V = V_0 \otimes_{\mathbb{F}_q} k$ for some $\mathbb{F}_q$-vector space V_0. This equips V with a $\mathrm{Gal}(k/\mathbb{F}_q)$-action, and in particular the Frobenius automorphism of k (given on k by $x \mapsto x^q$) yields a k-semilinear automorphism $\mathrm{Frob}\colon V \to V$. As this automorphism sends subspaces to subspaces, we obtain automorphisms

$$\mathrm{Frob}\colon X \to X \quad \text{and} \quad \mathrm{Frob}\colon Y \to Y.$$

Note that X and Y are defined over $\mathbb{F}_q$, and these automorphisms are the same as the respective (relative) Frobenii of X and Y over k. We write γ_{Frob} for the corresponding graph morphisms $X \to X \times_{\mathrm{Spec}(k)} X$ and $Y \to Y \times_{\mathrm{Spec}(k)} Y$.

For $w \in \Sigma_n$ and $t \in \mathbb{T}_X$, we define the spaces

$$X_w := O_w \times_{X \times_{\mathrm{Spec}(k)} X, \gamma_{\mathrm{Frob}}} X \quad \text{and} \quad Y_{w,t} := \tilde{O}_{w,t} \times_{Y \times_{\mathrm{Spec}(k)} Y, \gamma_{\mathrm{Frob}}} Y. \tag{6.2}$$

As γ_{Frob} admits a section, X_w is naturally a subscheme of X, parameterizing those families of flags that are in relative position w to their Frobenius twist. Similarly, $Y_{w,t}$ is naturally a subscheme of Y, parameterizing the families of marked flags in relative position (w, t) with their Frobenius twist. We have natural maps $Y_{w,t} \to X_w$. As a pair of marked flags $((\mathcal{F}_\bullet, e_\bullet), (cF'_\bullet, e'_\bullet))$ over S lies in the same GL_V orbit as $((\mathcal{F}_\bullet, t_\bullet e_\bullet), (\mathcal{F}'_\bullet, t'_\bullet e'_\bullet))$ for $t_\bullet, t'_\bullet \in \mathbb{G}^n_{m,X}(S)$ if and only if $t_\bullet = t'_\bullet$, we find that $Y_{w,t}$ is a Zariski torsor over X_w for an affine group scheme $\mathbb{T}^{\mathrm{Frob}}_w \times_k X_w$. Here, $\mathbb{T}_w = \mathrm{Res}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(\mathbb{G}_m) \times_{\mathbb{F}_q} k$ is the Weil restriction of the multiplicative group from $\mathbb{F}_{q^n}$ to $\mathbb{F}_q$. Hence, the S-valued points of $\mathbb{T}^{\mathrm{Frob}}_w$ are given by

$$\mathbb{T}^{\mathrm{Frob}}_w(S) = \{(t_1, \ldots, t_n) \in \mathbb{G}^n_m(S) \mid t^q_i = t_{w(i)}\}.$$

Furthermore, an element $g \in \mathrm{GL}_V(S)$ stabilizes $Y_{w,t}(S) \subset Y(S)$ if $g \in \mathrm{GL}^{\mathrm{Frob}}_V(S)$, so we obtain a $\mathrm{GL}^{\mathrm{Frob}}_V$ action on $Y_{w,t}$ and X_w. The morphism $Y_{w,t} \to X_w$ is equivariant for the $\mathrm{GL}^{\mathrm{Frob}}_V$-action.

One can show that the scheme X_w is smooth (of pure dimension $l(w)$, as the intersection in (6.2) is transverse; cf. [DL76]), $Y_{w,t}$ is smooth over k and affine over X_w. Thus, we have constructed the spaces in the commutative diagram

$$
\begin{array}{ccc}
Y_{w,t} & \longhookrightarrow & Y \cong \mathrm{GL}_V/U \\
{\scriptstyle \mathbb{T}^{\mathrm{Frob}}_w\text{-torsor}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathbb{T}_X\text{-torsor}} \\
X_w & \longhookrightarrow & X \cong \mathrm{GL}_V/B.
\end{array}
\tag{6.3}
$$

The interesting space is $Y_{w,t}$. One readily checks that it comes with commuting actions of $\mathrm{GL}^{\mathrm{Frob}}_V(k) = \mathrm{GL}_V(\mathbb{F}_q)$ and $\mathbb{T}^{\mathrm{Frob}}_w(k) = \mathbb{F}^\times_{q^n}$.

6.2 An Explicit Example of a Deligne–Lusztig Variety

We keep the notation from the previous subsection. That is, $k = \overline{\mathbb{F}}_q$, $V = V_0 \otimes_{\mathbb{F}_q} k$, X is the flag variety of V, and Y is the variety of marked flags. In this subsection, we fix $w = (1\ 2\ \ldots\ n) \in \Sigma_n$ and $t = (1, \ldots, 1) \in \mathbb{G}^n_{X,m}$, and give explicit descriptions of the resulting varieties appearing in the square (6.3). We write $\mathrm{DL}_V = Y_{w,t}$ in this situation.

First, note that $\mathbb{T}^{\mathrm{Frob}}_w(S) = \mathbb{G}_m(S)^{\mathrm{Frob}^n}$, implying that DL_V admits commuting actions by $\mathrm{GL}(V_0)$ and $\mathbb{T}^{\mathrm{Frob}}_w(k) = \mathbb{F}^\times_{q^n}$.

Lemma 6.2.1. *A pair of flags $(\mathcal{F}_\bullet, \mathcal{F}'_\bullet) \in (X \times X)(S)$ is in relative position $(1\ 2\ \ldots\ n)$ if and only if for all $i = 1, \ldots, n-1$ the condition $F_i + F'_i = F_{i+1}$ is satisfied. Here, the sum denotes the fppf-sheafification of the corresponding presheaf.*

Proof. The criterion may be checked fppf-locally. Here it follows quickly from Lemma 6.1.8 and the Bruhat decomposition, Proposition 6.1.7. □

For a linear form $\mu \in V^\vee$, we write $D^+(\mu)$ for the affine open subscheme of $\mathbb{P}(V)$ parameterizing lines in V that do not lie in the hyperplane defined by the equation $\mu(v) = 0$.

Proposition 6.2.2. *If $w = (1\ 2\ \ldots\ n)$, the morphism of functors defined on k-schemes S by*

$$\Phi(S)\colon X_w(S) \to \mathbb{P}(V)(S) \quad F_\bullet \mapsto F_1$$

yields an isomorphism

$$X_w \cong \bigcap_{\mu \in V_0^\vee} D^+(\mu) \subset \mathbb{P}(V).$$

That is, X_w parametrizes lines in V that do not lie inside any $\mathbb{F}_q$-rational hyperplane. In particular X_w is equal to a finite intersection of affine subschemes, hence an affine scheme itself.

Proof. We first show that the image of any family of flags $F_\bullet \in X_w(S)$ is inside $\cap_\mu D^+(\mu)(S)$. As the latter is an open subscheme, it suffices to show that any $s \in |S|$ maps into $\cap_\mu D^+(\mu)$, which is the case if any only if $F_1(s) = F_{1,s} \otimes_{\mathcal{O}_{S,s}} \kappa(s)$ does not lie inside any $\mathbb{F}_q$-rational hyperplane in $\kappa(s) \otimes_k V$. By Lemma 6.2.1, we find

$$F_1(s) \oplus \mathrm{Frob}(F_1(s)) \oplus \cdots \oplus \mathrm{Frob}^{n-1}(F_1(s)) = V \otimes_k \kappa(s),$$

so $F_1(s)$ cannot lie inside any non-trivial Frobenius-stable linear subspace of $V \otimes_k \kappa(s)$. The claim follows.

To see bijectivity of Φ, note that the inverse is, if well-defined, given by the morphism of functors Ψ given on components by

$$\Psi(S)\colon \bigcap_{\mu \in V_0^\vee} D^+(\mu)(S) \to X_w(S), \quad L \mapsto \left(L \oplus \mathrm{Frob}(L) \oplus \cdots \oplus \mathrm{Frob}^{i-1}L \right)_{i=1,\ldots,n}.$$

To see that this is indeed well defined, choose a basis $(e_1, \ldots, e_n)$ of V_0 and take any section $\mathcal{L} \in \cap_{\mu \in V_0^\vee} D^+(\mu)(S)$, interpreted as a locally direct summand of $f^*\tilde{V}$. It suffices to work fppf-locally on S, and we may choose a Zariski cover $S' = \mathrm{Spec}(R') \to S$ trivializing $\mathcal{L}$. Write L for the corresponding free rank 1 direct summand of R'^n. Then $L = \langle v \rangle$ for some $v = (v_1, \ldots, v_n) \in R'^n$. Now L constitutes a flag if and only if $(v, \mathrm{Frob}(v), \ldots, \mathrm{Frob}^{n-1}(v))$ is a basis for R'^n if and only if $\det(\mathrm{Frob}^{j-1}(v_i))_{i,j} \in R^\times$. The last condition is satisfied. Indeed, if not, we may choose a maximal ideal $\mathfrak{m} \in \mathrm{Spec}(R)$ containing $\det(\mathrm{Frob}^{j-1}(v_i))_{i,j}$. Let $\bar{v}$ denote the residue of v in $(R'/\mathfrak{m})^n$. Now, the subspace

$$\langle \bar{v}, \mathrm{Frob}(\bar{v}), \ldots, \mathrm{Frob}^{n-1}(\bar{v}) \rangle \subset (R'/\mathfrak{m})^n$$

is non-trivial and Frobenius-stable, and in particular contained in some $\mathbb{F}_q$-rational hyperplane. This contradicts $\mathcal{L} \in \cap_{\mu \in V_0^\vee} D^+(\mu)(S)$. $\qquad\square$

Proposition 6.2.3. *The map of functors $\mathrm{DL}_V \to \mathrm{Spec}(\mathrm{Sym}(V^\vee))$ given by $(F_\bullet, e_\bullet) \mapsto e_1$ yields an isomorphism of DL_V and the subfunctor of $\mathrm{Spec}(\mathrm{Sym}(V^\vee))$ given on affine schemes by*

$$\mathrm{Spec}(R) \mapsto \left\{ v \in R \otimes_k V \,\middle|\, \begin{array}{c} (v, \mathrm{Frob}\,v, \ldots, \mathrm{Frob}^{n-1}v) \text{ is a basis and} \\ v \wedge \cdots \wedge \mathrm{Frob}^{n-1}v = (-1)^{n-1}\mathrm{Frob}(v \wedge \cdots \wedge \mathrm{Frob}^{n-1}v) \end{array} \right\}.$$

This functor is readily seen to representable by an affine k-scheme, upon choosing a basis of

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$V_0 \cong \mathbb{F}_q^n$ *and abbreviating* $\boldsymbol{x} = (x_1, \ldots, x_n)$, *it is given by*

$$\mathrm{DL}_n := \mathrm{Spec}\left(\frac{k[x_1, \ldots, x_n]}{(\det D(\boldsymbol{x})^{q-1} - (-1)^{n-1})}\right), \quad \text{where} \quad D(\boldsymbol{x}) = \begin{pmatrix} x_1 & \cdots & x_1^{q^{n-1}} \\ \vdots & \ddots & \vdots \\ x_n & \cdots & x_n^{q^{n-1}} \end{pmatrix}.$$

