MATH 6397 Homework 4

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1. In this problem, we will designate son 1 as the row player and son 2 the column player. The move of each son i, s_i, is the amount of money they want to inherit. Assuming that each can only ask for an integer value, then each has 1,000,001 possible choices starting from \$0 up to the maximum possible inheritance \$1,000,000. We specify the following payoff scheme:

   - If $s_1 + s_2 \leq 1,000,000$, then we have the payoff to son 1 and son 2 respectively being $P_{s_1,s_2} = (s_1, s_2)$.
   - If $s_1 + s_2 > 1,000,000$, then $P_{s_1,s_2} = (0,0)$.

   In the payoff matrix P, arranging the possible moves naturally, that is in increasing order, it is clear that the only nonzero pairs are found in the upper triangular matrix for which $s_1 + s_2 \leq 1,000,000$. The payoff matrix P therefore has the form

$$
\begin{array}{c|ccccccc}
 & 0 & 1 & \cdots & s_2 & \cdots & 999999 & 1000000 \\
\hline
0 & 0,0 & 0,1 & \cdots & 0,s_2 & \cdots & 0,999999 & 0,1000000 \\
1 & 1,0 & 1,1 & \cdots & 1,s_2 & \cdots & 1,999999 & \\
\vdots & \vdots & \vdots & & \vdots & \iddots & & \\
s_1 & s_1,0 & s_1,1 & & s1,s2 & & & \\
\vdots & \vdots & \vdots & \iddots & & & \Big(0,0\Big) & \\
999999 & 999999,0 & 999999,1 & & & & & \\
1000000 & 1000000,0 & & & & & &
\end{array}
$$

   Hence, whenever the row player chooses a move s_1, the best possible response of the column player is the move $s_2 = 1,000,000 - s_1$. Similarly, when the column player chooses a move s_2, the best possible response of the row player is the move $s_1 = 1,000,000 - s_2$. Therefore there are multiple pure Nash equilibria, each taking the form (s_1, s_2) for which $s_1 + s_2 = 1,000,000$.

2. Consider the two strategies GRIM and ALLD. The payoff for cooperation (C) and defection (D) is

$$
\begin{array}{c|cc}
 & \text{C} & \text{D} \\
\hline
\text{C} & R,R & S,T \\
\text{D} & T,S & P,P
\end{array}
$$

   where $T > R > P > S$.

   (a) Below we detail the payoff of an iterated game with m rounds.

Round	GRIM	vs	ALLD
1	S		T
2	P		P
$\vdots$	$\vdots$		$\vdots$
m	P		P
Payoff	$S+(m-1)P$		$T+(m-1)P$

Round	GRIM	vs	GRIM
1	R		R
2	R		R
$\vdots$	$\vdots$		$\vdots$
m	R		R
Payoff	mR		mR

Round	ALLD	vs	ALLD
1	P		P
2	P		P
$\vdots$	$\vdots$		$\vdots$
m	P		P
Payoff	mP		mP

The payoff matrix for the iterated game is therefore

	GRIM	ALLD
GRIM	$mR,\ mR$	$S+(m-1)P,\ T+(m-1)P$
ALLD	$T+(m-1)P,\ S+(m-1)P$	$mP,\ mP$

(b) We have a symmetric game, so GRIM is stable against invasion by ALLD if $mR > T+(m-1)P$. Now, we have

$$\begin{aligned} m > \frac{T-P}{R-P} &\Leftrightarrow m(R-P) > T-P \\ &\Leftrightarrow mR > T-P+mP \\ &\Leftrightarrow mR > T\,(m-1)\,P \end{aligned}$$

as required.

(c) Now consider GRIM*. The payoff for GRIM* versus GRIM is computed below.

Round	GRIM*	vs	GRIM
1	R		R
2	R		R
$\vdots$	$\vdots$		$\vdots$
m	T		S
Payoff	$T+(m-1)R$		$S+(m-1)R$

Round	GRIM*	vs	GRIM*
1	R		R
2	R		R
$\vdots$	$\vdots$		$\vdots$
m	P		P
Payoff	$P+(m-1)R$		$P+(m-1)R$

The payoff matrix for the iterated game is therefore

	GRIM	GRIM*
GRIM	$mR,\ mR$	$S+(m-1)R,\ T+(m-1)R$
GRIM*	$T+(m-1)R,\ S+(m-1)R$	$P+(m-1)R,\ P+(m-1)R$

Looking at GRM* vs GRIM, since $T>S$, GRIM* dominates GRIM.

(d) Define the strategy GRIM** which defects starting 2 rounds before the last. The payoff of an iterated game between GRIM** and GRIM* is detailed below.

Round	GRIM**	vs	GRIM*
1	R		R
2	R		R
$\vdots$	$\vdots$		$\vdots$
$m-1$	T		S
m	P		P
Payoff	$P+T+(m-2)R$		$P+S+(m-2)R$

Again, since $T>S$, GRIM** dominates GRIM*.

(e) Following the argument above, we will arrive at ALLD.

