LINEAR ALGEBRA (YMA3710)

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U05-410

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1. Complex numbers, the definition and different forms for their representation. Operations on complex numbers: adding, subtracting, multiplying, dividing, finding exponents and roots.
2. Matrices and matrix arithmetics (linear operations, multiplication, transposition), properties of matrix operations.
3. The systems of linear equations, solving these systems by elimination methods.
4. The definition and properties of determinants. The rank of a matrix and finding it.
5. The inverse of a matrix, when it exists and how to find it.
6. The matrix form of a linear system, its solution, least square solution of a linear system.
7. Algebraic structures, axioms of a vector space. Examples of algebraic systems and vector spaces.

8. Linearly dependent sets of vectors. The basis of a vector space. The coordinates of a vector relative to a given basis. The theorem on the rank of a matrix.
9. Inner product spaces, the length of a vector, angle between two vectors. Examples.
10. The affine space, coordinates in an affine space. Euclidean spaces. Metric values in an Euclidean space.
11. Straight lines and a hyperplanes in n-dimensional Euclidean spaces. A distance between a point and a hyperplane.
12. The cross product of vectors and its properties. Solving the problems related to straight lines and planes in 3-dimensional Euclidean space.
13. A linear mapping and its matrix. Orthogonal transformations and orthogonal matrices. Finding eigenvalues and eigenvectors.
14. An overview of second-order curves. Quadratic forms and diagonalizing quadratic forms.

5 EAP = 130 hours

To get the credits one has to pass the examination.
Prerequisites for the examination: the student has to pass two tests on exercises. Maximal number of points for each test is 100. To pass the test, one has to get at least 51 points.

Concepts and relations are asked on the exam.
An alternative possibility to pass examination is to do it by theory tests during the semester. There will be 3 theory tests during the semester + homework on second-order curves.

The final grade of the course is computed by the sum of the points of tests of exercises and the exam or theory tests.

The theory tests

The theory tests are the easier way to answer the theory.

There will be 3 theory tests during the semester. Since these tests are voluntary, they will not be during the class. There will be agreed some other suitable time.

The 4. test is a homework on second-order curves. This will be at the end of the semester.

The theory tests are 60+60+40+40=200 points in total.
If You are satisfied with the grade calculated by the points, You bring the tests to the examination and get the grade for the tests.

The other possibility is to answer all the theory at the oral examination.

During the semester there will be

2 tests of exercise-solving (max 100 points). Test is passed by min 51 points.

4 theory tests (one of them is a homework on second-order curves).

If both exercise-solving tests are more than 51 points then the grade is calculated by the sum of points of all tests:

$$\begin{array}{ll} 202..241 & ''1'' \\ 242..281 & ''2'' \\ 282..321 & ''3'' \\ 322..361 & ''4'' \\ 362..400 & ''5'' \end{array}$$

In order to get the grade, the student has to present all the tests at the examination.

Complex numbers.

A complex number (on algebraic form) is an expression $a + bi$, where a and b are real numbers and i is imaginary unit.

Imaginary unit, denoted by i, is defined by the equation $i^2 = -1$.

Sometimes it is convenient to use a single letter, such as z, to denote a complex number. Thus we may write $z = a + bi$.

Definition. The real number a is called the real part of complex number $z = a + bi$. This is denoted $Re\, z = a$.

The real number b is called the imaginary part of the same complex number $z = a + bi$. This is denoted $Im\, z = b$.

If $b = 0$, then the complex number $a + bi$ reduces to $a + 0i = a$
Thus, for any real number a

$$a = a + 0i,$$

so the real numbers can be regarded as complex numbers with the imaginary part of zero.

If we have $a = 0$, then $a + bi = 0 + bi = bi$. These complex numbers are called pure imaginary numbers.

Equality of complex numbers.

Two complex numbers $a + bi$ and $c + di$ are equal if $a = c$ and $b = d$. This means that

$$a + bi = c + di \quad \Leftrightarrow \quad \begin{cases} a = c \\ b = d \end{cases}$$

The sum of two complex numbers $z_1 = a_1 + b_1 i$ end $z_2 = a_2 + b_2 i$ is a complex number $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$.