Proof. The subfunctor defined above is a sheaf for the fppf-topology, so it suffices to prove the claim fppf-locally. Let $S \to \mathrm{Spec}(k)$ be an arbitrary scheme. By Lemma 6.1.8, there is an affine fppf-cover $\mathrm{Spec}(R') = S' \to S$ and a basis $(v_1, \ldots, v_n)$ of $R' \otimes_k V$ such that

$$\begin{aligned} v_i &\equiv e_i \mod F_{i-1} && \text{and} && v_{i+1} \equiv \mathrm{Frob}(e_i) \mod \mathrm{Frob}(F_{i-1}) \quad \text{for } 1 \le i \le n-1, \\ v_n &\equiv e_n \mod F_{n-1} && \text{and} && v_1 \equiv \mathrm{Frob}(e_n) \mod \mathrm{Frob}(F_{n-1}). \end{aligned} \tag{6.4}$$

From here we quickly find

$$\mathrm{Frob}(v_1 \wedge v_2 \wedge \cdots \wedge v_n) = \mathrm{Frob}\, v_1 \wedge \cdots \wedge \mathrm{Frob}\, v_n = v_2 \wedge v_3 \wedge \cdots \wedge v_n \wedge v_1.$$

The equivalences in (6.4) also imply that for integers $2 \le m \le n$, we have $\mathrm{Frob}^{m-1} v_1 \equiv v_m$ mod $\mathrm{Frob}(F_{m-2})$. Also, we find $v_1 \equiv \mathrm{Frob}^n v_1 \mod \mathrm{Frob}(F_{n-1})$. Altogether, writing $v = v_1 = e_1$, this yields

$$\mathrm{Frob}(v \wedge \mathrm{Frob}(v) \wedge \cdots \wedge \mathrm{Frob}^{n-1} v) = (-1)^{n-1}(v \wedge \cdots \wedge \mathrm{Frob}^{n-1} v). \tag{6.5}$$

This shows that the map given in the statement of the proposition is well-defined, and it is clearly injective. Conversely, given any $v \in R' \otimes_k V$ such that $(v, \mathrm{Frob}\, v, \ldots, \mathrm{Frob}^{n-1} v)$ is a basis and v satisfies the equation (6.5), Gaussian elimination shows that the corresponding marked flag is in relative position $(w, 1)$ to its Frobenius-twist.

If we are given a basis of V_0, we may write $v = (x_1, \ldots, x_n)$ and identify $v \wedge \mathrm{Frob}\, v \wedge \ldots \mathrm{Frob}^{n-1} v$ with $\det\left[(x_i^{q^{j-1}})_{1 \le i, j \le n}\right]$. Thus, v gives a marked flag in DL_n if and only if

$$\det((x_i^{q^{j-1}})_{i,j})^{q-1} = (-1)^{n-1}.$$

This gives the representability statement of DL_n. $\qquad\square$

Note that DL_n has $q-1$ disjoint irreducible components, parameterized by the set of solutions $b \in k$ to the equation $z^{q-1} = (-1)^{n-1}$. For any such $b \in k$ we write

$$Y_b := \mathrm{Spec}\left(\frac{k[x_1, \ldots, x_n]}{(\det D(\underline{x}) - b)}\right), \tag{6.6}$$

and obtain

$$\mathrm{DL}_n = \coprod_{b^{q-1} = (-1)^{n-1}} Y_b.$$

Let us turn our attention to the group action of $\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times$ on DL_n. We defined it above as a left action, but we will need a right action. We obtain such a right action by precomposing

the left action with the anti-automorphism

$$\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^{\times} \to \mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^{\times}, \quad (g, d) \mapsto (g^t, d^{-1}),$$

where g^t denotes the transpose. The resulting action of $\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^{\times}$ on DL_n has the following explicit description.

Lemma 6.2.4. *Let $g \in \mathrm{GL}_n(\mathbb{F}_q)$ be an element with matrix entries $(a_{ij})_{1 \leq i,j \leq n}$. Then g acts on the global sections of DL_n from the left via*

$$x_i \mapsto \sum_{j=1}^{n} a_{ji} x_j \quad for \quad i = 1, \ldots, n.$$

Similarly, an element $d \in \mathbb{F}_{q^n}^{\times}$ acts via

$$x_i \mapsto d^{-1} x_i \quad for \quad i = 1, \ldots, n.$$

Through the q-th power Frobenius automorphism on $\overline{\mathbb{F}}_q$, we may also define an action of $\mathbb{Z}$ on DL_n, sending 1 to the automorphism given by $x_i \mapsto x_i$, $a \mapsto a^{-q}$. Note that this action is defined over $\mathbb{F}_q$, not over $\overline{\mathbb{F}}_q$. By construction, the action of $\mathbb{Z}$ commutes with the action of $\mathrm{GL}_n(\mathbb{F}_q)$, but to make it commute with the action of $\mathbb{F}_{q^n}^{\times}$, we have to restrict it to $n\mathbb{Z} \subset \mathbb{Z}$. Note that $g \in \mathrm{GL}_n(\mathbb{F}_q)$ induces a morphism of schemes $Y_b \to Y_{b \det(g)}$. Similarly, $\zeta \in \mathbb{F}_{q^{\times}}$ restricts to $Y_b \to Y_{b \mathrm{N}(\zeta)^{-1}}$, where $\mathrm{N} = \mathrm{N}_{\mathbb{F}_{q^n}/\mathbb{F}_q}$ denotes the norm map of the extension $\mathbb{F}_{q^n}/\mathbb{F}_q$. The action of $1 \in \mathbb{Z}$ sends Y_b to $Y_{(-1)^{n-1}b}$, therefore, the action of $n\mathbb{Z}$ stabilizes each component Y_b. In particular, the subgroup

$$(\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^{\times})^1 \times n\mathbb{Z} := \{(g, d, n) \in \mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^{\times} \times n\mathbb{Z} \mid \det(g) \mathrm{N}(d)^{-1} = 1\}$$

of $\mathrm{GL}_n \times \mathbb{F}_{q^n}^{\times} \times \mathbb{Z}$ stabilizes Y_b for every choice of b.

We write H_{DL} for the $\overline{\mathbb{Q}}_\ell$-vector space

$$H_{\mathrm{DL}} = H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell).$$

By the above, this is a representation of $\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^{\times} \times n\mathbb{Z}$. The following subsection is concerned with the study of this representation.

6.3   The Deligne–Lusztig Correspondence

In this subsection, we are concerned with the étale cohomology of the variety DL_n introduced in the previous subsection. As before, k denotes an algebraic closure of $\mathbb{F}_q$.

We give a short review of [DL76]. For a connected, reductive algebraic group G defined over $\mathbb{F}_q$ and a maximal Frobenius-stable torus $T \subset G$ contained in a Borel subgroup $B \subset G$, Deligne and Lusztig construct varieties with right G^{Frob}-actions $X_{T \subset B}$ and $\tilde{X}_{T \subset B}$, constituting a G^{Frob}-equivariant Galois covering

$$\tilde{X}_{T \subset B} \to X_{T \subset B}$$

with Galois group T^{Frob}. See (6.8) for explicit descriptions of these spaces. The space $\tilde{X}_{T \subset B}$ comes with commuting actions of G^{Frob} and T^{Frob}, and for characters θ of $T^{\mathrm{Frob}}(k)$, the main concern of [DL76] is the study of the resulting virtual representations

$$R_T^\theta = \sum_i (-1)^i H_c^i(\tilde{X}_{T \subset B}, \overline{\mathbb{Q}}_\ell)_\theta \in \mathcal{R}(T^{\mathrm{Frob}}(k) \times G^{\mathrm{Frob}}(k)). \tag{6.7}$$

Here, the subscript θ denotes the direct summand of $H_c^i(\tilde{X}_{T \subset B}, \overline{\mathbb{Q}}_\ell)$ where $T^{\mathrm{Frob}}(k)$ acts by θ. These representations depend only on the choice of the torus, not on the choice of the Borel sugroup ([DL76, Corollary 4.3]). To formulate the results of the theory precisely, we need the notion of regular position.

Definition 6.3.1 (Regular Position). Let T be a Frobenius-stable maximal torus of G and let θ be a character of $T^{\mathrm{Frob}}(k)$. We say that θ is in general position if it is not fixed by any non-trivial element of $\mathrm{W}_G(T)^{\mathrm{Frob}}$. Here, $\mathrm{W}_G(T)$ denotes the Weyl group $(\mathrm{N}_G(T)/T)$, which acts on T by conjugation.

Recall that for a finite group H, the Grothendieck group $\mathcal{R}(H)$ of finite-dimensional H-representations over $\overline{\mathbb{Q}}_\ell$ comes with a natural inner product. Indeed, any representation takes values in the maximal cyclotomic subfield $\cup_r \mathbb{Q}(\zeta_r) \subset \overline{\mathbb{Q}}_\ell$, which has the unique "complex conjugation" automorphism given by $\zeta_r \mapsto \zeta_r^{-1}$ for $r \in \mathbb{N}$. The inner product is now defined on finite-dimensional representations $\rho, \rho' \in (H\text{-Rep})$ as

$$\langle \rho, \rho' \rangle = \frac{1}{\#H} \sum_{h \in H} \mathrm{Tr}(\rho(h)) \overline{\mathrm{Tr}(\rho'(h))} \in \overline{\mathbb{Q}}_\ell.$$

This definition linearly extends to $\mathcal{R}(H)$, and the irreducible finite-dimensional representations of H give an orthogonal basis for $\mathcal{R}(H) \otimes \mathbb{Q}$. If ρ is an irreducible representation of H and R is an element in $\mathcal{R}(H) \otimes \mathbb{Q}$, we say that ρ occurs in R if $\langle \rho, R \rangle \neq 0$.

The following results of Deligne–Lusztig theory are important for our purposes.

Theorem 6.3.2 (Some Results of Deligne-Lusztig Theory).

1. [DL76, Corollary 1.22] *Let $Z \subset G$ denote the center of G. Then Z^{Frob} acts on $H_c^{n-1}(\tilde{X}_{T \subset B}, \overline{\mathbb{Q}}_\ell)_\theta$ through $\theta|_{Z^{\mathrm{Frob}}}$.*

2. [DL76, Corollary 7.3] *If (T, θ) is in general position, one of $\pm R_T^\theta$ is an irreducible representation.*

3. [DL76, Corollary 8.3] *If furthermore T is not contained in any proper Frobenius-stable parabolic subgroup, one of $\pm R_T^\theta$ is a cuspidal representation of $G^{\mathrm{Frob}}(k)$.*

4. [DL76, Corollary 9.9] *If furthermore $\tilde{X}_{T \subset G}$ (or equivalently, $X_{T \subset G}$) is affine, and we denote by $w \in \mathrm{W}_G(T)$ the relative position of B and $\mathrm{Frob}(B)$ (given by the Bruhat decomposition for G), we have*

$$H_c^i(\tilde{X}_{T \subset G}, \overline{\mathbb{Q}}_\ell)_\theta = 0 \quad \text{if} \quad i \neq l(w).$$

Here, $l(w)$ denotes the Coxeter-length of w.

