# Optimal Play for a Non-Ergodic Card Game

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## 1 Problem formulation

Consider the following card game [1]. A decision maker faces a deck $\mathcal{D} = \{x_1, x_2, \ldots, x_n\}$ containing $n$ cards, labeled $i \in \{1, 2, \ldots, n\}$. The value of card $x_i = i$. At every time step, the decision maker draws from the deck with uniform probability without replacement. Denote by $D_t$ the set of remaining cards in the deck at time $t$. The probability of drawing a card $x_i$, conditional on the remaining cards $D_t$, is given by the following PMF:

$$\mathbb{P}\Big(X = x_i \mid D_t\Big) = \begin{cases} \frac{1}{|D_t|} & \text{for } x_i \in D_t \\ 0 & \text{else} \end{cases}. \tag{1}$$

Let $x_t$ be the card drawn at time $t$. The decision maker sequentially draws cards, and faces the following choice. She can *pick* the drawn card $x_t$, and receive reward $r_t = x_t$. However, she must thereafter draw and discard $x_t$ cards from the deck, such that:

$$D_{t+1} \mid \text{Pick} = D_t \setminus \{y_1, y_2, \ldots, y_{x_t}, x_t\}, \tag{2}$$

where $y_j$ is drawn uniformly at random without replacement $y_j \sim U(D_t \setminus \{y_1, \ldots, y_{j-1}\})$. Alternatively, she can choose to *skip* the card, whereafter she receives reward $r_t = 0$, and can draw a new card: We then have:

$$D_{t+1} \mid \text{Skip} = D_t \setminus \{x_t\}. \tag{3}$$

The decision maker's information set at time $t$ is given by $\Omega_t = \{\mathcal{D}, D_t\}$, such that she can observe which cards are left in the deck. At $t = 0$, we have $D_0 = \mathcal{D}$. The decision maker's objective is to maximize her total expected sum of rewards:

$$R = \mathbb{E}\left[\sum_{t=0}^{\infty} r_t\right]. \tag{4}$$

Whenever a time step $\tau$ occurs where there are no cards remaining, such that $D_\tau = \emptyset$, all subsequent rewards are 0, i.e., $r_t = 0$ for all $t \geq \tau$. $D_\tau$ is an absorbing state. $D_\tau$ eventually occurs with $\mathbb{P} = 1$, at the latest at $t = n$, if the decision maker never picks any card. This is clearly suboptimal, as her total sum of rewards will be $R = 0$. She can easily improve upon this with the following policy:

$$\pi(x_t, D_t) = \begin{cases} \text{Pick} & \text{if } x_t = n \\ \text{Skip} & \text{if } x_t \neq n \end{cases}, \tag{5}$$

[1] Of course single player games are not games in the formal sense.

such that her total sum of rewards is $R = n$.

**THIS IS A NON-HOMOGENOUS, NON-ERGORIC MARKOV CHAIN? WHAT IS THE STRUCTURE, WHAT DOES THIS SAY ABOUT THE OPTIMAL POLICY?**.

## 2 Optimal play

Consider the following policy:

$$\pi^{\max}(x_t, D_t) = \begin{cases} \text{Pick} & \text{if } x_t = \max\{D_t\} \\ \text{Skip} & \text{if } x_t \neq \max\{D_t\} \end{cases}, \tag{6}$$

The policy $\pi^s$ simply searches for the largest card remaining in the deck and ends the game. Once this policy chooses *pick*, we enter the absorbing state $D_\tau$, ending the game. For any state $D_t$, the value of this policy is given by:

$$V^{\pi^s}(x_t, D_t) = \sum_t^{\infty} r_t = \max\{D_t\}, \tag{7}$$

We can think of an example of a deck $D^s$ for which the following policy is optimal, namely a deck for which, for every $x_i \in D^s, x_i > |D^s|$. Picking any card from $D^s$ will generate the absorbing state $D_\tau$, ending the game, so clearly it is optimal to search for the largest card in the deck and end the game.

Our proposition for the optimal policy is as follows:

**Proposition 2.1.** *The optimal policy for the card game is given by:*

$$\pi^m(x_t, D_t) = \arg \max_{a \in \{\textit{Pick, Skip}\}} \Big\{ r(x_t, a) + \mathbb{E}\Big[ V^{\pi^s}(x_{t+1}, D_{t+1}) \Big] \Big\}. \tag{8}$$

*Proof.* The Bellman equation for the card game is given by:

$$V(x_t, D_t) = \max_{a \in \{\text{Pick, Skip}\}} \Big\{ r(x_t, a) + \mathbb{E}\Big[ V(x_{t+1}, D_{t+1}) \; |a \Big] \Big\}. \tag{9}$$

