Explicit Aspects of Non-Abelian Lubin-Tate Theory

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Contents

1 Introduction

1.1 Notation

We denote the category of sets with (Set) and the category of (unital, commututative) rings with (Ring). If A is a ring, we write (A-Alg) for the category of A-algebras, and (A-Mod) for the category of A-modules.

If $f(T) = c_1T + c_2T^2 + \cdots \in A[\![T]\!]$, we write $f^k(T)$ for the k-fold self composite of f, that is

$$f^k(T) = \underbrace{f(f(\cdots(f}_{k-\text{fold}}(T))\cdots).$$

In order to not confuse this with taking multiplicative powers, we write

$$f(T)^k = \underbrace{f(T)f(T)\cdots f(T)}_{k-\text{fold}}.$$

1.2 Acknowledgements

2 Formal Modules

This section will serve as an introduction to formal groups and formal modules. Formal groups (or rather, formal group laws) were first introduced by Salomon Bochner in 1946 as a natural means of studying Lie Groups over fields of characteristic 0, cf. [Boc46]. The study of formal groups later became interesting for its own right, with pioneering works of Lazard [Laz55]. blabla

2.1 Basic Notions

As promised in the introduction, we begin by defining formal group laws. For now, let A be any ring.

Definition 2.1.1 (Formal Group Laws of arbitrary dimension)**.** A (commutative) formal group law of dimension n over R is a tuple of power series $F = (F_1, \dots, F_n)$ with

$$F_i(X_1, \dots, X_n, Y_1, \dots, Y_n) \in R[\![X_1, \dots, X_n, Y_1, \dots, Y_n]\!], \quad 1 \le i \le n$$

such that $F_i(\mathbf{X}, \mathbf{Y}) \equiv X_i + Y_i$ modulo degree ≥ 2 and the following equalities are satisfied:

1. $F(F(\mathbf{X}, \mathbf{Y}), \mathbf{Z}) = F(\mathbf{X}, F(\mathbf{Y}, \mathbf{Z}))$.
2. $F(\mathbf{X}, \mathbf{0}) = \mathbf{X}$.
3. $F(\mathbf{X}, \mathbf{Y}) = F(\mathbf{Y}, \mathbf{X})$.

Here, and in the sequel, we abbreviate $\mathbf{X} = (X_1, \dots, X_n)$, et cetera. Given a formal group F of dimension n and a formal group law G of dimension m, a morphism $F \to G$ is a m-tuple $f = (f_1, \dots, f_m)$ of power series $f_i \in R[\![X_1, \dots, X_n]\!]$ such that $f(0) = 0$ and

$$G(f(\mathbf{X}), f(\mathbf{Y})) = f(F(\mathbf{X}, \mathbf{Y})).$$

For any n-dimensional formal module F, the identity is given by the morphism id_F with components $\mathrm{id}_{F,i}(\mathbf{X}) = X_i$. Composition of morphisms is given by composition of tuples of power-series. This yields the category of formal modules of arbitrary dimension over R, which we denote by $(\mathrm{FGL}^{\mathrm{arb}}/R)$. We will mostly be concerned with the full subcategory of one-dimensional formal groups, which we denote by (FGL/R).

Lemma 2.1.2. *1. The set* $\mathrm{Hom}_{(\mathrm{FGL}^{\mathrm{arb}}/R)}(F, G)$ *is an abelian group with addition* $f + g = G(f, g)$*. In particular,* $(\mathrm{FGL}^{\mathrm{arb}}/R)$ *is pre-additive (cf. [Stacks, Tag 00ZY]).*

2. *Furthermore,* $(\mathrm{FGL}^{\mathrm{arb}}/R)$ *admits finite products. Thereby it is an additive category (cf. [Stacks, Tag 0104]). The unique final and initial object of* $(\mathrm{FGL}^{\mathrm{arb}}/R)$ *is the unique* 0*-dimensional formal* A*-module law.*

3. *In particular* $\mathrm{End}_{(\mathrm{FGL}^{\mathrm{arb}}/R)}(F)$ *is a (possibly non-commutative) ring.*

Example. Let us introduce the following two formal group laws.

- *The additive formal group law.* Write $\widehat{\mathbb{G}}_a$ for the formal group law with addition given by $\widehat{\mathbb{G}}_a(X,Y) = X + Y$.
- We write $\widehat{\mathbb{G}}$ for the formal group law associated with the with $\widehat{\mathbb{G}}(X,Y) = X+Y+XY$. Note that $\widehat{\mathbb{G}}(X,Y) = (X+1)(Y+1) - 1$

Next up is the definition of formal A-module laws. Naively, we would like to define formal A-module laws as formal group laws F with A-module structure, i.e. a morphism of rings $[\cdot]_F : A \to \mathrm{End}_{(\mathrm{FGL}^{\mathrm{arb}}/R)}(F)$. But there is a subtlety, which becomes evident after defining the Lie-algebra of a formal group law.

Definition 2.1.3 (Lie-algebra of formal group law)**.** Let $\mathrm{Lie} : (\mathrm{FGL}^{\mathrm{arb}}/R) \to (\mathrm{Ab})$ be the functor taking an n-dimensional formal group law F to the R-module

$$\mathrm{Lie}(F) = \mathrm{Hom}_{(R\text{-Mod})}\left(\frac{(X_1,\ldots,X_n)}{(X_1,\ldots,X_n)^2}, R\right)$$

Given an m-dimensional group law G and a morphism $f : F \to G$, $\mathrm{Lie}(f)$ is the induced morphism

$$\mathrm{Lie}(F) \to \mathrm{Lie}(G), \quad \psi \mapsto \left(S_j \mapsto \psi(\overline{f_j})\right) \in \mathrm{Hom}_{(R\text{-Mod})}\left(\frac{(X_1,\ldots,X_n)}{(X_1,\ldots,X_n)^2}, R\right),$$

where $\overline{f_j}$ is the reduction of f_j mod $(\mathbf{X})^2$.

We have a canonical basis on both sides, and writing $\mathrm{Lie}(F) = R^n$, $\mathrm{Lie}(G) \cong R^m$, the induced map $\mathrm{Lie}(f) : R^n \to R^m$ is given by multiplication with the matrix

$$\left(\frac{\partial f_i}{\partial X_j}(0)\right)_{\substack{1\le i\le m\\ 1\le j\le n}}.$$

Given a one-dimensional group law $F \in (\mathrm{FGL}/R)$, the condition that $F(X,Y) \equiv X + Y$ modulo degree ≥ 2 enforces that the induced map $\mathrm{End}(F) \xrightarrow{\mathrm{Lie}} \mathrm{End}(R)$ is a morphism of rings. If we are given $[-]_F : A \to \mathrm{End}_{(\mathrm{FGL}/R)}(F)$, this A-module structure on F yields an A-module structure on R, given by the composition

$$A \xrightarrow{[\cdot]_F} \mathrm{End}(F) \xrightarrow{\mathrm{Lie}} \mathrm{End}(R), \quad a \mapsto \mathrm{Lie}([a]_F)$$

This is a morphism of rings, and we obtain an A-algebra structure on R. This motivates the following definition.

Definition 2.1.4 (Formal A-Module Law of arbitrary dimension)**.** Let R be an A-algebra with structure morphism $j : A \to R$. A formal A-module law over R of dimension n is given by the data of a formal n-dimensional group law F over R and a morphism of rings

$$A \to \mathrm{End}_{(\mathrm{FGL}^{\mathrm{arb}}/R)}(F), \quad a \mapsto ([a]_{F,i})_{1\le i\le n} \in (R[\![X_1,\dots,X_n]\!])^n$$

such that $[a]_{F,i}(\mathbf{X}) \equiv j(a)X_i$ modulo terms of degree ≥ 2. Morphisms between formal A-modules of arbitrary dimension are morphisms of formal groups respecting the A-module structure. The resulting category is denoted $(A\text{-FML}^{\mathrm{arb}}/R)$. As before, the full subcategory of one-dimensional formal A modules over R is denoted $(A\text{-FML}/R)$.

Note that $(\mathrm{FGL}/R) \cong (\mathbb{Z}\text{-FML}/R)$. Slightly abusing notation, we usually do not explicitely mention the A-structure when referring to formal module laws, simply writing $F \in (A\text{-FML}/R)$, for example.

The following lemma explains the functoriality of the assignment $R \mapsto (A\text{-FML}^{\mathrm{arb}}/R)$.

Lemma 2.1.5. *The assignment $R \mapsto (A\text{-FML}^{\mathrm{arb}}/R)$ is functorial in the following sense. If $i : R \to R'$ is a morphism of A-algebras, we obtain a functor*

$$(A\text{-FML}^{\mathrm{arb}}/R) \to (A\text{-FML}^{\mathrm{arb}}/R'), \quad F \mapsto F \otimes_R R',$$

where $F \otimes_R R'$ is the formal A-module law obtained by applying i to the coefficients of the formal power series representing the A-module structure of F.

Note that every n-dimensional formal module law $F \in (A\text{-FML}^{\mathrm{arb}}/R)$ yields a functor

$$(R\text{-Alg}) \to (A\text{-Mod}), \quad S \mapsto \mathrm{Nil}(S)^n, \tag{2.1}$$

where $\mathrm{Nil}(S)^n$, the set of n-tuples of nilpotent elements of S, is equipped with addition and scalars given by

$$s_1 + s_2 = F(s_1, s_2) \in \mathrm{Nil}(S)^n, \quad as = [a]_F(s) \in \mathrm{Nil}(S)^n.$$

This construction yields a functor

$$(A\text{-FML}/R) \to \mathrm{Fun}((R\text{-Alg}), (A\text{-Mod})), \tag{2.2}$$

where Fun denotes the functor category.

Passing from discrete R-algebras to admissible R-algebras (cf. Definition A.0.1), this construction extends naturally to a functor

$$(A\text{-FML}/R) \to \mathrm{Fun}((R\text{-Adm}), (A\text{-Mod})), \quad F \mapsto \mathrm{Spf}\, R[\![\mathbf{T}]\!],$$

where we equip $\mathrm{Spf}\, R[\![\mathbf{T}]\!]$ with the structure of an A-module object using the endomorphisms coming from F. Following this line of thought leads naturally to the definition of formal modules.

Definition 2.1.6 (Formal Groups and Formal Modules.)**.** Given an A-scheme X, we define the category $(A\text{-FM}^{\mathrm{arb}}/X)$ as follows. Objects are A-module objects $\mathcal{F}$ in the category of formal schemes over X, having the property that there is a cover of X by Zariski-open affine subsets $U_i = \mathrm{Spec}(R_i)$ such that $\mathcal{F} \times_X U_i$ is isomorphic to $\mathrm{Spf}\, R_i[\![X_1, \ldots, X_n]\!]$ and the induced A-module structure on $\mathrm{Spf}\, R_i[\![X_1, \ldots, X_n]\!]$ yields a formal A-module law on R_i. Given $\mathcal{F}, \mathcal{G} \in (A\text{-FML}^{\mathrm{arb}}/X)$, a morphism $\phi : \mathcal{F} \to \mathcal{G}$ is the same as a morphism of A-module objects in the category of formal schemes over X. Again, we denote the full subcategory of one-dimensional formal A-modules over X by $(A\text{-FM}/X)$.

Remark. Formal schemes (over a base an A-scheme X, say) locally isomorphic to $\mathrm{Spf}\, \mathcal{O}_X(U)[\![\mathbf{T}]\!]$ are sometimes called Formal Lie Varieties . Equivalently to the definition above, we could reference have defined formal A-modules as A-module objects in the category of Formal Lie Varieties, such that the A-module structure on the tangent space at the identity agrees with the usual one.

Definition 2.1.7 (Coordinate)**.** Let $\mathcal{F}$ be a formal A-module over X. The choice of a cover $\sqcup_{i \in I} \mathrm{Spec}(R_i) \to X$ together with isomorphisms $\mathcal{F} \times_X \mathrm{Spec}(R_i) \cong \mathrm{Spf}(R_i[\![\mathbf{T}]\!])$ will be referred to as a coordinate of $\mathcal{F}$.

Of course there is a functor

$$\mathrm{FG} : (A\text{-FML}^{\mathrm{arb}}/R) \to (A\text{-FM}^{\mathrm{arb}}/R),$$

essentially forgetting the choice of module law. The observation of Lemma 2.1.5 remains valid in the category formal modules, a morphism $j : R \to R'$ yields a functor

$$(A\text{-FM}/R) \to (A\text{-FM}/R'), \quad \mathcal{F} \mapsto \mathcal{F} \otimes_R R'.$$

Definition 2.1.8 (Lie functor)**.** The functor Lie descends to a functor

$$\mathrm{Lie} : (A\text{-FM}^{\mathrm{arb}}/X) \to (\mathcal{O}_X\text{-QCoh}),$$

given by locally describing a formal A-module $\mathcal{F}$ via formal group laws and gluing the local data. Alternatively, it arises from sending sending a formal A-module $\mathcal{F}$ to $(\mathcal{I}/\mathcal{I}^2)^\vee$, where $\mathcal{I}$ is the ideal associated to the closed immersion $[0]_{\mathcal{F}} : X \to \mathcal{F}$.

Lemma 2.1.9. *A map $\phi : \mathcal{F} \to \mathcal{G}$ of formal A-modules (of arbitrary dimension) over X is an isomorphism if and only if the induced morphism of Lie algebras* $\mathrm{Lie}(\phi) : \mathrm{Lie}(\mathcal{F}) \to \mathrm{Lie}(\mathcal{G})$ *is an isomorphism.*

Proof. This is easily verified in the one-dimensional situation after choosing coordinates. The general case adds no complication. □

Example. The additive group law $\widehat{\mathbb{G}}_a$ extends to a formal A-module over an affine base $\mathrm{Spec}\, R$ by setting

$$[a]_{\widehat{\mathbb{G}}_a}(T) = aT$$

for $a \in A$. More generally, we obtain a formal A-module over an arbitrary base scheme X over A.

Over $\mathbb{Z}_p$, the formal group $\widehat{\mathbb{G}}$ extends to a formal $\mathbb{Z}_p$-module as follows. As a functor, $\widehat{\mathbb{G}}$ is isomorphic to the assignment

$$(\mathbb{Z}_p\text{-Adm}) \to (\text{Ab}), \quad S \mapsto 1 + S^{\circ\circ} \subset S^\times.$$

Here, we equipped $\mathbb{Z}_p$ with the p-adic topology. The subgroup $1 + S^{\circ\circ}$ naturally carries the structure of a $\mathbb{Z}_p$-module. Indeed, for $k \in \mathbb{N}$, we have

$$(1+s)^{p^k} = 1 + p^k s + \binom{p^k}{2} s^2 + \cdots + s^{p^k},$$

and given $s \in S^{\circ\circ}$, this is of the form $1 + o(1)$ as k gets large. In particular, if $x = a_0 + a_1 p + a_2 p^2 + \cdots \in \mathbb{Z}_p$, expressions of the form

$$(1+s)^x = \prod_{i=1}^{\infty} (1+s)^{a_k p^k}$$

make sense by Lemma A.0.3. This gives $\widehat{\mathbb{G}}_{m,\mathbb{Z}_p}$ the structure of a formal $\mathbb{Z}_p$-module. In the upcoming subsection, we discuss how this is the simplest example of a whole family of formal modules constructed by Lubin and Tate. In Section 3 we explain applications of these formal modules to local class field theory.

Definition 2.1.10 (Formal Module associated to R-module)**.** Suppose that M is a finite projective R-module. Then we write $\widehat{\mathbb{G}}_a \otimes M$ for the additive formal A-module associated to M over R. As a formal scheme, this formal module is given by

$$\widehat{\mathbb{G}}_a \otimes M \cong \operatorname{Spf} R[\![M^\vee]\!],$$

where $R[\![M^\vee]\!]$ denotes the completion of $\operatorname{Sym}_R(M^\vee)$ with respect to the ideal generated by $M^\vee$. The (formal) A-module structure is the canonical additive one. Note that $\operatorname{Lie}(\widehat{\mathbb{G}}_a \otimes M) = M$ by design. More generally, if X is a quasi-compact and quasi-separated A-scheme and $\mathcal{M}$ is a finite locally free quasi-coherent $\mathcal{O}_X$-module, this construction yields a formal A-module $\widehat{\mathbb{G}}_a \otimes \mathcal{M}$ over X.

Remark. If $R \to R'$ is a ring morphism that turns R' into a (say) finite free R -algebra, the above definition overloads the expression $\widehat{\mathbb{G}}_a \otimes_R R'$. In order to disambiguate, we usually denote the additive formal A-module over R' by $\widehat{\mathbb{G}}_{a,R'}$.

2.2 Lubin–Tate Formal Module Laws

We sketch the construction of a family of formal modules introduced by Lubin and Tate in [LT65].

Let A be a complete discrete valuation ring with finite residue field k, set $q = \#k$ and let $\varpi \in A$ be a choice of uniformizer. Write $\mathcal{F}_{\varpi,h}$ for the following set of power series

$$\mathcal{F}_\varpi := \{f \in \mathcal{O}_K[\![T]\!] \mid f \equiv \varpi T \pmod{T^2} \text{ and } f \equiv T^{q^n} \pmod{\varpi}\}.$$

The construction of the Lubin–Tate formal module laws is based on the following lemma, which is Lemma 1 in [LT65].

Lemma 2.2.1. *Let $f(T)$ and $g(T)$ be elements of $\mathcal{F}_{\varpi,h}$ and let $L(X_1,\dots,X_n)=\sum_{i=1}^n a_iX_i$ be a linear form with coefficients in A. Then there exists a unique series $F(X_1,\dots,X_n)$ with coefficients in A such that*

$$F(X_1,\dots,X_n)\equiv L(X_1,\dots,X_n)\pmod{T^2},$$
and
$$f(F(X_1,\dots,X_n))=F(g(X_1),\dots,g(X_n)).$$

As a direct consequence, we obtain the following useful result.

Lemma 2.2.2. *Let $f\in\mathcal{F}_{\varpi,h}$. Then there is a unique formal A-module law F_f over A with $[\varpi]_F(T)=f(T)$.*

Proof. In the above lemma, set $L(X,Y)=X+Y$ and $g=f$ to uniquely determine the power series F_f. The same Lemma yields unique power series $[a]_{F_f}(T)\in R[\![T]\!]$ by setting $L(T)=aT$, $g=f$. It is routine to check that $(F_f,([a]_f)_{a\in A})$ is a formal A-module law, cf. [LT65]. □

Definition 2.2.3 (Lubin–Tate Module Law)**.** We refer to module laws arising from the construction above as Lubin–Tate module laws.

Furthermore, attached to each $a\in\mathcal{O}_K$ and $f,g\in\mathcal{F}_{\varpi,h}$, we find unique $[a]_{f,g}(T)\in\mathcal{O}_K[\![T]\!]$ satisfying

$$[a]_{f,g}(T)\equiv aT\pmod{(T)^2}\quad\text{and}\quad f([a]_{f,g}(T))=[a]_{f,g}(g(T)).\tag{2.3}$$

We now have the following theorem.

Theorem 2.2.4 (Lubin–Tate Formal $\mathcal{O}_K$-Module Laws)**.** *Let K be a local field with ring of integers $\mathcal{O}_K$. For any choice of uniformizer $\varpi\in\mathcal{O}_K$ and any $f\in\mathcal{F}_{\varpi,h}$, the family of power series $(F_f,([a]_{f,f})_{a\in\mathcal{O}_K})$ gives rise to a formal $\mathcal{O}_K$-module law over $\mathcal{O}_K$. For $f,g\in\mathcal{F}_{\varpi,h}$, the formal $\mathcal{O}_K$-module laws F_f and F_g are canonically isomorphic, via the morphism induced by $[1]_{f,g}\in\mathcal{O}_K[\![T]\!]$.*

Proof. See Theorem 1 of [LT65] and the subsequent discussion. □

In particular, up to canonical isomorphism, there is only one Lubin–Tate formal $\mathcal{O}_K$-module law over $\mathcal{O}_K$ attached to the choice of the uniformizer $\varpi\in\mathcal{O}_K$.

Example. If $K=\mathbb{Q}_p$, this reconstructs the multiplicative formal $\mathbb{Z}_p$ module $\widehat{\mathbb{G}}$ constructed above. Indeed, we have

$$\mathcal{F}_p=\{f\in\mathbb{Z}_p[\![T]\!]\mid f(T)\equiv T^p \bmod p \text{ and } f(T)\equiv pT \bmod (T)^2\},$$

implying that $f(T)=(1+T)^p-1$ lies in $\mathcal{F}_p$. One quickly checks that

$$F_f(X,Y)=(1+X)(1+Y)-1=X+Y+XY\in\mathbb{Z}_p[\![X,Y]\!]$$

is the addition law associated to f, and that for $a \in \mathbb{Z}_p$, the power series

$$[a]_{f,f} = (1+T)^a - 1 \in \mathbb{Z}_p[\![T]\!]$$

satisfies the condition of (2.3).

2.3 Logarithms

Again, A is a complete discrete valuation ring with uniformizing parameter ϖ and finite residue field $k = A/\varpi A$. We write q for the cardinality of k and K for the field of fractions of A. Let R be a local, admissible A-algebra with structure map $i : A \to R$.

We review results from Sections 2 and 3 of [GH94]. Suppose that $F = (F_1, \dots, F_n)$ is a n-dimensional formal A-module law over R. We abbreviate $\mathbf{X} = (X_1, \dots, X_n)$, $\mathbf{Y} = (Y_1, \dots, Y_n)$, etc.

Definition 2.3.1 (Invariant Differentials)**.** The module $\omega(F)$ of invariant differentials is the submodule of the R-module of differentials

$$\Omega_{R[\![T_1,\dots,T_n]\!]/R} \cong \bigoplus_{i=1}^{n} R[\![T_1, \dots, T_n]\!]\, \mathrm{d}T_i,$$

consisting of those $\omega \in \omega(F)$ satisfying

$$\omega(F(\mathbf{X},\mathbf{Y})) = \omega(\mathbf{X}) + \omega(\mathbf{Y}) \quad \text{and} \quad \omega([a]_F(\mathbf{X})) = a\omega(\mathbf{X}). \tag{2.4}$$

for all $a \in A$.

It is possible to explicitly calculate a basis for the R-module $\omega(F)$, which we now explain. Let

$$A(\mathbf{X},\mathbf{Y}) \in \mathrm{Mat}_{n\times n}(R[\![\mathbf{X},\mathbf{Y}]\!])$$

denote the matrix $((\partial/\partial X_j)F_i(\mathbf{X},\mathbf{Y}))_{i,j}$, the derivative of $F(\mathbf{X},\mathbf{Y})$ with respect to $\mathbf{X}$. Set $B(\mathbf{Y}) = A(0,\mathbf{Y})$. Then B is a unit in $\mathrm{Mat}_{n\times n}\, R[\![\mathbf{Y}]\!]$; and we write $(C_{ij}(\mathbf{Y}))$ for the components of $B(\mathbf{Y})^{-1}$. We now construct

$$\omega_i := \sum_{j=1}^{n} C_{ij}(\mathbf{X})\, \mathrm{d}X_j \in \Omega_{R[\![\mathbf{X}]\!]/R}$$

for $1 \le i \le n$. By definition we have

$$C_{ij}(0) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

Checking that ω_i is an invariant differential is a matter of applying the chain rule.

Proposition 2.3.2. *The R-module $\omega(F)$ is free of rank n generated by invariant differentials $\omega_1, \omega_2, \dots, \omega_n$.*

Proof. This is [Hon70, Proposition 1.1]. $\square$

Example. The invariant differentials for $\widehat{\mathbb{G}}_a$ are spanned by the form $\mathrm{d}X$. The invariant differentials for $\widehat{\mathbb{G}}$ are spanned by the form $\omega_1(X) = \frac{1}{1+X}\,\mathrm{d}X$.

By the Proposition above and Equation (2.5), we may define a pairing

$$\omega(F) \times \mathrm{Lie}(F) \to R, \quad \langle X_i, \omega_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

This pairing is independent of the parameterization of F. In particular, it descends to a pairing defined for formal modules $\mathcal{F} \in (A\text{-FM}^{\mathrm{arb}}/R)$, and we have a natural isomorphism $\omega(\mathcal{F}) \cong \mathrm{Hom}_R(R, \mathrm{Lie}(\mathcal{F}))$.
Let $\widehat{\mathbb{G}}_a$ be the additive formal A-module over R. There is a map

$$\mathrm{d}_F : \mathrm{Hom}_{(A\text{-FML}/R)}(F, \widehat{\mathbb{G}}_{a,R}) \to \omega(F), \quad f \mapsto \mathrm{d}f(\mathbf{X}) \tag{2.6}$$

which is a map of R-modules if we equip the left hand side with the R-module structure coming from the natural action of $R \subset \mathrm{End}(\widehat{\mathbb{G}}_a)$.

Proposition 2.3.3. *1. If R is a flat A-algebra, the map d_F is injective.*

2. If R is a K-algebra, the map d_F is an isomorphism.

Proof. This is [GH94, Proposition 3.2]. □

Suppose now that $F \in (A\text{-FML}^{\mathrm{arb}}/R)$ is a formal module law of dimension n over a flat A-algebra R. Let $\omega_1, \ldots, \omega_n$ be the distinguished basis for $\omega(F)$ constructed above. By the previous proposition, there are unique power series $f_i(\mathbf{X}) \in (R \otimes_A K)[\![\mathbf{X}]\!]$ that furnish homomorphisms $F \otimes (R \otimes_A K) \to \widehat{\mathbb{G}}_{a,R\otimes_A K}$ of formal A-module laws and satisfying

$$\mathrm{d}_F f_i(\mathbf{X}) = \omega_i(\mathbf{X}) \in \omega(F).$$

Definition 2.3.4 (Logarithm and Exponential)**.** The induced morphism of formal group laws

$$f = (f_1, \ldots, f_n) : F \otimes (R \otimes_A K) \to \widehat{\mathbb{G}}_a^n \otimes_R (R \otimes K)$$

is called the logarithm attached to F, we write $\log_F(\mathbf{X}) \in ((R \otimes_A K)[\![\mathbf{X}]\!])^n$ for the corresponding collection of power series. The inverse of $\log_F(\mathbf{X})$ is called the exponential attached to F, denoted $\exp_F(\mathbf{X})$. We have $\mathrm{Lie}(\log_F) = \mathrm{Lie}(\exp_F) = \mathrm{id}$, so $\log_F$ and $\exp_F$ are isomorphisms.

Lemma 2.3.5. *Let F and G be formal A-module laws over R, with $\dim F = n$ and $\dim G = m$. Let $\phi : F \to G$ be a morphism. Then the diagram*

$$\begin{tikzcd}
F \otimes (R \otimes_A K) \arrow[r, "\log_F"] \arrow[d, "\phi"'] & \widehat{\mathbb{G}}_{a,R\otimes_A K} \otimes (\mathrm{Lie}(F) \otimes_A K) = \widehat{\mathbb{G}}^n_{a,R\otimes_A K} \arrow[d, "\mathrm{Lie}(\phi)"] \\
G \otimes (R \otimes_A K) \arrow[r, "\log_G"] & \widehat{\mathbb{G}}_{a,R\otimes_A K} \otimes (\mathrm{Lie}(G) \otimes_A K) = \widehat{\mathbb{G}}^m_{a,R\otimes_A K}
\end{tikzcd}$$

commutes. In particular, attached to any $\mathcal{F} \in (A\text{-FM}^{\mathrm{arb}}/R)$ comes a natural morphism

$$\log_{\mathcal{F}} : \mathcal{F} \otimes (R \otimes_A K) \to \widehat{\mathbb{G}}_{a,R\otimes_A K} \otimes (\mathrm{Lie}(\mathcal{F}) \otimes_R (R \otimes_A K)) .$$

Proof. The square commutes because of the equality

$$\mathrm{Hom}(\widehat{\mathbb{G}}^n_{a,R\otimes_A K}, \widehat{\mathbb{G}}^m_{a,R\otimes_A K}) = \mathrm{Hom}_{R\otimes_A K}((R \otimes_A K)^n, (R \otimes_A K)^m),$$

and the fact that $\mathrm{Lie}(\log_G \circ \phi \circ \exp_H) = \mathrm{Lie}(\phi)$. □

Lemma 2.3.6. *Let K be a local field with integers $\mathcal{O}_K$ and a choice of uniformizer $\varpi \in \mathcal{O}_K$, and let F be a Lubin-Tate $\mathcal{O}_K$-module law corresponding to some $f \in \mathcal{F}_\varpi$, cf. Theorem 2.2.4. Let S be an admissible $\mathcal{O}_K$-algebra and let $s \in S^{\infty}$ be an element such that the series $\log_{\mathcal{F}}(s)$ converges. Then we have $\log_F(s) = 0$ if and only if $[\varpi]^r_F(s) = 0$ for some $r \in \mathbb{N}$.*

Proof. Up to canonical isomorphism, F is a $\mathcal{O}_K$-module law with $[\varpi]_F(T) = \varpi T + T^q$. Now one may check that

$$\log_F(T) = \lim_{r\to\infty} \frac{[\varpi]^r_F(T)}{\varpi^r} = \prod_{i=1}^{\infty} \frac{[\varpi]^i_F(T)}{\varpi[\varpi]^{i-1}_F(T)},$$

where convergence is taken coefficient-wise. After inserting $s \in S^{\infty}$, we see that the product vanishes if and only if $[\varpi]^r_F(s) = 0$ for some $r \in \mathbb{N}$. □

2.4 Formal DVR-Modules over Fields of Finite Characteristic

As above, let A be a discrete valuation ring with uniformizer ϖ and finite residue field k; write q for the cardinality of k. Let K denote the field of fractions of A, and let R be a A-algebra.

We have seen in the previous section that if R is a field extension of K, then any morphism of formal group laws $f : F \to G$ over R is either 0 or an isomorphism, which makes the theory of formal A-modules over R rather simple. This situation changes if R is a field extension of k: there are homomorphisms of formal group laws $f : F \to G$ corresponding to power series $f(T) \in R[\![T]\!]$ with vanishing differential. The prototype of such homomorphisms are the relative Frobenii $f : F \to F^{(q)}$ corresponding to the monomial $f(T) = T^q$. Here, of course, $F^{(q)}$ is the formal group law obtained by raising each coefficient of $F(X, Y) \in R[\![X, Y]\!]$ to the q-th power.

We introduce the concept of height, which is in a sense an attempt to quantify the disorder introduced by the Frobenius homomorphisms. This leads to interesting invariants of formal A-modules over local A-algebras R.

Definition 2.4.1 (Height of morphisms of group laws)**.** Assume that R is a field extension of k and $f : F \to G$ is a morphism of formal groups laws over R, given by a formal series $f(T) \in R[\![T]\!]$. If $f = 0$, we say that f has infinite height. If $f \neq 0$, the height of f is defined as the largest integer h such that $f = g(T^{q^h})$ for some power series $g(T) = c_1T + c_2T^2 + \cdots \in R[\![T]\!]$ with $c_1 \neq 0$.

One readily checks that if $f : \mathcal{F} \to \mathcal{G}$ is a morphism of formal groups over a field extension R

of k, the height of f does not depend on the choices of group laws on $\mathcal{F}$ and $\mathcal{G}$. This allows us to define the height function attached to f.

Definition 2.4.2 (Height function). Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of formal groups over a scheme X. For a scheme-theoretic point $x \in |X|$, let f_x denote the base change of f to the residue field of x. The height function attached to f is the upper semicontinuous function.

$$\mathrm{ht}(f) : |X| \to \mathbb{Z}_{\geq 0} \cup \{\infty\}, \quad x \mapsto \mathrm{ht}(f_x). \tag{2.7}$$

It is not hard to see that the height function is additive, that is, we have

$$\mathrm{ht}(f \circ g) = \mathrm{ht}(f) + \mathrm{ht}(g).$$

Definition 2.4.3 (Isogeny). A morphism $f : \mathcal{F} \to \mathcal{G}$ of formal groups over a field k is called an isogeny if $\mathrm{Ker}(f)$ is a represented by a finite free k-scheme. More generally, a morphism of formal A-modules over a base scheme X is an isogeny if and only if $\mathrm{Ker}(f)$ is finite and locally free over X.

Isogenies can be described using the height function.

Lemma 2.4.4. *A morphism $f : \mathcal{F} \to \mathcal{G}$ is a isogeny if and only if the height function* $\mathrm{ht}(f)$ *is locally constant with values in $\mathbb{Z}_{\geq 0}$.*

Definition 2.4.5 (ϖ-divisible A-module). We say that a formal A-module H over X is ϖ-divisible if $[\varpi]_H$ is an isogeny. If X is connected, the height of H is the (constant) height of the endomorphism $[\varpi]_H : H \to H$.

We close this subsection with a discussion about the structure of formal $\mathcal{O}_K$-modules over separably closed field extensions k' of k.

Proposition 2.4.6. *Over k', any two formal $\mathcal{O}_K$-module laws of the same height are isomorphic.*

Proof. [Dri74, Proposition 1.7]. □

In particular, any formal $\mathcal{O}_K$-module of height h is isomorphic to the formal $\mathcal{O}_K$-module F_{norm} with $[\varpi]_{F_{\mathrm{norm}}}(T) = T^{q^h}$. We call this the normalized formal $\mathcal{O}_K$-module.

Proposition 2.4.7. *Suppose that $F \in (\mathcal{O}_K\text{-FML}/k')$. Then* $\mathrm{End}_{(A\text{-FM}/k')}(F)$ *is isomorphic to the maximal order of the central division algebra D over K of rank h^2 and invariant $\frac{1}{h}$.*

Proof. This is part of [Dri74, Proposition 1.7] as well. □

Lemma 2.4.8. *Let $f : F \to G$ be an isogeny of ϖ-divisible formal $\mathcal{O}_K$-module laws over k'. Then there is an integer $n \geq 0$ and an isogeny $g : G \to F$ with*

$$f \circ g = [\varpi^n]_{\mathcal{G}} \quad \textit{and} \quad g \circ f = [\varpi^n]_{\mathcal{F}}.$$

Proof. As the height is additive, we necessarily have $\mathrm{ht}(F) = \mathrm{ht}(G)$, thus by Lemma 2.4.6, we may assume that F and G are given by the normalized formal $\mathcal{O}_K$-module F_{norm}. Write $f(T) = g(T^{q^n})$ for some power series $h(T) = c_1T + c_2T^2 + \ldots$, where $c_1 \neq 0$ is a unit in k', and let $g(T) = h^{-1}(T)$ be the formal inverse of h. Now g is a morphism of formal $\mathcal{O}_K$-module laws satisfying $f \circ g(T) = g \circ f(T) = T^{q^n}$. The claim follows. □

2.5 Explicit Dieudonné Theory

Let $\mathcal{F}$ and $\mathcal{F}'$ be formal A-modules of dimension m and n respectively, over an affine base $\operatorname{Spec} R$, coming from formal module laws F and F'. We give an explicit description of $\operatorname{Ext}(\mathcal{F}, \mathcal{F}')$ in terms of terms of the Symmetric 2-Cocycles associated with F and F' (cf. Definition B.1.4). We also give a related explicit description of $\operatorname{RigExt}(F, \widehat{\mathbb{G}}_a)$ in terms of Quasi-Logarithms, cf. Definition 2.5.3.