5. [DL76, Theorem 9.8] *For a character θ in general position, the natural map*

$$H^i_c(Y_{T \subset B}, \overline{\mathbb{Q}}_\ell)_\theta \to H^i(Y_{T \subset B}, \overline{\mathbb{Q}}_\ell)_\theta$$

is an isomorphism.

Next, we explain how these results apply to the compactly supported étale cohomology of DL_n. We set $G = \mathrm{GL}_n$, and denote by B^{std} the standard Borel subgroup of upper triangular matrices, by T^{std} the standard torus of diagonal matrices and by U^{std} the standard unipotent subgroup of upper triangular matrices with diagonal entries equal to 1. We have $T^{\mathrm{std}} U^{\mathrm{std}} = B^{\mathrm{std}}$.

Choose a flag $F_\bullet \in X_w(k)$, where w is, as usual, $(1\ 2\ \ldots\ n) \in \Sigma_n \cong \mathrm{W}_G(T^{\mathrm{std}})$. Let $B \subset \mathrm{GL}_n$ denote the isotropy subgroup of $F_\bullet$, and let $T \subset B$ be a Frobenius-stable maximal torus in B. Write U for the unipotent radical of B. We have the explicit descriptions (cf. [DL76, Definition 1.17])

$$X_{T \subset B} = \{g \in \mathrm{GL}_n \mid g^{-1}\mathrm{Frob}(g) \in \mathrm{Frob}(U)\}/(T^{\mathrm{Frob}}(U \cap \mathrm{Frob}(U)))$$
$$\text{and} \tag{6.8}$$
$$\tilde{X}_{T \subset B} = \{g \in \mathrm{GL}_n \mid g^{-1}\mathrm{Frob}(g) \in \mathrm{Frob}(U)\}(U \cap \mathrm{Frob}(U)).$$

Given any marked flag $(\mathcal{F}'_\bullet, e'_\bullet) \in X(S)$, we write $g(\mathcal{F}'_\bullet, e'_\bullet) \in \mathrm{GL}_n /U^{\mathrm{std}}(S)$ for the corresponding section under the isomorphism of Lemma 6.1.6 (i.e., $g.(F^{\mathrm{std}}_\bullet, e^{\mathrm{std}}_\bullet) = (\mathcal{F}'_\bullet, e'_\bullet)$). Also, by [Mil17, Proposition 17.13], we may choose $h \in \mathrm{GL}_n(k)$ such that $h(T^{\mathrm{std}}, B^{\mathrm{std}})h^{-1} = (T, B)$. Furthermore, for any k-scheme S, we may identify $\mathbb{T}^{\mathrm{Frob}}_w$ with the subgroup

$$\{t \in T^{\mathrm{std}}(S) \mid \mathrm{ad}\, w^{-1}t = \mathrm{Frob}(t)\} \subset T^{\mathrm{std}}(S).$$

Using these identifications, [DL76, Proposition 1.19] implies that the map

$$\mathrm{DL}_n \to \tilde{X}_{T \subset B}, \quad (F'_\bullet, e'_\bullet) \mapsto g(F'_\bullet, e'_\bullet)h^{-1}$$

gives an isomorphism of $\mathrm{GL}_n^{\mathrm{Frob}}$-equivariant torsors

$$
\begin{array}{ccc}
\mathrm{DL}_n & \overset{\sim}{\longrightarrow} & \tilde{X}_{T \subset B} \\
{\scriptstyle \mathbb{T}^{\mathrm{Frob}}_w\text{-torsor}}\big\downarrow & & \big\downarrow{\scriptstyle T^{\mathrm{Frob}}\text{-torsor}} \\
X_w & \overset{\sim}{\longrightarrow} & X_{T \subset B}
\end{array}
$$

Given any representation θ of $\mathbb{T}^{\mathrm{Frob}}_w(k)$, we write

$$R_\theta = \sum_i (-1)^i H^i_c(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta.$$

By definition, we have $R_{\theta \circ \mathrm{ad}\, h} \cong R^\theta_{T \subset B}$, where h is chosen as above. If θ is a characterr of $\mathbb{T}^{\mathrm{Frob}}_w(k) \cong \mathbb{F}^\times_{q^n}$, the pair $(T, \theta \circ \mathrm{ad}\, h)$ is in regular position if and only if θ is regular, in the following sense.

Definition 6.3.3 (Regular Character on $\mathbb{F}_{q^n}^\times$). We say that a character $\theta\colon \mathbb{F}_{q^n}^\times \to \mathbb{C}^\times$ is regular if it does not factor through the norm morphism

$$N_{\mathbb{F}_{q^n}/\mathbb{F}_{q^m}}\colon \mathbb{F}_{q^n}^\times \to \mathbb{F}_{q^m}^\times$$

for any $m < n$.

Recall the definition of H_{DL} and the left action of $\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times \times n\mathbb{Z}$ defined in the previous subsection. The statements of Theorem 6.3.2 imply the following.

Theorem 6.3.4 (Deligne–Lusztig Correspondence). *Let θ be a regular character of $\mathbb{F}_{q^n}^\times$, and let $H_{\mathrm{DL},\theta} = H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta$ denote the direct summand of H_{DL} on which $\mathbb{F}_{q^n}^\times$ acts through θ.*

1. We have

$$H_{\mathrm{DL},\theta} = R_\theta \boxtimes \theta$$

as representations of $\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times$. The representation R_θ is irreducible and cuspidal, and its central character is given by $\theta|_{\mathbb{F}_q^\times}$ under the identification $\mathbb{F}_q^\times \cong Z_{\mathrm{GL}_n}$.

2. The natural map

$$H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta \to H^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta$$

is an isomorphism.

This finishes the discussion about the $\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times$-action on H_{DL}.

It remains to make the action of $n\mathbb{Z}$ explicit. The Frobenius automorphism on $\overline{\mathbb{F}}_q$ yields an action of $n\mathbb{Z}$ on H_{DL}. This action admits the following description.

Proposition 6.3.5. *Let θ be a regular character. The group $n\mathbb{Z}$ acts on $H_{\mathrm{DL},\theta}$ through the character $\gamma\colon n\mathbb{Z} \to \mathbb{Q}^\times$, given by*

$$\gamma(nm) = (-1)^{(n-1)m} q^{m\frac{n(n-1)}{2}}.$$

Proof. Note that, as the absolute Frobenius morphism $\mathrm{DL}_n \to \mathrm{DL}_n$ induces the identity on étale cohomology (cf. [Stacks, Tag 03SN]), the claim is equivalent to showing that pullback along the relative Frobenius

$$\mathrm{Frob}_{q^n}\colon \mathrm{DL}_n \to \mathrm{DL}_n^{(q^n)} = \mathrm{DL}_n, \quad x_i \mapsto x_i^q, \quad a \mapsto a \quad (a \in \overline{\mathbb{F}}_q)$$

induces multiplication of $(-1)^{n-1} q^{\frac{n(n-1)}{2}}$ on $H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta$. This result is essentially due to Digne–Michel, cf. [DM85, Remarque 3.14], but their proof appears to contain a sign error. This mistake was identified and corrected by Wang in [Wan14, Théorème 3.1.12]. $\square$

In conclusion, we obtain the following structural result about the $\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times \times n\mathbb{Z}$-representation H_{DL}.

Theorem 6.3.6 (Structure of H_{DL}). *Let $\theta\colon \mathbb{F}_{q^n}^\times \to \overline{\mathbb{Q}}_\ell^\times$ be a regular character. The $\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times \times n\mathbb{Z}$-representation $H_{\mathrm{DL},\theta}$ is given by*

$$H_{\mathrm{DL},\theta} = R_\theta \boxtimes \theta \boxtimes \gamma.$$

Here, R_θ is the irreducible cuspidal representation from Deligne–Lusztig theory (cf. Theorem 6.3.4), and γ is the character defined in the previous proposition.

Recall from Section 6.2 the definition of the component $Y_b \subset \mathrm{DL}_n$. As above, we write

$$(\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times)^1 \times n\mathbb{Z} := \{(g, d, n) \in \mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times \times n\mathbb{Z} \mid \det(g)\,\mathrm{N}(d) = 1\}.$$

This group stabilizes Y_b for every choice of b. We also write $(\mathbb{F}_{q^n}^\times)^1 = \{\zeta \in \overline{\mathbb{F}}_q \mid \mathrm{N}(\zeta) = 1\}$. We end this section with a remark on its étale cohomology of Y_b.

Lemma 6.3.7.

1. *There is a natural isomorphism of $\mathbb{F}_{q^n}^\times$-representations*

$$H^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell) \cong \mathrm{Ind}_{(\mathbb{F}_{q^n}^\times)^1}^{\mathbb{F}_{q^n}^\times} H^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell) = \bigoplus_\zeta H^{n-1}(Y_{b\,\mathrm{N}(\zeta)^{-1}}, \overline{\mathbb{Q}}_\ell),$$

   where ζ runs over representatives of the quotient $\mathbb{F}_{q^n}^\times/(\mathbb{F}_{q^n}^\times)^1$. The same statement holds with étale cohomology replaced by compactly supported étale cohomology.

2. *Let θ be a character of $\mathbb{F}_{q^n}^\times$. Then there is a natural $(\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times)^1 \times n\mathbb{Z}$-equivariant isomorphism*

$$H^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta \cong H^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)_\theta,$$

   where $H^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)_\theta$ denotes the $\mathrm{Res}_{(\mathbb{F}_{q^n}^\times)^1}^{\mathbb{F}_{q^n}^\times}(\theta)$-isotypic component of $H^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)$. The same statement holds with étale replaced by compactly supported étale cohomology.

3. *Let θ be a regular character of $\mathbb{F}_{q^n}^\times$. Then the natural morphism $H_c^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)_\theta \to H^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)_\theta$ is an isomorphism.*

Proof. The first statement is a direct consequence of the fact that any $\zeta \in \mathbb{F}_{q^n}^\times$ yields an isomorphism $Y_b \to Y_{b\,\mathrm{N}(\zeta)^{-1}}$ through the action of $\mathbb{F}_{q^n}^\times$ on DL_n. By Frobenius reciprocity, this implies

$$H^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta \cong \mathrm{Hom}_{(\mathbb{F}_{q^n}^\times\text{-Rep})}\left(\theta, \mathrm{Ind}_{(\mathbb{F}_{q^n}^\times)^1}^{\mathbb{F}_{q^n}^\times} H^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)\right)$$

$$= \mathrm{Hom}_{((\mathbb{F}_{q^n}^\times)^1\text{-Rep})}\left(\mathrm{Res}_{(\mathbb{F}_{q^n}^\times)^1}^{\mathbb{F}_{q^n}^\times} \theta, H^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)\right) = H^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)_\theta,$$

and the same statement for compactly supported étale cohomology. Explicitly, the above isomorphism is given by the restriction of the natural projection (given by the pull-back map)

$$H^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell) \to H^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)$$

to $H^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta$, and thereby clearly $(\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times)^1 \times n\mathbb{Z}$-equivariant. This shows the second statement. The final statement is now a direct consequence of part two in Theorem 6.3.4. Alternatively, a proof can be obtained from part 5 of Theorem 6.3.2 and the fact that Y_b is isomorphic to a Deligne–Lusztig variety for $\mathrm{SL}_n(\mathbb{F}_q)$. $\square$

7 Explicit Non-Abelian Lubin–Tate Theory for Depth Zero Supercuspidal Representations

In this final section, we explicitely calculate the representations arising in Theorem 3.2.2 for depth zero supercuspidal representations π of $\mathrm{GL}_n(E)$, which essentially are representations of $\mathrm{GL}_n(E)$ obtained by compact induction and character twist from Deligne–Lusztig theory. In particular, we give explicit descriptions of $\mathrm{rec}_E(\pi)$ and $\mathrm{JL}(\pi)$. The main input is the construction of an affinoid in the Lubin–Tate perfectoid space whose special fiber is isomorphic to the perfection of the Deligne–Lusztig variety constructed in Section 6.2. By the results of Section 5, we can relate the ℓ-adic cohomology of DL_n and the representation H_{LT}, which allows us to make some parts of H_{LT} explicit.

