Complex conjugate of z in polar notation.

If $z = a + ib$ is any complex number, then the complex conjugate of z is $\bar{z} = a - ib$. Geometrically, the conjugate of z is the reflection of z about the real axis.

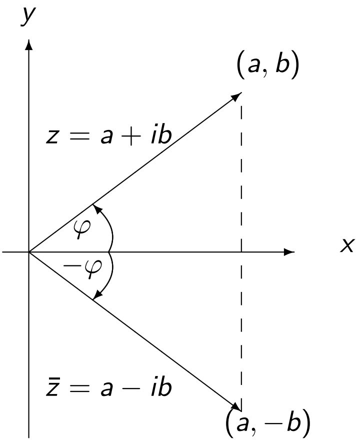

Figure 6. The complex conjugate on the complex plane.

If $z = r(\cos\varphi + i\sin\varphi)$ then $\bar{z} = r(\cos\varphi - i\sin\varphi)$.
Recalling the trigonometric identities $\sin(-\varphi) = -\sin\varphi$ and $\cos(-\varphi) = \cos\varphi$ we may write that

$$\bar{z} = r(\cos(-\varphi) + i\sin(-\varphi))$$

Adding and subtracting using polar form of complex number. Because the operation of addition of a complex number parallel the addition for vectors in $\mathbb{R}^2$, the geometric interpretation holds for complex numbers.

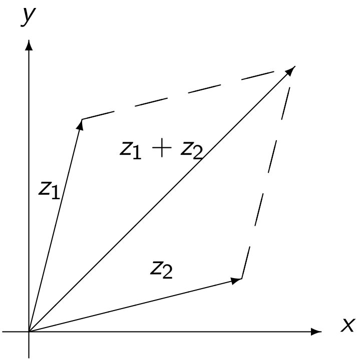

Figure 7. The sum of two complex numbers

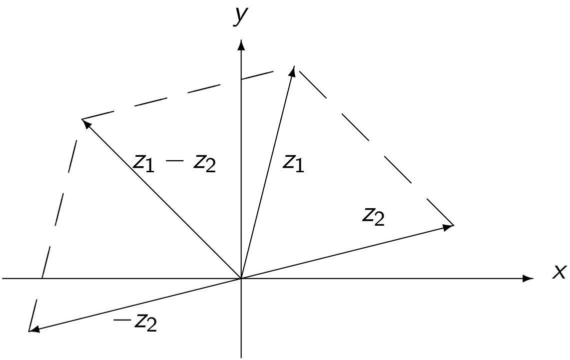

Figure 8. The difference of two complex numbers.

Multiplication and division using polar forms.

Let $z_1 = r_1(cos\varphi_1 + isin\varphi_1)$ and $z_2 = r_2(cos\varphi_2 + isin\varphi_2)$.
Multiplying, we obtain

$$\begin{aligned} z_1 z_2 =& r_1(cos\varphi_1 + isin\varphi_{)} r_2(cos\varphi_2 + isin\varphi_2) \\ =& r_1 r_2((cos\varphi_1 cos\varphi_2 - sin\varphi_1 sin\varphi_2) + i(cos\varphi_1 sin\varphi_2 + sin\varphi_1 \cos\varphi_2)) \\ =& r_1 r_2(cos(\varphi_1 + \varphi_2) + isin(\varphi_1 + \varphi_2)) \end{aligned}$$

We have

$$z_1 z_2 = r_1 r_2(cos(\varphi_1 + \varphi_2) + isin(\varphi_1 + \varphi_2)),$$

This is the polar form of the complex number with modulus $r_1 r_2$ and argument $\varphi_1 + \varphi_2$. Thus, the product of two complex numbers is obtained by multiplying their moduli and adding their arguments. Similarly can be shown, that the quotient of two complex numbers is obtained by dividing their moduli and subtracting their arguments (in the appropriate order). This means the following equation holds (for $z_2 \neq 0$):

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(cos(\varphi_1 - \varphi_2) + isin(\varphi_1 - \varphi_2))$$

deMoivre's formula.

Let n be positive integer and $z = r(\cos\varphi + i\sin\varphi)$, then

$$z^n = (r(cos\varphi + isin\varphi))^n = r^n(cos(n\varphi) + isin(n\varphi)).$$

In the special case where $r = 1$ we have $z = \cos\varphi + i\sin\varphi$ and the formula for powers becomes

$$\boxed{(cos\varphi + isin\varphi)^n = cos(n\varphi) + isin(n\varphi)}$$

which is called deMoivre's formula. This formula is used to calculate the sines and cosines of multiples of arguments.