Write $\mathbf{X}$ for the variables of F' and $\mathbf{Z}$ for the variables of F.

Definition 2.5.1 (Symmetric 1-Cochain)**.** A symmetric 1-cochain for the pair (F, F') is an n-tuple of power series $g = (g_1, \ldots, g_m)$, such that $g_i(\mathbf{Z}) \in R[\![\mathbf{Z}]\!]$ satisfying $g_i(0) = 0$ for all i. We write δg for the coboundary of g, that is, the pair $(\Delta g, (\delta_a g)_{a \in A})$, where

$$\Delta g = g(\mathbf{Z}_1) -_{F'} g(F(\mathbf{Z}_1, \mathbf{Z}_2)) +_{F'} g(\mathbf{Z}_2) \in (R[\![\mathbf{Z}_1, \mathbf{Z}_2]\!])^m$$

and

$$\delta_a g = [a]_{F'} g(\mathbf{Z}) -_{F'} g([a]_F(\mathbf{Z})) \in (R[\![\mathbf{Z}]\!])^m.$$

One readily checks that $\delta g \in \operatorname{SymCoc}^2(F, F')$.

Proposition 2.5.2. *Given two extensions $\mathcal{E}, \mathcal{E}' \in \operatorname{Ext}(\mathcal{F}, \mathcal{F}')$, write E, E' for the respective formal A-module laws coming from Lemma B.1.3, and write Δ_E and $\Delta_{E'}$ for the associated symmetric 2-cocycles (cf. Proposition B.1.5). There is a bijection*

$$\{g \in (R[\![\mathbf{Z}]\!])^m \mid g(0) = 0 \textit{ and } \delta g = \Delta_{E'} - \Delta_E\} \xrightarrow{\sim} \{\textit{Isomorphisms of extensions } E \to E'\}.$$

Explicitly, this bijection is given by sending g to the morphism $i_g \in \operatorname{Hom}_{(A\text{-}\mathrm{FML}^{\mathrm{arb}}/R)}(E, E')$, given by $i_g(\mathbf{X}, \mathbf{Z}) = (\mathbf{X} +_{F'} g(\mathbf{Z}), \mathbf{Z})$. In particular, there is a bijection

$$\operatorname{Ext}(\mathcal{F}, \mathcal{F}') \cong \frac{\operatorname{SymCoc}^2(F, F')}{\{\delta g \mid g \in (R[\![\mathbf{Z}]\!])^m \textit{ with } g(0) = 0\}}.$$

This bijection is an isomorphism of $\operatorname{End}(\mathcal{F}')$*-modules.*

For now, this finishes the study of $\operatorname{Ext}(\mathcal{F}, \mathcal{F}')$.

Assume now that $\mathcal{F}' = \widehat{\mathbb{G}}_a$, and that $\mathcal{F}$ comes from a one-dimensional formal A-module $F \in (A\text{-FML}/R)$. For the remainder of this subsection, we will be concerned with the R-module $\operatorname{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a)$. The notion of Quasi-Logarithms will play a major role.

Definition 2.5.3 (Quasi-Logarithms)**.** A power series $g(T) \in (R \otimes_A K)[\![T]\!]$ is called a Quasi-Logarithm for F, if $g(0) = 0$ and $g'(T)$, as well as all of the power series appearing in δg

(with $F' = \widehat{\mathbb{G}}_a$, cf. Definition 2.5.1) have coefficients in R. We define the R-module

$$\mathrm{QLog}(F) = \frac{\{g(T) \in (R \otimes_A K)[\![T]\!] \mid g \text{ is a quasi-logarithm for } F\}}{\{g(T) \in R[\![T]\!] \mid g(0) = 0\}}$$

Let $(\mathcal{E}, s) \in \mathrm{RigExt}(F, \widehat{\mathbb{G}}_a)$ be a rigidified extension. The splitting s yields an isomorphism $\omega(\mathcal{E}) \cong \omega(\widehat{\mathbb{G}}_a) \oplus \omega(\mathcal{F})$ on duals, giving an invariant differential $\omega_{\mathcal{E}} \in \omega(\mathcal{E})$ pulling back to $\mathrm{d}X$ on $\widehat{\mathbb{G}}_a$. Conversely, any such invariant differential $\omega_{\mathcal{E}}$ yields a splitting, so the choice of s is equivalent to the choice of ω_E, and we will henceforth write $(\mathcal{E}, \omega_{\mathcal{E}}) \in \mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a)$.

Theorem 2.5.4 (Classification of Rigidified Extensions in terms of Quasi-Logarithms)**.** *There is a bijection*

$$\{\textit{Quasi-logarithms for } F\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{Pairs } (E, \omega_E), \textit{ where } E \textit{ is an } A\textit{-module law} \\ \textit{fitting into an exact sequence} \\ 0 \to \widehat{\mathbb{G}}_a \xrightarrow{\alpha} E \xrightarrow{\beta} F \to 0 \\ \textit{with } \alpha(X) = (X, 0) \textit{ and } \beta(X,T) = T \textit{ and } \omega_E \\ \textit{is an invariant differential on } E \textit{ with } \alpha^*\omega_E = \mathrm{d}X. \end{array} \right\} \tag{2.8}$$

The map sends any quasi-logarithm $g(T) \in (R \otimes_A K)[\![T]\!]$ to the pair $(E_{\delta g}, \mathrm{d}(X + g(T))) \in \mathrm{RigExt}(F, \widehat{\mathbb{G}}_a)$. Here $E_{\delta g} \in \mathrm{Ext}(F, \widehat{\mathbb{G}}_a)$ is the extension corresponding to $\delta g \in \mathrm{SymCoc}^2(F, \widehat{\mathbb{G}}_a)$. Furthermore, given two rigidified extensions $(E, \omega_E), (D, \omega_D)$ with associated quasi-logarithms $g(T)$ and $h(T)$, there is a (unique) isomorphism $(E, \omega_E) \to (D, \omega_D)$ if and only if $h(T) - g(T) =: f(T)$ has coefficients in $R[\![T]\!]$. In this case, the isomorphism $i_f(X,T) \in \mathrm{Hom}_{(A\text{-}\mathrm{FML}^{\mathrm{arb}}/R)}(E, D)$ is given by $i_f(X,T) = (X + f(T), T)$. In particular, there is a canonical bijection

$$\mathrm{QLog}(F) \xrightarrow{\sim} \mathrm{RigExt}(F, \widehat{\mathbb{G}}_a).$$

This bijection is an isomorphism of R-modules.

Proof. We construct an inverse of the map in (2.8). Let (E, ω_E) be an element of the set on the right and let $(\Delta, (\delta_a)_{a \in A}) \in \mathrm{SymCoc}^2(F, \widehat{\mathbb{G}}_a)$ be the symmetric 2-cochain corresponding to E. Following Proposition 2.3.3, the datum of $\omega_E \in \omega(E)$ is equivalent to a morphism

$$f_E \in \mathrm{Hom}_{(A\text{-}\mathrm{FML}/R \otimes K)}(E \otimes_R (R \otimes_A K), \widehat{\mathbb{G}}_a) \quad \text{satisfying} \quad f_E(X,T) = X + g(T)$$

for some $g(T) \in (R \otimes_A K)[\![T]\!]$. The fact that f_E is a homomorphism implies that

$$\begin{aligned} X_1 + X_2 + \Delta(T_1, T_2) + g(F(T_1, T_2)) &= f_E(E((X_1, T_1), (X_2, T_2))) = \\ &= f_E(X_1, T_1) + f_E(X_2, T_2)) = X_1 + g(T_1) + X_2 + g(T_2), \end{aligned}$$

thereby $\Delta g = \Delta(T_1, T_2) \in R[\![T_1, T_2]\!]$. Similarly, we find $\delta_a g = \delta_a \in R[\![T]\!]$. Hence, $g(T)$ is a quasi-logarithm with $\delta g = (\Delta, (\delta_a)_a)$. This construction yields the desired inverse. The remaining statements are verified directly, also cf. [GH94, Section 8]. □

Now, let A be a complete, discrete valuation ring with uniformizing parameter ϖ and finite residue field k.

Proposition 2.5.5. *Let $\mathcal{F}$ be a one-dimensional formal A-module law over a flat, local A-algebra R, and suppose that $\mathcal{F}' = \widehat{\mathbb{G}}_a$. The short exact sequence of Proposition B.2.4 fits into a commutative diagram with exact rows and vertical maps isomorphisms induced by any choice of coordinate $\mathcal{F} = \mathrm{FG}(F)$.*

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{F}, \widehat{\mathbb{G}}_a) \arrow[r, hook, "\mathrm{d}_F"] \arrow[d] & \omega(\mathcal{F}) \arrow[r] \arrow[d] & \mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a) \arrow[r, two heads] \arrow[d] & \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a) \arrow[d] \\
\left\{\begin{matrix} f \in TR[\![T]\!] : \\ \delta f = 0 \end{matrix}\right\} \arrow[r, hook] & \left\{\begin{matrix} f \in (R \otimes_A K)[\![T]\!] : \\ \delta f = 0,\ f(0) = 0 \\ \textit{and } f'(T) \in R[\![T]\!] \end{matrix}\right\} \arrow[r] & \mathrm{QLog}(F) \arrow[r, two heads, "\delta"] & \frac{\mathrm{SymCoc}^2(F, \widehat{\mathbb{G}}_a)}{\{\delta g \mid g \in TR[\![T]\!]\}}
\end{tikzcd}$$

Proof. Injectivity of d_F is provided by Proposition 2.3.3, and related to the original exact sequence as $\mathrm{Hom}_R(\mathrm{Lie}(\mathcal{F}), \mathrm{Lie}(\widehat{\mathbb{G}}_a)) = \omega(\mathcal{F})$. Surjectivity of $\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a) \to \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a)$ comes from the fact that $\mathrm{Lie}(\mathcal{F})$ is projective. The first vertical map is an isomorphism by definition. The vertical arrow describing $\omega(F)$ is obtained by identifying the preimage of $\omega(F) \subseteq \omega(F \otimes_R (R \otimes_A K))$ under the isomorphism

$$\{f \in T(R\otimes_A K)[\![T]\!] \mid \delta f = 0\} = \mathrm{Hom}_{(A\text{-FML}/R\otimes_A K)}(F\otimes(R\otimes_A K), \widehat{\mathbb{G}}_a) \xrightarrow{\mathrm{d}_F} \omega(F\otimes_R(R\otimes_A K)).$$

All squares commute by construction. ☐

We admit the following facts from Section 9 of [GH94].

Proposition 2.5.6. *Let F be a formal A-module law of height h over a local, adic A-algebra R. Write $\mathcal{F}$ for the formal A-module coming from F. Then* $\mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a)$ *is a free R-module of rank $n-1$,* $\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a)$ *is a free R-module of rank n.*

Proof. This is Proposition 9.8 in [GH94]. The authors make use of a description of $\mathrm{Ext}(F, \widehat{\mathbb{G}}_a)$ in terms of deformation theory and combine it with a convenient normal form of formal A-modules, so called A-typical modules (we touch upon the theory in Section 2.6), to construct an explicit basis for the corresponding modules. ☐

As a corollary, the authors obtain

Lemma 2.5.7. *If $R \to R'$ is a homomorphism of local A-algebras, the induced maps of free R'-modules*

$$\mathrm{Ext}_R(\mathcal{F}, \widehat{\mathbb{G}}_a) \otimes_R R' \to \mathrm{Ext}_{R'}(\mathcal{F}, \widehat{\mathbb{G}}_a)$$
$$\mathrm{RigExt}_R(\mathcal{F}, \widehat{\mathbb{G}}_a) \otimes_R R' \to \mathrm{RigExt}_{R'}(\mathcal{F}, \widehat{\mathbb{G}}_a)$$

are isomorphisms.

Proof. [GH94, Corollary 9.13]. ☐

Definition 2.5.8 (The Dieudonné module of a formal A-module)**.** Given $\mathcal{F} \in (A\text{-FM}/R)$, we define

$$\mathrm{D}(\mathcal{F}) \coloneqq \mathrm{Hom}_R(\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a), R).$$

We call $\mathrm{D}(\mathcal{F})$ the (covariant) Dieudonné-module of $\mathcal{F}$.

Proposition 2.5.9 (Crystalline Nature of $\mathrm{D}(-)$)**.** *The assignment $\mathcal{F} \mapsto \mathrm{D}(\mathcal{F})$ yields a functor*

$$(A\text{-FM}/R) \to (R\text{-Mod}).$$

Given two formal A-modules $\mathcal{F}, \mathcal{G} \in (A\text{-FM}/R)$ and two morphisms ϕ, ψ from $\mathcal{F}$ to $\mathcal{G}$ such that the induced morphisms of their reductions to R/I agree, the induced morphisms $\mathrm{D}(\mathcal{F}) \to \mathrm{D}(\mathcal{G})$ agree.

Proof. □ !!!

2.6 Hazewinkel's Functional Equation Lemma and the Standard Formal Module Law

If, A is an integral domain and R is a flat A-module, the structure of a formal A-module F over R is uniquely determined by its logarithm $\log_H \in R \otimes_A K[\![T]\!]$. Indeed, we find

$$F(X, Y) = \exp_H(X + Y), \quad [a]_F(X) = \exp_H(aX).$$

It is therefore natural to wonder about conditions on power series $f \in (R \otimes_A K)[\![T]\!]$ ensuring that f is the logarithm of some formal group law. Hazewinkel found such a condition in his functional equation lemma.

Theorem 2.6.1 (Hazewinkel's Functional Equation Lemma)**.** *Let p be a prime and $q = p^e$. Given an inclusion of rings $B \subseteq L$, an ideal $\mathfrak{a} \subseteq B$ containing p, an endomorphism of rings $\sigma : L \to L$ and elements $s_1, s_2, \dots \in L$ subject to the conditions that*

$$\sigma(b) \equiv b^q \pmod{\mathfrak{a}} \text{ for all } b \in B \quad \text{and} \quad \sigma^r(s_i)\mathfrak{a} \subset B \text{ for all } r, s \geq 1.$$

Suppose now that $f \in L[\![T]\!]$ has $f'(0) \in L^\times$ and satisfies the functional equation condition

$$f(X) - \sum_{i=1}^{\infty} s_i(\sigma_*^i f)(X^{q^i}) \in B[\![X]\!].$$

Then we have

$$F(X, Y) = f^{-1}(f(X) + f(Y)) \in B[\![X, Y]\!],$$

where f^{-1} is the inverse power series as in Lemma 2.1.9. Also, if $g(Z) \in L[\![Z]\!]$ is another power series satisfying the same condition

$$g(Z) - \sum_{i=1}^{\infty} s_i(\sigma_*^i f)(Z^{q^i}) \in B[\![Z]\!],$$

then $f^{-1}(g(Z)) \in B[\![Z]\!]$. *Furthermore, if* $\alpha(T) \in B[\![T]\!]$ *and* $\beta(T) \in B[\![T]\!]$, *then*

$$\alpha(T) \equiv \beta(T) \pmod{\mathfrak{a}^r} \iff f(\alpha(T)) \equiv f(\beta(T)) \pmod{\mathfrak{a}^r} \tag{2.9}$$

Proof. A more general statement can be found in [Haz79, Section 2]. Proofs can be found in [Haz78, Sections 2 and 10]. □

Note that by construction, $F(X,Y)$ as defined above yields a (commutative) formal group law over B. Let B^σ denote the subring of elements in B fixed by σ. Then the second part of the Functional Equation Lemma implies that we even obtain formal B^σ-modules with $[b]_F(T) = f^{-1}(bf(T))$, as $bf(T)$ satisfies the same functional equation if $b \in B^\sigma$.
We now enter the situation where K is a local field with ring of integers $\mathcal{O}_K$ and uniformizer ϖ and use the Functional Equation Lemma to construct Lubin–Tate Formal Group Laws. A special role will play the power series

$$f(T) = \sum_{i=1}^{\infty} \frac{T^{q^{in}}}{\varpi^i} \in K[\![T]\!].$$

It satisfies the functional equation

$$f(T) = T + \frac{1}{\varpi} f(T^{q^n}),$$

which is a functional equation of the form above, with $B = \mathcal{O}_K$, $\mathfrak{a} = (\varpi)$, $L = K$, $s_1 = \varpi^{-1}$, $s_2 = s_3 = \cdots = 0$, $\sigma = \mathrm{id}_L$. Hence f arises as the logarithm of a formal $\mathcal{O}_K$-module law H over $\mathcal{O}_K$. The fact that $f^{-1}(X) = X - \frac{1}{\varpi}X^{q^n} + \ldots$ reveals $[\varpi]_H(T) \equiv \varpi T \bmod (T^2)$. Additionally, note that

$$f([\varpi]_H(T)) = \varpi f(T) = \varpi T + f(T^{q^n}) \equiv f(T^{q^n}) \pmod{\varpi}.$$

Hence, the equivalence in (2.9) implies that $[\varpi]_H(T) \equiv T^{q^n} \bmod \varpi$. So H is a Lubin–Tate formal $\mathcal{O}_K$-module law of height n, we call it the standard Lubin–Tate formal module law of height n.

Remark. The formal $\mathcal{O}_K$-module H is a member of the class of so called A-typical formal modules - formal A-modules F with logarithm of the form

$$\log_F(T) = \sum_{i=0}^{\infty} b_i X^{q^i}$$

for elements $b_0, b_1, \dots \in R \otimes_A K$ (cf. [Haz78, Definition 21.5.5 and Criterion 21.5.9]). If R is flat over A, every formal A-module over R is isomorphic to an A-typical one (cf. [Haz78, p. 21.5.6]). Most results about the standard $\mathcal{O}_K$-module H remain true for general $\mathcal{O}_K$-typical formal module laws.

We note the following.

Lemma 2.6.2. *Let* H *be the standard formal* $\mathcal{O}_E$*-module of height* n *over* $\mathcal{O}_{\breve{E}}$. *Let* $\zeta \in \breve{E}$ *be a* (q^n-1)*-th root of unity. Then* $[\zeta]_H(T) = \zeta T$ *is an automorphism of* H. *In particular,*

$\mathrm{End}(H)$ *naturally carries the structure of a* $\mathcal{O}_{E_n}$*-algebra, where* E_n *is the unramified extension of* E *with residue field* $\mathbb{F}_{q^n}$.

Proof. This is an immediate consequence of the equality $\zeta \log_H(T) = \log_H(\zeta T)$. We have

$$\zeta T = \exp_H(\log_H(\zeta T)) = \exp_H(\zeta \log_H(T)) = \exp_H([\zeta]_{\widehat{\mathbb{G}}_{a,\breve{E}}}(\log_H(T))).$$

As $\exp_H : \widehat{\mathbb{G}}_{a,\breve{E}} \to H \otimes_{\mathcal{O}_{\breve{E}}} \breve{E}$ is an isomorphism of formal modules, the claim follows. □

It will be convenient to make the terms in the exact sequence of Proposition 2.5.5 explicit for $\mathcal{F} = \mathrm{FG}(H)$. As H is of height $n > 0$, there is no non-trivial map $H \to \widehat{\mathbb{G}}_a$ and the sequence becomes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega(H) & \longrightarrow & \mathrm{RigExt}(H, \widehat{\mathbb{G}}_a) & \longrightarrow & \mathrm{Ext}(H, \widehat{\mathbb{G}}_a) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \left\{\begin{array}{c} g \in TK[\![T]\!] : \delta g = 0 \\ \text{and } g'(T) \in \mathcal{O}_K[\![T]\!] \end{array}\right\} & \longrightarrow & \mathrm{QLog}(H) & \xrightarrow{\delta} & \frac{\mathrm{SymCoc}^2(H, \widehat{\mathbb{G}}_a)}{\{\delta g | g \in T\mathcal{O}_K[\![T]\!]\}} & \longrightarrow & 0.
\end{array}$$

This allows us to formulate the following result.

Proposition 2.6.3. *The* R*-module* $\omega(H)$ *is free of rank* 1*, generated by* $f(T) = \log_H(T)$. $\mathrm{QLog}(H)$ *is free of rank n, generated by the classes of* $(f(T), \frac{1}{\varpi}f(T^q), \ldots, \frac{1}{\varpi}f(T^{q^{n-1}}))$. *Consequently, the short exact sequence above is given by*

$$0 \to \langle f(T) \rangle \to \left\langle f(T), \frac{1}{\varpi}f(T^q), \ldots, \frac{1}{\varpi}f(T^{q^{n-1}}) \right\rangle \xrightarrow{\delta} \left\langle \delta\left(\frac{1}{\varpi}f(T^q)\right), \ldots, \delta\left(\frac{1}{\varpi}f(T^{q^{n-1}})\right) \right\rangle \to 0.$$

Proof. A simple calculation shows that $\frac{1}{\varpi}f(T^{q^k})$ is a quasi-logarithm for $1 \leq k \leq n-1$. As $\delta f = 0$, we have $f(T) \in \mathrm{QLog}(F)$ as well. The claim is [GH94, Proposition 13.8] which is a special case of [ibid., Proposition 9.8]. □

2.7 The Universal Additive Extension

We follow [GH94, Section 11], and specialize to the situation where A is a complete discrete valuation ring with uniformizer ϖ and finite residue field of characteristic p and R is a local admissible A-algebra with residue field $\overline{\mathbb{F}}_q$.

Lemma 2.7.1. *Let* M *be a finite free module over* R. *Then there is a natural bijection, functorial in* M *and* $\mathcal{F}$

$$\mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a \otimes M) \cong \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a) \otimes_R M.$$

Proof. After choosing coordinates on $\mathcal{F}$, this follows directly from the description of Ext in terms of symmetric 2-cocycles, cf. Propositions B.1.5 and 2.5.2. □

Let $\mathcal{F}$ be a one-dimensional formal A-module over R. We put $\mathrm{M}(\mathcal{F}) := \mathrm{Hom}_R(\mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a), R)$, which is free of rank $n-1$, and write $\mathcal{V} = \widehat{\mathbb{G}}_a \otimes \mathrm{M}(\mathcal{F})$. Now, by the previous lemma,

$$\mathrm{Ext}(\mathcal{F}, \mathcal{V}) = \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a \otimes \mathrm{M}(\mathcal{F})) = \mathrm{End}_R(\mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a)).$$

Let $0 \to \mathcal{V} \to \mathcal{E} \to \mathcal{F} \to 0$ be the extension corresponding to the identity on the right. This class is unique up to unique isomorphism. Indeed, as R is a local ring we may choose formal module laws F and V giving rise to $\mathcal{F}$ and $\mathcal{V}$, and let E be the module law obtained from Lemma B.1.3. If $0 \to V \to E' \to F \to 0$ is another extension in this class, we have by construction a commutative square

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "i"] & F \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & V \arrow[r] & E' \arrow[r] & F \arrow[r] & 0, \end{tikzcd}$$

and by Proposition 2.5.2 we see that any other isomorphism i' making the diagram above commute differs from i by an element in $\mathrm{Hom}(F, V) = 0$.

Definition 2.7.2 (Universal Additive Extension)**.** The extension

$$0 \to \mathcal{V} \to \mathcal{E} \to \mathcal{F} \to 0$$

constructed above is called the universal additive extension of $\mathcal{F}$.

Proposition 2.7.3. *If N is a finite, free R-module, $\mathcal{G}' = \widehat{\mathbb{G}}_a \otimes N$ and*

$$0 \to \mathcal{G}' \to \mathcal{E}' \to F \to 0$$

is an extension of $\mathcal{F}$ by $\mathcal{G}'$, there are unique homomorphisms $i : \mathcal{E} \to \mathcal{E}'$ and $g' : \mathcal{V} \to \mathcal{G}'$ making the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{V} \arrow[r] \arrow[d, "g'"'] & \mathcal{E} \arrow[r] \arrow[d, "i"] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{G}' \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{F} \arrow[r] & 0 \end{tikzcd}$$

commute. In particular, we have $\mathcal{E}' = g'_*\mathcal{E}$.

Proof. As $\mathcal{V}$ and $\mathcal{G}'$ are additive, we have

$$\mathrm{Hom}(\mathcal{V}, \mathcal{G}') = \mathrm{Hom}_R(\mathrm{M}(\mathcal{F}), N) = \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a) \otimes N = \mathrm{Ext}(\mathcal{F}, \mathcal{G}').$$

This yields g'. Again, i is unique as by observations similar to Proposition 2.5.2, the difference of two morphisms $i, i' : \mathcal{E} \to \mathcal{E}'$ is given a morphism $\mathcal{F} \to \mathcal{G}'$, which has to be trivial. □

Lemma 2.7.4. *There is a natural isomorphism* $\mathrm{Lie}(\mathcal{E}) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a), R) = \mathrm{D}(\mathcal{F})$.

Proof. We show the equivalent statement $\omega(\mathcal{E}) = \mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a)$. Let $(\mathcal{E}', \omega_{\mathcal{E}'}) \in \mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a)$. Then by universality of $\mathcal{E}$, we obtain a unique homomorphism $i : \mathcal{E} \to \mathcal{E}'$. This yields a homomorphism of R-modules $\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a) \to \omega(\mathcal{E})$, sending a pair $(\mathcal{E}', \omega_{\mathcal{E}'})$ to $i^*\omega_{\mathcal{E}'}$. This morphism fits into the following commutative diagram, where the top row is the short exact sequence from Proposition 2.5.5 and the bottom row is the dual short exact sequence of $0 \to \mathrm{Lie}(\mathcal{V}) \to \mathrm{Lie}(\mathcal{E}) \to \mathrm{Lie}(\mathcal{F}) \to 0$.

$$\begin{tikzcd}
0 \arrow[r] & \omega(\mathcal{F}) \arrow[r] \arrow[d, equal] & \mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a) \arrow[r] \arrow[d] & \mathrm{Ext}(\mathcal{F}, \widehat{\mathbb{G}}_a) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \omega(\mathcal{F}) \arrow[r] & \omega(\mathcal{E}) \arrow[r] & \omega(\mathcal{V}) \arrow[r] & 0
\end{tikzcd}$$

Thereby, $\mathrm{RigExt}(\mathcal{F}, \widehat{\mathbb{G}}_a) \to \omega(\mathcal{E})$ is a natural isomorphism. $\square$

3 Local Class Field Theory following Lubin–Tate

Let K be a local field with residue field k, put $q = \#k$, and denote by $\nu_K : K \to \mathbb{Z} \cup \{\infty\}$ the valuation of K, normalized such that $\nu_K(\varpi) = 1$ for a uniformizer ϖ of K. The aim of this subsection is to describe the maximal abelian extension of a local field K.

The Local Kronecker-Weber theorem gives an explicit description of the abelianization of the absolute Galois group of K only in terms of K:

Theorem 3.0.1 (Local Kronecker-Weber)**.** *There is an isomorphism (canonical up to choice of a uniformizer $\varpi \in K$)*

$$\mathrm{Gal}(\overline{K}/K)^{\mathrm{ab}} \cong \mathrm{Gal}(K^{\mathrm{ab}}/K) \cong \mathcal{O}_K^\times \times \widehat{\mathbb{Z}}.$$

Here, K^{ab} denote the maximal abelian extension of K, which can (after choosing an algebraic closure of K) be described as $\overline{K}^{[G_K, G_K]}$.

The extension K^{ab} consists of two parts, we have $K^{\mathrm{ab}} = K^{\mathrm{rm}} \cdot K^{\mathrm{nr}}$. The field K^{nr}, the maximal unramified extension of K, has relatively simple structure. Describing the field K^{rm} (or rather, it's completion) is the hard part and it is here where we apply the theory of formal modules.

The valuation ν_K extends uniquely to $\overline{K}$, yielding a ϖ-adic norm on $\overline{K}$. Let C denote the completion with respect to this norm. An application of Krasner's Lemma implies that $\mathrm{Gal}(C/K) \cong \mathrm{Gal}(\overline{K}/K) =: G_K$. One readily checks that any $\sigma \in G_K$ yields a continuous automorphism $\mathcal{O}_C \to \mathcal{O}_C$, and we obtain a short exact sequence Ref

$$0 \to I_K \to G_K \to \mathrm{Gal}(\overline{k}/k) \to 0.$$

The subgroup $I_K \subset G_K$ is called the inertia subgroup of K, and we write $\breve{K}$ for the subfield of C fixed by I_K. In particular we have $\mathrm{Gal}(\breve{K}/K) \cong \mathrm{Gal}(\overline{k}/k)$. One readily confirms that $\breve{K}$ is complete with respect to the norm induced by K.

As the Galois group of any finite extension of k is cyclic, we find that $\mathrm{Gal}(\breve{K}/K)$ is abelian. In fact, it is isomorphic to $\widehat{\mathbb{Z}} = \lim_n(\mathbb{Z}/n\mathbb{Z})$. Hence K_∞ decomposes as $\breve{K} \cdot K_\varpi$ for some abelian, complete extension K_ϖ/K such that $K_\varpi \cap \breve{K} = K$. Now K_ϖ is the completion of

K^{rm}. Observe that

$$\mathrm{Gal}(K_\infty/K) \cong \mathrm{Gal}(K_\varpi/K) \times \mathrm{Gal}(\breve{K}/K) \cong \mathrm{Gal}(K_\varpi/K) \times \widehat{\mathbb{Z}},$$

so Theorem 3.0.1, the local Kronecker-Weber Theorem, is equivalent to showing that the Galois group of K_ϖ over K is isomorphic to $\mathcal{O}_K^\times$.

4 Non-Abelian Lubin-Tate Theory: An Overview

In the preceeding chapter we used formal $\mathcal{O}_F$-modules to understand the maximial abelian extension of a local field F. The hope of non-Abelian Lubin-Tate theory is to gain insight about the Abelian extensions of F by considering certain moduli spaces of formal $\mathcal{O}_F$-modules. More precisely, attached to a formal $\mathcal{O}_F$-module H_0 over $\overline{\mathbb{F}}_q$ (determined up to isomorphism by its height n), we attach a system of rigid spaces $\{M_K\}_{K \subset \mathrm{GL}_n(\mathcal{O}_F)}$, the so called Lubin-Tate Tower. For $l \neq p$, the system of l-adic compactly supported cohomology groups $\{H_c^i(M_K, \overline{\mathbb{Q}}_l)\}_K$ admits commuting actions by $\mathrm{GL}_n(F)$, W_F and $D^\times$, where the latter denotes the units of the central divison algebra $D = \mathrm{End}_{(\mathcal{O}_K\text{-FM}/\overline{\mathbb{F}}_q)}(H_0) \otimes \mathbb{Q}$. This yields a correspondence of representations of the respective groups, and Harris and Taylor showed in [HT01] that the cohomology of middle degree induces (a version of) the Local Langlands Correspondence for GL_n. Our goal is an explicit description of (a part of) this correspondence, and we obtain such a description by understanding (a part of) the Lubin-Tate tower explicitely.

4.1 Lubin–Tate Deformation Spaces

The aim of this section is to constuct, associated to a formal $\mathcal{O}_K$-module $H_0 \in (A\text{-FM}/k)$ and an integer m, a certain moduli problem $\mathcal{M}_{H_0,m}$, parametrizing "deformations of H_0 with Drinfeld level ϖ^m-structure". By results of Drinfeld [Dri74], these moduli problems turn out to be representable by formal schemes.

4.1.1 The Tower of Deformation Spaces

We mostly follow [Str08, Chapter 2] in the following exposition. Let $\mathcal{C}$ denote the category of complete, local, Noetherian $\mathcal{O}_{\breve{F}}$-algebras with residue field $\overline{\mathbb{F}}_q$. Let H_0 be a formal $\mathcal{O}_F$-module law over k and let R be an element of $\mathcal{C}$.

Definition 4.1.1 (Deformation)**.** A deformation of H_0 to R is a pair (H, ι) where H is a formal $\mathcal{O}_K$-module over R and ι is a quasi-isogeny

$$\iota : H_0 \dashrightarrow H \otimes_R \overline{\mathbb{F}}_q.$$

Two deformations (H, ι) and (H', ι') are isomorphic if there is an isomorphism $\tau : H \to H'$ with $\iota' \circ \tau = \iota$.

The Lubin-Tate space without level structure is the moduli space of such deformations. More precisely, we define it as the functor

$$\mathcal{M}_0 : \mathcal{C} \to (\mathrm{Set}), \quad R \mapsto \{\text{deformations } (H, \iota) \text{ of } H_0\}/ \simeq .$$

Note that we have a stratification

$$\mathcal{M}_0 = \coprod_{j \in \mathbb{Z}} \mathcal{M}_0^{(j)},$$

where $\mathcal{M}_0^{(j)}$ parametrizes deformations (H, ι) with $\mathrm{ht}(\iota) = j$.

Theorem 4.1.2 (Representability of $\mathcal{M}_0$)**.** *The functor $\mathcal{M}_0^{(0)}$ is representable by a ring $A_0 \in \mathcal{C}$, non-canonically isomorphic to*

$$\mathcal{O}_{\breve{F}}[\![u_1, \dots, u_{n-1}]\!] \in \mathcal{C}.$$

In particular, there is a universal deformation $(F^{\mathrm{univ}}, \iota^{\mathrm{univ}})$, with $F^{\mathrm{univ}} \in (\mathcal{O}_{\breve{F}}\text{-FM}/A_0)$.

Proof. This statement is due to Lubin–Tate, cf. [LT66]. They show representability of a slightly different moduli functor (classifying certain $\star$-deformations of a formal group law $H_0 \in (\mathcal{O}_F\text{-FML}/\overline{\mathbb{F}}_q))$). As quasi-isogenies of height 0 are isomorphisms, their result is equivalent to the one above. □

We next introduce variants of this moduli problem with a certain level structure.

Definition 4.1.3 (Drinfeld level ϖ^m-structure)**.** Let $H \in (\mathcal{O}_F\text{-FM}/R)$ be a ϖ-divisible formal $\mathcal{O}_F$-module of height $n > 0$ and let m be a non-negative integer. A Drinfeld level ϖ^m-structure on H is a morphism of $\mathcal{O}_F$-module objects

$$\phi : \underline{(\varpi^{-m}\mathcal{O}_F/\mathcal{O}_F)^n_{\mathrm{Spf}\, R}} \to H$$

in the category of formal schemes over $\mathrm{Spf}\, R$, such that after choosing a coordinate $H \cong \mathrm{Spf}\, R[\![T]\!]$, the power series $[\varpi]_H(T) \in R[\![T]\!]$ satisfies the divisibility constraint

$$\prod_{x \in (\varpi^{-1}\mathcal{O}_F/\mathcal{O}_F)^n} (T - \phi(x)) \;\Big|\; [\varpi]_H(T).$$

Definition 4.1.4 (Lubin–Tate Deformation Space with Level Structure)**.** Let $\mathcal{M}_m : \mathcal{C} \to (\mathrm{Set})$ be the functor assigning to $R \in \mathcal{C}$ the set

$$\mathcal{M}_m(R) := \{(H, \iota, \phi) \mid (H, \iota) \in \mathcal{M}_0(R) \text{ and } \phi \text{ a Drinfeld level } \varpi^m\text{-structure on } H\}/ \simeq .$$

By results of Drinfeld, the functor $\mathcal{M}_m$ is representable.