If we subtract from complex number $z_1 = a_1 + b_1 i$ a complex number $z_2 = a_2 + b_2 i$ we get $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i$.

To define a multiplication of complex numbers, we recall that $i^2 = -1$. Using this equation, we expand the product. For example

$$i(3 - 2i) - (3 + i)(5 + 3i)$$

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To define a multiplication of complex numbers, we recall that $i^2 = -1$. Using this equation, we expand the product. For example

$$i(3-2i) - (3+i)(5+3i)$$

$$= 3i - 2i^2 - (15 + 9i + 5i + 3i^2) = 3i + 2 - (12 + 14i) = -10 - 11i$$

Complex conjugate.

Definition. If $z = a + ib$ is any complex number, then the complex conjugate of z (also called the conjugate of z) is denoted by the symbol $\bar{z}$ and is defined by $\bar{z} = a - ib$.

For any complex numbers the following equations hold:
1) $\overline{(\bar{z})} = z$;
2) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$;
3) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$;
4) $Re\, z = \frac{1}{2}(z + \bar{z}), \quad Im\, z = \frac{1}{2i}(z - \bar{z})$.

Prove that for any complex number $\overline{(\bar{z})} = z$

The modulus of a complex number $z = a + ib$, denoted by $|z|$, is defined by $|z| = \sqrt{a^2 + b^2}$.

Division of Complex numbers.

If z_1 and z_2 are complex numbers and $z_2 \neq 0$ then there exists $\frac{z_1}{z_2}$ and it can be found as

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \frac{1}{|z_2|^2} z_1 \bar{z}_2.$$

For finding $\frac{z_1}{z_2}$ one has to multiply the numerator and denominator by the conjugate of the denominator.

Properties of complex numbers.

1) adding is commutative: $z_1 + z_2 = z_2 + z_1$,
2) adding is associative: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$,
3) there exists a zero element: 0 such that $z + 0 = 0 + z = z$,
4) exists an opposite element: $-z$ such that
$z + (-z) = (-z) + z = 0$,
5) multiplication is commutative: $z_1 z_2 = z_2 z_1$,
6) multiplication is associative: $(z_1 z_2) z_3 = z_1 (z_2 z_3)$,
7) the distribution laws hold:

$$\begin{cases} z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3, \\ (z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3, \end{cases}$$

8) exists a unit 1 such that $1z = z$,
9) every nonzero element z has an inverse: z^{-1} such that
$zz^{-1} = z^{-1} z = 1$

Prove that for any complex numbers z_1 and z_2

$$z_1 + z_2 = z_2 + z_1.$$

Geometric vectors

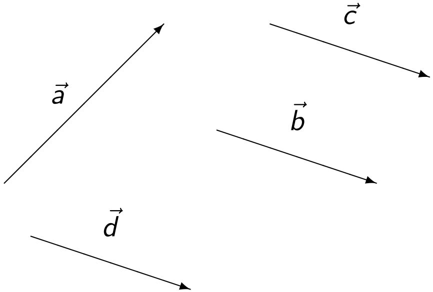


Figure 1. Vectors $\vec{b}$, $\vec{c}$ and $\vec{d}$ are equivalent.

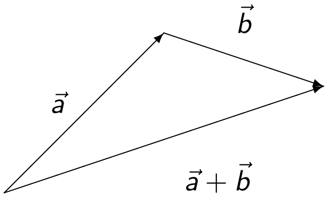


Figure 2. Finding the sum $\vec{a} + \vec{b}$.

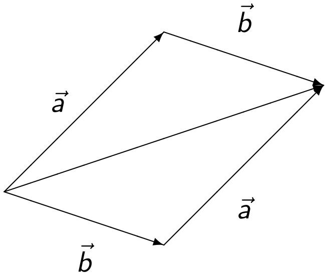


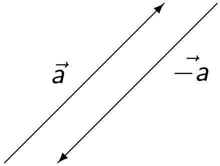
Figure 3. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.