3. Suppose that in an iterated game, the next round happens with probability δ.

(a) Let m be the number of rounds a game is played. Then m can be viewed as a Geometric random variable with a successful event being the game ending and a failure being the game continuing on to the next round.
So $m \sim Geo(p)$, with $p=1-\delta$. With this, $E[m]=\frac{1}{p}=\frac{1}{1-\delta}=1+\frac{\delta}{1-\delta}$.

(b) Using the results in item 2b., and replacing m with $E[m]$, we have the payoff matrix

	GRIM	ALLD
GRIM	$\frac{1}{1-\delta}R,\ \frac{1}{1-\delta}R$	$S+\frac{\delta}{1-\delta}P,\ T+\frac{\delta}{1-\delta}P$
ALLD	$T+\frac{\delta}{1-\delta}P,\ S+\frac{\delta}{1-\delta}P$	$\frac{1}{1-\delta}P,\ \frac{1}{1-\delta}P$

(c) We have a symmetric game, so GRIM is stable against invasion by ALLD if $\frac{1}{1-\delta}R > T +$

$\dfrac{\delta}{1-\delta}P$. Now, we have

$$\begin{aligned}
\delta > \frac{T-R}{T-P} &\Leftrightarrow \delta(T-P) > T-R \\
&\Leftrightarrow \delta T - T > \delta P - R \\
&\Leftrightarrow R > \delta P + T(1-\delta) \\
&\Leftrightarrow \frac{R}{1-\delta} > T + \frac{\delta}{1-\delta}P
\end{aligned}$$

as required.

4. Consider an iterated game with memory-1 strategies TFT, GRIM, and ALLC. We will denote by $M_{i,j}$, $i,j \in \{\text{T:=TFT, G:=GRIM, A:=ALLC}\}$, the transition matrix for the Markov chain describing the state-transition of the game when strategies i and j are played against each other.

   (a) Given the strategy probability vectors $P_T = [1-\varepsilon, \varepsilon, 1-\varepsilon, \varepsilon]$, $P_G = [1-\varepsilon, \varepsilon, \varepsilon, \varepsilon]$, and $P_A = [1-\varepsilon, 1-\varepsilon, 1-\varepsilon, 1-\varepsilon]$, the transition matrices are computed as follows:

$$M_{T,T} = \begin{bmatrix} (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon \\ (1-\varepsilon)\varepsilon & (1-\varepsilon)^2 & \varepsilon^2 & \varepsilon(1-\varepsilon) \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & (1-\varepsilon)\varepsilon & (1-\varepsilon)^2 \end{bmatrix}$$

$$M_{T,G} = \begin{bmatrix} (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & (1-\varepsilon)\varepsilon & (1-\varepsilon)^2 \\ (1-\varepsilon)\varepsilon & (1-\varepsilon)^2 & \varepsilon^2 & \varepsilon(1-\varepsilon) \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & (1-\varepsilon)\varepsilon & (1-\varepsilon)^2 \end{bmatrix}$$

$$M_{T,A} = \begin{bmatrix} (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon \\ (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon \end{bmatrix}$$

$$M_{G,G} = \begin{bmatrix} (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & (1-\varepsilon)\varepsilon & (1-\varepsilon)^2 \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & (1-\varepsilon)\varepsilon & (1-\varepsilon)^2 \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & (1-\varepsilon)\varepsilon & (1-\varepsilon)^2 \end{bmatrix}$$

$$M_{G,A} = \begin{bmatrix} (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon \end{bmatrix}$$

$$M_{A,A} = \begin{bmatrix} (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ (1-\varepsilon)^2 & (1-\varepsilon)\varepsilon & \varepsilon(1-\varepsilon) & \varepsilon^2 \end{bmatrix}$$

(b) Next, we compute the corresponsding stationary distributions $\pi_{i,j} = [x,y,z,w]$ for each pair $i,j \in \{$T,G,A$\}$, given the transition matrix $M_{i,j}$ computed in the previous item. For each computation, we solve the system $\pi_{i,j}M_{i,j} = \pi_{i,j}$ provided $x+y+z+w = 1$. For all computations, the MATLAB symbolic solver for linear systems was used, which is detailed in the attachment. The stationary distributions in terms of ε, as well as the limiting values as $\varepsilon \to 0$ are given below.

- $\pi_{T,T} = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$
  When $\varepsilon \to 0$, at equilibrium, there is equal chance of being in any of the states.

- $\pi_{T,G} = \begin{bmatrix} \dfrac{\varepsilon}{2\varepsilon+1} & \dfrac{2\varepsilon(1-\varepsilon)}{2\varepsilon+1} & \dfrac{\varepsilon}{2\varepsilon+1} & \dfrac{2\varepsilon^2-2\varepsilon+1}{2\varepsilon+1} \end{bmatrix} \xrightarrow[\varepsilon\to 0]{} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$
  When $\varepsilon \to 0$, at equilibrium, both players will be defecting.

- $\pi_{T,A} = \begin{bmatrix} 4\varepsilon^2-3\varepsilon-2\varepsilon^3+1 & 2\varepsilon^3-2\varepsilon^2+\varepsilon & 2\varepsilon\left(1+\varepsilon^2-2\varepsilon\right) & 2\varepsilon^2(1-\varepsilon) \end{bmatrix} \xrightarrow[\varepsilon\to 0]{} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$
  When $\varepsilon \to 0$, at equilibrium, both players will be cooperating.

