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This section will serve as an introduction to formal groups and formal modules. Formal groups (or rather, formal group laws) were first introduced by SALOMON BOCHNER in 1946 as a natural means of studying Lie Groups over fields of characteristic 0, cf. [1]. The study of formal groups later became interesting for its own right, with pioneering works of Lazard [3].

2.1 Formal Modules

As promised in the introduction, we begin by defining formal group laws.

Definition 2.1.1 (Formal Group Law)**.** Let R be a ring. A (commututative, one-dimensional) formal group law over R is a power series $F(X,Y) \in R[\![X,Y]\!]$ such that $F(X,Y) \equiv X+Y$ modulo terms of degree 2 and the following properties are satisfied:

- $F(F(X,Y),Z) = F(X,F(Y,Z))$,
- $F(X,Y) = F(Y,X)$,
- $F(X,0) = X$.

Given two formal group laws $F,G \in R[\![X,Y]\!]$, a morphism $f : F \to G$ is a power series $f \in R[\![T]\!]$ such that $f(0) = 0$ and $f(F(X,Y)) = G(f(X),f(Y))$. Such a series is an isomorphism if there is an inverse, that is, a power series $g \in R[\![T]\!]$ with $(f \circ g)(T) = T$. This yields the category of formal group laws over R, which we notate by (FGL/R).

The following statements about morphisms of formal group laws are useful and easily verified.

Lemma 2.1.2. *Let R be a ring and let $F,G \in R[\![X,Y]\!]$ be two formal group laws over R.*

1. *Given two morphisms $f,g : F \to G$, the power series $G(f(T),g(T)) \in R[\![T]\!]$ is a morphism of formal group laws $F \to G$. In particular,* $\mathrm{Hom}_{(\mathrm{FGL}/R)}(F,G)$ *is an abelian group for any two formal group laws F,G.*
2. *The abelian group* $\mathrm{End}_{(\mathrm{FGL}/R)}(F)$ *has a natural ring structure with multiplication given by concatenation.*
3. *A morphism $f = c_1T + c_2T^2 + \cdots \in R[\![T]\!]$ between F and G is an isomorphism if and only if $c_1 \in R^\times$.*

Example. Let us introduce the following two formal group laws.

- *The additive formal group law.* Write $\mathbb{G}_a$ for the formal group law with addition given by $\mathbb{G}_a(X,Y) = X+Y$.
- We write $\mathbb{G}_m$ for the formal group law associated with the with $\mathbb{G}_m(X,Y) = X+Y+XY$.

Next up is the definition of formal A-module laws. Naively, we'd like to say that an A-module law is the same as that of a formal group law F plus A-module structure, i.e. a morphism of rings $[\cdot]_F : A \to \mathrm{End}_{(\mathrm{FGL}/R)}(F)$. But there is a subtlety going on here: Let

$$\mathrm{Lie} : (\mathrm{FGL}/R) \to (\mathrm{Ab})$$

be the (constant) functor that sends $F \in (\mathrm{FGL}/R)$ to $(R, +)$, and morphisms $f : G \to H$ given by a formal power series $f = c_1T + c_2T^2 + \cdots \in R[\![T]\!]$ to the endomorphism of R given by multiplication with c_1. The condition that $F(X, Y) \equiv X + Y$ modulo degree 2 enforces that the induced map $\mathrm{End}(F) \to \mathrm{End}(R)$ is a morphism of rings. Now, the A-module structure on F yields an A-module structure on R, given by the concatenation

$$A \xrightarrow{[\cdot]_F} \mathrm{End}(F) \xrightarrow{\mathrm{Lie}} \mathrm{End}(R), \quad a \mapsto \mathrm{Lie}([a]_F)$$

This is a morphism of rings, and we obtain an A-algebra structure on R. We'd like the A-algebra structure on R to be uniform. This motivates the following definition.

Definition 2.1.3 (Formal A-module law)**.** Let A be a ring and R be an A-algebra with structure morphism $p : A \to R$. A (one-dimensional) A-module law over an R is a pair $(F, ([a]_F)_{a\in A})$, where $F \in R[\![X, Y]\!]$ is a formal group law and $[a]_F = p(a)X + c_2X^2 + \cdots \in R[\![X]\!]$ yield endomorphisms $F \to F$ such that the induced map

$$A \to \mathrm{End}(F), \quad a \mapsto [a]_F$$

is a morphism of rings.

