MATH 6397 Homework 1

Mark Jayson Cortez

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1. Let $X_1, X_2, ..., X_j, ...$ be independent exponential random variables with X_j having the parameter λ_{j-1}. Let $T_\infty := \lim_{n\to\infty} T_n$, where T_n is the time of the n^{th} arrival. Let X_n be such that $X_n = T_n - T_{n-1}$.

   Given $P_n(t)$, the probability of being at state n at time t, we have that $1 - \sum_{n=0}^{\infty} P_n(t)$ is the probability $P(T_\infty \le t)$ of an explosion to infinity at time t. We therefore see the equivalence, for any given time t

   $$\sum_{n=0}^{\infty} P_n(t) = 1 \Leftrightarrow P\left(T_\infty > t\right) = 1 \Leftrightarrow P\left(T_\infty = \infty\right) = 1$$

   We instead prove the equivalence $P\left(T_\infty = \infty\right) = 1 \Leftrightarrow \sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \infty$ in place of the original statement.

   ($\Longrightarrow$) We go by contraposition.

   Suppose $\sum_{n=0}^{\infty} \frac{1}{\lambda_n} < \infty$. Then

   $$E\left[T_\infty\right] = E\left[\sum_{n=1}^{\infty} X_n\right] = \sum_{n=1}^{\infty} E\left[X_n\right] = \sum_{n=1}^{\infty} \frac{1}{\lambda_{n-1}} = \sum_{n=0}^{\infty} \frac{1}{\lambda_n} < \infty.$$

   So $P\left(T_\infty = \infty\right) = 0$.

   ($\Longleftarrow$) Suppose $\sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \infty$. We work with the random variable $e^{-T_\infty} := \lim_{n\to\infty} e^{-T_n}$. We then have

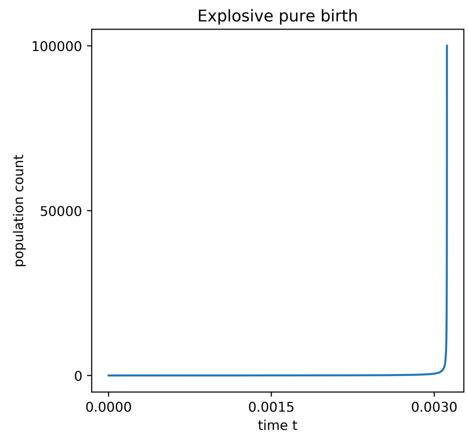
   $$\begin{aligned} E\left[e^{-T_\infty}\right] &= E\left[e^{-\sum_{n=1}^{\infty} X_n}\right] = E\left[\lim_{M\to\infty} \prod_{n=1}^{M} e^{-X_n}\right] \overset{MCT}{=} \lim_{M\to\infty} E\left[\prod_{n=1}^{M} e^{-X_n}\right] \overset{independence}{=} \lim_{M\to\infty} \prod_{n=1}^{M} E\left[e^{-X_n}\right] \\ &= \lim_{M\to\infty} \prod_{n=1}^{M} \frac{1}{1+\lambda_{n-1}^{-1}} = \left[\prod_{n=1}^{\infty}\left(1+\frac{1}{\lambda_{n-1}}\right)\right]^{-1} \end{aligned}$$

   By assumption, $\infty = \sum_{n=0}^{\infty} \frac{1}{\lambda_n} \le \prod_{n=0}^{\infty}\left(1+\frac{1}{\lambda_n}\right)$. Putting things together, we have

   $$E\left[e^{-T_\infty}\right] = \left[\prod_{n=1}^{\infty}\left(1+\frac{1}{\lambda_{n-1}}\right)\right]^{-1} = 0.$$

Therefore, $P\left(T_\infty = \infty\right) = P\left(e^{-T_\infty} = 0\right) = 1$, as needed.

