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Max von Consbruch

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This section will serve as an introduction to formal groups and formal modules. Formal groups (or rather, formal group laws) were first introduced by SALOMON BOCHNER in 1946 as a natural means of studying Lie Groups over fields of characteristic 0, cf. **Bochner1946FGrps**. The study of formal groups later became interesting for its own right, with pioneering works of Lazard **Lazard1955FGrps**.

2.1 Formal Modules

As promised in the introduction, we begin by defining formal group laws.

Definition 2.1.1 (Formal Group Law)**.** Let R be a ring. A (commututative, one-dimensional) formal group law over R is a power series $F(X,Y) \in R[\![X,Y]\!]$ such that $F(X,Y) \equiv X+Y$ modulo terms of degree 2 and the following properties are satisfied:

- $F(F(X,Y),Z) = F(X,F(Y,Z))$,
- $F(X,Y) = F(Y,X)$,
- $F(X,0) = X$.

Given two formal group laws $F,G \in R[\![X,Y]\!]$, a morphism $f : F \to G$ is a power series $f \in R[\![T]\!]$ such that $f(0) = 0$ and $f(F(X,Y)) = G(f(X),f(Y))$. Such a series is an isomorphism if there is an inverse, that is, a power series $g \in R[\![T]\!]$ with $(f \circ g)(T) = T$. This yields the category of formal group laws over R, which we notate by (FGL/R).

The following statements about morphisms of formal group laws are useful and easily verified.

Lemma 2.1.2. *Let R be a ring and let $F,G \in R[\![X,Y]\!]$ be two formal group laws over R.*

1. *Given two morphisms $f,g : F \to G$, the power series $G(f(T),g(T)) \in R[\![T]\!]$ is a morphism of formal group laws $F \to G$. In particular, $\mathrm{Hom}_{(\mathrm{FGL}/R)}(F,G)$ is an abelian group for any two formal group laws F,G.*
2. *The abelian group $\mathrm{End}_{(\mathrm{FGL}/R)}(F)$ has a natural ring structure with multiplication given by concatenation.*
3. *A morphism $f = c_1T + c_2T^2 + \cdots \in R[\![T]\!]$ between F and G is an isomorphism if and only if $c_1 \in R^\times$.*

Example. Let us introduce the following two formal group laws.

- *The additive formal group law.* Write $\widehat{\mathbb{G}}_a$ for the formal group law with addition given by $\widehat{\mathbb{G}}_a(X,Y) = X+Y$.

- We write $\widehat{\mathbb{G}}_m$ for the formal group law associated with the with $\widehat{\mathbb{G}}_m(X,Y) = X+Y+XY$. Note that $\widehat{\mathbb{G}}_m(X,Y) = (X+1)(Y+1)-1$

Next up is the definition of formal A-module laws. Naively, we'd like to say that an A-module law is the same as that of a formal group law F plus A-module structure, i.e. a morphism of rings $[\cdot]_F : A \to \mathrm{End}_{(\mathrm{FGL}/R)}(F)$. But there is a subtlety here: Let

$$\mathrm{Lie} : (\mathrm{FGL}/R) \to (\mathrm{Ab})$$

be the (constant) functor that sends $F \in (\mathrm{FGL}/R)$ to $(R,+)$, and morphisms $f : G \to H$ given by a formal power series $f = c_1T + c_2T^2 + \cdots \in R[\![T]\!]$ to the endomorphism of R given by multiplication with c_1. The condition that $F(X,Y) \equiv X+Y$ modulo degree 2 enforces that the induced map $\mathrm{End}(F) \to \mathrm{End}(R)$ is a morphism of rings. Now, the A-module structure on F yields an A-module structure on R, given by the concatenation

$$A \xrightarrow{[\cdot]_F} \mathrm{End}(F) \xrightarrow{\mathrm{Lie}} \mathrm{End}(R), \quad a \mapsto \mathrm{Lie}([a]_F)$$

This is a morphism of rings, and we obtain an A-algebra structure on R. We'd like the A-algebra structure on R to be uniform. This motivates the following definition.