Theorem 4.1.5 (Representability of the Lubin–Tate Deformation Space with Level Structure)**.** *The functor $\mathcal{M}_m^{(0)}$ is representable by a regular local ring $A_m \in \mathcal{C}$ of dimension $m - 1$.*

Proof. This is [Dri74, Proposition 4.3]. □

Note that for non-negative integers $m' \leq m$, we have natural morphisms $\mathcal{M}_m \to \mathcal{M}_{m'}$ by restricting the level structure. We obtain a tower $\{\mathcal{M}_m\}_{m\geq 0}$ of Lubin–Tate deformation spaces.

4.1.2 Group Actions on the Tower of Lubin–Tate Deformation Spaces

We describe actions of $\mathrm{GL}_n(F) \times D^\times$ on the tower $\{\mathcal{M}_m\}_{m\geq 0}$. More precisely, given an element $d \in D^\times$ and an element $g \in \mathrm{GL}_n(F)$, we construct, for sufficiently large $m \geq 0$, morphisms

$$d_m \colon \mathcal{M}_m^{(j)} \to \mathcal{M}_m^{(j')} \quad \text{and} \quad g_{m,m''} \colon \mathcal{M}_m^{(j)} \to \mathcal{M}_{m''}^{(j'')},$$

where $j' = j + \mathrm{val}_\varpi(\mathrm{Nrd}(d))$, $j'' = j - \mathrm{val}_\varpi(\det g)$ and $m'' = m - d$ is an integer differeing from m by an integer depending on g.

The action of $D^\times$ is easy to describe. Given $R \in \mathcal{C}$ and $d \in D^\times$, we put

$$(H, \iota, \phi).d = (H, \iota \circ d, \phi).$$

The group $\mathrm{GL}_n(F)$ acts in a less simple matter. The idea is to have $\mathrm{GL}_n(F)$ act on the level structure of triples $(H, \iota, \phi) \in \mathcal{M}_m(R)$. Akin to the action of $D^\times$, we would like to define the action as $(H, \iota, \phi).g = (H, \iota, \phi \circ g)$, but this only makes sense if $g \in \mathrm{GL}_n(\mathcal{O}_F)$. To nonetheless define an 'action' of $\mathrm{GL}_n(F)$, we allow ourselves to also change the underlying formal group. We need the following notion of quotients of formal $\mathcal{O}_F$-modules.

Definition 4.1.6 (Quotient of Formal Module by Finite Subgroup)**.** Let H be a formal $\mathcal{O}_F$-module law and let $G \subset R^\infty$ be a finite sub $\mathcal{O}_F$-module, where we equip R^∞ with the $\mathcal{O}_F$-module structure coming from H. Then, we define the quotient H/G as the following formal $\mathcal{O}_F$-module law. [How is this supposed to work please?? It should be something like

$$(H/G)(X, Y) = g(H(g^{-1}(X), g^{-1}(Y)), \quad \text{where} \quad g(T) = \prod_{a\in F} (T - a).$$

But this doesn't make sense.]

Lemma 4.1.7. *The quotient as above is a formal A-module. If $\#P = q^n$, the induced morphism $H \to H/G$ of formal module laws over R is of height n.*

Proof. □

Lemma 4.1.8. *Let H be a ϖ-divisible formal A-module over $R \in \mathcal{C}$ and let ϕ be a Drinfeld ϖ^m-level structure on H. Suppose that $P \subset (\varpi^{-m}\mathcal{O}_F/\mathcal{O}_F)^n$ is a submodule and that there is an injection*

$$\left(\varpi^{-m'}\mathcal{O}_F/\mathcal{O}_F\right)^n \to \frac{(\varpi^{-m}\mathcal{O}_F/\mathcal{O}_F)^n}{P}.$$

Then, the morphism

$$\phi' : \left(\varpi^{-m'}\mathcal{O}_F/\mathcal{O}_F\right)^n \to H/\phi(P)$$

is a Drinfeld ϖ^m-level structure.

Proof. This is [Dri74, Proposition 4.4]. $\square$

First assume that $g \in \mathrm{GL}_n(F)$ is such that $g^{-1} \in \mathrm{Mat}_{n\times n}(\mathcal{O}_F)$ and $g \in \varpi^{-d}\,\mathrm{Mat}_{n\times n}(\mathcal{O}_F)$ for some non-negative integer d. In this case, we construct for all integers $m \geq d$ a natural transformation

$$g_{m,m-d} : \mathcal{M}_m \to \mathcal{M}_{m-d}.$$

Note that $g\mathcal{O}_F^n \subset \varpi^{-d}\mathcal{O}_F^n$, and that multiplication with g yields an injection

$$(\varpi^{m-d}\mathcal{O}_F/\mathcal{O}_F)^n \xrightarrow{g} (\varpi^{-m}\mathcal{O}_F^n/g\mathcal{O}_F^n) = \frac{(\varpi^{-m}\mathcal{O}_F^n/\mathcal{O}_F^n)}{(g\mathcal{O}_F^n/\mathcal{O}_F^n)}.$$

Now, given a tuple $(H,\iota,\phi) \in \mathcal{M}_m(R)$, we put

$$(H,\iota,\phi).g = (H',\iota',\phi'),$$

where

$$H' = H/\phi(g\mathcal{O}_F^n/\mathcal{O}_F^n)$$

is a the quotient of H as in Definition 4.1.6,

$$\iota' \colon H_0 \to H \otimes \overline{\mathbb{F}}_q \to H' \otimes \overline{\mathbb{F}}_q$$

is the corresponding quasi-isogeny of height $(\mathrm{ht}(\iota) - \mathrm{val}_\varpi(\det g))$, and

$$\phi' \colon (\varpi^{m-d}\mathcal{O}_F/\mathcal{O}_F)^n \to H'$$

is the Drinfeld ϖ^{m-d}-level structure obtained by Lemma 4.1.8. For varying choices of d, this gives a system of maps compatible with the transition functions $\mathcal{M}_m \to \mathcal{M}_{m'}$. Indeed, given integers $m \geq d' \geq d$ with d as above, the triangle

$$\begin{tikzcd} \mathcal{M}_m \arrow[r, "g_{m,m-d}"] \arrow[dr, "g_{m,m-d'}"'] & \mathcal{M}_{m-d} \arrow[d] \\ & \mathcal{M}_{m-d'} \end{tikzcd}$$

commutes.

If $g \in \mathrm{GL}_n(F)$ is an arbitrary element, we may choose an integer r such that $(\varpi^{-r}g)^{-1} \in \mathrm{Mat}_{n\times n}(\mathcal{O}_F)$. We now pick $d > 0$ in a way that $\varpi^{-r}g \in \varpi^{-d}\,\mathrm{Mat}_{n\times n}(\mathcal{O}_F)$. Now, for $m \geq d$, we obtain natural transformations

$$g_{m,m-d}\colon \mathcal{M}_m \to \mathcal{M}_{m-d}, \quad (H,\iota,\phi).g = (H, \iota \circ \varpi^{-r}, \phi).(\varpi^{-r}g).$$

By the same reason as above, this yields, for varying choices of d, a compatible system of natural transformations. This construction is independent of the choice of r, as $(H,\iota,\phi).(\varpi \cdot \mathrm{id}) \simeq (H, \iota \circ [\varpi^{-1}], \phi)$. Furthermore, given $g, g' \in \mathrm{GL}_n(F)$ with suitable choices of integers

d, d', we obtain a commutative triangle

$$\begin{tikzcd} \mathcal{M}_m \arrow[r, "g_{m,m-d}"] \arrow[dr, "(gg')_{m,m-d-d'}"'] & \mathcal{M}_{m-d} \arrow[d, "g'_{m-d,m-d-d'}"] \\ & \mathcal{M}_{m-d-d'} \end{tikzcd}$$

This finishes the construction of the $\mathrm{GL}_n(F)$-action.
[Further observations: For $m' \leq m$, we have $A_{m'} = A_m^{K_{m'}}$. We may, for $K \subset \mathrm{GL}_n(\mathcal{O}_F)$ compact open and contained in K_m, define $A_K = A_m^K$. This gives refined tower $\{\mathcal{M}_K\}_K$.]

4.1.3 The Weil Descent Datum on the Deformation Space

We now fix an embedding $\breve{E} \hookrightarrow \mathbb{C}_p$ and consider for integers $m \geq 0$ the base change $\mathcal{M}_{m,\mathcal{O}_{\mathbb{C}_p}} = \mathcal{M}\widehat{\otimes}_{\mathcal{O}_{\breve{E}}}\mathcal{O}_{\mathbb{C}_p}$. We recall the notion of Weil descent data and make use of this notion to describe an action of the Weil group W_E on $\mathcal{M}_{m,\mathcal{O}_{\mathbb{C}_p}}$.

Let $\Phi \in \mathrm{Gal}(\breve{E}/E)$ be the automorphism corresponding to the q-th power Frobenius automorphism of the residue field of E. Given an $\mathcal{O}_{\breve{E}}$-algebra $\mathcal{O}_{\breve{E}} \xrightarrow{i} R$, we write Φ^*R for the $\mathcal{O}_{\breve{E}}$-algebra with structure morphism $\mathcal{O}_{\breve{E}} \xrightarrow{\Phi} \mathcal{O}_{\breve{E}} \xrightarrow{i} R$. The identity on R yields a morphism $R \to \Phi^*R$, which preserves only the $\mathcal{O}_E$-algebra structure. Note that Φ^*R admits the following equivalent descriptions as $\mathcal{O}_{\breve{E}}$-algebra:

$$\begin{tikzcd} \Phi^*R \arrow[r, "r\mapsto r\otimes 1", "\sim"'] & R\widehat{\otimes}_{\mathcal{O}_{\breve{E},\Phi^{-1}}}\mathcal{O}_{\breve{E}} \arrow[r, "(r,x)\mapsto r\otimes\Phi(x)", "\sim"'] & R\widehat{\otimes}_{\mathcal{O}_{\breve{E}},\mathrm{id}}\Phi^*\mathcal{O}_{\breve{E}} \\ & \mathcal{O}_{\breve{E}} \arrow[ul, "i\circ\Phi"] \arrow[u, "1\otimes\mathrm{id}"'] \arrow[ur, "1\otimes\Phi"'] & \end{tikzcd} \tag{4.1}$$

More generally, given any functor $\mathcal{G}\colon \mathcal{C} \to (\mathrm{Set})$, we denote by $\Phi^*\mathcal{G}$ the fiber product $\mathcal{G} \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathrm{Spf}(\Phi^*\mathcal{O}_{\breve{E}})$, interpreted as a functor $\mathcal{C} \to (\mathrm{Set})$ under the equality of categories $(\Phi^*\mathcal{O}_{\breve{E}}\text{-Adm}) = (\mathcal{O}_{\breve{E}}\text{-Adm})$. In this situation, we have the notion of Weil descent data (cf. [RZ96, Definition 3.45]).

Definition 4.1.9 (Weil Descent Datum)**.** A Weil Descent Datum for $\mathcal{G}$ is an isomorphism

$$\alpha\colon \mathcal{G} \xrightarrow{\sim} \Phi^*\mathcal{G}$$

of functors $\mathcal{C} \to (\mathrm{Set})$.

To make this definition a bit more tangible, we give the following example. [Is this example too boring?]

Example. 1. Suppose that $p \neq 2$ and let $\ell \neq p$ be a prime number. Write $\zeta_\ell \in \mathcal{O}_{\breve{\mathbb{Q}}_p}$ for an ℓ-th root of unity. Let $\mathcal{P}_{\zeta_\ell}$ be the functor parametrizing square roots of ζ_ℓ. One readily sees that $\mathcal{P}_{\zeta_\ell}$ is representable by [In what category?]$\mathrm{Spf}(R)$, where R denotes the the $\mathcal{O}_{\breve{\mathbb{Q}}_p}$-algebra $R = \frac{\mathcal{O}_{\breve{\mathbb{Q}}_p}[X]}{(X^2-\zeta_\ell)}$.

Now, $\Phi^*\mathcal{P}_{\zeta_\ell}$ is readily seen to be the functor parametrizing square roots of $\Phi^{-1}(\zeta_\ell) = \zeta_\ell^{p^{-1}}$, where p^{-1} denotes the inverse residue class of p mod ℓ. Hence, a Weil descent datum for $\mathcal{P}_{\zeta_\ell}$ is equvialent to a $\mathcal{O}_{\breve{\mathbb{Q}}_p}$-linear isomorphism of rings

$$\alpha\colon \Phi^*R = \frac{\mathcal{O}_{\breve{\mathbb{Q}}_p}[X]}{(X^2 - \zeta_\ell^{p^{-1}})} \to \frac{\mathcal{O}_{\breve{\mathbb{Q}}_p}[X]}{(X^2 - \zeta_\ell)} = R,$$

One easily finds that any such isomorphism must send X to aX, where a is a square root of $\zeta_\ell^{p^{-1}-1}$, and conversely any such square root yields a Weil descent datum. In particular, unlike the Lubin–Tate deformation space, there is no canonical Weil descent datum for $\mathcal{P}_{\zeta_\ell}$.

2. More generally, if $\mathcal{G}$ is representable by the formal spectrum of a ring formally of finite type over $\mathcal{O}_{\breve{E}}$, a Weil descent datum for $\mathcal{G}$ is an isomorphism of topological R-algebras

$$\Phi^*R \cong \frac{\mathcal{O}_{\breve{E}}[\![X_1,\dots,X_n]\!]}{(f_1^{\Phi^{-1}},\dots,f_n^{\Phi^{-1}})} \to \frac{\mathcal{O}_{\breve{E}}[\![X_1,\dots,X_n]\!]}{(f_1,\dots,f_n)} = R,$$

where $f_i^{\Phi^{-1}} \in \mathcal{O}_{\breve{E}}[\![X_1,\dots,X_n]\!]$ denotes the power series obtained by applying Φ^{-1} to the coefficients of f_i. This follows quickly from the descriptions of Φ^*R in (4.1).

An element $w \in \mathrm{W}_E$ with $w|_{\breve{E}} = \Phi^m$ for $m \in \mathbb{Z}$, induces a morphism

$$\mathcal{G} \times_{\mathrm{Spf}\,\mathcal{O}_{\breve{E}}} \mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p}) \xrightarrow{(1,w)} \Phi^{m,*}\mathcal{G} \times_{\mathrm{Spf}\,\mathcal{O}_{\breve{E}}} \mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p}).$$

If $\mathcal{G}$ admits a Weil descent datum, we obtain a commuatative square of isomorphisms of functors $(\mathrm{FSch}/\mathcal{O}_E)^{\mathrm{op}} \to (\mathrm{Set})$

$$\begin{array}{ccc} \mathcal{G} \times_{\mathcal{O}_{\breve{E}}} \mathcal{O}_{\mathbb{C}_p} & \xrightarrow{(\Phi^{-m,*}\alpha^{-m},1)} & \Phi^{-m,*}\mathcal{G} \times_{\mathcal{O}_{\breve{E}}} \mathcal{O}_{\mathbb{C}_p} \\ {\scriptstyle (1,\mathrm{Spf}(w))}\downarrow & & \downarrow{\scriptstyle (1,\mathrm{Spf}(w))} \\ \Phi^{m,*}\mathcal{G} \times_{\mathcal{O}_{\breve{E}}} \mathcal{O}_{\mathbb{C}_p} & \xrightarrow{(\alpha^{-m},1)} & \mathcal{G} \times_{\mathcal{O}_{\breve{E}}} \mathcal{O}_{\mathbb{C}_p}. \end{array} \tag{4.2}$$

Here, α^m denotes the isomorphism

$$\alpha^m : \mathcal{G} \xrightarrow{\alpha} \Phi^*\mathcal{G} \xrightarrow{\Phi^*\alpha} \Phi^{2,*}\mathcal{G} \xrightarrow{\Phi^{2,*}\alpha} \dots \xrightarrow{\Phi^{m-1,*}\alpha} \Phi^{m,*}\mathcal{G}$$

obtained by iterating the isomorphism α, and α^{-m} denotes the inverse of α^m. We write δ_w for the automorphism of $\mathcal{G} \times_{\mathcal{O}_{\breve{E}}} \mathcal{O}_{\mathbb{C}_p}$ described in the square (4.2). For $w_1, w_2 \in \mathrm{W}_E$, we have $\delta_{w_1 \circ w_2} = \delta_{w_1} \circ \delta_{w_2}$, so the assignment $w \mapsto \delta_{w^{-1}}$ defines a right-action of W_E on $\mathcal{G} \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathcal{O}_{\mathbb{C}_p}$.

Example. Again, suppose that the functor $\mathcal{G}$ is representable by the formal spectrum of some $\mathcal{O}_{\breve{E}}$ algebra R, and assume for simplicity that R admits a finite presentation

$$R = \frac{\mathcal{O}_{\breve{E}}[\![X_1,\dots,X_n]\!]}{(f_1,\dots,f_n)}.$$

Furthermore, assume that $\mathcal{G}$ admits a Weil descent datum α. Then by the example above, the inverse of $\Phi^{-m,*}\alpha^{-m}$ yields (upon choosing an embedding $\breve{E} \to \mathbb{C}_p$) an isomorphism of $\mathcal{O}_{\mathbb{C}_p}$-algebras

$$\rho^m : \frac{\mathcal{O}_{\mathbb{C}_p}[\![X_1,\dots,X_n]\!]}{(f_1^{\Phi^m},\dots,f_n^{\Phi^m})} \to \frac{\mathcal{O}_{\mathbb{C}_p}[\![X_1,\dots,X_n]\!]}{(f_1,\dots,f_n)},$$

uniquely determined by the images $\rho(X_i)$. Let $w \in \mathrm{W}_E$ be an element satisfying $w|_{\breve{E}} = \Phi^m$. Then δ_w corresponds on the level of global sections to an isomorphism

$$R\widehat{\otimes}_{\mathcal{O}_{\breve{E}}}\mathcal{O}_{\mathbb{C}_p} = \frac{\mathcal{O}_{\mathbb{C}_p}[\![X_1,\dots,X_n]\!]}{(f_1,\dots,f_n)} \xrightarrow{w} \frac{\mathcal{O}_{\mathbb{C}_p}[\![X_1,\dots,X_n]\!]}{(f_1^{\Phi^m},\dots,f_n^{\Phi^m})} \xrightarrow{\rho^m} \frac{\mathcal{O}_{\mathbb{C}_p}[\![X_1,\dots,X_n]\!]}{(f_1,\dots,f_n)} = R\widehat{\otimes}_{\mathcal{O}_{\breve{E}}}\mathcal{O}_{\mathbb{C}_p}.$$

That is, $\delta_w(a) = w(a)$ for $a \in \mathcal{O}_{\mathbb{C}_p}$ and $\delta_w(X_i) = \rho^m(X_i)$ for $i = 1,\dots,n$. The commutativity of the square (4.2) reflects the fact that $w(\rho^m(X_i)) = \Phi^m(\rho^m(X_i))$. Also note that δ_w does not respect the $\mathcal{O}_{\mathbb{C}_p}$- or $\mathcal{O}_{\breve{E}}$-structure, it is only an isomorphism of $\mathcal{O}_E$-algebras.

We define a Weil descent datum α on the functor $\mathcal{M}_m$ as follows. Given $R \in \mathcal{C}$ and $(H,\iota,\phi) \in \mathcal{M}_m(R)$, we put

$$\alpha(R)\,(H,\iota,\phi) = (H',\iota',\phi'),$$

where

- the formal group H' is equal to H, however interpreted as a formal module over Φ^*R. That is, $H' = H \otimes_R \Phi^*R = \Phi^*H$.
- the quasi-isogeny ι' is defined as

  $$H_0 \overset{\mathrm{Frob}_q^{-1}}{\dashrightarrow} H_0^{(-q)} = H_0 \otimes_{\overline{\mathbb{F}}_q,\mathrm{Frob}^{-1}} \overline{\mathbb{F}}_q \dashrightarrow H \otimes_{R,\Phi^{-1}} (R/\mathfrak{m}_R) = (\Phi^*H) \otimes_R \overline{\mathbb{F}}_q.$$

  Here Frob_q^{-1} denotes the inverse quasi-isogeny of the relative Frobenius morphism $\mathrm{Frob}_q : H^{(-q)} \to H$. In particular, $\mathrm{ht}(\iota') = \mathrm{ht}(\iota) - 1$.
- the level structure ϕ' is

  $$\underline{\left(\varpi^{-m}\mathcal{O}_E/\mathcal{O}_E\right)^m_{\mathrm{Spf}\,R}} \cong \Phi^*\underline{\left(\varpi^{-m}\mathcal{O}_E/\mathcal{O}_E\right)^m_{\mathrm{Spf}\,R}} \xrightarrow{\Phi^*\phi} \Phi^*H.$$

As explained above, this yields a right action of W_E on $\mathcal{M}_{m,\mathcal{O}_{\mathbb{C}_p}}$. Given $w \in \mathrm{W}_E$ such that $w|_{\breve{E}} = \Phi^m$ for $m \in \mathbb{Z}$, we find that δ_w restricts to an isomorphism $\mathcal{M}^{(j)}_{m,\mathcal{O}_{\mathbb{C}_p}} \to \mathcal{M}^{(j+m)}_{m,\mathcal{O}_{\mathbb{C}_p}}$.

Example. Suppose that $E'/\breve{E}$ is a finite extension. Then $R = \mathcal{O}_{E'} \in \mathcal{C}$, and we may describe the action of W_E on $\mathcal{M}_{m,\mathcal{O}_{\mathbb{C}_p}}(R)$ as follows. Given (an equivalence class of) a triple $(\mathcal{H},\iota,\phi) \in \mathcal{M}_m(R)$ such that $\mathcal{H} = \mathrm{FG}(H)$, and an element $w \in \mathrm{W}_E$ such that $w|_{\breve{E}} = \Phi^m$ for $m \in \mathbb{Z}$, we find

$$\delta_w(H,\iota,\phi) = (\mathrm{FG}(H^w), \iota^w \circ \mathrm{Frob}_q^m, \phi^w),$$

where H^w is the formal group law over E' obtained by applying $w|_{E'}$ to H coefficient-wise. Likewise,

$$\iota^w : H_0^{(q^m)} \dashrightarrow H^w \otimes \overline{\mathbb{F}}_q \quad \text{and} \quad \phi^w : \underline{\left(\varpi^{-m}\mathcal{O}_E/\mathcal{O}_E\right)^m_{\mathrm{Spf}\,R}} \to H^w$$

denote the corresponding "twisted" quasi-isogeny and level structure.

4.2 The Local Langlands Correspondence for the General Linear Group

The aim of this section is to review some of the results in [HT01]. Recall from the previous section the tower of deformation spaces $\{\mathcal{M}_K\}_{K\subset \mathrm{GL}_n(\mathcal{O}_F)}$. Let $M_K \in (\mathrm{Rig}/\breve{F})$ denote the (rigid) generic fiber of $\mathcal{M}_K \in (\mathrm{FSch}/\mathcal{O}_{\breve{F}})$. Fix an algebraic closure $\mathbb{C}_p$ of F and a prime number $\ell \neq p$. This section is concerned with the representation-theoretic aspects of the vector space

reference (Berthelot? Reference)

$$H_{\mathrm{LT}} := \lim_{K} H_c^{n-1}(M_K \otimes_{\breve{F}} \mathbb{C}_p, \overline{\mathbb{Q}}_l).$$

By functoriality of the generic fiber functor, the action of $D^\times \times \mathrm{GL}_n(F)$ on the tower $\{\mathcal{M}_K\}_{K\subset \mathrm{GL}_n(\mathcal{O}_F)}$ from the right yields an action of $D^\times \times \mathrm{GL}_n(F)$ on H_{LT} from the left.
For $K \subset \mathrm{GL}_n(\mathcal{O}_F)$, write $M_{K,\varpi^{\mathbb{Z}}}$ for the quotient of M_K by the action of the subgroup $\varpi^{\mathbb{Z}} \subset D^\times$. Writing $\mathcal{M}_m = \coprod_{\delta\in\mathbb{Z}} \mathcal{M}_m^{(\delta)}$ induces $M_K = \coprod_{\delta\in\mathbb{Z}} M_K^{(\delta)}$, and the action of ϖ induces for any $\delta \in \mathbb{Z}$ an isomorphism $M_K^{(\delta)} \cong M_K^{(\delta+n)}$. Hence, $M_{K,\varpi^{\mathbb{Z}}}$ is isomorphic to $\coprod_{0\le\delta\le n-1} M_K^{(\delta)}$.
Let $l \neq p$ be a prime number and fix an isomorphism $\overline{\mathbb{Q}}_l \cong \mathbb{C}$.

Definition 4.2.1 (Cohomology of the Lubin–Tate tower)**.** We write $H_{\mathrm{LT}} = \lim_K H_c^{n-1}(M_{K,\varpi^{\mathbb{Z}}} \otimes_{\breve{F}} C, \overline{\mathbb{Q}}_l)$.

Theorem 4.2.2 (Non-Abelian Lubin–Tate theory)**.** *Let π be an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$ whose central character is trivial on $\varpi^{\mathbb{Z}}$. We write $\mathrm{rec}_F(\pi)$ for the irreducible smooth representation of W_F corresponding to π undet the local Langlands correspondence, and $\mathrm{JL}(\pi)$ for the irreducible smooth representation of $D^\times$ corresponding to π under the local Jacquet-Langlands correspondence. Then we have*

$$H_{\mathrm{LT}}[\pi^\vee] = \pi^\vee \boxtimes \mathrm{JL}(\pi) \boxtimes \mathrm{rec}_F(\pi)(\tfrac{1-n}{2})$$

as representations of $\mathrm{GL}_n(F) \times D^\times \times W_F$.

Proof. □

We set

$$H_{\mathrm{LT}} := \lim_{K} H_c^{n-1}(M_{K,\varpi^{\mathbb{Z}}} \otimes_{\breve{F}} C, \overline{Q}_l) \quad \text{and} \quad H'_{\mathrm{LT}} := \lim_{K} H_c^{n-1}(M_K^{(0)} \otimes_{\breve{F}} C, \overline{Q}_l). \tag{4.3}$$

Also, we set

$$G := \mathrm{GL}_n(F) \times D^\times/\varpi^{\mathbb{Z}} \times W_F \quad \text{and} \quad G^1 := \{(g,d,\sigma) \in \mathrm{GL}_n(F) \times D^\times \times W_F \mid \det(g)^{-1}\,\mathrm{Nrd}(d)\mathrm{Art}_F^{-1}(\sigma) = 1\}. \tag{4.4}$$

Lemma 4.2.3. *The natural map $G^1 \to G$ is injective and realizes G^1 as a co-compact closed normal subgroup of G.*

Proof. The morphism $G^1 \to G$ is clearly injective, Further, the image of the natural homomorphism is isomorphic to the kernel of the map $\nu : G \to F^\times/\varpi^{n\mathbb{Z}}$, given by $\nu(g, \overline{d}, \sigma) = \det(g)^{-1}\operatorname{Nrd}(d)\operatorname{Art}_F^{-1}(\sigma)$. The claim follows. □

We have actions $G \curvearrowright H_{\mathrm{LT}}$ and $G^1 \curvearrowright H'_{\mathrm{LT}}$. [Again, this uses Weil-Descent Data; make this precise.]

Theorem 4.2.4 (Non-Abelian Lubin-Tate Theory)**.** *Let π be a irreducible supercuspidal representation of GL_n whose cetral character is trivial on $\varpi^{\mathbb{Z}}$. Then, as representations of $\mathrm{GL}_n(F) \times D^\times \times \mathrm{W}_F$, the $\pi^\vee$-supercuspidal part of H_{LT} has the form*

$$H_{\mathrm{LT},\pi^\vee} = \pi^\vee \boxtimes \mathrm{JL}(\pi) \boxtimes \mathrm{rec}_F(\pi)(\tfrac{1-n}{2}), \tag{4.5}$$

$\mathrm{JL}(\pi)$ is a representation of $D^\times$ and $\mathrm{rec}_F(\pi)$ is a representation of W_F. The assignments $\pi \mapsto \mathrm{JL}(\pi)$ and $\pi \mapsto \mathrm{rec}_F(\pi)$ satisfy the conditions imposed on the Jacquet–Langlands and local Langlands correspondeces for GL_n.

Lemma 4.2.5. *These actions are smooth.*

Proof. [TODO] □

Lemma 4.2.6. *The G-representation $\text{c-Ind}_{G^1}^G(H'_{\mathrm{LT}})$ is isomorphic to H_{LT}.*

Proof. [TODO] □

5 The Lubin–Tate Space at Infinite Level

In this section, we introduce, attached to a formal group law $H_0 \in (\mathcal{O}_E\text{-FML}/\overline{\mathbb{F}}_q)$ of height $n \in \mathbb{N}$, the infinite level deformation moduli problem

$$\mathcal{M}_\infty^{(0)} = \lim_{m \in \mathbb{N}} \left(\mathcal{M}_m^{(0)}\right).$$

We have seen in the previous section that $\mathcal{M}_m^{(0)}$ is representable by the formal spectrum of a local ring A_m, finite and étale over $\operatorname{Spf}(A_0) = \operatorname{Spf}(\mathcal{O}_{\breve{E}}[\![u_1, \dots, u_{n-1}]\!])$. In particular, $\mathcal{M}_\infty^{(0)}$ is represented by the formal spectrum of the completed colimit

$$A_\infty = (\operatorname{colim}_m A_m)^\wedge_{\mathfrak{m}}.$$

Here, $\mathfrak{m}$ denotes the image of the maximal ideal of A_0 (or of any A_m, it doesn't matter). Note that $\operatorname{colim}_m A_m$ is not Noetherian, so the completion along an arbitrary ideal $I \subset A_\infty$ is, in general, not I-adically complete (see [Stacks, Tag 05JA] for an example). However, we have seen that $\mathfrak{m}$ is finitely generated, so this pathology does not occur and $\mathcal{M}_\infty^{(0)} := \operatorname{Spf} A_\infty$ makes senes as a formal scheme.

This section is concerned with the study of the ring A_∞. The main results are summarized as follows. First, we review some of the constructions in Chapter 2 of [Wei16]. As a first step, making use of the determinants of formal $\mathcal{O}_E$-modules constructed in [Hed15], we obtain a natural homomorphism

$$\mathcal{O}_{\widehat{E^{\mathrm{ab}}}} \to A_\infty.$$

We then introduce the notion of universal covers of formal $\mathcal{O}_E$-modules; the passage from H to its universal cover $\tilde{H}$ can be interpreted as a sort of tilting procedure. Still following [Wei16], this notion and its relation with the moduli problem $\mathcal{M}^{(0)}_{\infty,\mathcal{O}_{\mathbb{C}_p}}$ makes it possible to construct an isomorphism

$$A_{\infty,\mathcal{O}_{\mathbb{C}_p}} \coloneqq A_\infty \widehat{\otimes}_{\mathcal{O}_{\widehat{E^{\mathrm{ab}}}}} \mathcal{O}_{\mathbb{C}_p} \cong \mathcal{O}_{\mathbb{C}_p}[\![X_1^{q^{-\infty}}, \dots, X_n^{q^{-\infty}}]\!]/(\Delta(X_1^{q^{-\infty}}, \dots, X_n^{q^{-\infty}})^{q^{-m}} - \tau^{q^{-m}} \mid m \in \mathbb{N})^-, \tag{5.1}$$

for certain elements $\Delta \in \mathcal{O}_{\breve{E}}[\![X_1^{q^{-\infty}}, \dots, X_n^{q^{-\infty}}]\!]$ and $\tau \in \mathcal{O}_{\mathbb{C}_p}$. The superscript minus denotes the completion of the ideal.

The main effort is taken in an description of the various group actions on $\mathcal{M}_\infty \times_{\mathrm{Spf}(\mathcal{O}_{\breve{E}})} \mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p})$ in terms of this isomorphism, following [IT20, Section 1.2], as well as an approximative description of Δ following [BW11, Section 2.10]. These results give the necessary information to observe the perfection of a Deligne–Lusztig variety as the special fiber of some affinoid inside the Lubin–Tate perfectoid space, in a certain way compatible with the respective group actions.

5.1 Determinants of Formal modules

In [Hed15], Hedayatzadeh constructs determinants of ϖ-divisible formal $\mathcal{O}_E$-modules over [properties]rings. We cite the result of main importance for us.

Theorem 5.1.1 (Determinants of Formal Modules). *ABCDE*

Proof. □

Let $\mathcal{F}_0 \in (\mathcal{O}_E\text{-FM}/\overline{\mathbb{F}}_q)$ be a formal module of height n and write $\wedge^n \mathcal{F}_0$ for the associated determinant module, that is, the formal $\mathcal{O}_E$ module with $\mathrm{D}(\wedge^n \mathcal{F}_0) = \wedge^n \mathrm{D}(\mathcal{F}_0)$. Write $\mathcal{M}_m$ for the Deformation space of $\mathcal{F}_0$ with Drinfeld level ϖ^m-structure, and write $\mathcal{M}_{m,\wedge}$ for the deformation space of $\wedge^n \mathcal{F}$. Following [Wei16], we sketch how this result can be used to construct a functor $\mathcal{M}_m \to \mathcal{M}_{m,\wedge}$.

For $R \in \mathcal{C}$ and $(\mathcal{F}, \iota) \in \mathcal{M}_0(R)$, we write δ_m for the induced universal multilinear and alternating morphisms

$$\delta_m \mathcal{F}[\varpi^m]^n \to \wedge^n \mathcal{F}[\varpi^m].$$

We now have the following result.

Lemma 5.1.2. *Let $(x_1, \dots, x_n) \in \mathcal{F}[\varpi^m]^n(R)$ be a Drinfeld level ϖ^m structure. Then*

$$\delta_m(x_1, \dots, x_n) \in \wedge^n \mathcal{F}[\varpi^m](R)$$

is a Drinfeld level ϖ^m structure.

Proof. This is [Wei16, Proposition 2.11]. □

In particular, we obtain the desired map

$$\mathcal{M}_m^{(0)}(R) \to \mathcal{M}_{m,\wedge}^{(0)}(R), \quad (\mathcal{F}, \iota, \phi) \mapsto (\wedge^n \mathcal{F}, \wedge^n \iota, \delta_m \circ \phi).$$

We also need the following result.

Lemma 5.1.3. *Let L/E be a separable extension of degree n and suppose that there is an action $\mathcal{O}_L \hookrightarrow \mathrm{End}(\mathcal{F})$ making $\mathcal{F}$ into a formal $\mathcal{O}_L$-module of height 1. Then, for all $m \geq 1$, the identity*

$$\lambda_m(\alpha x_1, \dots, \alpha x_n) = \mathrm{N}_{L/E}(\alpha) \lambda_m(x_1, \dots, x_n)$$

holds.

Proof. This is [Wei16, Lemma 2.12]. □

We remark that the Lemma above in particular applies to the standard formal $\mathcal{O}_E$-module H over $\mathcal{O}_{\breve{E}}$. We turn our attention to the determinant of the standard formal $\mathcal{O}_E$-module in the following example.

Example. The determinant $\wedge^n H$ is the formal $\mathcal{O}_E$-module over $\mathcal{O}_{\breve{E}}$ with logarithm given by the power series

$$f_\wedge(T) = \sum_{i=0}^{\infty} (-1)^{(n-1)i} \frac{T^{q^i}}{\varpi^i}.$$

We do not prove this, but we note that this can be witnessed on the corresponding Dieuodonné-modules: $\mathrm{D}(\wedge^n H)$ and $\wedge^n \mathrm{D}(H)$ are naturally isomorphic. By Theorem 2.6.1 we have that $[\varpi]_{(\wedge^n H)_0}(T) = (-1)^{n-1} T^q$.