7.1 The Special Affinoid and its Formal Model

We define the subspace $U \subset M'_{\infty, \mathbb{C}_\varpi}$ as the rational subset cut out by the inequalities $|X_i(x)| \leq |\varpi(x)|^{1/q^n-1}$ for $i = 1, \dots, n$. We construct a formal model $\mathcal{X} \in (\mathrm{FSch}/\mathcal{O}_{\mathbb{C}_\varpi})$ of U whose special fiber $\mathcal{X}_s = \mathcal{X} \times_{\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_\varpi})} \mathrm{Spec}(\overline{\mathbb{F}}_q)$ is isomorphic to the perfection of the Deligne–Lusztig variety DL_n constructed in Section 6.2. Furthermore, the formal model admits group actions $\mathcal{X}$ that are compatible with the group actions defined on DL_n and $M'_{\infty, \mathcal{O}_{\mathbb{C}_\varpi}}$.

Let H be the standard formal $\mathcal{O}_E$-module over $\mathcal{O}_{\breve{E}}$ and write E_n for the degree n unramified extension of E. The construction of the formal model $\mathcal{X}$ requires some choices. First, we fix a choice of basis $(\alpha_1, \dots, \alpha_n)$ of E_n over E, which in particular allows us to view E_n as a subfield of $\mathrm{Mat}_{n \times n}(E)$. By Lemma 1.5.5 and Proposition 1.3.9, E_n is also naturally a subfield of D.

Furthermore, we'll have to make the choice of a primitive element

$$y = (y_1, y_2, \dots) \in T_\varpi H(\mathcal{O}_{\mathbb{C}_\varpi}),$$

that is, an element with $y_1 \neq 0$ and $[\varpi]_H(y_1) = 0$. Write $\xi = \lambda(y)$ for the corresponding element of $\mathrm{Nilp}^\flat(\mathcal{O}_{\mathbb{C}_\varpi})$.

Lemma 7.1.1. *We have*

$$\log_H(y) = \sum_{i=-\infty}^{\infty} \frac{\xi^{q^{in}}}{\varpi} = 0 \quad and \quad |\xi|^{q^n-1} = |\varpi|.$$

Proof. The first identity follows from Lemma 1.11.11 and the fact that $\log_H(y_1)$ vanishes, cf. Lemma 1.4.4. For the second one, note that $[\varpi]_H(y_1) = 0$ and $[\varpi]_H(y_{k+1}) = y_k$ implies that $|y_k| = |\varpi|^{\frac{1}{q^{nk}(q^n-1)}}$. Hence, the claim follows by the equality $\xi = \lim_{k \to \infty} y_k^{q^{nk}}$; cf. Lemma 4.4.2. $\square$

Let $t := \delta(\alpha_1 y, \dots, \alpha_n y) \in (T_\varpi \wedge H)(\mathcal{O}_{\mathbb{C}_\varpi})$ and write $\tau \in \mathrm{Nilp}^\flat(\mathcal{O}_{\mathbb{C}_\varpi})$ for the corresponding system of q-th power roots.

Lemma 7.1.2. *We have*

$$\log_{\wedge H}(t) = \sum_{i=-\infty}^{\infty} (-1)^{i(n-1)} \frac{\tau^{q^i}}{\varpi^i} = 0 \quad \text{and} \quad |\tau|^{q-1} = |\varpi|.$$

Furthermore, we have the congruence

$$\tau \equiv \det(\alpha_i^{q^{j-1}})_{1 \le i,j \le n} \xi^{1+q+\cdots+q^{n-1}}$$

modulo the ideal generated by elements $z \in \mathcal{O}_{\mathbb{C}_\varpi}$ with $|z| < |\xi|^{1+q+\cdots+q^{n-1}}$.

Proof. We have $t_1 \ne 0$ and $[\varpi]_{\wedge H}(t) = 0$, so the first two assertions follow just as above in the proof of Lemma 7.1.1, appropiately adjusted. The congruence is a corollary of the approximation of Δ in Lemma 1.14.3. $\square$

We now define the formal model $\mathcal{X}$. We abbreviate with $(x_i^{q^{-m}})_{m \in \mathbb{N}_0}$ the system of q-th power roots $(X_i^{q^{-m}}/\xi^{q^{-m}})_{m \in \mathbb{N}_0}$ of elements in $\mathbb{C}_\varpi[\![X_1^{q^{-\infty}}, \ldots, X_n^{q^{-\infty}}]\!]$, and define systems of q-th power roots Δ' and τ' as

$$\Delta'(x_1, \ldots, x_n)^{q^{-m}} := (\xi^{q^{-m}})^{-(1+q+\cdots+q^{n-1})} \Delta^{q^{-m}}(\xi x_1, \ldots, \xi x_n) \in \mathbb{C}_\varpi[\![x_1^{q^{-\infty}}, \ldots, x_n^{q^{-\infty}}]\!]$$

$$\text{and}$$

$$\tau'^{q^{-m}} := (\xi^{q^{-m}})^{-(1+q+\cdots+q^{n-1})} \tau^{q^{-m}} \in \mathbb{C}_\varpi^\times.$$

Lemma 7.1.3. *The coefficients of Δ' are integral and we have*

$$\Delta'(x_1, \ldots, x_n)^{q^{-m}} \equiv (\det(x_i^{q^j})_{1 \le i,j \le n})^{q^{-m}} \mod \mathfrak{m}_{\mathbb{C}_\varpi}.$$

In particular, we have $\Delta'(x_1, \ldots, x_n)^{q^{-m}} \in \mathcal{O}_{\mathbb{C}_\varpi}\langle x_1^{q^{-\infty}}, \ldots, x_n^{q^{-\infty}}\rangle$. Also, $\tau'^{q^{-m}} \in \mathcal{O}_{\mathbb{C}_\varpi}^\times$.

Proof. The statement about Δ' follows directly from the approximation in Lemma 1.14.3. One easily checks $|\tau'| = 1$, implying the second statement. $\square$

We define

$$Q := \frac{\mathcal{O}_{\mathbb{C}_\varpi}\langle x_1^{q^{-\infty}}, \ldots, x_n^{q^{-\infty}}\rangle}{(\Delta'(x_1, \ldots, x_n)^{q^{-m}} - \tau'^{q^{-m}} \mid m \in \mathbb{N}_0)^-} \quad \text{and} \quad \mathcal{X} := \operatorname{Spf}(Q). \tag{7.1}$$

Proposition 7.1.4. *The formal scheme $\mathcal{X}$ is a formal model for U.*

Proof. By definition, U is isomorphic to the affinoid adic space $\operatorname{Spa}(B[\frac{1}{\varpi}], \tilde{B})$, where

$$B := \frac{\mathcal{O}_{\mathbb{C}_\varpi}\langle x_1^{q^{-\infty}}, \ldots, x_n^{q^{-\infty}}\rangle}{(\Delta^{q^{-m}}(\xi x_1, \ldots, \xi x_n) - \tau^{q^{-m}} \mid m \in \mathbb{N})^-}$$

and $\tilde{B} \subset B[\frac{1}{\varpi}]$ denotes the normalization of B inside $B[\frac{1}{\varpi}]$. As ξ is invertible in $B[\frac{1}{\varpi}]$, we have $Q[\frac{1}{\varpi}] = B[\frac{1}{\varpi}]$. Furthermore, the images of B and Q inside $B[\frac{1}{\varpi}]$ agree. The claim follows. $\square$

Proposition 7.1.5. *Let b^{1/q^m} be the residue class of τ'^{1/q^m} in $\mathcal{O}_{\mathbb{C}_\varpi}/\mathfrak{m}_{\mathbb{C}_\varpi}$. Then $b^{q-1} = (-1)^{n-1}$, and the special fiber of $\mathcal{X}_s := \mathcal{X} \times_{\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_\varpi})} \mathrm{Spec}(\overline{\mathbb{F}}_q)$ is isomorphic to Y_b^{perf}, the perfection of the component $Y_b \subset \mathrm{DL}_n$ defined in (6.6).*

Proof. We first show that $b^{q^{-m}(q-1)} = (-1)^{n-1}$ for $m \in \mathbb{Z}$. This follows from Lemma 7.1.2. Write $\overline{\alpha}_i \in \mathbb{F}_{q^n}$ for the residue classes of the elements α_i for $i = 1, \dots, n$. We have $\overline{\alpha}_i^{q^n} = \overline{\alpha}_i$, implying for $m \in \mathbb{Z}$ the equalities

$$b^{q^{-(m-1)}} = \det(\overline{\alpha}_i^{q^j})_{1 \le i,j \le n}^{q^{-m}} = (-1)^{n-1} \det(\overline{\alpha}_i^{q^{j-1}})_{1 \le i,j \le n}^{q^{-m}} = (-1)^{n-1} b^{q^{-m}}.$$

This gives $b^{q^{-m}(q-1)} = (-1)^{n-1}$, as desired. By Lemma 7.1.3, we find that $\mathcal{X}_s$ is equal to

$$\mathrm{Spec}\left(\frac{\overline{\mathbb{F}}_q[x_1^{q^{-\infty}}, \dots, x_n^{q^{-\infty}}]}{(\det(x_i^{q^{j-1}})^{q^{-m}} - b^{q^{-m}})} \right),$$

which is precisely the perfection of Y_b. $\qquad\square$

Proposition 7.1.6. *The formal model $\mathcal{X} = \mathrm{Spf}(Q)$ is flat over $\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_\varpi})$.*