For example for $n = 2$ we have

$$(cos\varphi + isin\varphi)^2 = cos(2\varphi) + isin(2\varphi).$$

On the other hand

$$\begin{aligned}(cos\varphi + isin\varphi)^2 =& cos^2\varphi + 2cos\varphi isin\varphi + i^2 sin^2\varphi = \\ =& (cos^2\varphi - sin^2\varphi) + i2cos\varphi sin\varphi\end{aligned}$$

While the left sides of the two equations are equal, the right hand sides also must be equal. Two complex numbers are equal, if the real parts are equal and their imaginary parts are equal, thus we have

$$\begin{cases} cos(2\varphi) & = cos^2\varphi - sin^2\varphi \\ sin(2\varphi) & = 2sin\varphi cos\varphi \end{cases}$$

Although we derived deMoivre's formula assuming n to be a positive integer, this formula is valid for all integers n.

Finding nth roots.

Let $z = a + ib$.

Definition. An nth root of z is any complex number w that satisfies the equation $w^n = z$.

Theorem. Every nonzero complex number has exactly n different nth roots.

Proof. Let $z \neq 0$, then we can represent z in form $z = r(cos\varphi + isin\varphi)$. Let $w = \rho(cos\psi + isin\psi)$ and we assume $w^n = z$ that means

$$\rho^n(cos(n\psi) + isin(n\psi)) = r(cos\varphi + isin\varphi).$$

Two complex numbers are equal only if
1) the moduli are equal: $\rho^n = r$ or $\rho = \sqrt[n]{r}$ ($\sqrt[n]{r}$ denotes the real positive nth root) and
2) the angles differ by a multiple of 2π: $n\psi = \varphi + 2k\pi$, that is $\psi = \frac{\varphi+2k\pi}{n}$, $k \in \mathbb{Z}$.
Although there are infinitely many values of k, some of roots we found may be duplicate of others. The two roots $w_s = w_t$, if $\psi_s = \psi_t + 2k\pi$, $k \in \mathbb{Z}$ Therefore, there are exactly n different roots:

$$\sqrt[n]{z} = \sqrt[n]{r}(cos\frac{\varphi + 2k\pi}{n} + isin\frac{\varphi + 2k\pi}{n}), \quad k = 0, 1, \ldots, n-1. \quad \square$$

Example. Find $\sqrt[3]{1} = \sqrt[3]{1+0i} = \sqrt[3]{1(cos0 + isin0)}$.

$$\sqrt[3]{1} = \sqrt[3]{1}(cos\frac{0+2k\pi}{3} + isin\frac{0+2k\pi}{3}), \quad k = 0, 1, 2$$

$$\sqrt[3]{1} = \begin{cases} cos0 + isin0 \\ cos\frac{2\pi}{3} + isin\frac{2\pi}{3} \\ cos\frac{4\pi}{3} + isin\frac{4\pi}{3} \end{cases} = \begin{cases} 1 \\ -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{cases}$$

Corollary. All the nth roots of z lie on the circle of radius $\rho = \sqrt[n]{r}$, and are equally spaced $\frac{2\pi}{n}$ radians apart .

Exponent form of a complex number.

In more detailed studies of complex numbers, complex exponents are defined, and it is proved that

$$e^{i\varphi} = cos\varphi + i\, sin\varphi$$

where e is a real number given approximately by $e = 2,71828...$ From the given equation it follows that the polar form of complex number $z = r(\cos\varphi + i\sin\varphi)$ can be written as

$$z = re^{i\varphi}.$$

Any complex number z can be written $z = re^{i\varphi}$, where r is the modulus of complex number and φ is its argument. This is called the exponent form of a complex number z.

For example $-1 = e^{i\pi}$, $2 + 2i = 2\sqrt{2}e^{i\frac{\pi}{4}}$.

If $z = re^{i\varphi}$, then its complex conjugate $\bar{z} = re^{-i\varphi}$.