Similarly to above, we obtain a category of formal A-module laws over R, which we denote by $(A\text{-FML}/R)$. Note that $(\mathrm{FGL}/R) \cong (\mathbb{Z}\text{-FML}/R)$. Slightly abusing notation, we usually do not explicitely mention the A-structure when referring to formal module laws, simply writing $F \in (A\text{-FML}/R)$, for example.
The following lemma explains a the functoriality of the assignment $R \mapsto (A\text{-FML}/R)$.

Lemma 2.1.4. *The assignment $R \mapsto (A\text{-FML}/R)$ is functorial in the following sense. If $p : R \to R'$ is a morphism of A-algebras, we obtain a functor*

$$(A\text{-FML}/R) \to (A\text{-FML}/R'), \quad F \mapsto p_*F,$$

where p_*F is the formal A-module law obtained by applying p to the coefficients of the formal power series representing addition and scalar multiplication of F. We sometimes write (with abuse of notation) $p_*F = F \otimes_R R'$.

Note that every formal module law $F \in (A\text{-FML}/R)$ yields a functor

$$(R\text{-Alg}) \to (A\text{-Mod}), \quad S \mapsto \mathrm{Nil}(S), \tag{2.1}$$

where $\mathrm{Nil}(S)$, the set of nilpotent elements of S, is equipped with addition and scalars given by

$$s_1 + s_2 = F(s_1, s_2) \in \mathrm{Nil}(S), \quad as = [a]_F(s) \in \mathrm{Nil}(S).$$

This construction yields a functor (with slight abuse of notation)

$$(A\text{-FML}/R) \to \mathrm{Fun}((R\text{-Alg}), (A\text{-Mod})), \tag{2.2}$$

where Fun denotes the functor category.

Passing from discrete R-algebras to admissible R-algebras, this construction extends naturally to a functor

$$\mathrm{Spf}^F : (A\text{-FML}/R) \to \mathrm{Fun}((R\text{-Adm}), (A\text{-Mod})), \quad F \mapsto \mathrm{Spf}\, R[\![T]\!],$$

where we equip $\mathrm{Spf}\, R[\![T]\!]$ with the structure of an A-module object using the endomorphisms coming from F. Following this line of thought leads naturally to the definition of formal modules.

Definition 2.1.5 (Formal Group and Formal Module.)**.** Let X be an A-scheme, and let let $\mathcal{F}$ be an A-module object in (FSch/X), the category of formal schemes over X. Suppose that there is a Zariski-covering $(\mathrm{Spec}(R_i))_{i\in I}$ of X with $\mathcal{F} \times_X U_i \cong \mathrm{Spf}(R_i[\![T]\!])$. If for every $i \in I$ the induced A-module structure on $\mathrm{Spf}(R_i[\![T]\!])$ comes from a formal A-module law F_i over R_i, we say that $\mathcal{F}$ is a formal A-module.

Definition 2.1.6 (Coordinate)**.** Let $\mathcal{F}$ be a formal A-module over X. The choice of a cover $\sqcup_{i\in I} \mathrm{Spec}(R_i) \to X$ together with isomorphisms $\mathcal{F} \times_X \mathrm{Spec}(R_i) \cong \mathrm{Spf}(R_i[\![T]\!])$ will be referred to as a coordinate of $\mathcal{F}$.

Of course there is a functor

$$(A\text{-FML}/R) \to (A\text{-FM}/R),$$

essentially forgetting the choice of module law. The observation of Lemma 2.1.4 translates to formal modules, a morphism $p : R \to R'$ yields a functor

$$p_* : (A\text{-FM}/R) \to (A\text{-FM}/R'), \quad \mathcal{F} \mapsto \mathcal{F} \otimes_R R'.$$

Example. The additive group law $\mathbb{G}_a$ extends to a formal A-module over an affine base $\mathrm{Spec}\, R$ by setting

$$[a]_{\mathbb{G}_a}(T) = aT$$

for $a \in A$. More generally, we obtain a formal A-module over an arbitrary base scheme.

The multiplicative formal group $\mathbb{G}_m$ does not have an obvious generalization to a formal A-module law for general A. In the case where A is the ring of integers of a local field, Lubin and Tate [4] construct such generalizations. This construction, and the application to local class fild theory, will be discussed in section 2.2.