Next, we demonstrate an explosive pure birth process. Here consider $\lambda_n = A(1+n^2)$. From the theorem above, since $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \sum_{n=1}^{\infty} \frac{1}{1+n^2}$ is convergent, the process is sure to explode in a finite time. See simulation below, with $A = 15$, where the population count reached 10^5 in 0.003 time units.



2. Consider a simple birth-death process, respectively with rates λ_n and μ_n, with constant immigration rate ν. The deterministic model for this process is given by $\dfrac{dn}{dt} = (\lambda - \mu)n + \nu$, and has solution

$$n(t) = \begin{cases} \frac{\exp((\lambda-\mu)t)((\lambda-\mu)n_0+\nu)-\nu}{\lambda-\mu} & \text{if } \lambda \neq \mu \\ \nu t + n_0 & \text{if } \lambda = \mu \end{cases}$$

when given $n(0) = n_0$. We wish to show that the mean of the stochastic model, which we will denote by $E[n(t)]$, is precisely this deterministic solution. For this we need the forward Kolmogorov equation

$$P_n{}'(t) = (\lambda(n-1)+\nu)P_{n-1}(t) + \mu(n+1)P_{n+1}(t) - ((\lambda+\mu)n+\nu)P_n(t).$$

First, consider the probability generating function for the stochastic model $f(s,t) = \sum_{n=0}^{\infty} P_n(t)s^n$, where $P_n(t)$ is the probability of being in state n at time t. Note that

- $f(1,t) = \sum_{n=0}^{\infty} P_n(t) = 1$
- $\dfrac{\partial f}{\partial t} = \sum_{n=0}^{\infty} P_n{}'(t)s^n$
- $\dfrac{\partial f}{\partial s} = \sum_{n=0}^{\infty} nP_n(t)s^{n-1}$, and $\left.\dfrac{\partial f}{\partial t}\right|_{s=1} = E[n(t)]$
- $\left.\dfrac{\partial}{\partial s}\dfrac{\partial f}{\partial t}\right|_{s=1} = \left.\dfrac{\partial}{\partial s}\sum_{n=0}^{\infty} P_n{}'(t)s^n\right|_{s=1} = \left.\sum_{n=0}^{\infty} nP_n{}'(t)s^{n-1}\right|_{s=1} = \sum_{n=0}^{\infty} nP_n{}'(t) = \dfrac{d}{dt}E[n(t)]$

Now,

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= \sum_{n=0}^{\infty} P_n{}'(t)\, s^n \\
&= \lambda \sum_{n=0}^{\infty} (n-1) P_{n-1}(t)\, s^n + \nu \sum_{n=0}^{\infty} P_{n-1}(t)\, s^n + \mu \sum_{n=0}^{\infty} (n+1) P_{n+1}(t)\, s^n - (\lambda+\mu) \sum_{n=0}^{\infty} n P_n(t)\, s^n - \nu \sum_{n=0}^{\infty} P_n(t)\, s^n \\
&= \lambda s^2 \frac{\partial f}{\partial s} + \nu s f(s,t) + \mu \frac{\partial f}{\partial s} - (\lambda+\mu)\, s \frac{\partial f}{\partial s} - \nu f(s,t) \\
&= \left[\lambda s^2 + \mu - (\lambda+\mu)\, s\right] \frac{\partial f}{\partial s} - v f(s,t)
\end{aligned}
$$

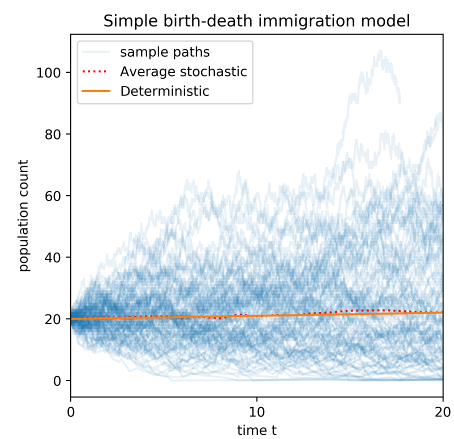
Next, we have

$$
\frac{\partial}{\partial s}\frac{\partial f}{\partial t} = \frac{\partial f}{\partial s}\left[2\lambda s - (\lambda+\mu)\right] + \left[\lambda s^2 + \mu - (\lambda+\mu)\, s\right] \frac{\partial^2 f}{\partial s^2} + \nu \left[(s-1)\frac{\partial f}{\partial s} + f(s,t)\right]
$$