Proof. We show that Q, as defined in (7.1), is a flat $\mathcal{O}_{\mathbb{C}_\varpi}$-algebra. For the moment, let us abbreviate $A = \mathcal{O}_{\mathbb{C}_\varpi}\langle x_1^{q^{-\infty}}, \dots, x_n^{q^{-\infty}}\rangle$, and let $I_m \subset A$ denote the ideal generated by the element $(\Delta'^{q^{-m}} - \tau'^{q^{-m}}) \in A$. Furthermore, we write $I = I_1 + I_2 + \cdots \subset A$ and $\overline{I}$ for the closure of I in A, so that $Q = A/\overline{I}$. We first show that the (non-complete) quotient A/I is $\mathcal{O}_{\mathbb{C}_\varpi}$-torsion free, which is equivalent to flatness as $\mathcal{O}_{\mathbb{C}_\varpi}$ is a valuation ring, cf. [Stacks, Tag 0539]. From the factorization

$$\Delta'^{q^{-m}} - \tau'^{q^{-m}} = (\Delta'^{q^{-(m+1)}} - \tau'^{q^{-(m+1)}})(\Delta'^{(q-1)q^{-(m+1)}} + \cdots + \tau'^{(q-1)q^{-(m+1)}}), \qquad (7.2)$$

we deduce that $I_m \subset I_{m+1}$. Now $A/I = \mathrm{colim}_{m \in \mathbb{N}} A/I_m$, and it suffices to check that A/I_m is flat for all $m \in \mathbb{N}$, which is equivalent to $\Delta'^{q^{-m}} - \tau'^{q^{-m}} \notin \mathfrak{m}_{\mathbb{C}_\varpi} A$ for all $m \in \mathbb{N}$. As the reduction

$$(A/I) \otimes_{\mathcal{O}_{\mathbb{C}_\varpi}} \overline{\mathbb{F}}_q = (A/\overline{I}) \otimes_{\mathcal{O}_{\mathbb{C}_\varpi}} \overline{\mathbb{F}}_q \cong \frac{\overline{\mathbb{F}}_q[x_1^{q^{-\infty}}, \dots, x_n^{q^{-\infty}}]}{(\det(x_i^{q^{j-1}})^{q^{-m}} - b^{q^{-m}} | m \in \mathbb{N})}$$

is not isomorphic to $A \otimes_{\mathcal{O}_{\mathbb{C}_\varpi}} \overline{\mathbb{F}}_q$, the reduction of I modulo $\mathfrak{m}_{\mathbb{C}_\varpi}$ cannot be trivial and there must be some $m \in \mathbb{N}$ such that $(\Delta'^{q^{-m}} - \tau'^{q^{-m}}) \notin \mathfrak{m}_{\mathbb{C}_\varpi} A$. By taking q^{m-1}-th powers, this readily implies $(\Delta' - \tau') \notin \mathfrak{m}_{\mathbb{C}_\varpi} A$, which, by the factorization in (7.2), implies that $(\Delta'^{q^{-m}} - \tau'^{q^{-m}}) \notin \mathfrak{m}_{\mathbb{C}_\varpi} A$ for every m. Hence, A/I is flat over $\mathcal{O}_{\mathbb{C}_\varpi}$.

The ultrametric inequality for the ϖ-adic metric on A implies that

$$I \cap \mathfrak{m}_{\mathbb{C}_\varpi} A = (0) \iff \overline{I} \cap \mathfrak{m}_{\mathbb{C}_\varpi} A = (0),$$

and it follows that $Q = A/\overline{I}$ is flat over $\mathcal{O}_{\mathbb{C}_\varpi}$ as well. This is what we wanted to show. $\quad\square$

7.2 Comparison of the Group Actions

In Section 3, we saw that the Lubin–Tate perfectoid space

$$M'_{\infty, \mathbb{C}_\varpi} = \mathcal{M}_\infty \times_{\mathrm{Spa}(\widehat{E}^{\mathrm{ab}}, \mathcal{O}_{\widehat{E}^{\mathrm{ab}}})} \mathrm{Spa}(\mathbb{C}_\varpi, \mathcal{O}_{\mathbb{C}_\varpi})$$

admits a right action by the group

$$G^1 \subset G = \mathrm{GL}_n(E) \times D^\times \times W_E,$$

given by those elements (g, d, σ) satisfying $\det(g) \operatorname{Nrd}(d)^{-1} \operatorname{Art}_E^{-1}(\sigma|_{\breve{E}}) = 1$. In this section, we construct a subgroup $J^1 \subset G^1$ that stabilizes the special affinoid U constructed above, and we further show that the action of J^1 on U extends to an action of J^1 on the formal model $\mathcal{X}$. This induces a right action on the special fiber $\mathcal{X}_s$ of $\mathcal{X}$, and as $\mathcal{X}_s \cong Y_b^{perf} \subset \mathrm{DL}_n^{perf}$, this yields an action of J^1 on the perfection of a part of the Deligne–Lusztig variety constructed in Section 6.2. Recall that the group

$$\overline{J}^1 = (\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times)^1 \times n\mathbb{Z}$$

acts on Y_b, in particular it also acts on Y_b^{perf}. We will show that the action of J^1 on Y_b^{perf} can be described in terms of the latter action, through a homomorphism $\Theta \colon J^1 \to \overline{J}^1$. These results lay the representation-theoretic ground for the comparison between the representations H_{LT} and H_{DL}.

We set

$$J := E^\times \mathrm{GL}_n(\mathcal{O}_E) \times \mathcal{O}_D^\times \times W_{E_n} \quad \text{and} \quad J^1 = J \cap G^1. \tag{7.3}$$

In addition, we define the morphism Θ as follows. For $\sigma \in W_{E_n}$ with $\sigma|_{\breve{E}} = \Phi^{n_\sigma}$ and $u_\sigma := \varpi^{-n_\sigma} \operatorname{Art}_{E_n}^{-1}(\sigma|_{\widehat{E}_n^{\mathrm{ab}}}) \in \mathcal{O}_{E_n}^\times$, we set

$$\Theta \colon J \to \mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times \times n\mathbb{Z}, \quad (\varpi^m g, d, \sigma) \mapsto (\overline{g}, \overline{d^{-1} u_\sigma^{-1}}, -n_\sigma). \tag{7.4}$$

It will be convenient to have a list of generators for the group J^1.

Lemma 7.2.1. *The group J^1 is generated by elements of the form*

- $(g, 1, 1)$ *for $g \in \mathrm{GL}_n(\mathcal{O}_E)$ with $\det g = 1$; for those elements we have $\Theta(g, 1, 1) = (\overline{g}, 1, 0)$*

- $(1, d, 1)$ *for $d \in \mathcal{O}_D^\times$ with $\operatorname{Nrd} d = 1$; for those elements we have $\Theta(1, d, 1) = (1, \overline{d}, 0)$*

- $(a, a, 1)$ *for $a \in E_n^\times$; for those elements we have $\Theta(a, a, 1) = (\overline{a}, \overline{a}, 0)$*

- $(1, \alpha^{-1}, \sigma)$ *for $\sigma \in I_{E_n}$ and $\alpha \in \mathcal{O}_{E_n}$ with $\operatorname{Art}_{E_n}^{-1}(\sigma|_{E_n^{\mathrm{ab}}}) = \alpha$; for those elements we have $\Theta(1, \alpha, \sigma) = (1, 1, 0)$.*

- $(1, \varpi^{-1}, \sigma)$ *for $\sigma \in W_{E_n}$ with $\operatorname{Art}_{E_n}^{-1}(\sigma|_{E_n^{\mathrm{ab}}}) = \varpi$; for those elements, we have $\Theta(1, \varpi^{-1}, \sigma) = (1, 1, -n)$.*

In particular, the image of J^1 under the homomorphism Θ is $\overline{J}^1$, so the induced action of J^1 on DL_n stabilizes Y_b.

For $(g, d, \sigma) \in G^1$, recall the definition of $\gamma_{g,i} \in \mathcal{O}_{\mathbb{C}_\varpi}[\![X_1^{q^{-\infty}}, \ldots, X_n^{q^{-\infty}}]\!]$ and $\delta_{d^{-1}} \in \mathcal{O}_{\mathbb{C}_\varpi}[\![X^{q^{-\infty}}]\!]$ from Lemma 4.4.5. We wish to show the following proposition.

Proposition 7.2.2. *The action of J^1 on $M'_{\infty, \mathbb{C}_\varpi}$ stabilizes U and extends to $\mathcal{X}$. The induced action on the special fiber $\mathcal{X}_s$ is compatible with the action of J^1 on Y_b.*

Proof. By the description of the group action of G^1 on $A_{\infty, \mathcal{O}_{\mathbb{C}_\varpi}}$ in Proposition 4.4.6, the induced actions on Q and Y_b^{perf} must take the form described in Figure 1.

Element	Action on $A_{\infty, \mathcal{O}_{\mathbb{C}_\varpi}}$	Action on Q	Action on Y_b^{perf}
$(g, 1, 1)$	$X_i \mapsto$ $\gamma_{g,i}(X_1, \ldots, X_n)$	$x_i \mapsto$ $\xi^{-1}\gamma_{g,i}(\xi x_1, \ldots, \xi x_n)$	$x_i \mapsto$ $(x_1, \ldots, x_n) \cdot \bar{g}_i$
$(1, d, 1)$	$X_i \mapsto \delta_{d^{-1}}(X_i)$	$x_i \mapsto \xi^{-1}\delta_{d^{-1}}(\xi x_i)$	$x_i \mapsto \bar{d}^{-1} x_i$
$(a, a, 1)$	trivial	trivial	trivial
$(1, \alpha^{-1}, \sigma)$	$a \mapsto \sigma(a)$	$a \mapsto \sigma(a)$, $x_i \mapsto \frac{\xi}{\sigma(\xi)} x_i$	trivial
$(1, \varpi, \sigma)$	$a \mapsto \sigma(a)$	$a \mapsto \sigma(a)$, $x_i \mapsto \frac{\xi}{\sigma(\xi)} x_i$	$a \mapsto \mathrm{Frob}_q^n(a)$

Figure 1: Description of the group actions.

To show that the actions described above fulfill the desired properties, it suffices to verify the following claims.

1. For $i = 1, \ldots, n$, the power series

$$\xi^{-1}\gamma_{g,i}(\xi x_1, \ldots, \xi x_n) \in \mathbb{C}_\varpi[\![x_1, \ldots x_n]\!]$$

   lies inside $\mathcal{O}_{\mathbb{C}_\varpi}\langle x_1, \ldots, x_n \rangle$ and reduces to $(x_1, \ldots, x_n) \cdot \bar{g}_i$ modulo $\mathfrak{m}_{\mathbb{C}_\varpi}$.

2. The power series
$$\xi^{-1}\delta_{d^{-1}}(\xi x_i) \in \mathbb{C}_\varpi[\![x_i]\!]$$
   lies inside $\mathcal{O}_{\mathbb{C}_\varpi}\langle x_i \rangle$ and reduces to $\bar{d}^{-1} x_i$ modulo $\mathfrak{m}_{\mathbb{C}_\varpi}$.

3. If $\sigma \in W_{E_n}$ lies inside the inertia subgroup I_{E_n}, the element $\xi/\sigma(\xi)$ reduces to 1 modulo $\mathfrak{m}_{\mathbb{C}_\varpi}$.

4. If $\sigma \in W_{E_n}$ satisfies $\mathrm{Art}_{E_n}^{-1}(\sigma|_{E_n^{\mathrm{ab}}}) = \varpi$, the element $\xi/\sigma(\xi)$ reduces to 1 modulo $\mathfrak{m}_{\mathbb{C}_\varpi}$.

