1 Quotient Stacks

We want to find out how stacks arise when taking quotients by group actions. To start things off, let's recall what a group action of presheafs is.

Definition 1.1 (Group action on a presheaf.)**.** Let $\mathcal{C}$ be a (locally small) category, let $X : \mathcal{C}^{\mathrm{op}} \to (\mathrm{Sets})$ be a presheaf of sets and let $G : \mathcal{C}^{\mathrm{op}} \to (\mathrm{Grps})$ be a presheaf of groups. A *left action* of G on X is a morphism $\sigma : G \times X \to X$ which behaves like a left group action on points. In this case, X is sometimes called a *presheaf of G-sets.*

Definition 1.2 (Quotient presheaf)**.** Given a presheaf of groups G and a sheaf of G-sets X, we can build the quotient presheaf (X/G), via the functor

$$S \mapsto X(S)/G(S).$$

Instead of quotient sets, we will look at quotient groupoids.

Definition 1.3 (Quotient Groupoid)**.** Let X be a set with a group action $G \curvearrowright X$. Then we define the *quotient groupoid* $[X/G]$ given by the category with objects $x \in X$ and $\mathrm{Hom}(x, x') = \{g \in G \mid gx = x'\}$.

One easily verifies (Exercise 0.4.7) that an action $G \curvearrowright X$ is free (i.e., has trivial stabilizers) iff $[X/G]$ is a set.
A Quotient prestack is a scheme-theoretic/relative version of the quotient groupoid. It behaves to quotient groupoids the same way quotient presheaves behave to quotient sets.
Also, let's switch to scheme world. From now on, $\mathcal{C} = (\mathrm{Sch}/S)$.

Definition 1.4 (Quotient prestack)**.** Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S. We define the *quotient prestack* $[U/G]^{\mathrm{pre}}$ as the category over (Sch/S) where the fiber category over an S-scheme T is the quotient groupoid $[U(T)/G(T)]$. In other words, the objects of $[U/G]$ are given by morphisms $T \to U$ over S. A morphism from $u' \in U(T')$ to $u \in U(T)$ is the data of a map $f : T' \to T$ and an element $g \in G(T')$ such that $u' = g \cdot (u \circ f)$. Composition of maps is given by $(g, f) \circ (g', f') = (g'g, f \circ f')$.

Exercise 2.3.15 a). $[U/G]^{\mathrm{pre}}$ is a prestack. There is not much to check. It is a category fibered over groupoids by definition, which is already a good sign.

1. *Pullbacks exist.* Given an object $(a \mapsto W) \in [U/G]^{\mathrm{pre}}$ (which is really just a morphism $W \to U$) and a morphism $f : W' \to W$, we simply set $f^*a = (a \circ f)$.

2. *Pullbacks are unique up to unique isomorphism.* This is a bit more involved, but the following diagram essentially gives the proof.

$$\begin{tikzcd}
b \arrow[r, dashed, "(g_2 g_1^{-1}, f')"'] \arrow[rr, bend left, "(g_2, f \circ f')"] \arrow[d, mapsto] & a' \arrow[r, "(g_1, f)"'] \arrow[d, mapsto] & a \arrow[d, mapsto] \\
V \arrow[r, "f'"] & W' \arrow[r, "f"] & W
\end{tikzcd}$$

Here, we start with an arrow $a' \to a$, which comes with an element $g_1 \in G(W)$ such that $a' = g \cdot (a \circ f)$. Similarly, an arrow $b \to a$ comes with an element $g_2 \in G(W)$ such that $b = g_2 \cdot (a \circ f \circ f')$. But now $b \to a$ factors through $a' \to a$, as

$$b = g_2(a \circ f \circ f') = g_2 g_1^{-1} \cdot g_1 \cdot (a \circ f) \circ f' = g_2 g_1^{-1} \cdot (a' \circ f'),$$

and this factorization is unique, as $(hg_1 = g_2 \iff h = g_2 g_1^{-1})$.

Note that if G acts freely on U (i.e., the action map $(\sigma, p_2) : G \times_S U \to U \times_S U$ is a monomorphism) if and only if $[U/G]^{\text{pre}}$ is equivalent to a presheaf. Indeed, in this case, the action maps are monomorphisms on points. Hence we have $[U(T)/G(T)] \cong (U(T)/G(T))$ for all T, as there are no non-trivial stabilizers. This is **Exercise 2.3.19.**.