and finally,

$$
\frac{d}{dt} E\left[n(t)\right] = \left.\frac{\partial}{\partial s}\frac{\partial f}{\partial t}\right|_{s=1} = (\lambda-\mu)\, E\left[n(t)\right] + \nu
$$

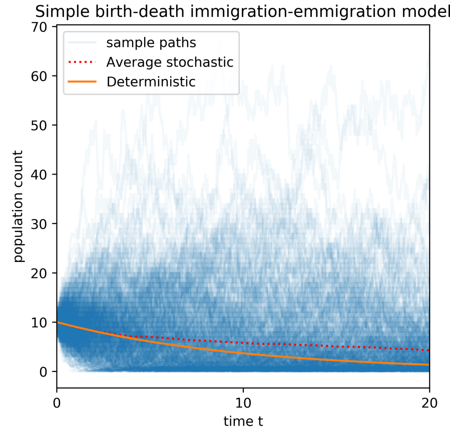
Since this coincides with the deterministic model, $E[n(t)]$ is precisely the solution of the deterministic model presented above. A simulation of 100 trajectories with $\lambda = \mu = 0.5$ and $\nu = 1$ below demostrates this.



Now to show that this isn't necessarily the case for a more general birth-death-immigration model, consider the a population model with both immigration and emmigration having growth, death, immigration, and emmigration parameters, respectively

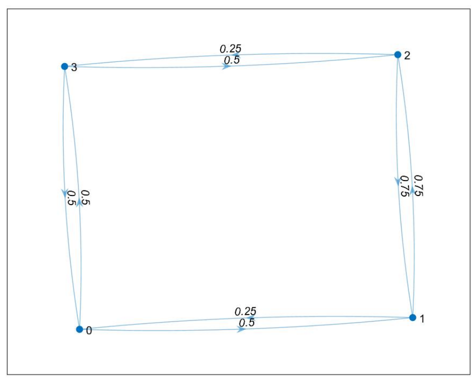
$$
\begin{aligned}
\lambda_n &= \lambda n \\
\mu_n &= \mu n \\
\nu_n &= \nu \\
w_n &= w.
\end{aligned}
$$

for positive constants λ, μ, ν, and w. Simulations with $\lambda = 0.9$, $\mu = 1$, $\nu = w = 1$ are shown below. Here 500 trajectories were simulated and averaged on the time interval $[0, 20]$.



3. Consider the Markov chain transition matrix $P = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 3/4 & 0 & 1/4 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}$.

   (a) The digraph of the chain is



   (b) A quick examination of $P = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 3/4 & 0 & 1/4 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}$ and $P^2 = \begin{bmatrix} 0.375 & 0 & 0.625 & 0 \\ 0 & 0.6875 & 0 & 0.3125 \\ 0.3125 & 0 & 0.6875 & 0 \\ 0 & 0.6250 & 0 & 0.375 \end{bmatrix}$ shows that for $i, j = 0, 1, 2, 3$, we have $p_{i,j}^{(m)} > 0$ for either $m = 1$ or 2. Hence the chain is irreducible.