The vector of length zero is called the zero vector and is denoted by $\vec{0}$. Since there is no natural direction for the zero vector, we shall agree that it can be assigned any direction that is convenient for the problem being considered.

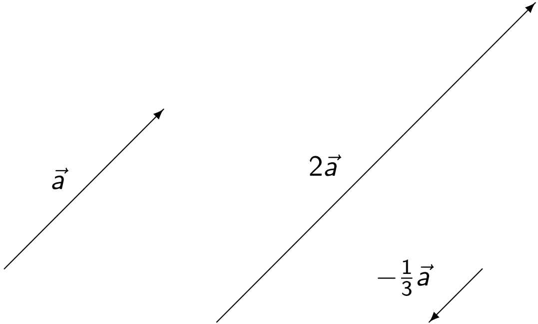
Zero vector has the property, that for every vector $\vec{a}$

$$\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}.$$

If $\vec{a}$ is any vector, then $-\vec{a}$, negative of $\vec{a}$, is defined to be the vector having the same magnitude as $\vec{a}$, but oppositely directed.



Any vector $\vec{a}$ has the property $\vec{a} + -\vec{a} = \vec{0}$. In addition $-\vec{0} = \vec{0}$.



Note that the vector $(-1)\vec{a}$ has the same length as $\vec{a}$, but is oppositely directed. Thus $(-1)\vec{a}$ is just the negative of $\vec{a}$, that is

$$(-1)\vec{a} = -\vec{a}.$$

Properties of geometric vectors

1) Commutative law for addition: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.
2) Associative law for addition

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}).$$

3) There exists of zero vector $\vec{0}$ so that

$$\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}.$$

4) For every vector $\vec{a}$ there exists an opposite vector $-\vec{a}$ such that

$$\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}.$$

5) Distributive law $a(\vec{a} + \vec{b}) = a\vec{a} + a\vec{b}$.
6) Distributive law $(a + b)\vec{a} = a\vec{a} + b\vec{a}$.
7) Associativity of scalar multiplication $a(b\vec{a}) = (ab)\vec{a}$.
8) Unitality $1\vec{a} = \vec{a}$.

Arithmetical vectors

n-dimensional arithmetical vector is a sequence of n numbers (we are dealing with real numbers)

$$\vec{a} = (a_1; a_2; \ldots, a_n) \quad ,$$

The numbers $a_1, a_2, \ldots$ are called the components of vector $\vec{a}$.
Two vectors $\vec{a} = (a_1; a_2; \ldots, a_n)$ and $\vec{b} = (b_1; b_2; \ldots, b_n)$ are called equal if $a_1 = b_1$, $a_2 = b_2$, $\ldots$, $a_n = b_n$.

Let $\vec{a} = (a_1; a_2; \ldots; a_n)$,$\vec{b} = (b_1; b_2; \ldots; b_n)$ then

$$\vec{a} + \vec{b} = (a_1 + b_1; a_2 + b_2; \ldots, a_n + b_n),$$

example:$(2; -1; 0; 5) + (-3; 9; 7; -5) = (-1; 8; 7; 0),$

$$a\vec{a} = (aa_1; aa_2; \ldots; aa_n),$$

example: $7(2; -1; 0; 5) = (14; -7; 0; 35).$

The zero vector in is $\vec{0} = (0; 0; \ldots; 0),$
the negative of $\vec{a} = (a_1; a_2; \ldots; a_n)$ is $-\vec{a} = (-a_1; -a_2; \ldots; -a_n).$

Properties of arithmetical vectors

1) Commutative law for addition: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.
2) Associative law for addition

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}).$$

3) There exists of zero vector $\vec{0}$ so that

$$\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}.$$

4) For every vector $\vec{a}$ there exists an opposite vector $-\vec{a}$ such that

$$\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}.$$

5) Distributive law $a(\vec{a} + \vec{b}) = a\vec{a} + a\vec{b}$.
6) Distributive law $(a + b)\vec{a} = a\vec{a} + b\vec{a}$.
7) Associativity of scalar multiplication $a(b\vec{a}) = (ab)\vec{a}$.
8) Unitality $1\vec{a} = \vec{a}$.