Instead of thinking of an object of $[U/G]^{\text{pre}}$ over a scheme T as a morphism $f : T \to U$, we could also think of it as a trivial G-bundle over T:

$$\begin{tikzcd} G \times T \arrow[r, "\tilde{f}"] \arrow[d, "p_2"] & U \\ T, & \end{tikzcd}$$

where $\tilde{f}$ sends (g, s) to $g \cdot f(s)$. Hence, $[U/G]^{\text{pre}}$ parametrizes the trivial G-bundles. In this setting, it is clear that the quotient prestack is not a stack: If we are given a cover $\{T_i \to T\}$ of T together with trivial G-bundles $G \times T_i \xrightarrow{\tilde{f}_i} X$ satisfying the cocycle conditions, the object we obtain from gluing these local data does not have to glue to a trivial G-bundle again. Rather, this naturally leads to the definition of *principal G-bundles.*

Definition 1.5 (Principal G-Bundle / Torsor)**.** Let $G \to S$ be an fppf affine group scheme. A *principal G-bundle* over an S-scheme T is a scheme P with an action of G, given by a morphism $\sigma : G \times_S P \to P$, such that $P \to T$ is a G-invariant[1] fppf morphism and

$$(\sigma, p_2) : G \times_S P \to P \times_T P, \quad (g, p) \mapsto (gp, p)$$

is an isomorphism. (As isomorphisms can be checked fpqc-locally on T, this just says that the action $G \curvearrowright U$ is faithfully free fpqc-locally. This means that locally, P is a trivial G-bundle.). If we equip (Sch/S) with some Grothendieck-topology (or more generally, if everything lives on a site), we say that P is a *G-Torsor* if for every object $T \in (\mathrm{Sch}/S)$ there is a covering $\{T_i \to T\}$ such that $P(T_i) \neq \emptyset$.

This motivates the definition of the quotient stack.

Definition 1.6 (Quotient stack)**.** We define the *Quotient Stack* $[U/G]$ as the prestack over (Sch/S) whose obects over an S scheme T are diagrams

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U \\ T & \end{tikzcd}$$

where $P \to T$ is a principal G-bundle and $P \to U$ is a G-equivariant morphism of schemes. A morphism $(P' \to T', P' \to U) \to (P \to T, P \to U)$ consists of a morphism $T' \to T$ and a G-equivariant morphism $P' \to P$ of schemes such that the diagram

$$\begin{tikzcd} P' \arrow[r] \arrow[d] \arrow[rr, bend left] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[r] \arrow[d] & U \\ T' \arrow[r] & T & \end{tikzcd}$$

is commutative and the left square is cartesian.

[1]Remember that a morphism $P \to T$ is called G-invariant if the diagram

$$\begin{tikzcd} G \times_S P \arrow[r, "\sigma"] \arrow[d, "p_2"] & P \arrow[d] \\ P \arrow[r] & T \end{tikzcd}$$

commutes.

Exercise 2.3.15 b). $[U/G]$ is a prestack. This is straight forward. Pullbacks are given by pullbacks, and one quickly verifies that indeed, if P is a principal G-bundle over T, $P \times_T T'$ is a principal G-bundle over T'. The fact that pullbacks are universal just comes down to the fact that in any (locally small) category, if we are given a square

$$\begin{tikzcd}
P'' \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & P \arrow[d] \\
T'' \arrow[r] & T' \arrow[r] & T
\end{tikzcd}$$

in which the right square is cartesian, the left square is cartesian iff the composition square is.

Interesting exercises

((I want to solve these at some point))
Exercise 2.3.22.

(a) Show that there is a morphism $p : U \to [U/G]^{\mathrm{pre}}$ and a 2-commutative diagram

$$\begin{tikzcd}
G \times_S U \arrow[r, "\sigma"] \arrow[d, "p_2"'] & U \arrow[d, "p"] \\
U \arrow[r, "p"'] \arrow[ur, phantom, "\overset{\alpha}{\swarrow}"] & {[U/G]^{\mathrm{pre}}}
\end{tikzcd}$$

(b) The morphism p is a categorical quotient among prestacks, i.e. for every 2-commutative diagram

$$\begin{tikzcd}
G \times_S U \arrow[r, "\sigma"] \arrow[d, "p_2"'] & U \arrow[d, "p"] \arrow[ddr, bend left, "\varphi"] & \\
U \arrow[r, "p"'] \arrow[ur, phantom, "\overset{\alpha}{\swarrow}"] \arrow[drr, bend right, "\varphi"'] & {[U/G]^{\mathrm{pre}}} \arrow[dr, phantom, "\overset{\tau}{\swarrow}"] & \\
& & \mathcal{Z}
\end{tikzcd}$$

of prestacks, there exists a morphism $\chi : [U/G]^{\mathrm{pre}} \to \mathcal{Z}$ and 2-isomorphisms compatible with α and τ such that everything commutes.

Exercise 2.3.27. a) The square in part a) of the previous exercise is cartesian (this should correspond to the fact that the group actions are simply transitive).
Exercise 2.4.16. Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S.

(a) Show that $[U/G]^{\mathrm{pre}}$ satisfies Axiom (1) of a stack over $(\mathrm{Sch}/S)_{et}$.

(b) Show that $[U/G]$ is isomorphic to the stackification of $[U/G]^{\mathrm{pre}}$ over $((\mathrm{Sch}/S))_{et}$, and that $[U/G]^{\mathrm{pre}} \to [U/G]$ is fully faithful.

Exercise 2.4.17. Show that $U \to [U/G]$ is a categorical quotient among stacks.

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