The first claim follows directly from the description of g^* in Lemma 4.4.5. Indeed, one quickly checks that modulo $\mathfrak{m}_{\mathbb{C}_\varpi}$, we have

$$\xi^{-1}\gamma_{g,i}^*(\xi x_1, \ldots, \xi x_n) \equiv \sum_{H_{j=1}}^n a_{ji}^{(0)} x_j = (x_1, \ldots, x_n).\bar{g}_i \quad \mathrm{mod} \ \mathfrak{m}_{\mathbb{C}_\varpi}.$$

Similarly one obtains the second claim.

The third claim follows as σ lies in the inertia subgroup, hence $\xi \equiv \sigma(\xi) \mod \mathfrak{m}_{\mathbb{C}_\varpi}$.

The fourth statement follows from classical Lubin–Tate theory and the construction of the Artin map in Section 2. Recall that $\xi = \lim_{k \to \infty} y_k^{q^{nk}}$, where $y = (y_1, y_2, \dots) \in T_\varpi H(\mathcal{O}_{\mathbb{C}_\varpi})$ is a primitive element. Thereby, each component of y lies inside the (totally ramified) Lubin–Tate extension $E_{n,\varpi}$ (this is an infinite extension, not to be confused with the finite subextensions $E_{n,\varpi,m}$). By construction of the Artin map, we have

$$\sigma|_{E_{n,\varpi}} = \mathrm{Art}_{E_n}(\varpi)|_{E_{n,\varpi}} = \mathrm{id}_{E_{n,\varpi}},$$

hence σ acts trivially on the system $(y_k)_{k \in \mathbb{N}}$. As σ is continuous, this implies $\sigma(\xi) = \xi$, concluding the proof. $\qquad\square$

7.3   The Explicit Correspondence

In this chapter we apply Mieda's Theorem 5.2.10 to the rational subspace U and its formal model $\mathcal{X}$. We show how this allows us to explicitly calculate certain parts of the local Langlands correspondence for GL_n over E. Fix, for the remainder of the section, an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$.

As usual, we let E_n be the degree n unramified extension of E. We need the following definitions.

Definition 7.3.1 (Regular Tame Character). We say that a character $\theta\colon E_n^\times \to \mathbb{C}^\times$ is *tame* if it is trivial on the subgroup $1 + \varpi\mathcal{O}_{E_n} \subset E_n^\times$. If θ is tame, then its restriction to $\mathcal{O}_{E_n}^\times$ factors as

$$\mathcal{O}_{E_n}^\times \to \mathbb{F}_{q^n}^\times \xrightarrow{\overline{\theta}} \mathbb{C}^\times.$$

In this case, we say that θ is *regular* if $\overline{\theta}$ is regular in the sense of Definition 6.3.3.

Fix a regular tame character $\theta\colon E_n^\times \to \mathbb{C}^\times$. The datum of θ can be used to construct representations of W_E and $D^\times$ and, using Deligne–Lusztig theory, a representation of $\mathrm{GL}_n(E)$. The construction is as follows.

- Let $\overline{\tau}_\theta$ be the character of W_{E_n} given by the composition

$$\mathrm{W}_{E_n} \twoheadrightarrow \mathrm{W}_{E_n}^{\mathrm{ab}} \xrightarrow{\mathrm{Art}_{E_n}^{-1}} E_n^\times \xrightarrow{\theta\delta^{n-1}} \mathbb{C}^\times$$

  where $\delta\colon \mathrm{W}_{E_n} \to \{\pm 1\}$ the character corresponding to $a \mapsto (-1)^{\mathrm{val}_{E_n}(a)}$ under the Artin map $\mathrm{Art}_{E_n}\colon E_n^\times \to \mathrm{W}_{E_n}^{\mathrm{ab}}$, and define $\tau_\theta := \text{c-Ind}_{\mathrm{W}_{E_n}}^{\mathrm{W}_E}(\overline{\tau}_\theta)$.

- Let $\overline{\rho}_\theta$ be the character on $E^\times\mathcal{O}_D^\times$ given by the composition

$$E^\times\mathcal{O}_D^\times \cong \varpi^{\mathbb{Z}} \times \mathcal{O}_D^\times \twoheadrightarrow \mathcal{O}_D^\times \times \mathbb{F}_{q^n} \xrightarrow{\theta|_{E^\times} \boxtimes \overline{\theta}} \mathbb{C}^\times$$

  and define $\rho_\theta := \text{c-Ind}_{E^\times\mathcal{O}_D^\times}^{D^\times}(\overline{\rho}_\theta)$.

- Let $\bar{\pi}_\theta$ be the representation of $E^\times \mathrm{GL}_n(\mathcal{O}_E)$ given by the composition

$$E^\times \mathrm{GL}_n(\mathcal{O}_E) \cong \varpi^\mathbb{Z} \times \mathrm{GL}_n(\mathcal{O}_E) \twoheadrightarrow \varpi^\mathbb{Z} \times \mathrm{GL}_n(\mathbb{F}_q) \xrightarrow{\theta|_{E^\times} \boxtimes R_{\bar{\theta}}} \mathrm{GL}(V_{\bar{\theta}}),$$

where $(R_{\bar{\theta}}, V_{\bar{\theta}})$ is the cuspidal representation corresponding to $\bar{\theta}$ under the Deligne–Lusztig correspondence laid out in Theorem 6.3.4. Define $\pi_\theta \coloneqq \mathrm{c\text{-}Ind}_{E^\times \mathrm{GL}_n(\mathcal{O}_E)}^{\mathrm{GL}_n(E)}(\bar{\pi}_\theta)$.

Lemma 7.3.2. *The representations π_θ, ρ_θ, and τ_θ are smooth. Furthermore, π_θ and ρ_θ are irreducible and π_θ is supercuspidal.*

Proof. By design, $\bar{\pi}_\theta$ is trivial on the compact open subgroup

$$1 + \varpi \, \mathrm{Mat}_{n \times n}(\mathcal{O}_E) \subset E^\times \mathrm{GL}_n(\mathcal{O}_E).$$

Similar statements hold for $\bar{\rho}_\theta$ and $\bar{\tau}_\theta$. Hence, π_θ, ρ_θ and τ_θ are also smooth. In the case that $n = 2$, the fact that π_θ is supercuspidal and irreducible is laid out in detail in [BH06, Section 11]. The general case should add no complication. Similarly, the irreducibility of ρ_θ is shown, again in the case $n = 2$, in [BH06, Section 54]. $\qquad\square$

We call a representation π *depth zero supercuspidal* if it is equal to π_θ for some regular tame character $\theta \colon E_n \to \mathbb{C}^\times$. The aim of this section is to prove the following statement.

Theorem 7.3.3 (Explicit Jacquet–Langlands and Local Langlands Correspondence for Depth Zero Supercuspidal Representations). *Let $\theta \colon E_n \to \mathbb{C}^\times$ be a regular tame character. The representation $\mathrm{JL}(\pi_\theta)$ of $D^\times$ and the representation $\mathrm{rec}_E(\pi_\theta)$ of W_E corresponding to π_θ under the Jacquet–Langlands and local Langlands correspondence take the form*

$$\mathrm{JL}(\pi_\theta) = \rho_\theta \quad \text{and} \quad \mathrm{rec}_E(\pi_\theta) = \tau_\theta.$$

The rest of this section will be devoted to the proof of this theorem. Recall that H_{DL} denotes the middle-degree ℓ-adic cohomology of DL_n, cf. Section 6.3. Fix a regular tame character

$$\theta \colon E_n \to \mathbb{C}^\times.$$

For now, we assume that θ is trivial on $\varpi^\mathbb{Z}$.

Lemma 7.3.4. *The morphism Θ from (7.4) makes J act on $H_{\mathrm{DL},\theta}$. If θ is trivial on $\varpi^\mathbb{Z}$, this representation is of the form*

$$(g, d, \sigma) \mapsto \pi_\theta(g) \otimes \bar{\rho}_{\theta^{-1}}(d) \otimes \tau_\theta^{-1}(\tfrac{1-n}{2})(\sigma).$$

This representation is smooth.

Proof. This follows directly from Theorem 6.3.6, and the fact that if θ is trivial on $\varpi^\mathbb{Z}$, the central characters of ρ_θ and ϖ_θ are also trivial on $\varpi^\mathbb{Z}$. $\qquad\square$

The input we get from Mieda's theory (Section 5) is the following.

Proposition 7.3.5. *If θ is trival on $\varpi^{\mathbb{Z}}$, then there is an injective morphism of J^1-represen tations*

$$\mathrm{Res}^J_{J^1}(H_{\mathrm{DL},\theta}) \hookrightarrow \mathrm{Res}^{G^1}_{J^1}(H'_{\mathrm{LT}}).$$

Proof. By the results of the previous two sections, $\mathcal{X} = \mathrm{Spf}(Q)$ is a formal model of the rational subset $U \subset M'_{\infty,\mathbb{C}_{\varpi}}$ (Proposition 7.1.4), Q is flat over $\mathcal{O}_{\mathbb{C}_{\varpi}}$ (Proposition 7.1.6), and there is a group action of J^1 on Y_b, which is compatible with the group action on $\mathcal{X}_s$ obtained from the group action of J^1 on $\mathcal{M}'_{\infty,\mathcal{O}_{\mathbb{C}_{\varpi}}}$ (Section 7.2). We are therefore in the situation of Mieda's Theorem 5.2.10, which we want to apply with $V = H_c^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)_\theta$. By part 3 of Lemma 6.3.7, the natural map $H_c^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)_\theta \to H^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)_\theta$ is an isomorphism. Hence the composition

$$H_c^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)_\theta \hookrightarrow H_c^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell) \to H^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)$$

is injective. Theorem 5.2.10 now yields a J^1-equivariant injective morphism

$$H_c^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)_\theta \hookrightarrow H'_{\mathrm{LT}}.$$

By part 2 of Lemma 6.3.7, the left-hand side J^1-equivariantly isomorphic to $H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta$. The claim follows. $\qquad\square$

Recall that

$$\overline{G} = \mathrm{GL}_n(E) \times (D^\times/\varpi^{\mathbb{Z}}) \times \mathrm{W}_E,$$

and that G^1 is naturally a co-compact and closed normal subgroup of $\overline{G}$ (Lemma 3.2.3).

Lemma 7.3.6. *Assume that θ is trivial on $\varpi^{\mathbb{Z}}$. Then, the injection from Proposition 7.3.5 naturally gives rise to an injective J-equivariant morphism*

$$H_{\mathrm{DL},\theta} \hookrightarrow \mathrm{Res}^{\overline{G}}_J H_{\mathrm{LT}}.$$

Proof. We construct a sequence of J-equivariant injections

$$H_{\mathrm{DL},\theta} \hookrightarrow \mathrm{Ind}^J_{J^1}(\mathrm{Res}^{G^1}_{J^1} H'_{\mathrm{LT}}) \xrightarrow{\sim} \mathrm{Res}^{G^1 J}_J(\mathrm{Ind}^{G^1 J}_{G^1} H'_{\mathrm{LT}}) \hookrightarrow \mathrm{Res}^{\overline{G}}_J H_{\mathrm{LT}}.$$

The first morphism. This morphism arises by adjunction from the morphism in Proposition 7.3.5, which is injective by Proposition B.10 and Lemma B.13.