The Euclidean inner product of arithmetical vectors

The Euclidean inner product $\vec{u} \cdot \vec{v}$ of arithmetical vectors $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{v} = (v_1, \ldots, v_n)$ is the number defined by

$$\vec{u} \cdot \vec{v} = u_1 v_1 + \cdots + u_n v_n = \sum_{k=1}^{n} u_k v_k.$$

For example the Euclidean inner product of vectors $\vec{u} = (1, -2, 0, 5)$ and $\vec{v} = (2, 1, 5, 0)$ is

$$\vec{u} \cdot \vec{v} = 1 \cdot 2 + (-2) \cdot 1 + 0 \cdot 5 + 5 \cdot 0 = 0.$$

We see that the Euclidean inner product may equal to zero even if both vectors are nonzero.

Complex numbers as ordered pairs of real numbers.

If we know the real numbers a and b, we know the complex number we are dealing with. Thus a complex number $z = a + bi$ can be regarded as an ordered pair of real numbers $(a; b)$. Geometrically, this can be viewed either as a point or a vector in the xy-plane.

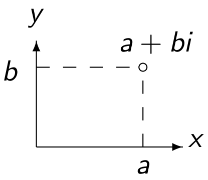


Figure 1.
Complex number as a point

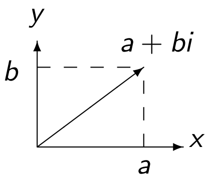


Figure 2.
Complex number as a vector

Complex numbers as points and as vectors on the Complex Plain

When complex numbers are represented geometrically in an xy-coordinate system, the x-axis is called the real axis, the y-axis is called the imaginary axis, and the plane is called the complex plane.

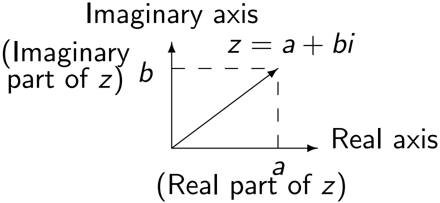


Figure 3. Complex number in the complex plane.

Polar form of a complex number.

If $z = a + bi$ is a nonzero complex number regarded on a complex plane, then $r = \sqrt{a^2 + b^2}$ is the length of vector z.

Obviously $r = |z|$.

Geometrically, the modulus of a complex number is the length of the vector on the complex plane.

φ measures the angle from the positive real axis to the vector z.

The angle φ is called an argument of z and is denoted by $\varphi = Argz$.

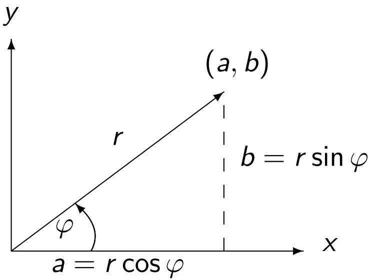


Figure 4. The modulus and argument of the complex number.

The following equalities hold for the argument of complex number:

$$cos\varphi = \frac{a}{r}, \quad sin\varphi = \frac{b}{r}$$

or $a = rcos\varphi$, $b = rsin\varphi$, so that $z = a + bi$ can be written as

$$z = r(cos\varphi + isin\varphi).$$

This is called a polar form of z.
For every nonzero complex number, we can find a pair of real numbers (r, φ) so that r is the modulus and φ is the argument of z
For complex number $z = 0$ we assume $r = 0$ and argument of z is not defined.