Definition 2.1.3 (Formal A-module law)**.** Let A be a ring and R be an A-algebra with structure morphism $p : A \to R$. A (one-dimensional) A-module law over an R is a pair $(F, ([a]_F)_{a\in A})$, where $F \in R[\![X,Y]\!]$ is a formal group law and $[a]_F = p(a)X + c_2X^2 + \cdots \in R[\![X]\!]$ yield endomorphisms $F \to F$ such that the induced map

$$A \to \mathrm{End}(F), \quad a \mapsto [a]_F$$

is a morphism of rings.

Similarly to above, we obtain a category of formal A-module laws over R, which we denote by $(A\text{-FML}/R)$. Note that $(\mathrm{FGL}/R) \cong (\mathbb{Z}\text{-FML}/R)$. Slightly abusing notation, we usually do not explicitely mention the A-structure when referring to formal module laws, simply writing $F \in (A\text{-FML}/R)$, for example.
The following lemma explains a the functoriality of the assignment $R \mapsto (A\text{-FML}/R)$.

Lemma 2.1.4. *The assignment $R \mapsto (A\text{-FML}/R)$ is functorial in the following sense. If $p : R \to R'$ is a morphism of A-algebras, we obtain a functor*

$$(A\text{-FML}/R) \to (A\text{-FML}/R'), \quad F \mapsto p_*F,$$

where p_*F is the formal A-module law obtained by applying p to the coefficients of the formal power series representing addition and scalar multiplication of F. We sometimes write (with abuse of notation) $p_*F = F \otimes_R R'$.

Note that every formal module law $F \in (A\text{-FML}/R)$ yields a functor

$$(R\text{-Alg}) \to (A\text{-Mod}), \quad S \mapsto \mathrm{Nil}(S), \tag{2.1}$$

where $\mathrm{Nil}(S)$, the set of nilpotent elements of S, is equipped with addition and scalars given by

$$s_1 + s_2 = F(s_1, s_2) \in \mathrm{Nil}(S), \quad as = [a]_F(s) \in \mathrm{Nil}(S).$$

This construction yields a functor (with slight abuse of notation)

$$(A\text{-FML}/R) \to \mathrm{Fun}((R\text{-Alg}), (A\text{-Mod})), \tag{2.2}$$

where Fun denotes the functor category.

Passing from discrete R-algebras to admissible R-algebras, this construction extends naturally to a functor

$$\mathrm{Spf}^F : (A\text{-FML}/R) \to \mathrm{Fun}((R\text{-Adm}), (A\text{-Mod})), \quad F \mapsto \mathrm{Spf}\, R[\![T]\!],$$

where we equip $\mathrm{Spf}\, R[\![T]\!]$ with the structure of an A-module object using the endomorphisms coming from F. Following this line of thought leads naturally to the definition of formal modules.

Definition 2.1.5 (Formal Group and Formal Module.)**.** Let X be an A-scheme, and let let $\mathcal{F}$ be an A-module object in (FSch/X), the category of formal schemes over X. Suppose that there is a Zariski-covering $(\mathrm{Spec}(R_i))_{i\in I}$ of X with $\mathcal{F} \times_X U_i \cong \mathrm{Spf}(R_i[\![T]\!])$. If for every $i \in I$ the induced A-module structure on $\mathrm{Spf}(R_i[\![T]\!])$ comes from a formal A-module law F_i over R_i, we say that $\mathcal{F}$ is a formal A-module.

Definition 2.1.6 (Coordinate)**.** Let $\mathcal{F}$ be a formal A-module over X. The choice of a cover $\sqcup_{i\in I} \mathrm{Spec}(R_i) \to X$ together with isomorphisms $\mathcal{F} \times_X \mathrm{Spec}(R_i) \cong \mathrm{Spf}(R_i[\![T]\!])$ will be referred to as a coordinate of $\mathcal{F}$.