Operations with complex numbers in the exponent form.
The exponent form of complex number is just a different notation of polar form:

$$
\begin{aligned}
z_1 z_2 =& (r_1 e^{i\varphi_1})(r_2 e^{i\varphi_2}) = (r_1 r_2) e^{i(\varphi_1+\varphi_2)}, \\
\frac{z_1}{z_2} =& \frac{r_1 e^{i\varphi_1}}{r_2 e^{i\varphi_2}} = \frac{r_1}{r_2} e^{i(\varphi_1-\varphi_2)} \\
z^n =& = (e^{i\varphi})^n = e^{in\varphi}, \\
\sqrt[n]{z} =& \sqrt[n]{re^{i\varphi}} = \sqrt[n]{r} e^{i\frac{\varphi+2k\pi}{n}}.
\end{aligned}
$$

This means that complex exponents follow the same laws as real exponents.

Definition of matrix

Let m and n be two fixed natural numbers: $m, n \in \mathbb{N}$.
$m \times n$-matrix is the table consisting of numbers in m rows and n columns. The numbers in this table are called the entries in the matrix.

Example.

$$A = \begin{pmatrix} 3 & 7 & -2 & 0.5 & 1 \\ 6 & -11 & 0 & 4 & 0 \end{pmatrix}$$

is a matrix with two rows and five columns.
The size of a matrix is described by the number of rows (horizontal lines) and columns (vertical lines) it contains.

The pair $m \times n$ is called the size of matrix A.

In a size description, the first number always denotes the number of rows and the second denotes the number of columns.

$m \times n$-matrix consists of m rows and n columns. Let $i = 1, \ldots, m$, $j = 1, \ldots, n$, $a_{ij} \in \mathbb{R}$. Then

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} = (a_{ij})_{m \times n}$$

is a $m \times n$-matrix.

We shall use capital letters to denote matrices and lowercase letters to denote numerical entries.

Entry a_{ij} is the element in row i and column j.

The entry in row i and column j of matrix A is also commonly denoted by the symbol $(A)_{ij}$.

If $m = n$, then A is a square matrix.
A matrix with n rows and n columns is called a square matrix of order n.

$$A = \begin{pmatrix} \mathbf{a_{11}} & a_{12} & \dots & a_{1n} \\ a_{21} & \mathbf{a_{22}} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & \mathbf{a_{nn}} \end{pmatrix}$$

The entries a_{11},a_{22}, $\dots$, a_{nn} are said to be on the main diagonal of the square matrix A.

When discussing matrices, it is common to refer to numerical quantities as scalars. Unless stated otherwise, scalars will be real numbers. The set of all $(m \times n)$-matrices with real entries is denoted by $\mathbb{R}^{m \times n}$.

Properties of scalars.

1) adding is commutative: $a_1 + a_2 = a_2 + a_1$,
2) adding is associative: $(a_1 + a_2) + a_3 = a_1 + (a_2 + a_3)$,
3) there exists a zero element: 0 such that for any a
$a + 0 = 0 + a = a$,
4) for any a exists an opposite $-a$ such that
$a + (-a) = (-a) + a = 0$,
5) multiplication is commutative: $a_1 a_2 = a_2 a_1$,
6) multiplication is associative: $(a_1 a_2) a_3 = a_1 (a_2 a_3)$,
7) the distribution laws hold:

$$\begin{cases} a_1(a_2 + a_3) = a_1 a_2 + a_1 a_3, \\ (a_1 + a_2) a_3 = a_1 a_3 + a_2 a_3, \end{cases}$$

8) exists a unit 1 such that for any a: $1a = a$,
9) every nonzero element a has an inverse: a^{-1} such that
$a a^{-1} = a^{-1} a = 1$

If A is a $m \times n$-matrix, we can single out the rows of the matrix. Every row is a $(1 \times n)$-matrix and it is called the row vector of matrix A.
Every column of an $m \times n$-matrix is a $(n \times 1)$-matrix called the column vector of A.
For example the matrix

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 & 4 \\ 2 & -1 & 0 & 5 \end{pmatrix}$$

has row vectors

$$B_{1R} = (1, 0, 3, 4), \quad B_{2R} = (2, -1, 0, 5) \in \mathbb{R}^4$$

and
column vectors

$$B_{C1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, B_{C2} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, B_{C3} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, B_{C4} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \in \mathbb{R}^2$$

Equality of matrices

Two matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{k\times l}$ are equal (denoted by $A = B$) if

1) they have the same size,
that means $m = k$ and $n = l$,

and

2) their corresponding entries are equal,
that means $a_{ij} = b_{ij}$ for all values of i and j
$(0 \leq i \leq m,\ 0 \leq j \leq n)$.