5.2 The Universal Cover

Assume that A is a discrete valuation ring with uniformizer ϖ, finite residue field k and field of fractions K. Write $q = \#k$. Let R be an admissible A-algebra admitting an ideal of definition I with $R/I = \overline{\mathbb{F}}_q$. Let $\mathcal{F}$ be a formal ϖ-divisible A-module over R of height n.

Definition 5.2.1 (The Universal Cover)**.** We denote by $\tilde{\mathcal{F}}$ the functor

$$\tilde{\mathcal{F}} \colon (R\text{-Adm}) \to \vec{K}, \quad S \mapsto \left\{ (x_1, x_2, \dots) \in \prod_{\mathbb{N}} \mathcal{F}(S) \mid \varpi(x_{i+1}) = x_i \right\}.$$

Note that for $S \in (R\text{-Adm})$, the set $\tilde{\mathcal{F}}(S)$ has a natural A-module structure with multiplication by ϖ an isomorphism, making it a K-vector space.

We remark that the Tate module

$$T_\varpi \mathcal{F} \colon (R\text{-Adm}) \to (A\text{-Mod}), \quad S \mapsto \{(x_1, x_2, \dots) \in \tilde{\mathcal{F}} \mid \varpi x_1 = 0\} \tag{5.2}$$

as well as the rational Tate module

$$V_\varpi \mathcal{F} \colon (R\text{-Adm}) \to \vec{K}, \quad S \mapsto \{(x_1, x_2, \dots) \in \tilde{\mathcal{F}} \mid \exists n \in \mathbb{N} : \varpi^n x_1 = 0\} \tag{5.3}$$

arise as subfunctors of $\tilde{\mathcal{F}}$.

5.2.1 Useful Calculations

Let p be a prime. Let R be a Noetherian local ring with maximal ideal I such that $p \in I$, R is complete with respect to the I-adic topology and $k_R := R/I$ is an algebraically closed field (necessarily of characteristic p). If q is a power of p, we write $\mathcal{P}_{R,q}$ for the set of power series $f \in R[\![T]\!]$ satisfying

$$f(T) \equiv g(T^q) \pmod{I} \tag{5.4}$$

for some power series $g(T) = c_1T + c_2T^2 + \cdots \in R[\![T]\!]$ with $c_1 \in R^\times$. If $q' > q$ is another power of p, we have injections $\mathcal{P}_{R,q} \hookrightarrow \mathcal{P}_{R,q'}$ given by sending $f(T)$ to its (q'/q)-fold self-composite $f^{q'/q}(T)$. Making use of these transition maps, we define

$$\mathcal{P}_R := \operatorname*{colim}_{n \in \mathbb{N}} \mathcal{P}_{R,p^n},$$

identifying any power series $f \in \mathcal{P}_{R,q}$ with its image in $\mathcal{P}_{R,q'}$ for higher p-powers q'. For any $f \in \mathcal{P}_{R,q}$, we define the functor

$$U_f : (R\text{-Adm}) \to (\text{Set}), \quad S \mapsto \left\{ (x_0, x_1, \dots) \in \prod_{\mathbb{N}} S^{\circ\circ} \mid f(x_{i+1}) = x_i \right\}.$$

This functor does, up to canonical isomorphism, only depend on the equivalence class of f in $\mathcal{P}_R$. We write $U_{0,f}$ for the base change of U_f to k_R, that is

$$U_{0,f} : (k_R\text{-Adm}) \to (\text{Set}), \quad S \mapsto \left\{ (x_0, x_1, \dots) \in \prod_{\mathbb{N}} S^{\circ\circ} \mid \overline{f}(x_{i+1}) = x_i \right\}.$$

Here, $\overline{f}$ is the image of f under the reduction map $R[\![T]\!] \to k_R[\![T]\!]$.

In the sequel, we denote R-algebras by S and write J for an ideal of definition containing the image of I (provided, for example, by A.0.2). Given an element $f \in \mathcal{P}_R$, we do not distinguish between f and a choice of a representative $\tilde{f} \in \mathcal{P}_{R,q}$ for some sufficiently large p-power.

The following observation lays the groundwork for many of the upcoming results.

Lemma 5.2.2. *Let f be any power series in $\mathcal{P}_R$. For any two elements $s_1, s_2 \in S$ with $s_1 \equiv s_2$ mod J such that $f(s_1)$ and $f(s_2)$ exist (for example if f is a polynomial or $s_1, s_2 \in S^{\circ\circ}$), we have*

$$f^k(s_1) \equiv f^k(s_2) \pmod{J^{k+1}}.$$

Here, f^k denotes k-fold composition of f.

Proof. We will show that if $s_1 \equiv s_2$ mod J^k, then $f(s_1) \equiv f(s_2)$ mod J^{k+1}, which suffices to prove the claim. We may write $s_2 = s_1 + r$ for some $r \in J^k$. By the assumptions on f there exist power series $g, h \in R[\![T]\!]$ such that h only has coefficients in I and $f(T) = g(T^q) + h(T)$. As I is finitely generated, say by elements $(r_1, \dots, r_l)$, we obtain a representation

$$f(s_1) - f(s_2) \in g(s_1^q) - g(s_2^q) + \sum_{i=1}^{l} r_i \left(h_i(s_1) - h_i(s_2) \right).$$

As r divides $(h_i(s_1) - h_i(s_2))$, we find $r_i(h_i(s_1) - h_i(s_2)) \in (r_i r) \subseteq J^{k+1}$. Also note that for any $s \in S$ and $n \in \mathbb{N}$,

$$(s+r)^{nq} = s^{nq} + nqrs^{nq-1}r + \cdots + r^{nq},$$

so after cancellation, all monomials of $g(s_1^q) - g(s_2^q)$ lie in (qr) or (r^2). This implies

$$g(s_1^q) - g\left((s_1 + r)^q\right) \in (qr) + (r^2) \subseteq J^{k+1},$$

and we are done. □

Lemma 5.2.3. *The natural reduction map*

$$U_f(S) \to U_f(S/J) = U_{0,f}(S/J)$$

is bijective.

Proof. We first show surjectivity. Given a sequence $(x_0, x_1, \dots) \in U_f(S/J)$, we can choose a sequence of arbitrary lifts $(y_0, y_1, \dots) \in \prod_{\mathbb{N}} S^{\circ\circ}$ and set

$$z_i = \lim_{r\to\infty} f^r(y_{i+r}).$$

The limit exists, because if $s \geq r$ are two non-negative integers, we calculate

$$f^{s-r}(y_{i+s}) \equiv \overline{f}^{s-r}(x_{i+s}) = x_{i+r} \equiv y_{i+r} \pmod{J},$$

implying by Lemma 5.2.2 that

$$f^s(y_{i+s}) \equiv f^r(y_{i+r}) \pmod{J^r}.$$

This shows that $(f^r(y_{i+r}))_{r\in\mathbb{N}}$ is a Cauchy-sequence for the J-adic topology on S, thereby convergent (cf. Lemma A.0.4). The sequence $(z_0, z_1, \dots)$ now lies in $U_f(S)$ and lifts $(x_0, x_1, \dots)$. It remains to show that the lift is unique. Suppose that $(z_0', z_1', \dots)$ is another lift. Then, for any $i, k \in \mathbb{N}$ we have $z_{i+k} \equiv z_{i+k}' \bmod J$, and another application of Lemma 5.2.2 shows that

$$z_i = f^k(z_{i+k}) \equiv f^k(z_{i+k}') = z_i' \pmod{J^k}.$$

Thereby $(z_i - z_i') \in \bigcap_{k\in\mathbb{N}} J^k = \{0\}$. Hence, the lift is unique. □

We write $\mathrm{Nilp}^\flat$ for the functor U_{T^q}. That is, $\mathrm{Nilp}^\flat(S) = \lim_{x\mapsto x^q} S^{\circ\circ}$ is the set of q-power compatible sequences with values in $S^{\circ\circ}$.

Lemma 5.2.4. *For any $f \in \mathcal{P}_R$, there is a canonical bijection $U_{0,f}(S/J) \to \mathrm{Nilp}^\flat(S/J)$. This bijection is functorial in S.* Use different S

Proof. By assumption on f we have $f(T) = g(T^q) \in k_R[\![T]\!]$ for some $g(T) = c_1T + c_2T^2 + \dots$ with $c_1 \neq 0$. For each coefficient c_i, let $d_i \in k_R$ be the unique element such that $d_i^q = c_i$. Let $h(T) \in k_R[\![T]\!]$ be the power series given by $d_1T + d_2T^2 + \dots$. Now $(h(T))^q = f(T)$, and we find that

$$U_f(S/J) \to \mathrm{Nilp}^\flat(S/J) : \quad (x_1, x_2, x_3, \dots) \mapsto (x_1, h(x_2), h(h(x_3)), \dots)$$

is a well-defined function, and functorial in S. For the inverse, let $h^{-1}(T) \in k_R[\![T]\!]$ be the unique power series with $h^{-1}(h(T)) = h(h^{-1}(T)) = T$, see Lemma 2.1.9. The map

$$\mathrm{Nilp}^\flat(S/J) \to U_f(S/J), \quad (x_1, x_2, \dots) \mapsto (x_1, h^{-1}(x_2), h^{-1}(h^{-1}(x_3)), \dots)$$

is well-defined as

$$f(h^{-1}(T)) = g((h^{-1}(T))^q) = (h(h^{-1}(T)))^q = T^q,$$

and it is readily seen to be inverse to the map constructed above. □

We collect results.

Proposition 5.2.5. *Given $f, g \in \mathcal{P}_R$, we have bijections, functorial in S,*

$$U_f(S) \to U_f(S/J) \to \mathrm{Nilp}^\flat(S/J) \to U_g(S/J) \to U_g(S). \tag{5.5}$$

Explicitly, the bijection $U_f(S) \to U_g(S)$ can be described as follows. Suppose that $f, g \in \mathcal{P}_{R,q}$ for some sufficiently large q. Let $h_f(T)$ and $h_g(T)$ be power series with coefficients in A such that

$$h_f(T)^q \equiv f(T) \pmod I \quad \textit{and} \quad h_g(T)^q \equiv g(T) \pmod I.$$

Write $h_g^{-1}(T)$ for the (formal) inverse power series of h_g. Now the isomorphism is given by the mapping

$$(x_0, x_1, \dots) \mapsto (y_0, y_1, \dots), \quad \textit{where} \quad y_i = \lim_{r\to\infty} g^r(h_g^{-(r+i)}(h_f^{r+i}(x_{i+r}))).$$

Here, the exponents are to be interpreted as iterated composition.

Proof. The first part follows directly from repeated application of the previous two Lemmas. The second part follows by tracing through the previous lemmas. □

5.2.2 Applications to the Universal Cover

Fix a coordinate $\mathcal{F} \cong \mathrm{Spf}(R[\![T]\!])$ so that $\mathcal{F} = \mathrm{FG}(F)$ for some A-module law $F \in (A\text{-FML}/R)$. Then $[\varpi]_F(T) \in \mathcal{P}_R$, and we obtain an isomorphism $\tilde{\mathcal{F}} \cong U_{[\varpi]_F} =: \tilde{F}$. Write $F_0 = F \otimes k_R$, and $\tilde{F}_0 = U_{0,[\varpi]_F}$.

Lemma 5.2.6. *We have an isomorphism*

$$\tilde{\mathcal{F}}_0 \cong \mathrm{Nilp}^\flat_{k_R}$$

of functors $(k_R\text{-Adm}) \to (\mathrm{Set})$

Proof. Any lift of $[\varpi]_{F_0}(T) \in k_R[\![T]\!]$ lies inside $\mathcal{P}_R$. Hence, the statement is an application of Lemma 5.2.4. □

Lemma 5.2.7. *Suppose that S is an admissible R-algebra admitting an ideal of definition J such that $\varpi \in J$. Then the natural reduction map*

$$\tilde{\mathcal{F}}(S) \to \tilde{\mathcal{F}}(S/J) = \mathcal{F}_0(S/J)$$

is an isomorphism.

Proof. After choosing a coordinate $\mathcal{F} = \mathrm{FG}(F)$, we have $[\varpi]_F \in \mathcal{P}_R$ and hence $\tilde{\mathcal{F}}(S) \cong U_{[\varpi]_F}$. Thereby the statement is given by Lemma 5.2.3. $\square$

The following is analogous to Proposition 5.2.5.

Proposition 5.2.8. *Let S be an admissible R-algebra with ideal of definition J such that $\phi(I) \subseteq J$. Then there are canonical isomorphisms (of sets)*

$$\tilde{\mathcal{F}}(S) \cong \tilde{\mathcal{F}}(S/J) = \tilde{\mathcal{F}}_0(S/J) \cong \mathrm{Nilp}^\flat(S/J) \cong \mathrm{Nilp}^\flat(S).$$

In particular, $\tilde{\mathcal{F}}(S)$ is, as a functor to (Set), *representable by* $\mathrm{Spf}(R[\![T^{q^{-\infty}}]\!])$.

We write λ for the isomorphism $\tilde{\mathcal{F}} \to \mathrm{Nilp}^\flat$, and $\lambda_i : \tilde{\mathcal{F}} \to (-)^{\infty}$ for projection on the i-th component. Similarly, we write $\mu : \mathrm{Nilp}^\flat \to \tilde{\mathcal{F}}$ for the inverse of λ and μ_i for the i-th component of μ.

By the proposition above, quasi-isogenies on $\mathcal{F}_0$ induce isomorphisms on $\tilde{\mathcal{F}}$. This will be used to construct an action of $D^\times$ on $\tilde{\mathcal{F}}$ below. The relative Frobenius morphism lifts as well.

Definition 5.2.9 (Relative Frobenius on $\tilde{\mathcal{F}}$)**.** Write $\Pi : \tilde{\mathcal{F}} \to \Phi^{-1,*}\tilde{\mathcal{F}}$ for the isomorphism coming from the Frobenius quasi-isogeny

$$\mathrm{Frob}_q : \mathcal{F}_0 \to \mathcal{F}_0^{(q)} = \Phi^{-1,*}\mathcal{F}_0.$$

5.2.3 Relation to the Deformation Space at Infinite Level

Let $(e_1, \ldots, e_n)$ denote the standard basis of $\mathcal{O}_E^n$. By Theorem 4.1.5, there is for every positive integer m a universal triple

$$(\mathcal{F}_m^{\mathrm{univ}}, \iota_m^{\mathrm{univ}}, \phi_m^{\mathrm{univ}}) \in \mathcal{M}_m^{(0)}(A_m),$$

[where the pair $(\mathcal{F}_m^{\mathrm{univ}}, \iota_m^{\mathrm{univ}}) = (\mathcal{F}^{\mathrm{univ}}, \iota^{\mathrm{univ}})$ can be chosen independently of m]and the Drinfeld level ϖ^m-structure

$$\phi_m^{\mathrm{univ}} : \underline{(\varpi^{-m}\mathcal{O}_E/\mathcal{O}_E)}^n \to \mathcal{F}^{\mathrm{univ}}$$

gives, evaluated at A_∞, rise to elements $x_i^{(m)} = \phi_m(e_i) \in \mathcal{F}^{\mathrm{univ}}(A_\infty)$ for $i = 1, \ldots, n$. This gives rise to an n-tuple of compatible systems

$$(x_1, \ldots, x_n) \in \tilde{\mathcal{F}}^{\mathrm{univ}}(A_\infty).$$

Now let $\mathcal{F}$ be an arbitrary deformation of $\mathcal{F}_0$ to $\mathcal{O}_{\breve{E}}$. By Proposition 5.2.8, we have isomorphisms

$$\tilde{\mathcal{F}}^{\mathrm{univ},n}(A_\infty) \cong \tilde{\mathcal{F}}_0^{\mathrm{univ},n}(A_\infty/I_\infty) \xrightarrow{\iota} \tilde{\mathcal{F}}_0(A_\infty/I_\infty) \cong \tilde{\mathcal{F}}^n(A_\infty).$$

This constructs a morphism of formal schemes over $\mathrm{Spf}\, R$

$$\mathcal{M}_\infty^{(0)} \to \tilde{\mathcal{F}}^n. \tag{5.6}$$

Theorem 5.2.10 (Structure of $\mathcal{M}_\infty$). *There is a cartesian square of formal schemes*

$$\begin{array}{ccc}\mathcal{M}_\infty^{(0)} & \xrightarrow{\det} & \mathcal{M}_{\wedge,\infty}^{(0)} \\ \downarrow & & \downarrow \\ \tilde{\mathcal{F}}^n & \xrightarrow{\delta} & \widetilde{\wedge^n\mathcal{F}}.\end{array}$$

Here, the vertical maps are the ones constructed above, the horizontal ones come from the determinant construction of [Hed15].

Proof. This is [Wei16, Theorem 2.17]. $\square$

To end this subsection, we note that, as the embeddings of $\mathcal{O}_{\breve{E}} \hookrightarrow \mathcal{O}_{\widehat{E}^{\mathrm{ab}}}$ are identified with $\mathcal{O}_E^\times$ by the Artin map, we have the decomposition

$$\mathcal{M}_\infty^{(0)} \times_{\operatorname{Spf}\mathcal{O}_{\breve{E}}} \operatorname{Spf}(\mathcal{O}_{\mathbb{C}_p}) = \coprod_{\alpha\in\mathcal{O}_E^\times} \mathcal{M}^\alpha_{\infty,\mathcal{O}_{\mathbb{C}_p}}. \tag{5.7}$$

Here $\mathcal{M}^\alpha_{\infty,\mathcal{O}_{\mathbb{C}_p}} := \mathcal{M}_\infty^{(0)} \times_{\operatorname{Spf}\mathcal{O}_{\breve{E}},\alpha} \operatorname{Spf}(\mathcal{O}_{\mathbb{C}_p})$. This also implies the decomposition

$$\mathcal{M}_{\infty,\varpi^{\mathbb{Z}}} \times_{\operatorname{Spf}\mathcal{O}_{\breve{E}}} \operatorname{Spf}(\mathcal{O}_{\mathbb{C}_p}) = \coprod_{j\in\mathbb{Z}/n\mathbb{Z}} \mathcal{M}_\infty^{(j)} \times_{\operatorname{Spf}\mathcal{O}_{\breve{E}}} \operatorname{Spf}(\mathcal{O}_{\mathbb{C}_p}) = \coprod_{\alpha\in E^\times/\varpi^n} \mathcal{M}^\alpha_{\infty,\mathcal{O}_{\mathbb{C}_p}}, \tag{5.8}$$

where $\mathcal{M}^\alpha_{\infty,\mathcal{O}_{\mathbb{C}_p}} = \mathcal{M}_\infty^{(j)} \times_{\operatorname{Spf}\mathcal{O}_{\breve{E}},\overline{\alpha}} \operatorname{Spf}\mathcal{O}_{\mathbb{C}_p}$ for $j \in \mathbb{Z}/n\mathbb{Z}$ equal to the residue of the ϖ-adic valuation of α and $\overline{\alpha} = \varpi^{-\mathrm{val}_\varpi(\alpha)}\alpha \in \mathcal{O}_E^\times$.

5.2.4 Reviewing the Group Actions

Write

$$G := \mathrm{GL}_n(E) \times D^\times \times \mathrm{W}_E,$$

and define a homomorphism

$$\theta\colon G \to E^\times, \quad (g,d,\sigma) \mapsto \det(g)\operatorname{Nrd}(d)^{-1}\mathrm{Art}_E^{-1}(\sigma).$$

Let G^1 be the preimage of $\mathcal{O}_E^\times$ under θ, so that G^1 acts on $\mathcal{M}_\infty^{(0)} \times_{\operatorname{Spf}\breve{E}} \operatorname{Spf}(\mathcal{O}_{\mathbb{C}_p})$. Given a ϖ-divisible formal A-module $\mathcal{F} \in (A\text{-FM}/R)$ of height n, we write $\mathcal{F}_{\mathcal{O}_{\mathbb{C}_p}} = \mathcal{F} \otimes \mathcal{O}_{\mathbb{C}_p}$ and describe a natural right action on $\tilde{\mathcal{F}}^n_{\mathcal{O}_{\mathbb{C}_p}}$ by the group such that the map $\mathcal{M}_\infty^{(0)} \to \tilde{\mathcal{F}}^n_{\mathcal{O}_{\mathbb{C}_p}}$, induced by the map constructed above is equivariant for the respective G^1-actions.
The action of G on $\tilde{\mathcal{F}}^n$ is easy to describe. For the action of $\mathrm{GL}_n(E)$, note that $\tilde{\mathcal{F}}$ carries the structure of a E-vector space object. Hence $\tilde{\mathcal{F}}^n$ obtains a natural right action by $\mathrm{GL}_n(E)$: an object $g \in \mathrm{GL}_n(E)$ with entries $g = (a_{ij})_{i,j}$ acts by matrix multiplication from the right, as in

$$(x_1,\dots,x_n).g = (y_1,\dots,y_n), \quad \text{where} \quad y_j = \sum_{i=1}^n a_{ij}x_i. \tag{5.9}$$

For the action of $D^\times$, note that by Proposition 5.2.8, we have a natural left action of $D^\times$ on

$\tilde{\mathcal{F}}$. Indeed, given $0 \neq d_0 \in \mathrm{End}_{(A\text{-FM}/\overline{\mathbb{F}}_q)}(F_0) = \mathcal{O}_D$, an integer $r \in \mathbb{Z}$ and any $S \in (R\text{-Adm})$ [Correct Category?], we let the element $d = \varpi^r d_0 \in D^\times$ act on $\tilde{\mathcal{F}}$ via the automorphism

$$\tilde{\mathcal{F}}(S) \to \tilde{\mathcal{F}}_0(S/J) \xrightarrow{\varpi^r} \mathcal{F}_0(S/J) \xrightarrow{d_0} \tilde{\mathcal{F}}_0(S/J) \to \tilde{\mathcal{F}}(S), \quad x \mapsto dx.$$

Note that multiplication by ϖ and d_0 commute (as ϖ lies in the center of the multiplicative monoid $\mathcal{O}_D$), so this yields a well-defined left action. We define the right action of $D^\times$ on $\tilde{\mathcal{F}}^n$ via

$$(x_1, \ldots, x_n).d = (d^{-1}x_1, \ldots, d^{-1}x_n).$$

The map $\Pi : \tilde{\mathcal{F}} \to \Phi^{-1,*}\tilde{\mathcal{F}}$ from Definition 5.2.9 equips $\mathcal{F}$ with the Weil descent datum

$$(\Phi^*\Pi)^{-1} : \mathcal{F} \to \Phi^*\mathcal{F},$$

and in particular yields an action of W_E on $\tilde{\mathcal{F}}^n_{\mathcal{O}_{\mathbb{C}_p}}$.
It is easy to see that the actions of $\mathrm{GL}_n(E)$ and W_E commute, and that both these actions commute with the Weil descent datum. Hence we obtain a right action by G on $\tilde{\mathcal{F}}^n_{\mathcal{O}_{\mathbb{C}_p}}$.

Proposition 5.2.11. *The morphism $\mathcal{M}_\infty^{(0)} \times_{\mathrm{Spf}\,\mathcal{O}_{\breve{E}}} \mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p}) \to \tilde{\mathcal{F}}^n_{\mathcal{O}_{\mathbb{C}_p}}$ in* (5.6) *is equivariant for the action of G^1 on both sides.*

Proof. It suffices to check G-equivariance of the induced map

$$\mathcal{M}^{(0)}_{\infty,\mathcal{O}_{\mathbb{C}_p}}(A_\infty \widehat{\otimes}_R \mathcal{O}_{\mathbb{C}_p}) \to \tilde{\mathcal{F}}^n_{\mathcal{O}_{\mathbb{C}_p}}(A_\infty \widehat{\otimes}_R \mathcal{O}_{\mathbb{C}_p}).$$

Here it suffices to show that the morphism $\mathcal{M}_\infty^{(0)}(A_\infty) \to \tilde{\mathcal{F}}^n(A_\infty)$ is equivariant for the action of $\mathrm{GL}_n(E) \times D^\times$ and that it preserves the Weil descent datum in the sense that the square

$$\begin{array}{ccc} \mathcal{M}_\infty^{(0)}(A_\infty) & \longrightarrow & \tilde{\mathcal{F}}^n(A_\infty) \\ \downarrow & & \downarrow \\ \Phi^*\mathcal{M}_\infty(A_\infty) & \longrightarrow & \Phi^*\tilde{\mathcal{F}}^n(A_\infty) \end{array}$$

commutes. [Todo: more explicit description of group action on A_∞.] $\square$

5.2.5 Making the Group Actions Explicit

We now choose a coordinates for $\mathcal{M}_\infty^{(0)}$ and make the group action of G^1 explicit in terms of these coordinates. Let H be the standard formal $\mathcal{O}_E$-module over $\breve{E}$ of height n. By the monomorphism constructed in (5.6) and Proposition 5.2.11, the action of G^1 on $\mathcal{M}_\infty^{(0)}$ is determined by the action of G^1 on $\tilde{H}^n$. As canonically $\tilde{H}^n \cong \mathrm{Nilp}^{\flat,n}$, the right action of G on $\tilde{H}^n$ is equivalent to a left action on the algebra

$$\Xi_n := \mathcal{O}_{\mathbb{C}_p}[\![X_1^{q^{-\infty}}, \ldots, X_n^{q^{-\infty}}]\!].$$

Our first aim is to make this action explicit.

We begin with an explicit description of the isomorphisms μ and λ of Proposition 5.2.8.

Lemma 5.2.12. *The bijections*

$$\lambda\colon \tilde{H}(S) \rightleftarrows \mathrm{Nilp}^\flat(S)\colon \mu, \quad (x_0, x_1, \dots) \leftrightarrows (y, y^{q^{-1}}, y^{q^{-2}}, \dots)$$

are, in either direction, given by the equations

$$y^{1/q^{ni}} = \lim_{r\to\infty} x_{r+i}^{q^{nr}} \quad \textit{and} \quad x_i = \lim_{s\to\infty} [\varpi^s]_H(y^{q^{-n(i+s)}}).$$

Proof. This follows directly from the fact that $[\varpi]_H(T) \equiv T^{q^n}$ modulo ϖ and the explicit description of the isomorphism in Proposition 5.2.5. $\square$

This Lemma allows us to make the $\mathcal{O}_E$-module structure on $\tilde{H}$ explicit. Let S be an adic $\mathcal{O}_{\breve{E}}$-algebra [correct category?].

Lemma 5.2.13. *The K-vector space structure on $\tilde{H}$ takes on the following form.*

- *Given two q-th power compatible systems $y_1, y_2 \in \mathrm{Nilp}^\flat(S)$ corresponding to compatible systems $\mu(y_1) = x_1$, $\mu(y_2) = x_2 \in \tilde{H}(S)$, the sum $x_1 + x_2 \in \tilde{H}(S)$ corresponds to the element $\lambda(x_1 + x_2) = y_1 +_H y_2 \in \mathrm{Nilp}^\flat(S)$, where*

  $$(y_1 +_H y_2)^{1/q^j} = \lim_{r\to\infty} H(y_1^{q^{-r}}, y_2^{q^{-r}})^{q^{r-j}}.$$

  If $G \in (A\text{-FML}/R)$ with $G \otimes \overline{\mathbb{F}}_q = H \otimes \overline{\mathbb{F}}_q$, the systems of q-th power roots $(y_1 +_H y_2)$ and $(y_1 +_G y_2)$ agree.

- *Similarly, given $a \in \mathcal{O}_E$ and $y \in \mathrm{Nilp}^\flat(S)$ with $\mu(y) = x \in \widetilde{H}(S)$, we have*

  $$a_H y = \lambda([a]_H(x)) = \lim_{r\to\infty} [a]_H(y^{q^{-r}})^{q^{r-j}}.$$

  For G as above, we have $[a]_H(x) = [a]_G(x)$.

Proof. The first statement follows directly from the lemma above, after tracing through the commutative diagram

$$\begin{tikzcd} & \widetilde{H}(S)^2 \arrow[r, "H(-,-)"] \arrow[d] & \widetilde{H}(S) \arrow[d] \arrow[dr, "\lambda"] & \\ \mathrm{Nilp}^\flat(S)^2 \arrow[r] \arrow[ur, "\mu"] & \widetilde{H}_0(S/J)^2 \arrow[r, "H_0(-,-)"] & \widetilde{H}_0(S/J) \arrow[r] & \mathrm{Nilp}^\flat(S). \end{tikzcd}$$

Similarly one proves the third statement. $\square$

We obtain the following description of the $K^\times \times D^\times$-action.

Corollary 5.2.14. *Let $a \in E$ and $d \in D^\times$ be elements so that*

- *the element a is, for some $l \in \mathbb{Z}$, of the form*

  $$a = \sum_{i=l}^{\infty} a_i \varpi^i \quad \textit{with} \quad a_i \in \mu_{q-1}(E) \cup \{0\}.$$

- *the element d is, for some $l' \in \mathbb{Z}$ and $\vartheta \in \mathrm{End}_{(\mathcal{O}_E\text{-FML}/\overline{\mathbb{F}}_q)}(H_0)$ the endomorphism given by $\vartheta(T) = T^q$, of the form*

$$d = \sum_{i=l'}^{\infty} d_i \vartheta^i \quad \text{with} \quad d_i \in \mu_{q^n-1}(E_n) \cup \{0\}.$$

Here E_n denotes the unramified extension of E with residue field $\mathbb{F}_{q^n}$.

Then, given any $x = (x_1, x_2, \dots) \in \tilde{H}(S)$ with $\lambda(x)^{1/q^i} = y^{1/q^i} \in \mathrm{Nilp}^\flat(S)$, we have the explicit descriptions

$$a_H(y)^{1/q^j} = \sum_{i=l}^{\infty}{}_H\, a_i y^{q^{ni-j}} \quad \text{and} \quad d_H(y)^{1/q^j} := \lambda(dx)^{1/q^j} = \sum_{i=l'}^{\infty}{}_H\, d_i y^{q^{i-j}}$$

Here the symbol $\sum_H$ denotes addition of q-th power root systems with respect to the addition in H, as defined in the previous lemma.

Proof. This is an immediate corollary of the Lemma above, Lemma 2.6.2 and the fact that $[\varpi]_H(T) \equiv T^{q^n} \pmod{\varpi}$. $\square$

It is now not hard to deduce an explicit description of the $\mathrm{GL}_n(E)$-action on $\tilde{H}^n(S)$, we write it down in terms of Ξ_n below. Finally, we describe the action of W_E on $(\tilde{H} \otimes \mathcal{O}_{\mathbb{C}_p})$ in terms of $\mathrm{Nilp}^\flat \times_{\mathrm{Spf}\, \mathcal{O}_{\breve{E}}} \mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p}) =: \mathrm{Nilp}^\flat_{\mathcal{O}_{\mathbb{C}_p}}$.

Lemma 5.2.15. *Let $\sigma \in \mathrm{W}_E$ and let $m \in \mathbb{Z}$ be an integer such that $\sigma|_{\breve{E}} = \Phi^m$. Let $(y, y^{1/q}, \dots)$ be an element in $\mathrm{Nilp}^\flat_{\mathcal{O}_{\mathbb{C}_p}}(S)$. Then*

$$\sigma.(y_1, y_2, \dots) = (\sigma(y)^{q^{-m}}, \sigma(y)^{q^{-(m+1)}}, \dots).$$

Proof. This follows from the definition of the Weil descent datum and the fact that the square

$$\begin{array}{ccc} \tilde{H}(S) & \xrightarrow{\sim} & \mathrm{Nilp}^\flat(S) \\ {\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle y \mapsto y^q} \\ \tilde{H}(S) & \xrightarrow{\sim} & \mathrm{Nilp}^\flat(S) \end{array}$$

is commutative. $\square$

We obtain the following description of the left action of G on Ξ_n.

Lemma 5.2.16. *Let $g = (a_{ij})_{i,j} \in \mathrm{GL}_n(E)$, $d \in D^\times$ and $\sigma \in \mathrm{W}_E$. Let $m \in \mathbb{Z}$ be such that $\sigma|_{\breve{E}} = \Phi^m$. Then the morphism $\Xi_n \to \Xi_n$ induced by the element $(g, d, \sigma) \in G$ is given by the composition of the morphsims*

- *$g^* \colon \Xi_n \to \Xi_n$ is the morphism of $\mathcal{O}_{\mathbb{C}_p}$ algebras given by $X_i \mapsto \sum_{j=1H}^{n} a_{ji\,H}(X_j)$. Writing $a_{ij} = \sum_{k=l}^{\infty} a_{ij}^{(k)} \varpi^k$ for $a_{ij}^{(k)}$ either vanishing or a $(q-1)$th root of unity and a sufficiently chosen integer k, we obtain, by Corollary 5.2.14, the description*

$$g^*(X_i) = \sum_{k=l}^{\infty}{}_H \sum_{j=1}^{n}{}_H\, a_{ji}^{(k)} X_j^{q^{nk}}$$

- $d^{-1,*}\colon \Xi_n \to \Xi_n$ *is the isomorphism of* $\mathcal{O}_{\mathbb{C}_p}$*-algebras given by* $X_i \mapsto d_H(X_i)$*. Writing* $d^{-1} = \sum_{k=l}^{\infty} d_i \vartheta^k$ *for a sufficiently small integer* k*, and* ϑ *and* d_i *as in Corollary 5.2.14, we obtain the description*
$$d^{-1,*}(X_i) = \sum_{k=l}^{\infty}{}_H\, d_k X_i^{1/q^k}.$$
- $\sigma^* : \Xi_n \to \Xi_n$ *is the isomorphism of* $\mathcal{O}_E$*-algebras given by* $X_i \mapsto X_i^{q^{-m}}$ *and* $a \mapsto \sigma(a)$ *for* $a \in \mathcal{O}_{\mathbb{C}_p}$*.*

Let us now turn our attention to the space $\mathcal{M}^{(0)}_{\infty,\mathcal{O}_{\mathbb{C}_p}}$. Recall the decomposition in (5.7) and write $\mathcal{M}^{(0),\alpha}_{\infty,\mathcal{O}_{\mathbb{C}_p}} = \mathrm{Spf}(A^{\alpha}_{\infty,\mathcal{O}_{\mathbb{C}_p}})$. By Proposition 5.2.8, we obtain the following description of $A^{\alpha}_{\infty,\mathcal{O}_{\mathbb{C}_p}}$ from Proposition 5.2.10.