Of course there is a functor

$$(A\text{-FML}/R) \to (A\text{-FM}/R),$$

essentially forgetting the choice of module law. The observation of Lemma 2.1.4 translates to formal modules, a morphism $p : R \to R'$ yields a functor

$$p_* : (A\text{-FM}/R) \to (A\text{-FM}/R'), \quad \mathcal{F} \mapsto \mathcal{F} \otimes_R R'.$$

Example. The additive group law $\widehat{\mathbb{G}}_a$ extends to a formal A-module over an affine base $\mathrm{Spec}\, R$ by setting

$$[a]_{\widehat{\mathbb{G}}_a}(T) = aT$$

for $a \in A$. More generally, we obtain a formal A-module over an arbitrary base scheme.

The formal group associated to $\widehat{\mathbb{G}}_m$ over $\mathbb{Z}_p$ is isomorphic to the functor

$$(\mathbb{Z}_p\text{-Adm}) \to (\mathrm{Ab}), \quad S \mapsto 1 + S^{\infty} \subset S^{\times}.$$

Here, we equipped $\mathbb{Z}_p$ with the p-adic topology. The subgroup $1 + S^{\infty}$ naturally carries the structure of a $\mathbb{Z}_p$-module. Indeed, for $k \in \mathbb{N}$, we have

$$(1+s)^{p^k} = 1 + p^k s + \binom{p^k}{2} s^2 + \cdots + s^{p^k},$$

and given $s \in S^{\infty}$, this is of the form $1 + o(1)$ as k gets large. In particular, if $x = a_0 + a_1 p + a_2 p^2 + \cdots \in \mathbb{Z}_p$, expressions of the form

$$(1+s)^x = \prod_{i=1}^{\infty} (1+s)^{a_k p^k}$$

make sense by lemma A.0.2. This gives $\widehat{\mathbb{G}}_{m,\mathbb{Z}_p}$ the structure of a formal $\mathbb{Z}_p$-module. In section 2.2, we will see that this is the simplest example of a whole family of formal modules constructed by Lubin and Tate, and discuss their applications to local class field theory.

2.1.1 Logarithms

Suppose that A is a discrete valuation ring with uniformizer π and that K is a field. We review results from **hopkins1994equivariant**. Suppose that F is a formal A-module law over an A-algebra R.

Definition 2.1.7 (Invariant Differentials for module laws.)**.** The module $\omega(F)$ of invariant differentials is the submodule of the module of differentials

$$\Omega_{R[\![T]\!]/R} \cong R[\![T]\!]\,\mathrm{d}T,$$

cut out by the condition that all $\omega \in \omega(F)$ satisfy

$$\omega(F(X,Y)) = \omega(X) + \omega(Y) \quad \text{and} \quad \omega([a]_F(X)) = a\omega(X). \tag{2.3}$$

for all $a \in A$.

It is possible to explicitely construct invariant differentials. Let $f(X,Y)$ denote $(\partial_x F)(X,Y)$, the derivative of $F(X,Y)$ with respect to X. Denote $g(Y) = f(0,Y)$. Then g is a unit in $R[\![Y]\!]$; and we construct $\omega_F(X) := \frac{1}{g(X)}\,\mathrm{d}X$. Checking that ω_F is indeed invariant is a matter of applying the chain rule.
All other invariant differentials are scalar multiples of ω_F.

Proposition 2.1.8. *1. The R-module $\omega(R)$ is free of rank* 1 *generated by ω_F*

2. There is a non-degenerate pairing $\omega(F) \times \mathrm{Lie}(F) \to R$.

Proof. Part one is **hopkins1994equivariant**, Proposition 2.2. ☐

Example. The invariant differentials for $\widehat{\mathbb{G}}_m$ are spanned by the form $\omega_1(X) = \frac{1}{1+X}\,\mathrm{d}X$.