The second morphism. The morphism is given by the inverse of the base change morphism constructed in Lemma B.14, which is applied with $H = G^1$, $N = J$. Note that G^1 is normal in $\overline{G}$, so the assumptions of the lemma are satisfied. As J is open in $\overline{G}$, the map is an isomorphism.

The third morphism. Since $G^1 J$ is open in $\overline{G}$, the unit of the adjunction $\text{c-Ind}^{\overline{G}}_{G^1 J} \dashv \mathrm{Res}^{\overline{G}}_{G^1 J}$ yields a monomorphism of $G^1 J$-representations

$$\mathrm{Ind}^{G^1 J}_{G^1} H'_{\mathrm{LT}} \to \mathrm{Res}^{\overline{G}}_{G^1 J}(\text{c-Ind}^{\overline{G}}_{G^1 J}(\mathrm{Ind}^{G^1 J}_{G^1} H'_{\mathrm{LT}})). \qquad (7.5)$$

As $G^1 J$ cocompact in $\overline{G}$, we have $\text{c-Ind}^{\overline{G}}_{G^1 J} = \mathrm{Ind}^{\overline{G}}_{G^1 J}$, so the right-hand side is isomorphic to $\mathrm{Res}^{\overline{G}}_{G^1 J}(\mathrm{Ind}^{\overline{G}}_{G^1} H'_{\mathrm{LT}}) \cong \mathrm{Res}^{\overline{G}}_{G^1 J}(H_{\mathrm{LT}})$ by Proposition B.12 and Proposition 3.2.4. Hence, applying $\mathrm{Res}^{G^1 J}_J$ to the morphism in (7.5) yields the desired map. $\qquad\square$

Lemma 7.3.7. *If θ is trivial on $\varpi^{\mathbb{Z}}$, the statements of Theorem 7.3.3 hold.*

Proof. We show that $H_{\mathrm{LT},\pi_\theta} = \pi_\theta \boxtimes \rho_\theta^\vee \boxtimes \tau_\theta^\vee(\frac{1-n}{2})$. The morphism constructed in Lemma 7.3.6 yields, by Frobenius reciprocity, a non-zero map of $\overline{G}$-representations

$$\mathrm{Ind}_J^{\overline{G}}(H_{\mathrm{DL},\theta}) \cong \pi_\theta \boxtimes \rho_{\theta^{-1}} \boxtimes \tau_\theta^\vee(\tfrac{1-n}{2}) \to H_{\mathrm{LT}}, \tag{7.6}$$

where we use that $\tau_{\theta^{-1}} = \tau_\theta^\vee$, which follows from Frobenius reciprocity. The image of this morphism lies inside $H_{\mathrm{LT},\pi_\theta}$, the π_θ-isotypic component of H_{LT}. As π_θ is supercuspidal and its central character is trivial on $\varpi^{\mathbb{Z}}$, non-Abelian Lubin–Tate theory (Theorem 3.2.2) gives a map

$$\pi_\theta \boxtimes \rho_{\theta^{-1}} \boxtimes \tau_\theta^\vee \to \pi_\theta \boxtimes \mathrm{JL}(\pi_\theta)^\vee \boxtimes \mathrm{rec}_E(\pi_\theta)^\vee.$$

Applying $\mathrm{Hom}(\pi_\theta, -)$, we obtain a nonzero map $\rho_{\theta^{-1}} \boxtimes \tau_\theta^\vee \to \mathrm{JL}(\pi_\theta)^\vee \boxtimes \mathrm{rec}_E(\pi_\theta)^\vee$. As $\rho_{\theta^{-1}}$ and $\mathrm{JL}(\pi_\theta)^\vee$ are irreducible, this implies $\rho_\theta \cong \mathrm{JL}(\pi_\theta)$. As $\mathrm{rec}_E(\pi_\theta)$ is irreducible and $\dim(\tau_\theta) = n = \dim(\mathrm{rec}_E(\pi_\theta))$, this also implies $\tau_\theta = \mathrm{rec}_E(\pi_\theta)$, concluding the proof. $\square$

Proof of Theorem 7.3.3. By the above lemma, all that is left to do is to check that the assumption on θ to be trivial on $\varpi^{\mathbb{Z}}$ can be removed. Let $\theta \colon E_n^\times \to \mathbb{C}^\times$ be an arbitrary regular tame character. Let $\xi \colon E^\times \to \mathbb{C}^\times$ be an unramified character (i.e., a character trivial on $\mathcal{O}_E^\times$) with $\xi(\varpi)^n = \theta(\varpi)$, and put $\theta_0 := \theta \cdot (\xi \circ N_{E_n/E})^{-1}$. Now θ_0 is tame and we have $\overline{\theta}_0 = \overline{\theta}$, so θ_0 is also regular. By construction, we have $\theta_0(\varpi) = 1$, so θ_0 is trivial on $\varpi^{\mathbb{Z}}$ and Lemma 7.3.7 implies that

$$\mathrm{JL}(\pi_{\theta_0}) = \rho_{\theta_0} \quad \mathrm{rec}_E(\pi_{\theta_0}) = \tau_{\theta_0}.$$

On the other hand, we have

$$\rho_\theta = \rho_{\theta_0} \otimes (\xi \circ \mathrm{Nrd}_{D/E}), \quad \pi_\theta = \pi_{\theta_0} \otimes (\xi \circ \det) \quad \text{and} \quad \tau_\theta = \tau_{\theta_0} \otimes (\xi \circ \mathrm{Art}_E^{-1});$$

the first two statements follow quickly from definition and the third one is a consequence of the functoriality of the Artin map, cf. Theorem 2.1.1.

As the Jacquet-Langlands and local Langlands correspondences are compatible with character twist, this gives

$$\mathrm{JL}(\pi_\theta) = \rho_{\theta_0} \otimes (\xi \circ \mathrm{Nrd}_{D/E}) \quad \text{and} \quad \mathrm{rec}_E(\pi_\theta) = \tau_{\theta_0} \otimes (\xi \circ \mathrm{Art}_E^{-1}).$$

The desired statement follows from the identities laid out above. $\square$

Appendix

A Topological Rings

To deal with the topological rings showing up, the notion of admissible rings will be convenient (taken from [Stacks, Tag 07E8]).

Definition A.1. Let A be a topological ring. We say that A is admissible if

- the element $0 \in A$ has a fundamental system of neighbourhoods consisting of ideals.

- there exists an ideal of definition, that is, an open ideal $I \subset A$ such that every open neighbourhood of 0 contains I^n for some n.

- it is complete, that is, the natural map

$$A \to \lim_{\substack{J \subset A \text{ open ideal}}} A/J$$

is an isomorphism.

We say that A is adic if it admits an ideal of definition I such that I^n is open for all n. Given a topological ring A, we denote the categories of admissible and adic A-algebras (algebras S with continuous morphism $A \to S$) by (A-Adm) and (A-Adic), respectively.

Lemma A.2. *Let $\phi \colon R \to S$ be a morphism of admissible rings, and let $I \subset R$ be a finitely generated ideal of definition for R. Then, there exists an ideal of definition J' of S such that $\phi(I) \subseteq J'$.*

Proof. Let J be an ideal of definition for S. Then the open ideal $J' = \phi(I) \cdot S + J$ works. Indeed, as I is finitely generated and J is open, there is some positive integer m with $\phi(I)^m \subset J$. If now U is any open neighborhood of 0 so that $J^r \subseteq U$ for some $r \in \mathbb{N}$, then $J'^{rm} \subseteq U$. $\square$

Lemma A.3. *Let S be an admissible ring, and let $(s_1, s_2, \dots)$ be a sequence with elements in S. Then $\sum_{i=1}^{\infty} s_i$ converges if and only if $\lim_{i \to \infty} s_i = 0$. In this case, the product $\prod_{i=1}^{\infty}(1 + s_i)$ exists in S.*

Proof. If the sum converges, $(s_i)_{i \in \mathbb{N}}$ has to be a null-sequence. The reverse implication and the convergence of the product follows after writing $S \cong \lim_J S/J$ for a system of open ideals $J \subset S$. $\square$

The topology on an admissible ring R with ideal of definition I is coarser than the I-adic topology on R.

Lemma A.4. *Let R be an admissible ring with ideal of definition I. Let R' be the same ring, but equipped with the I-adic topology. Then the identity map $R' \to R$ is continuous. In particular, if a sequence converges with respect to the I-adic topology, it also converges in R'.*

Proof. It suffices to check that open ideals of R are open in R'. Let $J \subset R$ an open ideal. By assumption, there is some n with $I^n \subset J$. But now, for any $x \in J$, we have $x + I^n \subset J$. Hence, J is open in R'. $\square$

B   Smooth Representations of Locally Profinite Groups

We review some aspects of the representation theory (over complex vector spaces) of locally profinite groups. If G is an arbitrary group, we denote the category of complex representations (that is, morphisms $G \to \mathrm{GL}(V)$, where V is a $\mathbb{C}$-vector space) as (G-Rep). At the slight cost of precision, we also allow ourselves to refer to an element of $\pi \colon G \to \mathrm{GL}(V) \in$ (G-Rep) by the underlying vector space V, or the pair (π, V).

Definition B.1 (Locally Profinite Group). A *locally profinite group* is a topological group G such that every open neighborhood of the identity in G contains a compact open subgroup of G.

For example, any discrete group is locally profinite and $\mathrm{GL}_n(E)$ is locally profinite for any non-Archimedian local field E. Quotients and closed subgroups of locally profinite groups are again locally profinite.

Throughout this section, if not stated otherwise, G is a locally profinite group and $H \subset G$ is a closed subgroup of G.

Definition B.2 (Smooth and Admissible Representations). A representation $\pi \colon G \to \mathrm{GL}(V) \in$ (G-Rep) is called *smooth* if for any $v \in V$, the stabilizer G_v of v is an open subgroup of G. We define (G-Rep$^{\mathrm{sm}}$), the category of smooth G-representations, as the full subcategory of (G-Rep) with objects given by smooth G-representations.

A smooth representation (π, V) is *admissible* if it is smooth and for every open compact subgroup $H \subseteq G$, the space V^H of H-invariants is finite-dimensional.

The forgetful functor (G-Rep$^{\mathrm{sm}}$) $\to$ (G-Rep) has a left adjoint, given by taking smooth parts.

Definition B.3 (Smooth Part of a Representation). Let $(\pi, V) \in$ (G-Rep). We write

$$V^{\mathrm{sm}} = \bigcup_{K \subseteq G} V^K,$$

where K runs over the compact open subgroups of G and $V^K \subseteq V$ denotes the subspace of elements fixed by K. Now V^{sm} is a G-stable subspace of V, and we write $(\pi^{\mathrm{sm}}, V^{\mathrm{sm}})$ for the induced representation $G \to \mathrm{GL}(V^{\mathrm{sm}})$ of π. We call π^{sm} the smooth part of π.