Corollary 5.2.17. *Let* $(\tau^{1/q^m})_{m\in\mathbb{N}} \in \mathcal{O}_{\widehat{E}^{\mathrm{ab}}}$ *be a* [primitive, make precise] *system of* q*-th power roots. Then we have*

$$A_\infty \cong \mathcal{O}_{\widehat{E}^{\mathrm{ab}}}[\![X_1^{q^{-\infty}},\dots,X_n^{q^{-\infty}}]\!]/(\delta^{q^{-m}} - \tau^{q^{-m}} \mid m \in \mathbb{N})^-.$$

In particular, as Δ *has coefficients in* $\mathcal{O}_E$*, this implies*

$$A^{\alpha}_{\infty,\mathcal{O}_{\mathbb{C}_p}} \cong \mathcal{O}_{\mathbb{C}_p}[\![X_1^{q^{-\infty}},\dots,X_n^{q^{-\infty}}]\!]/(\delta^{q^{-m}} - \sigma(\tau^{q^{-m}}) \mid m \in \mathbb{N})^-,$$

where σ *is the embedding* $\mathcal{O}_{\breve{E}} \hookrightarrow \mathcal{O}_{\widehat{E}^{\mathrm{ab}}}$ *corresponding to* α *under the Artin map (and a fixed choice of embedding* $\mathcal{O}_E \hookrightarrow \mathcal{O}_{\widehat{E}^{\mathrm{ab}}}$*). With* $\alpha = 1$*, this is the isomorphism* (5.1)*.*

Proposition 5.2.18. *The group* G^1 *is generated by elements of the form*

- $(a, a, 1) \in G$ *for* $a \in E^\times$*.*
- $(g, d, 1) \in G$ *such that* $\det(g)\,\mathrm{Nrd}(d)^{-1} \in \mathcal{O}_E^\times$*.*
- $(1, \vartheta^{-m}, \sigma)$ *for* $\sigma \in \mathrm{W}_E$ *with* $\sigma|_{\breve{E}} = \Phi^m$*.*

These elements act on $A_{\infty,\mathcal{O}_{\mathbb{C}_p}}$ *as follows.*

- $(a, a, 1)$ *acts trivially.*
- $(g, d, 1)$ *acts by the morphism of* $\mathcal{O}_{\mathbb{C}_p}$*-algebras*
$$A^{\alpha}_{\infty,\mathcal{O}_{\mathbb{C}_p}} \to A^{\det(g)^{-1}\mathrm{Nrd}(d)\alpha}_{\infty,\mathcal{O}_{\mathbb{C}_p}}, \quad X_i \mapsto (g, d^{-1})^*(X_i) \text{ for } i = 1,\dots,n.$$
- $(1, \vartheta^{-m}, \sigma)$ *acts by the morphism of* $\mathcal{O}_E$*-algebras* $A^{\alpha}_{\infty,\mathcal{O}_{\mathbb{C}_p}} \to A^{a_E(\sigma)\alpha}_{\infty,\mathcal{O}_{\mathbb{C}_p}}$ *given by* $X_i \mapsto X_i$ *for* $i = 1,\dots,n$ *and* $a \mapsto \sigma(a)$ *for* $a \in \mathcal{O}_{\mathbb{C}_p}$*.*

Proof. $\square$

In particular, there is a natural left action of $D^\times = \mathrm{End}_{(\mathcal{O}_E\text{-FM}/\overline{\mathbb{F}}_q)}(\mathcal{F}_0)[\varpi^{-1}]^\times$ on $\tilde{\mathcal{F}}$.

By the explicit description of λ in Lemma 5.2.13, we find that $\lambda_i(\Pi x) = \lambda_i(x)^q$ for $x \in \tilde{\mathcal{F}}(S)$ and $i \in \mathbb{N}_0$.

As $\tilde{\mathcal{F}}$ is a K-vector space object, the isomorphism $\lambda : \tilde{\mathcal{F}} \to \mathrm{Nilp}^\flat$ equips $\mathrm{Nilp}^\flat$ with the stucture of a K-vector space object. Upon choosing a suitable coordinate for $\mathcal{F}$, this structure admits the following description.

Finally, we remark how to express the logarithm map in terms of the isomorphism λ.

Lemma 5.2.19. *Let H be the standard formal $\mathcal{O}_K$-module of height n over $R = \mathcal{O}_{\breve{K}}$. We have a commutative diagram (cf. [BW11, Lemma 2.6.1])*

$$\begin{tikzcd}
(x_0, x_1, \dots) \arrow[d, mapsto] & \in & \widetilde{H}(S) \arrow[rr, "\lambda"] \arrow[dr, "\log_H"'] & & \mathrm{Nil}^\flat(S) \arrow[dl] & \ni & (y, y^{1/q}, \dots) \arrow[d, mapsto] \\
\sum_{i=0}^\infty \frac{x_0^{q^{ni}}}{\varpi^i} & & & S \otimes_{\mathcal{O}_K} K & & & \sum_{i=-\infty}^\infty \frac{y^{q^{ni}}}{\varpi^i}
\end{tikzcd}$$

With this terminology, we have $\log_H((\Pi^j x)_0) = \sum_{i=-\infty}^\infty \frac{y^{ni+j}}{\varpi^i}$.

Proof. This follows directly from the remark above. Let $x \in \widetilde{H}(S)$ and write $\lambda(x) = (y, y^{1/q}, \dots)$. We have $x_0 = \lim_{s\to\infty}[\varpi^s]_H(y^{-ns})$, hence

$$\log_H(x_0) = \lim_{s\to\infty}\left(\varpi^s \log_H(y^{1/q^{ns}})\right) = \lim_{s\to\infty}\left(\sum_{i=0}^\infty \frac{y^{q^{n(i-s)}}}{\varpi^{i-s}}\right) = \sum_{i=-\infty}^\infty \frac{y^{q^{ni}}}{\varpi^i}.$$

The second part is an immediate consequence. □

5.3 The Quasilogarithm Map

We keep the assumptions on A, R and S from the previous subsection. That is, A is a local ring with finite residue field and uniformizer ϖ, R is a local A-algebra with maximal ideal I complete with respect to the I-adic topology and algebraically closed residue field k_R, and S denotes an admissible R-algebra (where $R \to S$ is continuous with the I-adic topology on R) with ideal of definition $J \subseteq S$ containing the image of I.

The aim of this subsection is to define, attached to any ϖ-divisible formal A-module $\mathcal{F}$ over R, a map

$$\mathrm{qlog}_{\mathcal{F}} : \tilde{\mathcal{F}}(S) \to \mathrm{D}(\mathcal{F}) \otimes_R (S \otimes_A K),$$

called the quasi-logarithm map. We give an explicit description of this map if $\mathcal{F} = \mathrm{FG}(H)$ is the standard $\mathcal{O}_K$-module over $\mathcal{O}_{\breve{K}}$.

The construction of $\mathrm{qlog}_{\mathcal{F}}$ is as follows. Let $0 \to \mathcal{V} \xrightarrow{\psi} \mathcal{E} \xrightarrow{\phi} \mathcal{F} \to 0$ be the universal additive extension of $\mathcal{F}$. For any sequence $(x_1, x_2, \dots) \in \tilde{\mathcal{F}}(S)$, choose an arbitrary sequence $(y_1, y_2, \dots) \in \tilde{\mathcal{E}}(S)$ such that y_i is a lift of x_i under the map $\mathcal{E}(S) \to \mathcal{F}(S)$. Let y be the limit $y = \lim_{i\to\infty}[\varpi]^i_{\mathcal{E}}(y_i)$ and put

$$\mathrm{qlog}_{\mathcal{F}}((x_1, x_2, \dots)) = \log_{\mathcal{E}}(y) \in \mathrm{D}(\mathcal{F}) \otimes_R (S \otimes_A K).$$

Proposition 5.3.1. *This construction yields a well-defined map.*

Proof. We may assume that $\mathcal{F}$ and $\mathcal{V}$ come from formal module laws F and V, and we may furthermore assume that $\mathcal{E} = \mathrm{FG}(E)$ for an $\mathcal{O}_K$-module law E obtained by Lemma B.1.3. Now $(x_1, x_2, \dots)$ is a sequence in S^{∞} and $(y_1, y_2, \dots)$ is a sequence of elements in $(S^{\infty})^n$. It suffices to show that $y = \lim_{i\to\infty}[\varpi]^i_E(y_i)$ exists and that it is independent of the choice of lifts $(y_1, y_2, \dots)$. Both claims follow from the additivity of $\mathcal{V}$, implying that $[\varpi]_V(T) = \varpi T$. The sequence $([\varpi^i]_E(y_i))$ converges, as for positive integers $i \le j$, we have

$$[\varpi^i](y_i) - [\varpi^j](y_j) = [\varpi^i]([\varpi^{i-j}]y_j - y_i) \in \psi(\varpi^i (S^{\infty})^{n-1}) \subseteq J^i (S^{\infty})^n.$$

If $(y_1', y_2', \dots)$ is another sequence of lifts, put $y' = \lim_{i\to\infty}[\varpi^i]_E(y_i') \in S^{\infty}$. Now there exists some $z \in \mathcal{V}(S)$ such that $y - y' = \psi(z)$. But by construction $z \in \bigcap_{i\in\mathbb{N}} \varpi^i (S^{\infty})^{n-1} = 0$. $\square$

Let us now consider the case where $\mathcal{F} = \mathrm{FG}(H)$ comes from the standard formal $\mathcal{O}_K$-module of height n over $\mathcal{O}_{\breve{K}}$. Then from Proposition 2.5.6 we have the distinguished basis elements of $\mathrm{Ext}(H, \widehat{\mathbb{G}}_a)$ corresponding to the symmetric 2-cocycles δf_i, $1 \le i \le n-1$ where $f_i(T) = \frac{1}{\varpi} \log_H(T^{q^i})$. Also recall that, setting $f_0(T) = \log_H(T)$, the elements $(f_0, f_1, \dots, f_{n-1})$ freely generate $\mathrm{QLog}(H)$. The universal additive extension now corresponds to the symmetric 2-cocycle $(\delta f_1, \dots, \delta f_{n-1}) \in \mathrm{SymCoc}^2(H, V)$. We can make the quasi-logarithm map explicit.

Proposition 5.3.2. *Let* $x = (x_0, x_1, \dots) \in \widetilde{H}(S)$. *With respect to the basis* $(\log_H(T), \log_H(T^q), \dots, \log_H(T^{q^{n-1}}))$ *of* $\mathrm{QLog}(H) \otimes_{\mathcal{O}_K} K$*, the quasi-logarithm map is given by*

$$\mathrm{qlog}_H(x) = \Big(\log_H(x_0), \log_H((\Pi x)_0), \dots, \log_H((\Pi^{n-1}x)_0)\Big) \in (S \otimes K)^n.$$

Here, $\Pi x = ((\Pi x)_0, (\Pi x)_1, \dots)$ *is the image of* x *under* Π*, the automorphism of* $\widetilde{H}(S)$ *induced by the (relative) Frobenius quasi-isogeny on* H_0*, cf. Definition 5.2.9.*

We postpone the proof to state the following auxiliary result.

Lemma 5.3.3. *Let* $x = (x_0, x_1, \dots) \in \widetilde{H}(S)$. *For positive integers* i *and* j *we have*

$$\log_H((\Pi^j x)_i) = \lim_{r\to\infty} \varpi^r \log_H(x_{r+i}^{q^j}).$$

Proof. Tracing through the commutative square (with λ and μ the isomorphisms from the previous subsection)

$$\begin{array}{ccc} \widetilde{H}(S) & \xrightarrow{\lambda} & \mathrm{Nilp}^{\flat}(S) \\ \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle (y_i)_i \mapsto (y_i^q)_i} \\ \widetilde{H}(S) & \xleftarrow{\mu} & \mathrm{Nilp}^{\flat}(S), \end{array}$$

we find

$$(\Pi^j x)_i = \lim_{s\to\infty} \lim_{r\to\infty} \left([\varpi]^s_H(x_{r+s+i}^{q^{nr+j}})\right). \tag{5.10}$$

The claim follows after applying $\log_H$ and making repeated use of the functional equation $\log_H(T^{q^n}) = \varpi \log_H(T) + \varpi T$. $\square$

Proof of Proposition 5.3.2. Using the coordinates provided by $(\delta f_1, \dots, \delta f_{n-1})$, the universal additive extension of H is isomorphic to

$$0 \to \widehat{\mathbb{G}}_a^{n-1} \to E \to H \to 0,$$

where E is a module law with

$$[\varpi]_E(\mathbf{X}, T) = \Big(\varpi X_1 + (\delta_\varpi f_1)(T), \dots, \varpi X_{n-1} + (\delta_\varpi f_{n-1})(T), [\varpi]_H(T)\Big).$$

Beginning with $x = (x_0, x_1, \dots) \in \widetilde{H}(S)$, lifting to $(y_0, y_1, \dots) \in E(S)^{\mathbb{N}}$ and writing $y = \lim_{i\to\infty}[\varpi]_E^i(y_i)$, we find

$$y = \Big(\lim_{r\to\infty}(\delta_{\varpi^r} f_1)(x_r), \dots, \lim_{r\to\infty}(\delta_{\varpi^r} f_{n-1})(x_r), x_0\Big) \in E(S).$$

Now, Lemma 5.3.3 provides the equality

$$\lim_{r\to\infty} \delta_{\varpi^r} f_i(x_r) = \frac{1}{\varpi} \lim_{r\to\infty} \varpi^r \log_H(x_r^{q^{nr+i}}) - \frac{1}{\varpi} \log_H\left(x_0^{q^i}\right) = \frac{1}{\varpi}\left(\log_H((\Pi^i x)_0) - \log_H(x_0^{q^i})\right).$$

We need to calculate $\log_E(y)$, which calls for an explicit description of $\log_E : E \otimes (R \otimes_A K) \to (\widehat{\mathbb{G}}_a \otimes (R \otimes_A K))^n$. Tracing through the procedure provided in Subsection 2.3, we find

$$\log_E(\mathbf{X}, T) = \left(X_1 + \tfrac{1}{\varpi}\log_H(T^q), \dots, X_{n-1} + \tfrac{1}{\varpi}\log_H(T^{q^{n-1}}), \log_H(T)\right).$$

This representation is with respect to the basis $(f_1, \dots, f_{n-1}, f_0)$. The claim follows. $\square$

5.4 An Approximation of the Determinant Morphism

Let H be the standard formal $\mathcal{O}_K$-module over $\mathcal{O}_{\breve{K}}$ of height n. Write $\wedge H$ for the formal $\mathcal{O}_K$-module over $\mathcal{O}_{\breve{K}}$ with logarithm

$$\log_{\wedge H}(T) = \sum_{i=0}^{\infty} (-1)^{(n-1)i} \frac{T^{qi}}{\varpi^i}.$$

By Hazewinkel's integrality Lemma (cf. Theorem 2.6.1), such a module law exists. We have $\mathrm{D}(\wedge H) = \wedge^n \mathrm{D}(H)$. We follow [BW11, Theorem 2.10.3] to describe a map $\delta : \widetilde{H}^n \to \widetilde{\wedge H}$ making the square

$$\begin{array}{ccc} \widetilde{H}^n(S) & \xrightarrow{\quad\delta\quad} & \widetilde{\wedge H}(S) \\ {\scriptstyle \mathrm{qlog}_H \times \cdots \times \mathrm{qlog}_H}\downarrow & & \downarrow{\scriptstyle \mathrm{qlog}_{\wedge H}} \\ \mathrm{D}(H)^n \otimes (S \otimes_{\mathcal{O}_K} K) & \xrightarrow{\mathrm{det}} & \mathrm{D}(\wedge H) \otimes (S \otimes_{\mathcal{O}_K} K) \end{array} \tag{5.11}$$

commute.

Let $(s_1, \dots, s_n) \in \widetilde{H}(S)^n$, and write $x_i = \lambda(s_i) \in \mathrm{Nilp}^\flat(S)$, which are elements in S^∞ with distinguished q-power roots. Here $\lambda : \widetilde{H} \to \mathrm{Nilp}^\flat$ is the isomorphism from Section 5.2 with inverse $\mu = (\mu_0, \mu_1, \dots)$. We set

$$\delta_0(s_1, \dots, s_n) = \sum_{(a_1, \dots, a_n)} \varepsilon(a_1, \dots a_n) \mu_0(x_1^{q^{a_1}} \cdots x_n^{q^{a_n}}) \in \wedge H(S),$$

where

- The sum takes place in $\wedge H(S)$.
- The sum ranges over n-tuples $(a_1, \dots, a_n)$ of (possibly negative) integers satisfying $a_1 + \cdots + a_n = n(n-1)/2$, subject to the condition that each a_i occupies a distinct residue class modulo n.
- The expression $\varepsilon(a_1, \dots, a_n)$ denotes the sign of the permutation $i \mapsto a_{i+1} \pmod{n}$ of $(0, \dots, n-1)$.

Proposition 5.4.1. *The map δ_0 makes the diagram*

$$\begin{tikzcd} \tilde{H}^n(S) \arrow[r, "\delta_0"] \arrow[d, "\mathrm{qlog}_H^n"'] & \wedge H(S) \arrow[d, "\log_{\wedge H}"] \\ \mathrm{D}(H)^n \otimes (S \otimes K) \arrow[r, "\det"] & \mathrm{D}(\wedge H) \otimes (S \otimes K) \end{tikzcd}$$

commute. It is $\mathcal{O}_K$-multilinear and alternating.

Proof. This is part of the proof of [BW11, Theorem 2.10.3]. Commutativity follows from

$$\begin{aligned} \log_{\wedge H}(\delta_0(s_1, \dots, s_n)) &= \sum_{(a_1, \dots, a_n)} \varepsilon(\mathbf{a}) \log_{\wedge H} \mu_0(x_1^{q^{a_1}} \cdots x_n^{q^{a_n}}) \\ &= \sum_{(a_1, \dots, a_n)} \varepsilon(\mathbf{a}) \sum_{m \in \mathbb{Z}} (-1)^{(n-1)m} \frac{x_1^{q^{a_1+m}} \cdots x_n^{q^{a_n+m}}}{\varpi^m} = \det \left(\sum_{m \in \mathbb{Z}} \frac{x_i^{q^{mn+j-1}}}{\varpi^m} \right)_{\substack{1 \le i \le n, \\ 1 \le j \le n}}, \end{aligned}$$

which is equal to $\det(\mathrm{qlog}_H^n(s_1, \dots, s_n))$ by Proposition 5.3.2 and Lemma 5.2.19. The fact that δ_0 is multilinear and alternating ultimately follows from the corresponding properties of det, the fact that $\mathrm{Ker}(\log_H) = \wedge H[\varpi^\infty]$ (cf. Lemma 2.3.6) and topological considerations in the induced diagram in the category of adic spaces over $(\breve{K}, \mathcal{O}_{\breve{K}})$. □

This allows us to define the sought for morphism of functors $\delta : \widetilde{H}^n \to \widetilde{\wedge H}$.

Definition 5.4.2. Put $\delta_i(s_1, \dots, s_n) = \delta_0(\varpi^{-i} s_1, \dots, s_n)$. Then $\delta = (\delta_0, \delta_1, \dots)$ yields a map $\widetilde{H}^n \to \widetilde{\wedge H}$. It is K-multilinear and alternating.

Using the canonical identifications $\widetilde{H}^n \cong (\mathrm{Nilp}^\flat)^n$ and $\widetilde{\wedge H} \cong \mathrm{Nilp}^\flat$, the morphism δ yields a map $(\mathrm{Nilp}^\flat)^n \to \mathrm{Nilp}^\flat$, which in turn is the same as a power series

$$\Delta(X_1, \dots, X_n) \in \mathcal{O}_{\breve{K}}[\![X_1^{q^{-\infty}}, \dots, X_n^{q^{-\infty}}]\!]$$

together with distinguished q-th power roots. We have the following approximation of Δ, cf. [BW11, Lemma 2.10.4].

Lemma 5.4.3. *We have*

$$\Delta(X_1, \ldots, X_n) \equiv \det(X_i^{q^{j-1}})_{\substack{1 \leq i \leq n, \\ 1 \leq j \leq n}}$$

modulo terms of degree greater than $1 + q + \cdots + q^{n-1}$.

Proof. By Proposition 5.4.1 and the explicit description of the quasi-logarithm map in Proposition 5.3.2, we have the equality

$$\sum_{k=-\infty}^{\infty} (-1)^{(n-1)k} \frac{\Delta(X_1, \ldots, X_n)^{q^k}}{\varpi^k} = \det \left(\sum_{k=-\infty}^{\infty} \frac{X_i^{q^{nk+j-1}}}{\varpi^k} \right)_{1 \leq i,j \leq n}$$

of elements inside $\breve{E}[\![X_1^{q^{-\infty}}, \ldots, X_n^{q^{-\infty}}]\!]$ (equipped with the $(\varpi, X_1, \ldots, X_n)$-adic topology). The claim follows after comparing coefficients of the respective series. □

6 Mieda's Approach to the Explicit Local Langlands Correspondence

We give a brief summary of the content of results in [Mie16]. These results take the following form.

6.1 The Specialization Map

Let R be a complete discrete valuation ring with residue field $\overline{\mathbb{F}}_q$ and write E for the field of fractions of R. Let $\mathcal{Y}$ be a flat and topologically of finite type formal scheme over $\operatorname{Spf} R$. Utilizing a classical construction of Raynauld [Ray74], we can consider the corresponding rigid generic fiber $d(\mathcal{Y})$, which we may consider as an analytic adic space over $\operatorname{Spa}(E, R)$ by [Hub13, Section 1.9]. We denote by $\mathcal{Y}_s = \mathcal{Y} \times_{\operatorname{Spf}(R)} \operatorname{Spec} \overline{\mathbb{F}}_q$ the special fiber of $\mathcal{Y}$, which is of finite type over $\operatorname{Spec} \overline{\mathbb{F}}_q$. Recall that we may identify the étale sites $(\mathcal{Y}_{\mathrm{red}})_{\text{ét}}$ and $\mathcal{Y}_{s,\text{ét}}$ via the isomorphism induced by the closed immersion of formal schemes $\mathcal{Y}_s \hookrightarrow \mathcal{Y}_{\mathrm{red}}$.
By [Hub13, Lemma 3.5.1] we have a morphism of sites

$$\lambda_{\mathcal{Y}} : d(\mathcal{Y})_{\text{ét}} \to (\mathcal{Y}_{\mathrm{red}})_{\text{ét}} = \mathcal{Y}_{s,\text{ét}}.$$

For a positive integer m, let us fix the torsion ring $\Lambda = \mathbb{Z}/\ell^m\mathbb{Z}$ and write $\underline{\Lambda}_{d(\mathcal{Y})}$ (respectively $\underline{\Lambda}_{\mathcal{Y}_s}$) for the corresponding constant sheaf on $d(\mathcal{Y})_{\text{ét}}$ (respectively $\mathcal{Y}_{s,\text{ét}}$). Then pushforward along $\lambda_{\mathcal{Y}}$ induces a left-exact functor of Grothendieck Abelian categories

$$\lambda_{\mathcal{Y},*} : (\underline{\Lambda}_{d(\mathcal{Y})_{\text{ét}}}\text{-Mod}) \to (\underline{\Lambda}_{\mathcal{Y}_{s,\text{ét}}}\text{-Mod}),$$

which is right-adjoint to the exact pullback functor $\lambda_{\mathcal{Y}}^*$. This allows for the following definitions.

Definition 6.1.1 (Formal Nearby Cycle Functor and Specialization Map)**.** We denote by

$$R\Psi_{\mathcal{Y}} : D(d(\mathcal{Y})_{\text{ét}}, \underline{\Lambda}_{d(\mathcal{Y})}) \to D(\mathcal{Y}_{s,\text{ét}}, \underline{\Lambda}_{\mathcal{Y}_s})$$

the right derived functor of $\lambda_{\mathcal{Y},*}$, which we call the formal nearby cycle functor. The unit of the adjunction $\lambda_{\mathcal{Y}}^* \dashv R\Psi_{\mathcal{Y}}$ induces, evaluated at $\underline{\Lambda}_{\mathcal{Y}_s}$, a morphism

$$\mathrm{sp}^* : \underline{\Lambda}_{\mathcal{Y}_s} \to R\Psi_{\mathcal{Y}}(\lambda_{\mathcal{Y}}^* \underline{\Lambda}_{\mathcal{Y}_s}) = R\Psi_{\mathcal{Y}}(\underline{\Lambda}_{d(\mathcal{Y})}).$$

which we shall call the specialization map.

Our goal is to compare the compact supported cohomologies of $\mathcal{Y}_s$ and $d(\mathcal{Y})$. From classical results (cf. for example [Hub13, Corollary 0.7.9], which essentially deals with the case where $\mathcal{Y}$ is algebraizable), we might hope that if $\mathcal{Y}$ admits a suitable compactification, there is isomorphism

$$R\Gamma_c(\mathcal{Y}_s, R\Psi_{\mathcal{Y}}\underline{\Lambda}_{d(\mathcal{Y})}) \xrightarrow{\sim} R\Gamma_c(d(\mathcal{Y}), \Lambda) \tag{6.1}$$

inducing the morphism (which we also call specialization map)

$$\mathrm{sp}^* : R\Gamma_c(\mathcal{X}_s, \underline{\Lambda}_{\mathcal{Y}_s}) \to R\Gamma_c(d(\mathcal{X}), \underline{\Lambda}_{d(\mathcal{Y})}).$$

And indeed, in [Mie14, Corollary 4.29], Mieda constructs an isomorphism as in (6.1) if $\mathcal{Y}$ is pseudo-compactifiable over Spf A in the sense of [Mie14, Definition 4.24]. We do not explain this notion here, but we remark that all affine formal schemes that are topologically of finite type are pseudo-compactifiable (cf. [Mie14, Example 4.25]), which is the only important case for us.

Let us assume that $d(\mathcal{Y})$ is smooth. Then, by [Hub13, Theorem 7.3.4], we have the trace map

$$\mathrm{Tr}_{d(\mathcal{Y})} \colon R\Gamma_c(d(\mathcal{Y}), \underline{\Lambda}_{d(\mathcal{Y})}) \to \underline{\Lambda}_{d(\mathcal{Y})}(-d)[-2d].$$

(Here, and in the following, $\mathcal{F}(r)$ denotes $\mathcal{F} \otimes \mu_{\ell m}^{\otimes r}$ for any étale $\underline{\Lambda}$-sheaf $\mathcal{F}$ and any $r \in \mathbb{Z}$). Let us write $i : \mathcal{Y}_s \to \mathrm{Spec}(\overline{\mathbb{F}}_q)$ for the structure map. Then by the adjunction $Ri_! \dashv i^!$ and the isomorphism in (6.1), we obtain a map

$$\mathrm{cosp}^* \colon R\Psi_{\mathcal{Y}}\underline{\Lambda}_{d(\mathcal{Y})} \to i^! \underline{\Lambda}_{d(\mathcal{Y})}(-d)[-2d].$$

Furthermore, we have the trace map

$$\mathrm{Tr}_i : Ri_! \underline{\Lambda}_{\mathcal{Y}_s} \to \underline{\Lambda}_{\mathcal{Y}_s}(-d)[-2d].$$

[I was only able to find constructions of this in the case where i is smooth, and in this case it seems that the Gysin map, as defined below, is simply the identity on Λ. Is this bad?]

In this setting, we have the following result, which is Theorem 2.1 of [Mie16].

Theorem 6.1.2 (Mieda's theorem about the Specialization Map)**.** *The composite*

$$\underline{\Lambda}_{\mathcal{Y}_s} \xrightarrow{\mathrm{sp}^*} R\Psi_{\mathcal{Y}}\underline{\Lambda}_{d(\mathcal{Y})} \xrightarrow{\mathrm{cosp}^*} i^! \underline{\Lambda}_{\mathcal{Y}_s}(-d)[-2d]$$

is equal to the Gysin map with respect to i, that is, the map $\Lambda \to i^!\Lambda(-d)[-2d]$ obtained by the adjoint of the trace map.

Theorem 6.1.3 (Mieda's Theorem about the Specialization Map). *Let $\pi : \mathcal{Y}_s \to$*

Proof. $\square$

6.2 Application to the Lubin–Tate Tower

For our purposes the following results, taken from [Mie16, Corollary 4.6], suffice.

Theorem 6.2.1 (Mieda's Result for the Lubin–Tate Tower). *Let J be a subgroup of G^1 whose action on $M^{(0)}_{\infty,E}$ stabilizes U and extends to an action on $\mathcal{X}$. Assume that there exists an affine scheme Y of finite type over $\overline{\mathbb{F}}_q$ equipped with a right action of J such that there is an isomorphism*

$$\mathcal{X}_s = \operatorname{Spec}(A \otimes_R \overline{\mathbb{F}}_q) \xrightarrow{\sim} Y^{perf}$$

of schemes over $\operatorname{Spec} \overline{\mathbb{F}}_q$, equivariant for the action of J on both sides. Then we have the following.

1. *We have a J-equivariant homomorphism*

$$\mathrm{sp}^* : H^{n-1}_c(Y, \overline{\mathbb{Q}}_\ell) \to \operatorname*{colim}_K H^{n-1}_c(M^{(0)}_{K,E}, \overline{\mathbb{Q}}_\ell) =: H'_{\mathrm{LT}}.$$

2. *If Y is pure-dimensional and smooth over $\operatorname{Spec}(\overline{\mathbb{F}}_q)$ and V is a subspace of $H^{n-1}_c(Y, \overline{\mathbb{Q}}_\ell)$ such that the composite*

$$V \hookrightarrow H^{n-1}_c(Y, \overline{\mathbb{Q}}_\ell) \hookrightarrow H^{n-1}_c(Y, \overline{\mathbb{Q}}_\ell)$$

is injective, the composite

$$V \hookrightarrow H^{n-1}_c(Y, \overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{sp}^*} H'_{\mathrm{LT}}$$

is injective as well.

7 Deligne–Lusztig Theory for Depth Zero Representations

The aim of this section is to outline the construction of a correspondence between certain characters of $\mathbb{F}^\times_{q^n}$ (with values in $\mathbb{C}^\times$) and cuspidal representations of $\mathrm{GL}_n(\mathbb{F}_q)$. The correspondence we construct here is an instance of a more general theory developed by Deligne–Lusztig. In [DL76], they construct for any connected reductive algebraic group $G = G_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ and any Frobenius-stable maximal torus $T \subseteq G$ a correspondence associating to certain characters θ of T^{Frob} a virtual representation $R_{T,\theta}$ of G^{Frob}. These virtual representations arise from the ℓ-adic cohomology (with $\ell \neq p$) of a certain variety $\mathrm{DL}_{G,T}$ admitting commuting actions by G^{Frob} and T^{Frob}. In this section, we give explicit descriptions of the occuring spaces in the situation where $G = \mathrm{GL}_{V_0 \otimes \overline{\mathbb{F}}_q}$ for some n-dimensional $\mathbb{F}_q$-vector space V_0 and $T \subset G$ is a maximal Frobenius-stable torus with $T(\overline{\mathbb{F}}_q) = \mathbb{F}_{q^n}^\times$. The main theorems of the theory are stated as facts, proofs are omitted.

7.1 Deligne–Lusztig Varieties for the General Linear Group

We begin by introducing (full) flags and their classifying objects, flag varieties. Let k be a field and let V be a finite dimensional k-vector space of dimension n. We write $\tilde{V}$ for the corresponding quasi-coherent sheaf on $\operatorname{Spec} k$, and GL_V for the general linear group scheme of $\tilde{V}$.

Definition 7.1.1 (Flag Variety)**.** Let $X : (\mathrm{Sch}/k)^{\mathrm{op}} \to (\mathrm{Set})$ be the functor assigning to each k-scheme $f : S \to \operatorname{Spec} k$ the set

$$X(S) = \left\{ \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_{n-1} \subset f^*\tilde{V} \;\middle|\; \begin{array}{c} \mathcal{F}_i \text{ is, for all } i, \text{ a locally direct summand} \\ \text{of } f^*\tilde{V}, \text{ locally free of rank } i \end{array} \right\}.$$

Recall that a subsheaf $\mathcal{F}_i \subset f^*\tilde{V}$ is locally a direct summand if it is quasi-coherent, and for each $s \in S$ there is some neighbourhood U of s such that $\mathcal{F}_i|_U$ is a direct summand of $f^*\tilde{V}|_U$. The S-valued points of X are called families of flags over S.

Elements of $X(k)$ are called (full) flags. They are given by an increasing $n-1$-tuple of vector spaces

$$F_\bullet = (F_1 \subsetneq F_2 \subsetneq \ldots F_{n-1} \subsetneq V) \in X(k).$$

There are natural morphisms

$$\nu_i : X \to \mathrm{Grass}_{V,i}, \quad (\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{n-1} \subset f^*\tilde{V}) \mapsto (f^*\tilde{V}^\vee \twoheadrightarrow \mathcal{F}_i^\vee),$$

where $\mathrm{Grass}_{V,i}$ denotes the Grassmannian parametrizing surjections of $f^*\tilde{V}^\vee$ to locally free coherent modules of rank i, as defined in [Ric22].

Proposition 7.1.2. *The induced morphism of functors*

$$X \to \mathrm{Grass}_{V,1} \times_{\operatorname{Spec} k} \cdots \times_{\operatorname{Spec} k} \mathrm{Grass}_{V,n-1}$$

is representable by a closed embedding. In particular, as $\mathrm{Grass}_{V,d}$ is representable by a projective scheme for integers $1 \leq d \leq n-1$, the functor X is representable by a projective scheme.

Proof. Upon picking a basis of V, the claim can be checked directly on the standard affine cover of the Grassmannians, where the condition that $\mathcal{F}_i$ is contained in $\mathcal{F}_{i+1}$ is cut out by a polynomial equation. For representability of $\mathrm{Grass}_{V,d}$, cf. [Ric22, Theorem 5.1.4]. □

There is a natural GL_V-action on X, induced by the natural action of $\mathrm{GL}_V(S)$ on $f^*\tilde{V}$. Given a flag $F_\bullet \in X(k)$, we write $B_{F_\bullet} \subset \mathrm{GL}_V$ for the isotropy subgroup of $F_\bullet$ under this action. In [DL76], the authors work with schemes arising as quotients G/B where B is some Borel subgroup of a connected, reductive algebraic group G. The following proposition shows that X is isomorphic to the quotient of $\mathrm{GL}_V / B_{F_\bullet}$.