The conditions imposed on invariant differentials remind of those imposed on morphisms of A-module laws $F \to \widehat{\mathbb{G}}_a$. And indeed, there is a map

$$\mathrm{d}_F : \mathrm{Hom}_{(A\text{-FML}/R)}(F, \widehat{\mathbb{G}}_{a,R}) \to \omega(F), \quad f \mapsto \mathrm{d}f(X) \tag{2.4}$$

One may check that $\mathrm{End}(\widehat{\mathbb{G}}_{a,R}) \supseteq R$, turning d in a map of R-modules.

Proposition 2.1.9. *1. If R is a flat A-algebra, the map d_F is injective.*

2. If R is a K-algebra, the map d_F is an isomorphism.

Proof. **hopkins1994equivariant**, Chapter 3 [Everything is easy if K has characteristic 0, as we can integrate the differential forms. The proof in positive characteristic is a bit tricky; First it is shown that there is an isomorphism of formal goups $F \cong \widehat{\mathbb{G}}_a$, which is immediate. Then that there is a unique homomorphism $f : \widehat{\mathbb{G}}_a \to \widehat{\mathbb{G}}_a$ that maps to ω_F and behaves well with respect to the A-module structure on F.] PROOF □

In particular, if R is a K-algebra, the invariant differential $\omega_F(X)$ constructed above comes from a homomorphism $f(X) = X + c_2X^2 + \dots$, which is an isomorphism by lemma 2.1.2. This allows us to define the logarithm attached to F.

Definition 2.1.10 (Logarithm)**.** If R is a flat A-algebra, there is a unique power series

$$\log_F(X) = X + c_2X^2 + \dots \in (R \otimes_A K)[\![X]\!]$$

inducing an isomorphism $F \otimes (R \otimes K) \to \widehat{\mathbb{G}}_{a,R\otimes K}$. This power series is called the logarithm attached to F.

2.1.2 Formal DVR-Modules over Fields

As above, let A be a discrete valuation ring with uniformizer π and finite residue field k; write q for the cardinality of k. Let K denote the field of fractions of A.

We introduce the concept of height, which is an integer attached to morphisms of formal group laws over fields. The height of a formal A-module $\mathcal{F}$ over R will be defined as the height of it's endomorphism $[\pi]_{\mathcal{F}}$.

We have seen in the previous section that if R is a field extension of K, then any morphism of formal group laws $f : F \to G$ over R is either 0, in which case we say it has height ∞, or an isomorphism, in which case we say it has height 0. The height becomes interesting in positive characteristic.

We define the height over field extensions of the residue field.

Definition 2.1.11 (Height of morphisms of group laws)**.** Assume that R is a field extension of k and $f : F \to G$ is a morphism of formal groups laws over R, given by a formal series $f(T) \in R[\![T]\!]$. If $f = 0$, we say that f has infinite height. If $f \neq 0$, the height of f is defined as the largest integer h such that $f = g(T^{q^h})$ for some power series $g(T) = c_1T + c_2T^2 + \dots \in R[\![T]\!]$ with $c_1 \neq 0$.

One readily checks that if $f : \mathcal{F} \to \mathcal{G}$ is a morphism of formal groups over a field extension R of k, the height of f does not depend on the choices of group laws on $\mathcal{F}$ and $\mathcal{G}$. This allows us to define the height function attached to f.

Definition 2.1.12 (Height function)**.** Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of formal groups over a scheme X. For a (scheme-theoretic) point $x \in |X|$, let f_x denote the base-change of f to the residue field of x. The height function attached to f is the upper-semicontinuous function

$$\mathrm{ht}(f) : |X| \to \mathbb{Z}_{\geq 0} \cup \{\infty\}, \quad x \mapsto \mathrm{ht}(f_x). \tag{2.5}$$

It is not hard to see that the height function is additive, that is, we have

$$\mathrm{ht}(f \circ g) = \mathrm{ht}(f) + \mathrm{ht}(g).$$

Definition 2.1.13 (Isogeny)**.** A morphism $f : \mathcal{F} \to \mathcal{G}$ of formal groups over a field k is called an isogeny if $\mathrm{Ker}(f)$ is a represented by a finite free k-scheme. More generally, a morphism of formal A-modules over a base scheme X is an isogeny if and only if $\mathrm{Ker}(f)$ is finite and locally free over X.