2.1.1 Formal DVR-Modules over Fields of Characteristic 0

As above, let A be a discrete valuation ring with uniformizer π and finite residue field k. Let K denote the field of fractions of A.

2.1.2 Formal DVR-Modules over Residue Fields

Let $\mathbb{F}_q$ denote the finite field with $q = p^n$ elements.

Definition 2.1.7 (Frobenius)**.** Given a formal group F over $\mathbb{F}_q$, let ϕ denote the Frobenius endomorphism. This is the endomorphism given by $f(T) = T^q$ after choosing a coordinate on F.

Let F and G be formal group laws over $\mathbb{F}_q$, and let $f : F \to G$ be a non-zero morphism between F and G given by a formal power series $f(T) = c_1 T + c_2 T + \dots$.

Definition 2.1.8 (Height)**.** In the above situation, the height of f, denoted $\mathrm{ht}(f)$, is the greatest integer h such that f factors through $\phi^h : F \to F$. In case that $f = 0$, we write $\mathrm{ht}(f) = \infty$.

2.2 Application: Local Class Field Theory

Let K be a local field with residue field k, put $q = \#k$, and denote by $\nu_K : K \to \mathbb{Z} \cup \{\infty\}$ the valuation of K, normalized such that $\nu_K(\pi) = 1$ for a uniformizer π of K. The aim of this subsection is to describe the maximal abelian extension of a local field K.

The Local Kronecker-Weber theorem gives an explicit description of the abelianization of the absolute Galois group of K only in terms of K:

Theorem 2.2.1 (Local Kronecker-Weber)**.** *There is an isomorphism (canonical up to choice of a uniformizer $\pi \in K$)*

$$\mathrm{Gal}(\overline{K}/K)^{\mathrm{ab}} \cong \mathrm{Gal}(K^{\mathrm{ab}}/K) \cong \mathcal{O}_K^{\times} \times \widehat{\mathbb{Z}}.$$

Here, K^{ab} denote the maximal abelian extension of K, which can (after choosing an algebraic closure of K) be described as $\overline{K}^{[G_K, G_K]}$.

The extension K^{ab} consists of two parts, we have $K^{\mathrm{ab}} = K^{\mathrm{rm}} \cdot K^{\mathrm{nr}}$. The field K^{nr}, the maximal unramified extension of K, has relatively simple structure. Describing the field K^{rm} (or rather, it's completion) is the hard part and it is here where we apply the theory of formal modules.

The valuation ν_K extends uniquely to $\overline{K}$, yielding a π-adic norm on $\overline{K}$. Let C denote the completion with respect to this norm. An application of Krasner's Lemma implies that $\mathrm{Gal}(C/K) \cong \mathrm{Gal}(\overline{K}/K) =: G_K$. One readily checks that any $\sigma \in G_K$ yields a continuous Ref automorphism $\mathcal{O}_C \to \mathcal{O}_C$, and we obtain a short exact sequence

$$0 \to I_K \to G_K \to \mathrm{Gal}(\overline{k}/k) \to 0.$$

The subgroup $I_K \subset G_K$ is called the inertia subgroup of K, and we write $\breve{K}$ for the subfield of C fixed by I_K. In particular we have $\mathrm{Gal}(\breve{K}/K) \cong \mathrm{Gal}(\overline{k}/k)$. One readily confirms that $\breve{K}$ is complete with respect to the norm induced by K.

As the Galois group of any finite extension of k is cyclic, we find that $\mathrm{Gal}(\breve{K}/K)$ is abelian. In fact, it is isomorphic to $\widehat{\mathbb{Z}} = \lim_n(\mathbb{Z}/n\mathbb{Z})$. Hence K_∞ decomposes as $\breve{K} \cdot K_\pi$ for some abelian, complete extension K_π/K such that $K_\pi \cap \breve{K} = K$. Now K_π is the completion of K^{rm}. Observe that

$$\mathrm{Gal}(K_\infty/K) \cong \mathrm{Gal}(K_\pi/K) \times \mathrm{Gal}(\breve{K}/K) \cong \mathrm{Gal}(K_\pi/K) \times \widehat{\mathbb{Z}},$$

so Theorem 2.2.1, the local Kronecker-Weber Theorem, is equivalent to showing that the Galois group of K_π over K is isomorphic to $\mathcal{O}_K^\times$.