Proposition 7.1.3. *The morphism of schemes $\mu_{F_\bullet} : \mathrm{GL}_V \to X$, $g \mapsto g.F_\bullet$ yields an isomorphism $\mathrm{GL}_V / B_{F_\bullet} \to X$.*

Proof. We show that Zariski-locally, $\mu_{F_\bullet}$ induces an isomorphism $\mathrm{GL}_V(S)/B_{F_\bullet}(S) \to X(S)$. Let $(v_1, v_2, \ldots, v_n)$ be a basis of V such that each F_i is generated by the first i basis vectors. Given any k-scheme S and a family of flags $F'_\bullet \in X(S)$, there is a Zariski-cover $\phi : S' \to S$ (with structure map to $\operatorname{Spec} k$ denoted by f') trivializing all of the quotients F'_i/F'_{i-1} for $i = 1, \ldots, n$. Hence we may choose generators $w_i \in \Gamma(S', \phi^*(F'_i/F'_{i-1}))$, and lift them to elements $\tilde{w}_i \in \Gamma(S', f'^*\tilde{V})$. The global sections w_i generate $f'^*\tilde{V}$, and the $\mathcal{O}_{S'}$-linear map $f'^*v_i \mapsto w_i$ yields an element in $\mathrm{GL}_V(S')$, unique up to an element in $B_{F_\bullet}(S')$. Thereby $X(S') \cong \mathrm{GL}_V(S')/B_{F_\bullet}(S')$. □

Remark. The proof shows that the quotient sheaf GL_V / B,

Corollary 7.1.4. *The scheme X is smooth over k, of dimension $\frac{n(n-1)}{2}$.*

Proof. This follows as quotients of smooth algebraic groups by algebraic subgroups are smooth (cf. [Mil17, Corollary 5.26]), and the fact that

$$n^2 = \dim \mathrm{GL}_V = \dim B + \dim X = \frac{n(n+1)}{2} + \dim X,$$

cf. [Mil17, Proposition 5.23]. □

Write $\mathbb{T}_X$ for the sheaf assigning to an X-scheme $S \to X$ (corresponding to a family of flags $(\mathcal{F}_i)_i \in X(S)$) the group

$$\mathbb{T}_X(S) = \mathrm{Aut}_{\mathcal{O}_S}(\mathcal{F}_1/\mathcal{F}_0) \times \cdots \times \mathrm{Aut}_{\mathcal{O}_S}(\mathcal{F}_n/\mathcal{F}_{n-1}) = \mathbb{G}^n_{m,X}(S).$$

Definition 7.1.5 (Classifying Space of Marked Flags)**.** Let Y be the functor $(\mathrm{Sch}/X)^{\mathrm{op}} \to (\mathrm{Set})$ given by sending a morphism $S \to X$, corresponding to a family of flags $(\mathcal{F}_i)_i \in X(S)$, to the set

$$Y(S) = \left\{(e_1, \ldots, e_n) \mid e_i : \mathcal{O}_S \xrightarrow{\sim} \mathcal{F}_i/\mathcal{F}_{i-1} \text{ for } i = 1, \ldots, n-1\right\}.$$

Here, $\mathcal{F}_0$ is the zero-sheaf.

Just like X, the functor Y comes with a natural action by GL_V and the natural morphism

$$Y(S) \to X(S), \quad (\mathcal{F}_i, e_i)_i \mapsto (\mathcal{F}_i)_i \tag{7.1}$$

is equivariant for this action. One readily checks that Y is a sheaf on $(\mathrm{Sch}/X)_{\mathrm{Zar}}$, the big Zariski site of schemes over X. By design, it is a $\mathbb{T}_X$-torsor and thereby admits a Zariski-cover of open subfunctors isomorphic to $\mathbb{T}_X$. Hence it is representable (cf. [GW20, Theorem 8.9]), and the morphism $Y \to X$ is smooth and affine.

Furthermore, the scheme Y is also isomorphic to certain quotients of algebraic groups. Let $(F_\bullet, e_\bullet) \in Y(k)$ be a marked flag, and write $U_{F_\bullet, e_\bullet} \subset \mathrm{GL}_V$ for the (unipotent) isotropy subgroup of $(F_\bullet, e_\bullet)$ under the action of GL_V.

Lemma 7.1.6. *In this situation, $Y \cong \mathrm{GL}_V / U_{F_\bullet, e_\bullet}$.*

Proof. This can be shown using the same arguments as in the proof of Proposition 7.1.3. $\square$

If $(v_1, \dots, v_n)$ is a basis for V, we write $F(v_1, \dots, v_n)$ for the (marked) flag spanned by the vectors $(v_1, \dots, v_n)$. More generally, if S is a k-scheme $(v_1, \dots, v_n)$ is a tuple of elements in $\Gamma(S, f^*\tilde{V})$ such that the induced map $(v_1, \dots, v_n) : \mathcal{O}_S^n \to f^*\tilde{V}$ is an isomorphism (in which case we call $(v_i)_i$ a basis), we write $F(v_1, \dots, v_n)$ for the corresponding family of (marked) flags.

Recall the Bruhat decomposition for GL_V. Fixing a basis $(e_1, \dots, e_n)$ of V, we obtain an injection $\Sigma_n \hookrightarrow \mathrm{GL}_V$ (assigning to each $w \in \Sigma_n$ the corresponding permutation matrix), and a (marked) flag $F_\bullet^{\mathrm{std}} = F(e_1, \dots, e_n) \in X(k)$. For any such choice of a basis, we define O_w as the GL_V-orbit of the pair of flags $(F_\bullet^{\mathrm{std}}, w.F_\bullet^{\mathrm{std}}) \in (X \times X)(k)$. Note that this does not depend on the choice of basis. The Bruhat decomposition states that all GL_V-orbits inside $X \times X$ are of this form.

Proposition 7.1.7 (Bruhat Decomposition). *There is a decomposition of $X \times X$ into GL_V-stable locally closed subschemes*

$$X \times X = \dot{\bigcup_{w \in \Sigma_n}} O_w.$$

Each O_w is smooth of dimension $\dim(X) + l(w)$, where $l(w)$ denotes the Coxeter-length of w.

Proof. For each w, the scheme O_w is locally closed as orbits are locally closed by [Mil17, Proposition 1.65 b)] and smooth, as it is isomorphic to a quotient of GL_V. The remaining claims boil down to classical theory (in particular, the classical Bruhat decomposition), cf. [Mil17, Chapter 21], and elementary considerations about the dimensions of the isotropy subgroups of pairs $(F_\bullet, F'_\bullet) \in X_w(k)$. $\square$

Let $(F_\bullet, F'_\bullet) \in (X \times X)(S)$ be a pair of flags over a k-scheme S. We say that $(F_\bullet, F'_\bullet)$ is in relative position $w \in \Sigma_n$ if it lies inside the subset $O_w(S) \subset (X \times X)(S)$.

Similarly, we can characterize the GL_V-orbits in $Y \times_{\mathrm{Spec}\, k} Y$. For any choice of elements $w \in \Sigma_n \subset \mathrm{GL}_V(k)$ and $t \in \mathbb{T}_X(k)$, we define $\tilde{O}_{w,t}$ as the GL_V-orbit of the element

$$((F_\bullet^{\mathrm{std}}, e_\bullet^{\mathrm{std}}), (w.F_\bullet^{\mathrm{std}}, w.te_\bullet^{\mathrm{std}})) \in (Y \times Y)(k).$$

A pair of marked flags over a k-scheme S is said to be in relative position $(w, t) \in \Sigma_n \times (k^\times)^n$ if it lies inside $\tilde{O}_{w,t}$. The following proposition gives a convenient characterization of relative position.

Lemma 7.1.8. *1. A pair of families of flags $(F_\bullet, F'_\bullet) \in (X \times X)(S)$ is in relative position $w \in \Sigma_n$ if and only if there exists a Zariski-cover $\phi : S' \to S$ (with structure map to k denoted by f') and a basis $(v_1, \dots, v_n) \in \Gamma(S', f'^*\tilde{V})$ such that*

$$\phi^* F_i = \langle v_1, \dots, v_i \rangle \quad \textit{and} \quad \phi^* F'_i = \langle v_{w(1)}, \dots, v_{w(i)} \rangle \quad \textit{for all } i = 1, \dots, n-1.$$

2. A pair of families of marked flags $((F_\bullet, e_\bullet), (F'_\bullet, e'_\bullet)) \in (Y \times Y)(S)$ is in relative position (w, t) if and only if $(F_\bullet, F'_\bullet)$ is in relative position w and there is a basis as above *furthermore satisfying*

$$\phi^* e_i \equiv v_i \mod \phi^* F_{i-1} \quad \textit{and} \quad \phi^* e'_i \equiv t_{w(i)} v_{w(i)} \mod \phi^* F'_{w(i)-1} \quad \textit{for all } i = 1, \dots, n.$$

Here, ϕ^* denotes the natural pullback of sections $\Gamma(S, f^*\tilde{V}) \to \Gamma(S', f'^*\tilde{V})$.

Proof. This is a mere reformulation of what it means to be in the corresponding GL_V-orbits. Given any choice of 'standard' basis $(e_1, \dots, e_n)$ of V and a section $(F_\bullet, F'_\bullet)$ in the orbit of $(F_\bullet^{\mathrm{std}}, w.F_\bullet^{\mathrm{std}})$, we may choose S' such that there exists a $g \in \mathrm{GL}_V(S')$ satisfying $g.(F_\bullet^{\mathrm{std}}|_{S'}, w.F_\bullet^{\mathrm{std}}|_{S'}) = (F_\bullet, F'_\bullet)$. Now it is easily seen that the global sections $v_i = g(e_i) \in \Gamma(S', f^*\tilde{V})$ satisfy the desired conditions. Conversely, any such basis yields an element in $\mathrm{GL}_V(S')$. The same ideas lead to the second statement. Note that here we only need to lift the sections e_i to sections in $\phi^*\tilde{V}$, which is possible once S' is affine. $\square$

We now specialize to the case where $k = \overline{\mathbb{F}}_q$ is an algebraic closure of the finite field with q elements, and $V = V_0 \otimes_{\mathbb{F}_q} k$ for some $\mathbb{F}_q$-vector space V_0. This equips V with a $\mathrm{Gal}(k/\mathbb{F}_q)$-action, and in particular the Frobenius automorphism of k (given on k by $x \mapsto x^q$) yields a k-semilinear automorphism $\mathrm{Frob} : V \to V$. As this automorphism sends subspaces to subspaces, we obtain automorphisms

$$\mathrm{Frob} : X \to X \quad \text{and} \quad \mathrm{Frob} : Y \to Y.$$

Note that X and Y are defined over $\mathbb{F}_q$, and these automorphisms are the same as the respective (relative) frobenii of X and Y over k. We write γ_{Frob} for the corresponding graph-morphisms $X \to X \times_{\mathrm{Spec}\, k} X$ and $Y \to Y \times_{\mathrm{Spec}\, k} Y$.
For $w \in \Sigma_n$ and $t \in \mathbb{T}_X$, we define the spaces

$$X_w := O_w \times_{X \times_{\mathrm{Spec}\, k} X, \gamma_{\mathrm{Frob}}} X \quad \text{and} \quad Y_{w,t} := \tilde{O}_{w,t} \times_{Y \times_{\mathrm{Spec}\, k} Y, \gamma_{\mathrm{Frob}}} Y \tag{7.2}$$

As γ_{Frob} admits a section, X_w is naturally a subscheme of X, parametrizing those families of flags that are pointwise in relative position w to their Frobenius twist. Similarly, $Y_{w,t}$ is naturally a subscheme of Y, parametrizing families of marked flags in relative position (w, t) with their Frobenius twist. We have natural maps $Y_{w,t} \to X_w$. As a pair of marked flags $((F_\bullet, e_\bullet), (F'_\bullet, e'_\bullet))$ over S lies in the same GL_V-orbit as $((F_\bullet, t_\bullet e_\bullet), (F'_\bullet, t'_\bullet e'_\bullet))$ for $t_\bullet, t'_\bullet \in \mathbb{G}_{m,X}^n(S)$ if and only if $t_\bullet = t'_\bullet$, we find that $Y_{w,t}$ is a Zariski-torsor over X_w for an affine group scheme $\mathbb{T}_w^{\mathrm{Frob}} \times_k X_w$. Here, $\mathbb{T}_w = \mathrm{Res}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(\mathbb{G}_m) \times_{\mathbb{F}_q} k$ is the Weil restriction of the multiplicative group from $\mathbb{F}_{q^n}$ to $\mathbb{F}_q$. Hence, the S-valued points of $\mathbb{T}_w^{\mathrm{Frob}}$ are given by

$$\mathbb{T}_w^{\mathrm{Frob}}(S) = \{(t_1, \dots, t_n) \in \mathbb{G}_m^n(S) \mid t_i^q = t_{w(i)}\}.$$

Furthermore, an element $g \in \mathrm{GL}_V(S)$ stabilizes $Y_{w,t}(S) \subset Y(S)$ if $g \in \mathrm{GL}_V^{\mathrm{Frob}}(S)$, so we obtain a $\mathrm{GL}_V^{\mathrm{Frob}}$ action on $Y_{w,t}$ and X_w. The morphism $Y_{w,t} \to X_w$ is equivariant for the $\mathrm{GL}_V^{\mathrm{Frob}}$-action.
One can show that the scheme X_w is smooth (of pure dimension $l(w)$, as the intersection in (7.2) is transverse, cf. [DL76]), so $Y_{w,t}$ is smooth and affine over X_w. To this end, we have

constructed the spaces in the commutative diagram

$$\begin{tikzcd} Y_{w,t} \arrow[r, hook] \arrow[d, "\mathbb{T}_w^{\mathrm{Frob}}\text{-torsor}"'] & Y \cong \mathrm{GL}_V/U \arrow[d, "\mathbb{T}_X\text{-torsor}"] \\ X_w \arrow[r, hook] & X \cong \mathrm{GL}_V/B. \end{tikzcd} \tag{7.3}$$

The interesting space is $Y_{w,t}$. It comes with commuting (left-)actions of $\mathrm{GL}_V^{\mathrm{Frob}}(k) = \mathrm{GL}_V(\mathbb{F}_q)$ and $\mathbb{T}_w^{\mathrm{Frob}}(k) = \mathbb{F}_{q^n}^\times$.

7.2 An Explicit Example

We keep the notation from the previous subsection. That is, $k = \overline{\mathbb{F}}_q$, $V = V_0 \otimes_{\mathbb{F}_q} k$, X is the flag variety of V, and Y is the variety of marked flags. In this subsection, we fix $w = (1\ 2\ \ldots\ n) \in \Sigma_n$ and $t = (1, \ldots, 1) \in \mathbb{G}_{X,m}^n$, and give explicit descriptions of the resulting varieties appearing in the square (7.3). To clearify notation, we write $\mathrm{DL}_V = Y_{w,t}$ in this situation.

First, note that $\mathbb{T}_w^{\mathrm{Frob}}(S) = \mathbb{G}_m(S)^{\mathrm{Frob}^n}$, implying that DL_V admits commuting actions by $\mathrm{GL}(V_0)$ and $\mathbb{T}_w^{\mathrm{Frob}}(k) = \mathbb{F}_{q^n}^\times$.

Lemma 7.2.1. *A pair of flags $(F_\bullet, F'_\bullet) \in (X \times X)(S)$ is in relative position $(1\ 2\ \ldots\ n)$ if and only if for all $i = 1, \ldots, n-1$ the condition $F_i + F'_i = F_{i+1}$ is satisfied. Here, the sum denotes the Zariski-sheafification of the corresponding presheaf.*

Proof. It is easily seen that the criterion may be checked Zariski-locally. Here it follows quickly from Lemma 7.1.8 and the Bruhat decomposition, Proposition 7.1.7. □

For a linear form $\mu \in V^\vee$, we write $D^+(\mu)$ for the affine open subscheme of $\mathbb{P}(V)$ parametrizing lines in V that do not lie in the hyperplane defined by the equation $\mu(v) = 0$.

Proposition 7.2.2. *If $w = (1\ 2\ \ldots\ n)$, the morphism of functors defined on k-schemes S by*

$$\Phi(S) : X_w(S) \to \mathbb{P}(V)(S) \quad F_\bullet \mapsto F_1$$

yields an isomorphism

$$X_w \cong \bigcap_{\mu \in V_0^\vee} D^+(\mu) \subset \mathbb{P}(V).$$

That is, X_w parametrizes lines in V that do not lie inside any $\mathbb{F}_q$-rational hyperplane. In particular X_w is equal to a finite intersection of affine subschemes, hence an affine scheme itself.

Proof. We first show that the image of any family of flags $F_\bullet \in X_w(S)$ lies inside $\cap_\mu D^+(\mu)(S)$. As the latter is an open subscheme, it suffices to show that any $s \in |S|$ maps into $\cap_\mu D^+(\mu)$, which is the case if any only if $F_1(s) = F_{1,s} \otimes_{\mathcal{O}_{S,s}} \kappa(s)$ does not lie inside any $\mathbb{F}_q$-rational hyperplane in $\kappa(s) \otimes_k V$. By Lemma 7.2.1, we find

$$F_1(s) \oplus \mathrm{Frob}(F_1(s)) \oplus \cdots \oplus \mathrm{Frob}^{n-1}(F_1(s)) = V \otimes_k \kappa(s),$$

so $F_1(s)$ cannot lie inside any non-trivial Frobenius-stable linear subspace of $V \otimes_k \kappa(s)$. The claim follows.

To see bijectivity of Φ, note that the inverse is, if well-defined, given by the morphism of functors Ψ given on components by

$$\Psi(S) : \bigcap_{\mu \in V_0^\vee} D^+(\mu)(S) \to X_w(S), \quad L \mapsto \Big(L \oplus \mathrm{Frob}(L) \oplus \cdots \oplus \mathrm{Frob}^{i-1} L\Big)_{i=1,\ldots,n}.$$

To see that this is indeed well-defined, choose a basis $(e_1, \ldots, e_n)$ of V_0 and take any section $\mathcal{L} \in \cap_{\mu \in V_0^\vee} D^+(\mu)(S)$, interpreted as a locally direct summand of $f^*\tilde{V}$. It suffices to work locally on S, and we may pick for any $s \in |S|$ some open affine $\operatorname{Spec} R \subset S$ trivializing $\mathcal{L}$. Write L for the corresponding free rank 1 direct summand of R^n. Then $L = \langle v \rangle$ for some $v = (v_1, \ldots, v_n) \in R^n$. Now L constitutes a flag if and only if $(v, \mathrm{Frob}(v), \ldots, \mathrm{Frob}^{n-1}(v))$ is a basis for R^n if and only if $\det(\mathrm{Frob}^{j-1}(v_i))_{i,j} \in R^\times$. The last condition is satisfied. Indeed, if not, we may choose a maximal ideal $\mathfrak{m} \in \operatorname{Spec} R$ containing $\det(\mathrm{Frob}^{j-1}(v_i))_{i,j}$. Let $\overline{v}$ denote the residue of v in $(R/\mathfrak{m})^n$. Now, the subspace

$$\langle \overline{v}, \mathrm{Frob}(\overline{v}), \ldots, \mathrm{Frob}^{n-1}(\overline{v}) \rangle \subset (R/\mathfrak{m})^n$$

is non-trivial and Frobenius-stable, and in particular contained in some $\mathbb{F}_q$-rational hyperplane. This contradicts $\mathcal{L} \in \cap_{\mu \in V_0^\vee} D^+(\mu)(S)$. $\square$

We write $\Delta : \mathrm{Sym}(\bigwedge V^\vee) \to \mathrm{Sym}(V^\vee)$ for the morphism corresponding to the k-linear morphism

$$\bigwedge V^\vee \to \mathrm{Sym}(V^\vee), \quad \mu \mapsto \Big[v \mapsto \mu(v \wedge \mathrm{Frob}(v) \wedge \ldots \mathrm{Frob}^{n-1}(v))\Big].$$

Proposition 7.2.3. *The map of functors* $\mathrm{DL}_V \to \operatorname{Spec} \mathrm{Sym}(V^\vee)$ *given by* $(F_\bullet, e_\bullet) \mapsto e_1$ *yields an isomorphism of* DL_V *and the subfunctor of* $\operatorname{Spec}\mathrm{Sym}(V^\vee)$ *given on affine schemes by*

$$\operatorname{Spec} R \mapsto \left\{ v \in R \otimes_k V \;\middle|\; \begin{array}{c} (v, \mathrm{Frob}\, v, \ldots, \mathrm{Frob}^{n-1} v) \textit{ is a basis and} \\ v \wedge \cdots \wedge \mathrm{Frob}^{n-1} v = (-1)^{n-1} \mathrm{Frob}(v \wedge \cdots \wedge \mathrm{Frob}^{n-1} v) \end{array} \right\}.$$

Writing S_1 for the degree-1 part of $\mathrm{Sym}(\bigwedge V^\vee)$, this functor is readily seen to representable by the k-scheme

$$\mathrm{DL}_V := \operatorname{Spec}\left(\frac{\mathrm{Sym}(V^\vee)[\Delta(S_1 \setminus \{0\})^{-1}]}{(\mathrm{Frob}(\Delta(\lambda)) - (-1)^{n-1}\Delta(\lambda) \mid \lambda \in S_1)} \right).$$

Upon choosing a basis of $V_0 \cong \mathbb{F}_q^n$, this takes on the form

$$\mathrm{DL}_n := \operatorname{Spec}\left(\frac{k[x_1, \ldots, x_n]}{(\det D(\underline{x})^{q-1} - (-1)^{n-1})} \right), \quad \textit{where} \quad D(\underline{x}) = \begin{pmatrix} x_1 & \cdots & x_1^{q^{n-1}} \\ \vdots & \ddots & \vdots \\ x_n & \cdots & x_n^{q^{n-1}} \end{pmatrix}.$$

Proof. Let $S = \operatorname{Spec} R$ be an affine k-scheme and let $(F_\bullet, e_\bullet) \in \mathrm{DL}_V(S)$. By Lemma 7.1.8, there is a basis $(v_1,\dots,v_n)$ of $R\otimes_k V$ such that

$$\begin{aligned} v_i &\equiv e_i \mod F_{i-1} \quad &&\text{and} \quad v_{i+1}\equiv \mathrm{Frob}(e_i) \mod \mathrm{Frob}(F_{i-1}) \quad \text{for } 1\le i\le n-1,\\ v_n &\equiv e_n \mod F_{n-1} \quad &&\text{and} \quad v_1\equiv \mathrm{Frob}(e_n) \mod \mathrm{Frob}(F_{n-1}). \end{aligned} \tag{7.4}$$

From here we quickly find

$$\mathrm{Frob}(v_1\wedge v_2\wedge\cdots\wedge v_n)=\mathrm{Frob}\, v_1\wedge\cdots\wedge\mathrm{Frob}\, v_n = v_2\wedge v_3\wedge\cdots\wedge v_n\wedge v_1.$$

The equivalences in (7.4) also imply that for integers $2\le m\le n$, we have $\mathrm{Frob}^{m-1} v_1\equiv v_m \mod \mathrm{Frob}(F_{m-2})$. Also, we find $v_1\equiv \mathrm{Frob}^n v_1 \mod \mathrm{Frob}(F_{n-1})$. Altogether, writing $v=v_1=e_1$, this yields

$$\mathrm{Frob}(v\wedge\mathrm{Frob}(v)\wedge\cdots\wedge\mathrm{Frob}^{n-1}v)=(-1)^{n-1}(v\wedge\cdots\wedge\mathrm{Frob}^{n-1}v). \tag{7.5}$$

This shows that the map given in the statement of the proposition is well-defined. To see that it is bijective, note that it has an inverse. Indeed, given any $v\in R\otimes_k V$ such that $(v,\mathrm{Frob}\, v,\dots,\mathrm{Frob}^{n-1}v)$ is a basis and v satisfies the equation (7.5), Gaussian elimination shows that the corresponding marked flag is in relative position $(w,1)$ to its Frobenius-twist. If we are given a basis of V_0, we may write $v=(x_1,\dots,x_n)$ and identify $v\wedge\mathrm{Frob}\, v\wedge\dots\mathrm{Frob}^{n-1}v$ with $\det\left[(x_i^{q^{j-1}})_{1\le i,j\le n}\right]$. Thereby v gives a marked flag in DL_n if and only if

$$\det((x_i^{q^{j-1}})_{i,j})^{q-1}=(-1)^{n-1}.$$

This gives the representability statement of DL_n. □

Note that DL_n has $q-1$ disjoint irreducible components, parametrized by the set of solutions $b\in k$ to the equation $z^{q-1}=(-1)^{n-1}$. For any such $b\in k$ we write (in accordance with notation in [Mie16])

$$Y_b := \mathrm{Spec}\left(\frac{k[x_1,\dots,x_n]}{(\det D(\underline{x})-b)}\right), \tag{7.6}$$

and obtain $\mathrm{DL}_n=\sqcup_{b^{q-1}=(-1)^{n-1}} Y_b$.

The right action of $\mathrm{GL}_n(\mathbb{F}_q)\times\mathbb{F}_{q^n}^\times$ on DL_n has the following explicit description.

Lemma 7.2.4. *Let $g\in\mathrm{GL}_n(\mathbb{F}_q)$ be an element with matrix entries $(a_{ij})_{1\le i,j\le n}$. Then g acts on the global sections of DL_n from the left via*

$$x_i\mapsto\sum_{j=1}^n a_{ji}x_j \quad \textit{for} \quad i=1,\dots,n.$$

Similarly, an element $d\in\mathbb{F}_{q^n}^\times$ acts via

$$x_i\mapsto d^{-1}x_i \quad \textit{for} \quad i=1,\dots,n.$$

Through the q-th power Frobenius automorphism on $\overline{\mathbb{F}}_q$, we may also define an action of $\mathbb{Z}$ on DL_n, sending 1 to the automorphism given by $x_i\mapsto x_i$, $a\mapsto a^{-q}$. Note that this action is defined over $\mathbb{F}_q$, not over $\overline{\mathbb{F}}_q$. By construction, the action of $\mathbb{Z}$ commutes with the action of $\mathrm{GL}_n(\mathbb{F}_q)$, but in order to make it commute with the action of $\mathbb{F}_{q^n}^\times$, we have to restrict to $n\mathbb{Z} \subset \mathbb{Z}$.

Note that $g \in \mathrm{GL}_n(\mathbb{F}_q)$ induces a morphism of schemes $Y_b \to Y_{b\det(g)}$. Similarly, $\zeta \in \mathbb{F}_{q^\times}$ restricts to $Y_b \to Y_{b\mathrm{N}(\zeta)^{-1}}$, where $\mathrm{N} = \mathrm{N}_{\mathbb{F}_{q^n}/\mathbb{F}_q}$ denotes the norm map of the extension $\mathbb{F}_{q^n}/\mathbb{F}_q$. The action of $1 \in \mathbb{Z}$ sends Y_b to $Y_{(-1)^{n-1}b}$, therefore the action of $n\mathbb{Z}$ stabilizes each component Y_b. In particular, the subgroup

$$(\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times)^1 \times n\mathbb{Z} := \{(g, d, n) \in \mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times \times n\mathbb{Z} \mid \det(g)\mathrm{N}(d) = 1\}$$

of $\mathrm{GL}_n \times \mathbb{F}_{q^n}^\times \times \mathbb{Z}$ stabilizes Y_b for every choice of b.

We write H_{DL} for the $\overline{\mathbb{Q}}_l$-vector space

$$H_{\mathrm{DL}} = H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_l).$$

By the above, this is a representation of $\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times \times n\mathbb{Z}$. The following subsection is concerned with the study of this representation.

7.3 An Example of the Deligne–Lusztig Correspondence

In this subsection, we are concerned with the étale cohomology of the variety DL_n introduced in the previous subsection. As before, k denotes an algebraic closure of $\mathbb{F}_q$.

We give a short review of [DL76]. For a connected, reductive algebraic group G defined over $\mathbb{F}_q$ and a maximal Frobenius-stable torus $T \subset G$ contained in a Borel subgroup $B \subset G$, Deligne and Lusztig construct varieties with right G^{Frob}-actions $X_{T\subset B}$ and $\tilde{X}_{T\subset B}$, constituting a G^{Frob}-equivariant Galois covering

$$\tilde{X}_{T\subset B} \to X_{T\subset B}$$

with Galois group T^{Frob}. See (7.8) for explicit descriptions of these spaces. The space $\tilde{X}_{T\subset B}$ comes with commuting actions of G^{Frob} and T^{Frob}, and for characters θ of $T^{\mathrm{Frob}}(k)$, the main concern of [DL76] is the study of the resulting virtual representations

$$R_T^\theta = \sum_i (-1)^i H_c^i(\tilde{X}_{T\subset B}, \overline{\mathbb{Q}}_l)_\theta \in \mathcal{R}(T^{\mathrm{Frob}}(k) \times G^{\mathrm{Frob}}(k)). \tag{7.7}$$

Here, the subscript θ denotes the direct summand of $H_c^i(\tilde{X}_{T\subset B}, \overline{\mathbb{Q}}_l)$ where $T^{\mathrm{Frob}}(k)$ acts by θ. These representations do only depend on the choice of the torus, not on the choice of the Borel sugroup ([DL76, Corollary 4.3]). To formulate the results of the theory precisely, we need the notion of regular position.

Definition 7.3.1 (Regular Position)**.** Let T be a Frobenius-stable maximal torus of G and let θ be a character of $T^{\mathrm{Frob}}(k)$. We say that θ is in general position if it is not fixed by any non-trivial element of $\mathrm{W}_G(T)^{\mathrm{Frob}}$. Here $\mathrm{W}_G(T)$ denotes the Weyl group $(\mathrm{N}_G(T)/T)$, which acts on T by conjugation.

Recall that for a finite group H, the Grothendieck group $\mathcal{R}(H)$ of finite dimensional H-representations over $\overline{\mathbb{Q}}_l$ comes with a natural inner product. Indeed, any representation takes values in the maximal cyclotomic subfield $\cup_r \mathbb{Q}(\zeta_r) \subset \overline{\mathbb{Q}}_l$, which has the unique "complex conjugation" automorphism given by $\zeta_r \mapsto \zeta_r^{-1}$ for $r \in \mathbb{N}$. The inner product is now defined on finite-dimensional representations $\rho, \rho' \in (H\text{-Rep})$ as

$$\langle \rho, \rho' \rangle = \frac{1}{\#H} \sum_{h \in H} \mathrm{Tr}(\rho(h)) \overline{\mathrm{Tr}(\rho'(h))} \in \overline{\mathbb{Q}}_l.$$

This definition linearly extends to $\mathcal{R}(H)$, and the irreducible finite-dimensional representations of H give an orthogonal basis for $\mathcal{R}(H) \otimes \mathbb{Q}$. If ρ is an irreducible representation of H and R is an element in $\mathcal{R}(H) \otimes \mathbb{Q}$, we say that ρ occurs in R if $\langle \rho, R \rangle \neq 0$.

The following results of Deligne-Lusztig theory are important for our purposes.

Theorem 7.3.2 (Some Results of Deligne-Lusztig Theory)**.**

1. [DL76, Corollary 1.22] *Let $Z \subset G$ denote the center of G. Then Z^{Frob} acts on $H_c^{n-1}(\tilde{X}_{T \subset B}, \overline{\mathbb{Q}}_l)_\theta$ through $\theta|_{Z^{\mathrm{Frob}}}$.*

2. [DL76, Corollary 7.3] *If (T, θ) is in general position, one of $\pm R_T^\theta$ is an irreducible representation.*

3. [DL76, Corollary 8.3] *If furthermore T is not contained in any proper Frobenius-stable parabolic subgroup, one of $\pm R_T^\theta$ is a cuspidal representation of $G^{\mathrm{Frob}}(k)$.*

4. [DL76, Corollary 9.9] *If furthermore $\tilde{X}_{T \subset G}$ (or equivalently, $X_{T \subset G}$) is affine, and we denote by $w \in \mathrm{W}_G(T)$ the relative position of B and* $\mathrm{Frob}(B)$ *(given by the Bruhat decomposition for G), we have*

$$H_c^i(\tilde{X}_{T \subset G}, \overline{\mathbb{Q}}_l)_\theta = 0 \quad \textit{if} \quad i \neq l(w).$$

   Here, $l(w)$ denotes the Coxeter-length of w.

5. [DL76, Theorem 9.8] *For a character θ in general position, the natural map*

$$H_c^i(Y_{T \subset B}, \overline{\mathbb{Q}}_l)_\theta \to H^i(Y_{T \subset B}, \overline{\mathbb{Q}}_l)_\theta$$

   is an isomorphism.

We next explain how these results apply to the compactly supported étale cohomology of DL_n. We set $G = \mathrm{GL}_n$, and we denote by B^{std} the standard Borel subgroup of upper triangular matrices, by T^{std} the standard torus of diagonal matrices and by U^{std} the standard unipotent subgroup of upper triangular matrices with diagonal entries equal to 1. We have $T^{\mathrm{std}} U^{\mathrm{std}} = B^{\mathrm{std}}$.

Choose a flag $F_\bullet \in X_w(k)$, where w is, as usual, $(1\ 2\ \dots\ n) \in \Sigma_n \cong \mathrm{W}_G(T^{\mathrm{std}})$. Let $B \subset \mathrm{GL}_n$ denote the isotropy subgroup of $F_\bullet$, and let $T \subset B$ be a Frobenius-stable maximal torus in B. Write U for the unipotent radical of B. We have the explicit descriptions (cf. [DL76, Definition 1.17])

$$\begin{array}{c} X_{T\subset B} = \{g \in \mathrm{GL}_n \mid g^{-1}\mathrm{Frob}(g) \in \mathrm{Frob}(U)\}/(T^{\mathrm{Frob}}(U \cap \mathrm{Frob}(U))) \\ \text{and} \\ \tilde{X}_{T\subset B} = \{g \in \mathrm{GL}_n \mid g^{-1}\mathrm{Frob}(g) \in \mathrm{Frob}(U)\}(U \cap \mathrm{Frob}(U)). \end{array} \tag{7.8}$$

Given any marked flag $(F'_\bullet, e'_\bullet) \in X(S)$, we write $g(F'_\bullet, e'_\bullet) \in \mathrm{GL}_n / U^{\mathrm{std}}(S)$ for the corresponding section under the isomorphism of Proposition 7.1.6 (i.e., $g.(F^{\mathrm{std}}_\bullet, e^{\mathrm{std}}_\bullet) = (F'_\bullet, e'_\bullet)$). Also, by [Mil17, Proposition 17.13], we may choose $h \in \mathrm{GL}_n(k)$ such that $h(T^{\mathrm{std}}, B^{\mathrm{std}})h^{-1} = (T, B)$. Furthermore, for any k-scheme S, we may identify $\mathbb{T}_w^{\mathrm{Frob}}$ with the subgroup

$$\{t \in T^{\mathrm{std}}(S) \mid \mathrm{ad}\, w^{-1}t = \mathrm{Frob}(t)\} \subset T^{\mathrm{std}}(S).$$

Using these identifications, [DL76, Proposition 1.19] implies that the map

$$\mathrm{DL}_n \to \tilde{X}_{T\subset B}, \quad (F'_\bullet, e'_\bullet) \mapsto g(F'_\bullet, e'_\bullet)h^{-1}$$

gives an isomorphism of $\mathrm{GL}_n^{\mathrm{Frob}}$-equivariant torsors

$$\begin{tikzcd} \mathrm{DL}_n \arrow[d, "\mathbb{T}_w^{\mathrm{Frob}}\text{-torsor}"'] \arrow[r, phantom, "\xrightarrow{\sim}"] & \tilde{X}_{T\subset B} \arrow[d, "T^{\mathrm{Frob}}\text{-torsor}"] \\ X_w & X_{T\subset B}. \end{tikzcd}$$

Given any representation θ of $\mathbb{T}_w^{\mathrm{Frob}}(k)$, we write

$$R_\theta = \sum_i (-1)^i H^i_c(\mathrm{DL}_n, \overline{\mathbb{Q}}_l)_\theta.$$

By definition, we have $R_{\theta\circ\mathrm{ad}\, h} \cong R^\theta_{T\subset B}$, where h is chosen as above. If θ is a character of $\mathbb{T}_w^{\mathrm{Frob}}(k) \cong \mathbb{F}_{q^n}^\times$, the pair $(T, w \circ \mathrm{ad}\, h)$ is in regular position if and only if θ is regular, in the following sense.

Definition 7.3.3 (Regular Character on $\mathbb{F}_{q^n}^\times$)**.** We say that a character $\theta : \mathbb{F}_{q^n}^\times \to \mathbb{C}^\times$ is regular if it does not factor through the norm morphism

$$\mathrm{N}_{\mathbb{F}_{q^n}/\mathbb{F}_{q^m}} : \mathbb{F}_{q^n}^\times \to \mathbb{F}_{q^m}^\times$$

for any $m < n$.