Isogenies can be described using the height function.

Lemma 2.1.14. *A morphism $f : \mathcal{F} \to \mathcal{G}$ is a isogeny if and only if the height function* $\mathrm{ht}(f)$ *is locally constant with values in $\mathbb{Z}_{\geq 0}$.*

Definition 2.1.15 (π-divisible A-module)**.** We say that a formal A-module H over k is π-divisible if $[\pi]_H$ is an isogeny.

The following lemma allows us to invert quasi-isogenies.

Lemma 2.1.16. *Let $f : \mathcal{F} \to \mathcal{G}$ be an isogeny of π-divisible formal A-modules over a quasi-compact* [quasi-separated?]*A-scheme X. Then there is an integer $n \geq 0$ and an isogeny $g : \mathcal{G} \to \mathcal{F}$ with*

$$f \circ g = [\pi^n]_{\mathcal{G}} \quad \textit{and} \quad g \circ f = [\pi^n]_{\mathcal{F}}.$$

2.1.3 Deformations of Formal Modules and the Standard Formal Module

- Introduction to Hazewinkel's theory of A-typical formal modules
- The standard $\mathcal{O}_K$-module of height n: The unique formal $\mathcal{O}_K$-module law H over $\mathcal{O}_K$ with logaritm

$$\log_H(X) = \sum_{i=0}^{\infty} \frac{T^{q^{in}}}{\pi^i}.$$

2.1.4 The Dieudonné functor

- Definition using quasi-logarithms
- Definition with rigidified extensions as in **hopkins1994equivariant** (?)

2.1.5 Tate Modules and the Universal Cover

Let A be a ring and R be an A-algebra. Given $H \in (A\text{-FM}/R)$ and $a \in A$, we define the functor

$$\tilde{H}_a : (R\text{-Adm}) \to (A[\tfrac{1}{a}]\text{-Mod}), \quad S \mapsto \left\{ (x_1, x_2, \dots) \in \prod_{\mathbb{N}} H(S) \mid [a]_H(x_{i+1}) = x_i \right\}.$$

From now on assume that A is a discrete valuation ring with uniformizer π, finite residue field k and field of fractions K.

Definition 2.1.17 (The Universal Cover and Tate Module)**.** We omit π from notation and write $\tilde{H} = \tilde{H}_\pi$. This functor takes values in the category of K-vector spaces. Up to isomorphism, $\tilde{H}$ is uniquely determined. We call this functor the universal cover of H.
The Tate-Module $T_\pi H$ is the subfunctor of $\tilde{H}$ cut out out by the condition that $[\pi]_H(x_1) = 0$. Note that $T_\pi H$ does no longer carry the structure of a K-vector space, it is an A-module. The Rational Tate Module $V_\pi H$ is the subfunctor of $\tilde{H}$ cut out by the condition that x_1 has $[\pi]_H$-torsion. Equivalently, we have

$$V_\pi H(S) = T_\pi H(S) \otimes_A K.$$

Let R be an adic A-algebra such that A/I is a perfect field. Suppose that H is a formal A-module over R. Write H_0 for the reduction of H mod I.

- Results from **BoyarchenkoWeinstein2011MaxVar**, section 2.5

2.1.6 The Quasilogarithm Map

- See **BoyarchenkoWeinstein2011MaxVar**, section 2.6.

2.1.7 Determinants of Formal Modules

- "Functorial" description of the determinant. Either as in **BoyarchenkoWeinstein2011MaxVar**, or as in **weinstein2016semistable**.
- Construction.
- Approximations.