It follows that if we can show that

$$\arg \max_{a \in \{\text{Pick, Skip}\}} \Big\{ r(x_t, a) + \mathbb{E}\Big[ V(x_{t+1}, D_{t+1}) \Big] \Big\} = \arg \max_{a \in \{\text{Pick, Skip}\}} \Big\{ r(x_t, a) + \mathbb{E}\Big[ V^{\pi^{\max}}(x_{t+1}, D_{t+1}) \Big] \Big\}, \tag{10}$$

then we can be sure that our policy is the optimal policy:

$$\pi^m(x_t, D_t) = \pi^*(x_t, D_t). \tag{11}$$

We can rewrite the RHS as:

$$r(x_t, a) + \mathbb{E}\Big[ V^{\pi^{\max}}(x_{t+1}, D_{t+1}) \Big] = r(x_t, a) + \mathbb{E}\big[ \max\{D_{t+1}\} \big]. \tag{12}$$

We can order the elements of $D_t$ as $m_1 > m_2 > \cdots > m_{|D_t|}$, where $\max\{D_t\} = m_1$. Note that the total number of ways we can discard $x_t$ cards from a deck of size $|D_t|$ is given by

$$\binom{|D_t| - 1}{x_t}. \tag{13}$$

We are interested in the number of outcomes for which $\max\{D_t\} = m_j$. These are all events where all cards $m_i > m_j$ are discarded, and $m_j$ is not discarded. The number of events such that these conditions hold is given by

$$\binom{|D_t - 1| - j}{x_t - (j-1)}, \tag{14}$$

since we can randomize over $x_t - (j-1)$ cards (the cards smaller than $m_j$), and can draw them from $|D_t| - j$ cards. It follows that:

$$\mathbb{P}\Big(\max\{D_{T+1}\} = m_j \Big| \text{Pick } x_t\Big) = \mathbb{P}_j = \frac{\binom{|D_t|-1}{x_t}}{\binom{|D_t|-j-1}{x_t-(j-1)}}. \tag{15}$$

We can then expand the expected maximum after picking $x_t$ as:

$$\mathbb{E}\big[\max\{D_{t+1}\}\ |\text{Pick}x_t\big] = \sum_{j=1}^{x_t} \mathbb{P}_j \cdot m_j = \sum_{j=1}^{x_t} \frac{\binom{|D_t|-1}{x_t}}{\binom{|D_t|-j-1}{x_t-(j-1)}} m_j. \tag{16}$$

It follows that the decision makes should select action *Pick* iff:

$$x_t + \sum_{j=1}^{x_t} \frac{\binom{|D_t|-1}{x_t}}{\binom{|D_t|-j-1}{x_t-(j-1)}} m_j > m_1. \tag{17}$$

**WORK IN PROGRESS |**

We need to show that

$$x_t + \mathbb{E}\Big[V(x_{t+1}, D_{t+1})|\text{Pick}\Big] \geq \mathbb{E}\Big[V(x_{t+1}, D_{t+1})|\text{Skip}\Big] \implies x_t + \mathbb{E}\big[\max\{D_{t+1}\}|\text{Pick}\big] \geq \max\{D_t\} \tag{18}$$

and

$$x_t + \mathbb{E}\Big[V(x_{t+1}, D_{t+1})|\text{Pick}\Big] \leq \mathbb{E}\Big[V(x_{t+1}, D_{t+1})|\text{Skip}\Big] \implies x_t + \mathbb{E}\big[\max\{D_{t+1}\}|\text{Pick}\big] \leq \max\{D_t\}. \tag{19}$$

**Lemma 2.2.** *Under the optimal policy, any game ending card should be the maximum card left in the deck.*

Consider the first proposition, where the optimal policy would prescribe to Pick. We then have:

$$x_t + \mathbb{E}\Big[V(x_{t+1}, D_{t+1})|\text{Pick}\Big] \geq \mathbb{E}\Big[V(x_{t+1}, D_{t+1})|\text{Skip}\Big] \tag{20}$$

We know that, by the principle of optimality, the following two inequalities must hold:

$$\mathbb{E}\big[\max\{D_{t+1}\}|\text{Pick}\big] \leq \mathbb{E}\Big[V(x_{t+1}, D_{t+1})|\text{Pick}\Big], \tag{21}$$

$$\max\{D_t\} \leq \mathbb{E}\Big[V(x_{t+1}, D_{t+1})|\text{Skip}\Big], \tag{22}$$

such that we have

$$x_t + \mathbb{E}\Big[V(x_{t+1}, D_{t+1})|\text{Pick}\Big] \geq \mathbb{E}\Big[V(x_{t+1}, D_{t+1})|\text{Skip}\Big] \geq \max\{D_t\}. \tag{23}$$

In the case where $V_\pi^m$ and $V_\pi^*$ are the same, they trivially perscribe the same actions. We are therefore only interested in the case where inequalities are strict:

$$x_t + \mathbb{E}\Big[V(x_{t+1}, D_{t+1})|\text{Pick}\Big] \geq \mathbb{E}\Big[V(x_{t+1}, D_{t+1})|\text{Skip}\Big] > \max\{D_t\}. \tag{24}$$

□