The statements of Theorem 7.3.2 reduce to the following.

Theorem 7.3.4 (Deligne–Lusztig Correspondence)**.** *Let θ be a regular character of $\mathbb{F}_{q^n}^\times$, and let $H_{\mathrm{DL},\theta} = H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_l)_\theta$ denote the direct summand of H_{DL} on which $\mathbb{F}_{q^n}^\times$ acts through θ.*

1. *We have*

$$H_{\mathrm{DL},\theta} = R_\theta \boxtimes \theta$$

as representations of $\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times$. The representation R_θ is irreducible and cuspidal, and its central character is given by $\theta|_{\mathbb{F}_q^\times}$ under the identification $\mathbb{F}_q^\times \cong Z_{\mathrm{GL}_n}$.

2. The natural map

$$H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_l)_\theta \to H^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_l)_\theta$$

is an isomorphism.

This finishes the discussion about the $\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times$-action on H_{DL}.
It remains to make the action of $n\mathbb{Z}$ explicit. The Frobenius automorphism on $\overline{\mathbb{F}}_q$ yields an action of $n\mathbb{Z}$ on H_{DL}. This action admits the following partial description.

Proposition 7.3.5. *Let θ be a regular character. The subgroup $n\mathbb{Z} \subset \mathbb{Z}$ acts on $H_{\mathrm{DL},\theta}$ through the character $\gamma : n\mathbb{Z} \to \mathbb{Q}^\times$, given by*

$$\gamma(nm) = (-1)^{(n-1)m} q^{m\frac{n(n-1)}{2}}.$$

Proof. Note that, as the absolute Frobenius morphism $\mathrm{DL}_n \to \mathrm{DL}_n$ induces the identity on étale cohomology (cf. [Stacks, Tag 03SN]), the claim is equivalent to showing that pullback along the relative Frobenius

$$\mathrm{Frob}_{q^n} : \mathrm{DL}_n \to \mathrm{DL}_n^{(q^n)} = \mathrm{DL}_n, \quad x_i \mapsto x_i^q, \quad a \mapsto a \quad (a \in \overline{\mathbb{F}}_q)$$

induces multiplication of $(-1)^{n-1} q^{\frac{n(n-1)}{2}}$ on $H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta$. This result is essentially due to Digne–Michel, cf. [DM85, Remarque 3.14], but their proof contains a sign error. This mistake was identified and corrected by Wang in [Wan14, Théorème 3.1.12]. □

Concludingly, we obtain the following structural result about the $\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times \times n\mathbb{Z}$-representation H_{DL}.

Theorem 7.3.6 (Structure of H_{DL}). *Let $\theta : \mathbb{F}_{q^n}^\times \to \overline{\mathbb{Q}}_l^\times$ be a regular character. The $\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times \times n\mathbb{Z}$-representation $H_{\mathrm{DL},\theta}$ is given by*

$$H_{\mathrm{DL},\theta} = R_\theta \boxtimes \theta \boxtimes \gamma.$$

Here, R_θ is the irreducible cuspidal representation from Deligne–Lusztig theory (cf. Theorem 7.3.4), and γ is the character defined in the previous proposition.

8 Explicit Non-Abelian Lubin–Tate Theory for Depth Zero Supercuspidal Representations

In this final section, we explicitely calculate the representations arising in Theorem 4.2.4 for depth zero supercuspidal representations $\pi \in (\mathrm{GL}_n(F)\text{-Rep})$, which essentially are representations of $\mathrm{GL}_n(F)$ obtained by compact induction from Deligne–Lusztig theory. In particular, we give explicit descriptions of $\mathrm{rec}_E(\pi)$ and $\mathrm{JL}(\pi)$. The main input is the construction of an affinoid in the Lubin–Tate perfectoid space whose special fiber is isomorphic to the perfection of the Deligne–Lusztig variety constructed in Section 7.2. Mieda's results give a relation between the ℓ-adic cohomology of DL_n and the representation H_{LT}, which allows to make some parts of H_{LT} explicit.

8.1 The Special Affinoid and its Formal Model

We define the subspace $U \subset M^{(0)}_{\infty,\mathcal{O}_{\mathbb{C}_p}}$ as the rational subset cut out by the inequalities $|X_i| \leq |\varpi|^{1/q^n-1}$ for $i = 1, \dots, n$. We construct formal model $\mathcal{X} \in (\mathrm{FSch}/\mathcal{O}_{\mathbb{C}_p})$ of U whose special fiber $\mathcal{X}_s = \mathcal{X} \times_{\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p})} \mathrm{Spec}(\overline{\mathbb{F}}_q)$ is isomorphic to the perfection of the Deligne–Lusztig variety DL_n constructed in Section 7.2.

The formal model $\mathcal{X} = \mathrm{Spf}\, C_n$ is defined as follows. Let H be the standard formal $\mathcal{O}_E$-module over $\mathcal{O}_{\breve{E}}$ and let

$$y = (y_1, y_2, \dots) \in T_\varpi H(\mathcal{O}_{\mathbb{C}_p})$$

be a primitive element, that is, an element with $y_1 \neq 0$ and $[\varpi]_H(y_1) = 0$. Write $\xi = \lambda(y)$ for the corresponding element of $\mathrm{Nilp}^\flat(\mathcal{O}_{\mathbb{C}_p})$.

Lemma 8.1.1. *We have*

$$\log_H(y) = \sum_{i=-\infty}^{\infty} \frac{\xi^{q^i n}}{\varpi} = 0 \quad \textit{and} \quad |\xi|^{q^n-1} = |\varpi|.$$

Proof. The first identity follows from Lemma 5.2.19 and the fact that $\log_H(y_1)$ vanishes, cf. Lemma 2.3.6. For the second one, note that $[\varpi]_H(y_1) = 0$ and $[\varpi]_H(y_{k+1}) = y_k$ implies that $|y_k| = |\varpi|^{\frac{1}{q^{nk}(q^n-1)}}$. Hence, the claim follows by the equality $\xi = \lim_{k\to\infty} y_k^{q^{nk}}$, cf. Lemma 5.2.13. $\square$

Let E_n be the degree n unramified extension of E, and let $(\alpha_1, \dots, \alpha_n)$ be a basis of $\mathcal{O}_{E_n}$ over $\mathcal{O}_E$. Let $t := \delta(\alpha_1 y, \dots, \alpha_n y) \in (T_\varpi \wedge H)(\mathcal{O}_{\mathbb{C}_p})$ and write $\tau \in \mathrm{Nilp}^\flat(\mathcal{O}_{\mathbb{C}_p})$ for the corresponding system of q-th power roots. Also, the choice of $(\alpha_1, \dots, \alpha_n)$ lets us identify E_n as a subfield of $\mathrm{GL}_n(E)$ and $D^\times$.

Lemma 8.1.2. *We have*

$$\log_{\wedge H}(t) = \sum_{i=-\infty}^{\infty} (-1)^{i(n-1)} \frac{\tau^{q^i}}{\varpi^i} = 0 \quad \textit{and} \quad |\tau|^{q-1} = |\varpi|.$$

Furthermore, we have the congruence

$$\tau \equiv \det(\alpha_i^{q^j})\xi^{1+q+\dots+q^{n-1}}$$

modulo the ideal generated by elements $z \in \mathcal{O}_{\mathbb{C}_p}$ with with $|z| < |\xi|^{1+q+\dots+q^{n-1}}$.

Proof. We have $t \neq 0$ and $[\varpi]_{\wedge H}(t) = 0$, so the first two assertions follow just as above in the proof of Lemma 8.1.1, appropiately adjusted. The congruence is a corollary of the approximation of Δ in Lemma 5.4.3. $\square$

We now define the formal model $\mathcal{X}$. We abbreviate with $(x_i^{q^{-m}})_{m\in\mathbb{N}_0}$ the system of q-th power roots $(X_i^{q^{-m}}/\xi^{q^{-m}})_{m\in\mathbb{N}_0}$ of elements in $\mathcal{O}_{\mathbb{C}_p}[\![X_1^{q^{-\infty}}, \dots, X_n^{q^{-\infty}}]\!]$, and define systems of

q-th power roots Δ' and τ' as

$$\Delta'(x_1,\ldots,x_n)^{q^{-m}} := (\xi^{q^{-m}})^{-(1+q+\cdots+q^{n-1})}\Delta(\xi x_1,\ldots,\xi x_n) \in \mathcal{O}_{\mathbb{C}_p}[\![x_1^{q^{-\infty}},\ldots,x_n^{q^{-\infty}}]\!]$$
and
$$\tau'^{q^{-m}} := (\xi^{q^{-m}})^{-(1+q+\cdots+q^{n-1})}\tau^{q^{-m}} \in \mathcal{O}_{\mathbb{C}_p}^{\times}.$$

Lemma 8.1.3. *Let $\mathfrak{m}_{\mathbb{C}_p}$ denote the maximal ideal of $\mathcal{O}_{\mathbb{C}_p}$. We have*

$$\Delta'(x_1,\ldots,x_n)^{q^{-m}} \equiv (\det(x_i^{q^j})_{1\le i,j\le n})^{q^{-m}} \mod \mathfrak{m}_{\mathbb{C}_p}.$$

In particular, as Δ has coefficients in $\mathcal{O}_{\breve{E}}$, we have $\Delta'(x_1,\ldots,x_n)^{q^{-m}} \in \mathcal{O}_{\mathbb{C}_p}\langle x_1^{q^{-\infty}},\ldots,x_n^{q^{-\infty}}\rangle$. Also, $\tau'^{q^{-m}} \in \mathcal{O}_{\mathbb{C}_p}^{\times}$.

Proof. The statement for Δ follows directly from the approximation in Lemma 5.4.3. One easily checks $|\tau| = 1$, implying the second statement. $\square$

We define

$$\mathcal{X} := \operatorname{Spf}\left(\frac{\mathcal{O}_{\mathbb{C}_p}\langle x_1^{q^{-\infty}},\ldots,x_n^{q^{-\infty}}\rangle}{(\Delta'(x_1,\ldots,x_n)^{q^{-m}} - \tau'^{q^{-m}} \mid m \in \mathbb{N}_0)^{-}}\right)$$

Proposition 8.1.4. *The formal scheme $\mathcal{X}$ is a formal model for U.*

Proof of Proposition 8.1.4. [todo] $\square$

Proposition 8.1.5. *Let b^{1/q^m} be the residue class of τ'^{1/q^m} in $\mathbb{C}_p/\mathfrak{m}_{\mathbb{C}_p}$. Then $b^{q-1} = (-1)^{n-1}$, and the special fiber of $\mathcal{X}_s := \mathcal{X} \times_{\operatorname{Spf}\mathcal{O}_{\mathbb{C}_p}} \operatorname{Spec}(\overline{\mathbb{F}}_q)$ is isomorphic to Y_b^{perf}, the perfection of the component $Y_b \subset \mathrm{DL}_n$ defined in* (7.6).

Proof. We first show that $b^{q^{-m}(q-1)} = (-1)^{n-1}$ for $m \in \mathbb{Z}$. This follows from Lemma 8.1.2. Write $\overline{\alpha}_i \in \mathbb{F}_{q^n}$ for the residue classes of the elements α_i for $i = 1,\ldots,n$. We have $\overline{\alpha}_i^{q^n} = \overline{\alpha}_i$, implying for $m \in \mathbb{Z}$ the equalities

$$b^{q^{-(m-1)}} = \det(\overline{\alpha}_i^{q^j})_{1\le i,j\le n}^{q^{-m}} = (-1)^{n-1}\det(\overline{\alpha}_i^{q^{j-1}})_{1\le i,j\le n}^{q^{-m}} = (-1)^{n-1}b^{q^{-m}}.$$

This gives $b^{q^{-m}}(q-1) = (-1)^{n-1}$, as desired. By Lemma 8.1.3, we find that $\mathcal{X}_s$ is equal to

$$\operatorname{Spec}\left(\frac{\overline{\mathbb{F}}_q[x_1^{q^{-\infty}},\ldots,x_n^{q^{-\infty}}]}{(\det(x_i^{q^{j-1}})^{q^{-m}} - b^{q^{-m}}}\right),$$

which is precisely the perfection of Y_b. $\square$

We also have

Proposition 8.1.6. *The formal model $\mathcal{X}$ is flat over* $\operatorname{Spf}(\mathcal{O}_{\mathbb{C}_p})$.

Proof. [todo; this is well-documented in [Mie16]]. $\square$

8.2 Comparison of the Group Actions

In Section 4, we saw that the Lubin–Tate perfectoid space

$$M^{(0)}_{\infty,\mathbb{C}_p} = \mathcal{M}_\infty \times_{\mathrm{Spa}(\widehat{E}^{\mathrm{ab}},\mathcal{O}_{\widehat{E}^{\mathrm{ab}}})} \mathrm{Spa}(\mathbb{C}_p,\mathcal{O}_{\mathbb{C}_p})$$

admits a right action by the group

$$G^1 \subset G = \mathrm{GL}_n(E) \times D^\times \times W_E,$$

given by those elements (g,d,σ) satisfying $\det(g)\,\mathrm{Nrd}(d)^{-1}\mathrm{Art}_E^{-1}(\sigma|_{\breve{E}}) = 1$. In this section, we construct a subgroup $J^1 \subset G^1$ stabilizing the special affinoid U constructed above, and we furthermore show that the action of J^1 on U extends to an action of J^1 on the formal model $\mathcal{X}$. This induces a right action on the special fiber $\mathcal{X}_s$ of $\mathcal{X}$, and as $\mathcal{X}_s \cong Y_b^{perf} \subset \mathrm{DL}_n^{perf}$, this yields an action of J^1 on the perfection of a part of the Deligne–Lusztig variety constructed in Section 7.2. Recall that the group

$$\overline{J}^1 = (\mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times)^1 \times n\mathbb{Z}$$

acts on Y_b, and in particular on Y_b^{perf}. We will show that the action of J^1 on Y_b^{perf} factors through a certain homomorphism $\Theta : J^1 \to \overline{J}^1$. These results lay the representation-theoretic ground for the comparison between the representations H_{LT} and H_{DL}.

We set

$$J := F^\times\,\mathrm{GL}_n(\mathcal{O}_F) \times \mathcal{O}_D^\times \times \mathrm{W}_{F_n} \text{ and } J^1 = J \cap G^1. \tag{8.1}$$

Also, we define a morphism Θ as follows. For $\sigma \in \mathrm{W}_{E_n}$ with $\sigma|_{\breve{E}} = \Phi^{n_\sigma}$ and $u_\sigma \in \mathcal{O}_{E_n}^\times$ defined as $\varpi^{-n_\sigma}\mathrm{Art}_{E_n}^{-1}(\sigma|_{\widehat{E}_n^{\mathrm{ab}}})$, we set

$$\Theta\colon J \to \mathrm{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times \times n\mathbb{Z}, \quad (\varpi^m g, d, \sigma) \mapsto (\overline{g}, \overline{d^{-1}u_\sigma^{-1}}, n_\sigma).$$

It will be convenient to have a list of generators for the group J^1.

Lemma 8.2.1. *The group J^1 is generated by elements of the form*

- $(g,1,1)$ *for* $g \in \mathrm{GL}_n(\mathcal{O}_E)$ *with* $\det g = 1$*; for those elements we have* $\Theta(g,1,1) = (\overline{g},1,0)$
- $(1,d,1)$ *for* $d \in \mathcal{O}_D^\times$ *with* $\mathrm{Nrd}\,d = 1$*; for those elements we have* $\Theta(1,d,1) = (1,\overline{d},0)$
- $(a,a,1)$ *for* $a \in F_n^\times$*; for those elements we have* $\Theta(a,a,1) = (\overline{a},\overline{a},0)$
- $(1,\alpha^{-1},\sigma)$ *for* $\sigma \in I_{E_n}$ *and* $\alpha \in \mathcal{O}_{E_n}$ *with* $\mathrm{Art}_{E_n}(\alpha) = \sigma|_{\widehat{E}_n^{\mathrm{ab}}}$*; for those elements we have* $\Theta(1,\alpha,\sigma) = (1,1,0)$.
- $(1,\varpi^{-1},\sigma)$ *for* $\sigma \in W_{E_n}$ *with* $\mathrm{Art}_{E_n}^{-1}(\sigma|_{\widehat{E}_n^{\mathrm{ab}}}) = \varpi$*; for those elements we have* $\Theta(1,\varpi^{-1},\sigma) = (1,1,n)$.

In particular, the image of J^1 under the homomorphism Θ is $\overline{J}^1$, so the induced action of J^1 on DL_n stabilizes Y_b.

For $(g, d, \sigma) \in G^1$, recall the definition of $g^* \in \mathcal{O}_{\mathbb{C}_p}[\![X_1^{q^{-\infty}}, \dots, X_n^{q^{-\infty}}]\!]$ and $d^* \in \mathcal{O}_{\mathbb{C}_p}[\![X^{q^{-\infty}}]\!]$ from Proposition 5.2.16. We wish to show the following Proposition.

Proposition 8.2.2. *The action of J^1 on $M_{\infty,C}^{(0)}$ stabilizes U and extends to $\mathcal{X}$. The induced action on the special fiber $\mathcal{X}_s$ is compatible with the action of J^1 on Y_b.*

Proof. Note that by the description of the group action of G^1 on $A_{\infty,\mathcal{O}_{\mathbb{C}_p}}$ in Proposition 5.2.18, the induced actions on C_n and Y_b^{perf} must take the form described in Figure 1.

Element	Action on $A_{\infty,\mathcal{O}_{\mathbb{C}_p}}$	Action on C_n	Action on Y_b^{perf}
$(g,1,1)$	$(X_1,\dots,X_n) \mapsto$ $g^*(X_1,\dots,X_n)$	$(x_1,\dots,x_n) \mapsto$ $\xi^{-1}g^*(\xi x_1,\dots,\xi x_n)$	$(x_1,\dots,x_n) \mapsto$ $(x_1,\dots,x_n)\cdot \overline{g}$
$(1,d,1)$	$X_i \mapsto d^{-1,*}(X_i)$	$x_i \mapsto$ $\xi^{-1}d^{-1,*}(\xi x_i)$	$x_i \mapsto \overline{d}^{-1} x_i$
$(a,a,1)$	trivial	trivial	trivial
$(1,\alpha^{-1},\sigma)$	$a \mapsto \sigma(a)$	$a \mapsto \sigma(a)$, $x_i \mapsto \frac{\xi}{\sigma(\xi)}x_i$	trivial
$(1,\varpi,\sigma)$	$a \mapsto \sigma(a)$	$a \mapsto \sigma(a)$, $x_i \mapsto \frac{\xi}{\sigma(\xi)}x_i$	$a \mapsto a^{q^{-n}}$

Figure 1: Description of the group actions.

To show that the actions described above fulfill the desired properties, it suffices to show the following claims.

1. The power series
$$\xi^{-1}g^*(\xi x_1,\dots,\xi x_n) \in \mathbb{C}_p[\![x_1,\dots x_n]\!]$$
lies inside $\mathcal{O}_{\mathbb{C}_p}\langle x_1,\dots,x_n\rangle$ and reduces to $(x_1,\dots,x_n)\cdot\overline{g}$ modulo $\mathfrak{m}_{\mathbb{C}_p}$.

2. The power series
$$\xi^{-1}d^{-1,*}(\xi x_i) \in \mathbb{C}_p[\![x_i]\!]$$
lies inside $\mathcal{O}_{\mathbb{C}_p}\langle x_i\rangle$ and reduces to $\overline{d}^{-1}x_i$ modulo $\mathfrak{m}_{\mathbb{C}_p}$.

3. If $\sigma \in \mathrm{W}_{E_n}$ lies inside the inertia subgroup I_{E_n}, the element $\xi/\sigma(\xi)$ reduces to 1 modulo $\mathfrak{m}_{\mathbb{C}_p}$.

4. If $\sigma \in \mathrm{W}_{E_n}$ satisfies $\mathrm{Art}_{E_n}^{-1}(\sigma|_{\widehat{E}_n^{\mathrm{ab}}}) = \varpi$, the element $\xi/\sigma(\xi)$ reduces to 1 modulo $\mathfrak{m}_{\mathbb{C}_p}$.

The first claim follows directly from the description of g^* in Lemma 5.2.16. Indeed, writing $g = (g_1,\dots,g_n)$ for g_i the i-th column vector of g, one quickly checks that modulo $\mathfrak{m}_{\mathbb{C}_p}$, we have
$$\xi^{-1}g_i^*(\xi x_1,\dots,\xi x_n) \equiv \sum_{j=1}^{n}{}_H\, a_{ij}^{(0)}x_j = (x_1,\dots,x_n).\overline{g}_i \mod \mathfrak{m}_{\mathbb{C}_p}.$$

Similarly one obtains the second claim.
The third claim follows as σ lies in the inertia subgroup, hence $\xi \equiv \sigma(\xi) \mod \mathfrak{m}_{\mathbb{C}_p}$.

Finally, for the fourth statement, we use that by classical Lubin–Tate theory, σ acts trivially on the system $(y_k)_{k\in\mathbb{N}}$, implying that $\sigma(\xi)=\xi$. This concludes the proof. □

8.3 The Explicit Correspondence

Fix, for the remainder of the section, an isomorphism $\overline{\mathbb{Q}}_l \cong \mathbb{C}$ and a regular character $\theta : \mathbb{F}_{q^n}^\times \to \mathbb{C}^\times$. The datum of θ can be used to construct representations of W_F and $D^\times$ and, making use of Deligne–Lusztig theory, a representation of $\mathrm{GL}_n(F)$. We proceed as follows.

- Let $\overline{\tau}_\theta$ be the character of W_{F_n} given by the composition

$$\mathrm{W}_{F_n} \to \mathrm{W}_{E_n}^{\mathrm{ab}} \xrightarrow{\mathrm{Art}_{F_n}^{-1}} F_n^\times \cong \mathbb{Z}\times\mathcal{O}_{F_n}^\times \twoheadrightarrow \mathbb{F}_{q^n}^\times \xrightarrow{\theta} \mathbb{C}^\times$$

  and put $\tau_\theta = \text{c-Ind}_{\mathrm{W}_{F_n}}^{\mathrm{W}_F}(\overline{\tau}_\theta)$.

- Let $\overline{\rho}_\theta$ be the character on $F^\times\mathcal{O}_D^\times$ given by the composition

$$F^\times\mathcal{O}_D^\times \cong \varpi^{\mathbb{Z}}\times\mathcal{O}_D^\times \twoheadrightarrow \mathbb{F}_{q^n} \xrightarrow{\theta} \mathbb{C}^\times$$

  and let $\rho_\theta = \text{c-Ind}_{F^\times\mathcal{O}_D^\times}^{D^\times}(\overline{\rho}_\theta)$.

- Let $\overline{\pi}_\theta$ be the representation of $F^\times\,\mathrm{GL}_n(\mathcal{O}_F)$ arising from post-composing R_θ (cf. Theorem 7.3.4) with the composition

$$F^\times\,\mathrm{GL}_n(\mathcal{O}_F) \cong \varpi^{\mathbb{Z}}\times\mathrm{GL}_n(\mathcal{O}_F) \twoheadrightarrow \mathrm{GL}_n(\mathcal{O}_F) \twoheadrightarrow \mathrm{GL}_n(\mathbb{F}_q).$$

  Let $\pi_\theta = \text{c-Ind}_{F^\times\,\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(F)}(\overline{\pi}_\theta)$.

Lemma 8.3.1. *The representations $\overline{\pi}_\theta$, $\overline{\rho}_\theta$ and $\overline{\tau}_\theta$ are smooth, in particular π_θ, ρ_θ and τ_θ are smooth as well. Additionally, the representations π_θ and ρ_θ are irreducible, and π_θ is supercuspidal.*

Proof. By design, $\overline{\pi}_\theta$ is trivial on the compact open subgroup $1+\varpi\,\mathrm{Mat}_{n\times n}(\mathcal{O}_F)$ of $F^\times\,\mathrm{GL}_n(\mathcal{O}_F)$. Similar statements hold for $\overline{\rho}_\theta$ and $\overline{\tau}_\theta$. [why is ρ_θ irreducible? Why is π_θ supercuspidal and irreducible?] □

The aim of this section is to prove the following statement.

Theorem 8.3.2 (Explicit Non-Abelian Lubin–Tate Theory for Depth Zero Supercuspidal Representations). *The representation $\mathrm{JL}(\pi_\theta)$ of $D^\times$ and the representation $\mathrm{rec}_F(\pi_\theta)$ of W_F take the form*

$$\mathrm{JL}(\pi_\theta) = \rho_\theta \quad \textit{and} \quad \mathrm{rec}_F(\pi_\theta) = \mathrm{Ind}_{\mathrm{W}_{F_n}}^{\mathrm{W}_F}(\tau_\theta\,\delta^{n-1}),$$

where $\delta : \mathrm{W}_{F_n} \to \{\pm 1\}$ is the unramified quadratic character. This is the character corresponding to $a \mapsto (-1)^{\mathrm{val}_{F_n}(a)}$ under the isomorphism $\mathrm{Art}_{F_n} : F_n^\times \to \mathrm{W}_{E_n}^{\mathrm{ab}}$.

Recall that H_{DL} denotes the middle l-adic cohomology of DL_n, cf. Section 7.3.

Lemma 8.3.3. *The morphism Θ makes J act on $H_{\mathrm{DL},\theta}$. This representation is of the form*

$$(g,d,\sigma) \mapsto \overline{\pi}_\theta(g) \otimes \overline{\rho}_{\theta^{-1}}(d) \otimes \left(\overline{\tau}_\theta\, \delta^{n-1}\right)^{-1} (\tfrac{1-n}{2})(\sigma).$$

This representation is smooth.

Proof. This is a direct calculation. □

The input we get from Mieda's theory is the following.

Proposition 8.3.4. *There is an injective morphism of J^1-representations*

$$\mathrm{Res}^J_{J^1}(H_{\mathrm{DL},\theta}) \hookrightarrow \mathrm{Res}^{G^1}_{J^1}(H'_{\mathrm{LT}}).$$

Proof. This is [Mie16, Proposition 5.11]. □

Lemma 8.3.5. *The morphism in Proposition 8.3.4 naturally gives rise to an injective J-equivariant morphism*

$$H_{\mathrm{DL},\theta} \hookrightarrow \mathrm{Res}^G_J H_{\mathrm{LT}}.$$

Proof. We construct a sequence of J-equivariant injections

$$H_{\mathrm{DL},\theta} \hookrightarrow \mathrm{Ind}^J_{J^1}(\mathrm{Res}^J_{J^1} H_{\mathrm{DL},\theta}) \hookrightarrow \mathrm{Ind}^J_{J^1}(\mathrm{Res}^{G^1}_{J^1} H'_{\mathrm{LT}}) \xrightarrow{\sim} \mathrm{Res}^{G^1J}_J(\mathrm{Ind}^{G^1J}_{G^1} H'_{\mathrm{LT}}) \hookrightarrow \mathrm{Res}^G_J H_{\mathrm{LT}}.$$

The first morphism. This is the unit of the adjunction $\mathrm{Res}^J_{J^1} \dashv \mathrm{Ind}^J_{J^1}$ applied at $H_{\mathrm{DL},\theta}$, which is injective by Lemma C.0.13.

The second morphism. This is $\mathrm{Ind}^J_{J^1}$ applied to the injective morphism in Proposition 8.3.4. The resulting morphism is injective because $\mathrm{Ind}^J_{J^1}$ is exact, cf. Proposition C.0.10.

The third morphism. The morphism is given by the inverse of the base-change morphism constructed in Lemma C.0.14, which is applied with $H = G^1$, $N = J$. Note that G^1 is normal in G, so the assumptions of the Lemma are satisfied. As J is open in G, the map is an isomorphism.

The fourth morphism. Since G^1J is open in G, the unit of the adjunction $\text{c-Ind}^G_{G^1J} \dashv \mathrm{Res}^G_{G^1J}$ yields a monomorphism of G^1J-representations

$$\mathrm{Ind}^{G^1J}_{G^1} H'_{\mathrm{LT}} \to \mathrm{Res}^G_{G^1J}(\text{c-Ind}^G_{G^1J}(\mathrm{Ind}^{G^1J}_{G^1} H'_{\mathrm{LT}})). \tag{8.2}$$

As G^1J co-compact in G, we have $\text{c-Ind}^G_{G^1J} = \mathrm{Ind}^G_{G^1J}$, so the right-hand side is isomorphic to $\mathrm{Res}^G_{G^1J}(\mathrm{Ind}^G_{G^1} H'_{\mathrm{LT}}) \cong \mathrm{Res}^G_{G^1J}(H_{\mathrm{LT}})$ by Proposition C.0.12 and Lemma 4.2.6. Hence, applying $\mathrm{Res}^{G^1J}_J$ to the morphism in (8.2) yields the desired map. □

The morphism constructed in Lemma 8.3.5 yields, by Frobenius reciprocity, a non-zero map of G-representations

$$\mathrm{Ind}^G_J(H_{\mathrm{DL},\theta}) \cong \pi_\theta \boxtimes \rho_{\theta^{-1}} \boxtimes (\tau_\theta\, \delta^{n-1})^{-1}(\tfrac{1-n}{2}) \to H_{\mathrm{LT}}. \tag{8.3}$$

As π_θ is supercuspidal and its central character is trivial on $\varpi^{\mathbb{Z}}$, Theorem 4.2.4 yields a non-zero map

$$\rho_{\theta^{-1}} \boxtimes (\tau_\theta\, \delta^{n-1}) \to \mathrm{JL}(\pi_\theta)^\vee \boxtimes \mathrm{rec}_F(\pi_\theta)^\vee.$$

As $\rho_{\theta^{-1}}$ and $\mathrm{JL}(\pi_\theta)^\vee$ are irreducible, this implies $\mathrm{JL}(\pi_\theta) = \rho_{\theta^{-1}}^\vee = \rho_\theta$. As $\mathrm{rec}_F(\pi_\theta)$ is irreducible and $\dim(\tau_\theta) = n = \dim(\mathrm{rec}_F(\pi_\theta))$, this also implies $\tau_\theta\, \delta^{n-1} = \mathrm{rec}_F(\pi_\theta)$, concluding the proof of Theorem 8.3.2.

A Topological Rings

To deal with the topological rings showing up, the notion of admissible rings will be convenient (taken from [Stacks, Tag 07E8]).

Definition A.0.1. Let A be a topological ring. We say that A is admissible if

- The element $0 \in A$ has a fundamental system of neighbourhoods consisting of ideals.
- There exists an ideal of definition, that is, an open ideal $I \subset A$ such that every open neighbourhood of 0 contains I^n for some n.
- It is complete, that is, the natural map

  $$A \to \lim_{J \subset A \text{ open ideal}} A/J$$

  is an isomorphism.

We say that A is adic if it admits an ideal of definition I such that I^n is open for all n. Given a topological ring A, we denote the category of admissible and adic A-algebras (algebras S with continuous morphism $A \to S$) by (A-Adm) and (A-Adic), respectively.

[The following results might be not interesting enough to make it into the final draft]

Lemma A.0.2. *Let $\phi : R \to S$ be a morphism of admissible rings, and let $I \subset R$ be an admissible ideal. Then the ideal $J = \phi(I) \cdot S$ is an ideal of definition in S.*

Proof. Let U be an open ideal of S. By continuity of ϕ, it's preimage $U' = \phi^{-1}(U)$ is open in R. Hence there is some n with $I^n \subset U'$. But now

$$\phi(I)^n = \phi(I^n) \subseteq \phi(\phi^{-1}(U)) \subseteq U$$

and the claim follows. □

Lemma A.0.3. *Let S be an admissible ring, and let $(s_1, s_2, \dots)$ be a sequence with elements in S. Then $\sum_{i=1}^\infty s_i$ converges if and only if $\lim_{i \to \infty} s_i = 0$. In this case, the product $\prod_{i=1}^\infty (1+s_i)$ exists in S.*

Proof. If the sum converges, $(s_i)_{i \in \mathbb{N}}$ has to be a null-sequence. The reverse implication and the convergence of the product follows after writing $S \cong \lim_J S/J$ for a system of open ideals $J \subset S$. □

The topology on an admissible ring R with ideal of definition I is coarser than the I-adic topology on R

Lemma A.0.4. *Let R be an admissible ring with ideal of definition I. Let R' be the same ring, but equipped with the I-adic topology. Then the identity map $R' \to R$ is continuous. In particular, if a sequence converges with respect to the I-adic topology, it also converges in R'.*

Proof. It suffices to check that open ideals of R are open in R'. Let $J \subset R$ an open ideal. By assumption, there is some n with $I^n \subset J$. But now, for any $x \in J$, we have $x + I^n \subset J$. Hence, J is open in R'. $\square$

B Extensions of Formal Modules

In this section, we equip the category $(A\text{-FM}^{\text{arb}}/X)$, where A is any ring and X is a quasi-compact and quasi-separated A-scheme, with a notion of short exact sequences. We show that this gives $(A\text{-FM}^{\text{arb}}/X)$ the structure of an exact category in the sense of Quillen [Kel90, Appendix A]. We introduce functors

$$\begin{aligned} \operatorname{Ext}(-,-) &: (A\text{-FM}^{\text{arb}}/X)^{\text{op}} \times (A\text{-FM}^{\text{arb}}/X) \to (\text{Set}) \\ \operatorname{RigExt}(-,-) &: (A\text{-FM}^{\text{arb}}/X)^{\text{op}} \times (A\text{-FM}^{\text{arb}}/X) \to (\text{Set}), \end{aligned}$$

which send a pair $(\mathcal{F}, \mathcal{F}')$ to the set of equivalence classes of extensions (respectively rigidified extensions) of $\mathcal{F}$ by $\mathcal{F}'$.

B.1 The Category of Formal Modules is Exact

Before turning our attention to formal modules, we introduce the notion of exact categories, following [Kel90, Appendix A].

Definition B.1.1 (Exact Category)**.** Let $\mathcal{A}$ be an additive category, and let $\mathcal{E}$ be a class whose members are exact triples of objects connected by arrows

$$X \xrightarrow{i} Y \xrightarrow{d} Z,$$

where i is a kernel of d and d is a co-kernel of i. We call a morphism $i : X \to Y$ an inflation if it appears as first component of some $(i, d) \in E$, second components of such pairs are called deflations. We say that the pair $(\mathcal{A}, \mathcal{E})$ is an exact category if $\mathcal{E}$ is closed under isomorphisms and satisfies the following properties.

1. The identity $\mathrm{id}_0 : 0 \to 0$ is a deflation.

2. The composition of two deflations is a deflation.

3. For each $f \in \operatorname{Hom}_{\mathcal{A}}(Z', Z)$, there is a cartesian square

$$\begin{tikzcd} Y' \arrow[r, "d'"] \arrow[d, "f'"'] & Z' \arrow[d, "f"] \\ Y \arrow[r, "d"'] & Z \end{tikzcd}$$

such that d' is a deflation.

3^{op}. For each $f \in \mathrm{Hom}_{\mathcal{A}}(X, X')$, there is a co-cartesian square

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, "f'"] \\ X' \arrow[r, "i'"'] & Y' \end{tikzcd}$$

such that i' is an inflation.