2.2 Application: Local Class Field Theory

Let K be a local field with residue field k, put $q = \#k$, and denote by $\nu_K : K \to \mathbb{Z} \cup \{\infty\}$ the valuation of K, normalized such that $\nu_K(\pi) = 1$ for a uniformizer π of K. The aim of this subsection is to describe the maximal abelian extension of a local field K.
The Local Kronecker-Weber theorem gives an explicit description of the abelianization of the absolute Galois group of K only in terms of K:

Theorem 2.2.1 (Local Kronecker-Weber)**.** *There is an isomorphism (canonical up to choice of a uniformizer $\pi \in K$)*

$$\mathrm{Gal}(\overline{K}/K)^{\mathrm{ab}} \cong \mathrm{Gal}(K^{\mathrm{ab}}/K) \cong \mathcal{O}_K^\times \times \widehat{\mathbb{Z}}.$$

Here, K^{ab} denote the maximal abelian extension of K, which can (after choosing an algebraic closure of K) be described as $\overline{K}^{[G_K, G_K]}$.

The extension K^{ab} consists of two parts, we have $K^{\mathrm{ab}} = K^{\mathrm{rm}} \cdot K^{\mathrm{nr}}$. The field K^{nr}, the maximal unramified extension of K, has relatively simple structure. Describing the field K^{rm} (or rather, it's completion) is the hard part and it is here where we apply the theory of formal modules.

The valuation ν_K extends uniquely to $\overline{K}$, yielding a π-adic norm on $\overline{K}$. Let C denote the completion with respect to this norm. An application of Krasner's Lemma implies that $\mathrm{Gal}(C/K) \cong \mathrm{Gal}(\overline{K}/K) =: G_K$. One readily checks that any $\sigma \in G_K$ yields a continuous Ref
automorphism $\mathcal{O}_C \to \mathcal{O}_C$, and we obtain a short exact sequence

$$0 \to I_K \to G_K \to \mathrm{Gal}(\overline{k}/k) \to 0.$$

The subgroup $I_K \subset G_K$ is called the inertia subgroup of K, and we write $\breve{K}$ for the subfield of C fixed by I_K. In particular we have $\mathrm{Gal}(\breve{K}/K) \cong \mathrm{Gal}(\overline{k}/k)$. One readily confirms that $\breve{K}$ is complete with respect to the norm induced by K.

As the Galois group of any finite extension of k is cyclic, we find that $\mathrm{Gal}(\breve{K}/K)$ is abelian. In fact, it is isomorphic to $\widehat{\mathbb{Z}} = \lim_n(\mathbb{Z}/n\mathbb{Z})$. Hence K_∞ decomposes as $\breve{K} \cdot K_\pi$ for some abelian, complete extension K_π/K such that $K_\pi \cap \breve{K} = K$. Now K_π is the completion of K^{rm}. Observe that

$$\mathrm{Gal}(K_\infty/K) \cong \mathrm{Gal}(K_\pi/K) \times \mathrm{Gal}(\breve{K}/K) \cong \mathrm{Gal}(K_\pi/K) \times \widehat{\mathbb{Z}},$$

so Theorem 2.2.1, the local Kronecker-Weber Theorem, is equivalent to showing that the Galois group of K_π over K is isomorphic to $\mathcal{O}_K^\times$.