Theorem. Two nonzero complex numbers

$$z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$$

and

$$z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$$

are equal if

$$r_1 = r_2$$

and

$$\varphi_1 + \varphi_2 = 2k\pi \qquad (k \in \mathbb{Z}).$$

The argument of z is not uniquely determined because we can add or subtract any $2k\pi$, where k is an integer, to produce another value of the argument. However, there is only one value of the argument in radians that satisfies $-\pi < \varphi \leq \pi$. This is called the principal argument of z and is denoted by $\varphi = \mathit{argz}$. Any other argument of the same complex number can be produced as

$$\mathit{Argz} = \mathit{argz} + 2k\pi, \quad k \in \mathbb{Z}.$$

So for every nonzero complex number we can uniquely determine the values of r and $\varphi = \mathit{argz}$.

For example if $z = 2 + 2i$ then $r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$. To determine φ we use the figure 5 to obtain that $\varphi = \frac{\pi}{4}$ and

$$2 + 2i = 2\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}).$$

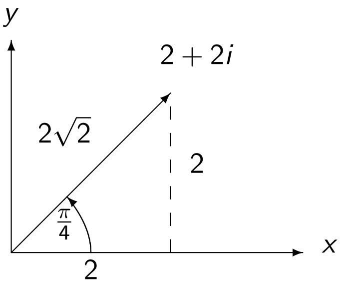


Figure 5. Finding the argument of complex number $z_1 = 2 + 2i$.

If $z = -1$, then $r = |z| = \sqrt{(-1)^2 + 0^2} = 1$, using the figure 6 we obtain $\varphi = \pi$, so $-1 = 1(\cos \pi + i \sin \pi)$.

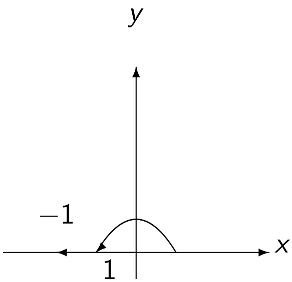


Figure 6. Finding the argument of complex number $z_2 = -1$.

The principal value of a nonzero complex number $z = a + ib$ can be calculated by the following formula:

$$\varphi = \begin{cases} \arctan \frac{b}{a}, & \text{if} \quad a > 0, \\ \frac{\pi}{2}, & \text{if} \quad a = 0, b > 0, \\ \arctan \frac{b}{a} + \pi, & \text{if} \quad a < 0, b > 0 \\ -\frac{\pi}{2}, & \text{if} \quad a = 0, b < 0, \\ \arctan \frac{b}{a} - \pi, & \text{if} \quad a < 0, b < 0 \end{cases}$$

Complex conjugate of z in polar notation.

If $z = a + ib$ is any complex number, then the complex conjugate of z is $\bar{z} = a - ib$. Geometrically, the conjugate of z is the reflection of z about the real axis.

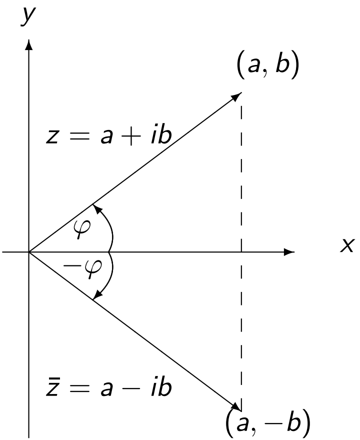


Figure 6. The complex conjugate on the complex plane.

If $z = r(\cos\varphi + i\sin\varphi)$ then $\bar{z} = r(\cos\varphi - i\sin\varphi)$.
Recalling the trigonometric identities $\sin(-\varphi) = -\sin\varphi$ and $\cos(-\varphi) = \cos\varphi$ we may write that

$$\bar{z} = r(\cos(-\varphi) + i\sin(-\varphi))$$

Adding and subtracting using polar form of complex number. Because the operation of addition of a complex number parallel the addition for vectors in $\mathbb{R}^2$, the geometric interpretation holds for complex numbers.