- $\pi_{G,G} = \begin{bmatrix} \dfrac{\varepsilon}{2} & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & 1+2\varepsilon^2-\dfrac{5\varepsilon}{2} \end{bmatrix} \xrightarrow[\varepsilon\to 0]{} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$
  When $\varepsilon \to 0$, at equilibrium, both players will be defecting.

- $\pi_{G,A} = \begin{bmatrix} \dfrac{1-\varepsilon}{3-2\varepsilon} & \dfrac{\varepsilon}{3-2\varepsilon} & \dfrac{2(1-\varepsilon)^2}{3-2\varepsilon} & \dfrac{2\varepsilon(1-\varepsilon)}{3-2\varepsilon} \end{bmatrix} \xrightarrow[\varepsilon\to 0]{} \begin{bmatrix} \dfrac{1}{3} & 0 & \dfrac{2}{3} & 0 \end{bmatrix}$
  When $\varepsilon \to 0$, at equilibrium, there is a third chance of both players cooperating, and two thirds chance of the row player cooperating and the column player defecting.

- $\pi_{A,A} = \begin{bmatrix} \varepsilon^2-2\varepsilon+1 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \end{bmatrix} \xrightarrow[\varepsilon\to 0]{} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$
  When $\varepsilon \to 0$, at equilibrium, both players will be cooperating.

(c) In the context of the Prisoner's dilemma defined in item 2, using the limiting stationary distributions computed above we now have the following payoff matrix which shows the expected payoff to the row player.

	TFT	GRIM	ALLC
TFT	$0.25(R+S+T+P)$	P	R
GRIM	P	P	$\dfrac{R+2T}{3}$
ALLC	R	$\dfrac{R+2S}{3}$	R

(d) Denoting the payoff to using strategy p against strategy q by $P(p,q)$ we analyze as follows.

- TFT is a Nash equilibrium if $P(T,T) \geq P(T,G)$ and $P(T,T) \geq P(T,A)$. Equivalently, we have $0.25(R+S+T+P) \geq R \iff S+T+P \geq 3R$.
- GRIM is a Nash equilibrium provided $P(G,G) \geq (G,T)$ and $P(G,G) \geq P(G,A)$. Equivalently, this is $P \geq \dfrac{R+2S}{3} \iff 3P \geq R+2S$.
- ALLC is NOT a Nash equilibrium since $P(A,A) = R < \dfrac{R+2T}{3} = P(G,A)$.

Attachment: MATLAB program for solving problem 4b.

```
syms x y z w e
%M transition probability appended by the probability dist. condition
M1=[(1-e)^2 (1-e)*e e*(1-e) e^2 1;
    e*(1-e) e^2 (1-e)^2 (1-e)*e 1;
    (1-e)*e (1-e)^2 e^2 e*(1-e) 1;
    e^2 e*(1-e) (1-e)*e (1-e)^2 1]

M2=[(1-e)^2 (1-e)*e e*(1-e) e^2 1;
    e^2 e*(1-e) (1-e)*e (1-e)^2 1;
    (1-e)*e (1-e)^2 e^2 e*(1-e) 1;
    e^2 e*(1-e) (1-e)*e (1-e)^2 1]

M3=[(1-e)^2 (1-e)*e e*(1-e) e^2 1;
    e*(1-e) e^2 (1-e)^2 (1-e)*e 1;
    (1-e)^2 (1-e)*e e*(1-e) e^2 1;
    e*(1-e) e^2 (1-e)^2 (1-e)*e 1]

M4=[(1-e)^2 (1-e)*e e*(1-e) e^2 1;
    e^2 e*(1-e) (1-e)*e (1-e)^2 1;
    e^2 e*(1-e) (1-e)*e (1-e)^2 1;
    e^2 e*(1-e) (1-e)*e (1-e)^2 1]

M5=[(1-e)^2 (1-e)*e e*(1-e) e^2 1;
    e*(1-e) e^2 (1-e)^2 (1-e)*e 1;
    e*(1-e) e^2 (1-e)^2 (1-e)*e 1;
    e*(1-e) e^2 (1-e)^2 (1-e)*e 1]

M6=[(1-e)^2 (1-e)*e e*(1-e) e^2 1;
    (1-e)^2 (1-e)*e e*(1-e) e^2 1;
    (1-e)^2 (1-e)*e e*(1-e) e^2 1;
    (1-e)^2 (1-e)*e e*(1-e) e^2 1]

%X stationary distribution
X1=linsolve(M1'-eye(5,4),[0;0;0;0;1])
X2=linsolve(M2'-eye(5,4),[0;0;0;0;1])
X3=linsolve(M3'-eye(5,4),[0;0;0;0;1])
X4=linsolve(M4'-eye(5,4),[0;0;0;0;1])
X5=linsolve(M5'-eye(5,4),[0;0;0;0;1])
X6=linsolve(M6'-eye(5,4),[0;0;0;0;1])
```