3 Non-Abelian Lubin-Tate Theory: An Overview

In the preceeding chapter we used formal $\mathcal{O}_K$-modules to understand the maximial abelian extension of a local field K. The hope of non-Abelian Lubin-Tate theory is to gain insight about the Abelian extensions of K by considering certain moduli spaces of formal $\mathcal{O}_K$-modules. More precisely, attached to a formal $\mathcal{O}_K$-module H_0 over $\overline{\mathbb{F}}_q$ (determined up to isomorphism by its height n), we attach a system of rigid spaces $\{M_K\}_{K \subset \mathrm{GL}_n(\mathcal{O}_K)}$, the so called Lubin-Tate Tower. For $l \neq p$, the system of l-adic compactly supported cohomology groups $\{H^i_c(M_K, \overline{\mathbb{Q}}_l)\}_K$ admits commuting actions by $\mathrm{GL}_n(K)$, W_K and $D^\times$, where the latter denotes the units of the central divison algebra $D = \mathrm{End}_{(\mathcal{O}_K\text{-FM}/\overline{\mathbb{F}}_q)}(H_0) \otimes \mathbb{Q}$. This yields a correspondence of representations of the respective groups, and Harris and Taylor showed in **HTShimura** that the cohomology of middle degree induces (a version of) the Local Langlands Correspondence for GL_n. Our goal is an explicit description of this correspondence, and we obtain such descriptions by understanding the Lubin-Tate tower explicitely. As it turns out, the the limit $\lim_K M_K$ is representable by a perfectoid space which is is easier to describe than its individual layers.

3.1 The Lubin-Tate Tower

3.1.1 Deformations of Formal Modules

We mostly follow **Strauch2008DefSp**, Chapter 2 for notation. Let $\mathcal{C}$ denote the category of local, Noetherian $\mathcal{O}_{\breve{K}}$-modules with distinguished isomorphisms $R/\mathfrak{m}_R \to \overline{\mathbb{F}}_q$. Let H_0 be a formal $\mathcal{O}_K$-module over $\overline{\mathbb{F}}_q$.

Definition 3.1.1 (Deformation)**.** Let $R \in \mathcal{C}$. A deformation of H_0 to R is a pair (H, ι) where H is a formal $\mathcal{O}_K$-module over R and ι is a quasi-isogeny

$$\iota : H_0 \dashrightarrow H \otimes_R \overline{\mathbb{F}}_q.$$

Two deformations (H, ι) and (H', ι') are isomorphic if there is an isomorphism $\tau : H \to H'$ with $\iota' \circ \tau = \iota$.

The Lubin-Tate space without level structure is the moduli space of such deformations. More precisely, we define it as the functor

$$\mathcal{M}_0 : \mathcal{C} \to (\text{Set}), \quad R \mapsto \{\text{deformations } (H, \iota) \text{ of } H_0\}/\text{iso}.$$

Theorem 3.1.2 (Representability of $\mathcal{M}_0$)**.** *The functor $\mathcal{M}_0$ is (non-canonically) representable, by the noetherian local ring*

$$A_0 \cong \mathcal{O}_{\breve{K}}[\![u_1, \dots, u_{n-1}]\!].$$

In particular, there is a universal deformation $(F^{\text{univ}}, \iota^{\text{univ}})$, with $F^{\text{univ}} \in (\mathcal{O}_{\breve{K}}\text{-FM}/A_0)$.

3.1.2 Deformations of Formal Modules with Drinfeld Level Structure

Definition 3.1.3 (Drinfeld level $\mathfrak{p}^m$-structure)**.** Let $R \in \mathcal{C}$ and $H \in (\mathcal{O}_K\text{-FM}/R)$. A Drinfeld level $\mathfrak{p}^m$-structure on H is a morphism of R-group schemes

$$(\mathfrak{p}^{-m}/\mathcal{O}_K)^{\oplus n} \to H(R)[\pi^m]$$

such that after choosing a coordinate $H \cong \operatorname{Spf} R[\![T]\!]$, the power series $[\pi]_H(T) \in R[\![T]\!]$ satisfies the divisibility constraint

$$\prod_{x \in (\mathfrak{p}^{-1}/\mathcal{O}_K)} (T - \phi(x)) \mid [\pi]_H(T).$$

The following examples might shed some light on this definition.

Example.