As above, suppose that A is any ring and X is a quasi-compact and quasi-separated A-scheme. Let $\mathcal{F}$, $\mathcal{E}$ and $\mathcal{F}'$ be formal A-modules over X. As a primer, we note that the category $(A\text{-FM}^{\mathrm{arb}}/X)$ is additive (essentially by Lemma 2.1.2).

Definition B.1.2 (Short Exact Sequence)**.** A pair of composable morphisms $\mathcal{F}' \to \mathcal{E} \to \mathcal{F}$ in $(A\text{-FM}^{\mathrm{arb}}/X)$ is called a short exact sequence if the induced sequence

$$0 \to \mathrm{Lie}(\mathcal{F}') \to \mathrm{Lie}(\mathcal{E}) \to \mathrm{Lie}(\mathcal{F}) \to 0$$

is a short exact sequence of $\mathcal{O}_X$-modules. In this case, we write

$$0 \to \mathcal{F}' \to \mathcal{E} \to \mathcal{F}' \to 0.$$

A pair of composable morphisms $F' \to E \to F$ in $(A\text{-FML}^{\mathrm{arb}}/R)$ is called an exact sequence if it is exact after passing to the respective formal modules.

Lemma B.1.3. *Let R be an A-algebra and let $F, F' \in (A\text{-FML}^{\mathrm{arb}}/R)$ be formal A-module laws of dimensions m and n, respectively. Write $\mathcal{F}', \mathcal{F} \in (A\text{-FM}^{\mathrm{arb}}/R)$ for the associated formal A-modules, and suppose that they fit into a exact sequence*

CONDIT ON R; NEED FORMAL IMPLICIT FUNCTION THEROE

$$0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{F} \to 0.$$

Write $\mathbf{X}$ for the variables of F' and $\mathbf{Z}$ for those of F. Then there exists a (non-canonical) coordinate on $\mathcal{E}$ giving rise to a formal A-module law E in the variables $(\mathbf{X}, \mathbf{Z})$ such that the induced morphisms of formal module laws are of the form $\alpha(\mathbf{X}) = (\mathbf{X}, 0)$, $\beta(\mathbf{X}, \mathbf{Z}) = \mathbf{Z}$. Furthermore, the formal A-module law E is of the form

$$\begin{gathered} E((\mathbf{X}_1, \mathbf{Z}_1), (\mathbf{X}_2, \mathbf{Z}_2)) = (F'(\mathbf{X}_1, \mathbf{X}_2) +_{F'} \Delta(\mathbf{Z}_1, \mathbf{Z}_2), F(\mathbf{Z}_1, \mathbf{Z}_2)) \\ \text{and} \\ [a]_E(\mathbf{X}, \mathbf{Z}) = ([a]_{F'}(\mathbf{X}) +_{F'} \delta_a(\mathbf{Z}), [a]_F(\mathbf{Z})). \end{gathered} \tag{B.1}$$

for some m-tuple of power series $\Delta \in (R[\![\mathbf{Z}_1, \mathbf{Z}_2]\!])^m$, $\delta_a \in (R[\![\mathbf{Z}]\!])^m$.

Proof. The construction of E is sketched in [GH94, Proposition 6.5]. We know that $\mathcal{E} \cong \mathrm{Spf}\, R[\![M]\!]$ for some free R-module M of rank $m + n$. As we have a short exact sequence on Lie-algebras, we may apply the formal implicit function theorem to obtain a section reference/

$\sigma : \mathcal{F} \to \mathcal{E}$ of $\beta : \mathcal{E} \to \mathcal{F}$. The datum of the morphisms α and σ is equivalent to morphisms

$$\alpha^\flat : R[\![M]\!] \to R[\![\mathbf{X}]\!] \quad \text{and} \quad \sigma^\flat : R[\![M]\!] \to R[\![\mathbf{Z}]\!]$$

on affines. Taking their sum, we obtain a morphism $R[\![M]\!] \to R[\![\mathbf{X}, \mathbf{T}]\!]$. On Lie-algebras, this morphism recovers the isomorphism $\mathrm{Lie}(\mathcal{E}) \cong \mathrm{Lie}(\mathcal{F}') \oplus \mathrm{Lie}(\mathcal{F})$ induced by $\mathrm{Lie}(\sigma)$. In particular, $\sigma^\flat + \alpha^\flat$ is an isomorphism in degree 1, hence an isomorphism. This yields the desired coordinate $\mathcal{E} \cong \operatorname{Spf} R[\![\mathbf{X}, \mathbf{Z}]\!]$. The fact about the structure of the formal A-module law E follows quickly from the fact that α and β are morphisms of formal A-module laws. $\square$

Let's turn our attention to the power series $(\Delta, (\delta_a)_{a \in A})$ appearing in the above Lemma. They satisfy certain conditions.

Definition B.1.4 (Symmetric 2-cocycles)**.** Let $\mathrm{SymCoc}^2(F, F')$ be the set of collections of power series $(\Delta, (\delta_a)_{a \in A})$ satisfying the following properties

- $\Delta(\mathbf{Z}_1, \mathbf{Z}_2) = \Delta(\mathbf{Z}_2, \mathbf{Z}_1)$
- $\Delta(\mathbf{Z}_2, \mathbf{Z}_3) +_{F'} \Delta(\mathbf{Z}_1, F(\mathbf{Z}_2, \mathbf{Z}_3)) = \Delta(F(\mathbf{Z}_1, \mathbf{Z}_2), \mathbf{Z}_3) +_{F'} \Delta(\mathbf{Z}_1, \mathbf{Z}_2)$
- $\delta_a(\mathbf{Z}_1) +_{F'} \delta_a(\mathbf{Z}_2) +_{F'} \Delta([a]_F(\mathbf{Z}_1), [a]_F(\mathbf{Z}_2)) = [a]_{F'}\Delta(\mathbf{Z}_1, \mathbf{Z}_2) +_{F'} \delta_a(F(\mathbf{Z}_1, \mathbf{Z}_2))$
- $\delta_a(\mathbf{Z}_1) +_{F'} \delta_b(\mathbf{Z}_1) +_{F'} \Delta([a]_F(\mathbf{Z}_1), [b]_F(\mathbf{Z}_1)) = \delta_{a+b}(\mathbf{Z}_1)$
- $[a]_{F'}\delta_b(\mathbf{Z}_1) +_{F'} \delta_a([b]_F(\mathbf{Z}_1)) = \delta_{ab}(\mathbf{Z}_1)$.

These objects are called symmetric 2-cocycles. The set $\mathrm{SymCoc}^2(F, F')$ is naturally a left-$\mathrm{End}(F')$-module.

Proposition B.1.5. *There is a bijection*

$$\mathrm{SymCoc}^2(F, F') \xrightarrow{\sim} \left\{ \begin{array}{c} A\text{-module laws } E \text{ on } R[\![\mathbf{X}, \mathbf{Z}]\!] \text{ fitting into an exact sequence} \\ 0 \to F' \xrightarrow{\alpha} E \xrightarrow{\beta} F \to 0 \\ \text{where } \alpha(\mathbf{X}) = (\mathbf{X}, 0) \text{ and } \beta(\mathbf{X}, \mathbf{Z}) = \mathbf{Z}. \end{array} \right\}$$

The map sends a pair $\{\Delta, (\delta_a)_a\}$ to the A-module law with structure defined following (B.1).

Proof. This is only a matter of calculation, cf. [GH94, Section 6]. $\square$

Lemma B.1.6. *If $\mathcal{F}'$, $\mathcal{E}$ and $\mathcal{F}$ are formal A-modules over a quasi-compact and quasi-separated A-scheme X, and α and β are morphisms such that $0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{F} \to 0$ is a short exact sequence of formal A-modules, α is a kernel of β and β is a cokernel of α.*

Proof. Let $\psi : \mathcal{G} \to \mathcal{E}$ be a morphism of formal A-modules such that the composition $\mathcal{G} \xrightarrow{\psi} \mathcal{E} \xrightarrow{\beta} \mathcal{F}$ is trivial. We have to show that there is a unique morphism $\overline{\psi} : \mathcal{G} \to \mathcal{F}'$ making the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r, "\alpha"] & \mathcal{E} \arrow[r, "\beta"] & \mathcal{F} \arrow[r] & 0 \\
& & \mathcal{G} \arrow[ul, dashed, "\exists! \overline{\psi}"] \arrow[u, "\psi"'] \arrow[ur, "0"'] & &
\end{tikzcd}$$

As $\overline{\psi}$ is unique, we may work locally X and assume that $X = \operatorname{Spec} R$ is affine and $\mathcal{F}'$, $\mathcal{F}$ and $\mathcal{G}$ all come from formal A-module laws. We may now assume that the short exact sequence is in the form of Lemma B.1.3. Write E, F, F', G for the formal A-module laws corresponding to $\mathcal{E}$, $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{G}$. Write $\mathbf{Y}$ for the variables of G. Now, as $\beta \circ \psi = 0$, the induced morphism of formal A-module laws $\psi : G \to E$ is of the form $\psi(\mathbf{Y}) = (\psi_1(\mathbf{Y}), 0)$, and we find that $\psi_1(\mathbf{Y}) \in (R[\![\mathbf{Y}]\!])^m$ yields a morphism of formal A-modules $G \to F'$. It is clearly unique.
Similar ideas show that β is a cokernel of α. $\square$

Lemma B.1.7. *The composition of two deflations of formal A-modules is a deflation.*

Proof. [Proof is simple application of Lemma B.1.3 but no time to write down] $\square$

Lemma B.1.8. *Let $0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{F} \to 0$ be a short exact sequence in $(A\text{-FML}^{\mathrm{arb}}/X)$. If $f \in \operatorname{Hom}_{(A\text{-FM}^{\mathrm{arb}}/X)} \mathcal{G} \to \mathcal{F}$ is a morphism of formal A-modules, then there is a formal A-module $f^*\mathcal{E}$ and a deflation $f^*\mathcal{E} \to \mathcal{G}$ fitting into a diagram with short exact sequences as rows*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r, "\alpha'"] \arrow[d, equal] & f^*\mathcal{E} \arrow[r, "\beta'"] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \mathcal{F}' \arrow[r, "\alpha"] & \mathcal{E} \arrow[r, "\beta"] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

The square on the right is cartesian.

Proof. Assume first that $X = \operatorname{Spec} R$ is affine and that $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{G}$ come from formal A-module laws over R. Then we assume to be in the situation of Lemma B.1.3, with $\mathcal{E}$ coming from a formal A-module law E. Using the induced morphism $f : G \to F$ of formal A-module laws, define the A-module law law f^*E via

$$f^*E((\mathbf{X}_1, \mathbf{Y}_1), (\mathbf{X}_2, \mathbf{Y}_2)) = (F'(\mathbf{X}_1, \mathbf{X}_2) +_{F'} \Delta(f(\mathbf{Y}_1), f(\mathbf{Y}_2)), G(\mathbf{Y}_1, \mathbf{Y}_2))$$
and
$$[a]_{f^*E}(\mathbf{X}, \mathbf{Y}) = ([a]_{F'}(\mathbf{X}) +_{F'} \delta_a(f(\mathbf{Y})), [a]_F(\mathbf{Y})).$$

Here, Δ and δ_a are the power series coming from E (cf. Lemma B.1.3). Now the top-row is exact with $\alpha'(\mathbf{X}) = (\mathbf{X}, 0)$ and $\beta'(\mathbf{X}, \mathbf{Y}) = \mathbf{Y}$. The morphism of A-module laws $f^*E \to E$ is given by $(\mathbf{X}, \mathbf{Y}) \mapsto (\mathbf{X}, f(\mathbf{Y}))$. One readily checks that

$$\begin{tikzcd}
f^*E \arrow[r, "\beta'"] \arrow[d] & G \arrow[d, "f"] \\
E \arrow[r, "\beta"] & F
\end{tikzcd}$$

is cartesian in the category of formal A-module laws over R. As the data of $\mathcal{E}$ glue, the power series defining f^*E glue to give a formal A-module $f^*\mathcal{E}$, satisfying all of the desired properties. $\square$

The dual statement is also true.

Lemma B.1.9. *Let $0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{F} \to 0$ be as above, and let $g \in \mathrm{Hom}_{(A\text{-FM}^{\mathrm{arb}}/X)}(\mathcal{F}', \mathcal{G}')$ be a morphism of formal A modules. There is a formal A-module $g_*\mathcal{E}$ over X and an inflation $\alpha' : \mathcal{G}' \to g_*\mathcal{E}$ fitting into a diagram with short exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r, "\alpha"] \arrow[d, "g"'] & \mathcal{E} \arrow[r, "\beta"] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{G}' \arrow[r, "\alpha'"] & g_*\mathcal{E} \arrow[r, "\beta'"] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

Proof. We proceed as in the proof of the previous lemma and assume that $X = \operatorname{Spec} R$ and that $\mathcal{F}'$, $\mathcal{F}$ and $\mathcal{G}$ come from formal A-module laws over R. Now E is a formal A-module law over R of the form described in Lemma B.1.3, and using the power series Δ and δ_a we define g_*E via

$$g_*E((\mathbf{Y}_1, \mathbf{Z}_1), (\mathbf{Y}_2, \mathbf{Z}_2)) = (G'(\mathbf{Y}_1, \mathbf{Y}_2) +_{G'} g(\Delta(\mathbf{Z}_1, \mathbf{Z}_2)), F(\mathbf{Z}_1, \mathbf{Z}_2))$$

and

$$[a]_{g_*E}(\mathbf{X}, \mathbf{Y}) = ([a]_{G'}(\mathbf{X}) +_{G'} g(\delta_a(\mathbf{Z})), [a]_F(\mathbf{Z})).$$

The morphism $E \to g_*E$ is given by $(\mathbf{X}, \mathbf{Z}) \mapsto (g(\mathbf{X}), \mathbf{Z})$. These data glue and give rise to a formal A-module $g_*\mathcal{E}$ over X satisfying the desired properties. □

As a consequence of the previous lemmas, we obtain

Proposition B.1.10. *Let S be a quasi-compact and quasi-separated S-scheme. Then the category $(A\text{-FML}^{\mathrm{arb}}/S)$, equipped with the notion of exact sequences from Definition B.1.2, is an exact category.*

The following calculation is convenient.

Lemma B.1.11. *We have natural isomorpisms*

$$\mathrm{Lie}(f^*\mathcal{E}) \cong \mathrm{Lie}(\mathcal{E}) \times_{\mathrm{Lie}(\mathcal{F})} \mathrm{Lie}(\mathcal{G}) \quad \textit{and} \quad \mathrm{Lie}(g_*\mathcal{E}) \cong \mathrm{Lie}(\mathcal{G}') \sqcup_{\mathrm{Lie}(\mathcal{F}')} \mathrm{Lie}(\mathcal{E}).$$

Proof. This is true locally, and the local descriptions descent to X. □

B.2 Extensions and Rigidified Extensions

We now introduce the functors Ext and RigExt. Let $\mathcal{F}$ and $\mathcal{F}'$ be formal A-modules over an A-scheme S.

Definition B.2.1 (Extension)**.** An extension of $\mathcal{F}$ by $\mathcal{F}'$ is a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{E} \to \mathcal{F} \to 0.$$

We say that this extension is equivalent to another extension

$$0 \to \mathcal{F}' \to \mathcal{E}' \to \mathcal{F} \to 0$$

if and only if there is an isomorphism $\mathcal{E} \to \mathcal{E}'$ making the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{F} \arrow[r] & 0 \end{tikzcd}$$

commute. We denote the set of equivalence classes of extensions of $\mathcal{F}$ by $\mathcal{F}'$ as $\mathrm{Ext}(\mathcal{F}, \mathcal{F}')$.

Proposition B.1.10 turns $\mathrm{Ext}(-,-)$ into a functor. In particular, $\mathrm{Ext}(\mathcal{F}, \mathcal{F}')$ carries the structure of a left-$\mathrm{End}(\mathcal{F}')$-module, with zero-object given by the canonical extension $\mathcal{F} \oplus \mathcal{F}'$.

Definition B.2.2 (Rigifidied Extension)**.** A rigidified extension of $\mathcal{F}$ by $\mathcal{F}'$ is a pair consisting of an extension

$$0 \to \mathcal{F}' \to \mathcal{E} \to \mathcal{F} \to 0$$

and a splitting s of the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Lie}(\mathcal{F}') \arrow[r] & \mathrm{Lie}(\mathcal{E}) \arrow[r] & \mathrm{Lie}(\mathcal{F}) \arrow[r] \arrow[l, bend left, "s"] & 0. \end{tikzcd}$$

We say that two rigidified extensions (E, s), (E', s') are isomorphic if there is an isomorphism $i : E \to E'$ of extensions such that $s' = \mathrm{Lie}(i) \circ s$. We denote the set of isomorphism classes of rigidified extensions by $\mathrm{RigExt}(\mathcal{F}, \mathcal{F}')$.

Lemma B.2.3. *The assignment $(\mathcal{F}, \mathcal{F}') \mapsto \mathrm{RigExt}(\mathcal{F}, \mathcal{F}')$ is functorial in both entries (contravariant in the first, covariant in the second).*

Proof. Given a morphism $f : \mathcal{G} \to \mathcal{F}$, the induced morphism $\mathrm{RigExt}(\mathcal{F}, \mathcal{F}') \to \mathrm{RigExt}(\mathcal{G}, \mathcal{F}')$ is given by sending the pair $(\mathcal{E}, s)$ to the pair $(f^*\mathcal{E}, s')$, where

$$s' : \mathrm{Lie}(\mathcal{G}) \to \mathrm{Lie}(f^*\mathcal{E}) \cong \mathrm{Lie}(\mathcal{E}) \times_{\mathrm{Lie}(\mathcal{F})} \mathrm{Lie}(\mathcal{G}), \quad x \mapsto ((s \circ \mathrm{Lie}(f))(x), x).$$

Here we used the description of $\mathrm{Lie}(f^*\mathcal{E})$ from Lemma B.1.11. Similarly, given a morphism $g : \mathcal{F}' \to \mathcal{G}'$, the induced morphism $\mathrm{RigExt}(\mathcal{F}, \mathcal{F}') \to \mathrm{RigExt}(\mathcal{F}, \mathcal{G}')$ sends $(\mathcal{E}, s)$ to $(g_*\mathcal{E}, \mathrm{Lie}(g') \circ s)$, where $g' : \mathcal{E} \to g_*\mathcal{E}$ is the canonical morphism. □

In particular, $\mathrm{RigExt}(-, \mathcal{F}')$ carries the structure of an $\mathrm{End}(\mathcal{F}')$-module, the zero-object is given by the equivalence class of the pair $(\mathcal{F}' \oplus \mathcal{F}, s_{\mathrm{triv}})$, where $s_{\mathrm{triv}} : \mathrm{Lie}(\mathcal{F}) \to \mathrm{Lie}(\mathcal{F}') \oplus \mathrm{Lie}(\mathcal{F})$ is the canonical inclusion.

Of course there is a natural transformation $\mathrm{RigExt}(-,-) \to \mathrm{Ext}(-,-)$, forgetting the splitting. It appears as the right-most term of an interesting exact sequence.

Proposition B.2.4. *There is an exact sequence of Abelian groups, functorial in $\mathcal{F}$ and $\mathcal{F}'$*

$$\mathrm{Hom}_{(A\text{-}\mathrm{FM}^{\mathrm{arb}}/S)}(\mathcal{F}, \mathcal{F}') \xrightarrow{\mathrm{Lie}} \mathrm{Hom}_{(\mathcal{O}_S\text{-}\mathrm{QCoh})}(\mathrm{Lie}(\mathcal{F}), \mathrm{Lie}(\mathcal{F}')) \to \mathrm{RigExt}(\mathcal{F}, \mathcal{F}') \to \mathrm{Ext}(\mathcal{F}, \mathcal{F}').$$

Proof. The kernel of $\mathrm{RigExt}(\mathcal{F}, \mathcal{F}') \to \mathrm{Ext}(\mathcal{F}, \mathcal{F}')$ is given (up to equivalence) by pairs of the form $(\mathcal{F}' \oplus \mathcal{F}, s)$, where s is a morphism of quasi-coherent $\mathcal{O}_S$-modules such that

$$\mathrm{Lie}(\mathcal{F}) \xrightarrow{s} \mathrm{Lie}(\mathcal{F}') \oplus \mathrm{Lie}(\mathcal{F}) \to \mathrm{Lie}(\mathcal{F})$$

is the identity. It is clear that these morphisms s correspond to morphisms $\mathrm{Lie}(\mathcal{F}) \to \mathrm{Lie}(\mathcal{F}')$. The kernel of $\mathrm{Hom}_{(\mathcal{O}_S\text{-QCoh})}(\mathrm{Lie}(\mathcal{F}), \mathrm{Lie}(\mathcal{F}')) \to \mathrm{RigExt}(\mathcal{F}, \mathcal{F}')$ is spanned by those pairs $(\mathcal{E}, s)$ that are in the same class as $(\mathcal{F}' \oplus \mathcal{F}, s_{\mathrm{triv}})$. Any such $\mathcal{E}$ fits into a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F}' \oplus \mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \psi} & & \| & & \\ 0 & \longrightarrow & \mathcal{F}' & \xrightarrow{\alpha} & \mathcal{E} & \xrightarrow{\beta} & \mathcal{F} & \longrightarrow & 0. \end{array}$$

Working locally, we assume that $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{F}'$ come from formal module laws E, F and F'. Now ψ is necessarily of the form $\psi(\mathbf{X}, \mathbf{Z}) = (\mathbf{X} +_{F'} g(\mathbf{Z}), \mathbf{Z})$. Hence, the power series g furnishes a morphism of formal module laws $F \to F'$. This construction descents to a morphism of formal A-modules $\mathcal{F} \to \mathcal{F}'$, and we have

$$s(x) = \mathrm{Lie}(\psi) \circ s_{\mathrm{triv}}(x) = \mathrm{Lie}(\alpha) \circ \mathrm{Lie}(g)(x) + x \in \mathrm{Lie}(\mathcal{E}).$$

This explains exactness on the left. $\square$

C Smooth Representations of Locally Profinite Groups

We review some aspects of the representation theory (over complex vector spaces) of locally profinite groups. If G is an arbitrary group, we denote the category of complex representations, (that is, morphisms $G \to \mathrm{GL}(V)$, where V is a $\mathbb{C}$-vector space) as (G-Rep). At the slight cost of precision, we also allow ourselves to refer to an element of $\pi : G \to \mathrm{GL}(V) \in$ (G-Rep) by the underlying vector space V, or the pair (π, V).

Definition C.0.1 (Locally Profinite Group)**.** A locally profinite group is a Hausdorff topological group such that there exists a neighbourhood of $1 \in G$ consisting of compact open subgroups.

Throughout this section, if not stated otherwise, G is a locally profinite group and $H \subset G$ is a closed subgroup of G.

Definition C.0.2 (Smooth Representation)**.** A smooth representation of G is a representation $\pi : G \to \mathrm{GL}(V) \in$ (G-Rep), such that for any $v \in V$, the stabilizer G_v of v is an open subgroup of G. We define (G-Rep$^{\mathrm{sm}}$), the category of smooth G-representations, as the full subcategory of (G-Rep) with objects given by smooth G-representations.

The forgetful functor (G-Rep) $\to$ (G-Rep$^{\mathrm{sm}}$) has a left adjoint, given by taking smooth parts.

Definition C.0.3 (Smooth Part of a Representation)**.** Let $(\pi, V) \in$ (G-Rep). We write

$$V^{\mathrm{sm}} = \bigcup_{K \subseteq G} V^K,$$

where K runs over the compact open subgroups of G and $V^K \subseteq V$ denotes the subspace of elements fixed by K. Now V^{sm} is a G-stable subspace of V, and we write $(\pi^{\mathrm{sm}}, V^{\mathrm{sm}})$ for the induced representation $G \to \mathrm{GL}(V^{\mathrm{sm}})$ of π. We call π^{sm} the smooth part of π.

Definition C.0.4 (Algebraic Induction)**.** Let G be any group and let H be a subgroup of G. We define the Algebraic Induction Functor $\mathrm{algInd}_H^G : (H\text{-Rep}) \to (G\text{-Rep})$ as follows. Given an H-representation $(\pi, V) \in (H\text{-Rep})$, consider the vector space

$$\mathrm{algInd}_H^G(V) = \{\phi : G \to V \mid \phi(hg) = \pi(h)g\}.$$

Now G acts naturally on $\mathrm{algInd}_H^G(V)$ by right-translation (that is, $g.\phi(x) = \phi(xg)$), and we write $(\mathrm{algInd}_H^G(\pi), \mathrm{algInd}_H^G(V))$ for the corresponding representation of G.

Remark. We have $\mathrm{algInd}_H^G(V) = \mathrm{Hom}_{(\mathbb{C}[H]\text{-Mod})}(\mathbb{C}[G], V)$. As $\mathbb{C}[G]$ has a natural $(\mathbb{C}[H], \mathbb{C}[G])$-bimodule structure, we obtain a natural left-G-action on $\mathrm{algInd}_H^G(V)$. This action is precisely the one described above.

Definition C.0.5 (Restriction Functor)**.** Let G be any group and let H be a subgroup of G. If $\pi : G \to \mathrm{GL}(V)$ is a representation of G, we define the restriction of π from G to H as

$$\mathrm{Res}_H^G(\pi) : H \hookrightarrow G \xrightarrow{\pi} \mathrm{GL}(V)$$

and call $\mathrm{Res}_H^G : (G\text{-Rep}) \to (H\text{-Rep})$ the restriction functor.

Lemma C.0.6. *Let G be any group and let H be any subgroup of G. Then Res_H^G is left-adjoint to algInd_H^G.*

Proof. By the Remark above, this statement readily reduces to the Tensor-Hom-Adjunction. $\square$

Lemma C.0.7. *If G is locally profinite and H is a closed subgroup, for any $(\pi, V) \in (G\text{-Rep})$ we have an H-equivariant split inclusion*

$$\mathrm{Res}_H^G(V^{\mathrm{sm}}) \subseteq \left(\mathrm{Res}_H^G(V)\right)^{\mathrm{sm}},$$

with equality if H is open. In particular, Res_H^G restricts to a functor

$$\mathrm{Res}_H^G : (G\text{-Rep}^{\mathrm{sm}}) \to (H\text{-Rep}^{\mathrm{sm}}).$$

Proof. The first part follows from

$$\mathrm{Res}_H^G(V^{\mathrm{sm}}) = \bigcup_{K \subset G} V^K \subseteq \bigcup_{K \subset G} V^{K \cap H} = \left(\mathrm{Res}_H^G(V)\right)^{\mathrm{sm}},$$

where K runs over the compact open subsets of G. This is an equality if H is open. The canonical splitting (sending everything outside the image to zero) is H-equivariant. $\square$

Definition C.0.8 (Smooth Induction)**.** We define the smooth indction functor $\mathrm{Ind}_H^G : (H\text{-Rep}^{\mathrm{sm}}) \to (G\text{-Rep}^{\mathrm{sm}})$ as the smooth part of the algebraic induction functor. That is, for any smooth representation $\pi : G \to \mathrm{GL}(V)$, we set

$$\mathrm{Ind}_H^G(\pi) := \left(\mathrm{algInd}_H^G(\pi)\right)^{\mathrm{sm}}.$$

Definition C.0.9 (Compact Induction)**.** Let $\pi : H \to \mathrm{GL}(V)$ be a smooth representation of H. Then we define $\text{c-Ind}_H^G(\pi)$, the compactly induced representation of π, as the subrepresentation of $\mathrm{Ind}_H^G(\pi)$ with underlying vector space

$$\{\phi \in \mathrm{Ind}_H^G(\pi) \mid \mathrm{Supp}(\phi) \subseteq G \text{ is compact in } H\backslash G\}.$$

This construction yields a functor $\text{c-Ind}_H^G : (H\text{-Rep}^{\mathrm{sm}}) \to (G\text{-Rep}^{\mathrm{sm}})$.

Note that if H is co-compact in G, we have $\text{c-Ind}_H^G = \mathrm{Ind}_H^G$.

Remark. If H is an open subgroup of G, the quotient $H\backslash G$ is discrete. Now given $(\pi, V) \in (H\text{-Rep}^{\mathrm{sm}})$, an element $\phi \in \mathrm{Ind}_H^G(\pi)$ lies in $\text{c-Ind}_H^G(\pi)$ if and only if the image of $\mathrm{Supp}(\phi)$ is finite in $H\backslash G$. In this case there is an isomorphism

$$\Psi : \text{c-Ind}_H^G(V) \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V, \quad \phi \mapsto \sum_{[g] \in \mathrm{Supp}(\phi)} g^{-1} \otimes \phi(g) \tag{C.1}$$

which does not depend on the choice of representative $g \in [g]$ as $(hg)^{-1} \otimes \phi(hg) = g^{-1} \otimes \phi(g)$. Giving $\mathbb{C}[G]$ the structure of an $(\mathbb{C}[G], \mathbb{C}[H])$-bimodule, the natural left-G-action on $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ is compatible with the one on $\text{c-Ind}_H^G(V)$ under the isomorphism Ψ.

Proposition C.0.10. *For H a closed subgroup of G, the functors algInd_H^G, Ind_H^G and c-Ind_H^G are exact.*

Proof. The statement for algInd_H^G is implied by the fact that $\mathbb{C}[G]$ is a free (thereby projective) $\mathbb{C}[H]$-module. For the remaining statements, see [BH06, p. 18f]. $\square$

Theorem C.0.11 (Smooth Frobenius Reciprocity)**.** *Let G be a locally profinite group and let $H \subseteq G$ be a closed subgroup. Then there is an adjunction*

$$\mathrm{Res}_H^G \dashv \mathrm{Ind}_H^G.$$

If H is additionally assumed to be open in G, there is an adjunction

$$\text{c-Ind}_H^G \dashv \mathrm{Res}_H^G.$$

In particular, if H is co-compact and open in G, Ind_H^G is both left- and right-adjoint to Res_H^G.

Proof. Making use of the remarks above, both adjunctions are the Tensor-Hom-Adjunction in disguise. For the adjunction $\text{c-Ind}_H^G \dashv \mathrm{Res}_H^G$, this is immediate. For the second we observe that

$$\mathrm{Hom}_{(H\text{-Rep}^{\mathrm{sm}})}(\mathrm{Res}_H^G V, W) \cong \mathrm{Hom}_{(G\text{-Rep})}(V, \mathrm{algInd}_H^G(W)) = \mathrm{Hom}_{(G\text{-Rep}^{\mathrm{sm}})}(V, \mathrm{Ind}_H^G(W)).$$

Here the first isomorphism is by Tensor-Hom-adjunction, the second equality uses that V is a smooth representation of G. $\square$

Proposition C.0.12. *Let I be a closed subgroup of H. There is a natural isomorphism $\mathrm{Ind}_H^G \circ \mathrm{Ind}_I^H \xrightarrow{\sim} \mathrm{Ind}_I^G$. The same statement is true for compact and algebraic induction.*

Proof. Trivially, $\mathrm{Res}_I^G = \mathrm{Res}_I^H \circ \mathrm{Res}_H^G$. The claim follows as the functors in question are adjoints to the left or the right hand side of this equation, thereby isomorphic. $\square$

Lemma C.0.13. *Let H be a closed subgroup of G. The functor Res_H^G is faithful. Equivalently, the unit $\mathrm{id}_{(G\text{-}\mathrm{Rep}^{\mathrm{sm}})} \to \mathrm{Ind}_H^G \circ \mathrm{Res}_H^G$ of the adjunction $\mathrm{Res}_H^G \dashv \mathrm{Ind}_H^G$ is injective on components. If H is additionally assumed to be an open subgroup of G, The functor c-Ind_H^G is faithful. Equivalently, the components of the unit $\mathrm{id}_{(H\text{-}\mathrm{Rep}^{\mathrm{sm}})} \to \mathrm{Res}_H^G \circ \text{c-Ind}_H^G$ coming from the adjunction $\text{c-Ind}_H^G \dashv \mathrm{Res}_H^G$ are injective.*

Proof. Faithfulness of Res_H^G is clear. For faithfulness of c-Ind_H^G, note that the unit of the adjunction $\text{c-Ind}_H^G \dashv \mathrm{Res}_H^G$ is given on components $(\pi, V) \in (H\text{-}\mathrm{Rep}^{\mathrm{sm}})$ by the map $v \mapsto \phi_v$, where $\phi_v \in \text{c-Ind}_H^G(V)$ is defined as

$$\phi_v : G \to V, \quad g \mapsto \begin{cases} \pi(g)v & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

The resulting morphism $V \to \mathrm{Res}_H^G(\text{c-Ind}_H^G(V))$ is injective. Now all claims follow since faithfulness of the left-adjoint is equivalent to the unit being a monomorphism on components, cf. [Rie17, Lemma 4.5.13]. $\square$

Remark. For the sake of completeness, we note that the unit of the adjunction $\mathrm{Res}_H^G \dashv \mathrm{Ind}_H^G$ is given on components $(\pi, V) \in (G\text{-}\mathrm{Rep}^{\mathrm{sm}})$ by

$$V \to \mathrm{Ind}_H^G(\mathrm{Res}_H^G(V)), \quad v \mapsto \psi_v; \quad \text{where} \quad \psi_v(g) = \pi(g)v.$$

The following Lemma is an instance of base-change.

Lemma C.0.14. *Let H and N be closed subgroups of G satisfying $NH = HN$. Let (π, V) be a smooth representation of H. Then there is a natural split monomorphism of N-representations*

$$\mathrm{Res}_N^{HN}(\mathrm{Ind}_H^{HN}\pi) \to \mathrm{Ind}_{H\cap N}^N(\mathrm{Res}_{H\cap N}^H \pi).$$

If N is open in HN, this map is an isomorphism.

Proof. One quickly checks that the map

$$\mathrm{Res}_N^{HN}(\mathrm{algInd}_H^{HN} V) \to \mathrm{algInd}_{H\cap N}^N(\mathrm{Res}_{H\cap N}^H V)$$

given by sending $\phi : HN \to V$ to its restriction $\phi|_N$, is an isomorphism. Now the claim follows by taking smooth parts and applying Lemma C.0.7. $\square$

Remark. There are multiple ways to construct the map above. Applying $\mathrm{Ind}_H^{HN}(-)$ to the unit of the adjunction $\mathrm{Res}_{H\cap N}^H \dashv \mathrm{Ind}_{H\cap N}^H$ yields for any $\pi \in (H\text{-}\mathrm{Rep}^{\mathrm{sm}})$ a natural morphism

$$\mathrm{Ind}_H^{HN}(\pi) \to \mathrm{Ind}_{H\cap N}^H \mathrm{Res}_{H\cap N}^H(\pi) \cong \mathrm{Ind}_N^{HN} \mathrm{Ind}_{H\cap N}^N \mathrm{Res}_{H\cap N}^H(\pi),$$

which is equivalent to a map

$$\mathrm{Res}_N^{HN}(\mathrm{Ind}_H^{HN}\pi) \to \mathrm{Ind}_{H\cap N}^N(\mathrm{Res}_{H\cap N}^H \pi).$$

This gives the same map as in the proof. The dual construction (starting with the co-unit) also yields the same map.

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