Optimal Play for Gregoris' Card Game

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1 Problem formulation

Consider the following card game ¹. A decision maker faces a deck $\mathcal{D} = \{x_1, x_2, \dots, x_n\}$ containing n cards, labeled $i \in \{1, 2, \dots, n\}$. The value of card $x_i = i$. At every time step, the decision maker draws from the deck with uniform probability without replacement. Denote by D_t the set of remaining cards in the deck at time t. The probability of drawing a card x_i , conditional on the remaining cards D_t , is given by the following PMF:

$$\mathbb{P}\left(X = x_i \mid D_t\right) = \begin{cases} \frac{1}{|D_t|} & \text{for } x_i \in D_t \\ 0 & \text{else} \end{cases} .$$
(1)

Let x_t be the card drawn at time t. The decision maker sequentially draws cards, and faces the following choice. She can pick the drawn card x_t , and receive reward $r_t = x_t$. However, she must thereafter draw and discard x_t cards from the deck, such that:

$$D_{t+1} \mid \text{Pick} = D_t \setminus \{y_1, y_2, \dots, y_{r_t}, x_t\},$$
 (2)

where y_j is drawn uniformly at random without replacement $y_j \sim U(D_t \setminus \{y_1, \dots, y_{j-1}\})$. Alternatively, she can choose to *skip* the card, whereafter she receives reward $r_t = 0$, and can draw a new card: We then have:

$$D_{t+1} \mid \text{Skip} = D_t \setminus \{x_t\}. \tag{3}$$

The decision maker's information set at time t is given by $\Omega_t = \{\mathcal{D}, D_t\}$, such that she can observe which cards are left in the deck. At t = 0, we have $D_0 = \mathcal{D}$. The decision maker's objective is to maximize her total expected sum of rewards:

$$R = \mathbb{E}\left[\sum_{t=0}^{\infty} r_t\right]. \tag{4}$$

Whenever a time step τ occurs where there are no cards remaining, such that $D_{\tau} = \emptyset$, all subsequent rewards are 0, i.e., $r_t = 0$ for all $t \geq \tau$. D_{τ} is an absorbing state. D_{τ} eventually occurs with $\mathbb{P} = 1$, at the latest at t = n, if the decision maker never picks any card. This is clearly suboptimal, as her total sum of rewards will be R = 0. She can easily improve upon this with the following policy:

$$\pi(x_t, D_t) = \begin{cases} \text{Pick} & \text{if } x_t = n\\ \text{Skip} & \text{if } x_t \neq n \end{cases}$$
(5)

¹Of course single player games are not games in the formal sense.

such that her total sum of rewards is R = n.

THIS IS A NON-HOMOGENOUS, NON-ERGORIC MARKOV CHAIN? WHAT IS THE STRUCTURE, WHAT DOES THIS SAY ABOUT THE OPTIMAL POLICY?.

2 Optimal play

Consider the following policy:

$$\pi^{s}(x_{t}, D_{t}) = \begin{cases} \operatorname{Pick} & \text{if } x_{t} = \max\{D_{t}\}\\ \operatorname{Skip} & \text{if } x_{t} \neq \max\{D_{t}\} \end{cases}, \tag{6}$$

The policy π^s simply searches for the largest card remaining in the deck and ends the game. Once this policy chooses pick, we enter the absorbing state D_{τ} , ending the game. For any state D_t , the value of this policy is given by:

$$V^{\pi^s}(x_t, D_t) = \sum_{t=0}^{\infty} r_t = \max\{D_t\},$$
 (7)

We can think of an example of a deck D^s for which the following policy is optimal, namely a deck for which, for every $x_i \in D^s$, $x_i > |D^s|$. Picking any card from D^s will generate the absorbing state D_{τ} , ending the game, so clearly it is optimal to search for the largest card in the deck and end the game.

Our proposition for the optimal policy is as follows:

Proposition 2.1. The optimal policy for Gregoris' card game is given by:

$$\pi^{m}(x_{t}, D_{t}) = \arg \max_{a \in \{Pick, Skip\}} \left\{ r(x_{t}, a) + \mathbb{E} \left[V^{\pi^{s}}(x_{t+1}, D_{t+1}) \right] \right\}.$$
 (8)

Proof. The Bellman equation for the card game is given by:

$$V(x_t, D_t) = \max_{a \in \{\text{Pick, Skip}\}} \left\{ r(x_t, a) + \mathbb{E}\left[V(x_{t+1}, D_{t+1})\right] \right\}. \tag{9}$$

It follows that if we can show that

$$\max_{a \in \{\text{Pick, Skip}\}} \left\{ r(x_t, a) + \mathbb{E}\left[V(x_{t+1}, D_{t+1})\right] \right\} = \max_{a \in \{\text{Pick, Skip}\}} \left\{ r(x_t, a) + \mathbb{E}\left[V^{\pi^s}(x_{t+1}, D_{t+1})\right] \right\}, (10)$$

then we can be sure that our policy is the optimal policy:

$$\pi^{m}(x_{t}, D_{t}) = \pi^{*}(x_{t}, D_{t}). \tag{11}$$