- $\widehat{\mathbb{G}}_m$

- Things over $\mathbb{F}_q$.
- Drinfeld Level
- Moduli Problem + Representability
- The Lubin-Tate Tower

3.1.3 The Group actions on the Tower and its Cohomology

- Action By $D^\times$ and GL_n
- Action by W_K via Weil descent Datum.

3.2 The Local Langlands Correspondence for the General Linear Group

3.3 The Lubin-Tate Perfectoid Space

4 Mieda's Approach to the Explicit Local Langlands Correspondence

5 The Explicit Local Langlands Correspondence for Depth Zero Supercuspidal Representations

5.1 The Special Affinoid

5.2 Deligne-Lusztig Theory for Depth Zero Representations

5.3 Proof

A Topological Rings

To deal with the topological rings showing up, the notion of admissible rings will be convenient (taken from **stacks-project**, Tag 07E8).

Definition A.0.1. Let A be a topological ring. We say that A is admissible if

- The element $0 \in A$ has a fundamental system of neighbourhoods consisting of ideals.
- There exists an ideal of definition, that is, an ideal $I \subset A$ such that every open neighbourhood of 0 contains I^n for some n.

- It is complete, that is, the natural map

$$A \to \lim_{J \subset A \text{ open ideal}} A/J$$

is an isomorphism.

We say that A is adic if it admits an open ideal of definition. Given a topological ring A, we denote the category of admissible and adic A-algebras (algebras S with continuous morphism $A \to S$) by (A-Adm) and (A-Adic), respectively.

Lemma A.0.2. *Let S be an admissible ring, and let $(s_1, s_2, \dots)$ be a sequence with elements in S. Then $\sum_{i=1}^{\infty} s_i$ converges if and only if $\prod_{i=1}^{\infty}(1+s_i)$ converges if and only if $\lim_{i\to\infty} s_i = 0$.*

Proof. If sum and product converge, $(s_i)_{i\in\mathbb{N}}$ has to converge to zero. The reverse implication follows after writing $S \cong \lim_J S/J$ for a system of open ideals $J \subset S$. $\square$

References

Bochner1946FGrps S. Bochner. "Formal Lie Groups". In: *Annals of Mathematics* 47.2 (1946), pp. 192–201. ISSN: 0003486X. URL: `http://www.jstor.org/stable/1969242` (visited on 03/30/2024).

BoyarchenkoWeinstein2011MaxVar Mitya Boyarchenko and Jared Weinstein. "Maximal varieties and the local Langlands correspondence for GL(n)". In: *Journal of the American Mathematical Society* 29 (Sept. 2011). DOI: `10.1090/jams826`.

hopkins1994equivariant Michael J Hopkins and Benedict H Gross. "Equivariant vector bundles on the Lubin-Tate moduli space". In: *Contemporary Mathematics* 158 (1994), pp. 23–23.

HTShimura Michael Harris and Richard Taylor. *The Geometry and Cohomology of Some Simple Shimura Varieties.(AM-151), Volume 151.* Vol. 151. Princeton university press, 2001.

Lazard1955FGrps Michel Lazard. "Sur les groupes de Lie formels à un paramètre". fr. In: *Bulletin de la Société Mathématique de France* 83 (1955), pp. 251–274. DOI: `10.24033/bsmf.1462`. URL: `http://www.numdam.org/articles/10.24033/bsmf.1462/`.

stacks-project The Stacks Project Authors. *Stacks Project.* `https://stacks.math.columbia.edu`. 2018.

Strauch2008DefSp Matthias Strauch. "Deformation spaces of one-dimensional formal modules and their cohomology". In: *Advances in Mathematics* 217.3 (2008), pp. 889–951. ISSN: 0001-8708. DOI: `https://doi.org/10.1016/j.aim.2007.07.005`. URL: `https://www.sciencedirect.com/science/article/pii/S0001870807002149`.

weinstein2016semistable Jared Weinstein. "Semistable models for modular curves of arbitrary level". In: *Inventiones mathematicae* 205 (2016), pp. 459–526.