Addition of matrices

We can add only matrices having the same size. Matrices of different sizes cannot be added.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$-matrices. The sum of matrices A and B is the $m \times n$-matrix $C = (c_{ij})$, where

$$c_{ij} = a_{ij} + b_{ij} \quad \text{for all values of } i \text{ and } j.$$

In other words, the sum of matrices is obtained by adding the corresponding entries.

Example.

$$\begin{pmatrix} 1 & 0 & 3 & 4 \\ 2 & -1 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 5 & 2 \\ 2 & 1 & 4 & 3 \end{pmatrix} =$$

$$\begin{pmatrix} 1+2 & 0+1 & 3+5 & 4+2 \\ 2+2 & -1+1 & 0+4 & 5+3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 8 & 6 \\ 4 & 0 & 4 & 8 \end{pmatrix}$$

Zero matrix

A matrix, all of whose entries are zero, is called a zero matrix and denoted by Θ.

$$\Theta = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

The size of zero matrix is meant from the context in which the symbol appears. Sometimes the zero matrix is denoted by 0. Then one should also use the context to figure out whether this is the real number 0 or zero matrix.

Theorem

(Adding a zero matrix) For every matrix A

$$A + \Theta = A = \Theta + A$$

(Θ is the zero matrix of corresponding size).

Proof. Let A and Θ be $m \times n$- matrices.

For every $1 \leq i \leq m$ and $1 \leq j \leq n$ we have

$$\begin{aligned}
(A + \Theta)_{ij} &= a_{ij} + (\Theta)_{ij} = a_{ij} + 0 \\
&= a_{ij} \\
&= 0 + a_{ij} = (\Theta)_{ij} + a_{ij} = (\Theta + A)_{ij}.
\end{aligned}$$

Therefore $A + \Theta = \Theta + A$. $\square$

Scalar multiple of matrix
If $A = (a_{ij})$ is an $m \times n$-matrix and c is any scalar, then the product cA is $m \times n$-matrix where

$$(cA)_{ij} = ca_{ij} \quad \text{for all values of } i \text{ and } j.$$

This means, the product cA is obtained by multiplying each entry of A by c.

Example 1.

$$3\begin{pmatrix}1 & 0 & 3 & 4\\ 2 & -1 & 0 & 5\end{pmatrix} = \begin{pmatrix}3\cdot 1 & 3\cdot 0 & 3\cdot 3 & 3\cdot 4\\ 3\cdot 2 & 3\cdot(-1) & 3\cdot 0 & 3\cdot 5\end{pmatrix} = \begin{pmatrix}3 & 0 & 9 & 12\\ 6 & -3 & 0 & 15\end{pmatrix}$$

Example 2.

$$-1\begin{pmatrix}1 & 0 & 3 & 4\\ 2 & -1 & 0 & 5\end{pmatrix} = \begin{pmatrix}-1 & 0 & -3 & -4\\ -2 & 1 & 0 & -5\end{pmatrix}$$

Opposite matrix

The opposite of matrix A is matrix

$$-A = (-1)A.$$

This means $(-A)_{ij} = -a_{ij}$.

For real numbers we have $a_{ij} - a_{ij} = 0$. This leads to the equality of matrices $A + (-A) = \Theta$.

The same way one may obtain the following properties of matrix operations.

Some properties of matrix arithmetic

Let $A, B, C \in \mathbb{R}^{m\times n}$ and $a, b \in \mathbb{R}$. Then the following properties hold:

1) Commutative law for adding $A + B = B + A$.
2) Associative law for adding $(A + B) + C = A + (B + C)$.
3) There exists a zero matrix Θ such that $A + \Theta = A = \Theta + A$.
4) For any matrix A there exists an opposite matrix $-A$ such that $A + (-A) = \Theta = (-A) + A$.
5) Distributive law $a(A + B) = aA + aB$.
6) Distributive law $(a + b)A = aA + bA$.
7) Associativity of scalar multiple $a(bA) = (ab)A$.
8) Unitality $1A = A$.

Proof of $A + B = B + A$.

Matrices $A + B$ and $B + A$ are equal, if there are equal their sizes and corresponding elements.