Definition B.4 (Algebraic Induction). Let G be any group and let H be a subgroup of G. We define the *algebraic induction functor* $\mathrm{algInd}_H^G \colon$ (H-Rep) $\to$ (G-Rep) as follows. Given an H-representation $(\pi, V) \in$ (H-Rep), consider the vector space

$$\mathrm{algInd}_H^G(V) = \{\phi \colon G \to V \mid \phi(hg) = \pi(h)g\}.$$

Now G acts naturally on $\mathrm{algInd}_H^G(V)$ by right-translation (that is, $g.\phi(x) = \phi(xg)$), and we write $(\mathrm{algInd}_H^G(\pi), \mathrm{algInd}_H^G(V))$ for the corresponding representation of G.

Remark. We have $\mathrm{algInd}_H^G(V) = \mathrm{Hom}_{(\mathbb{C}[H]\text{-Mod})}(\mathbb{C}[G], V)$. As $\mathbb{C}[G]$ has a natural $(\mathbb{C}[H], \mathbb{C}[G])$-bimodule structure, we obtain a natural left-G-action on $\mathrm{algInd}_H^G(V)$. This action is precisely the one described above.

Definition B.5 (Restriction Functor). Let G be any group and let H be a subgroup of G. If $\pi\colon G \to \mathrm{GL}(V)$ is a representation of G, we define the restriction of π from G to H as

$$\mathrm{Res}_H^G(\pi)\colon H \hookrightarrow G \xrightarrow{\pi} \mathrm{GL}(V)$$

and call $\mathrm{Res}_H^G\colon (G\text{-Rep}) \to (H\text{-Rep})$ the restriction functor.

Lemma B.6. *Let G be any group and let H be any subgroup of G. Then Res_H^G is left-adjoint to algInd_H^G.*

Proof. By the Remark above, this statement readily reduces to the Tensor-Hom-Adjunction. $\square$

Lemma B.7. *If G is locally profinite and H is a closed subgroup, for any $(\pi, V) \in (G\text{-Rep})$ we have an H-equivariant split inclusion*

$$\mathrm{Res}_H^G(V^{\mathrm{sm}}) \subseteq \left(\mathrm{Res}_H^G(V)\right)^{\mathrm{sm}},$$

with equality if H is open. In particular, Res_H^G restricts to a functor

$$\mathrm{Res}_H^G\colon (G\text{-Rep}^{\mathrm{sm}}) \to (H\text{-Rep}^{\mathrm{sm}}).$$

Proof. The first part follows from

$$\mathrm{Res}_H^G(V^{\mathrm{sm}}) = \bigcup_{K \subset G} V^K \subseteq \bigcup_{K \subset G} V^{K \cap H} = \left(\mathrm{Res}_H^G(V)\right)^{\mathrm{sm}},$$

where K runs over the compact open subsets of G. This is an equality if H is open. The canonical splitting (sending everything outside the image to zero) is H-equivariant. $\square$

Definition B.8 (Smooth Induction). We define the *smooth induction functor*

$$\mathrm{Ind}_H^G\colon (H\text{-Rep}^{\mathrm{sm}}) \to (G\text{-Rep}^{\mathrm{sm}})$$

as the smooth part of the algebraic induction functor. That is, for any smooth representation $\pi\colon G \to \mathrm{GL}(V)$, we set

$$\mathrm{Ind}_H^G(\pi) := \left(\mathrm{algInd}_H^G(\pi)\right)^{\mathrm{sm}}.$$

Definition B.9 (Compact Induction). Let $\pi\colon H \to \mathrm{GL}(V)$ be a smooth representation of H. Then we define $\mathrm{c\text{-}Ind}_H^G(\pi)$, the *compactly induced* representation of π, as the subrepresentation of $\mathrm{Ind}_H^G(\pi)$ with underlying vector space

$$\{\phi \in \mathrm{Ind}_H^G(\pi) \mid \mathrm{Supp}(\phi) \subseteq G \text{ is compact in } H\backslash G\}.$$

This construction yields a functor $\mathrm{c\text{-}Ind}_H^G\colon (H\text{-Rep}^{\mathrm{sm}}) \to (G\text{-Rep}^{\mathrm{sm}})$.

Note that if H is cocompact in G, we have $\mathrm{c\text{-}Ind}_H^G = \mathrm{Ind}_H^G$.

Remark. If H is an open subgroup of G, the quotient $H\backslash G$ is discrete. Now, given $(\pi, V) \in (H\text{-Rep}^{\mathrm{sm}})$, an element $\phi \in \mathrm{Ind}_H^G(\pi)$ lies in $\mathrm{c\text{-}Ind}_H^G(\pi)$ if and only if the image of $\mathrm{Supp}(\phi)$ is

finite in $H\backslash G$. In this case there is an isomorphism

$$\Psi\colon \text{c-Ind}_H^G(V) \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V, \quad \phi \mapsto \sum_{[g]\in\text{Supp}(\phi)} g^{-1} \otimes \phi(g) \tag{.7}$$

which does not depend on the choice of representative $g \in [g]$ as $(hg)^{-1} \otimes \phi(hg) = g^{-1} \otimes \phi(g)$. Giving $\mathbb{C}[G]$ the structure of an $(\mathbb{C}[G], \mathbb{C}[H])$-bimodule, the natural left-G-action on $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ is compatible with the one on c-Ind$_H^G(V)$ under the isomorphism Ψ.

Proposition B.10. *For H a closed subgroup of G, the functors* algInd$_H^G$, Ind$_H^G$ *and* c-Ind$_H^G$ *are exact.*

Proof. The statement for algInd$_H^G$ is implied by the fact that $\mathbb{C}[G]$ is a free (thereby projective) $\mathbb{C}[H]$-module. For the remaining statements, see [BH06, p. 18f]. $\square$

Theorem B.11 (Smooth Frobenius Reciprocity)**.** *Let G be a locally profinite group and let $H \subseteq G$ be a closed subgroup. Then there is an adjunction*

$$\text{Res}_H^G \dashv \text{Ind}_H^G .$$

If H is additionally assumed to be open in G, there is an adjunction

$$\text{c-Ind}_H^G \dashv \text{Res}_H^G .$$

In particular, if H is co-compact and open in G, the functor Ind$_H^G$ *is both left- and right-adjoint to* Res$_H^G$.

Proof. Making use of the remarks above, both adjunctions are the Tensor-Hom-Adjunction in disguise. For the adjunction c-Ind$_H^G \dashv$ Res$_H^G$, this is immediate. For the second we observe that

$$\text{Hom}_{(H\text{-Rep}^{\text{sm}})}(\text{Res}_H^G V, W) \cong \text{Hom}_{(G\text{-Rep})}(V, \text{algInd}_H^G(W)) = \text{Hom}_{(G\text{-Rep}^{\text{sm}})}(V, \text{Ind}_H^G(W)).$$

Here the first isomorphism is by Tensor-Hom-adjunction. The second equality uses that V is a smooth representation of G. $\square$

Proposition B.12. *Let I be a closed subgroup of H. There is a natural isomorphism* Ind$_H^G \circ$ Ind$_I^H \xrightarrow{\sim}$ Ind$_I^G$. *The same statement is true for compact and algebraic induction.*

Proof. Trivially, Res$_I^G =$ Res$_I^H \circ$ Res$_H^G$. The claim follows as the functors in question are adjoints to the left or the right hand side of this equation, thereby isomorphic. $\square$

Lemma B.13. *Let H be a closed subgroup of G. The functor* Res$_H^G$ *is faithful. Equivalently, the unit* id$_{(G\text{-Rep}^{\text{sm}})} \to$ Ind$_H^G \circ$ Res$_H^G$ *of the adjunction* Res$_H^G \dashv$ Ind$_H^G$ *is injective on components. If H is additionally assumed to be an open subgroup of G, The functor* c-Ind$_H^G$ *is faithful. Equivalently, the components of the unit* id$_{(H\text{-Rep}^{\text{sm}})} \to$ Res$_H^G \circ$ c-Ind$_H^G$ *coming from the adjunction* c-Ind$_H^G \dashv$ Res$_H^G$ *are injective.*

Proof. Faithfulness of Res_H^G is clear. For faithfulness of c-Ind_H^G, note that the unit of the adjunction c-$\mathrm{Ind}_H^G \dashv \mathrm{Res}_H^G$ is given on components $(\pi, V) \in (H\text{-Rep}^{\mathrm{sm}})$ by the map $v \mapsto \phi_v$, where $\phi_v \in$ c-$\mathrm{Ind}_H^G(V)$ is defined as

$$\phi_v \colon G \to V, \quad g \mapsto \begin{cases} \pi(g)v & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

The resulting morphism $V \to \mathrm{Res}_H^G(\text{c-}\mathrm{Ind}_H^G(V))$ is injective. Now all claims follow since faithfulness of the left-adjoint is equivalent to the unit being a monomorphism on components, cf. [Rie17, Lemma 4.5.13]. □

Remark. For the sake of completeness, we note that the unit of the adjunction $\mathrm{Res}_H^G \dashv \mathrm{Ind}_H^G$ is given on components $(\pi, V) \in (G\text{-Rep}^{\mathrm{sm}})$ by

$$V \to \mathrm{Ind}_H^G(\mathrm{Res}_H^G(V)), \quad v \mapsto \psi_v; \quad \text{where} \quad \psi_v(g) = \pi(g)v.$$

The following Lemma is an instance of base-change.

Lemma B.14. *Let H and N be closed subgroups of G satisfying $NH = HN$. Let (π, V) be a smooth representation of H. Then there is a natural split monomorphism of N-representations*

$$\mathrm{Res}_N^{HN}(\mathrm{Ind}_H^{HN} \pi) \to \mathrm{Ind}_{H\cap N}^N(\mathrm{Res}_{H\cap N}^H \pi).$$

If N is open in HN, this map is an isomorphism.

Proof. One quickly checks that the map

$$\mathrm{Res}_N^{HN}(\mathrm{algInd}_H^{HN} V) \to \mathrm{algInd}_{H\cap N}^N(\mathrm{Res}_{H\cap N}^H V)$$

given by sending $\phi \colon HN \to V$ to its restriction $\phi|_N$, is an isomorphism. Now the claim follows by taking smooth parts and applying Lemma B.7. □

Remark. There are multiple ways to construct the map above. Applying $\mathrm{Ind}_H^{HN}(-)$ to the unit of the adjunction $\mathrm{Res}_{H\cap N}^H \dashv \mathrm{Ind}_{H\cap N}^H$ yields for any $\pi \in (H\text{-Rep}^{\mathrm{sm}})$ a natural morphism

$$\mathrm{Ind}_H^{HN}(\pi) \to \mathrm{Ind}_{H\cap N}^H \mathrm{Res}_{H\cap N}^H(\pi) \cong \mathrm{Ind}_N^{HN} \mathrm{Ind}_{H\cap N}^N \mathrm{Res}_{H\cap N}^H(\pi),$$

which is equivalent to a map

$$\mathrm{Res}_N^{HN}(\mathrm{Ind}_H^{HN} \pi) \to \mathrm{Ind}_{H\cap N}^N(\mathrm{Res}_{H\cap N}^H \pi).$$

This gives the same map as in the proof. The dual construction (starting with the co-unit) also yields the same map.

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