   Since the chain is irreducible and has a finite state space, then it is positive recurrent. Also, it can be verified that for n even, $p_{0,0}^{(n)} \to 1/3$, $p_{1,1}^{(n)} \to 2/3$, $p_{2,2}^{(n)} \to 2/3$, and $p_{3,3}^{(n)} \to 1/3$. And so, for $i = 0, 1, 2, 3$, $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$. The positivity can be verified from mean return times, m_j, having values $m_0 = 6$, $m_1 = 3$, $m_2 = 3$, and $m_3 = 6$, which were obtained through the unique stationary distribution computed in the next item.

   Every state has period 2 since every move from any state is reversible. Also using the initial computations about P and P^2 above, we see that for each i, $p_{ii}^{(n)} = 0$ for n odd and $p_{ii}^{(n)} > 0$ for n even, which supports the claim of periodicity with period 2.

(c) To find the stationary distribution, we solve the system $\pi P = \pi$, or equivalently find the left eigevector of P associated with the eigenvalue 1, provided $\sum_{n=0}^{3} \pi_i = 1$. It is easy to verify that $\pi = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$ satisfies this, and is therefore the stationary distribution.

4. For the given Poisson process, consider the following random variables. Let X be the time before a protein arrives, and Y be the time before a promoter becomes unbound. We have $X \sim Exp(\lambda)$ and $Y \sim Exp\left(\frac{1}{\mu}\right)$. In the long run, the fraction of time that the promoter is unoccupied is the probability $P(X > Y)$.

   Since X and Y are independent, their joint density is given by $f_{X,Y}(x,y) = \lambda \exp(-\lambda x) \frac{1}{\mu} \exp\left(-\frac{1}{\mu} y\right)$.

$$
\begin{aligned}
P(X > Y) &= \int_0^\infty \int_0^x \lambda \exp(-\lambda x) \frac{1}{\mu} \exp\left(-\frac{1}{\mu} y\right) dy\, dx \\
&= \int_0^\infty \lambda \exp(-\lambda x) \int_0^x \frac{1}{\mu} \exp\left(-\frac{1}{\mu} y\right) dy\, dx \\
&= \int_0^\infty \lambda \exp(-\lambda x) \left(1 - \exp\left(-\frac{x}{\mu}\right)\right) dx \\
&= \lambda \left\{ \lim_{x\to\infty} \left[\frac{\exp(-\lambda x)}{-\lambda} + \frac{\exp\left(-\left(\lambda + \frac{1}{\mu}\right) x\right)}{\lambda + \frac{1}{\mu}} \right] - \left[-\frac{1}{\lambda} + \frac{1}{\lambda + \frac{1}{\mu}} \right] \right\} \\
&= \lambda \left(\frac{1}{\lambda} - \frac{1}{\lambda + \frac{1}{\mu}} \right) = 1 - \frac{\lambda\mu}{\lambda\mu + 1} = \frac{1}{1 + \lambda\mu}
\end{aligned}
$$

   as required.

5. Suppose the arrival of action potentials (AP) is described by a Poisson process with parameter λ. Consider the counting process $N(s)$ for the number of AP arriving before some time s.

   For some time t and $d \in \mathbb{R}^+$, let $B_d(t)$ be an interval of length d properly containg t. Because the counting process $N(s)$ is memoryless, the probability that there are n arrivals within $B_d(t)$ also is given by a Poisson distribution, but has parameter λd. Denote by $M_d(t)$ the number of arrivals within the interval $B_d(t)$. So from above, $M_d(t) \sim Poisson(\lambda d)$.

   Define the random variable $T(t)$ as the minimum distance between a fixed time t and the arrival time of an AP. Given a positive real number d, we have

$$
P(T(t) > d) = P(M_d(t) = 0) = \frac{(\lambda d)^0}{0!} e^{-\lambda d} = e^{-\lambda d}
$$

   Then the cumulative distribution function of $T(t)$ is given by $F_{T(t)}(d) = 1 - e^{-\lambda d}$, which is the distribution function of an exponential random variable with parameter λ. Therefore

$$
T(t) \sim Exp(\lambda),
$$

   and that $E[(T(t)] = \frac{1}{\lambda}$.