1) Let A be $m \times n$-matrix. Then B must be $m \times n$-matrix.
The sizes of matrices $A + B$ and $B + A$ are both $m \times n$.

2) The entry of matrix $A + B$ in row i and column j is $a_{ij} + b_{ij}$.
The entry of matrix $B + A$ in row i and column j is $b_{ij} + a_{ij}$.

The numbers a_{ij} and b_{ij} are real, therefore $a_{ij} + b_{ij} = b_{ij} + a_{ij}$.
Therefore the entries of matrices $A + B$ and $B + A$ in row i and in column j are equal.

This is true for every i $(1 \leq i \leq m)$ and every j $(1 \leq j \leq n)$, therefore matrices $A + B$ and $B + A$ are equal.

Transpose of a matrix

We define one matrix operation that has no analogue in the real numbers.

Definition. If A is any $m \times n$-matrix, then the transpose of A, denoted by A^T, is defined to be the $n \times m$- matrix $B = (b_{ij})$ where $b_{ij} = a_{ji}$. The rows of transposed matrix A^T are the columns of A (the same order) and the columns of A^T are the rows of A. For example

$$\begin{pmatrix} 1 & 0 & 3 & 4 \\ 2 & -1 & 0 & 5 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 0 \\ 4 & 5 \end{pmatrix}$$

Transpose of a row vector is a column vector and vice versa. For example

$$\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$$

Properties of transpose

Let A and B be the matrices of the same size and let c be a scalar. Then

1)$(A^T)^T = A$,

2) $(cA)^T = cA^T$,

3) $(A+B)^T = A^T + B^T$.

Proof of $(A+B)^T = A^T + B^T$.
There is defined $A+B$, therefore matrices A and B must be of the same size. Let A and B be $m \times n$-matrices. Then $A+B$ is also of the size $m \times n$, $(A+B)^T$ is $n \times m$-matrix.
A^T and B^T are $n \times m$-matrices. Their sum has the same size. Therefore the sizes of the matrices on the left hand side and the right hand side are both $n \times m$.

The entries of the row i and column j are respectively.

$$((A+B)^T)_{ij} = (A+B)_{ji} = (A)_{ji} + (B)_{ji},$$
$$((A^T + B^T))_{ij} = (A^T)_{ij} + (B^T)_{ij} = (A)_{ji} + (B)_{ji}.$$

We see that the right hand sides of these equations are equal, therefore the entries of matrices $((A+B)^T)_{ij}$ and $((A^T + B^T))_{ij}$ are equal. This is true for every i and j, therefore the matrices $(A+B)^T$ and $A^T + B^T$ are equal.

Symmetric matrices

A matrix A is called symmetric if $A = A^T$.

Symmetric matrices must be square matrices.

A matrix $A = (a_{ij})$ is symmetric if and only if $a_{ij} = a_{ji}$ for all values of i and j.

Example: a matrix

$$F = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ 3 & 6 & 8 & 9 \\ 4 & 7 & 9 & 10 \end{pmatrix}$$

is symmetric.

It is easy to recognize the symmetric matrix by inspection: The entries on the main diagonal may be arbitrary, but the "mirror images" of entries across the main diagonal must be equal. This follows from the fact that transposing a square matrix can be accomplished by interchanging entries that are symmetrically positioned about the main diagonal.

Skew-symmetric matrices

A matrix A is called skew-symmetric if $A = -A^T$.
Skew-symmetric matrices must be square matrices.
A matrix $A = (a_{ij})$ is skew-symmetric if and only if $a_{ij} = -a_{ji}$ for all values of i and j.
Example: a matrix

$$G = \begin{pmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 4 & 5 \\ -2 & -4 & 0 & 6 \\ -3 & -5 & -6 & 0 \end{pmatrix}$$

is skew-symmetric.
The entries on the main diagonal of an skew-symmetric matrix must be zero, the "mirror images" of entries across the main diagonal must be opposite values.

Exercises

1. a) Find a nonzero 3×3 matrix A such that $A^T = A$
b) Find a nonzero 3×3 matrix A such that $A^T = -A$

2. Show that if B is a square matrix then $B + B^T$ is symmetric and $B - B^T$ is skew-symmetric.

3. Prove the properties of matrix arithmetic involving adding, multiplication by a scalar and matrix transpose.