3 Non-Abelian Lubin-Tate Theory: An Overview

In the preceeding chapter we used formal $\mathcal{O}_K$-modules to understand the maximial abelian extension of a local field K. The hope of non-Abelian Lubin-Tate theory is to gain insight about the Abelian extensions of K by considering certain moduli spaces of formal $\mathcal{O}_K$-modules. More precisely, attached to a formal $\mathcal{O}_K$-module H_0 over $\overline{\mathbb{F}}_q$ (determined up to isomorphism by its height n), we attach a system of rigid spaces $\{M_K\}_{K\subset \mathrm{GL}_n(\mathcal{O}_K)}$, the so called Lubin-Tate Tower. For $l \neq p$, the system of l-adic compactly supported cohomology groups $\{H_c^i(M_K, \overline{\mathbb{Q}}_l)\}_K$ admits commuting actions by $\mathrm{GL}_n(K)$, W_K and $D^\times$, where the latter denotes the units of the central divison algebra $D = \mathrm{End}_{(\mathcal{O}_K\text{-FM}/\overline{\mathbb{F}}_q)}(H_0) \otimes \mathbb{Q}$. This yields a correspondence of representations of the respective groups, and Harris and Taylor showed in [2] that the cohomology of middle degree induces (a version of) the Local Langlands Correspondence. Our goal is an explicit description of this correspondence, and we hope to obtain such a description by understanding the Lubin-Tate tower explicitely. As it turns out, the the limit $\lim_K M_K$ is representable by a perfectoid space which is is easier to describe than its individual layers.

3.1 The Lubin-Tate Tower

3.1.1 Deformations of Formal Modules

We mostly follow [5, Chapter 2] for notation. Let $\mathcal{C}$ denote the category of local, Noetherian $\mathcal{O}_{\breve{K}}$-modules with distinguished isomorphisms $R/\mathfrak{m}_R \to \overline{\mathbb{F}}_q$. Let H_0 be a formal $\mathcal{O}_K$-module over $\overline{\mathbb{F}}_q$.

Definition 3.1.1 (Deformation). Let $R \in \mathcal{C}$. A deformation of H_0 to R is a pair (H, ι) where H is a formal $\mathcal{O}_K$-module over R and ι is a quasi-isogeny

$$\iota : H_0 \dashrightarrow H \otimes_R \overline{\mathbb{F}}_q.$$

Two deformations (H, ι) and (H', ι') are isomorphic if there is an isomorphism $\tau : H \to H'$ with $\iota' \circ \tau = \iota$.

The Lubin-Tate space without level structure is the moduli space of such deformations. More precisely, we define it as the functor

$$\mathcal{M}_0 : \mathcal{C} \to (\mathrm{Set}), \quad R \mapsto \{\text{deformations } (H, \iota) \text{ of } H_0\}/\mathrm{iso}.$$

Theorem 3.1.2 (Representability of $\mathcal{M}_0$). *Proof.* □

- Deformations
- Representability of $\mathcal{M}_0$.

3.1.2 Deformations of Formal Modules with Drinfeld Level Structure

- Drinfeld Level
- Moduli Problem + Representability
- The Lubin-Tate Tower

3.1.3 The Group actions on the Tower and its Cohomology

- Action By $D^\times$ and GL_n
- Action by W_K via Weil descent Datum.

3.2 The Local Langlands Correspondence for the General Linear Group

3.3 The Lubin-Tate Perfectoid Space

4 Mieda's Approach to the Explicit Local Langlands Correspondence

5 The Explicit Local Langlands Correspondence for Depth Zero Supercuspidal Representations

5.1 The Special Affinoid

5.2 Deligne-Lusztig Theory for Depth Zero Representations

5.3 Proof

A Topological Rings

To deal with the topological rings showing up, the notion of admissible rings will be convenient (taken from [Stacks, Tag 07E8]).

Definition A.0.1. Let A be a topological ring. We say that A is admissible if

- The element $0 \in A$ has a fundamental system of neighbourhoods consisting of ideals.

- There exists an ideal of definition, that is, an ideal $I \subset A$ such that every open neighbourhood of 0 contains I^n for some n.
- It is complete, that is, the natural map

$$A \to \lim_{J \subset A \text{ open ideal}} A/J$$

  is an isomorphism.

We say that A is adic if it admits an open ideal of definition. Given a topological ring A, we denote the category of admissible and adic A-algebras (algebras S with continuous morphism $A \to S$) by (A-Adm) and (A-Adic), respectively.