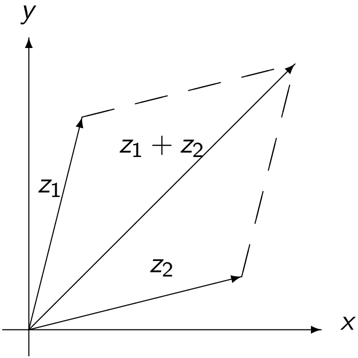


Figure 7. The sum of two complex numbers

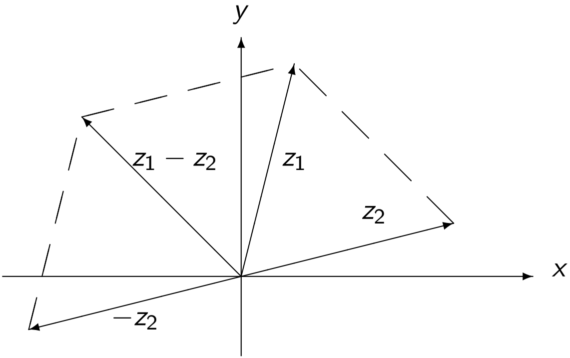


Figure 8. The difference of two complex numbers.

Multiplication and division using polar forms.

Let $z_1 = r_1(cos\varphi_1 + isin\varphi_1)$ and $z_2 = r_2(cos\varphi_2 + isin\varphi_2)$. Multiplying, we obtain

$$\begin{aligned} z_1 z_2 =& r_1(cos\varphi_1 + isin\varphi_{)} r_2(cos\varphi_2 + isin\varphi_2) \\ =& r_1 r_2((cos\varphi_1 cos\varphi_2 - sin\varphi_1 sin\varphi_2) + i(cos\varphi_1 sin\varphi_2 + sin\varphi_1 \cos\varphi_2)) \\ =& r_1 r_2(cos(\varphi_1 + \varphi_2) + isin(\varphi_1 + \varphi_2)) \end{aligned}$$

We have

$$z_1 z_2 = r_1 r_2(cos(\varphi_1 + \varphi_2) + isin(\varphi_1 + \varphi_2)),$$

This is the polar form of the complex number with modulus $r_1 r_2$ and argument $\varphi_1 + \varphi_2$. Thus, the product of two complex numbers is obtained by multiplying their moduli and adding their arguments. Similarly can be shown, that the quotient of two complex numbers is obtained by dividing their moduli and subtracting their arguments (in the appropriate order). This means the following equation holds (for $z_2 \neq 0$):

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(cos(\varphi_1 - \varphi_2) + isin(\varphi_1 - \varphi_2))$$

Powers of complex number.

Let n be positive integer and $z = r(\cos\varphi + i\sin\varphi)$, then

$$z^n = (r(cos\varphi + isin\varphi))^n = r^n(cos(n\varphi) + isin(n\varphi)).$$

In the special case where $r = 1$ we have $z = \cos\varphi + i\sin\varphi$ and the formula for powers becomes

$$\boxed{(cos\varphi + isin\varphi)^n = cos(n\varphi) + isin(n\varphi)}$$

which is called deMoivre's formula. This formula is used to calculate the sines and cosines of multiples of arguments.

For example for $n = 2$ we have

$$(cos\varphi + isin\varphi)^2 = cos(2\varphi) + isin(2\varphi).$$

On the other hand

$$\begin{aligned}(cos\varphi + isin\varphi)^2 =& cos^2\varphi + 2cos\varphi isin\varphi + i^2 sin^2\varphi = \\ =& (cos^2\varphi - sin^2\varphi) + i2cos\varphi sin\varphi\end{aligned}$$

While the left sides of the two equations are equal, the right hand sides also must be equal. Two complex numbers are equal, if the real parts are equal and their imaginary parts are equal, thus we have

$$\begin{aligned}cos(2\varphi) &= cos^2\varphi - sin^2\varphi \\ sin(2\varphi) &= 2sin\varphi cos\varphi\end{aligned}$$

Although we derived deMoivre's formula assuming n to be a positive integer, this formula is valid for all integers n.

Exercises

1. Prove the properties of complex numbers.
2. Show that $z = \bar{z}$ if and only if z is real.